SYNGE-WEINSTEIN THEOREMS IN RIEMANNIAN GEOMETRY

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Abstract. We give an exposition of the proof of a few results in global Riemannian geometry due to Synge and Weinstein using variations of the energy integral.

1. Introduction

One of the big refrains of modern Riemannian geometry is that curvature determines topology. Recall, for instance, the basic Cartan-Hadamard theorem that a complete, simply connected Riemannian manifold of nonnegative curvature is diffeomorphic to $\mathbb{R}^n$ under the exponential map. We proved this basically by showing that $\exp_p$ is nonsingular under the hypothesis of nonnegative curvature (using Jacobi fields) and that it was thus a covering map (the latter part was relatively easy). More difficult, and relevant to the present topic, was the Bonnet-Myers theorem, which asserted the *compactness* of a complete Riemannian manifold with bounded-below, positive Ricci curvature. The proof there showed that a long enough geodesic could not minimize energy (by using the second variation formula—recall that the second variation formula is intimately connected with curvature), and therefore could not minimize length. Since the distance between two points in a complete Riemanninan manifold is the length of the shortest geodesic between them (Hopf-Rinow!), this implied a bound on the diameter.

Today, however, we're going to assume at the outset that the manifold in question is already compact. One of the theorems will be that *a compact, even-dimensional orientable manifold of positive curvature is simply connected*. In particular, there is no metric of everywhere positive sectional curvature on the torus $\mathbb{T}^2$.

How will we do this? Well, first consider the universal cover $\tilde{M} \to M$. The covering transformations of $\tilde{M}$ are all smooth, and we can endow $\tilde{M}$ with a metric in a natural way such that these are isometries, and $\tilde{M}$ has positive curvature—hence, by comppleteness (a covering manifold of a complete manifold is also complete, easy exercise) and the Bonnet-Myers theorem, $\tilde{M}$ is compact. It is also orientable since we can pull back the M-orientation. If M is not simply connected, then we can find a nontrivial covering transformation $f : \tilde{M} \to \tilde{M}$.

But, we will show, using the that an *isometry of a compact, oriented, even-dimensional manifold* admits a fixed point. In particular, f does, which means that it is the identity, contradiction.

2. The statement

We will now begin work on the more general fixed-point theorem.

So, we're going to start with a compact oriented n-dimensional Riemannian manifold M of positive sectional curvature and an isometry $f : M \to M$.

Theorem 1 (Weinstein). *Suppose M is as above and f preserves orientation if n is even and reverses orientation if n is odd. Then f has a fixed point.*

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The hypothesis about the dimension seems a little odd, but it comes from linear algebra used in the proof.

3. The strategy

Here is the strategy. Let $d : M \times M \to \mathbb{R}_{\geq 0}$ be the metric. By compactness, there is $p \in M$ such that $d(p, f(p))$ is minimal. Assuimng this minimum is nonzero, we consider the minimal geodesic γ from p to $f(p)$ and construct a variation γ_s of it joining points $p_s \to f(p_s)$. By its construction and the second variation formula, we will show that $E(\gamma_s) < E(\gamma)$ for s small, which contradicts minimality.

So, how are we going to whisk this variation out of thin air? We will construct a parallel vector field V on γ, perpendicular to γ', and let

$$\gamma_s(t) = \exp_{\gamma(t)}(sV(t)).$$

In order that γ_s connects $p_s := \gamma_s(0) = \exp_p(sV(0))$ to $f(p_s)$, we need $f_*(V(0)) = V(1)$ (assuming γ is parametrized by $[0, 1]$).

4. Construction of the vector field

Proposition 1. *There exists a parallel vector field V on γ, perpendicular to γ', such that $f_*(V(0)) = V(1)$.*

The first step, paradoxically enough, will be to prove that γ' itself satisfies these conditions (except orthogonality), in other words that:

Lemma 1. $f_*(\gamma'(0)) = \gamma'(1)$.

Proof. Now $f \circ \gamma$ is a geodesic starting at $f(p)$, and if we show that the piecewise smooth broken geodesic $c = \gamma + f \circ \gamma$ (concatenation) is actually smooth, we will have established the first step.

Pick some point p^* in the middle of γ. Then $d(p^*, f(p^*)) \geq d(p, f(p))$. But there is a path c_0 from p^* to $f(p^*)$ of the same length $d(p, f(p))$, namely c traversed starting at p^* to $f(p^*)$. For instance, we could take $p^* = \gamma(0.5)$ and then traverse the curve c from 0.5 to 1.5, for a total distance of $||\gamma'(0)|| = \text{length}(\gamma) = d(p^*, f(p^*))$. This means that c_0 is smooth, hence so is c; the only point in doubt was at $t = 1$. In particular the left and right-hand derivatives match, so $f_*(\gamma'(0)) = \gamma'(1)$. □

Proof of the proposition. There was, in fact, method to this madness. We are now going to use this fact and linear algebra to construct the vector field V. So, the goal is to find some vector $V_0 \in T_p(M)$ such that the transformation $T : T_p(M) \to T_p(M)$ obtained by first applying f_* (and sending to $T_{f(p)}(M)$) and then parallel translating back along γ has an eigenvector perpendicular to $\gamma'(0)$—which we just proved is a fixed point. Then the parallel field extending V_0 can be taken as our V, which proves the lemma.

Now consider the subspace $W = \{\gamma'(0)\}^\perp \subset T_p(M)$. Now T is an isometry so fixes W, and W is of dimension one smaller. Also T (and hence $T|_W$) preserves (resp. reverses) orientation if $\dim W = n-1$ is odd (resp. even). By now invoking the following result from linear algebra, such a vector falls into our lap. □

Lemma 2 (Linear algebra)**.** *Let $T : W \to W$ be an orthogonal linear transformation of a real vector space W. Suppose A fixes orientation if $\dim W$ is odd and reverses it if $\dim W$ is even. Then T has a nontrivial fixed point.*

This will be proved later (in the appendix). Anyway, we now can use Proposition 1.

5. The second variation formula

5.1. **The approach.** Recall that we have defined the variation $\gamma_s(t) = \exp_{\gamma(t)}(sV(t))$; by what has been discussed, $f(\gamma_s(0)) = \gamma_s(1)$ for all s. In particular, we have paths between p_s and $f(p_s)$. Recall also the *energy* $E(c) = \int \langle c', c' \rangle$ of a piecewise-smooth path c; we shall use this in the sequel because it is easier to work with than the length (which has annoying square roots). Now

$$\frac{d}{ds} E(\gamma_s) = 0$$

because $E(\gamma_s)$ has a minimum at $s = 0$. Indeed, $E(\gamma) = d(p, f(p))^2$—since $\gamma = \gamma_0$ moves at constant speed, being a geodesic—and $E(\gamma_s) \geq d(p_s, f(p_s))^2$ by Schwarz's inequality. When we prove

$$\frac{d^2}{ds^2} E(\gamma_s) < 0$$

it will follow that there is some $s \neq 0$ small with $p_s \neq p$ but

$$\boxed{d(p_s, f(p_s))^2 \leq E(\gamma_s) < E(\gamma) = d(p, f(p)),}$$

contradiction.

5.2. **Proof of the variation formula.** First, let us recall a more general version of the second variation formula and a sketch of the proof. Let $\gamma : [0,1] \to M$ be a geodesic, γ_s a smooth variation of γ (*not necessarily fixing endpoints*) with variation vector field $V = \frac{\partial E}{\partial s}|_{s=0}$. Then

$$\frac{1}{2} E'(s) = \int \langle \frac{D}{ds} \frac{d}{dt} \gamma_s, \frac{d}{dt} \gamma_s \rangle = \int \langle \frac{D}{dt} \frac{d}{ds} \gamma_s, \frac{d}{dt} \gamma_s \rangle$$

This becomes (where, by abuse of notation γ' denotes differentiation w.r.t. t)

$$\frac{1}{2} \frac{d}{ds} E(s) = \int \frac{d}{dt} \langle \frac{d}{ds} \gamma_s, \frac{d}{dt} \gamma_s \rangle - \int \langle \frac{d}{ds} \gamma_s, \frac{D^2}{dt^2} \gamma_s \rangle$$

i.e.

$$\boxed{\frac{1}{2} \frac{d}{ds} E(s) = \langle \frac{d}{ds} \gamma_s, \gamma_s' \rangle_0^1 - \int \langle \frac{d}{ds} \gamma_s, \frac{D^2}{dt^2} \gamma_s \rangle.}$$

Differentiate with respect to s again:

$$\frac{1}{2} \frac{d^2}{ds^2} E(s) = \langle \frac{D^2}{ds^2} \gamma_s, \gamma_s' \rangle_0^1 + \langle \frac{d}{ds} \gamma_s, \frac{D}{dt} \frac{d}{ds} \gamma_s \rangle_0^1 - \frac{d}{ds} \int \langle \frac{d}{ds} \gamma_s, \frac{D^2}{dt^2} \gamma_s \rangle.$$

We shall now analyze each term separately. The first two terms become

$$\langle \frac{D^2}{ds^2} \gamma_s |_{s=0}, \gamma' \rangle_0^1 + \langle V, \frac{DV}{dt} \rangle_0^1.$$

The last term becomes

$$- \int \langle \frac{D^2}{ds^2} \gamma_s, \frac{D^2}{dt^2} \gamma_s \rangle - \int \langle \frac{d}{ds} \gamma_s, \frac{D}{ds} \frac{D}{dt} \gamma_s' \rangle$$

Since γ_0 is a geodesic, evaluation at $s = 0$ of this yields

$$- \int \langle V, \frac{D}{ds} \frac{D}{dt} \gamma_s' \rangle = - \int \langle V, \frac{D}{dt} \frac{D}{ds} \gamma' \rangle - \int \langle V, R(\gamma', V, \gamma') \rangle$$

which in total yields

$$\boxed{\frac{1}{2}\frac{d^2}{ds^2}E(s)|_{s=0} = -\int \langle V, V'' - R(\gamma', V)\gamma'\rangle + \langle \frac{D^2}{ds^2}\gamma_s, \gamma_s'\rangle_0^1 + \langle \frac{d}{ds}\gamma_s, V'\rangle_0^1.}$$

This is the version of the second variation formula that we shall use.

6. Computation of the variation

Now let's apply the formula to the γ_s constructed in the proof of Weinstein's theorem. Fortunately, most of the mess clears up. By parallelism, $V' = V'' = 0$, so all that we are left with is

$$\frac{1}{2}\frac{d^2}{ds^2}E(s)|_{s=0} = \int \langle V, R(\gamma', V)\gamma') = -||\gamma'|| \int K(\gamma(t), \text{span}\{\gamma'(t), V(t)\})dt < 0$$

by hypothesis on the sectional curvature and since γ', V are orthogonal. It now follows, as discussed previously, that $d(p, f(p))$ is not minimal, which gives a contradiction.

7. Consequences

Theorem 2 (Synge)**.** *Let M be a compact n-dimensional Riemannian manifold of positive curvature.*

(1) *If n is even and M is oriented, then M is simply connected.*
(2) *If n is odd, then M is orientable.*

Proof. We have already discussed case a) in the introduction. In case b), if M is not orientable, then there is an orientable *double cover* $\tilde{M} \to M$. The manifold $\tilde{M}$ is compact, has an induced Riemannian metric of positive curvature, and has an orientation-reversing covering transformation f when considered as a convering space of M. This transformation f must thus have no fixed points, which contradicts Weinstein's theorem. □

8. Appendix: Proof of the linear algebra lemma

For convenience, we restate the lemma:

Let $T : W \to W$ be an orthogonal linear transformation of W. Suppose A fixes orientation if $\dim W$ *is odd and reverses it if* $\dim W$ *is even. Then T has a nontrivial fixed point.*

Proof. First, in either case, the nonreal eigenvalues of A occur in conjugate pairs, so the product of nonreal eigenvalues is positive. All the real eigenvalues are ± 1 since T is orthogonal.

(1) $\dim W$ is odd. Then $\det A = 1$ and A has an odd number of real eigenvalues; they thus cannot all be -1.
(2) $\dim W$ is even. Then $\det A = -1$ and A has an even number of real eigenvalues; they thus cannot all be -1.

□

References

[1] Manfredo do Carmo. *Riemannian Geometry*. Birkhauser, 1992.

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