Journal of Mathematical Sciences and Applications, 2016, Vol. 4, No. 1, 4-13
Available online at http://pubs.sciepub.com/jmsa/4/1/2

DOI:10.12691/jmsa-4-1-2

On Logarithmic Prequantization of Logarithmic Poisson Manifolds

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Abstract In this article, we are going to introduce and study a new class of differential manifold, called logarithmic Poisson Manifold. We also introduced the notion of Logarithmic Poisson-Lichnerowicz cohomology and applied it to the study of prequantization of logarithmic-Poisson structures.

Keywords: *logarithmic Poisson Manifold,Logarithmic Poisson-Lichnerowicz cohomology, prequantization, logarithmic-Poisson structures*

Cite This Article: Joseph Dongho, and Shuntah Roland Yotcha, "On Logarithmic Prequantization of Logarithmic Poisson Manifolds." *Journal of Mathematical Sciences and Applications*, vol. 4, no. 1 (2016): 4-13. doi: 10.12691/jmsa-4-1-2.

1. Introduction

Symplectic geometry was discovered in 1780 by Joseph Louis Lagrange when he considered the orbital element of planet in solar system as non constants variables and defined the brackets of two such elements. From symplectic manifold, Poisson defined his brackets as tool for classical dynamics. Charles Gustave Jacob Jacobie realized the importance of those bracket and elucidated their algebraic property. Sophus Lie and other authors began the study of their geometry. Connection of Poisson geometry with members of areas including harmonic analytic, mechanics of particles and continua; completely integrable systems, justify his recent development. It is interested to recall that numbers of proprieties and results in this theory was developed in the case of differential manifold. Too few authors have worked in the case of singular varieties. Recently, Huebschmann in[5] study the algebraic point of view; A. Polishchuk in [7] study the Poisson brackets in algebraic framework.

In 2002, Ryushi Goto, with the aim of generalised the approach to the symplectic Atiyah class to the construction of the invariants of knots defined the Log Symplectic manifold and studied several examples in [2].

In the first section of this paper, we recall the concepts of logarithmic derivation and logarithmic form, as well as free divisor, all of them due to Kyogi Saito [1] and the definition of Log Symplectic manifold. We end the section with the notion of Log Poisson brackets on such varieties.

In the second part, we define the concept of Logarithmic multi-vector field. In the aim to define the notion of Log Poisson manifold the section will end with the notion of Log Schuten brackets.

In the third part, we define the notion of Log Poisson manifold and give some main examples. We introduce the Lichenerowicz Log Poisson cohomology and Log Poisson Chern class of a log complex line bundle over Log Poisson manifold; using the concept of contravariant log derivative who generalize the concept of Contravariant derivative given by I. Vaisman in [6].

Section fourth is devoted to the formulation of the integrality Log prequantization condition of Log Poisson manifold.

2. Log-Symplectic Poisson Structure

In this section, we recall the notion of Log-symplectic manifold introduce in 2002 by [2] and we construct Poisson structure induce by a Log symplectic structure.

2.1. Log symplectic manifold. Recall that; a logarithmic q-forme is a multilineal skew-symmetric function

$$\omega : T_M(-\log D)\times T_M(-\log D)\times ...T_M(-\log D)\to \mathcal{O}_M$$
$$\left(\delta_1;...;\delta_q\right)\mapsto \omega\left(\delta_1;...;\delta_q\right)$$

A logarithmic 0-form are germs of $\mathcal{O}_M$. Sheaf of germs of log form $\Omega_M^*(\log D)$ are constructed for singular or normal crossing divisor by [1]. The coherent sheaf $\Omega_M^1(\log D)$ is the dual of a sheaf of germs of logarithmic vector fields $T_M(-\log D)$. Hence $\Omega_M^1(\log D)$ is reflexive, i.e., normal and torsion-free. If δ is local section of $T_M(-\log D)$, then $\delta h\in h\mathcal{O}_M$ where $D=\{h=0\}$. The sheaf Ω_M^* of smooth differential forms is subsheaf of $\Omega_M^*(\log D)$. The following will be very useful in what follow.

Lemma 2.1. *$T_M(-\log D)$ is Lie subalgebra of $Der_{\mathbb{C}}(\mathcal{O}_M)$*

Proof. It is enough to prove that $T_M(-\log D)$ is closed with Lie bracket of vector field. Let $\delta_1;\delta_2$ be two logarithmic vectors field along D. For all $f \in \mathcal{O}_M$, $[\delta_1;\delta_2]f = \delta_1(\delta_2.f) - \delta_2(\delta_1.f)$. Therefor:

$$\begin{aligned}[\delta_1;\delta_2]h &= \delta_1(\delta_2.h) - \delta_2(\delta_1.h)\\ &= \delta_1(ah) - \delta_2(bh)\\ &= h\delta_1(a) + a\delta_1(h) - h\delta_2(b) - b\delta_2(h)\\ &= h(\delta_1(a) + aa' - \delta_2(b) - bb') \in (h)\mathcal{O}_M\end{aligned}$$

The lemma allows us to define the following mapping $d:\Omega_M^*(\log D) \to \Omega_M^*(\log D)$ by

$$\begin{aligned}d\omega(a_1;\ldots,a_{q+1}) &= \sum_{i=1}(-1)^{i+1} a_i\omega(a_1,\ldots;\breve{a}_i;\ldots a_{q+1})\\ &+\sum_{i<j}(-1)^{i+j}\omega([a_i;a_j],a_1,\ldots,\breve{a}_i,\ldots,\breve{a}_j,\ldots a_{q+1})\end{aligned} \tag{2.1}$$

It is easy to prove that $d^2 = 0$.

In particular, for all f; germs of $\mathcal{O}_M$, $df: T_M(-\log D) \to \mathcal{O}_M$ is $\mathcal{O}_M$ -linear map on $T_X(-\log D)$ with values in $\mathcal{O}_M$. For all germ δ of $T_M(\log D)$, $df(\delta) = \delta \bullet f$; where $\bullet$ is locally $T_M(-\log D)$-module structure on $\mathcal{O}_M$. If $\xi \in \Omega(\log D)$ i.e. $\xi: T_M(-\log D) \to \mathcal{O}_M$ $\mathcal{O}_M$ -linear, then:

$$(d\xi)(a_1;a_2) = a_1\xi(a_2) - a_2\xi(a_1) - \xi([a_1;a_2]).$$

In this paper, $df := \delta \bullet f$. A logarithmic form ω is close if $d\omega = 0$.

Definition 2.2. *The complex $(\Omega_M^*(\log D), d)$ is logarithmic De Rham complex. The corresponding cohomology is called logarithmic De Rham cohomology and denoted $H_{dR-\log}(M)$*

The locally $T_M(-\log D)$ -module structure of $\mathcal{O}_M$ could be extended to $\Omega_M^*(\log D)$ as follow:

for all $a \in T_M(-\log D)$, $m \in \mathcal{O}_M$, $\mathcal{L}_a m = am$ and $(\mathcal{L}_a\xi)(x) = a\xi(x) - \xi([a;x])$ for all $\xi \in \Omega_M^*(\log D)$ and $x \in T_M(-\log D)$.

More generally, for all a, $\mathcal{L}_a = \iota_a \circ d + d \circ \iota_a$ where ι is contraction of logarithmic form by logarithmic vector field.

A simple calculation give us

$$\mathcal{L}_a\mathcal{L}_b - \mathcal{L}_b\mathcal{L}_a = \mathcal{L}_{[a;b]}. \tag{2.2}$$

It follow that $\mathcal{L}: T_M(-\log D) \to End_{\mathcal{O}_M}(\Omega_M^*(\log D))$ is Lie homomorphism. In [1] it is prove that each germs ω of $\Omega_M^q(\log D)$ have the form $\omega = \frac{dh}{h} \wedge \eta + \xi$ where η, ξ are smooth forms. η is called residue form of ω and denoted $Res_{D_{red}}(\omega)$. Since $T_M(-\log D)$ is the dual of the $\Omega_M^1(\log D)$, there exist an isomorphism

$$\lambda: T_M(-\log D) \cong \mathcal{H}om_{\mathcal{O}_M}(\Omega_M^1(\log D);\mathcal{O}_M).$$

λ induce an isomorphism

$$\begin{aligned}\lambda: \mathcal{X}_M^q(-\log D) &= \overset{q}{\wedge} T_M(-\log D)\\ &\cong \mathcal{H}om_{\mathcal{O}_M}\left(\overset{q}{\wedge}\Omega_M^1(-\log D;\mathcal{O}_M)\right).\end{aligned}$$

Definition 2.3. *The sheaf $\mathcal{X}_M^*(-\log D)$ is called sheaf of logarithmic $*$-vector field.*

Definition 2.4. [2] *A Log symplectic manifold is a triple (M, D, ω) where M is complex manifold, D is reduced divisor of M and ω is logarithmic 2-form satisfy*:

$$d\omega = 0 \tag{2.3}$$

$$\overbrace{\omega \wedge \ldots \wedge \omega}^{n\ times} \neq 0 \in H^{2n}(M, \Omega^{2n}([D])) \tag{2.4}$$

The singular part of reduced divisor D is denoted by D_{sing}. Then on the smooth part $D_{red} = D - D_{sing}$ we have the residue 1 form $\eta = Res_{D_{red}}\omega \in \Omega_{D_{red}}^1$. We also have a log generalization of Darboux's theorem:

Lemma 2.5. [2]*(Log DARBOUX THEOREM).*

Let (M, D, ω) be a log symplectic manifold. There exist holomorphic coordinates $(z_0,\ldots,z_{2n-1})$ of a neighborhood of each smooth point of D_{red} such that ω is given by

$$\omega = \frac{dz_0}{z_0} \wedge dz_1 + dz_2 \wedge dz_3 + \ldots + dz_{2n-2} \wedge dz_{2n-1} \tag{2.5}$$

where $\{z_0 = 0\} = D$. We refer to those coordinates as log Darboux coordinates.

In log Darboux coordinates, $Res(\omega) = dz_1$ which is closed. Here, we have an integrable and $2n-2$ dimensional leaves on $D_{red}: \{v \in TD_{red} \mid \eta(v) = 0\}$. We have another useful lemma:

Lemma 2.6. [2] *The log symplectic form ω defines a symplectic structure on each leaf.*

2.2. Examples of log symplectic manifold. In litterature, there exist more examples of such manifold we refer the reader to [2] for some one.

(1) [2] Let X be a complex surface with a reduced divisor D. If the class $[D]$ is anticanonical class $\mathcal{K}_X^*$, then the triple $(X, D;\omega)$ is log symplectic manifold with log simplectic form $\omega \in K([D])$.

(2) On Toda lattice of dimension two $P = \{(a,b) \in \mathbb{R}^4, a^i > 0, i = 1;2\}$ the symplectic form $\omega = -\frac{da_1}{a_1} \wedge db_1 - \frac{da_2}{a_2} \wedge db_2$.

Induced on $\hat{P} := \left\{ (a,b) \in \mathbb{R}^4, a^i \geq 0, i = 1;2 \right\}$ a structure of log symplectic manifold

2.3. Hamiltonian and log Hamiltonian vector field. The important role played by Hamiltinian vector field and the corresponding Hamiltonian in classical mechanic force us to answer the question that is it possible to define the notion of Hamiltonian vector field on symplectic manifold? The fact that such manifold are not smooth, complicate concept of vector field; seem tangent vector that do not always exist at each point.

2.3.1. *General construction of symplectic Poisson structure*. Begin by recalling the general concept of Hamiltonian operator. The reader could refer to[9] for more explanation. Let ϑ a Lie algebra with Lie structure $[a,b] = ab - ba$ (the commutator) and $\mathcal{M}$ an ϑ-module. If ω is a fixed 2-form, In the space $\vartheta \oplus \mathcal{M}$, consider the subspace

$$\mathcal{G} = \left\{ (a,m) : \omega(a,x) = -xm, \ for\ all\ x \in \vartheta \right\}$$

Introduce on $\vartheta \oplus \mathcal{M}$ the following operations:

$$\left[(a_1,m_1),(a_2,m_2) \right]_1 = \left([a_1,a_2], \frac{1}{2}(a_1 m_2 - a_2 m_1) \right) \quad (2.6)$$

$$\left[(a_1,m_1),(a_2,m_2) \right]_2 = \left([a_1,a_2], \omega(a_1,a_2) \right) \quad (2.7)$$

On $\mathcal{G}, [,]_1 = [,]_2 =: [,]$. The following theorem prove that when ω is closed, $(\mathcal{G},[,])$ is Lie algebra.

Theorem 2.7. [9] *If* $d\omega = 0$, *then* $\mathcal{G}$ *is closed relative to* $[,]$, *and hence a Lie algebra with respect to it.*

By construction, we have the following commutative diagrams.

$$\begin{tikzcd}
& \vartheta & \\
\mathcal{G} \arrow[ur, "p_1"] \arrow[r, hook, "i"] \arrow[dr, "p_2"'] & \vartheta \bigoplus \mathcal{M} \arrow[u, "q_1"'] \arrow[d, "q_2"] & \\
& \mathcal{M} &
\end{tikzcd}$$

We denote $N := p_2(\mathcal{G})$. By definition, $m \in N$ there exist $a \in \vartheta$ such that $(a,m) \in \mathcal{G}$ i.e., for all $x \in \vartheta$, $\omega(a,x) = -xm$.

We recall that for all $a \in \vartheta$, $\iota_a \omega = \omega(a,-) : \vartheta \to \mathcal{M}$. Then, for all $x \in \vartheta, \iota_a \omega(x) = \omega(a,x)$. Therefor, $m \in N$ $\iota_a \omega(x) = -xm$ for all $x \in \vartheta$.

Remark 2.8.

(1) If $a, a' \in \vartheta$ and $\iota_a \omega(x) = \iota_{a'} \omega(x)$ for all $x \in \vartheta$, then $\omega(a - a', x) = 0$ for all $x \in \vartheta$. If ω is non degenerated, (it is the case for log symplectic structure), $a = a'$

(2) Let $m_1, m_2 \in N$. Chose $a_1, a_2 \in \vartheta$ such that $(a_1,m_1),(a_2,m_2) \in \mathcal{G}$

We set

$$\{m_1, m_2\}_\omega = \omega(a_1, a_2) \quad (2.8)$$

Il $a_1, b \in \vartheta$ such that $(a_1,m_1),(b,m_1) \in \mathcal{G}$. Since $\mathcal{G}$ is claused under addition, $(a_1 - b, 0) \in \mathcal{G}$ i.e., $\omega(a_1 - b, a_2) = -a_2 0 = 0$ i.e., $\omega(a_1,a_2) = \omega(b,a_2)$. Therefor, $\{m_1,m_2\}_\omega = \omega(a_1,a_2) = \omega(b,a_2)$. This prove that the operation didn't depend on the choice of a_1 and a_2.

Corollary 2.9. *If* $d\omega = 0$ *then the operation* $\{,\}_\omega$ *is Poisson structure on N*

Definition 2.10. *The operation defined by equation (2.8) is called Poisson symplectic structure when* ω *is symplectic form.*

2.3.2. *Application to the construction of log symplectic Poisson structure*. *If* ω *is a log symplectic structure, then* $d\omega = 0$ *and* ω *is non degenerated. Consider* $Lg := \left\{ \log h_i^{k_i} f_i, f_i \in \mathcal{O}_M^*; k_i \in \mathbb{N}^*, \prod h_i^{k_i} = h \right\}$.

For all $g \in Lg, dg = k_i dlogh_i + d \log f_i \in \Omega_M^1(\log D)$. We consider $\mathcal{A}_D$ the sheaf of algebra generated by $Lg \cup \mathcal{O}_M$. We have the following lemma:

Lemma 2.11. $\mathcal{A}_D$ *is sheaf of* $T_M(-\log D)$*-module*

Proof. Let $m \in \mathcal{A}_D; dm \in \Omega_M^1(\log D) = \left(T_M(-\log D) \right)^*$ Then for all $\delta \in T_M(-\log D), dm(\delta) := \langle dm, \delta \rangle \in \mathcal{O}_M$. Define $* : T_M(-\log D) \times \mathcal{A}_D \to \mathcal{A}_D$ by $\delta * m := dm(\delta)$.

Set

$$\mathcal{P} := \left\{ \begin{array}{l} (a,m) \in T_M(\log D) \oplus \mathcal{A}_D; \omega(a,x) = -xm \\ for\ all\ x \in T_M(-\log D) \end{array} \right\}$$

The image of $\mathcal{P}$ by second projection map p_2 is $\mathcal{A}_D$. It follow from remark [2.8] that for all $m \in \mathcal{A}_D$ there exist $a \in T_M(\log D)$ such that $(a,m) \in \mathcal{P}$.

Let $m, n \in \mathcal{A}_D$ the corresponding operation defined by equation (2.8) is:

$$\{m,n\}_\omega = \omega(a,b) \quad (2.9)$$

Where a, b are such that $(a,m)(b,n) \in \mathcal{P}$.

From corollary [2.9], it come that the pairs $\left(\mathcal{A}_D, \{;\}_\omega \right)$ is Poisson algebra.

According to the definition on Poisson manifold, the log symplectic manifold (M, D, ω) become a Poisson manifold. Since the Poisson structure is coming from a symplectic structure, we will call this **log symplectic Poisson manifold**.

Definition 2.12. *The operation (2.9) is log symplectic Poisson structure and the algebra* $\mathcal{A}_D$ *is called log symplectic Poisson algebra.*

2.3.3. *Hamiltonian vector field and Hamiltonian operator*. In the last section, we have seen that for all $m \in \mathcal{A}_D$, there exist an unique $a \in T_M(-\log D)$ such that $(a,m) \in \mathcal{P}$. Therefore, we have the following definitions:

Definition 2.13. (1) *The pairs* $(a,m) \in \mathcal{P}$ *is called log Hamiltonian pairs associated to* ω

(2) *In each log Hamiltonian pairs* (a,m), *when* $m \in \mathcal{O}_M$, *a is called Hamiltonian vector field and m is his Hamiltonian*.

(3) *If* $m \in L_g$ *and* (a,m) *is log Hamiltonian pairs, then a is called log Hamiltonian vector field and m is his hamiltonian*.

Let (a,m), be a log Hamiltonian pairs. For all $x, \omega(a,x) = -xm := -dm(x)$. Then $\iota_a \omega = -dm$. Therefore, *a* is Hamiltonian vector field on *M* iff there exist $m \in \mathcal{O}_M$ such that $\iota_a \omega = -dm$. It is log Hamiltonian vector field on *M* iff there exist $f \in \mathcal{O}_M^*$ such that $f \iota_a \omega = -df$.

In the goal of defining the notion of Hamiltonian operator, set $\Omega(\log \mathcal{A}_D)$ the subsheaf of $\Omega_M^1(\log D)$ *d*-image of $\mathcal{A}_D$.

In general, if ϑ and $\mathcal{M}$ are at in section one, and if Ω^1 is the d-image of $\mathcal{M}$, then an linear operator $H : \Omega^1 \to \vartheta$ is saying skew-symmetric if for any $\xi_1, \xi_2 \in \Omega^1$,

$$\xi_1(H\xi_2) = -\xi_2(H\xi_1). \tag{2.10}$$

Let $H : \Omega^1 \to \vartheta$ be a skew-symmetric operator. Consider $\omega_H(a_1,a_2) = (H^1 a_2)(a_1)$, $a_1, a_2 \in ImH$. It's prove in [9] that ω_H is well define 2-form on ImH.

Definition 2.14. [9] *A skew-symmetric operator* $H : \Omega^1 \to \vartheta$ *is said to be Hamiltonian if*:

a) *ImH is subalgebra of Lie algebra #*

b) ω_H *is closed in* ImH .

Now, giving a log symplectic manifold (M,D,ω), $d\omega = 0$ and ω is non degenerated. Therefore, for all $a \in T_M(-\log D)$, $\omega(a,-) : T_M(-\log D) \to \mathcal{O}_M \subset \mathcal{A}_D$ which is an element of $\Omega_M^1(\log D)$.

Then ω induce an isomorphism $\varphi_\omega : T_M(-\log D) \to \Omega_M(\log D)$. We could state the following lemma.

Lemma 2.15. *The operator* $H := \varphi_\omega^{-1}$ *is an Hamiltonian operator from* $\Omega(-\log \mathcal{A}_D)$ *to* $T_M(-\log D)$.

Definition 2.16. *The map H in* [2.15] *is called log Hamiltonian operator*.

Remark 2.17.

For all $a \in T_M(-\log D), \varphi_\omega(a) := \omega(a,-)$ and for $\alpha \in \Omega(\log \mathcal{A}_D)$

$\omega(H(\alpha),-) = \alpha$ therefore, for all $m \in \mathcal{A}_D$, $\omega(Hdm, x) = -dm(x)$ for all $x \in T_M(-\log D)$. Consequently, for al $m \in \mathcal{O}_M$, Hdm is the Hamiltonian vector field of Hamiltonian m and for each $m \in L_p$, Hdm is the log Hamiltonian vector field associated to the log Hamiltonian m.

It follows that the log symplectic Poisson structure is defined by:

$$\{m,n\}_\omega = \omega(Hdm, Hdn) \tag{2.11}$$

2.4. Some properties of log symplectic Poisson structure.

Lemma 2.18. *The bracket* $\{-,-\}_\omega$ *is logarithmic derivation on each component*.

Proof. From lemma 2.5, at each point of D_{red}, there exist holomorphic coordinates $(z_0,\ldots,z_{2n-1})$ such that $\omega = \frac{dz_0}{z_0} \wedge dz_1 + dz_2 \wedge dz_3 + \ldots + dz_{2n-2} \wedge dz_{2n-1}$. For all $m \in \mathcal{A}_D$, we have by a simple calculate:

$$\begin{aligned} H(dm) &= z_0\left(\partial_{z_1} m \partial_{z_0} - \partial_{z_0} m \partial_{z_1}\right) \\ &\quad + \sum_{k=1}^{n-1} \partial_{z_{2k+1}} m \partial_{z_{2k}} - \partial_{z_{2K}} m \partial_{z_{2k+1}} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &(Hdm)\omega = \partial_{z_1} m d_{Z_0} \\ &-\sum_{k=1}^{n-1} \partial_{z_{2K+1}} m \partial_{z_{2k}} - \partial_{z_{2k}} m \partial_{z_{2K+1}} \end{aligned} \tag{2.13}$$

For $m = h = z_0, H(dH) = -z_0 \partial_{z_1}$. Then $H(d_{z_0})((Hdm)\omega) = z_0 \partial_{z_0} m.$

Therefore, $\{m,h\}_\omega = z_0 \partial_{z_0}$ for all $m \in \mathcal{O}_M$.

Corollary 2.19. *There exist holomorphic local coordinates* $(z_0,\ldots,z_{2n+1})$ *of each point of* D_{red} *such that*

$$\{m,n\}_\omega = \sum_{k=1}^{n-1} \partial_{z_{2K}} m \partial_{z_{2k+1}} n - \partial_{z_{2k+1}} m \partial_{z_{2k}} n + z_0 \begin{pmatrix} \partial_{z_0} m \partial_{z_1} n \\ -\partial_{z_1} m \partial_{z_0} n \end{pmatrix}$$

Corollary 2.20. *For each log symplectic Poisson structure* $\{-,-\}_\omega$ *Poisson* $\{-,-\}_\omega$ *there exist a unique* $\pi_\omega \in \overset{2}{\wedge} T_M(-\log D)$ *such that*

$$\textit{for all } m,n \in \mathcal{A}_D, \{m,n\}_\omega = \pi_\omega(dm,dn).$$

Definition 2.21. π_ω is log symplectic Poisson bivector.

If *A* and *B* are two matrix of the same dimension denote $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then the matrix of π_ω is

$$M_{\pi_\omega} = \begin{pmatrix} 0 & z_0 \\ -z_0 & 0 \end{pmatrix} \oplus \left(\bigoplus_{k-1}^{n-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

3. On a Generalization of Log Symplectic Poisson Bracket: Log Poisson Structure

3.1. Logarithmic multivector field. It is well knowed that the dual of sheaf $\Omega_M(\log D)$ is the sheaf $T_M(-\log D)$ of logarithmic vector field. Let $\mathcal{X}_M(-\log D)$ is the module of sections of $T_M(-\log D)$. $\mathcal{X}_M(-\log D)$ is Lie subalgebra of $\mathcal{X}_M$ which play an important role in the notion of Poison Manifold in the point of view of A. Lichenerowicz. In the goal of constructing the notion of Log Poisson Manifold, we shall define the notion of a multi vector field. First of all, we recall that for all $z \in M$ and $p \in N$ a logarithmic contravariant antisymmetric p-vector field is a *p*-linear map $\delta : \Omega_M^1(\log D) \times \ldots \times \Omega_M^1(\log D) \to \mathbb{C}$. The set of logarithmic contravariant antisymmetric *p*-vector field at a point $z \in M$ is denoted $\overset{P}{\wedge}(T_{M,z}(-\log D))$. We denote $T_M(-\log D) = \bigcup_{z \in M} \overset{P}{\wedge}(T_{M,z}(-\log D))$.

$T_M(-\log D)$ is subbundle of tangent bundle TM in the smooth case. We will called $T_M(-\log D)$ the logarithmic tangent bundle and we denoted $\mathcal{X}_M^p(-\log D)$ the module of his section. Element δ of $\mathcal{X}_M^p(-\log D)$ is $\mathcal{O}_M$ -linear combination of homogeny of degree *p*; i.e; $\delta = f^{i_1 \ldots ip} \wedge \ldots \wedge \delta_{i_p}$ with $\delta_i \in \mathcal{X}_M^1(-\log D)$.

Definition 3.1. Element of $\mathcal{X}_M^p(-\log D)$ are called log *p*-vector field.

Exemple 3.2.

(1) [10] Let M_n be a complex manifold of dimension n and D a normal crossing divisor of M define by $D = \{h = z_1 \ldots z_r = 0\}$ where $(z_1, \ldots, z_n)$ local system of holomorphic coordinates of M_n. The set $S_1 = \left\{ z_1\partial_{z_1}, \ldots, z_r\partial_{z_r}, \partial_{z_{r+1}, \ldots, }\partial_{z_n} \right\}$ form a basis of $\mathcal{X}_M^p(-\log D)$. From S1 we deduce a basis $S_2 = \{z_i z_j \partial_{z_i} \wedge \partial_{z_j}\}_{i<j\le r} \bigcup \{z_i \partial_{z_i} \wedge \partial_{z_j}\}_{i \le r; j > r} \bigcup \{\partial_{z_i} \wedge \partial_{z_j}\}_{i<j\le r}{}_{r<i<j; n-r>2}$ of $\mathcal{X}_M^2(-\log D)$. In the same way, we construct basis of $\mathcal{X}_M^p(-\log D)$.

(2) On $\mathbb{C}P^1$ denoted by D the closed sphere of radius 1. The bivector $\pi = -\frac{i}{2}(1 + z\bar{z}) z\bar{z} \partial_z \wedge \partial_z$ is log 2-vector field

we shall remark that if $(z_1, \ldots, z_n)$ is local coordinates of M and $(z_1, \ldots, z_r)$ is the corresponding local coordinates of divisor D, for all $p < n - r$ then $\mathcal{X}_{D_{red}}^p \neq \{0\}$. Therefore, we could then state the following lemma.

Lemma 3.3. *Let M be a complex manifold of dimension n and D a reduced divisor of M. For each* $r < p < n - r$ *and each local section* δ *of* $\mathcal{X}_M^p(-\log D)$, *there is* $\delta_1, \delta_2 \in \mathcal{X}_M^p(-\log D)$. *such that*:

(1) $\delta_1(x) = 0$ *for all* $x \in D$

(2) $\delta_2 \omega = 0$ *for all* $\omega \in \Omega_D^p$ D

(3) $\delta = \delta_1 + \delta_2$

Proof. Let $(z_1, \ldots, z_n)$ is local holomorphic coordinates of M and $(z_1, \ldots, z_r)$ is the corresponding local coordinates of divisor D. Set

$$S_1 = \left\{ z_1\partial_{z_1}, \ldots, z_r\partial_{z_r}, \partial_{z_{r+1}, \ldots, }\partial_{z_n} \right\}$$

and

$$S_p = \left\{ \delta_{i_1} \wedge \ldots \wedge \delta_{i_p} \right\}_{i_1 \ldots < i_p, \delta_{i_k} \in S_1}.$$

S_p is a generator of $\mathcal{X}_M^p(-\log D)$.

Since $r < p < n - r$ there is $i_{k_1}, \ldots, i_{k_p} \in \{r+1, \ldots, n\}$ such that

$\delta_{i_{k_1}} \wedge \ldots \wedge \delta_{i_{k_p}} \in S_p$. However, we know that there is family $f^{i_1 \ldots i_p}$ of holomorphic functions such that

$$\delta = f^{i_1 \ldots i_p} \delta_{i_1} \wedge \ldots \wedge \delta_{i_p}.$$

We set $\delta_2 = f^{i_{k_1}, \ldots, i_{k_p}} \delta_{i_{k_1}} \wedge \ldots \wedge \delta_{i_{k_p}}$ and $\delta_1 = \delta - \delta_2$.

Definition 3.4. *Let* δ *be a log p-vector field.*

(1) *if* $\delta = \delta_1 + \delta_2$ *then* δ_2 *is called smooth p-vector field associated to* δ

(2) δ *is saying log Euler along D if* $\delta_2 = 0$.

3.2. Logarithmic super-algebra of Lie. Always in the main fine the definition of log Poisson manifold using the Lichenerowicz methods, we shall construct on the algebra $(\mathcal{X}_M(-\log D), \wedge)$ a Lie structure. Of cause, we shall define the notion of logarithmic Schouten bracket on $(\mathcal{X}_M(-\log D), \wedge)$. To do that, we shall define an important notion of interior product of log vector field with logarithmic form. It is the main of the following definition.

Definition 3.5. *Let* ω *be a logarithmic* $(q+p)$*-form and* δ *a log p-vector field. The interior product* $i_\delta \omega$ *of* δ *with* ω *is define as follows*:

- *If* $p = 0$, *let* $i_\delta \omega = \delta \omega$
- *If* $p > 0$ *and* $\delta = \delta_1 \wedge \ldots \wedge \delta_p$ *where* $\delta_i \in \mathcal{X}_M^1(-\log D)$ *then*

$$(i_\delta \omega)(\delta_{p+1}, \ldots, \delta_{p+q}) = \omega(\delta_1, \ldots, \delta_{p+q}) \tag{3.1}$$

When $q=0$ and M is finial dimension manifold, the equation 3.1 is an isomorphism between $\mathcal{X}_M^p(-\log D)$ and $\Omega_M^p(\log D)$.

One check that

$$\mathrm{i}_{\delta\wedge\eta}\omega = \mathrm{i}_\eta \mathrm{i}_\delta \omega. \tag{3.2}$$

In the goal to construct an hyper Lie structure on the associative, supercomutative algebra $(\mathcal{X}_M(-\log D),\wedge)$ we recall the following useful proposition.

Proposition 3.6. *Let g be a Lie algebra. There is an unique bracket on $\wedge g$ which extend the Lie bracket on g and such that if $A\in\wedge^a g$, $B\wedge^a g$, $C\wedge^a g$ then,*

(1) $[A,B] = -(-1)^{a-1}(b-1)[B,A]$

(2) $[A,B\wedge B]\wedge C + (-1)^{(a-1)b} B\wedge[A,C]$

(3) $$(-1)^{a-1}(c-1)[A,[B,C]] + (-1)^{b-1}(a-1)\begin{bmatrix} B,[C,A] \\ +(-1)^{c-1}(b-1)C[A,B] \end{bmatrix}$$

in the above section, we have proven that $\mathcal{X}_M(-\log D)$ endowed with the Lie-Jacobi structure is Lie algebra; we can then apply the above theorem on it and its became a Gestenhaber algebra. The corresponding hyper Lie structure we be called Log Schouten bracket. Precisely, we can follow the usual prove of Schouten Theorem to prove what follow.

Theorem 3.7. ***(Log Schouten Bracket Theorem.)***

There is a unique bilinear operator $[-,-]_{\log}:\mathcal{X}_M^*(-\log D)\times\mathcal{X}_M^*(-\log D)\to\mathcal{X}_M^*(-\log D)$ such that

(1) *It is biderivation of degree -1, that is it is bilinear, $\deg[\delta_1,\delta_2]_{\log} = \deg\delta_1+\deg\delta_2-1$ and for all $\delta_1;\delta_2;\delta_3\in\mathcal{X}_M^*(-\log D)$,* $[\delta_1,\delta_2\wedge\delta_2]_{\log} = [\delta_1,\delta_2]_{\log}\wedge\delta_3 + (-1)^{(\deg\delta_1-1)\deg\delta_2}\delta_2\wedge[\delta_1,\delta_3]_{\log}$

(2) It is determined on $\mathcal{A}_D$ and $\mathcal{X}_M^1(-\log D)$ by:

(a) $[f,g]_{\log}=0$

(b) $[\delta,f]_{\log}=\delta(f)$

(c) $[\delta_1,\delta_2]_{\log}$ *is the Jacobi-Lie bracket of logarithmic vector field*

(3) $[\delta_1,\delta_2]_{\log} = (-1)^{\deg\delta_2\deg\delta_1}[\delta_2,\delta_1]_{\log}$.

Definition 3.8. *The operation $[-,-]_{\log}$ of theorem* [3.7] *is called logarithmic Schouten bracket.*

In what follows we will denoted $[-,-]_s = [-,-]_{\log}$.

From theorem [3.7] we will deduce the following lemma.

Lemma 3.9. $(\mathcal{X}_M(-\log D),[-,-]_s)$ *is super-Lie algebra*.

3.3. Log Poisson structures. We can now give the definition of logarithmic Poisson structure.

Definition 3.10. A log-Poisson structure is local section π of the sheaf $\mathcal{X}_M^2(-\log D)$ such that $[\pi,\pi]_s=0$.

We deduce the definition of log-Poisson manifold.

Definition 3.11. *A log-Poisson manifold is a triple (M,D,π) where M is complex manifold, D is reduced free divisor of M and π is log-Poisson structure along D.*

It follows from definition of $\mathcal{X}_M^2(-\log D)$ that log Euler Poisson structure degenerate on divisor D.

3.4. Examples of log-Poisson manifold. log-Poisson structure appear in many way. We will give some of them.

(1) **log-symplectic manifold**. Each log-symplectic manifold is log-Poisson maifold.

(2) **Singular Toda lattice.**

Let $P:=\{(a,b)\in\mathbb{C}^{2n} / \mathrm{Re}(a_i)\geq 0, Ima_i\geq 0; i=1,\dots,n\}$ and consider the bracket

$$\{f,g\}(a,b) = \left[(\partial_a f)^T,(\partial_b f)^T\right] W\begin{bmatrix}\partial_a g\\ \partial_a g\end{bmatrix}$$

where $W=\begin{bmatrix}0 & -A\\ -A & 0\end{bmatrix}$ and $A=diag(a_1,\dots,a_n)$. The log bivector associated $\pi=\sum_{i=0}^n a_i\partial_{a_i}\wedge\partial_{b_i}$ is log-Poisson structure on P who become a log-Poisson manifold.

4. The Lichenerowicz-log Poisson Cohomology of Log-Poisson Manifold

Let (M,D,π) be a Log Poisson manifold. As in the case of Poisson manifold, we introduce hamiltonian vector field on X by setting, for any $f\in\mathcal{O}_X$; $X_f:=\pi^\#(df)=df^\#$. Follow [4] or simply use the definition of sharp map $\pi^\#$, we have $X_f=[\pi,f]_s$ The jacobian identity is equivalent to $[\pi,\pi]_s=0$.

The map $\pi^\#:\Omega_M^1(\log D)\to T_M(-\log D)$ is the anchor of algebroide with the following Lie bracket

$$\{\alpha,\beta\} = \mathcal{L}_{\pi^\#(\alpha)}\beta - \mathcal{L}_{\pi^\#(\beta)}\alpha - d(\pi(\alpha,\beta)) \tag{4.1}$$

Therefore, bracket satisfy the following equality

$$\{\alpha,f\beta\} = f\{\alpha,\beta\} + \left(\pi^\#(\alpha)f\right)\beta \tag{4.2}$$

The existence of this Lie-braked on $\Omega_M(\log D)$ allowed us to define the following exterior product $\partial_{\log}:\mathcal{X}_M^k(-\log D)\to\mathcal{X}_M^{k+1}(-\log D)$ by

$$\begin{aligned}&\left(\partial_{\log}P\right)(\alpha_1,\dots,\alpha_{k+1})\\ &=\sum_{i=1}^{k+1}(-1)^{k+1}\pi^\#(\alpha_i)P(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_{k+1})\\ &+\sum_{i<j}(-1)^{i+j}P\left(\{\alpha_i,\alpha_j\},\alpha_1,\dots,\hat{\alpha}_i,\dots,\hat{\alpha}_j,\dots,\alpha_{k+1}\right)\end{aligned} \tag{4.3}$$

It is clear that $\partial_{\log}^2 = 0$. Therefore, $\left(\mathcal{X}_M^*(-\log D), \partial_{\log}\right)$ constituted a chain complex calling log Poisson complex.

Definition 4.1. *We call Lichenerowiecz-log Poisson Cohomology (L-log P cohomology) of* (M, D, π) *the cohomology of* $\left(\mathcal{X}_M^*(-\log D), \partial_{\log}\right)$

It is denoted by $H_{L-\log P}^*(M, D, \pi)$ or simply $H_{L-\log P}^*(M)$ and for any $k \in \mathbb{N}$.

$$H_{L-\log P}^*(M) := \frac{Ker\left(\partial_{\log} : \mathcal{X}_M^k(-\log D) \to \mathcal{X}_M^{k+1}(-\log D)\right)}{Im\left(\partial_{\log} : \mathcal{X}_M^{k-1}(-\log D) \to \mathcal{X}_M^k(-\log D)\right)}. \quad (4.4)$$

The cohomology class of any element $P \in Ker\left(\partial_{\log} : \mathcal{X}_M^k(-\log D) \to \mathcal{X}_M^{k+1}(-\log D)\right)$ will be denoted by $[P]^{\log}$.

By definition of anchor map $\pi^\#$, we have

$$\pi^\#\{\alpha, \beta\} = \left[\pi^\#(\alpha), \pi^\#(\beta)\right] \quad (4.5)$$

where $[,]$ is the Lie bracket on $\mathcal{X}_M^1(-\log D)$.

In the Poisson Manifold, $\pi^\#$ induce cohomological homomorphism which in quasi-isomorphism in symplectic case. We are interest to know if or not the result trow in log Poisson manifold.

Let $\omega \in \Omega_M^k(\log D)$. Define $\pi^\#$ on $\Omega_M^k(\log D)$ by

$$\pi^\#(\omega)(\alpha_1, \ldots, \alpha_k) = (-1)^k \omega\left(\pi^\#(\alpha_1), \ldots, \pi^\#(\alpha_k)\right) \quad (4.6)$$

It follows that;

$$\begin{aligned}
&\partial_{\log}\left(\pi^\#(\omega)\right)(\alpha_1, \ldots, \alpha_{k+1}) \\
&= \sum_{i=1}^{k+1} (-1)^{i+1} \pi^\#(\alpha_i)\left(\pi^\#(\omega)\right)(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}) \\
&+ \sum_{i<j} (-1)^{i+j} \pi^\#(\omega)\left(\{\alpha_i, \alpha_j\}, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_{k+1}\right) \\
&= \sum_{i=1}^{k+1} (-1)^{i+1+k} \pi^\#(\alpha_i)\, \omega\begin{pmatrix} \pi^\#(\alpha_1), \ldots, \\ \pi^\#(\hat{\alpha}_i), \ldots, \pi^\#(\alpha_{k+1}) \end{pmatrix} \\
&+ \sum_{i<j} (-1)^{i+j+k}\, \omega\begin{pmatrix} \pi^\#\left(\{\alpha_i, \alpha_j\}\right), \pi^\#(\alpha_1), \ldots, \\ \pi^\#(\hat{\alpha}_i), \ldots, \pi^\#(\hat{\alpha}_j), \ldots, \pi^\#(\alpha_{k+1}) \end{pmatrix}
\end{aligned}$$

since

$$\begin{aligned}
&\pi^\# \circ d\omega(\alpha_1, \ldots, \alpha_{k+1}) \\
&= (-1)^{k+1} d\omega\left(\pi^\#(\alpha_1), \ldots, \pi^\#(\alpha_{k+1})\right),
\end{aligned}$$

then

$$\begin{aligned}
&\pi^\# \circ d\omega(\alpha_1, \ldots, \alpha_{k+1}) \\
&= \sum_{i=1}^{k+1} (-1)^{i+2+k} \pi^\#(\alpha_i)\, \omega\begin{pmatrix} \pi^\#(\alpha_1), \ldots, \\ \pi^\#(\hat{\alpha}_i), \ldots, \pi^\#(\alpha_{k+1}) \end{pmatrix} \\
&+ \sum_{i<j} (-1)^{i+j+k+1}\, \omega\begin{pmatrix} \left[\pi^\#(\alpha_i), \pi^\#(\alpha_j)\right], \pi^\#(\alpha_1), \ldots, \\ \pi^\#(\hat{\alpha}_i), \ldots, \pi^\#(\hat{\alpha}_j), \ldots, \pi^\#(\alpha_{k+1}) \end{pmatrix}
\end{aligned}$$

and then $\partial_{\log} \circ \pi^\# = -\pi^\# \circ d$

We have prove the following lemma

Lemma 4.2. *The homomorphism* $\pi^\#$ *is chain map,*

$$\partial_{\log} \circ \pi^\# = -\pi^\# \circ d$$

Hence we deduce

Proposition 4.3. *If* $H_{dR-\log}^*(M)$ *is the logarithmic de Rham cohomology, then the homomorphism* $\pi^\# : \left(\Omega_M^*(\log D), d\right) \to \left(\mathcal{X}_M^*(-\log D), \partial_{\log}\right)$ *induces a homomorphism in cohomology, also denoted by* $\pi^\#$,

$$\begin{aligned}
\pi^\# : H_{dR-\log}^*(M) &\to H_{L-\log P}^*(M) \\
[\alpha] &\mapsto \left[\pi^\#(\alpha)\right]^{\log}
\end{aligned}$$

5. Log Poisson Chern Class of Complex Line Bundle over a Log Poisson Manifold

Let (M, D, π) be a log Poisson manifold. $p : L \to M$ a complex line bundle over $M, \Gamma(L)$ the space of global cross section of $p : L \to M$ and $End_{\mathbb{C}}\left(\Gamma(L)\right)$ the space of the complex linear endomorphism of $\Gamma(L)$. In order to define the notion of log prequantization, we shall define the notion of log contravariant derivative.

Definition 5.1. *A log contravariant derivative* $D^{\log}$ *on* $p : L \to M$ *is* $\mathbb{R}-$*linear mapping* $\Omega_M(\log D) \to End_{\mathbb{C}}\left(\Gamma(L)\right)$ *such that*:

$$D_\alpha^{\log}(fs) = fD_\alpha^{\log} s + \left(\pi^\#(\alpha) f\right) s \quad (5.1)$$

for all $\alpha \in \Omega_M(\log D)$ *and* s *a local section of* L.

Equivelently, $D_\alpha^{\log}$ is log contravariant derivative if for all $\alpha, \beta \in \Omega_M(\log D)$, $D_{\alpha+\beta}^{\log} = D_\alpha^{\log} + D_\beta^{\log}; D_{f\alpha}^{\log} = fD_\alpha^{\log}$ and $D_\alpha^{\log}(fs) = fD_\alpha^{\log} s + \left(\pi^\#(\alpha) f\right) s$.

We say that $D_\alpha^{\log}$ is hermitian or compatible with hermitian metric h on $p : L \to M$ if for all $\alpha \to \Omega_M(\log D)$, $s_1, s_2 \in \Gamma(L) \pi^\#(\alpha)\left(h(s_1, s_2)\right) = h\left(D_\alpha^{\log} s_1, s_2\right) + h\left(s_1, D_\alpha^{\log} s_2\right)$.

Remark 5.2. *If ∇ is the connection on $p: L \to M$ with logarithmic singularities along D, $D_\alpha = \nabla_{\pi^\#(\alpha)}$ is log contravariant derivative on $p: L \to M$.*

Definition 5.3. *The curvature of a log contravariant derivative $D_\alpha^{\log}$ on $p: L \to M$ is a mapping $C_D : \Omega_M(\log D) \times \Omega_M(\log D) \to End_{\mathbb{C}}(\Gamma(L))$; define for all $\alpha, \beta \in \Omega_M(\log D)$,*

$$C_D(\alpha,\beta) = D_\alpha^{\log} \circ D_\beta^{\log} - D_\beta^{\log} \circ D_\alpha^{\log} - D_{\{\alpha,\beta\}}^{\log} \quad (5.2)$$

We have the following proposition

Proposition 5.4. *C_D is $\mathcal{O}_M$ -bilinear skew- symmetric.*

Proof. For all $\alpha, \beta \in \Omega_M(\log D)$,

$$\begin{aligned} C_D(\beta,\alpha)s &= \left(D_\beta^{\log} \circ D_\alpha^{\log} - D_\alpha^{\log} \circ D_\beta^{\log} - D_{\{\beta,\alpha\}}^{\log}\right)s \\ &= -\left(D_\alpha^{\log} \circ D_\beta^{\log} - D_\beta^{\log} \circ D_\alpha^{\log} - D_{\{\alpha,\beta\}}^{\log}\right)s \\ &= -C_D(\alpha,\beta) \end{aligned}$$

Let f be a section of $\mathcal{O}_M$, we have

$$\begin{aligned} & C_D(f\alpha,\beta)s \\ &= \left(D_{f\alpha}^{\log} \circ D_\beta^{\log} - D_\beta^{\log} \circ D_{f\alpha}^{\log} - D_{\{f\alpha,\beta\}}^{\log}\right)s \\ &= fD_\alpha^{\log} \circ D_\beta^{\log} - D_\beta^{\log}(fD_\alpha s) - D^{\log}_{f\{\alpha,\beta\} + \left(\pi^\#(\beta)f\right)\alpha} S \\ &= fD_\alpha^{\log} \circ D_\beta^{\log} s - fD_\beta^{\log}(D_\alpha s) - \left(\pi^\#(\beta)f\right)D_\alpha^{\log} s \\ & -fD_{\{\alpha,\beta\}}^{\log} s + \left(\pi^\#(\beta)f\right)D_\alpha^{\log} s \\ &= f\left(D_{f\alpha}^{\log} \circ D_\beta^{\log} - D_\beta^{\log} \circ D_{f\alpha}^{\log} - D_{\{f\alpha,\beta\}}^{\log}\right)s \\ &= fC_D(\alpha,\beta)s \end{aligned}$$

From the above result and the fact that $p: L \to M$ is complex line bundle, there exists a globelley defined complex bivector field $\prod = \prod_1 + i\prod_2$ on M with $\prod_1, \prod_2 \in \wedge^2 \mathcal{X}_M(-\log D)$ such that for all $\alpha, \beta \in \Omega_M(\log D)$, and $s \in \Gamma(L)$

$$C_D(\alpha,\beta) = \prod(\alpha,\beta)s \quad (5.3)$$

By linearity, the cohomology operator $\partial_{\log}$ on the log multivector fields on M by setting for any

$$P, Q \in \mathcal{X}_M^k(-\log D), \partial_{\log}(P + iQ) = \partial_{\log}P + i\partial_{\log}Q.$$

then $\partial_{\log}^2 = 0$ Consequently, $\left(\Omega_{\mathbb{C},M}(\log D), \partial_{\log}\right)$ is a chain complex whose cohomology will be called the complex Lichnerowicz-log Poisson of (M, D, π) and will be denoted by $H^*_{\mathbb{C}L-\log P}(M, D, \pi)$ or $H^*_{\mathbb{C}L-\log P}(M)$.

Proposition 5.5. Let $p: L \to M$ be a complex line bundle over a log Poisson manifold (M, D, π) $D^{\log}$ a log contravariant derivative on $p: L \to M$; C_D the curvature of $D^{\log}$ and $\prod$ the complex bivector field on M associated to C_D. Then

i) $\prod$ *define a cohomology class* $[\prod]^{\log}$ *in* $H^2_{\mathbb{C}L-\log P}(M)$

ii) $[\prod]^{\log}$ *does not depend of the log contravariant derivative* $D^{\log}$

iii) *In the case where* $D^{\log}$ *is compatible with a hermitian metric h on* $p: L \to M$, $\prod$ *is purely imaginary*.

Proof.

i) Let s be a nowhere vanishing local section of $p: L \to M$. Since the complex dimension of the fiber of $p: L \to M$ is 1, we may associate to s as follows. It is clear that for any logarithmic 1-form α on M $\frac{D_\alpha s}{s}$. is a complex function on M and the application $\alpha \mapsto \frac{D_\alpha s}{s}$ is $\mathbb{C}$ -linear Hence, there exists a unique complex local vecto field $X = X_1 + iX_2$ on M with X_1, X_2 local real logarithmic vector fields in M such that, for all $\alpha\Omega_M(\log D)$

$$D_\alpha^{\log} s = \langle \alpha, X \rangle s$$

we have that $\prod = \partial_{\log} X$.

$$\begin{aligned} & \prod(\alpha,\beta)s = C_D(\alpha,\beta) \\ &= \left(D_\alpha^{\log} \circ D_\beta^{\log} - D_\beta^{\log} \circ D_\alpha^{\log} - D_{\{\alpha,\beta\}}^{\log}\right)s \\ &= D_\alpha^{\log}(\langle \beta, X \rangle s) - D_\beta^{\log}(\langle \alpha, X \rangle s) - \langle \{\alpha,\beta\}, X \rangle s \\ &= \langle \alpha, X \rangle \langle \beta, X \rangle s + \pi^\#(\langle \alpha, X \rangle)s - \langle \beta, X \rangle \langle \alpha, X \rangle s \\ & -\pi^\#(\langle \beta, X \rangle)s - \langle \{\alpha,\beta\}, X \rangle s \\ &= \pi^\#(\alpha)(\langle \beta, X \rangle)s - \pi^\#(\beta)(\langle \alpha, X \rangle)s - \langle \{\alpha,\beta\}, X \rangle s \\ &= \partial_{\log} X(\alpha,\beta)s. \end{aligned}$$

Hence for all $\alpha, \beta \in \Omega_M(\log D)$, $\prod(\alpha,\beta) = \partial_{\log} X(\alpha,\beta)$ then $\prod = \partial_{\log} X$

Consequently, $\partial_{\log} \prod = \partial_{\log}^2 X = 0$ wich mean that $\prod$ define a cohomology class $[\prod]^{\log}$ in $H^2_{\mathbb{C}L-\log P}(M)$.

ii) Let D' be another log contravariant derivative on $p: L \to M$ having curvature C'_D and X' corresponding local complex log vector field. We denote by $\prod'$ the corresponding complex log bivector field on M. We obtain $\prod' - \prod = \partial_{\log} X' - \partial_{\log} X$ i.e; $\prod' = \prod + \partial_{\log}(X' - X)$

Now, for any $\alpha \in \Omega_M(\log D)$ the mapping $D'_\alpha = D_\alpha \in End_{\mathbb{C}}(\Gamma(L))$

Therefore there exists a global defined complex log vector field Y such that for all $s \in \Gamma(L)\left(D'_\alpha - D^{\log}_\alpha\right)s = \langle \alpha, Y\rangle s$

then $\langle \alpha, Y\rangle s = sD'_\alpha - D^{\log}_\alpha s = \langle \alpha, X'\rangle + \langle \alpha, X\rangle s$

$\langle \alpha, Y\rangle = \langle \alpha, X' - X\rangle$

$Y = X' - X$

i.e; $Y = X' - X$

hence $\prod' = \prod + \partial_{\log}(X' - X) = \prod + \partial_{\log} Y$

$[\prod']^{\log} = [\prod]^{\log}$.

iii) Assume that $D^{\log}$ is compatible with a hermitian metric h on $p: L \to M$ and let (e) be a local orthogonal basis of $\Gamma(L)$ then for all $\alpha \in \Omega_M(\log D)$ we have

$\pi^{\#}(\alpha)(h(e,e)) = h\left(D^{\log}_\alpha e, e\right) + h\left(e, D^{\log}_\alpha e\right)$

i.e; $0 = h(\langle \alpha, X\rangle e, e) + h(e, \langle \alpha, X\rangle e)$

i.e; $\langle \alpha, X\rangle + \overline{\langle \alpha, X\rangle} = 0$

i.e; $X + \overline{X} = 0$

hence X is purely imaginary and because of $\prod = \partial_{\log} X$ we conclude that $\prod$ is purely imaginary.

From the property iii) of the theorem, it follows that $\frac{1}{2\pi i}[\prod]^{\log} \in H^2_{L-\log P}(M)$. Therefore, we have the following definitions.

Definition 5.6. *$\frac{1}{2\pi i}[\prod]^{\log}$ is the first real log Poisson-Chern class of $p: L \to M$.*

6. Prequantization of Log Poisson Structure

Let $(M; D; \pi)$ be a Log Poisson manifold and $p: L \to M$ a hermitian line bundle over M with a log contravariant derivative $D^{\log}$ with curvature C_D. We define a representation of Lie algebra $(\mathcal{O}_M, \{-,-\})$ on $End_{\mathbb{C}}(\Gamma(L))$ by associating to each $f \in \mathcal{O}_M$ a complex endomorphism $\hat{f}$ of $\Gamma(L)$ that is defined for any local section s of $\Gamma(L)$ by

$$\hat{f}s = D^{\log}_{df} s + 2\pi i f s \tag{6.1}$$

It is well know that $(End_{\mathbb{C}}(\Gamma(L)), [-,-])$ is Lie algebra where $[-,-]$ denote the usual communtator on $End_{\mathbb{C}}(\Gamma(L))$

Proposition 6.1. *The representation $\wedge: \mathcal{O}_M \to End_{\mathbb{C}}(\Gamma(L))$ is a homomorphism of Lie algebra iff*

$$C_D(df, dg) = -2\pi i\{f, g\} \tag{6.2}$$

Proof. It is simply computation using (6.1)

Definition 6.2. *We say that Log Poisson manifold M, is prequantizable if there exist an hermitian complex line bundle $p: L \to M$ such that operator $\wedge$ make sence on $End_{\mathbb{C}}(\Gamma(L))$ and satisfy* (6.2)

Therefore, the prequantization problem of Log Poisson manifold has solution iff there exist a Hermitian line bundle equipped with a log contravariant derivative $D^{\log}$ whose the curvature C_D satisfy

$$C_D(df, dg) = -2\pi i \pi \tag{6.3}$$

We can now give a prequantization condition of Log Poisson manifold. Let (M, D, π) be a Log Poisson manifold with $D = D_{reg} \cup D_{sing}$ a reduced free divisor of M. Denote $\bar{M} = M - D_{sing}$. We have the following sequence:

$H^*(\bar{M}, \mathbb{Z}) \hookrightarrow H^*(\bar{M}, \mathbb{R}) \hookrightarrow H^*(M; \Omega^*_M(\log D))$

$H^*_{LP-\log P}(M)$, where the last map is induce by the sharp map. This allow us to think about an integrable and real Log Poisson cohomology class.

Proposition 6.3. *The Log Poisson manifold (M, D, π) is prequantizable if and only if, there exist a logarithmic vector field A on M and closed logarithmic 2-form Φ on M, which represents an integral cohomology class of M, such that the following relation holds on M*

$$\pi + \partial_{\log} A = \pi^{\#}(\Phi) \tag{6.4}$$

Proof. Suppose that there exist a logarithmic vector field A and closed logarithmic 2-form Φ on (M, D, π) such that 6.4 is true on M. Then, there exists hermitian complex line bundle $p: L \to M$ over M equipped with a hermitian connection ∇ with logarithmic pole along D having as curvature 2-form the purely imaginary closed 2-form $-2\pi I \pi \Phi$. Using ∇, we define a log contravariant derivative $D^{\log}: \Omega^1_M(\log D) \to End_{\mathbb{C}}(\Gamma(L))$ on $p: L \to M$ as follows: for all $\alpha \in \Omega^1_M(\log D)$ and $s \in \Gamma(L)$,

$$D^{\log}_\alpha s = \nabla_{\pi^{\#}(\alpha)} s + 2\pi i \langle \alpha, A\rangle s \tag{6.5}$$

It is easy to see that $D^{\log}$ is Hermitian and his curvature satisfy 6.3.

Conversely, we suppose that (M, π, D) is Prequatizable. Then there exist a Hermitian complex line bundle $p: L \to M$ with Hermitian log contravariant derivative $D^{\log}$ whose curvature C_D verify 6.3. Consequently

$$\pi = \frac{i}{2\pi i} C_D = \frac{i}{2\pi i} \prod \tag{6.6}$$

where $\prod$ is the purely imaginary, $\partial_{\log}$ -closed, logarithmic bivector associated to C_D. Let ∇ be the Hermitian connection on $p : L \to M$ with curvature 2-form Ω. So, $\Phi = \frac{i}{2\pi i}\Omega$ is real closed logarithmic 2-form on M and represents the first real Chern class $c_1(L,\mathbb{R})$ which is integral, i.e., $c_1(L,\mathbb{R}) = [\Phi]$. Now we consider the Hermitian log contravariant derivative $\bar{D}$ on $p : L \to M$ defined by ∇. Let $\prod$ be the purely imaginary logarithmic bivector field on M associated to C_D. We have $\pi^{\#}([\Phi]) = \left[\frac{i}{2\pi i}\right]\bar{\prod} \Leftrightarrow \left[\pi^{\#}(\Phi)\right] = \left[\frac{i}{2\pi i}\bar{\prod}\right]$. Therefore, $\left[\bar{\prod}\right] = \left[\prod\right]$, which means that there exists a purely imaginary logarithmic vector field B on M such that $\bar{\prod} = \prod + \partial_{\log} B.$ Then

$$\frac{i}{2\pi i}\bar{\prod} = \frac{i}{2\pi i}\prod + \frac{i}{2\pi i}\partial_{\log} B \Leftrightarrow \pi^{\#}(\Phi) = \pi + \partial_{\log}\left(\frac{i}{2\pi i} B\right).$$

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