MICHAEL'S SELECTION THEOREM IN A SEMILINEAR CONTEXT

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ABSTRACT. We establish versions of Michael's Selection Theorem and Tietze's Extension Theorem in the category of semilinear maps.

INTRODUCTION

Michael's Selection Theorem [11] is an important foundational result in non-linear functional analysis, which has found numerous applications in analysis and topology; see, e.g., [6, 15, 16] and the references in [21]. This theorem is concerned with set-valued maps. To state it, we first introduce some notation and terminology used throughout this paper. Let X, Y be sets. We write 2^Y for the power set of Y, and we use the notation $T\colon X \rightrightarrows Y$ to denote a map $T\colon X \to 2^Y$, and call such T a **set-valued map.** Let $T\colon X \rightrightarrows Y$ be a set-valued map. The **domain** of T ($\operatorname{dom}(T)$) is the set of $x \in X$ with $T(x) \neq \emptyset$, and the **graph** of T is the subset

$$\Gamma(T) := \big\{(x,y) \in X \times Y : y \in T(x)\big\}$$

of $X \times Y$. Note that every map $f\colon X \to Y$ gives rise to a set-valued map $X \rightrightarrows Y$ with domain X, whose graph is the graph

$$\Gamma(f) := \big\{(x,y) \in X \times Y : y = f(x)\big\}$$

of the map f. A **selection** of T is a map $f\colon X \to Y$ with $\Gamma(f) \subseteq \Gamma(T)$.

Suppose now that X and Y come equipped with topologies. Then T is called **lower semicontinuous (l.s.c.)** if, for every $x \in X$ and open subset V of Y with $V \cap T(x) \neq \emptyset$, there is an open neighborhood U of x such that for all $V \cap T(x') \neq \emptyset$ for all $x' \in U$. Lower semicontinuity is a necessary condition for a set-valued map $X \rightrightarrows Y$ to have *continuous* selections going through each prescribed point (x,y) on its graph. Suppose now that X is paracompact (i.e., every open cover of X has a locally finite open refinement) and Y is a Banach space. The Michael Selection Theorem says that then *every l.s.c. set-valued map $T\colon X \rightrightarrows Y$ with domain X such that $T(x)$ is convex and closed, for all $x \in X$, has a continuous selection.* (See, e.g., [6, Theorem 1.16] or [4, Section 9.1] for a proof.)

In the course of our work adapting the arguments of Glaeser [9] and Klartag-Zobin [10] for the C^1-case of the Whitney Extension Problem to maps which are definable in an o-minimal expansion of the real field [3], we established a version of the Michael Selection Theorem suitable for this context; see [3, Theorem 4.1]. This may be seen as a *constructive* selection principle for the case where the topological spaces X, Y and the set-valued map T are *tame* in a certain sense (and the stipulated continuous selection of T is also required to be tame). See [7] for an introduction to this kind of "tame topology."

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In the present paper we further restrict the class of spaces and set-valued maps under consideration, and adapt the Michael Selection Theorem to *semilinear* set-valued maps. The category of semilinear sets and maps is somewhat more flexible than the category of polyhedral sets and maps often considered in mathematical programming (see, e.g., [17]). To define semilinearity, let us fix an ordered field R. (For example, R could be the ordered field $\mathbb{Q}$ of rationals, or the ordered field $\mathbb{R}$ of real numbers.) An ordered R-linear space is a vector space V over R equipped with a linear ordering $\leq$ making V an ordered additive group, such that for all $\lambda \in R$ and $x \in V$, the implication $\lambda > 0$ & $x > 0 \Rightarrow \lambda x > 0$ holds. The ordered field R, considered as vector space over itself, with its usual ordering, is an ordered R-linear space. In fact, any ordered field extension of R is an ordered R-linear space in an obvious way. The ordered $\mathbb{Q}$-linear spaces are precisely the divisible ordered abelian groups, made into vector spaces over $\mathbb{Q}$ in the natural way.

Let $V \neq \{0\}$ be an ordered R-linear space. We equip V with the order topology (whose basis of open sets are given by the open intervals in V), and each cartesian power V^m with the corresponding product topology. An **affine function** on V^m is a function $f\colon V^m \to V$ of the form

$$f(v_1, \dots, v_m) = \lambda_1 v_1 + \cdots + \lambda_m v_m + a,$$

where $\lambda_1, \dots, \lambda_m \in R$ and $a \in V$ are fixed. A **basic semilinear set** in V^m is a set of the form

$$S = \big\{x \in V^m : f_i(x) \geq 0 \text{ for } i = 1, \dots, M,\ g_j(x) > 0 \text{ for } j = 1, \dots, N\big\},$$

with $f_1, \dots, f_M$ and $g_1, \dots, g_N$ affine functions on V^m. We allow the case where $M = 0$ or $N = 0$: a basic semilinear set in V^m of the form above with $N = 0$ is said to be a **polyhedral set** in V^m. A **semilinear set** in V^m is a finite union of basic semilinear sets in V^m. Although the complement (in V^m) of a basic semilinear set in V^m is, in general, not basic semilinear, the complement of a semilinear set in V^m is also semilinear. In fact, letting $\mathcal{S}_m$ denote the collection of semilinear sets in V^m, the family $(\mathcal{S}_m)_{m \in \mathbb{N}}$ has the following closure properties:

(1) $\mathcal{S}_m$ is a boolean algebra of subsets of V^m;
(2) if $S \in \mathcal{S}_m$, then $V \times S$ and $S \times V$ belong to $\mathcal{S}_{m+1}$;
(3) for $1 \leq i < j \leq m$ we have $\{(v_1, \dots, v_m) \in V^m : v_i = v_j\} \in \mathcal{S}_m$;
(4) if $S \in \mathcal{S}_{m+1}$, then $\pi(S) \in \mathcal{S}_m$, where $\pi\colon V^{m+1} \to V^m$ denotes the projection onto the first m coordinates.

That is, $(\mathcal{S}_m)$ is the family of definable sets in a certain structure in the sense of first-order logic (see [7, Chapter I, §7]), and thus the class of semilinear sets is readily seen to be stable under a number of natural topological and geometric operations: for example, if $S \subseteq V^m$ is semilinear, then so are the closure, interior, and boundary of S. Here, (1)–(3) are straightforward, whereas an explicit proof of (4) may be given using the Fourier-Motzkin elimination procedure [22, Chapter 1]. This procedure leading to (4) is uniform in V; employing logic terminology, (4) expresses that the theory of non-trivial ordered R-linear spaces admits quantifier elimination. As a particular consequence, if S is a basic semilinear set in V^m as above, V^* an ordered R-linear space extending V, and

$$S^* = \big\{x \in (V^*)^m : f_i^*(x) \geq 0 \text{ for } i = 1, \dots, M,\ g_j^*(x) > 0 \text{ for } j = 1, \dots, N\big\}$$

the semilinear set in $(V^*)^m$ defined by the unique extensions f_i^*, g_j^* of f_i respectively g_j to affine functions on $(V^*)^m$, then $S^* \neq \emptyset \iff S \neq \emptyset$. (Again, using logic terminology: V^* is an elementary extension of V.) Although this fact is used in Section 4 below, in this paper we mainly work with $V = R$, and we refer to the semilinear sets in R^m, for varying m, as semilinear sets over R.

Let now $T\colon X \rightrightarrows R^n$ be a set-valued map with domain $X \subseteq R^m$. We say that T is semilinear (over R) if its graph $\Gamma(T)$, as a subset of $R^m \times R^n = R^{m+n}$, is semilinear over R. (The domain X of T is then also semilinear, as are the values $T(x)$ of T, for each $x \in X$.) Similarly, a map $f\colon X \to R^n$ is said to be semilinear (over R) if its graph $\Gamma(f) \subseteq R^{m+n}$ is. Standard examples of semilinear (in fact, piecewise polyhedral) set-valued maps are the metric projections onto a polyhedral subset of $\mathbb{R}^m$ with respect to a polyhedral norm on $\mathbb{R}^m$; see [8, Section 5]. Our main theorem is the following:

Theorem (Semilinear Michael Selection Theorem)**.** *Let $T\colon X \rightrightarrows R^n$ be a l.s.c. semilinear set-valued map whose domain $X \subseteq R^m$ is closed and bounded, such that $T(x)$ is closed and convex for each $x \in X$. Then T has a continuous semilinear selection.*

The proof of this theorem uses (an adaptation of) a selection principle for polyhedral set-valued maps from [8], as well as the following fact, which is possibly of independent interest, and shown in Section 3 below:

Proposition (Semilinear Tietze Extension Theorem)**.** *Every continuous semilinear map $X \to R^n$, where $X \subseteq R^m$ is bounded, extends to a continuous semilinear map $R^m \to R^n$.*

The proof of this proposition, in turn, rests on a representation theorem for continuous piecewise affine functions (the "piecewise affine Pierce-Birkhoff Conjecture") from [13, 14]. The assumption on X to be bounded is necessary in both the theorem and the proposition above (cf. Examples 3.4 and 4.12 below).

Organization of the paper. In Section 1, after some preliminaries on convex functions, we give a useful description of the semilinear convex sets; as to be expected, they are very simple, see Theorem 1.18. Our description is also valid in the broader context of convex sets definable in an o-minimal expansion of an ordered R-linear space, and since this might be useful elsewhere, we conduct our analysis on this level of generality; thus in this section only, we assume that the reader has had some exposure to the basics of o-minimality on the level of [7]. In the rest of this paper, we again work in the semilinear situation. In Section 2 we focus on polyhedral sets, and then prove the Semilinear Tietze Extension Theorem in Section 3 and the Semilinear Michael Selection Theorem in Section 4.

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Corrections. We would like to point out two typographical errors that were introduced in Section 3 of [2] during the production process: On p. 481, the claim that the set C_ϵ is bounded should be replaced by the claim "$x_0 \in C$." On p. 482, the reference to Lemma 1.3 in the proof of Corollary 3.1 should be replaced by a reference to Lemma 1.13.

Conventions and notations. Throughout this paper, m and n will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. We let R be an ordered field. If $f, g\colon X \to R$ (where X is a set), then we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and we set

$$(f, g) := \big\{(x,t) \in X \times R : f(x) < t < g(x)\big\}.$$

For a set $S \subseteq R^n$ we denote by $\operatorname{cl} S = \operatorname{cl}(S)$ the closure, by $\partial S = \partial(S) := \operatorname{cl}(S) \setminus S$ the frontier, and by $\operatorname{int} S = \operatorname{int}(S)$ the interior of S. We denote the supremum norm on R^n by $||\cdot||_\infty$, so $||x||_\infty = \max\big\{|x_1|, \dots, |x_n|\big\}$ for $x = (x_1, \dots, x_n) \in R^n$.

1. Convex Sets and Functions Definable in o-minimal Structures

The only non-empty closed convex subsets of $\mathbb{R}$ are the singletons and the line segments $[a, b]$ where $-\infty \le a < b \le \infty$. Put differently, for every closed convex subset $E \neq \emptyset$ of $\mathbb{R}$, there is a semilinear cell C (in the sense of [7]) such that $E = \operatorname{cl}(C)$. In this section, we show that a suitable generalization of this fact holds true for closed convex sets definable in o-minimal structures. (See Corollary 1.20 below.) We first discuss some general properties of convex functions and affine maps. We let λ, μ, ρ (possibly with decorations) range over R.

1.1. Convex functions.

Let X be an R-linear space. Recall that a subset E of X is called **convex** if for all $x, y \in E$ and λ with $0 < \lambda < 1$, we have $(1-\lambda)x + \lambda y \in E$. In this subsection, we let $f\colon E \to R$ be a function where $E \subseteq X$. One says that f is **convex** if its **epigraph**

$$\operatorname{epi}(f) := \big\{(x,t) \in E \times R : t \ge f(x)\big\}$$

is a convex subset of the R-linear space $X \times R$, and **concave** if $-f$ is convex, i.e., if the **hypograph**

$$\operatorname{hyp}(f) := \big\{(x,t) \in E \times R : t \le f(x)\big\}$$

of f is convex. Note that if the function f is convex or concave, then the domain E of f is convex. Every linear combination $\lambda f + \mu g$ of two convex (concave) functions $f, g\colon E \to R$ with $\lambda, \mu \ge 0$ is convex (concave, respectively). We also declare the constant function $+\infty$ on E to be convex and the constant function $-\infty$ on E to be concave. The **strict epigraph** of f is the set

$$\operatorname{epi}_s(f) := \big\{(x,t) \in E \times R : t > f(x)\big\}$$

and the **strict hypograph** of f is

$$\operatorname{hyp}_s(f) := \big\{(x,t) \in E \times R : t < f(x)\big\}.$$

We often implicitly use the following equivalences, whose (easy) proofs we leave to the reader:

Lemma 1.1. *Suppose E is convex. Then*

$$\begin{aligned}
f \text{ is convex} \quad &\Longleftrightarrow \quad \operatorname{epi}_s(f) \text{ is convex} \\
&\Longleftrightarrow \quad \begin{cases} f\big((1-\lambda)\cdot x + \lambda\cdot y\big) \le (1-\lambda)\cdot f(x) + \lambda \cdot f(y) \\ \text{for all } x, y \in E \text{ and } 0 < \lambda < 1; \end{cases} \\
f \text{ is concave} \quad &\Longleftrightarrow \quad \operatorname{hyp}_s(f) \text{ is convex} \\
&\Longleftrightarrow \quad \begin{cases} f\big((1-\lambda)\cdot x + \lambda\cdot y\big) \ge (1-\lambda)\cdot f(x) + \lambda \cdot f(y) \\ \text{for all } x, y \in E \text{ and } 0 < \lambda < 1, \end{cases}
\end{aligned}$$

From the previous lemma, we obtain:

Corollary 1.2. *Let $C \subseteq X \times R$ be convex. Let $\pi\colon X \times R \to X$ be the natural projection, and let $f\colon E := \pi(C) \to R$ satisfy*

$$f(x) = \inf C_x = \inf\{t \in R : (x,t) \in C\} \quad \text{for each } x \in E. \tag{1.1}$$

Then f is convex.

Proof. Note that E is convex. Let $x, y \in E$ and $0 < \lambda < 1$. Then for all $s \in C_x$, $t \in C_y$ we have $(1-\lambda)(x,s) + \lambda(y,t) \in C$, so in particular $f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)s + \lambda t$ by the hypothesis on f. Thus, again using the assumption on f, we obtain $f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y)$. Hence f is convex by the previous lemma. □

Remark. If in the previous corollary, instead of assuming that f satisfies (1.1), we assume that $f(x) = \sup C_x$ for each $x \in E$, then f is concave.

Corollary 1.3. *Let $C = (f,g)$ where $f, g\colon E \to R$. Then C is a convex subset of $X \times R$ if and only if f is convex and g is concave.*

Proof. The forward direction is immediate from the previous corollary and the remark following it. Since $C = \operatorname{epi}_s(f) \cap \operatorname{hyp}_s(g)$, the backward direction follows from Lemma 1.1. □

The following technical lemmas are used in the subsections below:

Lemma 1.4. *Suppose f is convex, and let $y_1, y_2 \in E$. Set $x = \frac{y_1+y_2}{2}$ and*

$$M = \max\big\{\, |f(y_1) - f(x)|\,, |f(y_2) - f(x)|\, \big\}.$$

Let $0 \leq \lambda \leq 1$, and for $i = 1,2$ let $z_i = (1-\lambda)\cdot x + \lambda \cdot y_i \in E$. Then

$$|f(z_i) - f(x)| \leq \lambda M.$$

Proof. Let $z = z_1 = (1-\lambda)\cdot x + \lambda\cdot y_1$. By convexity,

$$f(z) - f(x) \leq (1-\lambda)\cdot f(x) + \lambda\cdot f(y_1) - f(x) = \lambda\cdot\big(f(y_1) - f(x)\big).$$

Since $x = \frac{1}{1+\lambda}z + \frac{\lambda}{1+\lambda}y_2$, we also have $f(x) \leq \frac{1}{1+\lambda}f(z) + \frac{\lambda}{1+\lambda}f(y_2)$ and thus $(1+\lambda)\cdot f(x) \leq f(z) + \lambda\cdot f(y_2)$, so

$$\lambda\cdot\big(f(x) - f(y_2)\big) \leq f(z) - f(x).$$

Therefore, $|f(z) - f(x)| \leq \lambda M$. For $z = z_2$ one argues similarly. □

Before we give the next lemma, we introduce some useful notation:

Notation. For $x, y \in X$, define the convex subsets

$$(x,y) := \big\{(1-\lambda)\cdot x + \lambda\cdot y : 0 < \lambda < 1\big\},$$
$$[x,y] := \big\{(1-\lambda)\cdot x + \lambda\cdot y : 0 \leq \lambda \leq 1\big\}$$

of X. Note that taking $x = y$, we obtain $(x,x) = [x,x] = \{x\}$.

Lemma 1.5. *Let $x, y \in X$ be distinct and $h\colon (x,y) \to R$ be convex such that $h(z) \leq 0$ for all $z \in (x,y)$. If $h(z) = 0$ for some $z \in (x,y)$, then $h(z) = 0$ for all $z \in (x,y)$.*

Proof. Let $z \in (x, y)$ with $h(z) = 0$, and let $z' \neq z$ be another element of (x, y); we claim that $h(z') = 0$. Suppose $z' \in (x, z)$ (the case $z' \in (z, y)$ being similar). Take an arbitrary $z'' \in (z, y)$ and λ such that $0 < \lambda < 1$ and $z = (1 - \lambda)z' + \lambda z''$. Then

$$0 = h(z) \leq (1 - \lambda)h(z') + \lambda h(z'') \leq 0$$

and hence $h(z') = h(z'') = 0$. □

1.2. **Affine maps.** In this subsection we let X, Y be R-linear spaces. Let V be an affine subspace of X, that is, $V - v_0$ is an R-linear subspace of X, for one (equivalently, every) $v_0 \in V$. Note that V is convex, and if $x_1, \ldots, x_n \in V$ and $\lambda_1, \ldots, \lambda_n$ satisfy $\sum_{i=1}^n \lambda_i = 1$, then $\lambda_1 x_1 + \cdots + \lambda_n x_n \in V$. We say that a map $\varphi\colon V \to Y$ is **affine** if

$$\varphi\big((1 - \lambda) \cdot x + \lambda \cdot y\big) = (1 - \lambda) \cdot \varphi(x) + \lambda \cdot \varphi(y) \quad \text{ for all } x, y \in V \text{ and all } \lambda.$$

Thus every affine map $V \to R$ is convex. Note that if V is an ordered R-linear space with $\dim_R V < \infty$, then an affine function on V^m (as defined in the introduction) is the same as an affine map $V \to R$.

Lemma 1.6. *Let $\varphi\colon X \to Y$. Then*

(1) *φ is R-linear iff φ is affine and $\varphi(0) = 0$;*
(2) *φ is affine iff there is an R-linear map $A\colon X \to Y$ and $b \in Y$ with $\varphi(x) = A(x) + b$ for all x; in this case, A and b are unique.*

We leave the proof of this lemma to the reader.

Corollary 1.7. *Let V be an affine subspace of X and $\varphi\colon V \to Y$ be affine. Then*

$$\varphi\left(\sum_{i=1}^n \lambda_i x_i\right) = \sum_{i=1}^n \lambda_i \varphi(x_i) \quad \textit{for } x_i \in V \textit{ and } \lambda_i \textit{ with } \sum_{i=1}^n \lambda_i = 1.$$

Proof. Fix $v_0 \in V$ and put $Z := V - v_0$, a linear subspace of X. Define $\psi\colon Z \to Y$ by $\psi(z) = \varphi(z + v_0)$. Then ψ is affine, hence by part (2) of the previous lemma, there is a linear map $A\colon Z \to Y$ and $b \in Z$ with $\psi(z) = A(z) + b$ for all $z \in Z$. Thus $\varphi(x) = A(x - v_0) + b$ for all $x \in V$, and the claim follows. □

Every affine map $V \to Y$ extends to an affine map $X \to Y$. We call an affine bijection $X \to X$ an **affine transformation** of X. One easily verifies that if $f\colon E \to R$, where $E \subseteq X$, is convex (concave), then the pullback

$$f^* := f \circ \varphi\colon E^* := \varphi^{-1}(E) \to R$$

of f under an affine transformation φ of X is also convex (concave, respectively).

Lemma 1.8. *Let V be an affine subspace of X and $\varphi\colon V \to Y$. Then φ is affine if and only if*

$$\varphi\big((1 - \lambda) \cdot x + \lambda \cdot y\big) = (1 - \lambda) \cdot \varphi(x) + \lambda \cdot \varphi(y) \quad \textit{for all } x, y \in V \textit{ and } 0 < \lambda < 1. \tag{1.2}$$

Proof. The "only if" direction being obvious, suppose (1.2) holds, and let $x, y \in V$, $z = (1-\lambda)x+\lambda y$; we want to show that $\varphi(z) = (1-\lambda)\varphi(x)+\lambda\varphi(y)$. For $0 \leq \lambda \leq 1$, this holds by (1.2). Suppose $\lambda > 1$. Then $0 < \frac{1}{\lambda} < 1$ and $y = \frac{1}{\lambda}z + \left(1 - \frac{1}{\lambda}\right)x$, consequently $\varphi(y) = \frac{1}{\lambda}\varphi(z) + \left(1 - \frac{1}{\lambda}\right)\varphi(x)$ and thus $\varphi(z) = (1 - \lambda)\varphi(x) + \lambda\varphi(y)$. The case $\lambda < 0$ can be treated in a similar way, expressing x in terms of y, z. □

The intersection of any family of affine subspaces of X is either empty or itself an affine subspace of X. Thus given any subset S of X, there is a smallest affine subspace of X which contains S, denoted by $\operatorname{aff}(S)$ and called the **affine hull** of S. It is easily verified that $\operatorname{aff}(S)$ consists precisely of the linear combinations $\lambda_1 x_1 + \cdots + \lambda_n x_n$ of elements $x_1, \dots, x_n$ of S with $n \geq 1$ and $\lambda_1 + \cdots + \lambda_n = 1$. It is easy to see that if S is a non-empty open subset of X, then $\operatorname{aff}(S) = X$. The affine hull of a convex subset can be described concisely as follows:

Lemma 1.9. *Let $E \subseteq X$ be convex. Then*

$$\operatorname{aff}(E) = \{\lambda x - \mu y : \ \lambda, \mu \geq 0, \ \lambda - \mu = 1, \ x, y \in E\}.$$

Proof. The inclusion "$\supseteq$" being obvious, let $z \in \operatorname{aff}(E)$. Take $z_1, \dots, z_n \in E$ and $\lambda_1, \dots, \lambda_n \neq 0$ $(n \geq 1)$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $z = \lambda_1 z_1 + \cdots + \lambda_n z_n$. Let

$$I = \{1, \dots, n\}, \quad I^+ = \{i \in I : \lambda_i > 0\}, \quad I^- = I \setminus I^+.$$

Clearly $I^+ \neq \emptyset$. If $I^- = \emptyset$, then $z \in E$, hence $z = \lambda z - \mu z$ with $\lambda := 1$, $\mu := 0$. So suppose $I^- \neq \emptyset$; then $z = \lambda x - \mu y$ where

$$\lambda := \sum_{i \in I^+} \lambda_i, \quad \mu := -\sum_{i \in I^-} \lambda_i, \quad x := \sum_{i \in I^+} \frac{\lambda_i}{\lambda} z_i \in E, \quad y := \sum_{i \in I^-} \frac{-\lambda_i}{\mu} z_i \in E.$$

$\square$

Next, a result about constructing affine functions which generalizes Lemma 1.8. We fix a map $\varphi \colon E \to Y$ where $E \subseteq X$.

Proposition 1.10. *Suppose E is convex and*

$$\varphi\big((1-\lambda) \cdot x + \lambda \cdot y\big) = (1-\lambda) \cdot \varphi(x) + \lambda \cdot \varphi(y) \quad \textit{for all } x, y \in E \textit{ and } 0 < \lambda < 1.$$

Then there exists a unique affine $\Phi \colon \operatorname{aff}(E) \to Y$ with $\Phi{\restriction}E = \varphi$.

Proof. We first show uniqueness: by the previous lemma, each $z \in \operatorname{aff}(E)$ can be expressed as $z = \lambda x - \mu y$ where $\lambda, \mu \geq 0$, $\lambda - \mu = 1$, and $x, y \in E$, and so if $\Phi \colon \operatorname{aff}(E) \to Y$ is an affine extension of φ, then $\Phi(z) = \lambda\varphi(x) - \mu\varphi(y)$. For existence, we first let $\lambda, \mu, \lambda', \mu' \geq 0$ with $\lambda - \mu = \lambda' - \mu' = 1$ and $x, y, x', y' \in E$ with $\lambda x - \mu y = \lambda' x' - \mu' y'$, and show that $\lambda\varphi(x) - \mu\varphi(y) = \lambda'\varphi(x') - \mu'\varphi(y')$. To see this set $\delta := \lambda' + \mu = \lambda + \mu'$ and note that $\delta > 0$ (since otherwise $\lambda = \lambda' = \mu = \mu' = 0$, which is impossible). Thus $\frac{\lambda'}{\delta} + \frac{\mu}{\delta} = \frac{\lambda}{\delta} + \frac{\mu'}{\delta}$ and so

$$\frac{\lambda'}{\delta}\varphi(x) + \frac{\mu}{\delta}\varphi(y) = \varphi\left(\frac{\lambda'}{\delta}x + \frac{\mu}{\delta}y\right) = \varphi\left(\frac{\lambda}{\delta}x + \frac{\mu'}{\delta}y\right) = \frac{\lambda}{\delta}\varphi(x) + \frac{\mu'}{\delta}\varphi(y),$$

and this implies $\lambda\varphi(x) - \mu\varphi(y) = \lambda'\varphi(x') - \mu'\varphi(y')$ as claimed. We may now define $\Phi \colon \operatorname{aff}(E) \to Y$ by setting $\Phi(z) := \lambda\varphi(x) - \mu\varphi(y)$, where $x, y \in E$ and $\lambda, \mu \geq 0$ are chosen arbitrarily such that $\lambda - \mu = 1$ and $z = \lambda x - \mu y$. Note that $\Phi{\restriction}E = \varphi$. To finish the proof, by Lemma 1.8, it suffices to show that

$$\Phi\big((1-\rho) \cdot z + \rho \cdot z'\big) = (1-\rho) \cdot \Phi(z) + \rho \cdot \Phi(z') \quad \text{for all } z, z' \in \operatorname{aff}(E) \text{ and } 0 < \rho < 1.$$

Write $z = \lambda x - \mu y$, $z' = \lambda' x' - \mu' y'$ with $x, y, x', y' \in E$ and $\lambda, \lambda', \mu, \mu' \geq 0$, $\lambda - \mu = \lambda' - \mu' = 1$. Then

$$(1-\rho) \cdot z + \rho \cdot z' = (1-\rho)\lambda x + \rho\lambda' x' - \big((1-\rho)\mu y + \rho\mu' y'\big).$$

Put $\alpha := (1-\rho)\lambda + \rho\lambda'$ and $\beta := (1-\rho)\mu + \rho\mu'$. Then $\alpha, \beta \geq 0$ and $\alpha - \beta = 1$. Suppose $\alpha, \beta > 0$. Then

$$(1-\rho)\cdot z + \rho \cdot z' = \alpha\left[\frac{(1-\rho)\lambda}{\alpha}x + \frac{\rho\lambda'}{\alpha}x'\right] - \beta\left[\frac{(1-\rho)\mu}{\beta}y + \frac{\rho\mu'}{\beta}y'\right]$$

where the expressions in square brackets are elements of E. Therefore

$$\begin{aligned}
\Phi\big((1-\rho)\cdot z + \rho\cdot z'\big) &= \alpha f\left(\tfrac{(1-\rho)\lambda}{\alpha}x + \tfrac{\rho\lambda'}{\alpha}x'\right) - \beta f\left(\tfrac{(1-\rho)\mu}{\beta}y + \tfrac{\rho\mu'}{\beta}y'\right) \\
&= \alpha\left[\tfrac{(1-\rho)\lambda}{\alpha}\Phi(x) + \tfrac{\rho\lambda'}{\alpha}\Phi(x')\right] - \beta\left[\tfrac{(1-\rho)\mu}{\beta}\varphi(y) + \tfrac{\rho\mu'}{\beta}\varphi(y')\right] \\
&= (1-\rho)\big(\lambda\varphi(x) - \mu\varphi(y)\big) + \rho\big(\lambda'\varphi(x') - \mu'\varphi(y')\big) \\
&= (1-\rho)\Phi(z) + \rho\Phi(z').
\end{aligned}$$

In the case where $\alpha = 0$ (equivalently, $\lambda = \lambda' = 0$) or $\beta = 0$ (equivalently, $\mu = \mu' = 0$), one argues similarly. $\square$

We say that φ as above is **affine** if there is an affine $\Phi\colon X \to Y$ with $\Phi\restriction E = \varphi$. By the preceding proposition, we obtain:

Corollary 1.11. *Let $f\colon E \to R$ where E is a convex subset of X. Then*

$$\begin{aligned}
\Gamma(f) \textit{ is convex} \quad &\Longleftrightarrow \quad \begin{cases} f\big((1-\lambda)\cdot x + \lambda\cdot y\big) = (1-\lambda)\cdot f(x) + \lambda\cdot f(y) \\ \textit{for all } x, y \in E \textit{ and } 0 < \lambda < 1 \end{cases} \\
&\Longleftrightarrow \quad f \textit{ is both convex and concave} \\
&\Longleftrightarrow \quad f \textit{ is affine.}
\end{aligned}$$

1.3. **Continuity of convex functions.** We view R as an ordered R-linear space, and we construe R as model-theoretic structure in the (one-sorted) language of ordered R-linear spaces (see [7, Chapter 1, §7]). In the rest of this section we also fix a definably complete expansion $\boldsymbol{R}$ of R. "Definable" always means "definable in $\boldsymbol{R}$, allowing for parameters." For $x, y \in R^n$, the map $\lambda \mapsto (1-\lambda)x + \lambda y\colon R \to R^n$ is definable (since R is commutative), so in particular, the convex subsets $[x, y]$ and (x, y) of R^n are definable, and every definable convex subset of R^n is definably path-connected. Similarly, every affine subspace V of the R-linear space R^n and every affine map $V \to R^m$ is definable. Next, we show a basic fact about convex definable functions; it is an analogue of a well-known result about real-valued convex functions on subsets of $\mathbb{R}^n$ (see [18, Theorem 10.1]).

Proposition 1.12. *Let E be a subset of an affine subspace V of R^n which is open in the subspace topology of V, and $f\colon E \to R$ be convex and definable. Then f is continuous.*

Proof. We proceed by induction on $d = \dim(V)$. The case $d = 0$ being trivial, assume this lemma is true for some value of d, and suppose $\dim(V) = d+1$. After replacing E, V and f by $E^* = A^{-1}(E)$, $V^* = A^{-1}(V)$, and $f^* = f \circ A$, for a suitable affine transformation A of R^n, we may assume $V = R^{d+1} \times \{0\}^{n-d-1}$, and then reduce to the case that $n = d+1$. Let $e_1, \dots, e_n$ be the standard basis of R^n, and let $\lambda_1, \dots, \lambda_n$ range over R. For $\delta \in R^{>0}$ and $x \in R^n$, let

$$C_\delta(x) := \left\{ x + \sum_{i=1}^n \lambda_i \cdot e_i : |\lambda_i| < \delta \text{ for } i = 1, \dots, n \right\}.$$

Note that $C_\delta(x)$ is definable, and

$$\operatorname{cl}(C_\delta(x)) = \left\{ x + \sum_{i=1}^{n} \lambda_i \cdot e_i : |\lambda_i| \leq \delta \text{ for } i = 1, \ldots, n \right\},$$

$$\partial(C_\delta(x)) = \left\{ x + \sum_{i=1}^{n} \lambda_i \cdot e_i \in \operatorname{cl}(C_\delta(x)) : \max_{i=1,\ldots,n} |\lambda_i| = \delta \right\}.$$

Let $x \in E$; we claim that f is continuous at x. Since E is open, we can take $\delta > 0$ such that $C_{2\delta}(x) \subseteq E$. Let $\square$ range over $\{+, -\}$ and $j = 1, \ldots, n$. Set

$$C_j^{\square} := \left\{ x + \sum_{i=1}^{n} \lambda_i \cdot e_i \in \partial(C_\delta(x)) : \lambda_j = \square\, \delta \right\},$$

$$E_j^{\square} := \left\{ x + \sum_{i=1}^{n} \lambda_i \cdot e_i \in C_{2\delta}(x) : \lambda_j = \square\, \delta \right\}.$$

Obviously, each $E_j^{\square}$ is a relatively open convex subset of a d-dimensional affine subspace of R^n, which contains $C_j^{\square}$. By induction hypothesis, $f\restriction E_j^{\square}$, and hence also $f\restriction C_j^{\square}$, are continuous. Since $C_j^{\square}$ is closed and bounded, $f\restriction C_j^{\square}$ is bounded (cf. [12, Proposition 1.10]). Therefore $f\restriction\partial(C_\delta(x))$ is bounded, since $\partial(C_\delta(x))$ is the union of the sets $C_j^{\square}$. Now let $y_1, y_2 \in \partial(C_\delta(x))$ be such that $x \in (y_1, y_2)$, so $x = \frac{y_1+y_2}{2}$. By Lemma 1.4, for $0 \leq \lambda \leq 1$ we have

$$\begin{aligned}
\left| f\big((1-\lambda)\cdot x + \lambda \cdot y_1\big) - f(x) \right|, \left| f\big((1-\lambda)\cdot x + \lambda \cdot y_2\big) - f(x) \right| \\
\leq \lambda \cdot \max\left\{ |f(y_1) - f(x)|, |f(y_2) - f(x)| \right\} \\
\leq \lambda \cdot \max\left\{ |f(y) - f(x)| : y \in \partial C_\delta(x) \right\}.
\end{aligned}$$

Since

$$C_\delta(x) = \bigcup_{\substack{y_1, y_2 \in \partial(C_\delta(x)) \\ x \in (y_1, y_2)}} (y_1, y_2),$$

this yields that f is continuous at x. $\square$

By replacing f by $-f$, we also get the following corollary:

Corollary 1.13. *If E is an open subset of an affine subspace of R^n and f is concave, then f is continuous.*

1.4. **Convex cell decomposition.** We now assume that our expansion $\boldsymbol{R}$ of R is o-minimal. Given a definable subset E of R^n, we set

$$\begin{aligned}
\mathrm{C}(E) &:= \{ f \colon E \to R : f \text{ is definable and continuous} \}, \\
\mathrm{C}_\infty(E) &:= \mathrm{C}(E) \cup \{-\infty, +\infty\},
\end{aligned}$$

where $+\infty$, $-\infty$ are the constant functions on E with values $+\infty$, $-\infty$, respectively. For a definable $E \subseteq R^n$, we also let

$$\begin{aligned}
\mathrm{Aff}(E) &:= \{ f \colon E \to R : f \text{ is affine} \} \subseteq \mathrm{C}(E), \\
\mathrm{Aff}_\infty(E) &:= \mathrm{Aff}(E) \cup \{-\infty, +\infty\} \subseteq \mathrm{C}_\infty(E).
\end{aligned}$$

We refer to [7, Chapter 3] for the definition of cells, and the Cell Decomposition Theorem in o-minimal structures. We now define a particular kind of cell:

Definition 1.14. We define $(i_1,\dots,i_n)$**-convex cells** in R^n, where $(i_1,\dots,i_n)$ is a sequence of 0's and 1's, by induction on n: The unique $(\ )$-convex cell in R^0 is R^0. Suppose $(i_1,\dots,i_n)$-convex cells in R^n have been defined already; then

(1) an $(i_1,\dots,i_n,0)$-convex cell in R^{n+1} is the graph of some $f\in\operatorname{Aff}(D)$, where D is an $(i_1,\dots,i_n)$-convex cell in R^n;
(2) an $(i_1,\dots,i_n,1)$-convex cell is a set
$$(f,g)=\big\{(x,t)\in D\times R : f(x)<t<g(x)\big\}$$
where D is an $(i_1,\dots,i_n)$-convex cell in R^n and $f,g\in\mathrm{C}_\infty(D)$ are such that f is convex, g is concave, and $f<g$.

A straightforward induction on n, using Corollaries 1.3 and 1.11, shows that these special cells are precisely the convex cells:

Lemma 1.15. *Let C be an $(i_1,\dots,i_n)$-cell in R^n. Then C is convex if and only if C is an $(i_1,\dots,i_n)$-convex cell.*

We now show an important property of convex cells:

Lemma 1.16. *Let $C\subseteq R^n$ be a convex cell. Then for all distinct elements x, y of C, there is $\delta\in R^{>0}$ such that*
$$\big(x-\delta(y-x),y+\delta(y-x)\big)=\big((1+\delta)x-\delta y,\ -\delta x+(1+\delta)y\big)\subseteq C.$$

Proof. We proceed by induction on n. The case $n=0$ being trivial, assume the statement is true for a given value of n. Let C be an $(i_1,\dots,i_{n+1})$-convex cell. Set $D=\pi(C)$, where π is the projection $R^{n+1}\to R^n$ onto the first n coordinates. First, suppose $i_{n+1}=0$. Then $C=\Gamma(f)$ where $f\in\operatorname{Aff}(D)$. Let $(x,f(x))\neq(y,f(y))$ be elements of C. Then $x\neq y$, hence by induction hypothesis, we can take $\delta_0>0$ in R such that
$$\big(x-\delta_0(y-x),y+\delta_0(y-x)\big)\subseteq D.$$
In particular, setting $\delta:=\delta_0/2$,
$$x_0:=x-\delta(y-x),\quad y_0:=y+\delta(y-x)$$
are elements of $\pi(C)$, and so
$$\begin{aligned}(1+\delta)\,(x,f(x))-\delta\,(y,f(y))&=(x_0,f(x_0)),\\ \delta\,(x,f(x))+(1+\delta)\,(y,f(y))&=(y_0,f(y_0)).\end{aligned}$$
Since C is convex, $\big((x_0,f(x_0)),(y_0,f(y_0))\big)\subseteq C$. Now suppose $i_{n+1}=1$. Let $f,g\in\mathrm{C}_\infty(D)$ be such that $f<g$ and $C=(f,g)$. We assume that $f,g\in\mathrm{C}(D)$, since the cases where $f\equiv-\infty$ or $g\equiv+\infty$ are similar, and simpler. Let $(x,s)\neq(y,t)$ be elements of C. Suppose first that $x=y$; we may then assume $s<t$. Take $\delta>0$ in R such that $\delta(t-s)<s-f(x),g(x)-t$; then
$$f(x)<(1+\delta)s-\delta t<-\delta s+(1+\delta)t<g(x)$$
as required. Now suppose $x\neq y$. By induction hypothesis, take $\delta_0\in R^{>0}$ such that
$$\big(x-\delta_0(y-x),y+\delta_0(y-x)\big)\subseteq D.$$
Let
$$\epsilon:=\min\big\{\,|s-f(x)|\,,|g(x)-s|\,,|t-f(y)|\,,|g(y)-t|\,\big\}\in R^{>0}.$$

Since f, g are continuous at both x and y, we can take $0 < \delta_1 < \delta_0$ such that

$$\begin{aligned} f(x') < s - \epsilon < s + \epsilon < g(x') \qquad & \text{for } x' \in \big((1+\delta_1)x - \delta_1 y, x\big) \text{ and} \\ f(y') < t - \epsilon < t + \epsilon < g(y') \qquad & \text{for } y' \in \big(y, -\delta_1 x + (1+\delta_1)y\big). \end{aligned}$$

If $s = t$, let $\delta := \delta_1/2$. Otherwise, take $0 < \delta \le \frac{\delta_1}{2}$ such that $\delta|s - t| \le \epsilon$. Let $0 < \lambda < \delta$; we need to show that

$$(1+\lambda)(x, s) - \lambda(y, t) \in C, \quad -\lambda(x, s) + (1+\lambda)(y, t) \in C.$$

Now $-\epsilon < \lambda(s - t) < \epsilon$ and $x' := (1+\lambda)x - \lambda y \in \big((1+\delta_1)x - \delta_1 y, x\big)$, so

$$f(x') < s - \epsilon < s + \lambda(s - t) = (1+\lambda)s - \lambda t < s + \epsilon < g(x'),$$

hence $(1+\lambda)(x, s) - \lambda(y, t) = (x', (1+\lambda)s - \lambda t) \in C$. Similarly one shows that $-\lambda(x, s) + (1+\lambda)(y, t) \in C$. □

Corollary 1.17. *Let $C \subseteq R^{n+1}$ be convex and definable such that $D = \pi(C)$ is a cell in R^n, where $\pi\colon R^{n+1} \to R^n$ is the projection onto the first n coordinates.*

(1) *If C_x is not bounded from above for some $x \in D$, then C_x is not bounded from above for all $x \in D$.*
(2) *If C_x is not bounded from below for some $x \in D$, then C_x is not bounded from below for all $x \in D$.*
(3) *If C_x is a singleton for some $x \in D$, then C_x is a singleton for all $x \in D$.*

Proof. To show (1), let $x \in D$ be such that C_x is not bounded from above, and let $y \in D$, $y \ne x$. Let $b \in R$ be given; we show that C_y contains an element $\ge b$. By the previous lemma, take $\delta \in R^{>0}$ with $\big(x - \delta(y - x), y + \delta(y - x)\big) \subseteq D$. Let $z \in \big(y, y + \delta(y - x)\big)$, and take $s \in R$ with $(z, s) \in C$. Next, let λ satisfy $y = (1 - \lambda)z + \lambda x$; then $0 < \lambda < 1$. Take $t \in C_x$ with $t \ge \frac{1}{\lambda}\big(b - (1 - \lambda)s\big)$. Since C is convex, we have $\big((z, s), (x, t)\big) \subseteq C$. Hence $(1 - \lambda)s + \lambda t \in C_y$ with $(1 - \lambda)s + \lambda t \ge (1 - \lambda)s + \lambda\left(\frac{1}{\lambda}\big(b - (1 - \lambda)s\big)\right) = b$, as required. This shows (1), and (2) follows in a similar way.

For (3), suppose C_x is a singleton for some $x \in D$. By (1) and (2), C_x is bounded for each $x \in D$, so we may define $f, g\colon D \to R$ by $f(x) = \inf C_x$ and $g(x) = \sup C_x$. Then f is convex and g is concave, by Corollary 1.2 and the remark following it, so $h := f - g$ is convex, and $h(x) \le 0$ for all $x \in D$. Let now $x \in D$ be such that $|C_x| = 1$ (so $h(x) = 0$), and let $y \in D$, $y \ne x$. By the previous lemma, take $\delta \in R^{>0}$ with $\big(x - \delta(y - x), y + \delta(y - x)\big) \subseteq D$. Then applying Lemma 1.5 to the restriction of h to $\big(x - \delta(y - x), y + \delta(y - x)\big)$ shows that $h(y) = 0$, hence $|C_y| = 1$. □

At the beginning of this section, we claimed that every definable, closed, convex and non-empty subset of R^n is the closure of a cell. Now, the precise statement and its proof will be given.

Theorem 1.18. *Let $E \subseteq R^n$ be definable, convex, and non-empty. Then there is a convex cell C in R^n such that $C \subseteq E \subseteq \mathrm{cl}(C)$, and there exist an affine transformation T of R^n and an open cell D in R^d, where $d = \dim(E)$, such that $T(C) = D \times \{0\}^{n-d}$.*

In the proof of this theorem, we use the following lemma. We let $\pi\colon R^{n+1} \to R^n$ again be the projection onto the first n coordinates.

Lemma 1.19. *Let $T'\colon R^n \to R^n$ be a definable continuous bijection, and let*

$$T\colon R^{n+1} \to R^{n+1}, \quad T(x,t) = (T'(x),t) \text{ for } x \in R^n,\ t \in R.$$

Let C be a cell in R^{n+1}. If $T'(\pi(C))$ is a cell in R^n, then $T(C)$ is a cell in R^{n+1}.

Proof. Set $C' := \pi(C)$, and suppose that $T'(C')$ is a cell. If $C = \Gamma(f)$ where $f \in \mathrm{C}(C')$, then we have $T(C) = \Gamma(F)$ where $F := \big(f \circ (T')^{-1}\big) \restriction T'(C') \in \mathrm{C}(T'(C'))$. Similarly, if $C = (f,g)$ where $f,g \in \mathrm{C}_\infty(C')$ with $f < g$, then $T(C) = (f',g')$ where $f' := \big(f \circ (T')^{-1}\big) \restriction T'(C')$ and $g' := \big(g \circ (T')^{-1}\big) \restriction T'(C')$ are elements of $\mathrm{C}_\infty(T'(C'))$ with $f' < g'$. □

Proof of Theorem 1.18. We proceed by induction on n. If $n = 0$, then this is trivial. Assume this theorem holds for a certain value of n, and let $E \neq \emptyset$ be a definable convex subset of R^{n+1} and $d = \dim(E)$. Then $E' := \pi(E)$ is a definable convex non-empty subset of R^n, so by induction hypothesis, there is a convex cell $C' \subseteq R^n$ such that $C' \subseteq E' \subseteq \mathrm{cl}(C')$ and an affine transformation T' of R^n with $T'(C') = D' \times \{0\}^{n-d'}$, where $d' = \dim E'$ and D' is an open cell in $R^{d'}$. Let us first assume that T' is the identity, so

$$C' = D' \times \{0\}^{n-d'} \subseteq E' = \pi(E) \subseteq \mathrm{cl}(C') = \mathrm{cl}(D') \times \{0\}^{n-d'}.$$

We are going to show that there is a convex cell C in R^{n+1} with $C \subseteq E \subseteq \mathrm{cl}(C)$ and $\pi(C) = C'$ and an affine transformation T of R^{n+1} such that $T(C) = D \times \{0\}^{n+1-d}$ where $d = \dim(E)$ and D is an open cell in R^d.

Define $f\colon C' \to R \cup \{-\infty\}$ and $g\colon C' \to R \cup \{+\infty\}$ by

$$\begin{aligned} f(x) &= \inf E_x = \inf\{t \in R : (x,t) \in E\} \\ g(x) &= \sup E_x = \sup\{t \in R : (x,t) \in E\} \qquad (x \in C'). \end{aligned}$$

Note that for $x \in C'$, we have $f(x) \leq g(x)$, and $f(x) = -\infty$ if and only if E_x is not bounded from below, $g(x) = +\infty$ if and only if E_x is not bounded from above. Hence parts (1) and (2) of Corollary 1.17 imply the following two claims.

Claim 1. *Suppose $f(x) = -\infty$ for some $x \in C'$. Then $f \equiv -\infty$.*

Claim 2. *Suppose $g(x) = +\infty$ for some $x \in C'$. Then $g \equiv +\infty$.*

These two claims and Corollary 1.1 and the remark after it immediately yield:

Claim 3. *f is convex and g is concave.*

By part (3) of Corollary 1.17 we have:

Claim 4. *If $f(x) = g(x)$ for some $x \in C'$, then $f = g$.*

By Claim 4, we either have $f = g$ or $f < g$, so we can now distinguish two cases:

Case 1. $f = g$.

Then, by Claim 3, f is both convex and concave, and hence affine, by Corollary 1.11. Thus $E = \Gamma(f)$ is itself a convex cell, of dimension d'. Let $F\colon R^n \to R$ be an affine extension of f, and define $T\colon R^{n+1} \to R^{n+1}$ by

$$T(x,t) := \big(x, t - F(x)\big) \quad \text{for } x \in R^n,\ t \in R.$$

Then T is an affine transformation of R^{n+1} with $T(E) = C' \times \{0\} = D' \times \{0\}^{n+1-d'}$.

Case 2. $f < g$.

By Claims 1–3, Proposition 1.12 and its Corollary 1.13, we have $f, g \in \mathrm{C}_\infty(C')$, so $C := (f, g)$ is a convex cell of dimension $d = d' + 1$ with $C \subseteq E \subseteq \mathrm{cl}(C)$. Now define $T\colon R^{n+1} \to R^{n+1}$ by

$$T(x_1, \dots, x_{d-1}, x_d, x_{d+1}, \dots, x_n, x_{n+1}) := (x_1, \dots, x_{d-1}, x_{n+1}, x_{d+1}, \dots, x_n, x_d).$$

Then T is an R-linear automorphism of R^n with $T(C) = (f', g') \times \{0\}^{n+1-d}$ where $f', g'\colon D' \to R$ are defined by $f'(x') = f(x', 0, \dots, 0)$ and $g'(x') = g(x', 0, \dots, 0)$.

This finishes the inductive step in the case where $T' = \mathrm{id}$. In the general case, define an affine transformation T of R^{n+1} by $T(x, t) := (T'(x), t)$ for $x \in R^n$, $t \in R$, and consider the definable convex set $\tilde{E} := T(E) \neq \emptyset$. Then $T'(E') = \pi(\tilde{E})$, so

$$D' \times \{0\}^{n-d'} = T'(C') \subseteq T'(E') = \pi(\tilde{E}) \subseteq \mathrm{cl}(T'(C')) = \mathrm{cl}(D') \times \{0\}^{n-d'}.$$

Hence by the above applied to $\tilde{E}$ instead of E, there is a convex cell $\tilde{C}$ in R^{n+1} with $\tilde{C} \subseteq \tilde{E} \subseteq \mathrm{cl}(\tilde{C})$ and $\pi(\tilde{C}) = T'(C')$ and an affine transformation $\tilde{T}$ of R^{n+1} such that $\tilde{T}(\tilde{C}) = D \times \{0\}^{n+1-d}$ where $d = \dim(\tilde{E}) = \dim(E)$ and D is an open cell in R^d. Then the convex subset $C := T^{-1}(\tilde{C})$ of R^{n+1} satisfies $C \subseteq E \subseteq \mathrm{cl}(C)$ and $(\tilde{T} \circ T)(C) = D \times \{0\}^{n+1-d}$, and by Lemma 1.19, C is a cell in R^{n+1}. $\square$

The first statement in the following corollary was shown by Scowcroft [19, Theorem A.9] for semilinear sets using techniques specific to that context.

Corollary 1.20. *Let E be a non-empty definable convex subset of R^n. Then E and its closure $\mathrm{cl}(E)$ have the same interior. Moreover, if E is closed, then E is the closure of a convex cell in R^n, and if E is open, then E is the image of an open convex cell under an affine transformation of R^n.*

Proof. This follows immediately from Theorem 1.18 and the following observation, whose proof we leave to the reader: if C is an open cell in R^n, then $\mathrm{int}(C) = \mathrm{int}\,\mathrm{cl}(C)$, and so if E a definable subset of R^n with $C \subseteq E \subseteq \mathrm{cl}(C)$, then $\mathrm{int}(E) = \mathrm{int}\,\mathrm{cl}(E)$. $\square$

Here is another useful consequences of Theorem 1.18. Recall that for a definable subset E of R^n and $x \in E$, one says that E is of **local dimension** d **at** x **(**$\dim_x(E) = d$**)** if there exists a definable open neighborhood V of x in R^n such that $\dim(E \cap U) = d$ for every definable open neighborhood U of x in V.

Corollary 1.21. *Let E be a non-empty definable convex subset of R^n of dimension d. Then $\dim_x E = d$ for each $x \in E$, and $\dim \mathrm{aff}(E) = d$.*

The previous corollary is used in the next section in combination with the following observation:

Lemma 1.22. *Let E be a definable subset of R^n with $\dim_x E = d$ for all $x \in E$. Let $\mathscr{C}$ be a finite collection of definable subsets of E with $E = \bigcup \mathscr{C}$. Then*

$$E = \bigcup \big\{ E \cap \mathrm{cl}(C) : C \in \mathscr{C},\ \dim(C) = d \big\}.$$

Proof. Suppose we have $x \in E$ with $x \notin \mathrm{cl}(C)$ for all $C \in \mathscr{C}$ with $\dim(C) = d$. Take $\delta > 0$ in R such that $B_\delta(x) \cap C = \emptyset$ for all such C. Then

$$B_\delta(x) \cap E = \bigcup \big\{ B_\delta(x) \cap E \cap \mathrm{cl}(C) : C \in \mathscr{C},\ \dim(C) < d \big\}$$

and hence $\dim(B_\delta(x) \cap E) < d$, a contradiction to $\dim_x E = d$. $\square$

2. Polyhedral Sets

In the rest of this paper, we work in the semilinear context. In this section, we recall the definition of polyhedral subsets of R^n and give a proof of the well-known fact (see [1, 19]) that every closed convex semilinear subset of R^n is polyhedral. Indeed, we will indicate two proofs of this result. The first one is based on Corollary 1.20 from the previous section and Theorem 2.3 below, a variant of the Cell Decomposition Theorem adapted to the semilinear situation, which will also be used in the following sections. The second one, which is perhaps more direct and additionally yields some useful uniformities, rests on an observation about unions of polyhedral sets from [5].

2.1. Affine cell decomposition.

We begin by introducing some definitions.

Definition 2.1. Let $(i_1,\dots,i_n)\in\{0,1\}^n$. We define $(i_1,\dots,i_n)$**-affine cells** in R^n by induction on n as follows: The unique $(\ )$-affine cell in R^0 is R^0. Suppose $(i_1,\dots,i_n)$-affine cells in R^n have been defined already; then

(1) an $(i_1,\dots,i_n,0)$-affine cell is the graph $\Gamma(f)$ of some $f\in\operatorname{Aff}(D)$, where D is an $(i_1,\dots,i_n)$-affine cell;
(2) an $(i_1,\dots,i_n,1)$-affine cell is a set (f,g) where D is an $(i_1,\dots,i_n)$-affine cell and $f,g\in\operatorname{Aff}_\infty(D)$ with $f<g$.

Each $(i_1,\dots,i_n)$-affine cell is an $(i_1,\dots,i_n)$-convex cell in R^n, as defined in Definition 1.14. We say that a semilinear subset of R^n is an **affine cell** if it is an $(i_1,\dots,i_n)$-affine cell for some $(i_1,\dots,i_n)$.

Definition 2.2. Let E be a semilinear subset of R^n. An **affine cell decomposition of** E is a finite partition of E into affine cells. We say that an affine cell decomposition $\mathscr{C}$ of E is **compatible** with a given subset E' of E if, for each $C\in\mathscr{C}$, either $C\subseteq E'$ or $C\cap E'=\emptyset$.

Every semilinear function $f\colon E\to R$ is piecewise affine, i.e., there are disjoint semilinear subsets $E_1,\dots,E_N$ of E such that $E=E_1\cup\dots\cup E_N$ and $f{\restriction}E_i$ is affine for $i=1,\dots,N$. (See [7, Chapter 1, Corollary 7.6].) The following theorem strengthens this remark. (See also [19, Corollaries A.2 and A.3].)

Theorem 2.3.

(I_n) *Let $E,E_1,\dots,E_N$ be semilinear subsets of R^n such that $E_i\subseteq E$ for all $i=1,\dots,N$. Then there is an affine cell decomposition of E compatible with $E_1,\dots,E_N$.*
(II_n) *Let $f\colon E\to R$ be a semilinear function where E is a semilinear subset of R^n. Then there is an affine cell decomposition $\mathscr{C}$ of E such that $f{\restriction}C$ is affine for every $C\in\mathscr{C}$.*

Proof. We will use the same strategy as in the proof of Cell Decomposition Theorem (see [7, Chapter 3]):

$$(\mathrm{I}_1),\quad (\mathrm{I}_n)\Rightarrow(\mathrm{II}_n),\quad \text{and}\quad (\mathrm{I}_n)+(\mathrm{II}_n)\Rightarrow(\mathrm{I}_{n+1}).$$

Here (I_1) is obvious. To show $(\mathrm{I}_n)\Rightarrow(\mathrm{II}_n)$, let $f\colon E\to R$ be semilinear. By the above remark, take disjoint semilinear subsets $E_1,\dots,E_N$ of E such that $E=E_1\cup\dots\cup E_N$ and $f{\restriction}E_i$ is affine for $i=1,\dots,N$. Applying (I_n) to each E_i now yields (II_n). Next, we show $(\mathrm{I}_n)+(\mathrm{II}_n)\Rightarrow(\mathrm{I}_{n+1})$. Thus, suppose (I_n) and (II_n) hold, and let $E,E_1,\dots,E_N$ be semilinear subsets of R^{n+1} with $E_i\subseteq E$ for all $i=1,\dots,N$. First, by the Cell Decomposition Theorem, we get a cell decomposition $\mathscr{D}$ of E compatible with $E_1, \dots, E_N$. For $D = \Gamma(f) \in \mathscr{D}$ where $f \in \mathrm{Aff}(\pi(D))$, by (II$_n$) there is an affine cell decomposition $\mathscr{C}_D$ of $\pi(D)$ such that $f{\restriction}C$ is affine for every $C \in \mathscr{C}_D$. Suppose $D = (f, g) \in \mathscr{D}$ where $f, g \in \mathrm{Aff}_\infty(\pi(D))$ and $f < g$. Applying (II$_n$) to both f and g, there are affine cell decompositions $\mathscr{C}_1$ and $\mathscr{C}_2$ of $\pi(D)$ such that $f{\restriction}C$ is affine for every $C \in \mathscr{C}_1$ and $g{\restriction}C$ is affine for all $C \in \mathscr{C}_2$. By (I$_n$), we can refine those affine cell decompositions and get an affine cell decomposition $\mathscr{C}_D$ of $\pi(D)$ such that $f{\restriction}C$ and $g{\restriction}C$ are affine for every $C \in \mathscr{C}_D$.

Now, by (I$_n$), there is an affine cell decomposition $\mathscr{C}'$ of R^n which is compatible with all cells in $\bigcup_{D \in \mathscr{D}} \mathscr{C}_D$. Then

$$\mathscr{C} := \big\{ D \cap (C' \times R) : D \in \mathscr{D},\ C' \in \mathscr{C}' \big\}$$

is an affine cell decomposition of E compatible with $E_1, \dots, E_N$. □

Corollary 2.4. *Let $f\colon E \to R$ be semilinear and continuous, where $E \subseteq R^n$ is convex. Then f is Lipschitz, that is, there is some $L \in R^{>0}$ such that*

$$|f(x) - f(y)| \le L||x - y||_\infty \quad \text{for all } x, y \in E.$$

Proof. By (II$_n$) in Theorem 2.3, let $\mathscr{C}$ be an affine cell decomposition of E such that $f{\restriction}C$ is affine for every $C \in \mathscr{C}$. Define a graph whose vertex set is $\mathscr{C}$, with vertices $C \neq C'$ connected by an edge if $C \cap \mathrm{cl}(C') \neq \emptyset$ or $\mathrm{cl}(C) \cap C' \neq \emptyset$. Since E is definably connected, this graph is connected, by [7, Chapter 3, (2.19), Exercise 5]. Together with the fact that each affine function is Lipschitz, this yields the claim. □

As a consequence of the preceding corollary, every continuous semilinear function $E \to R$ is uniformly continuous, and hence (see, e.g., the argument in [2, proof of Lemma 1.7]) extends (uniquely) to a continuous semilinear function $\mathrm{cl}(E) \to R$.

Corollary 2.5. *Let $f\colon E \to R$ be semilinear and continuous, where $E \subseteq R^n$ is convex. Then there is a finite set $\mathscr{C}$ of disjoint affine cells of dimension $\dim(E)$ such that $E = \bigcup\{E \cap \mathrm{cl}(C) : C \in \mathscr{C}\}$ and for each $C \in \mathscr{C}$, $f{\restriction}E \cap \mathrm{cl}(C)$ is affine.*

Proof. By (II$_n$) in Theorem 2.3, let $\mathscr{C}'$ be an affine cell decomposition of E such that $f{\restriction}C$ is affine for every $C \in \mathscr{C}'$; by continuity, also $f{\restriction}E \cap \mathrm{cl}(C)$ is affine for every $C \in \mathscr{C}'$. By Corollary 1.21, we have $\dim_x E = \dim E$ for each $x \in E$. Hence by Lemma 1.22, we have $E = \bigcup\{E \cap \mathrm{cl}(C) : C \in \mathscr{C}\}$. □

2.2. **Polyhedral sets.** A subset E of R^n is said to be **polyhedral** if E is the intersection of finitely many closed halfspaces of R^n; that is, if there exists an $l \times n$-matrix A over R, for some $l \in \mathbb{N}$, and a column vector $c \in R^l$ such that

$$E = \{x \in R^n : Ax \ge c\}.$$

Here $\ge$ denotes the coordinate-wise ordering of R^l. We say that a map $E \to R^m$ ($E \subseteq R^n$) is **polyhedral** if its graph, viewed as a subset of $R^n \times R^m = R^{n+m}$, is polyhedral.

The proof of the following basic facts about polyhedral sets is left to the reader:

Lemma 2.6.

(1) *The closure of an affine cell is polyhedral.*
(2) *Finite intersections of polyhedral subsets of R^n are polyhedral.*
(3) *If E is a polyhedral subset of R^n, then $E \times R$ is a polyhedral subset of R^{n+1}.*

See [22, Chapter 1] for more on polyhedral sets. The following can be shown using the Fourier-Motzkin elimination procedure; see [22, Section 1.2]:

Lemma 2.7. *Let $E \subseteq R^n$ be polyhedral. Then $T(E)$ is also polyhedral, for each affine map $T\colon R^n \to R^m$.*

The next lemma shows that the closure of a basic semilinear set is polyhedral (see [1, Lemma 3.6] for a different proof).

Lemma 2.8. *Let $f_1,\dots,f_M,g_1,\dots,g_N\colon R^n \to R$ be affine, and suppose*

$$E = \big\{x \in R^n : f_i(x) \geq 0 \text{ for } i = 1,\dots,M,\ g_j(x) > 0 \text{ for } j = 1,\dots,N\big\}$$

is non-empty. Then

$$\operatorname{cl}(E) = \big\{x \in R^n : f_i(x) \geq 0 \text{ for } i = 1,\dots,M,\ g_j(x) \geq 0 \text{ for } j = 1,\dots,N\big\}.$$

Proof. The inclusion "$\subseteq$" being obvious, let $x \in R^n$ such that $f_i(x) \geq 0$ for $i = 1,\dots,M$ and $g_j(x) \geq 0$ for $j = 1,\dots,N$. Let $y \in E$.

Claim. *Let $f\colon R^n \to R$ be an affine function with $f{\restriction}[x,y] \not\equiv 0$. Then f has at most one zero in $[x,y]$; i.e. $|\{z \in [x,y] : f(z) = 0\}| \leq 1$.*

(To see this, suppose $z_1, z_2 \in [x,y]$ satisfy $f(z_1) = f(z_2) = 0$ and $z_1 \neq z_2$. Then $f\big((1-\lambda)z_1 + \lambda z_2\big) = (1-\lambda)f(z_1) + \lambda f(z_2) = 0$ for every $\lambda \in R$; in particular, $f{\restriction}[x,y] \equiv 0$.)

By Intermediate Value Theorem and the above claim, $\emptyset \neq V \cap (x,y) \subseteq E$ for every neighborhood V of x. Therefore, $x \in \operatorname{cl}(E)$. □

Obviously, every polyhedral subset of R^n is semilinear, closed, and convex. Next we will show that the converse of the above statement is also true. First, some preliminary observations.

Lemma 2.9. *Let D be a polyhedral subset of R^n and $f\colon D \to R$ be a semilinear convex function. Then $\operatorname{epi}(f)$ is polyhedral.*

Proof. By Proposition 1.12, f is continuous. Hence by Corollary 2.5 and Lemma 2.6, (1), we can take affine functions $g_1\colon D_1 \to R,\dots,g_N\colon D_N \to R$, where each D_i is a non-empty polyhedral subset of R^n with $\dim(D_i) = \dim(D)$, such that $D = D_1 \cup \dots \cup D_N$ and

$$f(x) = \begin{cases} g_1(x) & \text{if } x \in D_1; \\ \vdots & \vdots \\ g_N(x) & \text{if } x \in D_N. \end{cases}$$

We extend each g_i to an affine function $R^n \to R$, also denoted by g_i. By parts (2) and (3) of Lemma 2.6, it is enough to show that, for all $(x,t) \in D \times R$, we have

$$t \geq f(x) \iff t \geq g_i(x) \text{ for } i = 1,\dots,N.$$

This follows immediately from:

Claim. *$g_i(x) \geq g_j(x)$ for all $i,j = 1,\dots,N$ and $x \in D_i$.*

To see this, let $i,j \in \{1,\dots,N\}$ and $x \in D_i$ with $g_i(x) < g_j(x)$. Since $\dim(D) = \dim(D_j)$, we can pick $x' \neq x_0$ in D_j such that $x_0 \in [x,x']$. Write $x_0 = (1-\lambda)\cdot x + \lambda\cdot x'$

where $0 \leq \lambda < 1$. Then

$$\begin{aligned} f(x_0) = g_j\big((1-\lambda)\cdot x + \lambda \cdot x'\big) &= (1-\lambda)\cdot g_j(x) + \lambda \cdot g_j(x') \\ &> (1-\lambda)\cdot g_i(x) + \lambda \cdot g_j(x') \\ &= (1-\lambda)\cdot f(x) + \lambda \cdot f(x'), \end{aligned}$$

which contradicts the convexity of f. □

Corollary 2.10. *Let D be a polyhedral subset of R^n and $f\colon D \to R$ be a semilinear concave function. Then* $\operatorname{hyp}(f)$ *is polyhedral.*

This follows by applying Lemma 2.9 to $-f$ in place of f.

Theorem 2.11. *Let E be the closure of a semilinear convex cell in R^n. Then E is polyhedral.*

Proof. We proceed by induction on n. The case $n = 0$ is trivial. Suppose this theorem is true for a certain value of n, and let E be the closure of a semilinear convex cell in R^{n+1}. Let $\pi\colon R^{n+1} \to R^n$ be the projection onto the first n coordinates. Then $\pi(E)$ is the closure of a semilinear convex cell C in R^n, hence polyhedral, by inductive hypothesis. Suppose first that $E = \operatorname{cl}(\Gamma(f))$ where $f \in \operatorname{Aff}(C)$. Then f extends to an affine function $\pi(E) = \operatorname{cl}(C) \to R$, denoted by $\tilde{f}$, and $E = \Gamma(\tilde{f})$. Since affine functions are both convex and concave, by Lemma 2.9 and Corollary 2.10, $E = \operatorname{epi}(\tilde{f}) \cap \operatorname{hyp}(\tilde{f})$ is polyhedral. Next assume $E = \operatorname{cl}((f,g))$ where $f, g \in \mathrm{C}(C)$ are convex and concave, respectively, and $f < g$. Let $\widetilde{f}, \widetilde{g}\colon \pi(E) = \operatorname{cl}(C) \to R$ be the continuous extensions of f and g, respectively. (Corollary 2.4.) Then $\tilde{f}$ is convex and $\tilde{g}$ is concave, hence $E = \operatorname{epi}(\tilde{f}) \cap \operatorname{hyp}(\tilde{g})$ is polyhedral, again by Lemma 2.9 and Corollary 2.10. The cases where $E = \operatorname{cl}((f, +\infty))$ for some convex $f \in \mathrm{C}(C)$, $E = \operatorname{cl}((-\infty, g))$ for some concave $g \in \mathrm{C}(C)$, or $E = \operatorname{cl}(C) \times R$, are similar. □

Combining Corollary 1.20 with the preceding theorem now gives:

Corollary 2.12. *The polyhedral subsets of R^n are precisely the closed convex semilinear subsets of R^n.*

2.3. **Semilinear families of closed convex sets.** The previous corollary is also shown in [1, 19], with different proofs. Alternatively, it may be deduced from a fact about polyhedral sets proved in [5]. We now outline how this is done, since the argument helps to exhibit some uniformities used later. To formulate the main theorem of [5], for $i = 1, \dots, N$ let A_i be an $l_i \times n$-matrix over R, where $l_i \in \mathbb{N}$, with rows $a_{i1}, \dots, a_{il_i} \in R^n$, and let $c_i = (c_{i1}, \dots, c_{il_i})^{\mathrm{tr}} \in R^{l_i}$. Let

$$E_i = \{x \in R^n : A_i x \geq c_i\} = \{x \in R^n : \langle a_{ij}, x\rangle \geq c_{ij} \text{ for } j = 1, \dots, l_i\}$$

be the polyhedral sets corresponding to A_i, c_i. Here and below,

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n \quad \text{for } x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in R^n.$$

Given a subset E of R^n, a row vector $a \in R^n$, and $c \in R$, we say that an inequality $\langle a, x\rangle \geq c$ is **valid** for E if $\langle a, x\rangle \geq c$ holds for all $x \in E$. Let now (B_i, d_i) be the $m_i \times (n+1)$-matrix (for some $m_i \in \{0, \dots, l_i\}$) whose rows are those rows (a_{ij}, c_{ij}) of (A_i, c_i) such that $\langle a_{ij}, x\rangle \geq c_{ij}$ is valid for $E_1 \cup \cdots \cup E_N$. Consider now the polyhedral subset

$$E := \{x \in R^n : B_1 x \geq d_1 \;\&\; \cdots \;\&\; B_N x \geq d_N\}$$

of R^n. Then:

Proposition 2.13 ([5, Theorem 3]). *The closed semilinear set $E_1 \cup \cdots \cup E_N$ is convex if and only if $E = E_1 \cup \cdots \cup E_N$.*

In [5], this proposition is stated and proved only for the case where $R = \mathbb{R}$ is the usual ordered field of real numbers, but the proof given there, including the proofs of the basic properties of polyhedral sets used therein (such as Motzkin's Theorem [22, Theorem 1.2]), go through for an arbitrary ordered field R. From Lemma 2.8 in combination with Proposition 2.13, we immediately obtain a uniform version of Corollary 2.12:

Corollary 2.14. *Let $\{E_y\}_{y \in R^M}$ be a semilinear family of closed convex subsets E_y of R^n. Then there are $l \times n$-matrices $A_1, \dots, A_N$ over R, for some $N, l \in \mathbb{N}$, $N \geq 1$, such that for each $y \in R^M$ there are $i \in \{1, \dots, N\}$ and $c \in R^l$ with*

$$E_y = \{x \in R^n : A_i x \geq c\}.$$

3. The Semilinear Tietze Extension Theorem

Tietze's Extension Theorem is one of the most well-known theorems in basic topology: specialized to closed subsets of $\mathbb{R}^n$, it says that *every continuous function on a closed subset of $\mathbb{R}^n$ has an extension to a continuous function $\mathbb{R}^n \to \mathbb{R}$.* In this special case, an explicit construction of this extension can be given which preserves definability in a given expansion of the ordered field of real numbers. For this see, e.g., [2, Section 6]. However, neither the construction given there nor the one in [7, Chapter 8], which is specific to the o-minimal context, preserves semilinearity of functions; the goal of this section is to specify such a construction. Our main tool for this is the following theorem by Ovchinnikov [13, 14] on the representation of continuous piecewise affine functions on closed convex semilinear sets. We say that a semilinear subset E of R^n is a **closed domain** in R^n if E is the closure of a non-empty open subset of R^n (i.e., if E is closed and of dimension n).

Theorem 3.1 (Ovchinnikov, [13, Theorem 4.2]). *Let $f : E \to R$ be a continuous function on a closed convex domain E in R^n, let $\mathscr{C}$ be a finite set of closed domains in R^n with $E = \bigcup \mathscr{C}$, and for each $C \in \mathscr{C}$ let $f_C : R^n \to R$ be an affine function with $f \restriction C = f_C \restriction C$. Then there is a family $\{\mathscr{C}_i\}_{i \in I}$ of subsets of $\mathscr{C}$ such that*

$$f(x) = \max_{i \in I} \min_{C \in \mathscr{C}_i} f_C(x) \qquad \text{for } x \in E.$$

In [13, 14], this theorem is proved under the assumption that R is the ordered field of real numbers (and not assuming that closed domains are semilinear, or even definable in some o-minimal expansion of the ordered field of reals); however, the proof given there goes through under the hypotheses stated above. Theorem 3.1 combined with Corollary 2.5 immediately implies the following special case of Tietze's Extension Theorem.

Corollary 3.2. *Every semilinear continuous function $E \to R$ on a closed convex domain E in R^n extends to a semilinear continuous function $R^n \to R$.*

In this section we show:

Theorem 3.3 (Semilinear Tietze's Theorem). *Let $f : E \to R^m$ be a continuous semilinear map, where $E \subseteq R^n$ is bounded. Then f extends to a continuous semilinear map $R^n \to R^m$.*

Before we turn to the proof of this theorem, we want to point out that the boundedness condition on the domain E of f is necessary:

Example 3.4. Let

$$E = \big\{(x,y) \in R^2 : x \leq 0\big\} \cup \big\{(x,y) \in R^2 : x \geq 1\big\}$$

and define $f\colon E \to R$ by

$$f(x,y) = \begin{cases} y & \text{if } x \geq 1; \\ -y & \text{if } x \leq 0. \end{cases}$$

Then f has no extension to a semilinear continuous functions $R^2 \to R$. (This follows immediately from (II_2) in Theorem 2.3 and the observation that if $g\colon [a,b] \times (c,+\infty) \to R$ is affine, where $a,b,c \in R$, then $g(a,t) = g(b,t)$ for all $t > c$.)

We precede the proof of Theorem 3.3 by a recapitulation of some definitions and basic facts concerning simplexes and complexes from [7, Chapter 8].

Let $a_0,\dots,a_d \in R^n$. The affine hull of $\{a_0,\dots,a_d\}$ is

$$\operatorname{aff}(\{a_0,\dots,a_d\}) = \left\{ \sum_{i=0}^{d} \lambda_i a_i : \lambda_0,\dots,\lambda_d \in R^{\geq 0},\ \sum_{i=0}^{d} \lambda_i = 1 \right\}.$$

A tuple $a_0,\dots,a_d$ of elements of R^n is said to be **affine independent** if the affine hull of $\{a_0,\dots,a_d\}$ has dimension d; equivalently, if the d vectors $a_1-a_0,\dots,a_d-a_0$ are linearly independent. Such an affine independent tuple $a_0,\dots,a_d \in R^n$ is said to span the simplex

$$(a_0,\dots,a_d) := \left\{ \sum_{i=0}^{d} \lambda_i a_i : \lambda_i \in R^{>0} \text{ for } i=0,\dots,d,\ \sum_{i=0}^{d} \lambda_i = 1 \right\}.$$

We call $(a_0,\dots,a_d)$ a d**-simplex** in R^n. The closure of $(a_0,\dots,a_d)$ in R^n is denoted by $[a_0,\dots,a_d]$, so

$$[a_0,\dots,a_d] = \left\{ \sum_{i=0}^{d} \lambda_i a_i : \lambda_i \in R^{\geq 0} \text{ for } i=0,\dots,d,\ \sum_{i=0}^{d} \lambda_i = 1 \right\}.$$

One calls $a_0,\dots,a_d$ the vertices of $(a_0,\dots,a_d)$. A **face** of $(a_0,\dots,a_d)$ is a simplex spanned by a non-empty subset of $\{a_0,\dots,a_d\}$. Then distinct non-empty subsets of $\{a_0,\dots,a_d\}$ span disjoint faces, and $[a_0,\dots,a_d]$ is the union of the faces of $(a_0,\dots,a_d)$.

Definition 3.5. A **complex** in R^n is a finite collection K of simplexes in R^n such that for all $\sigma_1,\sigma_2 \in K$, either

(1) $\operatorname{cl}(\sigma_1) \cap \operatorname{cl}(\sigma_2) = \emptyset$, or
(2) $\operatorname{cl}(\sigma_1) \cap \operatorname{cl}(\sigma_2) = \operatorname{cl}(\tau)$ for some common face τ of σ_1 and σ_2.

A subset of a complex K in R^n is itself a complex in R^n, called a **subcomplex** of K. Given a complex K, we let $|K|$ denote the union of the simplexes in K, called the **polyhedron** spanned by K, and we let $\operatorname{Vert}(K)$ be the set of vertices of the simplexes in K.

The polyhedron spanned by a complex in R^n is a bounded semilinear subset of R^n, and conversely, each bounded semilinear subset of R^n is the polyhedron of a complex in R^n; more generally, we have:

Proposition 3.6 (see [7, Chapter 8, (2.14), Exercise 2])**.** *Let $E_1, \dots, E_N$ be semilinear subsets of a bounded semilinear set $E \subseteq R^n$. Then there is a complex K in R^n such that $E = |K|$ and each E_i is a union of simplexes in K.*

A complex K is said to be **closed** if it contains all faces of each of its simplexes. Equivalently, a complex K in R^n is closed iff $|K|$ is closed in R^n.

Let $S \subseteq R^n$. We denote by $\operatorname{conv}(S)$ the **convex hull** of S, that is, the smallest convex subset of R^n that contains S. It is easy to see that $\operatorname{conv}(S)$ consists of all sums $\lambda_1 s_1 + \cdots + \lambda_m s_m$ where $\lambda_i \in R^{>0}$, $s_i \in S$ for $i = 1, \dots, m$, $m \geq 1$, with $\lambda_1 + \cdots + \lambda_m = 1$. Thus if S is finite, then $\operatorname{conv}(S)$ is semilinear. For example, if $a_0, \dots, a_d \in R^n$ are affine independent, then $\operatorname{conv}\{a_0, \dots, a_d\} = [a_0, \dots, a_d]$.

Corollary 3.7. *Let E be a closed, bounded, and semilinear subset of R^n. Then $\operatorname{conv}(E)$ is also closed, bounded, and semilinear.*

Proof. By Proposition 3.6 take a complex K with $|K| = E$. Then K is closed, so $\operatorname{Vert}(K) \subseteq E$ and $\operatorname{conv}(E) = \operatorname{conv}(\operatorname{Vert}(K))$ is semilinear. □

Let K be a complex in R^n and $f\colon E \to R$, where $E \subseteq R^n$. We say that K is **compatible with** f if $|K| = E$ and $f{\restriction}\sigma$ is affine for every $\sigma \in K$. By Theorem 2.3, (II_n) and Proposition 3.6:

Lemma 3.8. *Let E be a bounded semilinear set, $E_1, \dots, E_N \subseteq E$ be semilinear, and $f\colon D \to R$ be semilinear, where $D \subseteq E$. Then there is a complex K in R^n such that $|K| = E$ and each E_i is a union of simplices of K, and a subcomplex of K which is compatible with f.*

The following is a mild generalization of [7, Chapter 8, (1.6)]:

Lemma 3.9. *Let K be a complex in R^n and $f_0\colon \operatorname{Vert}(K) \to R$. Then there is a unique $f\colon |K| \to R$ which extends f_0 such that K is compatible with f. This extension f of f_0 is continuous and semilinear.*

Next we show:

Lemma 3.10. *Let E be a closed and bounded semilinear subset of R^n and $f\colon E \to R$ be continuous and semilinear. Then f has an extension to a continuous semilinear map $\operatorname{conv}(E) \to R$.*

Proof. By Lemma 3.8, let K be a complex in R^n such that $|K| = \operatorname{conv}(E)$ which contains a subcomplex compatible with f. Define $F_0\colon \operatorname{Vert}(K) \to R$ by

$$F_0(x) = \begin{cases} f(x) & \text{if } x \in E; \\ 0 & \text{otherwise.} \end{cases}$$

By the previous lemma, the unique extension of F_0 to a map $F\colon \operatorname{conv}(E) \to R$ such that K is compatible with F has the required properties. □

We now give the proof of Theorem 3.3. We will prove this theorem by induction on n, the case $n = 0$ being obvious. Suppose we have shown the theorem for some value of n, and let E be a bounded semilinear subset of R^{n+1} and $f\colon E \to R^m$ be continuous and semilinear. We may assume that $m = 1$. By the remark following Corollary 2.4 we may assume that E is closed, and by Lemma 3.10 we may assume that E is convex, and then further by Corollary 3.2, that $\dim(E) \leq n$. Next, by Theorem 1.18, after replacing f by $f^* = f \circ T\colon T^{-1}(E) \to R$, for a suitable affine transformation T of R^{n+1}, we reduce to the case that $E \subseteq R^n \times \{0\}$. Let $\pi\colon R^{n+1} \to R^n$ be the projection onto the first n coordinates, and define $g\colon \pi(E) \to R$ by $g(x) = f(x, 0)$. By the induction hypothesis, there is a continuous semilinear function $G\colon R^n \to R$ such that $G{\restriction}E = g$. Then $F\colon R^{n+1} \to R$ defined by $F(x,t) = G(x)$ for every $x \in R^n$ is a continuous semilinear extension of f. □

4. The Semilinear Michael Selection Theorem

Throughout this section, we fix a semilinear set-valued map $T\colon X \rightrightarrows R^n$ with domain $X \subseteq R^m$. We say that T has closed convex values if $T(x)$ is closed and convex, for each $x \in X$. In this section, we focus on semilinear set-valued maps with closed convex values. First, we look back to Theorem 2.3, and show an analogue of part (II_n) of this theorem for semilinear set-valued maps with closed convex values. We then study a particular selection of T, the *least norm selection* of T. Finally, we apply the results obtained so far to prove our semilinear version of Michael's Selection Theorem.

4.1. **Semilinear set-valued maps with closed convex values.** We say that T is **polyhedral** if $\Gamma(T)$ is a polyhedral subset of $R^m \times R^n = R^{m+n}$. (Note that then T has closed convex values, and by Lemma 2.7, the domain E of T is automatically polyhedral.) We let $\mathrm{cl}(T)$ be the set-valued map $R^m \rightrightarrows R^n$ whose graph is the closure of the graph $\Gamma(T)$ of T.

Lemma 4.1. *Suppose that T has closed convex values. Then there is an affine cell decomposition $\mathscr{C}$ of X such that for every $C \in \mathscr{C}$:*

(1) $\mathrm{cl}(T{\restriction}C)\colon \mathrm{cl}(C) \rightrightarrows R^n$ *is polyhedral, and*
(2) $\mathrm{cl}(T{\restriction}C){\restriction}C = T{\restriction}C$.

Proof. By Theorem 2.3 and Corollary 2.14, we may assume that there exists an $l \times m$-matrix A such that for each $x \in E$ there is some $c \in R^l$ with

$$T(x) = \{y \in R^n : Ay \geq c\}.$$

By Definable Choice [7, (1.2), (i)], let $f\colon X \to R^l$ be semilinear such that

$$T(x) = \big\{y \in R^n : Ay \geq f(x)\big\} \qquad \text{for every } x \in X.$$

Next, by Theorem 2.3, let $\mathscr{C}$ be an affine cell decomposition of X such that $f{\restriction}C$ is affine for every $C \in \mathscr{C}$. For each $C \in \mathscr{C}$, the closure $\mathrm{cl}(C)$ of C is polyhedral, and denoting by $\tilde{f}$ the extension of f to an affine map $\mathrm{cl}(C) \to R^n$, we have

$$\Gamma\big(\mathrm{cl}(T{\restriction}C)\big) = \mathrm{cl}\left(\Gamma(T{\restriction}C)\right) = \big\{(x,y) \in \mathrm{cl}(C) \times R^n : Ay \geq \tilde{f}(x)\big\}.$$

Thus $\mathrm{cl}(T{\restriction}C)$ is polyhedral, and for $x \in C$ we have $\mathrm{cl}(T{\restriction}C)(x) = T(x)$. □

4.2. **The least norm selection.** Let A be an $l \times n$-matrix over R, with rows $a_1, \dots, a_l \in R^n$. For a non-empty subset $J = \{j_1, \dots, j_m\}$ of $\{1, \dots, l\}$, $j_1 < \cdots < j_m$, we let A_J denote the $m \times n$-matrix with rows $a_{j_1}, \dots, a_{j_m}$. Similarly, for a column vector $b \in R^l$ and $\emptyset \neq J \subseteq \{1, \dots, l\}$, viewing b as an $l \times 1$-matrix, we define $b_J \in R^m$ where $m = |J|$. We let

$$E := \{x \in R^n : Ax \geq b\}$$

be the polyhedral subset of R^n defined by A, b, and assume $E \neq \emptyset$. For a differentiable function $f\colon R^n \to R$ and $x \in R^n$, we set $\nabla f(x) := \Big(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\Big)$, viewed as a column vector in R^n.

Lemma 4.2 (Kuhn-Tucker conditions)**.** *Suppose that R is real closed. Let $f\colon R^n \to R$ be a convex differentiable semialgebraic function, and $x_0 \in E$. Then*

$$f(x_0) = \inf\big\{f(x) : x \in E\big\}$$

if and only if there exist $J \subseteq \{1,\dots,l\}$, $J \neq \emptyset$, and $w \in R^l$ such that

$$\nabla f(x_0) = (A_J)^{\mathrm{tr}} w_J, \quad w_J \geq 0, \quad A_J x_0 = b_J.$$

Proof. For $R = \mathbb{R}$ (and without assuming that f is semialgebraic), this holds by the Kuhn-Tucker Theorem [18, Corollary 28.3.1]. The lemma is a consequence of this fact and the completeness of the theory of real closed ordered fields ("Tarski Principle"). □

In the following, we let R^* be the real closure of the ordered field R. For each polyhedral set $E = \{x \in R^n : Ax \geq b\}$ as above we denote by

$$E^* := \{x \in (R^*)^n : Ax \geq b\}$$

the polyhedral subset of $(R^*)^n$ defined by the same data A, b (so $E = E^* \cap R^n$). The fact (mentioned in the introduction) that the ordered R-linear space R^* is an elementary extension of the ordered R-linear space R implies that E^* only depends on E (and not on the particular choice of A and b defining E).

We denote by $||\cdot||$ the Euclidean norm on $(R^*)^n$, and by d the corresponding metric on $(R^*)^n$. Given $x \in (R^*)^n$ and a non-empty semialgebraic $S \subseteq (R^*)^n$, let

$$d(x,S) := \inf\{d(x,y) : y \in S\} \in (R^*)^{\geq 0}$$

denote the distance of x to S. If $S \subseteq (R^*)^n$ is non-empty, semialgebraic, closed, and convex, then for each $x \in (R^*)^n$ there is a unique $y \in S$ such that $d(x,y) = d(x,S)$. (See, e.g., [2, Corollary 1.11].) In particular, there is a unique element $\mathrm{lns}(E^*)$ of E^* such that $||\,\mathrm{lns}(E^*)|| = d(0,E^*)$.

Corollary 4.3. *Let $x_0 \in E^*$. Then $x_0 = \mathrm{lns}(E^*)$ if and only if there exist $J \subseteq \{1,\dots,l\}$, $J \neq \emptyset$, and $w \in R^l$ such that*

$$2x_0 = (A_J)^{\mathrm{tr}} w_J, \quad w_J \geq 0, \quad A_J x_0 = b_J. \tag{4.1}$$

In particular, $\mathrm{lns}(E^*) \in E$.

Proof. The convex differentiable semialgebraic function $f\colon (R^*)^n \to R^*$ defined by $f(x_1,\dots,x_n) = \sum_{i=1}^n x_i^2$ satisfies $\nabla f(x) = 2x$ for $x \in (R^*)^n$. We have $||x_0|| = d(0,E^*)$ if and only if $f(x_0) = \inf\{f(x) : x \in E^*\}$, and by Lemma 4.2, this is equivalent to the existence of a non-empty $J \subseteq \{1,\dots,l\}$ and $w \in (R^*)^l$ such that (4.1) holds. The rest now follows from this and R^* being an elementary extension of R. □

Thus if T has closed convex values, then we may define $\mathrm{lns}_T\colon X \to R^n$ (the **least norm selection** of T) by

$$\mathrm{lns}_T(x) = \mathrm{lns}(T(x)^*) = \text{ the unique } y \in T(x) \text{ such that } d(0,y) = d(0,T(x)^*).$$

By the corollary above, we obtain the following (perhaps slightly surprising) result:

Corollary 4.4. *The map lns_T is semilinear.*

In general, lns_T is not continuous, even if T is l.s.c. (Consider the semilinear set-valued map $T\colon R \rightrightarrows R$ with $T(x) = [0,1]$ for $x \neq 0$ and $T(0) = \{1\}$.) However, for polyhedral T, we do have:

Theorem 4.5 (Fenzel-Li)**.** *If T is polyhedral, then* lns_T *is continuous.*

This is shown in [8, Theorem 4.4] for the case $R = \mathbb{R}$. In the rest of this subsection, we give a proof of a slight generalization of this theorem in our general context. (See Lemmas 4.7 and 4.9 below.)

Definition 4.6. We call T **upper semicontinuous** (u.s.c.) or **closed** if $\Gamma(T)$ is closed in $X \times R^m$, and we say that T is **continuous** if T is both l.s.c. and u.s.c.

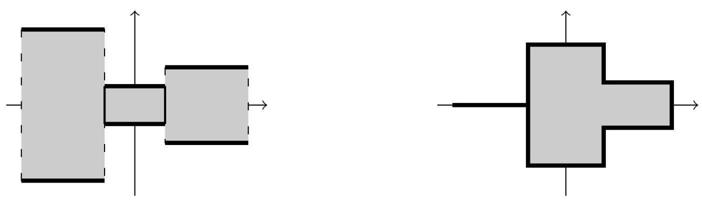

FIGURE 1. lower semicontinuous set-valued map (left); closed set-valued map (right).

Lemma 4.7. *If T is polyhedral, then T is continuous.*

Proof. Assume T is polyhedral. Every polyhedral set is closed, hence clearly T is closed. To prove the lower semicontinuity of T, it is sufficient to find, for every $x \in X$ and $y \in T(x)$, a semilinear map $F_{x,y}\colon X \to R^n$ with $\Gamma(F_{x,y}) \subseteq \Gamma(T)$ and $F_{x,y}(x') \to y$ as $x' \to x$ in X. Since T is polyhedral, the domain X of T is also polyhedral. For $\delta \in R$, $\delta > 0$, we let

$$\overline{B}_\delta(x) := \big\{ y \in R^m : ||x - y||_\infty \leq \delta \big\}$$

be the closed ball of radius δ around x; note that $\overline{B}_\delta(x)$ is polyhedral. Replacing X by $X \cap \overline{B}_1(x)$, we may assume that X is bounded. By Definable Choice, let $f\colon X \to R^n$ be a semilinear map such that $\Gamma(f) \subseteq \Gamma(T)$. By Theorem 2.3 and Proposition 3.6, let K be a complex in R^m such that $X = |K|$ and $\{x\} \in K$, and $f{\restriction}\sigma$ is affine for every $\sigma \in K$. Let

$$K' := \big\{ (a_1, \dots, a_d) \in K : (x, a_1, \dots, a_d) \in K,\ a_i \neq x \text{ for } i = 1, \dots, d \big\},$$

a subcomplex of K. For each $\sigma \in K'$, let $F_\sigma\colon \mathrm{conv}(\sigma \cup \{x\}) \to R^n$ be the unique affine extension of $f{\restriction}\sigma$ with $F_\sigma(x) = y$. Since $\Gamma(T)$ is convex, $\Gamma(F_\sigma) \subseteq \Gamma(T)$. Take some $\delta > 0$ such that

$$X \cap \overline{B}_\delta(x) \subseteq \bigcup \big\{ \mathrm{conv}(\sigma \cup \{x\}) : \sigma \in K' \big\}$$

and for every $x' \in X$ with $0 < \|x - x'\| \leq \delta$, there exists a unique $\sigma \in K'$ such that $x' \in \mathrm{conv}(\sigma \cup \{x\})$. Define $F_{x,y}\colon X \to R^n$ by

$$F_{x,y}(x') := \begin{cases} F_\sigma(x'), & \text{if } \|x - x'\|_\infty \leq \delta \text{ and } x' \in \mathrm{conv}(\sigma \cup \{x\}),\ \sigma \in K'; \\ f(x'), & \text{otherwise.} \end{cases}$$

Then $F_{x,y}$ is a semilinear map with $\Gamma(F_{x,y}) \subseteq \Gamma(T)$ and $F_{x,y}(x') \to y$ as $x' \to x$ in X, as required. $\square$

Remark. A stronger result can be deduced from [20, Section 2]: if T is polyhedral, then T is Lipschitz continuous with respect to the Hausdorff distance.

Lemma 4.8. *Suppose $m = 1$ and the domain of T is $(0,1)$, and let $(0,y) \in \mathrm{cl}(\Gamma(T))$. Then there is a semilinear continuous $f\colon (0,\epsilon) \to R^n$, for some $\epsilon > 0$, such that $f(t) \in T(t)$ for all $t \in (0,\epsilon)$ and $\lim\limits_{t\to 0^+} f(t) = y$.*

This is shown as in [3, proof of Lemma 4.2].

Lemma 4.9. *Suppose T is continuous. Then lns_T is continuous.*

Proof. Let $x_0 \in X$ and $\gamma\colon (0,1) \to X$ such that $\lim\limits_{t\to 0^+} \gamma(t) = x_0$; we need to show that $\lim\limits_{t\to 0^+} \mathrm{lns}_T(\gamma(t)) = \mathrm{lns}_T(x_0)$.

Claim. *Let $\epsilon > 0$. Then $\|\mathrm{lns}_T(\gamma(t))\| \leq \|\mathrm{lns}_T(x_0)\| + \epsilon$ as $t \to 0^+$.*

Proof of claim. Since T is l.s.c., by Lemma 4.8, after replacing γ by a suitable reparametrization of $\gamma{\restriction}(0,\epsilon_0)$, for some $\epsilon_0 \in (0,1)$, we obtain a semilinear continuous function $h\colon \gamma\big((0,1)\big) \to R^n$ such that $h(\gamma(t)) \in T(\gamma(t))$ for $t \in (0,1)$ and $\lim\limits_{t\to 0^+} h(\gamma(t)) = \mathrm{lns}_T(x_0)$. Thus $\|h(\gamma(t)) - \mathrm{lns}_T(x_0)\| \leq \epsilon$ as $t \to 0^+$, and by the definition of lns_T,

$$\|\mathrm{lns}_T(\gamma(t))\| \leq \|h(\gamma(t))\| \qquad \text{for all } t \in (0,1),$$

and the claim follows. □

By the claim, the limit $y_0 = \lim\limits_{t\to 0^+} \mathrm{lns}_T(\gamma(t))$ exists in R^n, and $\|y_0\| \leq \|\mathrm{lns}_T(x_0)\|$. Since T is closed, we have $y_0 \in T(x_0)$ and thus $y_0 = \mathrm{lns}_T(x_0)$. □

Theorem 4.5 now follows from Lemmas 4.7 and 4.9.

4.3. Statement and proof of the Semilinear Michael Selection Theorem.

We now prove the Semilinear Michael Selection Theorem from the introduction, whose statement we repeat here for the convenience of the reader:

Theorem 4.10 (Semilinear Michael Selection Theorem)**.** *Suppose T is l.s.c. with closed convex values, and the domain E of T is closed and bounded. Then T has a continuous semilinear selection.*

In the proof, we employ the following notation: given a map $g\colon X \to R^n$, define $T - g\colon X \rightrightarrows R^n$ by

$$(T-g)(x) = T(x) - g(x) = \big\{y \in R^n : y + g(x) \in T(x)\big\} \qquad \text{for every } x \in X.$$

It is easy to verify that if T is l.s.c. (u.s.c.) and g is continuous, then $T - g$ is l.s.c. (u.s.c., respectively). Moreover, if f is a selection for $T - g$, then $f + g$ is a selection for T.

Proof. We prove this theorem by induction on $d = \dim(X)$. If $d = 0$, then X is a finite set and this case is obvious. Suppose the theorem holds for all semilinear set-valued maps satisfying the hypotheses, on a domain of dimension $< d$. By Lemmas 4.1 and 4.7, let $\mathscr{C}$ be a cell decomposition of X such that $\mathrm{cl}(T{\restriction}C)$ is continuous and $\mathrm{cl}(T{\restriction}C){\restriction}C = T{\restriction}C$, for all $C \in \mathscr{C}$. Define

$$\mathscr{C}_0 := \{C \in \mathscr{C} : \dim(C) = d\}, \quad X' := \bigcup \big\{\, \mathrm{cl}(C) : C \in \mathscr{C} \setminus \mathscr{C}_0\big\}.$$

By induction hypothesis, take a continuous semilinear selection f of $T{\restriction}X'$. Next, apply the Semilinear Tietze Extension Theorem 3.3 to get a continuous semilinear map $g\colon R^m \to R^n$ such that $g\restriction X' = f$. Replacing T by $T-g$, we may assume that $g=0$ and $0 \in T(x)$ for every $x \in X'$.

Next, consider $\mathrm{lns}_T\colon X \to R^n$; by Corollary 4.4, lns_T is semilinear. To finish the proof, it remains to show that lns_T is continuous. Let $C \in \mathscr{C}_0$. Then $\mathrm{cl}(T\restriction C)$ is continuous and $0 \in T(x)$ for every $x \in \partial C \subseteq X'$. By Lemma 4.9, $\mathrm{lns}_{\mathrm{cl}(T\restriction C)}$ is continuous. Since $T(x) = \mathrm{cl}(T\restriction C)(x)$ for $x \in C$, we have $\mathrm{lns}_T(x) = \mathrm{lns}_{\mathrm{cl}(T\restriction C)}(x)$ for every $x \in C$. For $x \in \partial C$, since T is l.s.c., we have $\mathrm{lns}_T(x) = 0 = \mathrm{lns}_{\mathrm{cl}(T\restriction C)}(x)$. Therefore, $\mathrm{lns}_T \restriction \mathrm{cl}(C) = \mathrm{lns}_{\mathrm{cl}(T\restriction C)}$ and so, $\mathrm{lns}_T \restriction \mathrm{cl}(C)$ is semilinear and continuous. Since $\mathrm{lns}_T \restriction X' = 0$, $\mathscr{C}_0$ is finite, and $\partial C \subseteq X'$ for every $C \in \mathscr{C}_0$, lns_T is semilinear and continuous. $\square$

Theorem 4.10 immediately implies (see [3, proof of Corollary 4.3]):

Corollary 4.11. *Let T be as in the previous theorem, and let X_0 be a closed semilinear subset of X. Then every continuous semilinear selection of $T\restriction X_0$ extends to a continuous semilinear selection of T. In particular, given distinct $x_1, \dots, x_N \in X$ and $y_i \in T(x_i)$ for $i = 1, \dots, N$, there exists a continuous semilinear selection f of T with $f(x_i) = y_i$ for $i = 1, \dots, N$.*

The following example shows that there do exist l.s.c. semilinear set-valued maps with closed and convex values which do not admit continuous semilinear selections.

Example 4.12. Let $T\colon R^2 \to R$ be the semilinear set-valued map with

$$T(x,y) = \begin{cases} \{y\} & \text{if } x \geq 1; \\ R & \text{if } 0 < x < 1; \\ \{-y\} & \text{if } x \leq 0. \end{cases}$$

Then T does not admit a continuous semilinear selection, since any such selection would be a an extension of the function f from Example 4.12 to a continuous semilinear function $R^2 \to R$.

References

[1] C. Andradas, R. Rubio and M. P. Vélez, *An algorithm for convexity of semilinear sets over ordered fields,* manuscript (2006).

[2] M. Aschenbrenner, A. Fischer, *Definable versions of theorems by Kirszbraun and Helly*, Proc. Lond. Math. Soc. **102** (3), 2011, 468–502.

[3] M. Aschenbrenner, A. Thamrongthanyalak, *Whitney's Extension Problem in o-minimal structures,* preprint (2013).

[4] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, Systems & Control: Foundations & Applications, vol. 2, Birkhäuser Boston, Inc., Boston, MA, 1990.

[5] A. Bemporad, K. Fukuda, F. D. Torrisi, *Convexity recognition of the union of polyhedra,* Comput. Geom. **18** (2001), no. 3, 141–154.

[6] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis,* vol. 1, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[7] L. van den Dries, *Tame Topology and O-Minimal Structures*, London Math. Soc. Lecture Note Ser., vol. 248, Cambridge University Press, Cambridge, 1998.

[8] M. Finzel, W. Li, *Piecewise affine selections for piecewise polyhedral multifunctions and metric projections,* J. Convex Anal. **7** (2000), no. 1, 73–94.

[9] G. Glaeser, *Étude de quelques algèbres tayloriennes,* J. Analyse Math. **6** (1958), 1–124.

[10] B. Klartag, N. Zobin, *C^1-extensions of functions and stabilization of Glaeser refinements*, Rev. Mat. Iberoam. **23** (2007), no. 2, 635–669.

[11] E. Michael, *Continuous selections,* I, Ann. of Math. (2) **63** (1956), 361–382.

[12] C. Miller, *Expansions of dense linear orders with the intermediate value property,* J. Symbolic Logic **66** (2001), no. 4, 1783–1790.

[13] S. Ovchinnikov, *Discrete piecewise linear functions,* European J. Combin. **31** (2010), no. 5, 1283–1294.

[14] ———, *Max-Min representations of piecewise linear functions,* Beiträge Algebra Geom. **43** (2002), no. 1, 297–302.

[15] D. Repovš, P. V. Semenov, *Ernest Michael and theory of continuous selections,* Topology Appl. **155** (2008), no. 8, 755–763.

[16] ———, *Michael's theory of continuous selections. Development and applications*, Russian Math. Surveys **49** (1994), no. 6, 157–196.

[17] S. M. Robinson, *Some continuity properties of polyhedral multifunctions,* Math. Programming Stud. **14** (1981), 206–214.

[18] R. Tyrrell Rockafellar, *Convex Analysis,* Princeton Mathematical Series, vol. 28, Princeton University Press, Princeton, NJ, 1970.

[19] P. Scowcroft, *A representation of convex semilinear sets,* Algebra Universalis **62** (2009), no. 2-3, 289–327.

[20] D. W. Walkup, R. J.-B. Wets, *A Lipschitzian characterization of convex polyhedra,* Proc. Amer. Math. Soc. **23** (1969), no. 1, 167–173.

[21] D. Yost, *There can be no Lipschitz version of Michael's selection theorem,* in: S. Choy et al. (eds.), *Proceedings of the Analysis Conference, Singapore 1986,* pp. 295–299, North-Holland Mathematics Studies, vol. 150, North-Holland Publishing Co., Amsterdam, 1988.

[22] G. M. Ziegler, *Lectures on Polytopes,* Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

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