CHAPTER 1

Euler Equations and Related Hyperbolic Conservation Laws

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Abstract

Some aspects of recent developments in the study of the Euler equations for compressible fluids and related hyperbolic conservation laws are analyzed and surveyed. Basic features and phenomena including convex entropy, symmetrization, hyperbolicity, genuine nonlinearity, singularities, BV bound, concentration and cavitation are exhibited. Global well-posedness for discontinuous solutions, including the BV theory and the L^∞ theory, for the one-dimensional Euler equations and related hyperbolic systems of conservation laws is described. Some analytical approaches including techniques, methods and ideas, developed recently, for solving multidimensional steady problems are presented. Some multidimensional unsteady problems are analyzed. Connections between entropy solutions of hyperbolic conservation laws and divergence-measure fields, as well as the theory of divergence-measure fields, are discussed. Some further trends and open problems on the Euler equations and related multidimensional conservation laws are also addressed.

Keywords: Adiabatic, Clausius–Duhem inequality, Compensated compactness, Compressible fluids, Conservation laws, Divergence-measure fields, Entropy solutions, Euler equations, Finite difference schemes, Free boundary approaches, Gauss–Green formula, Genuine nonlinearity, Geometric fluids, Glimm scheme, Hyperbolicity, Ill-posedness, Isentropic, Isothermal, Lax entropy inequality, Potential flow, Self-similar, Singularity, Supersonic shocks, Supersonic vortex sheets, Traces, Transonic shocks, Multidimension, Well-posedness

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1. Introduction

Hyperbolic conservation laws, quasilinear hyperbolic systems in divergence form, are one of the most important classes of nonlinear partial differential equations, which typically take the following form:

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^d, \tag{1.1}$$

where $\nabla_{\mathbf{x}} = (\partial_{x_1}, \dots, \partial_{x_d})$ and

$$\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_d) : \mathbb{R}^n \to \left(\mathbb{R}^n\right)^d$$

is a nonlinear mapping with $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 1, \dots, d$.

Consider plane wave solutions

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \mathbf{x} \cdot \omega) \quad \text{for } \omega \in \mathcal{S}^{d-1}.$$

Then $\mathbf{w}(t, \xi)$ satisfies

$$\partial_t \mathbf{w} + \big(\nabla \mathbf{f}(\mathbf{w}) \cdot \omega\big)\, \partial_\xi \mathbf{w} = 0,$$

where $\nabla = (\partial_{w_1}, \dots, \partial_{w_n})$.

In order that there is a stable plane wave solution, it requires that, for any $\omega \in \mathcal{S}^{d-1}$,

$$\big(\nabla \mathbf{f}(\mathbf{w}) \cdot \omega\big)_{n \times n} \text{ have } n \text{ real eigenvalues } \lambda_i(\mathbf{w}; \omega) \text{ and be diagonalizable,}$$
$$1 \leqslant i \leqslant n. \tag{1.2}$$

Based on this, we say that system (1.1) is hyperbolic in a state domain $\mathcal{D}$ if condition (1.2) holds for any $\mathbf{w} \in \mathcal{D}$ and $\omega \in \mathcal{S}^{d-1}$.

The simplest example for multidimensional hyperbolic conservation laws is the following scalar conservation law

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathbf{f}(u) = 0, \quad u \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d, \tag{1.3}$$

with $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$ nonlinear. Then

$$\lambda(u, \omega) = \mathbf{f}'(u) \cdot \omega.$$

Therefore, any scalar conservation law is hyperbolic.

As is well known, the study of the Euler equations in gas dynamics gave birth to the theory of hyperbolic conservation laws so that the system of Euler equations is an archetype of this class of nonlinear partial differential equations. In general, the Euler equations for compressible fluids in $\mathbb{R}^d$ are a system of $d+2$ conservation laws

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot \mathbf{m} = 0 & \text{(Euler 1755–1759)}, \\ \partial_t \mathbf{m} + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right) + \nabla_{\mathbf{x}} p = 0 & \text{(Cauchy 1827–1829)}, \\ \partial_t E + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{m}}{\rho}(E+p)\right) = 0 & \text{(Kirchhoff 1868)} \end{cases} \tag{1.4}$$

for $(t,\mathbf{x}) \in \mathbb{R}^{d+1}_+$, $\mathbb{R}^{d+1}_+ = \mathbb{R}_+ \times \mathbb{R}^d := (0,\infty) \times \mathbb{R}^d$. System (1.4) is closed by the constitutive relations

$$p = p(\rho, e), \qquad E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e. \tag{1.5}$$

In (1.4) and (1.5), $\tau = 1/\rho$ is the deformation gradient (specific volume for fluids, strain for solids), $\mathbf{v} = (v_1, \dots, v_d)^\top$ is the fluid velocity with $\rho\mathbf{v} = \mathbf{m}$ the momentum vector, p is the scalar pressure and E is the total energy with e the internal energy which is a given function of (τ, p) or (ρ, p) defined through thermodynamical relations. The notation $\mathbf{a}\otimes\mathbf{b}$ denotes the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$. The other two thermodynamic variables are temperature θ and entropy S. If (ρ, S) are chosen as the independent variables, then the constitutive relations can be written as

$$(e, p, \theta) = \big(e(\rho, S), p(\rho, S), \theta(\rho, S)\big) \tag{1.6}$$

governed by

$$\theta\, \mathrm{d}S = \mathrm{d}e + p\, \mathrm{d}\tau = \mathrm{d}e - \frac{p}{\rho^2}\, \mathrm{d}\rho. \tag{1.7}$$

For a polytropic gas,

$$p = R\rho\theta, \qquad e = c_{\mathrm{v}}\theta, \qquad \gamma = 1 + \frac{R}{c_{\mathrm{v}}} \tag{1.8}$$

and

$$p = p(\rho, S) = \kappa\rho^\gamma \mathrm{e}^{S/c_{\mathrm{v}}}, \quad e = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}\mathrm{e}^{S/c_{\mathrm{v}}}, \tag{1.9}$$

where $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas, $c_{\mathrm{v}} > 0$ is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent and $\kappa > 0$ can be any positive constant by scaling.

As shown in Section 2.4, no matter how smooth the initial data is, the solution of (1.4) generally develops singularities in a finite time. Then system (1.4) is complemented by the Clausius–Duhem inequality

$$\partial_t(\rho S) + \nabla_{\mathbf{x}} \cdot (\mathbf{m}S) \geqslant 0 \quad \text{(Clausius 1854, Duhem 1901)} \tag{1.10}$$

in the sense of distributions in order to single out physical discontinuous solutions, so-called *entropy solutions*.

When a flow is isentropic, that is, entropy S is a uniform constant S_0 in the flow, then the Euler equations for the flow take the following simpler form

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot \mathbf{m} = 0, \\ \partial_t \mathbf{m} + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho}\right) + \nabla p = 0, \end{cases} \tag{1.11}$$

where the pressure is regarded as a function of density, $p = p(\rho, S_0)$, with constant S_0. For a polytropic gas,

$$p(\rho) = \kappa \rho^\gamma, \quad \gamma > 1, \tag{1.12}$$

where $\kappa > 0$ can be any positive constant under scaling. This system can be derived from (1.4) as follows. It is well known that, for smooth solutions of (1.4), entropy $S(\rho, \mathbf{m}, E)$ is conserved along fluid particle trajectories, i.e.,

$$\partial_t(\rho S) + \nabla_{\mathbf{x}} \cdot (\mathbf{m} S) = 0. \tag{1.13}$$

If the entropy is initially a uniform constant and the solution remains smooth, then (1.13) implies that the energy equation can be eliminated, and entropy S keeps the same constant in later time. Thus, under the constant initial entropy, a smooth solution of (1.4) satisfies the equations in (1.11). Furthermore, it should be observed that solutions of system (1.11) are also a good approximation to solutions of system (1.4) even after shocks form, since the entropy increases across a shock to third order in wave strength for solutions of (1.4), while in (1.11) the entropy is constant. Moreover, system (1.11) is an excellent model for the isothermal fluid flow with $\gamma = 1$ and for the shallow water flow with $\gamma = 2$.

In the one-dimensional case, system (1.4) in Eulerian coordinates is

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p\right) = 0, \\ \partial_t E + \partial_x \left(\frac{m}{\rho}(E + p)\right) = 0 \end{cases} \tag{1.14}$$

with $E = \frac{1}{2}\frac{m^2}{\rho} + \rho e$. The system above can be rewritten in Lagrangian coordinates in one-to-one correspondence as long as the fluid flow stays away from vacuum $\rho = 0$,

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t \left(e + \frac{v^2}{2}\right) + \partial_x (pv) = 0 \end{cases} \tag{1.15}$$

with $v = m/\rho$, where the coordinates (t, x) are the Lagrangian coordinates, which are different from the Eulerian coordinates for (1.14); for simplicity of notation, we do not distinguish them. For the isentropic case, systems (1.14) and (1.15) reduce to

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x\left(\frac{m^2}{\rho} + p\right) = 0 \end{cases} \tag{1.16}$$

and

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \end{cases} \tag{1.17}$$

respectively, where pressure p is determined by (1.12) for the polytropic case, $p = p(\rho) = \tilde{p}(\tau)$ with $\tau = 1/\rho$. The solutions of (1.16) and (1.17), even for entropy solutions, are equivalent (see [52,332]).

This chapter is organized as follows. In Section 2 we exhibit some basic features and phenomena of the Euler equations and related hyperbolic conservation laws such as convex entropy, symmetrization, hyperbolicity, genuine nonlinearity, singularities and BV bound. In Section 3 we describe some aspects of a well-posedness theory and related results for the one-dimensional isentropic, isothermal and adiabatic Euler equations, respectively. In Sections 4–7 we discuss some samples of multidimensional models and problems for the Euler equations with emphasis on the prototype models and problems that have been solved or expected to be solved rigorously at least for some cases. In Section 8 we discuss connections between entropy solutions of hyperbolic conservation laws and divergence-measure fields, as well as the theory of divergence-measure fields to construct a good framework for studying entropy solutions. Some analytical approaches including techniques, methods, and ideas, developed recently, for solving multidimensional problems are also presented.

2. Basic features and phenomena

In this section we exhibit some basic features and phenomena of the Euler equations and related hyperbolic conservation laws.

2.1. *Convex entropy and symmetrization*

A function $\eta : \mathcal{D} \to \mathbb{R}$ is called an entropy of system (1.1) if there exists a vector function $\mathbf{q} : \mathcal{D} \to \mathbb{R}^d$, $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_d)$, satisfying

$$\nabla \mathbf{q}_i(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u}), \quad i = 1, \ldots, d. \tag{2.1}$$

An entropy $\eta(\mathbf{u})$ is called a convex entropy in $\mathcal{D}$ if

$$\nabla^2 \eta(\mathbf{u}) \geqslant 0 \quad \text{for any } \mathbf{u} \in \mathcal{D}$$

and a strictly convex entropy in $\mathcal{D}$ if

$$\nabla^2\eta(\mathbf{u}) \geqslant c_0 I$$

with a constant $c_0 > 0$ uniform for $\mathbf{u} \in \mathcal{D}_1$ for any $\mathcal{D}_1 \subset \overline{\mathcal{D}}_1 \Subset \mathcal{D}$, where I is the $n \times n$ identity matrix. Then the correspondence of (1.10) in the context of hyperbolic conservation laws is the Lax entropy inequality

$$\partial_t\eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \leqslant 0 \tag{2.2}$$

in the sense of distributions for any C^2 convex entropy–entropy flux pair $(\eta, \mathbf{q})$.

THEOREM 2.1. *A system in* (1.1) *endowed with a strictly convex entropy* η *in a state domain* $\mathcal{D}$ *must be symmetrizable and hence hyperbolic in* $\mathcal{D}$.

PROOF. Taking ∇ of both sides of the equations in (2.1) with respect to $\mathbf{u}$, we have

$$\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u}) + \nabla\eta(\mathbf{u})\nabla^2\mathbf{f}_i(\mathbf{u}) = \nabla^2\mathbf{q}_i(\mathbf{u}), \quad i = 1, \ldots, d.$$

Using the symmetry of the matrices

$$\nabla\eta(\mathbf{u})\nabla^2\mathbf{f}_i(\mathbf{u}) \quad \text{and} \quad \nabla^2\mathbf{q}_i(\mathbf{u})$$

for fixed $i = 1, 2, \ldots, d$, we find that

$$\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u}) \text{ is symmetric.} \tag{2.3}$$

Multiplying (1.1) by $\nabla^2\eta(\mathbf{u})$, we get

$$\nabla^2\eta(\mathbf{u})\,\partial_t\mathbf{u} + \sum_{i=1}^{d} \nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u})\nabla_{x_i}\mathbf{u} = 0. \tag{2.4}$$

The fact that the matrices $\nabla^2\eta(\mathbf{u}) > 0$ and $\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u}), i = 1, 2, \ldots, d$, are symmetric implies that system (1.1) is symmetrizable. Notice that any symmetrizable system must be hyperbolic, which can be seen as follows.

Since $\nabla^2\eta(\mathbf{u}) > 0$ for $\mathbf{u} \in \mathcal{D}$, then the hyperbolicity of (1.1) is equivalent to the hyperbolicity of (2.4), while the hyperbolicity of (2.4) is equivalent to that, for any $\omega \in \mathcal{S}^{d-1}$,

$$\text{all zeros of the determinant } \left|\lambda\nabla^2\eta(\mathbf{u}) - \nabla^2\eta(\mathbf{u})\nabla\mathbf{f}(\mathbf{u}) \cdot \omega\right| \text{ are real.} \tag{2.5}$$

Since $\nabla^2\eta(\mathbf{u})$ is real symmetric and positive definite, there exists a matrix $C(\mathbf{u})$ such that

$$\nabla^2\eta(\mathbf{u}) = C(\mathbf{u})C(\mathbf{u})^\top.$$

Then the hyperbolicity is equivalent to that, for any $\omega \in S^{d-1}$, the eigenvalues of the following matrix

$$C(\mathbf{u})^{-1}\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}(\mathbf{u})\cdot\omega\big(C(\mathbf{u})^{-1}\big)^\top \tag{2.6}$$

are real, which is true since the matrix is real and symmetric. This completes the proof. □

REMARK 2.1. This theorem is particularly useful to determine whether a large physical system is symmetrizable and hence hyperbolic, since most of physical systems from continuum physics are endowed with a strictly convex entropy. In particular, for system (1.4),

$$(\eta_*, \mathbf{q}_*) = (-\rho S, -\mathbf{m}S) \tag{2.7}$$

is a strictly convex entropy–entropy flux pair when $\rho > 0$ and $p > 0$; while, for system (1.11), the mechanical energy and energy flux

$$(\eta_*, \mathbf{q}_*) = \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e(\rho), \frac{\mathbf{m}}{\rho}\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e(\rho) + p(\rho)\right)\right) \tag{2.8}$$

is a strictly convex entropy–entropy flux pair when $\rho > 0$ for polytropic gases. For multidimensional hyperbolic systems of conservation laws without a strictly convex entropy, it is possible to enlarge the system so that the enlarged system is endowed with a globally defined, strictly convex entropy. See [29,111,113,275,295].

REMARK 2.2. The observation that systems of conservation laws endowed with a strictly convex entropy must be symmetrizable is due to Godunov [155–157], Friedrich and Lax [140] and Boillat [22]. See also [284].

REMARK 2.3. This theorem has many important applications in the energy estimates. Basically, the symmetry plays an essential role in the following situation: For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\begin{aligned} &2\mathbf{u}^\top\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\,\partial_{x_k}\mathbf{u} \\ &\quad = \partial_{x_k}\big(\mathbf{u}^\top\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\mathbf{u}\big) - \mathbf{u}^\top\,\partial_{x_k}\big(\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\big)\mathbf{u} \end{aligned} \tag{2.9}$$

for $k = 1, 2, \dots, d$. This is very useful to make energy estimates for various problems.

There are several direct, important applications of Theorem 2.1 based on the symmetry property of system (1.1) endowed with a strictly convex entropy such as (2.9). We list three of them below.

2.1.1. *Local existence of classical solutions.* Consider the Cauchy problem for a general hyperbolic system (1.1) with a strictly convex entropy η whose Cauchy data is

$$\mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{2.10}$$

THEOREM 2.2. *Assume that* $\mathbf{u}_0 : \mathbb{R}^d \to \mathcal{D}$ *is in* $H^s \cap L^\infty$ *with* $s > d/2 + 1$. *Then, for the Cauchy problem* (1.1) *and* (2.10), *there exists a finite time* $T = T(\|\mathbf{u}_0\|_s, \|\mathbf{u}_0\|_{L^\infty}) \in (0, \infty)$ *such that there is a unique bounded classical solution* $\mathbf{u} \in C^1([0, T] \times \mathbb{R}^d)$ *with*

$$\mathbf{u}(t, \mathbf{x}) \in \mathcal{D} \quad \textit{for } (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^d$$

and

$$\mathbf{u} \in C\big([0, T]; H^s\big) \cap C^1\big([0, T]; H^{s-1}\big).$$

Kato [184,185] first formulated and applied a basic idea in the semigroup theory to yield the local existence of smooth solutions to (1.1).

The proof of this theorem in [241] relies solely on the elementary linear existence theory for symmetric hyperbolic systems with smooth coefficients via a classical iteration scheme (cf. [101]) by using the symmetry of system (1.1), especially (2.9). In particular, for all $\mathbf{u} \in \mathcal{D}$, there is a positive definite symmetric matrix $A_0(\mathbf{u}) = \nabla^2 \eta(\mathbf{u})$ that is smooth in $\mathbf{u}$ and satisfies

$$c_0 \mathbf{I} \leqslant A_0(\mathbf{u}) \leqslant c_0^{-1} \mathbf{I} \tag{2.11}$$

with a constant $c_0 > 0$ uniform for $\mathbf{u} \in \mathcal{D}_1$, for any $\mathcal{D}_1 \subset \overline{\mathcal{D}}_1 \Subset \mathcal{D}$, such that $A_i(\mathbf{u}) = A_0(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u})$ is symmetric. Moreover, a sharp continuation principle was also provided: For $\mathbf{u}_0 \in H^s$ with $s > d/2 + 1$, the interval $[0, T)$ with $T < \infty$ is the maximal interval of the classical H^s existence for (1.1) if and only if either

$$\big\|(\mathbf{u}_t, D\mathbf{u})(t, \cdot)\big\|_{L^\infty} \to \infty \quad \text{as } t \to T,$$

or

$$\mathbf{u}(t, \mathbf{x}) \text{ escapes every compact subset } K \Subset \mathcal{D} \quad \text{as } t \to T.$$

The first catastrophe in this principle is associated with the formation of shock waves and vorticity waves, among others, in the smooth solutions, and the second is associated with a blow-up phenomenon such as focusing and concentration.

In [246], Makino, Ukai and Kawashima established the local existence of classical solutions of the Cauchy problem with compactly supported initial data for the multidimensional Euler equations, with the aid of the theory of quasilinear symmetric hyperbolic systems; in particular, they introduced a symmetrization which works for initial data having either compact support or vanishing at infinity. There are also discussions in [48] on the local existence of smooth solutions of the three-dimensional Euler equations (1.4) by using an identity to deduce a time decay of the internal energy and the Mach number.

The local existence and stability of classical solutions of the initial–boundary value problem for the multidimensional Euler equations can be found in [182,189,191] and the references cited therein.

2.1.2. *Stability of Lipschitz solutions, rarefaction waves, and vacuum states in the class of entropy solutions in L^∞*

THEOREM 2.3. *Assume that system* (1.1) *is endowed with a strictly convex entropy η on compact subsets of $\mathcal{D}$. Suppose that $\mathbf{v}$ is a Lipschitz solution of* (1.1) *on* $[0, T)$, *taking values in a convex compact subset K of $\mathcal{D}$, with initial data $\mathbf{v}_0$. Let $\mathbf{u}$ be any entropy solution of* (1.1) *on* $[0, T)$, *taking values in K, with initial data $\mathbf{u}_0$. Then*

$$\int_{|\mathbf{x}|<R} |\mathbf{u}(t,\mathbf{x}) - \mathbf{v}(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \leqslant C(T) \int_{|\mathbf{x}|<R+Lt} |\mathbf{u}_0(\mathbf{x}) - \mathbf{v}_0(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}$$

holds for any $R > 0$ and $t \in [0, T)$, with $L > 0$ depending solely on K and the Lipschitz constant of $\mathbf{v}$.

The main point for the proof of Theorem 2.3 is to use the relative entropy–entropy flux pair (cf. [105])

$$\alpha(\mathbf{u},\mathbf{v}) = \eta(\mathbf{u}) - \eta(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{u} - \mathbf{v}), \tag{2.12}$$

$$\beta(\mathbf{u},\mathbf{v}) = \mathbf{q}(\mathbf{u}) - \mathbf{q}(\mathbf{v}) - \nabla\eta(\mathbf{v})\big(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})\big) \tag{2.13}$$

and to calculate and find

$$\partial_t \alpha(\mathbf{u},\mathbf{v}) + \nabla_{\mathbf{x}} \cdot \beta(\mathbf{u},\mathbf{v}) \leqslant -\big\{\partial_t\big(\nabla\eta(\mathbf{v})\big)(\mathbf{u} - \mathbf{v}) + \nabla_{\mathbf{x}}\big(\nabla\eta(\mathbf{v})\big)\big(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})\big)\big\}.$$

Since $\mathbf{v}$ is a classical solution, we use the symmetry property of system (1.1) with the strictly convex entropy η to have

$$\begin{aligned}
\partial_t\big(\nabla\eta(\mathbf{v})\big) &= (\partial_t \mathbf{v})^\top \nabla^2\eta(\mathbf{v}) \\
&= -\sum_{k=1}^{d} (\partial_{x_k}\mathbf{v})^\top \big(\nabla\mathbf{f}_k(\mathbf{v})\big)^\top \nabla^2\eta(\mathbf{v}) \\
&= -\sum_{k=1}^{d} (\partial_{x_k}\mathbf{v})^\top \nabla^2\eta(\mathbf{v}) \nabla\mathbf{f}_k(\mathbf{v}).
\end{aligned}$$

Therefore, we have

$$\partial_t \alpha(\mathbf{u},\mathbf{v}) + \nabla_{\mathbf{x}} \cdot \beta(\mathbf{u},\mathbf{v}) \leqslant -\sum_{k=1}^{d} (\partial_{x_k}\mathbf{v})^\top \nabla^2\eta(\mathbf{v}) Q\mathbf{f}_k(\mathbf{u},\mathbf{v}),$$

where

$$Q\mathbf{f}_k(\mathbf{u},\mathbf{v}) = \mathbf{f}_k(\mathbf{u}) - \mathbf{f}_k(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{u} - \mathbf{v}).$$

Integrating over a set

$$\{(\tau, \mathbf{x}) \colon 0 \leqslant \tau \leqslant t \leqslant T, |\mathbf{x}| \leqslant R + L(t-\tau)\}$$

with the aid of the Gauss–Green formula in Section 8 and choosing $L > 0$ large enough yields the expected result.

Some further ideas have been developed to show the stability of planar rarefaction waves and vacuum states in the class of entropy solutions in L^∞ for the multidimensional Euler equations by using the Gauss–Green formula in Section 8.

THEOREM 2.4. *Let $\omega \in \mathcal{S}^{d-1}$. Let*

$$\mathbf{R}(t, \mathbf{x}) = (\hat{\rho}, \hat{\mathbf{m}})\left(\frac{\mathbf{x} \cdot \omega}{t}\right)$$

be a planar solution, consisting of planar rarefaction waves and possible vacuum states, of the Riemann problem

$$\mathbf{R}|_{t=0} = \begin{cases} (\rho_-, \hat{\mathbf{m}}_-), & \mathbf{x} \cdot \omega < 0, \\ (\rho_+, \hat{\mathbf{m}}_+), & \mathbf{x} \cdot \omega > 0, \end{cases}$$

with constant states $(\rho_\pm, \hat{\mathbf{m}}_\pm)$. Suppose $\mathbf{u}(t, \mathbf{x}) = (\rho, \mathbf{m})(t, \mathbf{x})$ is an entropy solution in L^∞ of (1.11) *that may contain vacuum. Then, for any $R > 0$ and $t \in [0, \infty)$,*

$$\int_{|\mathbf{x}|<R} \alpha(\mathbf{u}, \mathbf{R})(t, \mathbf{x})\, \mathrm{d}\mathbf{x} \leqslant \int_{|\mathbf{x}|<R+Lt} \alpha(\mathbf{u}, \mathbf{R})(0, \mathbf{x})\, \mathrm{d}\mathbf{x},$$

where $L > 0$ depends solely on the bounds of the solutions $\mathbf{u}$ and $\mathbf{R}$, and

$$\alpha(\mathbf{u}, \mathbf{R}) = (\mathbf{u} - \mathbf{R})^\top \left(\int_0^1 \nabla^2 \eta_*\big(\mathbf{R} + \tau(\mathbf{u} - \mathbf{R})\big)\, \mathrm{d}\tau\right)(\mathbf{u} - \mathbf{R})$$

with $\eta_*(\mathbf{u}) = E \equiv \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e(\rho)$.

REMARK 2.4. Theorem 2.3 is due to Dafermos [110] (also see [111]). Theorem 2.4 is due to Chen and Chen [56], where a similar theorem was also established for the adiabatic Euler equations (1.4) with appropriate chosen entropy; also see [55] and [70].

REMARK 2.5. For multidimensional hyperbolic systems of conservation laws with partially convex entropies and involutions, see [111]; also see [24,106].

REMARK 2.6. For distributional solutions to the Euler equations (1.4) for polytropic gases, it is observed in Perthame [269] that, under the basic integrability condition

$$\rho, E, \rho\mathbf{v} \cdot \mathbf{x}, |\mathbf{v}|E \in L^1_{\mathrm{loc}}\big(\mathbb{R}_+; L^1(\mathbb{R}^d)\big)$$

and the condition that entropy $S(t, \mathbf{x})$ has an upper bound, the internal energy decays in time and, furthermore, the only time-decay on the internal energy suffices to yield the time-decay of the density. Also see [48].

2.1.3. *Local existence of shock front solutions.* Shock front solutions, the simplest type of discontinuous solutions, are the most important discontinuous nonlinear progressing wave solutions in compressible Euler flows and other systems of conservation laws. For a general multidimensional hyperbolic system of conservation laws (1.1), shock front solutions are discontinuous piecewise smooth entropy solutions with the following structure:

(i) there exist a C^2 time–space hypersurface $\mathcal{S}(t)$ defined in $(t, \mathbf{x})$ for $0 \leqslant t \leqslant T$ with time–space normal $(\mathbf{n}_t, \mathbf{n}_\mathbf{x}) = (\mathbf{n}_t, \mathbf{n}_1, \ldots, \mathbf{n}_d)$ and two C^1 vector-valued functions, $\mathbf{u}^+(t, \mathbf{x})$ and $\mathbf{u}^-(t, \mathbf{x})$, defined on respective domains $\mathcal{D}^+$ and $\mathcal{D}^-$ on either side of the hypersurface $\mathcal{S}(t)$, and satisfying

$$\partial_t \mathbf{u}^\pm + \nabla \cdot \mathbf{f}(\mathbf{u}^\pm) = 0 \quad \text{in } \mathcal{D}^\pm; \tag{2.14}$$

(ii) the jump across the hypersurface $\mathcal{S}(t)$ satisfies the Rankine–Hugoniot condition

$$\left\{\mathbf{n}_t(\mathbf{u}^+ - \mathbf{u}^-) + \mathbf{n}_\mathbf{x} \cdot \left(\mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-)\right)\right\}\Big|_{\mathcal{S}} = 0. \tag{2.15}$$

For the quasilinear system (1.1), the surface $\mathcal{S}$ is not known in advance and must be determined as a part of the solution of the problem; thus the equations in (2.14) and (2.15) describe a multidimensional, highly nonlinear, free-boundary value problem for the quasilinear system of conservation laws.

The initial data yielding shock front solutions is defined as follows. Let $\mathcal{S}_0$ be a smooth hypersurface parametrized by α, and let $\mathbf{n}(\alpha) = (\mathbf{n}_1, \ldots, \mathbf{n}_d)(\alpha)$ be a unit normal to $\mathcal{S}_0$. Define the piecewise smooth initial data for respective domains $\mathcal{D}_0^+$ and $\mathcal{D}_0^-$ on either side of the hypersurface $\mathcal{S}_0$ as

$$\mathbf{u}_0(\mathbf{x}) = \begin{cases} \mathbf{u}_0^-(\mathbf{x}), & \mathbf{x} \in \mathcal{D}_0^-, \\ \mathbf{u}_0^+(\mathbf{x}), & \mathbf{x} \in \mathcal{D}_0^+. \end{cases} \tag{2.16}$$

It is assumed that the initial jump in (2.16) satisfies the Rankine–Hugoniot condition, i.e., there is a smooth scalar function $\sigma(\alpha)$ so that

$$-\sigma(\alpha)\left(\mathbf{u}_0^+(\alpha) - \mathbf{u}_0^-(\alpha)\right) + \mathbf{n}(\alpha) \cdot \left(\mathbf{f}\left(\mathbf{u}_0^+(\alpha)\right) - \mathbf{f}\left(\mathbf{u}_0^-(\alpha)\right)\right) = 0, \tag{2.17}$$

and that $\sigma(\alpha)$ does not define a characteristic direction, i.e.,

$$\sigma(\alpha) \neq \lambda_i\left(\mathbf{u}_0^\pm\right), \quad \alpha \in \overline{\mathcal{S}_0}, 1 \leqslant i \leqslant n, \tag{2.18}$$

where λ_i, $i = 1, \ldots, n$, are the eigenvalues of (1.1). It is natural to require that $\mathcal{S}(0) = \mathcal{S}_0$.

Consider the three-dimensional full Euler equations in (1.4), away from vacuum, which can be rewritten in the form

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \quad \mathbf{x} \in \mathbb{R}^3, t > 0, \\ \partial_t(\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0, \\ \partial_t E + \nabla_{\mathbf{x}}\big(\mathbf{v}(E + p)\big) = 0, \end{cases} \tag{2.19}$$

with piecewise smooth initial data

$$(\rho, \mathbf{v}, E)|_{t=0} = \begin{cases} \big(\rho_0^-, \mathbf{v}_0^-, E^+\big)(\mathbf{x}), & \mathbf{x} \in \mathcal{D}_0^-, \\ \big(\rho_0^+, \mathbf{v}_0^+, E^+\big)(\mathbf{x}), & \mathbf{x} \in \mathcal{D}_0^+. \end{cases} \tag{2.20}$$

THEOREM 2.5. *Assume that $\mathcal{S}_0$ is a smooth hypersurface in $\mathbb{R}^3$ and that $(\rho_0^+, \mathbf{v}_0^+, E_0^+)(\mathbf{x})$ belongs to the uniform local Sobolev space $H_{\mathrm{ul}}^s(\mathcal{D}_0^+)$, while $(\rho_0^-, \mathbf{v}_0^-, E_0^-)(\mathbf{x})$ belongs to the Sobolev space $H^s(\mathcal{D}_0^-)$, for some fixed $s \geqslant 10$. Assume also that there is a function $\sigma(\alpha) \in H^s(\mathcal{S}_0)$ so that* (2.17) *and* (2.18) *hold, and the compatibility conditions up to order $s-1$ are satisfied on $\mathcal{S}_0$ by the initial data, together with the entropy condition*

$$\mathbf{v}_0^+ \cdot \mathbf{n}(\alpha) + \sqrt{p_\rho\big(\rho_0^+, S_0^+\big)} < \sigma(\alpha) < \mathbf{v}_0^- \cdot \mathbf{n}(\alpha) + \sqrt{p_\rho\big(\rho_0^-, S_0^-\big)}, \tag{2.21}$$

and the Majda stability condition

$$1 + \big(p(\rho_0^+) - p(\rho_0^-)\big)\frac{(\rho_0^-)^2 p_\rho(\rho_0^-, S_0^-) p_S(\rho_0^-, S_0^-)}{\theta_0^-} \\ - \big(\rho_0^-\big)^3\big(p(\rho_0^+) - p(\rho_0^-)\big)p_\rho\big(\rho_0^-, S_0^-\big) > 0. \tag{2.22}$$

Then there is a C^2 hypersurface $\mathcal{S}(t)$ together with C^1 functions $(\rho^\pm, \mathbf{v}^\pm, E^\pm)(t, \mathbf{x})$ defined for $t \in [0, T]$, with T sufficiently small, so that

$$(\rho, \mathbf{v}, E)(t, \mathbf{x}) = \begin{cases} \big(\rho^-, \mathbf{v}^-, E^-\big)(t, \mathbf{x}), & (t, \mathbf{x}) \in \mathcal{D}^-, \\ \big(\rho^+, \mathbf{v}^+, E^+\big)(t, \mathbf{x}), & (t, \mathbf{x}) \in \mathcal{D}^+, \end{cases} \tag{2.23}$$

is the discontinuous shockfront solution of the Cauchy problem (2.19) *and* (2.20) *satisfying* (2.14) *and* (2.15). *In particular, the condition in* (2.22) *is always satisfied for shocks of any strength for polytropic gas with $\gamma > 1$ and for sufficiently weak shocks for general equations of state.*

In Theorem 2.5, the uniform local Sobolev space $H_{\mathrm{ul}}^s(\mathcal{D}_0^+)$ is defined as follows: A vector function $\mathbf{u}$ is in H_{ul}^s, provided that there exists some $r > 0$ so that

$$\max_{\mathbf{y} \in \mathbb{R}^d} \|w_{r,\mathbf{y}} \mathbf{u}\|_{H^s} < \infty$$

with

$$w_{r,\mathbf{y}}(\mathbf{x}) = w\left(\frac{\mathbf{x}-\mathbf{y}}{r}\right),$$

where $w \in C_0^\infty(\mathbb{R}^d)$ is a function so that $w(\mathbf{x}) \geqslant 0$, $w(\mathbf{x}) = 1$ when $|\mathbf{x}| \leqslant 1/2$ and $w(\mathbf{x}) = 0$ when $|\mathbf{x}| > 1$.

REMARK 2.7. Theorem 2.5 is taken from [240]. The compatibility conditions in Theorem 2.5 are defined in [240] and needed in order to avoid the formation of discontinuities in higher derivatives along other characteristic surfaces emanating from $\mathcal{S}_0$: Once the main condition in (2.17) is satisfied, the compatibility conditions are automatically guaranteed for a wide class of initial data functions. Further studies on the local existence and stability of shock front solutions can be found in [239–241]. The uniform time of existence of shock front solutions in the shock strength was obtained in [249]. Also see [21] for further discussions.

The idea of the proof is similar to that for Theorem 2.2 by using the existence of a strictly convex entropy and the symmetrization of (1.1), but the technical details are quite different due to the unusual features of the problem considered in Theorem 2.5 (see [240]). The shock front solutions are defined as the limit of a convergent classical iteration scheme based on a linearization by using the theory of linearized stability for shock fronts developed in [239]. The technical condition $s \geqslant 10$, instead of $s > 1 + d/2$, is required because pseudo-differential operators are needed in the proof of the main estimates. Some improved technical estimates regarding the dependence of operator norms of pseudo-differential operators on their coefficients would lower the value of s. For more details, see [240].

2.2. *Hyperbolicity*

There are many examples of $n \times n$ hyperbolic systems of conservation laws for $\mathbf{x} \in \mathbb{R}^2$ which are strictly hyperbolic; that is, they have simple characteristics. However, for $d = 3$, there are no strictly hyperbolic systems if $n \equiv 2 \pmod 4$ or, even more generally, $n \equiv \pm 2, \pm 3, \pm 4 \pmod 8$ as a corollary of Theorem 2.6. Such multiple characteristics influence the propagation of singularities.

THEOREM 2.6. *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *be three matrices such that*

$$\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

has real eigenvalues for any real α, β *and* γ. *If* $n \equiv \pm 2, \pm 3, \pm 4 \pmod 8$, *then there exist* α_0, β_0 *and* γ_0 *with* $\alpha_0^2 + \beta_0^2 + \gamma_0^2 \neq 0$ *such that*

$$\alpha_0\mathbf{A} + \beta_0\mathbf{B} + \gamma_0\mathbf{C} \tag{2.24}$$

is degenerate, that is, there are at least two eigenvalues of matrix (2.24) *which coincide.*

PROOF. We prove only the case $n \equiv 2 \pmod 4$.

1. Denote $\mathcal{M}$ the set of all real $n \times n$ matrices with real eigenvalues, and $\mathcal{N}$ the set of nondegenerate matrices (that have n distinct real eigenvalues) in $\mathcal{M}$. The normalized eigenvectors $\mathbf{r}_j$ of $\mathbf{N}$ in $\mathcal{N}$, i.e.,

$$\mathbf{N}\mathbf{r}_j = \lambda_j \mathbf{r}_j, \qquad |\mathbf{r}_j| = 1, \quad j = 1, 2, \dots, n,$$

are determined up to a factor ± 1.

2. Let $\mathbf{N}(\theta), 0 \leqslant \theta \leqslant 2\pi$, be a closed curve in $\mathcal{N}$. If we fix $\mathbf{r}_j(0)$, then $\mathbf{r}_j(\theta)$ can be determined uniquely by requiring continuous dependence on θ.

Since

$$\mathbf{N}(2\pi) = \mathbf{N}(0),$$

then

$$\mathbf{r}_j(2\pi) = \tau_j \mathbf{r}_j(0), \quad \tau_j = \pm 1.$$

Clearly,

(i) each τ_j is a homotopy invariant of the closed curve,

(ii) each $\tau_j = 1$ when $\mathbf{N}(\theta)$ is constant.

3. Suppose now that the theorem is false. Then

$$\mathbf{N}(\theta) = \mathbf{A}\cos\theta + \mathbf{B}\sin\theta$$

is a closed curve in $\mathcal{N}$ and

$$\lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_n(\theta).$$

Since $\mathbf{N}(\pi) = -\mathbf{N}(0)$, we have

$$\lambda_j(\pi) = -\lambda_{n-j+1}(0),$$
$$\mathbf{r}_j(\pi) = \rho_j \mathbf{r}_{n-j+1}(0), \quad \rho_j \pm 1.$$

4. Since the ordered basis

$$\{\mathbf{r}_1(\theta), \mathbf{r}_2(\theta), \dots, \mathbf{r}_n(\theta)\}$$

is defined continuously, it retains its orientation. Then the ordered bases

$$\{\mathbf{r}_1(0), \mathbf{r}_2(0), \dots, \mathbf{r}_n(0)\} \quad \text{and} \quad \{\rho_1\mathbf{r}_n(0), \rho_2\mathbf{r}_{n-1}(0), \dots, \rho_n\mathbf{r}_1(0)\}$$

have the same orientation.

For the case $n \equiv 2 \pmod 4$, reversing the order reverses the orientation of an ordered basis, which implies

$$\prod_{j=1}^{n} \rho_j = -1.$$

Then there exists k such that

$$\rho_k \rho_{n-k+1} = -1. \tag{2.25}$$

5. Since $\mathbf{N}(\theta + \pi) = -\mathbf{N}(\theta)$, then

$$\lambda_j(\theta + \pi) = -\lambda_{n-j+1}(\theta),$$

which implies

$$\mathbf{r}_j(2\pi) = \rho_j \mathbf{r}_{n-j+1}(\pi) = \rho_j \rho_{n-j+1} \mathbf{r}_{n-j+1}(0).$$

Therefore, we have

$$\tau_j = \rho_j \rho_{n-j+1}.$$

Then (2.25) implies

$$\tau_k = 1,$$

which yields that the curve

$$\mathbf{N}(\theta) = \mathbf{A}\cos\theta + \mathbf{B}\sin\theta$$

is not homotopic to a point.

6. Suppose that all matrices of form $\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$, belong to $\mathcal{N}$. Then, since the sphere is simply connected, the curve $\mathbf{N}(\theta)$ could be contracted to a point, contracting $\tau_k = -1$. This completes the proof. □

REMARK 2.8. The proof is taken from [201] for the case $n \equiv 2 \pmod 4$. The proof for the more general case $n \equiv \pm 2, \pm 3, \pm 4 \pmod 8$ can be found in [138].

Consider the isentropic Euler equations (1.11).

When $d = 2, n = 3$, the system is strictly hyperbolic with three real eigenvalues $\lambda_- < \lambda_0 < \lambda_+$,

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \qquad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{p'(\rho)}, \quad \rho > 0.$$

The strict hyperbolicity fails at the vacuum states when $\rho = 0$.

However, when $d = 3, n = 4$, the system is no longer strictly hyperbolic even when $\rho > 0$ since the eigenvalue

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3$$

has double multiplicity, although the other eigenvalues

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \pm \sqrt{p'(\rho)}$$

are simple when $\rho > 0$.

Consider the adiabatic Euler equations (1.4).

When $d = 2, n = 4$, the system is nonstrictly hyperbolic since the eigenvalue

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2$$

has double multiplicity; however,

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{\frac{\gamma p}{\rho}}$$

are simple when $\rho > 0$.

When $d = 3, n = 5$, the system is again nonstrictly hyperbolic since the eigenvalue

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3$$

has triple multiplicity; however,

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \pm \sqrt{\frac{\gamma p}{\rho}}$$

are simple when $\rho > 0$.

2.3. *Genuine nonlinearity*

The jth-characteristic field of system (1.1) in $\mathcal{D}$ is called genuinely nonlinear if, for each fixed $\omega \in S^{d-1}$, the jth eigenvalue $\lambda_j(\mathbf{u}; \omega)$ and the corresponding eigenvector $\mathbf{r}_j(\mathbf{u}; \omega)$ determined by

$$(\nabla \mathbf{f}(\mathbf{u}) \cdot \omega)\mathbf{r}_j(\mathbf{u}; \omega) = \lambda_j(\mathbf{u}; \omega)\mathbf{r}_j(\mathbf{u}; \omega)$$

satisfy

$$\nabla_{\mathbf{u}} \lambda_j(\mathbf{u}; \omega) \cdot \mathbf{r}_j(\mathbf{u}; \omega) \neq 0 \quad \text{for any } \mathbf{u} \in \mathcal{D}, \omega \in S^{d-1}. \tag{2.26}$$

The jth-characteristic field of system (1.1) in $\mathcal{D}$ is called linearly degenerate if

$$\nabla_{\mathbf{u}}\lambda_j(\mathbf{u};\omega)\cdot\mathbf{r}_j(\mathbf{u};\omega)\equiv 0 \quad \text{for any } \mathbf{u}\in\mathcal{D}. \tag{2.27}$$

Then we immediately have the following theorem.

THEOREM 2.7. *Any scalar quasilinear conservation law in $\mathbb{R}^d$, $d\geqslant 2$, is never genuinely nonlinear in all directions.*

It is because, in this case,

$$\lambda(u;\omega)=\mathbf{f}'(u)\cdot\omega, \qquad r(u;\omega)=1$$

and

$$\lambda'(u;\omega)r(u;\omega)\equiv\mathbf{f}'(u)\cdot\omega$$

which is impossible to make this never equal to zero in all directions.

A multidimensional version of genuine nonlinearity for scalar conservation laws is

$$\left|\left\{u\colon \tau+\mathbf{f}'(u)\cdot\omega=0\right\}\right|=0 \quad \text{for any } (\tau,\omega)\in\mathcal{S}^d,$$

which is a generalization of (2.26).

Under this generalized nonlinearity, the following have been established:

(i) solution operators are compact as $t>0$ in [224] (also see [64,314]),
(ii) decay of periodic solutions [65,128],
(iii) initial and boundary traces of entropy solutions [82,329],
(iv) BV structure of L^∞ entropy solutions [112].

For systems with $n=2m$, $m\geqslant 1$ odd, and $d=2$, using a topological argument, we have the following theorem.

THEOREM 2.8. *Every real, strictly hyperbolic quasilinear system for $n=2m$, $m\geqslant 1$ odd, and $d=2$ is linearly degenerate in some direction.*

PROOF. We prove only for the case $m=1$.

1. For fixed $\mathbf{u}\in\mathbb{R}^n$, define

$$\mathbf{N}(\theta;\mathbf{u})=\nabla\mathbf{f}_1(\mathbf{u})\cos\theta+\nabla\mathbf{f}_2(\mathbf{u})\sin\theta.$$

Denote the eigenvalues of $\mathbf{N}(\theta;\mathbf{u})$ by $\lambda_\pm(\theta;\mathbf{u})$,

$$\lambda_-(\theta;\mathbf{u})<\lambda_+(\theta;\mathbf{u})$$

with

$$\mathbf{N}(\theta;\mathbf{u})\mathbf{r}_\pm(\theta;\mathbf{u})=\lambda_\pm(\theta;\mathbf{u})\mathbf{r}_\pm(\theta;\mathbf{u}), \qquad \left|\mathbf{r}_\pm(\theta;\mathbf{u})\right|=1. \tag{2.28}$$

This still leaves an arbitrary factor ± 1, which we fix arbitrarily at $\theta = 0$. For all other $\theta \in [0, 2\pi]$, we require $\mathbf{r}_\pm(\theta; \mathbf{u})$ to vary continuously with θ.

2. Since $\mathbf{N}(\theta + \pi; \mathbf{u}) = -\mathbf{N}(\theta; \mathbf{u})$,

$$\lambda_+(\theta + \pi; \mathbf{u}) = -\lambda_-(\theta; \mathbf{u}), \qquad \lambda_-(\theta + \pi; \mathbf{u}) = -\lambda_+(\theta; \mathbf{u}).$$

It follows from this and $|\mathbf{r}_\pm| = 1$ that

$$\mathbf{r}_+(\theta + \pi; \mathbf{u}) = \sigma_+ \mathbf{r}_-(\theta; \mathbf{u}), \qquad \mathbf{r}_-(\theta + \pi; \mathbf{u}) = \sigma_- \mathbf{r}_+(\theta; \mathbf{u}), \tag{2.29}$$

where $\sigma_\pm = 1$ or -1.

3. Since $\mathbf{r}_\pm(\theta; \mathbf{u})$ were chosen to be continuous functions of θ, we find that

(i) $\sigma_\pm$ are also continuous functions of θ and, thus, they must be constant since $\sigma_\pm = \pm 1$;

(ii) the orientation of the ordered basis $\{\mathbf{r}_-(\theta; \mathbf{u}), \mathbf{r}_+(\theta; \mathbf{u})\}$ does not change and, hence, the bases

$$\{\mathbf{r}_-(0; \mathbf{u}), \mathbf{r}_+(0; \mathbf{u})\} \quad \text{and} \quad \{\mathbf{r}_-(\pi; \mathbf{u}), \mathbf{r}_+(\pi; \mathbf{u})\}$$

have the same orientation.

Therefore, by (2.29),

$$\{\mathbf{r}_-(0; \mathbf{u}), \mathbf{r}_+(0; \mathbf{u})\} \quad \text{and} \quad \{\sigma_- \mathbf{r}_+(0; \mathbf{u}), \sigma_+ \mathbf{r}_-(0; \mathbf{u})\}$$

have the same orientation. Then

$$\sigma_+ \sigma_- = -1$$

and

$$\mathbf{r}_+(2\pi; \mathbf{u}) = \sigma_+ \mathbf{r}_-(\pi; \mathbf{u}) = \sigma_+ \sigma_- \mathbf{r}_+(0, \mathbf{u}) = -\mathbf{r}_+(0, \mathbf{u}). \tag{2.30}$$

Similarly, we have

$$\mathbf{r}_-(2\pi; \mathbf{u}) = -\mathbf{r}_-(0; \mathbf{u}). \tag{2.31}$$

4. Since the eigenvalues $\lambda_\pm(\theta; \mathbf{u})$ are periodic functions of θ with period 2π for fixed $\mathbf{u} \in \mathbb{R}^2$, so are their gradients. Then

$$\nabla_{\mathbf{u}} \lambda_\pm(2\pi; \mathbf{u}) \mathbf{r}_\pm(2\pi; \mathbf{u}) = -\nabla_{\mathbf{u}} \lambda(0; \mathbf{u}) \mathbf{r}_\pm(0; \mathbf{u}).$$

Noticing that

$$\nabla \lambda_\pm(\theta; \mathbf{u}) \mathbf{r}_\pm(\theta; \mathbf{u})$$

vary continuously with θ for any fixed $\mathbf{u} \in \mathbb{R}^2$, we conclude that there exist $\theta_\pm \in (0, 2\pi)$ such that

$$\nabla\lambda_\pm(\theta_\pm; \mathbf{u})\mathbf{r}_\pm(\theta_\pm; \mathbf{u}) = 0.$$

This completes the proof. □

REMARK 2.9. The proof of Theorem 2.8 is from [202].

REMARK 2.10. Quite often, linear degeneracy results from the loss of strict hyperbolicity. For example, even in the one-dimensional case, if there exists $j \neq k$ such that

$$\lambda_j(\mathbf{u}) = \lambda_k(\mathbf{u}) \quad \text{for all } u \in K,$$

then Boillat [23] proved that the jth- and kth-characteristic families are linearly degenerate.

For the isentropic Euler equations (1.11) with $d = 2, n = 3$,

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \qquad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{p'(\rho)},$$

and

$$\mathbf{r}_0 = (-\omega_2, \omega_1, 0)^\top, \qquad \mathbf{r}_\pm = \left(\pm\omega_1, \pm\omega_2, \frac{\rho}{\sqrt{p'(\rho)}}\right)^\top,$$

which implies

$$\nabla\lambda_0 \cdot \mathbf{r}_0 \equiv 0$$

and

$$\nabla\lambda_\pm \cdot \mathbf{r}_\pm = \pm\frac{\rho p''(\rho) + 2p'(\rho)}{2p'(\rho)} = \pm\frac{\gamma + 1}{2} \neq 0.$$

For the adiabatic Euler equations (1.4) with $d = 2, n = 4$,

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \qquad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{\frac{\gamma p}{\rho}}$$

and

$$\mathbf{r}_0 = (-\omega_2, \omega_1, 0, 1)^\top, \qquad \mathbf{r}_\pm = \left(\pm\omega_1, \pm\omega_2, \sqrt{\gamma p \rho}, \rho\sqrt{\frac{\rho}{\gamma p}}\right)^\top,$$

which implies

$$\nabla\lambda_0 \cdot \mathbf{r}_0 \equiv 0$$

and

$$\nabla\lambda_\pm \cdot \mathbf{r}_\pm = \pm\frac{\gamma+1}{2} \neq 0.$$

2.4. *Singularities*

For the one-dimensional case, singularities include the formation of shock waves and the development of vacuum states, at least for gas dynamics. For the multidimensional case, the situation is much more complicated: besides shock waves and vacuum states, singularities may include vorticity waves, focusing waves, concentration waves, complicated wave interactions, among others.

Consider the Cauchy problem of the Euler equations in (1.4) for polytropic gases in $\mathbb{R}^3$ with smooth initial data

$$(\rho, \mathbf{v}, S)|_{t=0} = (\rho_0, \mathbf{v}_0, S_0)(\mathbf{x}) \quad \text{with } \rho_0(\mathbf{x}) > 0 \text{ for } \mathbf{x} \in \mathbb{R}^3 \tag{2.32}$$

satisfying

$$(\rho_0, \mathbf{v}_0, S_0)(\mathbf{x}) = (\bar{\rho}, 0, \overline{S}) \quad \text{for } |\mathbf{x}| \geqslant R,$$

where $\bar{\rho} > 0$, $\overline{S}$ and R are constants. The equations in (1.4) possess a unique local C^1-solution $(\rho, \mathbf{v}, S)(t, \mathbf{x})$ with $\rho(t, \mathbf{x}) > 0$ provided that the initial data (2.32) is sufficiently regular (Theorem 2.2). The support of the smooth disturbance $(\rho_0(\mathbf{x}) - \bar{\rho}, \mathbf{v}_0(\mathbf{x}), S_0(\mathbf{x}) - \overline{S})$ propagates with speed at most $\sigma = \sqrt{p_\rho(\bar{\rho}, \overline{S})}$ (the sound speed), that is,

$$(\rho, \mathbf{v}, S)(t, \mathbf{x}) = (\bar{\rho}, 0, \overline{S}) \quad \text{if } |\mathbf{x}| \geqslant R + \sigma t. \tag{2.33}$$

The proof of this essential fact of finite speed of propagation for the three-dimensional case can be found in [181], as well as in [299], established through local energy estimates.

Take $\bar{p} = p(\bar{\rho}, \overline{S})$. Define

$$\begin{aligned} P(t) &= \int_{\mathbb{R}^3} \left(p(t, \mathbf{x})^{1/\gamma} - \bar{p}^{1/\gamma}\right) \mathrm{d}\mathbf{x} \\ &= \kappa^{1/\gamma} \int_{\mathbb{R}^3} \left(\rho(t, \mathbf{x}) \exp\left(\frac{S(t, \mathbf{x})}{\gamma c_v}\right) - \bar{\rho} \exp\left(\frac{\overline{S}}{\gamma c_v}\right)\right) \mathrm{d}\mathbf{x}, \\ F(t) &= \int_{\mathbb{R}^3} \rho \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{x} \, \mathrm{d}\mathbf{x}, \end{aligned}$$

which, roughly speaking, measure the entropy and the radial component of momentum. The following theorem on the formation of singularities in solutions of (1.4) and (2.32) is due to Sideris [300].

THEOREM 2.9. *Suppose that* $(\rho, \mathbf{v}, S)(t, \mathbf{x})$ *is a* C^1*-solution of* (1.4) *and* (2.32) *for* $0 < t < T$ *and*

$$P(0) \geqslant 0, \tag{2.34}$$

$$F(0) > \alpha\sigma R^4 \max_{\mathbf{x}} \rho_0(\mathbf{x}), \qquad \alpha = \frac{16\pi}{3}. \tag{2.35}$$

Then the lifespan T *of the* C^1*-solution is finite.*

PROOF. Set

$$M(t) = \int_{\mathbb{R}^3} \big(\rho(t,\mathbf{x}) - \bar{\rho}\big)\,\mathrm{d}\mathbf{x}.$$

Combining the equations in (1.4) with (2.33) and using the integration by parts, one has

$$M'(t) = -\int_{\mathbb{R}^3} \nabla \cdot (\rho\mathbf{v})\,\mathrm{d}\mathbf{x} = 0,$$

$$P'(t) = -\kappa^{1/\gamma} \int_{\mathbb{R}^3} \nabla \cdot \left(\rho\mathbf{v}\exp\left(\frac{S}{\gamma c_{\mathrm{v}}}\right)\right) \mathrm{d}\mathbf{x} = 0,$$

which implies

$$M(t) = M(0), \qquad P(t) = P(0) \tag{2.36}$$

and

$$\begin{aligned} F'(t) &= \int_{\mathbb{R}^3} \mathbf{x} \cdot (\rho\mathbf{v})_t\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} \big(\rho|\mathbf{v}|^2 + 3(p - \bar{p})\big)\,\mathrm{d}\mathbf{x} \\ &= \int_{B(t)} \big(\rho|\mathbf{v}|^2 + 3(p - \bar{p})\big)\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{2.37}$$

where $B(t) = \{\mathbf{x} \in \mathbb{R}^3\colon\ |\mathbf{x}| \leqslant R + \sigma t\}$. From Hölder's inequality, (2.34) and (2.36), one has

$$\begin{aligned} \int_{B(t)} p\,\mathrm{d}\mathbf{x} &\geqslant \frac{1}{|B(t)|^{\gamma-1}} \left(\int_{B(t)} p^{1/\gamma}\,\mathrm{d}\mathbf{x}\right)^{\gamma} \\ &= \frac{1}{|B(t)|^{\gamma-1}} \left(P(0) + \int_{B(t)} \bar{p}^{1/\gamma}\,\mathrm{d}\mathbf{x}\right)^{\gamma} \\ &\geqslant \int_{B(t)} \bar{p}\,\mathrm{d}\mathbf{x}, \end{aligned}$$

where $|B(t)|$ denotes the volume of ball $B(t)$. Therefore, by (2.37),

$$F'(t) \geqslant \int_{\mathbb{R}^3} \rho|\mathbf{v}|^2\,d\mathbf{x}. \tag{2.38}$$

By the Cauchy–Schwarz inequality and (2.36),

$$\begin{aligned} F(t)^2 &= \left(\int_{B(t)} \rho\mathbf{v}\cdot\mathbf{x}\,d\mathbf{x}\right)^2 \\ &\leqslant \int_{B(t)} \rho|\mathbf{v}|^2\,d\mathbf{x} \int_{B(t)} \rho|\mathbf{x}|^2\,d\mathbf{x} \\ &\leqslant (R+\sigma t)^2 \int_{B(t)} \rho|\mathbf{v}|^2\,d\mathbf{x}\left(M(t) + \int_{B(t)} \bar{\rho}\,d\mathbf{x}\right) \\ &\leqslant (R+\sigma t)^2 \int_{B(t)} \rho|\mathbf{v}|^2\,d\mathbf{x}\left(\int_{B(t)} \big(\rho_0(\mathbf{x}) - \bar{\rho}\big)\,d\mathbf{x} + \int_{B(t)} \bar{\rho}\,d\mathbf{x}\right) \\ &\leqslant \frac{4\pi}{3}(R+\sigma t)^5 \max_{\mathbf{x}} \rho_0(\mathbf{x}) \int_{B(t)} \rho|\mathbf{v}|^2\,d\mathbf{x}. \end{aligned}$$

Then (2.38) implies that

$$F'(t) \leqslant \left(\frac{4\pi}{3}(R+\sigma t)^5 \max_{\mathbf{x}} \rho_0(\mathbf{x})\right)^{-1} F(t)^2. \tag{2.39}$$

Since $F(0) > 0$ by (2.35), $F(t)$ remains positive for $0 < t < T$, as a consequence of (2.38). Dividing by $F(t)^2$ and integrating from 0 to T in (2.39) yields

$$F(0)^{-1} > F(0)^{-1} - F(T)^{-1} \geqslant (\alpha\sigma \max \rho_0)^{-1}\big(R^{-4} - (R+\sigma T)^{-4}\big).$$

Thus,

$$(R+\sigma T)^4 < \frac{R^4 F(0)}{F(0) - \alpha\sigma R^4 \max \rho_0}.$$

This completes the proof. □

REMARK 2.11. The proof is taken from [86], which is a refinement of Sideris [299]. The method of the proof above applies equally well in one and two space dimensions. In the isentropic case, the condition $P(0) \geqslant 0$ reduces to $M(0) \geqslant 0$.

REMARK 2.12. To illustrate a way in which conditions (2.34) and (2.35) may be satisfied, we consider the initial data

$$\rho_0 = \bar{\rho}, \qquad S_0 = \overline{S}.$$

Then $P(0) = 0$, and (2.35) holds if

$$\int_{|\mathbf{x}|<R} \mathbf{v}_0(x) \cdot \mathbf{x}\, d\mathbf{x} > \alpha\sigma R^4.$$

Comparing both sides, one finds that the initial velocity must be supersonic in some region relative to the sound speed at infinity. The formation of a singularity (presumably a shock wave) is detected as the disturbance overtakes the wave front forcing the front to propagate with supersonic speed.

The formation of singularities occurs even without condition of largeness such as (2.35). For example, if $S_0(x) \geqslant \overline{S}$ and, for some $0 < R_0 < R$,

$$\begin{aligned} &\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-1}\big(|\mathbf{x}| - r\big)^2\big(\rho_0(\mathbf{x}) - \bar{\rho}\big)\, d\mathbf{x} > 0, \\ &\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-3}\big(|\mathbf{x}|^2 - r^2\big)\rho_0(\mathbf{x})\mathbf{v}_0(\mathbf{x}) \cdot \mathbf{x}\, d\mathbf{x} \geqslant 0 \quad \text{for } R_0 < r < R, \end{aligned} \tag{2.40}$$

then the lifespan T of the C^1-solution of (1.4) and (2.32) is finite. The assumptions in (2.40) mean that, in an average sense, the gas must be slightly compressed and outgoing directly behind the wave front. For the proof in [300], some important technical points were adopted from [298] on the nonlinear wave equations in three dimensions.

REMARK 2.13. For the multidimensional Euler equations for compressible fluids with smooth initial data that is a small perturbation of amplitude ε from a constant state, the lifespan of smooth solutions is at least $O(\varepsilon^{-1})$ from the theory of symmetric hyperbolic systems [139,183]. Results on the formation of singularities show that the lifespan of a smooth solution is no better than $O(\varepsilon^{-2})$ in the two-dimensional case [276] and $O(e^{\varepsilon^{-2}})$ [300] in the three-dimensional case. See [2,301,302] for additional discussions in this direction. Also see [246] and [279] for a compressible fluid body surrounded by the vacuum.

2.5. *BV bound*

For one-dimensional strictly hyperbolic systems, Glimm's theorem [145] indicates that, as long as $\|\mathbf{u}_0\|_{BV}$ is sufficiently small, the solution $\mathbf{u}(t, x)$ satisfies the following stability estimate

$$\big\|\mathbf{u}(t, \cdot)\big\|_{BV} \leqslant C\|\mathbf{u}_0\|_{BV}. \tag{2.41}$$

Even more strongly, for two solutions $\mathbf{u}(t, x)$ and $\mathbf{v}(t, x)$ obtained by either the Glimm scheme, wave-front tracking method or vanishing viscosity method with small total variation,

$$\big\|\mathbf{u}(t, \cdot) - \mathbf{v}(t, \cdot)\big\|_{L^1(\mathbb{R})} \leqslant C\big\|\mathbf{u}(0, \cdot) - \mathbf{v}(0, \cdot)\big\|_{L^1(\mathbb{R})}.$$

See [20,33,111,167,204] and the references cited therein.

The recent great progress for entropy solutions for one-dimensional hyperbolic systems of conservation laws based on BV estimates and trace theorems of BV fields naturally arises the expectation that a similar approach may also be effective for multidimensional hyperbolic systems of conservation laws, that is, whether entropy solutions satisfy the relatively modest stability estimate

$$\left\|\mathbf{u}(t,\cdot)\right\|_{BV} \leqslant C\|\mathbf{u}_0\|_{BV}. \tag{2.42}$$

Unfortunately, this is not the case.

Rauch [278] showed that the necessary condition for (2.42) to be held is

$$\nabla\mathbf{f}_k(\mathbf{u})\nabla\mathbf{f}_l(\mathbf{u}) = \nabla\mathbf{f}_l(\mathbf{u})\nabla\mathbf{f}_k(\mathbf{u}) \quad \text{for all } k,l = 1,2,\ldots,d. \tag{2.43}$$

The analysis above suggests that only systems in which the commutativity relation (2.43) holds offer any hope for treatment in the BV framework. This special case includes the scalar case $n = 1$ and the one-dimensional case $d = 1$. Beyond that, it contains very few systems of physical interest.

An example is the system with fluxes

$$\mathbf{f}_k(\mathbf{u}) = \phi_k\big(|\mathbf{u}|^2\big)\mathbf{u}, \quad k = 1,2,\ldots,d,$$

which governs the flow of a fluid in an anisotropic porous medium. However, the recent study in [34] and [7] shows that, even in this case, the space BV is not a well-posed space, and (2.42) fails.

Even for the one-dimensional systems whose strict hyperbolicity fails or initial data is allowed to be of large oscillation, the solution is no longer in BV in general. However, some bounds in L^∞ or L^p may be achieved. One of such examples is the isentropic Euler equations (1.16), for which we have

$$\left\|\mathbf{u}(t,\cdot)\right\|_{L^\infty} \leqslant C\|\mathbf{u}_0\|_{L^\infty}.$$

See [75] and the references cited therein. However, for the multidimensional case, entropy solutions generally do not have even the relatively modest stability

$$\left\|\mathbf{u}(t,\cdot) - \bar{\mathbf{u}}\right\|_{L^p} \leqslant C_p\|\mathbf{u}_0 - \bar{\mathbf{u}}_0\|_{L^p}, \quad p \neq 2, \tag{2.44}$$

based on the linear theory by Brenner [31].

In this regard, it is important to identify good analytical frameworks for studying entropy solutions of multidimensional conservation laws (1.1), which are not in BV, or even in L^p. The most general framework is the space of divergence-measure fields, formulated recently in [67,69,83,84], which is based on the following class of entropy solutions:

(i) $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$ or $L^p(\mathbb{R}^{d+1}_+)$, $1 \leqslant p \leqslant \infty$;

(ii) for any convex entropy–entropy flux pair $(\eta,\mathbf{q})$,

$$\partial_t\eta(\mathbf{u}) + \nabla_\mathbf{x}\cdot\mathbf{q}(\mathbf{u}) \leqslant 0$$

in the sense of distributions, as long as $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))(t, \mathbf{x})$ is a distributional field.

According to the Schwartz lemma, we have

$$\mathrm{div}_{(t,\mathbf{x})}\big(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})\big) \in \mathcal{M},$$

which implies that the vector field

$$\big(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})\big)(t, \mathbf{x})$$

is a divergence measure field. We will discuss a theory of such fields in Section 8.

3. One-dimensional Euler equations

In this section, we present some aspects of a well-posedness theory and related results for the one-dimensional Euler equations.

3.1. *Isentropic Euler equations*

Consider the Cauchy problem for the isentropic Euler equations (1.16) with initial data

$$(\rho, m)|_{t=0} = (\rho_0, m_0)(x), \tag{3.1}$$

where (ρ_0, m_0) is in the physical region $\{(\rho, m)\colon \rho \geqslant 0, |m| \leqslant C_0\rho\}$ for some $C_0 > 0$. The pressure function $p(\rho)$ is a smooth function in $\rho > 0$ (nonvacuum states) satisfying

$$p'(\rho) > 0, \qquad \rho p''(\rho) + 2p'(\rho) > 0 \quad \text{when } \rho > 0, \tag{3.2}$$

and

$$p(0) = p'(0) = 0, \qquad \lim_{\rho\to 0} \frac{\rho p^{(j+1)}(\rho)}{p^{(j)}(\rho)} = c_j > 0, \quad j = 0, 1. \tag{3.3}$$

More precisely, we consider a general situation of the pressure law that there exist a sequence of exponents

$$1 < \gamma := \gamma_1 < \gamma_2 < \cdots < \gamma_J \leqslant \frac{3\gamma - 1}{2} < \gamma_{J+1}$$

and a function $P(\rho)$ such that

$$\begin{aligned} &p(\rho) = \sum_{j=1}^{J} \kappa_j \rho^{\gamma_j} + \rho^{\gamma_{J+1}} P(\rho), \\ &\limsup_{\rho\to 0}\big(\big|P(\rho)\big| + \big|\rho^3 P'''(\rho)\big|\big) < \infty, \end{aligned} \tag{3.4}$$

for some κ_j, $j = 1, \dots, J$, with $\kappa_1 > 0$. For a polytropic gas obeying the γ-law (1.12), or a mixed ideal polytropic fluid,

$$p(\rho) = \kappa_1 \rho^{\gamma_1} + \kappa_2 \rho^{\gamma_2}, \quad \kappa_2 > 0,$$

the pressure function clearly satisfies (3.2)–(3.4).

System (1.16) is strictly hyperbolic at the nonvacuum states $\{(\rho, v)\colon \rho > 0, |v| < \infty\}$, and strict hyperbolicity fails at the vacuum states $\{(\rho, v)\colon \rho = 0, |v| < \infty\}$.

Let $(\eta, q)\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ be an entropy–entropy flux pair of system (1.16). An entropy $\eta(\rho, m)$ is called a weak entropy if $\eta = 0$ at the vacuum states.

In the coordinates (ρ, v), any weak entropy function $\eta(\rho, v)$ is governed by the second-order linear wave equation

$$\begin{cases} \eta_{\rho\rho} - k'(\rho)^2 \eta_{vv} = 0, & \rho > 0, \\ \eta|_{\rho=0} = 0, \end{cases} \tag{3.5}$$

with $k(\rho) = \int_0^\rho p'(s)/s \, \mathrm{d}s$.

In the Riemann invariant coordinates $\mathbf{w} = (w_1, w_2)$ defined as

$$w_1 = v + \int_0^\rho \frac{\sqrt{p'(y)}}{y} \, \mathrm{d}y, \qquad w_2 = v - \int_0^\rho \frac{\sqrt{p'(y)}}{y} \, \mathrm{d}y, \tag{3.6}$$

any entropy function $\eta(\mathbf{w})$ is governed by

$$\eta_{w_1 w_2} + \frac{\Lambda(w_1 - w_2)}{w_1 - w_2} (\eta_{w_1} - \eta_{w_2}) = 0, \tag{3.7}$$

where

$$\Lambda(w_1 - w_2) = -\frac{k(\rho) k''(\rho)}{k'(\rho)^2} \quad \text{with } \rho = k^{-1}\left(\frac{w_1 - w_2}{2}\right). \tag{3.8}$$

The corresponding entropy flux function $q(\mathbf{w})$ is

$$q_{w_j}(\mathbf{w}) = \lambda_i(\mathbf{w}) \eta_{w_j}(\mathbf{w}), \quad i \neq j. \tag{3.9}$$

In general, any weak entropy–entropy flux pair (η, q) can be represented by

$$\eta(\rho, v) = \int_{\mathbb{R}} \chi(\rho, v; s) a(s) \, \mathrm{d}s, \qquad q(\rho, v) = \int_{\mathbb{R}} \sigma(\rho, v; s) b(s) \, \mathrm{d}s, \tag{3.10}$$

for any continuous function $a(s)$ and related function $b(s)$, where the weak entropy kernel and entropy flux kernel are determined by

$$\begin{cases} \chi_{\rho\rho} - k'(\rho)^2 \chi_{vv} = 0, \\ \chi(0, v; s) = 0, \qquad \chi_\rho(0, v; s) = \delta_{v=s} \end{cases} \tag{3.11}$$

and

$$\begin{cases} \sigma_{\rho\rho} - k'(\rho)^2 \sigma_{vv} = \frac{p''(\rho)}{\rho} \chi_v, \\ \sigma(0, v; s) = 0, \qquad \sigma_\rho(0, v; s) = v\delta_{v=s}, \end{cases} \tag{3.12}$$

respectively, with $\delta_{v=s}$ the delta function concentrated at the point $v = s$.

The equations in (3.5)–(3.9) and (3.11)–(3.12) belong to the class of Euler–Poisson–Darboux-type equations. The main difficulty comes from the singular behavior of $\Lambda(w_1 - w_2)$ near the vacuum. In view of (3.8), the derivative of $\Lambda(w_1 - w_2)$ in the coefficients of (3.7) may blow up like $(w_1 - w_2)^{-(\gamma-1)/2}$ when $w_1 - w_2 \to 0$ in general, and its higher derivatives may be more singular, for which the classical theory of Euler–Poisson–Darboux equations does not apply (cf. [19,341,342]). However, for a gas obeying the γ-law,

$$\Lambda(w_1 - w_2) = \lambda := \frac{3-\gamma}{2(\gamma-1)},$$

the simplest case, which excludes such a difficulty. In particular, for this case, the weak entropy kernel is

$$\chi(\rho, v; s) = \big[\big(w_1(\rho, v) - s\big)\big(s - w_2(\rho, v)\big)\big]_+^\lambda.$$

A mathematical theory for dealing with such a difficulty for the singularities of weak entropy kernel and entropy flux kernel can be found in [74,75].

A bounded measurable function $\mathbf{u}(t, x) = (\rho, m)(t, x)$ is called an entropy solution of (1.16) and (3.1)–(3.3) in $\mathbb{R}^2_+$ if $\mathbf{u}(t, x)$ satisfies the following:

(i) there exists $C > 0$ such that

$$0 \leqslant \rho(t, x) \leqslant C, \qquad \big|m(t, x)\big| \leqslant C\rho(t, x); \tag{3.13}$$

(ii) the entropy inequality

$$\partial_t \eta(\rho, m) + \partial_x q(\rho, m) \leqslant 0 \tag{3.14}$$

holds in the sense of distributions in $\mathbb{R}^2_+$ for any convex weak entropy–entropy flux pair $(\eta, q)(\rho, m)$.

Notice that $\eta(\rho, m) = \pm\rho, \pm m$ are trivial convex entropy functions so that (3.14) automatically implies that $(\rho, m)(t, x)$ is a weak solution in the sense of distributions.

THEOREM 3.1. *Consider the Euler equations* (1.16) *with* (3.2)–(3.4). *Let* $(\rho^h, m^h)(t, x)$, $h > 0$, *be a sequence of functions satisfying the following conditions*:

(i) *there exists* $C > 0$ *such that*

$$0 \leqslant \rho^h(t, x) \leqslant C, \qquad \big|m^h(t, x)\big| \leqslant C\rho^h(t, x) \quad \textit{for a.e. } (t, x); \tag{3.15}$$

(ii) *for any weak entropy–entropy flux pair* (η, q),

$$\partial_t \eta(\rho^h, m^h) + \partial_x q(\rho^h, m^h) \quad \text{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+). \tag{3.16}$$

Then the sequence $(\rho^h, m^h)(t,x)$ *is compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, *that is, there exist* $(\rho, m) \in L^\infty$ *and a subsequence* (*still denoted*) $(\rho^h, m^h)(t,x)$ *such that*

$$(\rho^h, m^h) \to (\rho, m) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2_+) \text{ as } h \to 0,$$

and

$$0 \leqslant \rho(t,x) \leqslant C, \qquad |m(t,x)| \leqslant C\rho(t,x).$$

REMARK 3.1. The compactness framework in Theorem 3.1 was established to replace the BV compactness framework (the Helly theorem), for which the detailed proof can be found in [75]. For a gas obeying the γ-law, the case $\gamma = (N+2)/N$, $N \geqslant 5$ odd, was first treated by DiPerna [123], and the general case $1 < \gamma \leqslant 5/3$ for usual gases was first completed by Chen [50] and Ding, Chen and Luo [115]. The cases $\gamma \geqslant 3$ and $5/3 < \gamma < 3$ were treated via kinetic formulation by Lions, Perthame and Tadmor [223] and Lions, Perthame and Souganidis [222], respectively.

In order to establish Theorem 3.1, it requires to establish the corresponding reduction theorem: A Young measure satisfying the Tartar commutation relations

$$\begin{aligned}&\langle \nu_{(t,x)}, \eta_1 q_2 - \eta_2 q_1 \rangle \\ &\quad = \langle \nu_{(t,x)}, \eta_1 \rangle \langle \nu_{(t,x)}, q_2 \rangle - \langle \nu_{(t,x)}, \eta_2 \rangle \langle \nu_{(t,x)}, q_1 \rangle \quad \text{for a.e. } (t,x),\end{aligned} \tag{3.17}$$

for all weak entropy–entropy flux pairs is a Dirac mass. These conditions are derived by the method of compensated compactness, especially the div–curl lemma (see [318,319] and [258,260]). The proof was based on new properties of *cancellation of singularities* of the entropy kernel χ and the entropy flux kernel σ in the following combination

$$E(\rho, v; s_1, s_2) := \chi(\rho, v; s_1)\sigma(\rho, v; s_2) - \chi(\rho, v; s_2)\sigma(\rho, v; s_1),$$

the fractional derivative technique first introduced in [50,115], and an important observation that the following identity is an elementary consequence of the symmetric form of (3.17)

$$\begin{aligned}&\langle \nu_{(t,x)}, \chi(\rho, v; s_1) \rangle \langle \nu_{(t,x)}, \partial_{s_2}^{\lambda+1} \partial_{s_3}^{\lambda+1} E(\rho, v; s_2, s_3) \rangle \\ &\quad + \langle \nu_{(t,x)}, \partial_{s_2}^{\lambda+1} \chi(\rho, v; s_2) \rangle \langle \nu_{(t,x)}, \partial_{s_3}^{\lambda+1} E(\rho, v; s_3, s_1) \rangle \\ &\quad + \langle \nu_{(t,x)}, \partial_{s_3}^{\lambda+1} \chi(\rho, v; s_3) \rangle \langle \nu_{(t,x)}, \partial_{s_2}^{\lambda+1} E(\rho, v; s_1, s_2) \rangle = 0\end{aligned} \tag{3.18}$$

for all s_1, s_2 and s_3, where the derivatives are understood in the sense of distributions (also see [222,223]). It was proved that, when $s_2, s_3 \to s_1$, the second and the third terms converge in the weak-star sense of measures to the *same* term but with opposite sign. The first term is more *singular* and contains the products of functions of bounded variation by bounded measures, which are known to depend upon regularization. The first term in (3.18) converges to a nontrivial limit which is determined explicitly. Finally, the genuine nonlinearity on $p(\rho)$ is required to conclude that the Young measure ν either reduces to a Dirac mass or is supported on the vacuum line.

This compactness framework has successfully been applied for proving the convergence of the Lax–Friedrichs scheme [50,115], the Godunov scheme [116], the vanishing viscosity method [68,222] and for establishing the compactness of solution operators and the decay of periodic solutions [65,75]. Also see the references cited in [86]. In particular, we have the following theorem.

THEOREM 3.2 (Existence, compactness and decay). *Assume that there exists $C_0 > 0$ such that the initial data $(\rho_0, m_0)(x)$ satisfies*

$$0 \leqslant \rho_0(x) \leqslant C_0, \qquad \left|m_0(x)\right| \leqslant C_0\rho_0(x).$$

Then

(i) *there exists a global solution $(\rho, m)(t, x)$ of the Cauchy problem for* (1.16) *satisfying* (3.13), *for some C depending only on C_0 and γ, and* (3.14) *in the sense of distributions for any convex weak entropy–entropy flux pairs (η, q);*

(ii) *the solution operator $(\rho, m)(t, \cdot) = S_t(\rho_0, m_0)(\cdot)$, determined by* (i), *is compact in $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ for $t > 0$;*

(iii) *furthermore, if the initial data $(\rho_0, m_0)(x)$ is periodic with period P, then there exists a global periodic solution $(\rho, m)(t, x)$ with* (3.13) *such that $(\rho, m)(t, x)$ asymptotically decays in L^1 to*

$$\frac{1}{|P|}\int_P (\rho_0, m_0)(x)\,\mathrm{d}x.$$

REMARK 3.2. All the results for entropy solutions to (1.16) in Eulerian coordinates can be presented equivalently as the corresponding results for entropy solutions to (1.17) in Lagrangian coordinates (see [52] and [332]).

REMARK 3.3. If the initial data is in BV and has small oscillation, or $(\gamma - 1)TV(\rho_0, m_0)$ is sufficiently small, away from vacuum, the solution is in BV; see [118,145,263]. In the direction of relaxing the requirement of small total variation for (1.16), see [117,287,322, 323,349]. For extensions to initial–boundary value problems, see [68,229,264,315].

REMARK 3.4. Consider the isentropic Euler equations (1.16) in nonlinear elasticity with $p = -\sigma(\tau) \in C^2(\mathbb{R})$, $\sigma'(\tau) > 0$, satisfying that

$$\operatorname{sign}\bigl((\tau - \hat{\tau})\sigma''(\tau)\bigr) \geqslant 0, \tag{3.19}$$

$$\text{there is no interval in which } \sigma \text{ is affine,} \tag{3.20}$$

and there exists a positive integer $m \in \mathcal{Z}_+$ such that, in an interval $(\hat{\tau}, \hat{\tau}+\delta)$ or $(\hat{\tau}-\delta, \hat{\tau})$ for some $\delta > 0$,

$$\sum_{k=1}^{m} \left|\sigma^{(2k)}(\tau)\right| \neq 0. \tag{3.21}$$

Then the existence, compactness and decay of entropy solutions in L^∞ has been established in [78], and the results have been extended to the equations of motion of viscoelastic media with memory in [59,78]. In the simplest model for common rubber, condition (3.19) implies that the stress σ as a function of the strain τ switches from concave in the compressive mode $\tau < \hat{\tau}$ to convex in the expansive mode $\tau > \hat{\tau}$.

3.2. *Isothermal Euler equations*

For the isothermal Euler equation (1.16) with $\gamma = 1$, we have entropy–entropy flux pairs in the form

$$\begin{aligned} \eta &= \rho^{1/(1-\xi^2)} \exp\left\{\frac{\xi}{1-\xi^2}\frac{m}{\rho}\right\}, \\ q &= \left(\frac{m}{\rho}+\xi\right)\rho^{1/(1-\xi^2)} \exp\left\{\frac{\xi}{1-\xi^2}\frac{m}{\rho}\right\}, \end{aligned} \tag{3.22}$$

which satisfy

$$\eta_{\rho\rho}\eta_{mm} - \eta_{\rho m}^2 = \frac{\xi^4}{(1-\xi^2)^3}\rho^{2\xi^2/(1-\xi^2)-2} \exp\left\{\frac{2\xi}{1-\xi^2}\frac{m}{\rho}\right\} \quad \text{for } \xi \in \mathbb{R}. \tag{3.23}$$

Then η is a weak and convex entropy for any $\xi \in (-1, 1)$.

We have the following similar compensated compactness framework for this case.

THEOREM 3.3. *Assume that* $(\rho^h, m^h)(t, x), h > 0$, *is a sequence of functions satisfying the following conditions*:

(i) *there exists some constant* $C > 0$ *such that*

$$0 \leqslant \rho^h(t,x) \leqslant C, \qquad \left|m^h(t,x)\right| \leqslant \rho^h(t,x)\left(C + \left|\ln \rho^h(t,x)\right|\right) \quad a.e.;$$

(ii) *the sequence of entropy dissipation measures*

$$\partial_t \eta\left(\rho^h, m^h\right) + \partial_x q\left(\rho^h, m^h\right) \quad \textit{is compact in } H^{-1}_{\text{loc}}\left(\mathbb{R}^2_+\right)$$

for any entropy–entropy flux pair (η, q) *in* (3.22) *with* $\xi \in (-1, 1)$.

Then the sequence $(\rho^h, m^h)(t,x)$ *is compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, *that is, there exist* $(\rho, m) \in L^\infty$ *and a subsequence* (*still denoted*) (ρ^h, m^h) *such that*

$$\left(\rho^h, m^h\right) \to (\rho, m) \quad \textit{in } L^1_{\mathrm{loc}}\left(\mathbb{R}^2_+\right) \textit{ as } h \to 0,$$

and

$$0 \leqslant \rho(t,x) \leqslant C, \qquad \left|m(t,x)\right| \leqslant \rho(t,x)\left(C + \left|\ln \rho(t,x)\right|\right).$$

REMARK 3.5. This compactness framework was first established in [172]. Another proof was also provided recently in [205] by employing the approach in [75].

For strictly hyperbolic systems with smooth fluxes, the H^{-1}-compactness condition is easy to be obtained, due to the uniform boundedness of approximate solutions and Murat's lemma [259], provided that the system has a strictly convex entropy. Similar to the isentropic case, it is not clear for the case $\gamma = 1$ whether the strong entropy–entropy flux pairs satisfy the H^{-1}-compactness condition. Furthermore, for the isothermal case, the propagation speed may not be finite due to the presence of vacuum and the entropy equation is not of EPD type, which is different from the isentropic case.

The key point in the proof of [172] is to establish the commutation relations for not only the *weak* entropy–entropy flux pairs but also the *strong* ones by using the *analytic extension theorem* even though it is not known whether strong entropy–entropy flux pairs satisfy the H^{-1}-compactness condition. To achieve this, formula (3.22) of entropies parameterized by a complex variable ξ was used, which includes both weak and strong entropies determined by the value of ξ. It was shown that, for any $\xi \in (-1, 1)$, the weak entropy–entropy flux pair satisfies the H^{-1}-compactness condition. Therefore, the commutation relations are satisfied for these weak entropy–entropy flux pairs. It was observed that the two sides of the commutation relations are regular in ξ and are analytic functions with respect to ξ, which implies that the commutation relations exactly hold for the whole complex variable except two points $(-1, 0)$ and $(1, 0)$ by using the analytic extension theorem. Noting that the entropies are strong if $|\xi| > 1$ (see (3.22)), therefore, the commutation relations hold for these weak and strong entropy–entropy flux pairs so that the H^{-1}-compactness condition for strong entropy–entropy flux pairs can be bypassed. Since both weak and strong entropy–entropy flux pairs are applied to the commutation relations, the reduction theorem for the corresponding Young measure was obtained as that in the strictly hyperbolic case in [124,290], which implies the compensated compactness framework. The proof of [205] employed the approach described in Section 3.1 for the isentropic case by using only the weak entropy–entropy flux pairs.

As an application of Theorem 3.3, we have the following theorem.

THEOREM 3.4 (Existence). *Assume that the initial data satisfies*

$$0 \leqslant \rho_0(x) \leqslant C_0, \qquad \left|m_0(x)\right| \leqslant \rho_0(x)\left(C_0 + \left|\log \rho_0(x)\right|\right) \quad \textit{a.e.} \tag{3.24}$$

for some constant $C_0 > 0$. *Then there exists a global entropy solution of* (1.16) *and* (3.1) (*with* $\gamma = 1$) *satisfying*

$$0 \leqslant \rho(t,x) \leqslant C, \qquad |m(t,x)| \leqslant \rho(t,x)\big(C + |\log \rho(t,x)|\big) \quad a.e., \tag{3.25}$$

where $C > 0$ *depends only on* C_0.

REMARK 3.6. The convergence of the viscosity method was established in [172]. Unlike the isentropic case, the eigenvalues of the system are no longer bounded near vacuum (which may grow with the speed $|\ln \rho|$), the construction of shock capturing scheme is more delicate since the Courant–Friedrichs–Lewy stability condition may fail for standard shock capturing schemes. In [77], such a shock capturing scheme was successfully developed and its strong convergence was established by introducing a cut-off technique to modify the approximate density functions and adjust the ratio of the time and space mesh sizes to construct the shock capturing scheme.

REMARK 3.7. Away from vacuum, the first result on the existence of BV solutions with large initial data was obtained in Nishida [262] by using the Glimm scheme [145] for $\gamma = 1$. Poupaud, Rascle and Vila [274] made further simplification and improved the results of [262] to the isothermal Euler–Poisson system. The existence result in Theorem 3.4 allows the initial data (ρ_0, m_0) only in L^∞, which may even contain vacuum.

3.3. *Adiabatic Euler equations*

For the full Euler equations in gas dynamics (1.15) with the following Cauchy problem

$$(\tau, v, S)|_{t=0} = (\tau_0, v_0, S_0)(x), \tag{3.26}$$

the following existence theorem holds which is due to Liu [232] (also see [85] and [321]).

THEOREM 3.5. *Let* $K \subset \{(\tau, v, S)\colon \tau > 0\}$ *be a compact set in* $\mathbb{R}_+ \times \mathbb{R}^2$, *and let* $N \geqslant 1$ *be any positive constant. Then there exists a constant* $C_0 = C_0(K, N)$, *independent of* $\gamma \in (1, 5/3]$, *such that, for every initial data* $(\tau_0, v_0, S_0)(x) \in K$ *with* $TV_{\mathbb{R}}(\tau_0, v_0, S_0) \leqslant N$, *when*

$$(\gamma - 1) TV_{\mathbb{R}}(\tau_0, v_0, S_0) \leqslant C_0$$

for any $\gamma \in (1, 5/3]$, *the Cauchy problem* (1.15) *and* (3.26) *has a global entropy solution* $(\tau, v, S)(t,x)$ *which is bounded and satisfies*

$$TV_{\mathbb{R}}(\tau, v, S)(t, \cdot) \leqslant C\, TV_{\mathbb{R}}(\tau_0, v_0, S_0)$$

for some constant $C > 0$ *independent of* γ.

In the direction of relaxing the requirement of small total variation for (1.15), see [268,287,322,323]. For extensions to initial–boundary value problems, see [68,229, 264,315].

For the decay of entropy solutions in BV_{loc} with periodic data or compact support, see [111,119,121,149,225,226]; also see [65] for entropy solutions only in L^∞. For additional further discussions and references to the Glimm scheme, see [111,235,294]; also see [85].

Furthermore, we have the following theorem.

THEOREM 3.6. *If the initial data functions $\mathbf{u}_0(x)$ and $\mathbf{v}_0(x)$ have sufficiently small total variation and $\mathbf{u}_0 - \mathbf{v}_0 \in L^1(\mathbb{R})$, then, for the corresponding exact Glimm, or wave-front tracking, or vanishing viscosity solutions $\mathbf{u}(t,x)$ and $\mathbf{v}(t,x)$ of the Cauchy problem* (1.1) *and* (2.10) ($d = 1$), *there exists a constant $C > 0$ such that*

$$\left\|\mathbf{u}(t,\cdot) - \mathbf{v}(t,\cdot)\right\|_{L^1(\mathbb{R})} \leqslant C\|\mathbf{u}_0 - \mathbf{v}_0\|_{L^1(\mathbb{R})} \quad \text{for all } t > 0. \tag{3.27}$$

An immediate consequence of this theorem is that the whole sequence of approximate solutions constructed by the Glimm scheme, as well as the wave-front tracking method and the vanishing viscosity method, converges to a unique entropy solution of (1.1) and (2.10) ($d = 1$) as the mesh size or the viscosity coefficient tends to zero. See also [32] for the uniqueness of limits of Glimm's random choice method. The details of the proof of Theorem 3.6 can be found in [20,33,236,238]. In the direction relaxing the requirement of small total variation for (1.1), see [207,208].

For other discussions and extensive references about the L^1-stability of BV entropy solutions and related problems, we refer to [33,111,167,204].

Furthermore, the uniqueness and stability of Riemann solutions in the class of entropy solutions with large variation satisfying only one entropy inequality for the strictly convex physical entropy S has been established in [70] as follows.

THEOREM 3.7. *Let $\mathbf{u}(t,x) = (\tau, v, e + v^2/2)(t,x)$ be an entropy solution of* (1.15) *and* (3.26) *in $\Pi_T := \{(t,x)\colon\ 0 \leqslant t \leqslant T\}$ for some $T \in (0,\infty)$, which belongs to $BV_{\mathrm{loc}}(\Pi_T; \mathcal{D})$ with $\mathcal{D} \subset \{(\tau, v, e + v^2/2)\colon\ \tau > 0\} \subset \mathbb{R}^3$ bounded. Let $\mathbf{R}(x/t)$ be the classical Riemann solution with Riemann data $\mathbf{R}_0(x)$.*

(i) *If $\mathbf{u}_0 = \mathbf{R}_0$, then*

$$\mathbf{u}(t,x) = \mathbf{R}\left(\frac{x}{t}\right) \quad \text{for a.e. } (t,x) \in \Pi_T.$$

(ii) *If $\mathbf{u}_0 - \mathbf{R}_0 \in L^1 \cap L^\infty \cap BV_{\mathrm{loc}}(\mathbb{R})$, then*

$$\operatorname*{ess\,lim}_{t\to\infty} \int_{-L}^{L} \left|\mathbf{u}(t,\xi t) - \mathbf{R}(\xi)\right| \mathrm{d}\xi = 0 \quad \text{for any } L > 0; \tag{3.28}$$

that is, the Riemann solution $\mathbf{R}(x/t)$ is asymptotically stable in the sense (3.28) *with respect to the corresponding initial perturbation in $L^1 \cap L^\infty \cap BV_{\mathrm{loc}}(\mathbb{R})$.*

We now consider the 3×3 system of Euler equations (1.15) in Lagrangian coordinates in thermoelasticity with the following class of constitutive relations for the new state vector (τ, S) with the form

$$\begin{aligned} e &= \int_0^{\tau+\alpha S} \sigma(w)\,\mathrm{d}w + \beta S, \\ p &= -\sigma(\tau + \alpha S), \\ \theta &= \alpha\sigma(\tau + \alpha S) + \beta, \end{aligned} \tag{3.29}$$

where $\sigma(w)$ is a function with $\sigma'(w) > 0$, and α and β are positive constants. The model (3.29) is quite special. Even so, when we are dealing with solutions in which (τ, S) do not deviate far from some constant values $(\bar{\tau}, \overline{S})$, we may obtain a reasonable approximation for general constitutive relations (see [58])

$$e = \hat{e}(\tau, S), \qquad p = -\hat{\sigma}(\tau, S), \qquad \theta = \hat{\theta}(\tau, S) \tag{3.30}$$

satisfying the conditions

$$\hat{\sigma} = \hat{e}_\tau, \qquad \hat{\theta} = \hat{e}_S. \tag{3.31}$$

We also assume that, for some $\bar{w}$,

$$\sigma''(w) + 4\frac{\alpha\sigma'(w)^2}{\alpha\sigma(w) + \beta} \begin{cases} \leqslant 0 & \text{if } w < \bar{w}, \\ \geqslant 0 & \text{if } w > \bar{w}, \end{cases} \tag{3.32}$$

and

$$\sigma''(w) \neq 0 \quad \text{for } w > \bar{w}, \tag{3.33}$$

or there exists $\hat{w} > \bar{w}$ such that $\sigma(w)$ satisfies conditions (3.19)–(3.21) with $\hat{w}$ replacing $\hat{v}$.

Consider the Cauchy problem for (1.15) with initial data

$$(w, v, S)|_{t=0} = (w_0, v_0, S_0)(x) \tag{3.34}$$

for $w = \tau + \alpha S$.

THEOREM 3.8. *Assume*

$$(w_0, v_0)(x) \in \left\{(w, v)\colon \left| v \pm \int_{\hat{w}}^{w} \sqrt{\sigma'(\omega)}\,\mathrm{d}\omega \right| \leqslant C_0 \right\}$$

and $S_0(x) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R})$. *Then*

(i) *there exists a distributional solution*

$$(w, v, S)(t,x) \in L^\infty\big(\mathbb{R}^2_+; \mathbb{R}^2\big) \times \mathcal{M}_{\mathrm{loc}}\big(\mathbb{R}^2_+; \mathbb{R}\big)$$

of (1.15) *and* (3.34) *satisfying*

$$\begin{aligned} &S_t(t,x) \in \mathcal{M}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big), \qquad \theta\big(w(t,x)\big) \geqslant 0, \\ &|S|\big([0,T_0] \times \big\{|x| \leqslant cT_0\big\}\big) \leqslant CT_0^2 \end{aligned} \tag{3.35}$$

for any $c, T_0 > 0$, *with* $C > 0$ *independent of* T_0. *Moreover,* $(w, v, S)(t,x)$ *satisfies the entropy condition*

$$\partial_t \eta(w,v) + \partial_x q(w,v) \leqslant 0, \qquad S_t \geqslant 0 \tag{3.36}$$

in the sense of distributions for any C^2 *entropy–entropy flux pair* $(\eta, q)(w,v)$ *of the system*

$$\partial_t w - \partial_x v = 0, \qquad \partial_t v - \partial_x \sigma(w) = 0,$$

for which the following strong convexity condition holds:

$$\begin{aligned} &\theta\eta_{ww} - \alpha\sigma'(w)\eta_w \geqslant 0, \\ &\theta\eta_{vv} - \alpha\eta_w \geqslant 0, \\ &\big(\theta\eta_{ww} - \alpha\sigma'(w)\eta_w\big)(\theta\eta_{vv} - \alpha\eta_w) - \eta_{ww}^2 \geqslant 0; \end{aligned}$$

(ii) *any sequence* $(w^h, v^h)(t,x)$ *that is uniformly bounded in* $h > 0$ *and satisfies* (3.36) *is compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ *when* $t > 0$;

(iii) *furthermore, if the initial data* $(w_0, v_0, S_0)(x)$ *is periodic with period* P, *then there exists a periodic entropy solution* $(\tau, v, S)(t,x)$ *of* (1.15) *and* (3.34) *with period* P *satisfying*

$$(v, \tau + \alpha S) \in L^\infty\big(\mathbb{R}^2_+\big),$$

(3.35) *and* (3.36). *Moreover, the velocity* $v(t,x)$, *the pressure* $p(w(t,x))$ *and the temperature* $\theta(w(t,x))$ *asymptotically decay in* L^1 *to*

$$\bar{v} = \frac{1}{|P|}\int_P v_0(x)\,\mathrm{d}x$$

and

$$\begin{aligned} \tilde{p} &= p\left(\Theta^{-1}\left(\frac{1}{|P|}\int_P \Theta\big(w_0(x)\big)\,\mathrm{d}x\right)\right), \\ \tilde{\theta} &= \theta\left(\Theta^{-1}\left(\frac{1}{|P|}\int_P \Theta\big(w_0(x)\big)\,\mathrm{d}x\right)\right), \end{aligned}$$

respectively, where $\Theta(w) = \beta w + \alpha \int_0^w \sigma(\omega)\,\mathrm{d}\omega$.

REMARK 3.8. The first existence theorem for global entropy solutions for (1.15) and (3.29)–(3.33) was established in [58]. The existence result was extended in [65] and [78] to the existence, compactness and decay of entropy solutions of (1.15) and (3.29)–(3.33) under the weaker conditions (3.19)–(3.21) with $\hat{w}$ replacing $\hat{v}$.

REMARK 3.9. An interesting feature here is that, because of linear degeneracy of the second characteristic field of (1.15) and (3.29)–(3.33), one cannot expect the decay of all components of the solutions. However, some important quantities such as the velocity, the pressure, and the temperature do decay as $t \to \infty$.

4. Multidimensional Euler equations and related models

Multidimensional problems for the Euler equations are extremely rich and complicated. Some great developments and progress have been made in the recent decades through strong and close interdisciplinary interactions and diverse approaches including

(i) experimental data,
(ii) large and small scale computing by a search for effective numerical methods,
(iii) asymptotic and qualitative modeling,
(iv) rigorous proofs for prototype problems and an understanding of the solutions.

In some sense, the developments and progress made by using approach (iv) are behind those by using the other approaches (i)–(iii) (see [150]); however, most scientific problems are considered to be solved satisfactorily only after approach (iv) is achieved.

In this section, together with Sections 5–7, we give some samples of multidimensional models and problems for the Euler equations with emphasis on those prototype models and problems that have been solved or expected to be solved rigorously at least for some cases.

Since the multidimensional problems are so complicated in general, a natural strategy to attack these problems as a first step is to study

(i) simpler nonlinear models with strong physical motivations,
(ii) special, concrete nonlinear physical problems.

Meanwhile, extend the results and ideas from the first step to study

(i) the Euler equations in gas dynamics and elasticity,
(ii) more general problems,
(iii) nonlinear systems that the Euler equations are the main subsystem or describe the dynamics of macroscopic variables such as Navier–Stokes equations, MHD equations, combustion equations, Euler–Poisson equations, kinetic equations especially including the Boltzmann equation, among others.

In this section we first focus on some samples of multidimensional models for the Euler equations and related multidimensional hyperbolic conservation laws.

4.1. *The potential flow equation*

This approximation is well known in transonic aerodynamics, beyond the isentropic approximation (1.11) from (1.4). Denote

$$\mathrm{D}_t = \partial_t + \sum_{k=1}^{d} v_k\, \partial_{x_k},$$

the convective derivative along fluid particle trajectories. From (1.4), we have

$$\mathrm{D}_t S = 0 \tag{4.1}$$

and, by taking the curl of the momentum equations and using vector identities,

$$\mathrm{D}_t\left(\frac{\omega}{\rho}\right) = \frac{\omega}{\rho}\cdot\nabla\mathbf{v} + \frac{p_S(\rho, S)}{\rho^3}\nabla\rho\times\nabla S. \tag{4.2}$$

The identities in (4.1) and (4.2) imply that a smooth solution of (1.4) which is both isentropic and irrotational at time $t = 0$ remains isentropic and irrotational for all later time, as long as this solution stays smooth. Then the conditions $S = S_0 = \mathrm{const}$ and $\mathrm{curl}\,\mathbf{v} = 0$ are reasonable for smooth solutions.

For a smooth irrotational solution of (1.4), we integrate the d-momentum equations in (1.11) through Bernoulli's law

$$\partial_t\mathbf{v} + \frac{1}{2}\nabla\left(|\mathbf{v}|^2\right) + \nabla i(\rho) = 0,$$

where $i'(\rho) = p_\rho(\rho, S_0)/\rho$.

On a simply connected space region, the condition $\mathrm{curl}\,\mathbf{v} = 0$ implies that there exists Φ such that

$$\mathbf{v} = \nabla\Phi.$$

Then we have

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\nabla\Phi) = 0, \\ \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + i(\rho) = K, \end{cases} \tag{4.3}$$

where K is the Bernoulli constant, which is usually determined by the boundary conditions if such conditions are prescribed. From the second equation in (4.3), we have

$$\rho(\mathrm{D}\Phi) = i^{-1}\left(K - \left(\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2\right)\right).$$

Then system (4.3) can be rewritten as the following time-dependent potential flow equation of second order

$$\partial_t \rho(\mathrm{D}\Phi) + \nabla \cdot \big(\rho(\mathrm{D}\Phi)\nabla\Phi\big) = 0. \tag{4.4}$$

For a steady solution $\Phi = \varphi(\mathbf{x})$, i.e., $\partial_t \Phi = 0$, we obtain the celebrated steady potential flow equation of aerodynamics

$$\nabla \cdot \big(\rho(\nabla\varphi)\nabla\varphi\big) = 0. \tag{4.5}$$

In applications in aerodynamics, (4.3) or (4.4) is used for discontinuous solutions, and the empirical evidence is that entropy solutions of (4.3) or (4.4) are fairly good approximations to entropy solutions for (1.4) provided that

(i) the shock strengths are small,
(ii) the curvature of shock fronts is not too large,
(iii) there is a small amount of vorticity in the region of interest.

The advantages of equation (4.4), or equivalently (4.3), as the simplest multidimensional prototype conservation laws include (cf. [242])

(i) unidirectional plane wave solutions of (4.4) reduce to solutions of a 2×2 system of conservation laws with the structure of a wave equation,

(ii) the linear structure of (4.4) is strictly hyperbolic with characteristics defined by a single light cone in several space variables,

(iii) under reasonable thermodynamic assumptions such as an ideal gas law (1.12), the system for (4.4) is genuinely nonlinear in all wave directions simultaneously and the corresponding multidimensional shock fronts are uniformly stable,

(iv) this system has the vorticity waves removed unlike (1.4) and (1.11). Such vorticity waves are linearly degenerate wave fields but represent an enormous source of instability in multidimension through Kelvin–Helmhotz instability.

The model (4.4) or (4.3) is an excellent model to capture multidimensional shock waves by ignoring vorticity waves, while the model (the incompressible Euler equations) in Section 4.2 is an excellent model to capture multidimensional vorticity waves by ignoring shock waves in fluid flow.

4.2. *Incompressible Euler equations*

In the homogeneous case, the incompressible Euler equations take the form

$$\begin{cases} \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0. \end{cases} \tag{4.6}$$

This can formally be obtained from (1.11) by setting $\rho = 1$ as the equation of state and regarding p as an unknown function. As indicated above, the model (4.6) excludes the appearance of shock waves in fluid flow to capture multidimensional vorticity waves.

In the inhomogeneous case, the incompressible Euler equations are

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \partial_t (\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0. \end{cases} \tag{4.7}$$

These models can be obtained by formal asymptotics for low Mach number expansions from the compressible Euler equations. For more details, see [95,98,166,220,221,243] and the references cited therein.

4.3. *The transonic small disturbance equation*

A further simpler model than the potential flow equation in transonic aerodynamics is the unsteady transonic small disturbance equation or so-called the two-dimensional inviscid Burgers equation (see [97]),

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2} u^2\right) + \partial_y v = 0, \\ \partial_y u - \partial_x v = 0, \end{cases} \tag{4.8}$$

or in the form of Zabolotskaya–Khokhlov equation [346],

$$\partial_t (\partial_t u + u\, \partial_x u) + \partial_{yy} u = 0. \tag{4.9}$$

The equations in (4.8) describe the potential flow field near the reflection point in weak shock reflection, which determines the leading-order approximation of geometric optical expansions; and it can also be used to formulate asymptotic equations for the transition from regular to Mach reflection for weak shocks. See [173–175,252] and the references cited therein.

Equation (4.9) arises in many different situations. It was first derived by Timman in the context of transonic flows [325]. In nonlinear acoustics, it was derived by Zabolotskaya and Khokhlov [346] and is used to describe the diffraction of nonlinear acoustic beams [164]. Motivated by the experiments of Sturtevant and Kulkarny [310] on the focusing of shocks, Cramer and Seebass [102] used (4.9) to study caustics in nearly planar sound waves. The same equation arises as a weakly nonlinear equation for cusped caustics [174]. Hunter [173] also showed that (4.8) describes high-frequency waves near singular rays.

4.4. *Pressure-gradient equations*

The inviscid fluid motions are driven mainly by the pressure gradient and the fluid convection (i.e., transport). As for modeling, it is natural to study first the effect of the two driving factors separately. Such an idea has also been used by Argarwal and Halt [1] to formulate a flux-splitting scheme in numerical computations for airfoil flows.

Separating the pressure gradient from the Euler equations, we first have the pressure-gradient system

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t(\rho u) + \partial_x p = 0, \\ \partial_t(\rho v) + \partial_y p = 0, \\ \partial_t(\rho E) + \partial_x(up) + \partial_y(vp) = 0. \end{cases} \tag{4.10}$$

We may choose $\rho = 1$. Setting

$$p = (\gamma - 1)P, \qquad t = \frac{s}{\gamma - 1},$$

then we have the following pressure-gradient equations

$$\begin{cases} \partial_s u + \partial_x P = 0, \\ \partial_s v + \partial_y P = 0, \\ \partial_s(\ln P) + \partial_x u + \partial_y v = 0. \end{cases} \tag{4.11}$$

Eliminating the velocity (u, v), we obtain the following nonlinear wave equation for P:

$$\partial_{ss}(\ln P) - \Delta P = 0. \tag{4.12}$$

Although system (4.11) is obtained from the splitting idea, system (4.11) is a good approximation to the full Euler equations, especially when the velocity (u, v) is small and the adiabatic gas exponent $\gamma > 1$ is large (see [357]). This can be achieved by the formal expansion in terms of $\varepsilon = 1/(\gamma - 1)$

$$\begin{cases} \rho = \rho_1 + \varepsilon\rho_2 + O(\varepsilon^2), \\ (u, v) = \varepsilon(u_1, v_1) + O(\varepsilon^2), \\ p = \varepsilon p_1 + O(\varepsilon^2). \end{cases}$$

Plugging the expansion into the Euler equations (1.4), we first compare the order of ε^2 and have

$$\partial_t \rho_1 = 0,$$

and so we may choose $\rho_1 = 1$. We then compare the order of ε and have

$$\begin{cases} \partial_t u_1 + \partial_x p_1 = 0, \\ \partial_t v_1 + \partial_y p_1 = 0, \\ \partial_t\left(\frac{p_1}{\gamma-1}\right) + p_1\,\partial_x u_1 + p_1\,\partial_y v_1 = 0. \end{cases} \tag{4.13}$$

Set

$$p_1 = (\gamma - 1)P, \qquad t = \frac{1}{\gamma - 1}\tau.$$

Then we have

$$\begin{cases} \partial_s u_1 + \partial_x P = 0, \\ \partial_s v_1 + \partial_y P = 0, \\ \partial_s(\ln P) + \partial_x u_1 + \partial_y v_1 = 0, \end{cases}$$

which is the same as (4.11) that leads to (4.12).

4.5. *Pressureless Euler equations*

With the pressure-gradient equations (4.11), the convection (i.e., transport) part of fluid flow forms the pressureless Euler equations

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho u v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(\rho v^2) = 0, \\ \partial_t(\rho E) + \partial_x(\rho u E) + \partial_y(\rho v E) = 0. \end{cases} \tag{4.14}$$

This system also models the motion of free particles which stick under collision; see [30,127,348]. In general, solutions of (4.14) become measure solutions.

System (4.14) has been analyzed extensively; for example, see [26,27,30,127,161,172, 210–212,273,296,335] and the references cited therein. In particular, the existence of measure solutions of the Riemann problem was first presented in [26] for the one-dimensional case, and a connection of (4.14) with adhesion particle dynamics and the behavior of global weak solutions with random initial data were discussed in [127]. It has also been shown that δ-shocks and vacuum states do occur in the Riemann solutions even in the one-dimensional case. Since the two eigenvalues of the transport equations coincide, the occurrence of δ-shocks and vacuum states as $t > 0$ can be regarded as a result of resonance between the two characteristic fields. Such phenomena can also be regarded as the phenomena of concentration and cavitation in solutions to the Euler equations for compressible fluids as the pressure vanishes. It has shown in [79] for $\gamma > 1$ and [209] for $\gamma = 1$ that, as the pressure vanishes, any two-shock Riemann solution to the Euler equations tends to a δ-shock solution to (4.14) and the intermediate densities between the two shocks tend to a weighted δ-measure that forms the δ-shock. By contrast, any two-rarefaction-wave Riemann solution of the Euler equations has been shown in [79] to tend to a two-contact-discontinuity solution to (4.14), whose intermediate state between the two contact discontinuities is a vacuum state, even when the initial data stays away from the vacuum. Some numerical results exhibiting the formation process of δ-shocks and vacuum states have also been presented in [79].

4.6. *Euler equations in nonlinear elastodynamics*

The equations of nonlinear elastodynamics provide another excellent example of the rich special structure one encounters when dealing with hyperbolic systems of conservation laws. In three space dimensions, the state vector is $(\mathbf{v}, \mathbf{F})$, where $\mathbf{v} \in \mathbb{R}^3$ is the velocity vector and $\mathbf{F}$ is the 3×3 matrix-valued deformation gradient constrained by the requirement $\det \mathbf{F} > 0$. The system of conservation laws, which express the integrability conditions between $\mathbf{v}$ and $\mathbf{F}$ and the balance of linear momentum, reads

$$\begin{cases} \partial_t F_{i\alpha} - \partial_{x_\alpha} v_i = 0, & i, \alpha = 1, 2, 3, \\ \partial_t v_j - \sum_{\beta=1}^{3} \partial_{x_\beta} S_{j\beta}(F) = 0, & j = 1, 2, 3. \end{cases} \tag{4.15}$$

The symbol $\mathbf{S}$ stands for the *Piola–Kirchhoff stress tensor*, which is determined by the (scalar-valued) *strain energy function* $\sigma(\mathbf{F})$,

$$S_{j\beta}(\mathbf{F}) = \frac{\partial \sigma(\mathbf{F})}{\partial F_{j\beta}}.$$

System (4.15) is hyperbolic if and only if

$$\sum_{i,j=1}^{3} \sum_{\alpha,\beta=1}^{3} \frac{\partial^2 \sigma(\mathbf{F})}{\partial F_{i\alpha}\, \partial F_{j\beta}} \xi_i \xi_j n_\alpha n_\beta > 0 \tag{4.16}$$

for any vectors $\boldsymbol{\xi}, \mathbf{n} \in \mathbf{S}^3$.

System (4.15) is endowed with an entropy–entropy flux pair

$$\eta = \sigma(\mathbf{F}) + \frac{1}{2}|\mathbf{v}|^2, \qquad q_\alpha = -\sum_{j=1}^{3} v_j S_{j\alpha}(\mathbf{F}).$$

However, the laws of physics do not allow $\sigma(\mathbf{F})$, and thereby η, to be convex functions. Indeed, convexity of σ would violate the principle of *material frame indifference*

$$\sigma(\mathbf{OF}) = \sigma(\mathbf{F}) \quad \text{for all } \mathbf{O} \in SO(3),$$

and would also be incompatible with the natural requirement that $\sigma(\mathbf{F}) \to \infty$ as $\det \mathbf{F} \downarrow 0$ or $\det \mathbf{F} \uparrow \infty$ (see [106]). Consequently, the useful results on the local existence of classical solutions to the Cauchy problem and the uniqueness of classical solutions in the context of weak solutions that are available for hyperbolic systems of conservation laws endowed with a convex entropy in Section 2.1 are not directly applicable to system (4.15).

The failure of σ to be convex is also the main source of complication in elastostatics, where one is seeking to determine equilibrium configurations of the body by minimizing the total strain energy $\int \sigma(\mathbf{F})$. The following alternative conditions, weaker than convexity and physically reasonable, are relevant in that context [13]:

(i) *polyconvexity*,

$$\sigma(\mathbf{F}) = g\big(\mathbf{F}, \mathbf{F}^*, \det \mathbf{F}\big),$$

where $\mathbf{F}^*$ is the adjugate of $\mathbf{F}$ (the matrix of cofactors of $\mathbf{F}$), $\mathbf{F}^* = (\det \mathbf{F})\mathbf{F}^{-1}$, and $g(\mathbf{F}, \mathbf{G}, w)$ is a convex function of 19 variables,

(ii) *quasiconvexity* in the sense of Morrey [254],

(iii) *rank-one convexity*, expressed by (4.16).

It is known that convexity $\Rightarrow$ polyconvexity $\Rightarrow$ quasiconvexity $\Rightarrow$ rank-one convexity, however, none of the converse statements is generally valid. It is important to investigate the relevance of the above conditions in elastodynamics. A first start was made in [106] where it was shown that rank-one convexity suffices for the local existence of classical solutions, quasiconvexity yields the uniqueness of classical solutions in the context of the class of entropy-admissible weak solutions, and polyconvexity renders the system symmetrizable (also see [275]).

To achieve this for polyconvexity, one of the main ideas is to enlarge system (4.15) with the state vector $(\mathbf{v}, \mathbf{F})$ into a large, albeit equivalent, system for the new state vector $(\mathbf{v}, \mathbf{F}, \mathbf{F}^*, w)$ with $w = \det \mathbf{F}$

$$\partial_t w = \sum_{\alpha=1}^{3} \sum_{i=1}^{3} \partial_{x_\alpha}\big(F^*_{\alpha i} v_i\big), \tag{4.17}$$

$$\partial_t F^*_{\gamma k} = \sum_{\alpha,\beta=1}^{3} \sum_{i,j=1}^{3} \partial_{x_\alpha}(\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk} F_{j\beta} v_i), \quad \gamma, k = 1, 2, 3, \tag{4.18}$$

where $\varepsilon_{\alpha\beta\gamma}$ and ε_{ijk} denote the standard permutation symbols. Then the enlarged system with 21 equations, which consists of (4.15) augmented by (4.17) and (4.18), is endowed a uniformly convex entropy

$$\eta = \sigma\big(\mathbf{F}, \mathbf{F}^*, w\big) + \frac{1}{2}|\mathbf{v}|^2$$

so that the local existence of classical solutions and the stability of Lipschitz solutions may be inferred directly from Theorem 2.3. See [111,113,275] for more details.

4.7. *The Born–Infeld system in electromagnetism*

The Born–Infeld system is a nonlinear version of Maxwell equations,

$$\begin{cases} \partial_t B + \operatorname{curl} \frac{\partial W}{\partial D} = 0, \\ \partial_t D - \operatorname{curl} \frac{\partial W}{\partial B} = 0, \end{cases} \tag{4.19}$$

where $W:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ is the given energy density. The Born–Infeld model corresponds to the special case

$$W_{\rm BI}(B,D)=\sqrt{1+|B|^2+|D|^2+|P|^2}.$$

When W is strongly convex (i.e., $D^2W>0$), system (4.19) is endowed with a strictly convex entropy, which implies that the system is symmetric and hyperbolic and, therefore, the Cauchy problem is locally well posed in H^s for $s>5/2$. However, $W_{\rm BI}$ is not convex for a large enough field.

As in Section 4.6, the Born–Infeld model is enlarged from 6 to 10 equations in [29], by adjunction of the conservation laws satisfied by $P:=B\times D$ and W so that the augmented system turns out to be a set of conservation laws in the unknowns

$$(h,B,D,P)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3,$$

endowed with a strongly convex entropy, which is symmetric and hyperbolic,

$$\begin{cases}\partial_t h+\operatorname{div}P=0,\\ \partial_t B+\operatorname{curl}\left(\frac{P\times B+D}{h}\right)=0,\\ \partial_t D+\operatorname{curl}\left(\frac{P\times D-B}{h}\right)=0,\\ \partial_t P+\operatorname{Div}\left(\frac{P\otimes P-B\otimes B-D\otimes D-I}{h}\right)=0,\end{cases}$$

where I is the 3×3 identity matrix. The physical region is

$$\left\{(h,B,D,P):\ P=D\times B,h=\sqrt{1+|B|^2+|D|^2+|P|^2}>0\right\}.$$

Also see [295] for another enlarged system consisting of 9 scalar evolution equations in 9 unknowns (B,D,P), where P stands for the relaxation of the expression $D\times B$.

4.8. *Lax systems*

Let $f(\mathbf{u})$ be an analytic function of a single complex variable $\mathbf{u}=u+v\mathrm{i}$. We impose on the complex valued function $\mathbf{u}=\mathbf{u}(t,z)$, $z=x+y\mathrm{i}$, and the real variable t the following nonlinear partial differential equation

$$\partial_t\bar{\mathbf{u}}+\partial_z f(\mathbf{u})=0, \tag{4.20}$$

where the bar denotes the complex conjugate and $\partial_z=\frac{1}{2}(\partial_x-\mathrm{i}\,\partial_y)$. Then we can express this equation in terms of the real and imaginary parts of $\mathbf{u}$ and $\frac{1}{2}f(\mathbf{u})=a(u,v)+b(u,v)\mathrm{i}$. Then (4.20) gives

$$\begin{cases}\partial_t u+\partial_x a(u,v)+\partial_y b(u,v)=0,\\ \partial_t v-\partial_x b(u,v)+\partial_y a(u,v)=0.\end{cases}\tag{4.21}$$

In particular, when $f(\mathbf{u}) = \mathbf{u}^2 = u^2 + v^2 + 2uv\mathrm{i}$, system (4.20) is called the complex Burger equation, which becomes

$$\begin{cases} \partial_t u + \frac{1}{2}\,\partial_x\big(u^2 + v^2\big) + \partial_y(uv) = 0, \\ \partial_t v - \partial_x(uv) + \frac{1}{2}\,\partial_y\big(u^2 + v^2\big) = 0. \end{cases} \tag{4.22}$$

System (4.21) is a symmetric hyperbolic system of conservation laws with a strictly convex entropy

$$\eta(u, v) = u^2 + v^2,$$

so that local well posedness of classical solutions can be inferred directly from Theorem 2.3; see [202] for more details. For the one-dimensional case, this system is an archetype of hyperbolic systems of conservation laws with umbilic degeneracy, which has been analyzed in [72,286] and the references cited therein.

5. Multidimensional steady supersonic problems

Multidimensional steady problems for the Euler equations are fundamental in fluid dynamics. In particular, understanding of these problems will help us to understand the asymptotic behavior of evolution solutions for large time, especially global attractors. One of the excellent sources of steady problems is Courant–Friedrichs' book [100].

In this section we first discuss some of recent developments in the study of two-dimensional steady supersonic problems.

The two-dimensional steady Euler flows are governed by

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x\big(\rho u^2 + p\big) + \partial_y(\rho uv) = 0, \\ \partial_x(\rho uv) + \partial_y\big(\rho v^2 + p\big) = 0, \\ \partial_x\big(u(E + p)\big) + \partial_y\big(v(E + p)\big) = 0, \end{cases} \tag{5.1}$$

where (u, v) is the velocity and E is the total energy, and the constitutive relations among the thermodynamical variables ρ, p, e, θ and S are determined by (1.5)–(1.9). For the barotropic (isentropic or isothermal) case

$$p = p(\rho) = \frac{\kappa\rho^\gamma}{\gamma}, \quad \gamma \geqslant 1,$$

and then the first three equations in (5.1) form a self-contained system, the Euler system for steady barotropic fluids. The quantity

$$c = \sqrt{p_\rho(\rho, S)}$$

is defined as the sonic speed and, for polytropic gases, $c = \sqrt{\gamma p/\rho}$.

System (5.1) governing a supersonic flow (i.e., $u^2+v^2>c^2$) has all real eigenvalues and is hyperbolic, while system (5.1) governing a subsonic flow (i.e., $u^2+v^2<c^2$) has complex eigenvalues and is both elliptic–hyperbolic mixed and composite.

5.1. *Wedge problems involving supersonic shocks*

The mathematical study of two-dimensional steady supersonic flows past wedges whose vertex angles are less than the critical angle can date back to the 1940s since the stability of such flows is fundamental in applications (cf. [100] and [336]). Local solutions around the wedge vertex were first constructed in [162,219,285] and the references cited therein. Global potential solutions have been constructed in [89–91] when the wedge has some convexity or the wedge is a small perturbation of the straight wedge with fast decay in the flow direction and in [353,354] for piecewise smooth curved wedges that are a small perturbation of the straight wedge.

As indicated in Section 4.1, the potential flow equation is an excellent model for the flow containing only weak shocks since it approximates to the isentropic Euler equations up to third order in shock strength. For the flow containing shocks of large strength, the full Euler equations (5.1) are required to govern the physical flow. For the wedge problem, when the vertex angle is large, the flow contains a large shock front emanating from the wedge vertex and, for this case, the Euler equations should take the position to describe the physical flow. Thus it is important to study the two-dimensional steady supersonic flows governed by the Euler equations for the wedge problem with a large vertex angle. When a wedge is straight and the wedge vertex angle is less than the critical angle ω_{crit}, there exists a supersonic shock front emanating from the wedge vertex so that the constant states on both sides of the shock are supersonic; the critical angle condition is necessary and sufficient for the existence of the supersonic shock. This can be seen through the shock polar (see Figures 1 and 2; also see [88,100]).

Consider two-dimensional steady supersonic Euler flows past two-dimensional Lipschitz curved wedges whose vertex angles are less than the critical angle ω_{crit}, along which the total variation of the tangent angle functions is suitably small. More specifically,

(i) there exists a Lipschitz function $g\in \mathrm{Lip}(\mathbb{R}_+)$ with $g'\in BV(\mathbb{R}_+)$ and $g(0)=0$ such that $\omega_0:=\arctan(g'(0+))<\omega_{\text{crit}}$,

$$\begin{aligned}&TV\{g'(\cdot);\mathbb{R}_+\}\leqslant\varepsilon\quad\text{for some constant }\varepsilon>0,\\ &\Omega:=\{(x,y)\colon\ y>g(x),x\geqslant 0\},\qquad \Gamma:=\{(x,y)\colon\ y=g(x),x\geqslant 0\}\end{aligned}\tag{5.2}$$

and $\mathbf{n}(x\pm)=(-g'(x\pm),1)/\sqrt{(g'(x\pm))^2+1}$ are the outer normal vectors to Γ at points $x\pm$, respectively (see Figure 3);

(ii) the uniform upstream flow $U_-=(\rho_-,u_-,0,p_-)$ satisfies

$$u_->c_-:=\sqrt{\frac{\gamma p_-}{\rho_-}}$$

so that a strong supersonic shock emanates from the wedge vertex.

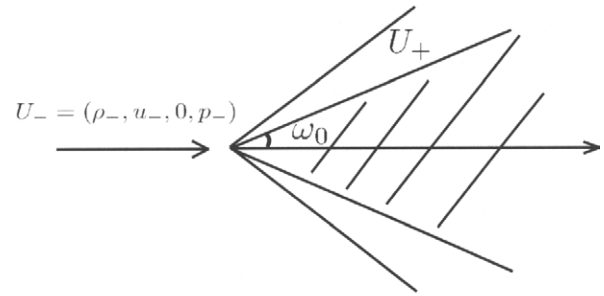

Fig. 1. Supersonic shock emanating from the wedge vertex.

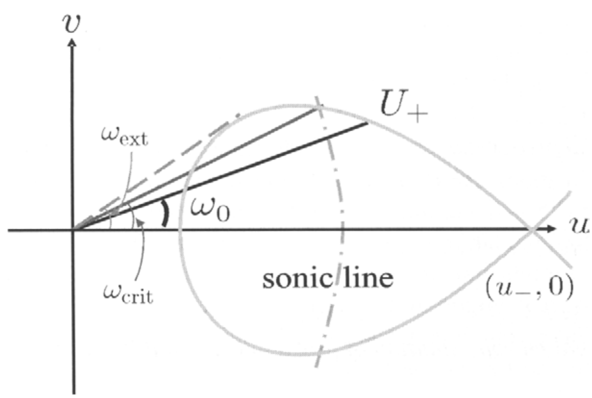

Fig. 2. Shock polar in the (u, v)-plane.

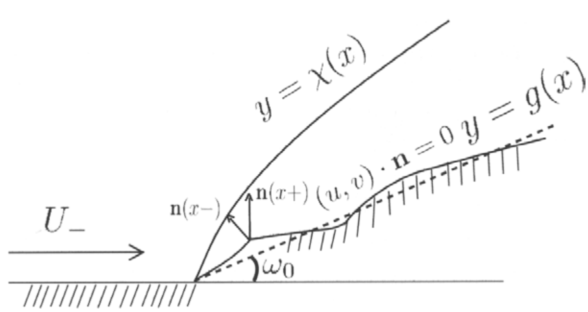

Fig. 3. Supersonic flow past a curved wedge.

With this setup, the wedge problem can be formulated into the following problem of initial–boundary value type for system (5.1)

$$\textit{Cauchy condition}: \quad U|_{x=0} = U_-; \tag{5.3}$$

$$\textit{boundary condition}: \quad (u, v) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{5.4}$$

DEFINITION 1 (Entropy solutions). A function $U = U(x, y) \in BV(\Omega)$ is called an entropy solution of problem (5.1) and (5.3)–(5.4) provided that

(i) U is a weak solution of (5.1): U satisfies the equations in the sense of distributions and the Cauchy and boundary conditions (5.3) and (5.4) in the trace sense,

(ii) U satisfies the entropy inequality in the sense of distributions,

$$\partial_x(\rho u S) + \partial_y(\rho v S) \geqslant 0, \tag{5.5}$$

that is, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\varphi \geqslant 0$,

$$\int_\Omega (\rho u S \varphi_x + \rho v S \varphi_y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_0^\infty \rho_- u_- S_- \varphi(0, y)\,\mathrm{d}y. \tag{5.6}$$

Then we have the following theorem.

THEOREM 5.1 (Existence and stability). *There exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, if* (5.2) *holds for* $\varepsilon \leqslant \varepsilon_0$, *there exists a pair of functions*

$$U \in BV\big(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}_+\big), \qquad \sigma \in BV(\mathbb{R}_+; \mathbb{R})$$

with $\chi = \int_0^x \sigma(s)\,\mathrm{d}s \in \mathrm{Lip}(\mathbb{R}_+; \mathbb{R}_+)$ *such that*

(i) *U is a global entropy solution of problem* (5.1) *and* (5.3)–(5.4) *in* Ω *with*

$$TV\big\{U(x,\cdot)\colon \big[g(x), -\infty\big)\big\} \leqslant C\,TV\big(g'(\cdot)\big) \quad \textit{for every } x \in \mathbb{R}_+,$$
$$(u, v)\cdot \mathbf{n}|_{y=g(x)} = 0 \qquad \textit{in the trace sense};$$

(ii) *the curve* $y = \chi(x)$ *is a strong shock front with* $\chi(x) > g(x)$ *for any* $x > 0$ *and*

$$U|_{\{y>\chi(x)\}} = U_-, \qquad \sqrt{u^2+v^2}\big|_{\{g(x)<y<\chi(x)\}} < u_-;$$

(iii) *there exist constants* p_∞ *and* σ_∞ *such that*

$$\lim_{x\to\infty} \sup\big\{\big|p(x,y) - p_\infty\big|\colon g(x) < y < \chi(x)\big\} = 0,$$
$$\lim_{x\to\infty} \big|\sigma(x) - \sigma_\infty\big| = 0$$

and

$$\lim_{x\to\infty} \sup\left\{\left|\arctan\left(\frac{v(x,y)}{u(x,y)}\right) - \omega_\infty\right|\colon g(x) < y < \chi(x)\right\} = 0,$$

where $\omega_\infty = \lim_{x\to\infty} \arctan(g'(x+))$.

This theorem has been established in [88]. It indicates that, under the BV perturbation of the wedge boundary as long as the wedge vertex angle is less than the critical angle, the strong shock front emanating from the wedge vertex is nonlinearly stable in structure globally, although there may be many weak shocks and vortex sheets between the wedge boundary and the strong shock front. This asserts that any supersonic shock for the wedge problem is nonlinearly stable.

In order to establish this theorem, we first developed a modified Glimm scheme whose mesh grids are designed to follow the slope of the Lipschitz wedge boundary, which are not standard rectangle mesh grids, so that the lateral Riemann building blocks contain only one shock or rarefaction wave emanating from the mesh points on the boundary. Such a design makes the BV estimates more convenient for the Glimm approximate solutions. Then careful interaction estimates were made. One of the essential estimates is the estimate of the strength δ_1 of the reflected 1-waves in the interaction between the 4-strong shock front and weak waves $(\alpha_1, \beta_2, \beta_3, \beta_4)$, that is,

$$\delta_1 = \alpha_1 + K_{s1}\beta_4 + O(1)|\alpha_1|\big(|\beta_2| + |\beta_3|\big) \quad \text{with } |K_{s1}| < 1.$$

The second essential estimate is the interaction estimate between the wedge boundary and weak waves.

Based on the construction of the modified Glimm scheme and interaction estimates, we successfully identified a Glimm-type functional to incorporate the curved wedge boundary and the strong shock front naturally and to trace the interactions not only between the wedge boundary and weak waves but also between the strong shock front and weak waves. In particular, the Glimm-type functional on the mesh curve J is defined by

$$F(J) = C_*\big|\sigma^J - \sigma_0\big| + L(J) + KQ(J).$$

Here the linear part measuring the total variation is

$$L(J) = K_0L_0(J) + L_1(J) + K_2L_2(J) + K_3L_3(J) + K_4L_4(J)$$

with

$$L_0(J) = \sum\big\{\big|\omega(C_l)\big|:\ C_l \in \Omega_J\big\},$$

$$L_j(J) = \sum\big\{|\alpha_j|:\ \alpha_j \text{ crosses } J\big\}, \quad 1 \leqslant j \leqslant 4,$$

and the quadratic part measuring the potential wave interaction is

$$Q(J) := \sum\big\{|\alpha||\beta|:\ \alpha, \beta \text{ interacting waves crossing } J\big\},$$

where Ω_J is the set of the mesh corner points lying in J and the boundary, σ^J stands for the speed of the strong shock crossing J, the constants K, C_*, K_0, K_2, K_3 and K_4 can be appropriately chosen with the aid of the important fact that $|K_{s1}| < 1$ so that the identified

Glimm functional monotonically decreases in the flow direction. Another essential estimate is to trace the approximate strong shocks in order to establish the nonlinear stability and asymptotic behavior of the strong shock emanating from the wedge vertex under the BV wedge perturbation.

Condition (5.2) can be relaxed by combining the analysis in [88] with the argument in [322,323]. The existence and stability of transonic flows past a curved wedge is under investigation with the aid of free boundary approaches (see Section 6.3).

For *the cone problem*, the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone with small vertex angle was established upon the perturbation of the obstacle in [203]. It would be interesting to combine the analysis in [203] with the argument in [88] to study the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone with arbitrary vertex angle. Other related results and analysis for this problem can be seen in [92,93] and the references cited therein.

5.2. *Stability of supersonic vortex sheets*

Another natural problem is the stability of supersonic vortex sheets above Lipschitz walls along which the total variation of the tangent angle functions is suitably small. More precisely,

(i) there exists a Lipschitz function $g \in \mathrm{Lip}(\mathbb{R}_+; \mathbb{R})$ with $g(0) = 0$, $g'(0+) = 0$, $\lim_{x\to\infty} \arctan(g'(x+)) = 0$, and $g' \in BV(\mathbb{R}_+; \mathbb{R})$ such that

$$\begin{aligned} &TV\big(g'(\cdot)\big) \leqslant \varepsilon \quad \text{for some constant } \varepsilon > 0, \\ &\Omega = \big\{(x, y)\colon\ y > g(x), x \geqslant 0\big\}, \qquad \Gamma = \big\{(x, y)\colon\ y = g(x), x \geqslant 0\big\}, \end{aligned} \tag{5.7}$$

and $\mathbf{n}(x\pm) = (-g'(x\pm), 1)/\sqrt{(g'(x\pm))^2 + 1}$ are the outer normal vectors to Γ at points $x\pm$, respectively (see Figure 4);

(ii) the upstream flow consists of one supersonic straight vortex sheet $y = y_0 > 0$ and two constant vectors $U_0 = (\rho_0, u_0, 0, p_0)$ when $y > y_0 > 0$ and $U_1 = (\rho_1, u_1, 0, p_0)$ when $0 < y < y_0$ satisfying

$$u_1 > u_0 > 0, \qquad u_i > c_i, \quad i = 0, 1,$$

where $c_i = \sqrt{\gamma p_i/\rho_i}$ is the sonic speed of states U_i, $i = 0, 1$.

With this setup, the vortex sheet problem can be formulated into the following problem of initial–boundary value type for system (5.1):

$$\textit{Cauchy condition:} \qquad U|_{x=0} = \begin{cases} U_0, & 0 < y < y_0, \\ U_1, & y > y_0; \end{cases} \tag{5.8}$$

$$\textit{boundary condition:} \quad (u, v) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{5.9}$$

The stability of supersonic vortex sheets has been studied by classical linearized stability analysis, large-scale numerical simulations, and asymptotic analysis. In particular,

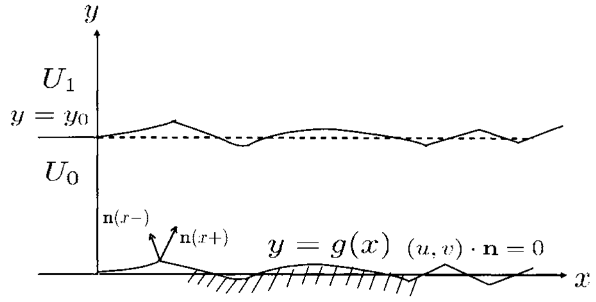

Fig. 4. Stability of the supersonic vortex sheet.

the nonlinear development of instabilities of supersonic vortex sheets has been predicted at high Mach number as time evolves; see [11,339] and the references cited therein. Motivated by the phenomenon of evolution instabilities, we are interested in whether steady supersonic vortex sheets, as time-asymptotics, are stable under a BV perturbation of the Lipschitz walls. In contrast with the prediction of instability in time, it has been proved that steady supersonic vortex sheets, as time-asymptotics, are stable in structure globally, even under the BV perturbation of the Lipschitz walls in [87].

THEOREM 5.2 (Existence and stability). *There exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, if* (5.7) *holds for* $\varepsilon \leqslant \varepsilon_0$, *there exists a pair of functions*

$$U \in BV(\mathbb{R}_+;\mathbb{R}), \qquad \chi \in \mathrm{Lip}(\mathbb{R}_+;\mathbb{R}_+)$$

with $\chi(0) = y_0$ *such that*

(i) U *is a global entropy solution of problem* (5.1) *and* (5.8)–(5.9) *in* Ω *with*

$$TV\{U(x,\cdot)\colon [g(x),\infty)\} \leqslant C\,TV(g'(\cdot)) \quad \text{for every } x \in [0,\infty),$$
$$(u,v)\cdot \mathbf{n}|_{y=g(x)} = 0 \quad \text{in the trace sense;}$$

(ii) *the curve* $\{y = \chi(x)\}$ *is a strong supersonic vortex sheet with* $\chi(x) > g(x)$ *for any* $x > 0$ *and*

$$\big|U|_{\{g(x)<y<\chi(x)\}} - U_0\big| \leqslant C\varepsilon, \qquad \big|U|_{\{y>\chi(x)\}} - U_1\big| \leqslant C\varepsilon;$$

(iii) *there exist constants* p_∞ *and* χ_∞ *such that*

$$\lim_{x\to\infty} \sup\big\{|p(x,y) - p_\infty|\colon g(x) < y < \chi(x)\big\} = 0,$$
$$\lim_{x\to\infty} |\chi(x) - \chi_\infty| = 0$$

and

$$\lim_{x\to\infty} \sup\left\{\left|\arctan\left(\frac{v(x,y)}{u(x,y)}\right)\right|\colon y > g(x)\right\} = 0.$$

This theorem indicates that the strong supersonic vortex sheets are nonlinearly stable in structure globally under the BV perturbation of the Lipschitz wall, although there may be many weak shocks and supersonic vortex sheets away from the strong vortex sheet.

In order to establish this theorem, as in Section 5.1, we first developed a modified Glimm scheme whose mesh grids are designed to follow the slope of the Lipschitz boundary, which are not standard rectangle mesh grids, so that the lateral Riemann building blocks contain only one wave emanating from the mesh points on the boundary. For this case, one of the essential estimates is the estimate of the strength δ_1 of the reflected 1-wave in the interaction between the 4-weak wave α_4 and the strong vortex sheet from below, that is,

$$\delta_1 = K_{01}\alpha_4, \quad |K_{01}| < 1.$$

Another essential estimate is the estimate of the strength δ_4 of the reflected 4-wave in the interaction between the 1-weak wave β_1 and the strong vortex sheet from above is also less than one, that is,

$$\delta_4 = K_{11}\beta_1, \quad |K_{11}| < 1.$$

The third essential estimate is the interaction estimate between the boundary and weak waves.

Based on the construction of the modified Glimm scheme and the new interaction estimates, we successfully identified a Glimm-type functional by both incorporating the Lipschitz wall and the strong vortex sheet naturally and tracing the interactions not only between the boundary and weak waves but also between the strong vortex sheet and weak waves so that the Glimm-type functional monotonically decreases in the flow direction. Another essential estimate is to trace the approximate supersonic vortex sheets in order to establish the nonlinear stability and asymptotic behavior of the strong vortex sheet under the BV boundary perturbation. For more details, see [87].

6. Multidimensional steady transonic problems

In this section we discuss another important class of multidimensional steady problems: transonic problems. In the last decade, a program has been initiated on the existence and stability of multidimensional transonic shocks, and some new analytical approaches including techniques, methods and ideas have been developed. We focus here on the potential flow equation for the velocity potential $\varphi : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, which is a second-order nonlinear equation of mixed elliptic–hyperbolic type,

$$\operatorname{div}\big(\rho\big(|\nabla\varphi|^2\big)\nabla\varphi\big) = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \tag{6.1}$$

where the density $\rho(q^2)$ is

$$\rho\big(q^2\big) = \big(1 - \theta q^2\big)^{1/(\gamma-1)}$$

with adiabatic exponent $\gamma > 1$. Equation (6.1) is elliptic at $\nabla\varphi$ with $|\nabla\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0$$

and hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0.$$

We are interested in compressible potential flows with shocks. Let Ω^+ and Ω^- be open subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \emptyset, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega}, \qquad \mathcal{S} = \partial\Omega^+ \cap \Omega.$$

Let $\varphi \in C^{0,1}(\Omega)$ be a weak solution of (6.1) and in $C^1(\overline{\Omega^\pm})$ so that $\nabla\varphi$ experiences a jump across $\mathcal{S}$ that is a $(d-1)$-dimensional smooth surface. Then φ satisfies the following Rankine–Hugoniot conditions on $\mathcal{S}$

$$[\varphi]_{\mathcal{S}} = 0, \qquad \left[\rho\left(|\nabla\varphi|^2\right)\nabla\varphi \cdot \mathbf{n}\right]_{\mathcal{S}} = 0, \tag{6.2}$$

where $\mathbf{n}$ is the unit normal to $\mathcal{S}$ from Ω^- to Ω^+, and the bracket denotes the difference between the values of the function along $\mathcal{S}$ on the $\Omega^\pm$ sides. Moreover, a function $\varphi \in C^1(\overline{\Omega^\pm})$, which satisfies $|\nabla\varphi| \leqslant \sqrt{2/(\gamma-1)}$, (6.2), and equation (6.1) in $\Omega^\pm$, respectively, is a weak solution of (6.1) in the whole domain Ω. Set $\varphi^\pm = \varphi|_{\Omega^\pm}$. Then we can also write (6.2) as

$$\varphi^+ = \varphi^- \quad \text{on } \mathcal{S} \tag{6.3}$$

and

$$\rho\left(\left|\nabla\varphi^+\right|^2\right)\nabla\varphi^+ \cdot \mathbf{n} = \rho\left(\left|\nabla\varphi^-\right|^2\right)\nabla\varphi^- \cdot \mathbf{n} \quad \text{on } \mathcal{S}. \tag{6.4}$$

Note that the function

$$\Phi(p) := \left(1 - \frac{\gamma-1}{2}p^2\right)^{1/(\gamma-1)} p \tag{6.5}$$

is continuous on $[0, \sqrt{2/(\gamma-1)}\,]$ and satisfies

$$\Phi(p) > 0 \quad \text{for } p \in \left(0, \sqrt{\frac{2}{\gamma-1}}\right), \qquad \Phi(0) = \Phi\left(\sqrt{\frac{2}{\gamma-1}}\right) = 0, \tag{6.6}$$

$$0 < \Phi'(p) < 1 \quad \text{on } (0, c_*), \qquad \Phi'(p) < 0 \quad \text{on } \left(c_*, \sqrt{\frac{2}{\gamma-1}}\right), \tag{6.7}$$

$$\Phi''(p) < 0 \qquad \text{on } (0, c_*], \tag{6.8}$$

where $c_* = \sqrt{2/(\gamma+1)}$ is the sonic speed, for which a flow is called supersonic if $|\nabla\varphi| > c_*$ and subsonic if $|\nabla\varphi| < c_*$.

Suppose that $\varphi \in C^1(\overline{\Omega^{\pm}})$ is a weak solution satisfying

$$|\nabla\varphi| < c_* \quad \text{in } \Omega^+, \qquad |\nabla\varphi| > c_* \quad \text{in } \Omega^-, \qquad \nabla\varphi^{\pm} \cdot \mathbf{n}|_{\mathcal{S}} > 0. \tag{6.9}$$

Then φ is a *transonic shock solution* with *transonic shock* $\mathcal{S}$ dividing Ω into the *subsonic region* Ω^+ and the *supersonic region* Ω^- and satisfying the physical entropy condition (see [100])

$$\rho\big(|\nabla\varphi^-|^2\big) < \rho\big(|\nabla\varphi^+|^2\big) \quad \text{along } \mathcal{S}. \tag{6.10}$$

Note that (6.1) is elliptic in the subsonic region and hyperbolic in the supersonic region.

Let $(x_1, \mathbf{x}')$ be the coordinates in $\mathbb{R}^d$, where $x_1 \in \mathbb{R}$ and $\mathbf{x}' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. Fix $\mathbf{V}_0 \in \mathbb{R}^d$, and let

$$\varphi_0(\mathbf{x}) := \mathbf{V}_0 \cdot \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^d.$$

If $|\mathbf{V}_0| \in (0, c_*)$ (resp. $|\mathbf{V}_0| \in (c_*, \sqrt{2/(\gamma-1)})$), then $\varphi_0(\mathbf{x})$ is a subsonic (resp. supersonic) solution in $\mathbb{R}^d$, and $\mathbf{V}_0 = \nabla\varphi_0$ is its velocity.

Let $q_0^- > 0$ and $\mathbf{V}_0' \in \mathbb{R}^{d-1}$ be such that the vector $\mathbf{V}_0^- := (q_0^-, \mathbf{V}_0')$ satisfies $|\mathbf{V}_0^-| > c_*$. Then, using the properties of function (6.5), we conclude from (6.6)–(6.8) that there exists a unique $q_0^+ > 0$ such that

$$\begin{aligned}&\left(1 - \frac{\gamma-1}{2}\big(|q_0^+|^2 + |\mathbf{V}_0'|^2\big)\right)^{1/(\gamma-1)} q_0^+ \\ &\quad = \left(1 - \frac{\gamma-1}{2}\big(|q_0^-|^2 + |\mathbf{V}_0'|^2\big)\right)^{1/(\gamma-1)} q_0^-.\end{aligned} \tag{6.11}$$

The entropy condition (6.10) implies $q_0^+ < q_0^-$. By denoting $\mathbf{V}_0^+ := (q_0^+, \mathbf{V}_0')$ and defining functions

$$\varphi_0^{\pm}(\mathbf{x}) := V_0^{\pm} \cdot \mathbf{x} \quad \text{on } \mathbb{R}^d,$$

then φ_0^+ (resp. φ_0^-) is a subsonic (resp. supersonic) solution. Furthermore, from (6.4) and (6.11), the function

$$\begin{aligned}\varphi_0(\mathbf{x}) &:= \min\big(\varphi_0^-(\mathbf{x}), \varphi_0^+(\mathbf{x})\big) \\ &= \begin{cases} \mathbf{V}_0^+ \cdot \mathbf{x}, & \mathbf{x} \in \Omega_0^- := \{\mathbf{x} \in \mathbb{R}^d\colon x_1 < 0\}, \\ \mathbf{V}_0^- \cdot \mathbf{x}, & \mathbf{x} \in \Omega_0^+ := \{\mathbf{x} \in \mathbb{R}^d\colon x_1 > 0\}, \end{cases}\end{aligned} \tag{6.12}$$

is a plane transonic shock solution in $\mathbb{R}^d$, Ω_0^- and Ω_0^+ are respectively its subsonic and supersonic regions, and $\mathcal{S} = \{x_1 = 0\}$ is a transonic shock. Note that, if $\mathbf{V}_0' = 0$, the velocities $V_0^{\pm}$ are orthogonal to the shock $\mathcal{S}$ and, if $\mathbf{V}_0' \neq 0$, the velocities are not orthogonal to $\mathcal{S}$.

In order to deal with multidimensional transonic shocks in an unbounded domain Ω, we define the following weighted Hölder seminorms and norms in a domain $\mathcal{D} \subset \mathbb{R}^d$.

Let $\mathbf{x} \to \delta_{\mathbf{x}}$ be a given nonnegative function defined on $\mathcal{D}$, which will be specified in each case we consider below. Let $\delta_{\mathbf{x},\mathbf{y}} := \min(\delta_{\mathbf{x}}, \delta_{\mathbf{y}})$ for $\mathbf{x}, \mathbf{y} \in \mathcal{D}$. For $k \in \mathbb{R}$, $\alpha \in (0, 1)$ and $m \in \mathcal{Z}_+$, we define

$$[u]^{(k)}_{m,0,\mathcal{D}} = \sum_{|\beta|=m} \sup_{\mathbf{x}\in\mathcal{D}} \big(\delta_{\mathbf{x}}^{m+k} |\mathrm{D}^\beta u(\mathbf{x})|\big),$$

$$[u]^{(k)}_{m,\alpha,\mathcal{D}} = \sum_{|\beta|=m} \sup_{\mathbf{x},\mathbf{y}\in\mathcal{D},\mathbf{x}\neq\mathbf{y}} \left(\delta_{\mathbf{x},\mathbf{y}}^{m+\alpha+k} \frac{|\mathrm{D}^\beta u(\mathbf{x}) - \mathrm{D}^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}\right), \tag{6.13}$$

$$\|u\|^{(k)}_{m,0,\mathcal{D}} = \sum_{j=0}^{m} [u]^{(k)}_{j,0,\mathcal{D}}, \qquad \|u\|^{(k)}_{m,\alpha,\mathcal{D}} = \|u\|^{(k)}_{m,0,\mathcal{D}} + [u]^{(k)}_{m,\alpha,\mathcal{D}},$$

where $\mathrm{D}^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$, $\beta = (\beta_1, \dots, \beta_d)$ is a multiindex with $\beta_j \geqslant 0, \beta_j \in \mathcal{Z}$ and $|\beta| = \beta_1 + \cdots + \beta_d$. We denote by $\|u\|_{m,\alpha,\mathcal{D}}$ the (nonweighted) Hölder norms in a domain $\mathcal{D}$, i.e., the norms defined as above with $\delta_{\mathbf{x}} = \delta_{\mathbf{x},\mathbf{y}} = 1$.

6.1. *Transonic shock problems in* $\mathbb{R}^d$

We now consider multidimensional perturbations of the uniform transonic shock solution (6.12) in the whole space $\mathbb{R}^d$ with $d \geqslant 3$.

Since it suffices to specify the supersonic perturbation φ^- only in a neighborhood of the unperturbed shock surface $\{x_1 = 0\}$, we introduce domains

$$\Omega := (-1, \infty) \times \mathbb{R}^{d-1}, \qquad \Omega_1 := (-1, 1) \times \mathbb{R}^{d-1}.$$

Note that we expect the subsonic region Ω^+ to be close to the half-space $\Omega_0^+ = \{x_1 > 0\}$. We use the norms in (6.13) with the weight function

$$\delta_{\mathbf{x}} = 1 + |\mathbf{x}|$$

and consider the following problem.

PROBLEM 6.1. Given a supersonic solution $\varphi^-(\mathbf{x})$ of (6.1) in Ω_1 satisfying that, for some $\alpha > 0$,

$$\|\varphi^- - \varphi_0^-\|^{(d-1)}_{2,\alpha,\Omega_1} \leqslant \sigma \tag{6.14}$$

with $\sigma > 0$ small, find a transonic shock solution $\varphi(\mathbf{x})$ in Ω such that

$$\Omega^- \subset \Omega_1, \qquad \varphi(\mathbf{x}) = \varphi^-(\mathbf{x}) \quad \text{in } \Omega^-,$$

where $\Omega^- := \Omega \setminus \Omega^+$ and $\Omega^+ := \{\mathbf{x} \in \Omega\colon |\nabla\varphi(\mathbf{x})| < c_*\}$, and

$$\varphi = \varphi^-, \qquad \partial_{x_1}\varphi = \partial_{x_1}\varphi^- \quad \text{on } \{x_1 = -1\}, \tag{6.15}$$

$$\lim_{R\to\infty} \|\varphi - \varphi_0^+\|_{C^1(\Omega^+\setminus B_R(0))} = 0. \tag{6.16}$$

Condition (6.15) determines that the solution has supersonic upstream, while condition (6.16) determines, in particular, that the uniform velocity state at infinity in the downstream direction is equal to the unperturbed downstream velocity state. The additional requirement in (6.16) that $\varphi \to \varphi_0^+$ at infinity within Ω^+ fixes the position of shock at infinity. This allows us to determine the solution of Problem 6.1 uniquely.

Then we have the following theorem (see [62]).

THEOREM 6.1. *Let* $|(q_0^-, \mathbf{V}_0')| \in (c_*, \sqrt{2/(\gamma-1)})$ *and* $q_0^+ \in (0, c_*)$ *satisfy* (6.11), *and let* $\varphi_0(\mathbf{x})$ *be the transonic shock solution* (6.12). *Then there exist positive constants* σ_0, C_1 *and* C_2 *depending only on* d, γ, α, $|\mathbf{V}_0'|$ *and* q_0^- *such that, for every* $\sigma \leqslant \sigma_0$ *and any supersonic solution* $\varphi^-(\mathbf{x})$ *of* (6.1) *satisfying the conditions stated in Problem* 6.1, *there exists a unique solution* $\varphi(\mathbf{x})$ *of Problem* 6.1 *satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+}^{(d-2)} \leqslant C_1\sigma \tag{6.17}$$

with Ω^+ *defined in Problem* 6.1. *In addition,*

$$\Omega^+ = \{x_1 > f(\mathbf{x}')\}, \tag{6.18}$$

where $f : \mathbb{R}^{d-1} \to \mathbb{R}$ *satisfies*

$$\|f\|_{2,\alpha,\mathbb{R}^{d-1}}^{(d-2)} \leqslant C_2\sigma, \tag{6.19}$$

that is, the shock surface

$$\mathcal{S} = \{(x_1, \mathbf{x}')\colon x_1 = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\}$$

is in $C^{2,\alpha}$ *and converges at infinity, with an appropriate algebraic rate, to the hyperplane*

$$\mathcal{S}_0 = \{x_1 = 0\}.$$

Moreover, there exist a nonnegative nondecreasing function $\Psi \in C([0,\infty))$ *satisfying* $\Psi(0) = 0$ *and a constant* σ_0 *depending only on* d, γ, α, $|\mathbf{V}_0'|$ *and* q_0^- *such that, if* $\sigma < \sigma_0$ *and smooth supersonic solutions* $\varphi^-(\mathbf{x})$ *and* $\hat\varphi^-(\mathbf{x})$ *of* (6.1) *satisfy* (6.14), *the unique solutions* $\varphi(\mathbf{x})$ *and* $\hat\varphi(\mathbf{x})$ *of Problem* 6.1 *for* $\varphi^-(\mathbf{x})$ *and* $\hat\varphi^-(\mathbf{x})$, *respectively, satisfy*

$$\|f_\varphi - f_{\hat\varphi}\|_{2,\alpha,\mathbb{R}^{d-1}}^{(d-2)} \leqslant \Psi\left(\|\varphi^- - \hat\varphi^-\|_{2,\alpha,\Omega_1}^{(d-1)}\right), \tag{6.20}$$

where $f_\varphi(\mathbf{x}')$ *and* $f_{\hat\varphi}(\mathbf{x}')$ *are the free boundary functions of* $\varphi(\mathbf{x})$ *and* $\hat\varphi(\mathbf{x})$ *in* (6.18), *respectively.*

This existence result can be extended to the case that the regularity of the steady perturbation φ^- is only $C^{1,1}$. That is, (6.14) can be replaced by

$$\|\varphi^- - \varphi_0^-\|_{1,1,\Omega_1}^{(d-1)} \leqslant \sigma. \tag{6.21}$$

Another related problem is the stability of transonic shocks near a spherical transonic shock, which can be established by following similar arguments (see [60,62]).

6.2. *Nozzle problems involving transonic shocks*

We now consider multidimensional transonic shocks in the following infinite nozzle Ω with arbitrary smooth cross-sections

$$\Omega = \Psi(\Lambda \times \mathbb{R}) \cap \{x_1 > -1\}, \tag{6.22}$$

where $\Lambda \subset \mathbb{R}^{d-1}$ is an open bounded connected set with a smooth boundary, and $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth map, which is close to the identity map. For simplicity, we assume that

$$\partial\Lambda \text{ is in } C^{[d/2]+3,\alpha}, \qquad \|\Psi - I\|_{[d/2]+3,\alpha,\mathbb{R}^d} \leqslant \sigma \tag{6.23}$$

for some $\alpha \in (0, 1)$ and small $\sigma > 0$, where $[s]$ is the integer part of s, $I : \mathbb{R}^d \to \mathbb{R}^d$ is the identity map, and $\partial_l\Omega := \Psi(\mathbb{R} \times \partial\Lambda) \cap \{x_1 > -1\}$. Such nozzles especially include the slowly varying de Laval nozzles [100,336]. For concreteness, we also assume that there exists $L > 1$ such that

$$\Psi(\mathbf{x}) = \mathbf{x} \quad \text{for any } \mathbf{x} = (x_1, \mathbf{x}') \text{ with } x_1 > L, \tag{6.24}$$

that is, the nozzle slowly varies in a bounded domain as the de Laval nozzles.

In the two-dimension case, the domain Ω defined above has the following simple form

$$\Omega = \{(x_1, x_2) \colon x_1 > -1, b^-(x_2) < x_2 < b^+(x_2)\},$$

where $\|b^\pm - b_\infty^\pm\|_{4,\alpha,\mathbb{R}} \leqslant \sigma$ and $b^\pm \equiv b_\infty^\pm$ on $[L, \infty)$ for some constants $b_\infty^\pm$ satisfying $b_\infty^+ > b_\infty^-$.

For the multidimensional case, the geometry of the nozzles is much richer.

Note that our setup implies that $\partial\Omega = \overline{\partial_o\Omega} \cup \partial_l\Omega$ with

$$\partial_l\Omega := \Psi[(-\infty, \infty) \times \partial\Lambda] \cap \{(x_1, \mathbf{x}') \colon x_1 > -1\},$$
$$\partial_o\Omega := \Psi((-\infty, \infty) \times \Lambda) \cap \{(x_1, \mathbf{x}') \colon x_1 = -1\}.$$

Then our transonic nozzle problem can be formulated into the following form.

PROBLEM 6.2 (Transonic nozzle problem). Given the supersonic upstream flow at the entrance $\partial_o\Omega$,

$$\varphi = \varphi_e^-, \qquad \varphi_{x_1} = \psi_e^- \quad \text{on } \partial_o\Omega, \tag{6.25}$$

the slip boundary condition on the nozzle boundary $\partial_l\Omega$,

$$\nabla\varphi \cdot \mathbf{n} = 0 \quad \text{on } \partial_l\Omega, \tag{6.26}$$

and the uniform subsonic flow condition at the infinite exit $x_1 = \infty$,

$$\|\varphi(\cdot) - \omega x_1\|_{C^1(\Omega\cap\{x_1>R\})} \to 0 \quad \text{as } R \to \infty \text{ for some } \omega \in (0, c_*), \tag{6.27}$$

find a multidimensional transonic flow φ of problem (6.1) and (6.25)–(6.27) in Ω.

The standard local existence theory of smooth solutions for the initial–boundary value problem (6.25) and (6.26) for second-order quasilinear hyperbolic equations implies that, as σ is sufficiently small in (6.23) and (6.30), there exists a supersonic solution φ^- of (6.1) in

$$\Omega_2 := \{-1 \leqslant x_1 \leqslant 1\},$$

which is a C^{l+1} perturbation of $\varphi_0^- = q_0^- x_1$: For any $\alpha \in (0, 1]$,

$$\|\varphi^- - \varphi_0^-\|_{l,\alpha,\Omega_2} \leqslant C_0\sigma, \quad l = 1, 2, \tag{6.28}$$

for some constant $C_0 > 0$, and satisfies

$$\nabla\varphi^- \cdot \mathbf{n} = 0 \quad \text{on } \partial_l\Omega_2, \tag{6.29}$$

provided that (φ_e^-, ψ_e^-) on $\partial_o\Omega$ satisfies

$$\|\varphi_e^- - q_0^- x_1\|_{H^{s+l}} + \|\psi_e^- - q_0^-\|_{H^{s+l-1}} \leqslant \sigma, \quad l = 1, 2, \tag{6.30}$$

for some integer $s > d/2 + 1$ and the compatibility conditions up to order $s + 1$, where the norm $\|\cdot\|_{H^s}$ is the Sobolev norm with $H^s = W^{s,2}$.

Then we have the following theorem (see [61]).

THEOREM 6.2. *Let $q_0^- \in (c_*, \sqrt{2/(\gamma-1)})$ and $q_0^+ \in (0, c_*)$ satisfy* (6.11), *and let φ_0 be the transonic shock solution* (6.12) *with $\mathbf{V}' = 0$. Then there exist $\sigma_0 > 0$, C_1 and C_2, depending only on $d, \alpha, \gamma, q_0^-, \Lambda$ and L, such that, for every $\sigma \in (0, \sigma_0)$, any map Ψ satisfying* (6.23) *and* (6.24), *and any supersonic upstream flow (φ_e^-, ψ_e^-) on $\partial_o\Omega$ satisfying* (6.30) *with $l = 1$, there exists a solution $\varphi \in C^{0,1}(\Omega)$ of Problem* 6.2 *satisfying*

$$\Omega^+(\varphi) = \{x_1 > f(\mathbf{x}')\}, \qquad \Omega^-(\varphi) = \{x_1 < f(\mathbf{x}')\}, \tag{6.31}$$

$$\|\varphi - \varphi_0^-\|_{1,\alpha,\Omega^-} \leqslant C_1\sigma, \qquad \|\nabla\varphi - q_0^+ e_1\|_{0,0,\Omega^+} \leqslant C_2\sigma.$$

Moreover, this solution satisfies $\varphi \in C^{0,1}(\Omega) \cap C^\infty(\Omega^+)$ *and the following properties.*

(i) *The constant* ω *in* (6.27) *must be* q^+,

$$\omega = q^+, \tag{6.32}$$

where q^+ *is the unique solution in the interval* $(0, c_*)$ *of the equation*

$$\rho\big((q^+)^2\big)q^+ = Q^+ \tag{6.33}$$

with

$$Q^+ = \frac{1}{|\Lambda|}\int_{\partial_o\Omega} \rho\big(|\nabla_{\mathbf{x}'}\varphi_e^-|^2 + (\psi_e^-)^2\big)\psi_e^- \, \mathrm{d}\mathcal{H}^{d-1}.$$

Thus, φ *and* q^+ *satisfy*

$$\|\varphi - q^+x_1\|_{C^1(\Omega\cap\{x_1>R\})} \to 0 \quad \text{as } R \to \infty \tag{6.34}$$

and

$$|q^+ - q_0^+| \leqslant C_2\sigma. \tag{6.35}$$

(ii) *The function* $f(\mathbf{x}')$ *in* (6.31) *satisfies*

$$\|f\|_{1,\alpha,\mathbb{R}^{d-1}} \leqslant C_2\sigma, \tag{6.36}$$

and the surface $S = \{(f(\mathbf{x}'), \mathbf{x}') \colon \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$ *is orthogonal to* $\partial_l\Omega$ *at every intersection point.*

(iii) *Furthermore,* $\varphi \in C^{1,\alpha}(\overline{\Omega^+})$ *with*

$$\|\varphi - q^+x_1\|_{1,\alpha,\Omega^+} \leqslant C_2\sigma. \tag{6.37}$$

In addition, if the supersonic uniform flow (φ_e^-, ψ_e^-) *on* $\partial_o\Omega$ *satisfies* (6.30) *with* $l = 2$, *then the solution* $\varphi \in C^{2,\alpha}(\overline{\Omega^+})$ *with*

$$\|\varphi - q^+x_1\|_{2,\alpha,\Omega^+} \leqslant C_2\sigma,$$

and the solution with a transonic shock is unique and stable with respect to the nozzle boundary and the smooth supersonic upstream flow at the entrance.

When the initial data $(\varphi_e^-, \psi_e^-) \equiv (-\psi_e^-, \psi_e^-)$ is constant and the nozzle

$$\Omega \cap \{-1 \leqslant x_1 \leqslant -1 + \varepsilon\} = [-1, -1+\varepsilon] \times \Lambda \quad \text{for some } \varepsilon > 0$$

as the de Laval nozzles, then the compatibility conditions are automatically satisfied. In fact, in this case, $\varphi^-(\mathbf{x}) = \psi_e^- x_1$ is a solution near $x_1 = -1$ in the nozzle.

When $d = 2$, condition (6.30) for the supersonic upstream flow (φ_e^-, ψ_e^-) on $\partial_o\Omega$ in Theorem 6.2 can be replaced by the C^3-condition,

$$\|\varphi_e^- + q_0^-\|_{C^3} + \|\psi_e^- - q_0^-\|_{C^2} \leqslant \sigma, \tag{6.38}$$

which can be achieved by following the arguments in [216]. For the isothermal gas $\gamma = 1$, the same results can be obtained by following similar arguments.

The techniques have been extended and applied to the nozzle problem for the full Euler equations in [57].

Other transonic problems include the stability of transonic flows past infinite nonsmooth wedges or cones which are under investigation with the aid of the approaches which will be discussed in Section 6.3.

A further problem is subsonic flow past an airfoil or an obstacle. Shiffman [297], Bers [18] and Finn and Gilbarg [136] studied subsonic (elliptic) solutions of (6.1) outside an obstacle when the upstream flows are sufficiently subsonic; also see [125]. Morawetz in [250] first showed that the flows of (6.1) past an obstacle may contain transonic shocks in general. An important problem is to construct global entropy solutions of the airfoil problem (see [251,253] and [141]).

6.3. *Free boundary approaches*

We now describe two of the free boundary approaches for Problems 6.1 and 6.2, developed recently in [60–62].

6.3.1. *Free boundary problems.* The transonic shock problems can be formulated into a one-phase free boundary problem for a nonlinear elliptic equation: Given $\varphi^- \in C^{1,\alpha}(\overline{\Omega})$, find a function φ that is continuous in Ω and satisfies

$$\varphi \leqslant \varphi^- \quad \text{in } \overline{\Omega}, \tag{6.39}$$

equation (6.1), the ellipticity condition in the noncoincidence set $\Omega^+ = \{\varphi < \varphi^-\}$, the free boundary condition (6.4) on the boundary $S = \partial\Omega^+ \cap \Omega$, as well as the prescribed conditions on the fixed boundary $\partial\Omega$ and at infinity. These conditions are different in different problems, for example, conditions (6.15) and (6.16) for Problem 6.1 and (6.25)–(6.27) for Problem 6.2.

The free boundary is the location of the shock, and the free boundary conditions (6.3) and (6.4) are the Rankine–Hugoniot conditions in (6.2). Note that condition (6.39) is motivated by the similar property (6.12) of unperturbed shocks; and (6.39), locally on the shock, is equivalent to the entropy condition (6.10). Condition (6.39) transforms the transonic shock problem, in which the subsonic region Ω^+ is determined by the gradient condition $|\nabla\varphi(\mathbf{x})| < c_*$, into a free boundary problem in which Ω^+ is the noncoincidence set.

In order to solve this free boundary problem, equation (6.1) is modified to be uniformly elliptic and then the free boundary condition (6.4) is correspondingly modified. Then this modified free boundary problem is solved. Since φ^- is a small $C^{1,\alpha}$ perturbation of φ_0^-, the solution φ of the free boundary problem is shown to be a small $C^{1,\alpha}$ perturbation of the given subsonic shock solution φ_0^+ in Ω^+. In particular, the gradient estimate implies that φ in fact satisfies the original free boundary problem, hence the transonic shock problem, Problem 6.1 (Problem 6.2, respectively).

The modified free boundary problem does not directly fit into the variational framework of Alt and Caffarelli [4] and Alt, Caffarelli and Friedman [5], as well as the regularization framework of Berestycki, Caffarelli and Nirenberg [16]. Also, the nonlinearity of the free boundary problem makes it difficult to apply the Harnack inequality approach of Caffarelli [38]. In particular, a boundary comparison principle for positive solutions of nonlinear elliptic equations in Lipschitz domains is not available yet for the equations that are not homogeneous with respect to $\nabla^2 u$, ∇u and u, which, however, is our case.

6.3.2. *Iteration approach.* The first approach we developed in [60,61] is an iteration scheme based on the nondegeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound. Our iteration process is as follows: Suppose the domain Ω_k^+ is given so that $S_k := \partial\Omega_k^+ \setminus \partial\Omega$ is $C^{1,\alpha}$. Consider the oblique derivative problem in Ω_k^+ obtained by rewriting the (modified) equation (6.1) and free boundary condition (6.4) in terms of the function $u := \varphi - \varphi_0^+$. Then the problem has the following form:

$$\begin{aligned} &\operatorname{div} \mathbf{A}(\mathbf{x}, \nabla u) = F(\mathbf{x}) && \text{in } \Omega_k^+ := \{u > 0\}, \\ &\mathbf{A}(\mathbf{x}, \nabla u) \cdot \mathbf{n} = G(\mathbf{x}, \mathbf{n}) && \text{on } \mathcal{S} := \partial\Omega_k^+ \setminus \partial\Omega, \end{aligned} \tag{6.40}$$

plus the fixed boundary conditions on $\partial\Omega_k^+ \cap \partial\Omega$ and the conditions at infinity. The equation is quasilinear, uniformly elliptic, $\mathbf{A}(\mathbf{x}, 0) \equiv 0$, while $G(\mathbf{x}, \mathbf{n})$ has a certain structure. Let $u_k \in C^{1,\alpha}(\overline{\Omega_k^+})$ be the solution of (6.40). Then $\|u_k\|_{1,\alpha,\Omega_k^+}$ is estimated to be small if the perturbation is small, where appropriate weighted Hölder norms are actually needed in the unbounded domains. The function $\varphi_k := \varphi_0^+ + u_k$ from Ω_k^+ is extended to Ω so that the $C^{1,\alpha}$ norm of $\varphi_k - \varphi_0^+$ in Ω is controlled by $\|u_k\|_{1,\alpha,\Omega_k^+}$. Define

$$\Omega_{k+1}^+ := \{\mathbf{x} \in \Omega\colon \varphi_k(\mathbf{x}) < \varphi^-(\mathbf{x})\}$$

for the next step. Note that, since $\|\varphi_k - \varphi_0^+\|_{1,\alpha,\Omega}$ and $\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega}$ are small, we have

$$|\nabla\varphi^-| - |\nabla\varphi_k| \geqslant \delta > 0 \quad \text{in } \Omega,$$

and this nondegeneracy implies that $S_{k+1} := \partial\Omega_{k+1}^+ \setminus \partial\Omega$ is $C^{1,\alpha}$ and its norm is estimated in terms of the data of the problem.

The fixed point Ω^+ of this process determines a solution of the free boundary problem since the corresponding solution φ satisfies $\Omega^+ = \{\varphi < \varphi^-\}$ and the Rankine–Hugoniot condition holds on $\mathcal{S} := \partial\Omega^+ \cap \Omega$.

On the other hand, the elliptic estimates alone are not sufficient to get the existence of a fixed point, because the right-hand side of the boundary condition in problem (6.40) depends on the unit normal **n** of the free boundary. One way is to require the orthogonality of the flat shocks so that

$$\rho\left(\left|\nabla\varphi_0^+\right|^2\right)\nabla\varphi_0^+ = \rho\left(\left|\nabla\varphi_0^-\right|^2\right)\nabla\varphi_0^- \quad \text{in } \Omega \tag{6.41}$$

to obtain better estimates for the iteration and to prove the existence of a fixed point. Note that (6.41) is a vector identity, and the Rankine–Hugoniot condition (6.4) is the normal part of (6.41) on the unperturbed free boundary S_0.

The uniqueness and stability of solutions for the transonic shock problems are obtained by using the regularity and nondegeneracy of solutions.

For more details, see [60,61].

6.3.3. *Partial hodograph approach.* The second approach we developed in [60,62] is a partial hodograph procedure, with which we can handle the existence and stability of multidimensional transonic shocks that are not nearly orthogonal to the flow direction. One of the main ingredients in this new approach is to employ a partial hodograph transform to reduce the free boundary problem to a co-normal boundary value problem for the corresponding nonlinear second-order elliptic equation of divergence form in unbounded domains and then develop techniques to solve the co-normal boundary value problem in the unbounded domain. To achieve this, the strategy is to construct first solutions in the intersection domains between the physical unbounded domain under consideration and a series of half-balls with radius R, then make uniform estimates in R, and finally send $R \to \infty$. It requires delicate a priori estimates to achieve this. A uniform bound in a weighted L^∞ norm can be achieved by both employing a comparison principle and identifying a global function with the same decay rate as the fundamental solution of the elliptic equation with constant coefficients which controls the solutions. Then, by scaling arguments, the uniform estimates can be obtained in a weighted Hölder norm for the solutions, which lead to the existence of a solution in the unbounded domain with some decay rate at infinity. For such decaying solutions, a comparison principle holds, which implies the uniqueness for the co-normal problem. Finally, by the gradient estimate, the limit function can be shown to be a solution of the multidimensional transonic shock problem, and then the existence result can be extended to the case that the regularity of the steady perturbation is only $C^{1,1}$. We can further prove that the multidimensional transonic shock solution is stable with respect to the $C^{2,\alpha}$ supersonic perturbation.

When the regularity of the steady perturbation is $C^{3,\alpha}$ or higher, that is,

$$\left\|\varphi^- - \varphi_0^-\right\|_{3,\alpha,\Omega_1}^{(d-1)} \leqslant \sigma, \tag{6.42}$$

we introduced another simpler approach to deal with the existence and stability problem.

We also extend the approach by using the partial hodograph transform in the radial direction in the polar coordinates to establish the existence and stability of multidimensional transonic shocks near spheres in $\mathbb{R}^d$, $d \geqslant 3$. The case $d = 2$ can also be handled with similar approaches.

Another approach can be found in [40,41,357].

7. Multidimensional unsteady problems

In this section, we introduce some sample multidimensional time-dependent problems with a simplifying feature that the data (domain and/or the initial data) coupled with the structure of the underlying equations obey certain geometric structure so that the multidimensional problems can be reduced to lower dimensional problems with more complicated couplings. Different types of geometric structure call for different techniques.

The Euler equations for compressible fluids with geometric structure describe many important fluid flows, including spherically symmetric flow and self-similar flow. Such geometric flows are motivated by many physical problems, such as shock diffraction, supernovae formation in stellar dynamics, inertial confinement fusion, and underwater explosions. For the initial data with large amplitude having geometric structure, the physical insight we seek is

(a) whether the solution has the same geometric structure globally,

(b) whether the solution blows up to infinity in a finite time, for example, the density near the origin for spherically symmetric flow.

These questions are not easily understood in physical experiments and numerical simulations, especially for the blow-up, because of the limited capacity of available instruments and computers.

7.1. *Spherically symmetric solutions*

The first problem is the study of singularity at the origin for the Euler equations for isentropic or adiabatic fluids under spherical symmetry in $\mathbb{R}^d, d \geqslant 2$. The singularity at the origin makes the problem truly multidimensional. The central difficulty of this problem in the unbounded domain is the singularity at the origin and the reflection of waves from infinity and their strengthening as they move radially inwards.

Consider the Cauchy problem for (1.11),

$$(\rho, \mathbf{m})|_{t=0} = \big(\rho_0(\mathbf{x}), \mathbf{m}_0(\mathbf{x})\big) \tag{7.1}$$

with the following geometric structure,

$$\big(\rho_0(\mathbf{x}), \mathbf{m}_0(\mathbf{x})\big) = \left(\rho_0\big(|\mathbf{x}|\big), m_0\big(|\mathbf{x}|\big)\frac{\mathbf{x}}{|\mathbf{x}|}\right), \tag{7.2}$$

where $m_0(x)$ is a scalar function of $x = |\mathbf{x}| \geqslant 0$. Such a problem describes dynamic behavior of many physical problems with spherically symmetric initial structure such as explosion waves in air and other media [100,336]. Motivated by the physical experiments, we look for the solutions with spherical symmetry

$$\rho(t, \mathbf{x}) = \rho\big(t, |\mathbf{x}|\big), \qquad \mathbf{m}(t, \mathbf{x}) = m\big(t, |\mathbf{x}|\big)\frac{\mathbf{x}}{|\mathbf{x}|}. \tag{7.3}$$

The function $(\rho, \mathbf{m})(t, x)$, $x = |\mathbf{x}|$, is determined by the one-dimensional isentropic Euler equations with geometric source terms

$$\begin{cases} \partial_t \rho + \partial_x m = -\frac{d-1}{x} m, & x > 0, \\ \partial_t m + \partial_x\left(\frac{m^2}{\rho} + p(\rho)\right) = -\frac{d-1}{x}\frac{m^2}{\rho}. \end{cases} \tag{7.4}$$

It is evident that the density ρ blows up as $|\mathbf{x}| \to 0$ in general, for instance, for the focusing case. One of the challenging open problems is to understand the order of singularity

$$\rho\left(t, |\mathbf{x}|\right) \sim |\mathbf{x}|^{-\alpha}$$

for bounded Cauchy data.

On the other hand, a criterion was observed in [54] for L^∞ Cauchy data functions of arbitrarily large amplitude to guarantee the global existence of L^∞ spherically symmetric solutions which model outgoing blast waves and large-time asymptotic solutions.

THEOREM 7.1. *Consider the Cauchy problem for the Euler equations* (1.11) *with spherically symmetric initial data* (7.1) *and* (7.2). *Assume that the initial data satisfies*

$$0 \leqslant \int_0^{\rho_0(\mathbf{x})} \frac{\sqrt{p'(s)}}{s}\, \mathrm{d}s \leqslant \frac{|\mathbf{m}_0(\mathbf{x})|}{\rho_0(\mathbf{x})} \leqslant C_0 < \infty \tag{7.5}$$

for some constant $C_0 > 0$. *Then there exists a global entropy solution* $(\rho, \mathbf{m})(t, \mathbf{x}) \in L^\infty$ *of the Cauchy problem* (1.11) *and* (7.1)–(7.3) *satisfying*

$$0 \leqslant \rho(t, \mathbf{x}) \leqslant C, \qquad 0 \leqslant \left|\mathbf{m}(t, \mathbf{x})\right| \leqslant C\rho(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{7.6}$$

for some constant $C > 0$ *and*

$$\frac{1}{T}\int_0^T (\rho, \mathbf{m})(t, \mathbf{x})\, \mathrm{d}t \to 0 \quad \textit{for a.e. } \mathbf{x} \in \mathbb{R}^d \textit{ when } T \to \infty. \tag{7.7}$$

PROOF. This theorem was established in [54] by developing the fractional Godunov scheme through system (7.4). The proof is divided into five steps, and we now briefly describe them for the case of polytropic gases with $p = \kappa\rho^\gamma$, $1 < \gamma \leqslant 2$.

Step 1. *Construction of approximate solutions via the fractional-step Godunov scheme.* Partition $\mathbb{R}_+$ by the sequence $t_k = kh$, $k \in \mathcal{Z}_+$, with mesh size h and partition $\mathbb{R}_+$ into cells with the jth cell centered at $x_j = jl$, $j \in \mathcal{Z}_+$, with mesh size l. Denote $\mathbf{u}^h = (\rho^h, m^h)$ as the approximate solutions satisfying the inequality

$$\Lambda \equiv \max_{i=1,2}\left(\sup\left|\lambda_i\left(\mathbf{u}^h\right)\right|\right) \leqslant \frac{l}{4h} \leqslant 2\Lambda. \tag{7.8}$$

We will prove that $\mathbf{u}^h(t,x)$ have a uniform bound with respect to h so that it is possible to construct $\mathbf{u}^h(t,x)$ satisfying (7.8).

Assume that $\mathbf{u}^h(t,x)$ have been defined for $t < kh$. Then we define

$$\mathbf{u}^h(kh+0,x) = \mathbf{u}_j^n \equiv \frac{1}{l}\int_{(j-1/2)l}^{(j+1/2)l} \mathbf{u}^h(kh-0,x)X^h(x)\,\mathrm{d}x,$$
$$\left(j-\frac{1}{2}\right)l \leqslant x \leqslant \left(j+\frac{1}{2}\right)l, \tag{7.9}$$

where $X^h(x)$ is the characteristic function on $[Nl, \frac{1}{h}]$, where $N = N(C_0) > 0$ is some large constant depending only on $C_0 > 0$, which is solely determined by the initial data (see (7.18)).

In the strip $kh \leqslant t < (k+1)h$, $jl < x \leqslant (j+1)l$, $j,k \in \mathcal{Z}_+$, we define

$$\begin{cases} \rho^h(t,x) = \rho_0^h(t,x)\left(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\right)_+, \\ m^h(t,x) = m_0^h(t,x)\left(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\right)_+, \end{cases} \tag{7.10}$$

where $\mathbf{u}_0^h(t,x)$ are the Riemann solutions of (1.16) with initial data $(\mathbf{u}_j^k, \mathbf{u}_{j+1}^k)$ with respect to $x = (j+1/2)l$ at $t = kh$.

From this, we define the fractional step Godunov scheme

$$\mathbf{u}_j^{k+1} = \frac{1}{l}\int_{(j-1/2)l}^{(j+1/2)l} \mathbf{u}^h(kh-0,x)X^h(x)\,\mathrm{d}x. \tag{7.11}$$

In this way, for $kh \leqslant t < (k+1)h$, $k \geqslant 0$ integers, we have

$$\begin{cases} w_1^h(t,x) \\ \quad = \frac{m_0^h(t,x)}{\rho_0^h(t,x)} + \left(\rho_0^h(t,x)\right)^{(\gamma-1)/2}\left(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\right)_+^{(\gamma-1)/2}, \\ w_2^h(t,x) \\ \quad = \frac{m_0^h(t,x)}{\rho_0^h(t,x)} - \left(\rho_0^h(t,x)\right)^{(\gamma-1)/2}\left(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\right)_+^{(\gamma-1)/2}, \end{cases} \tag{7.12}$$

where (w_1, w_2) are the Riemann invariants introduced in (3.6).

Step 2. *L^∞ estimate for the approximate solutions.* There exists $C = C(C_0) > 0$, independent of h, such that

$$0 \leqslant \rho^h(t,x) \leqslant C, \qquad 0 \leqslant m^h(t,x) \leqslant C\rho^h(t,x), \quad (t,x) \in \mathbb{R}_+^2. \tag{7.13}$$

In order to show this estimate, we first need some properties of the Riemann solutions for the homogeneous Euler equations (1.16) with initial Riemann data

$$\mathbf{u} = \begin{cases} \mathbf{u}_- \equiv (\rho_-, m_-), & x < x_0,\, x_0 > 0, \\ \mathbf{u}_+ \equiv (\rho_+, m_+), & x > x_0, \end{cases} \tag{7.14}$$

and lateral Riemann data

$$\begin{cases} \mathbf{u}|_{t=0} = \mathbf{u}_+, & x > 0, \\ m|_{x=0} = 0, & t > 0, \end{cases} \tag{7.15}$$

where $\rho_\pm \geqslant 0$ and $m_\pm$ are the constants with $|m_\pm/\rho_\pm| < \infty$. The discontinuity in the weak solutions of (1.16) satisfies the Rankine–Hugoniot condition

$$\sigma(\mathbf{u} - \mathbf{u}_0) = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_0), \tag{7.16}$$

where σ is the propagation speed of the discontinuity, $\mathbf{u}_0$ and $\mathbf{u}$ are the corresponding left and right state, respectively. A discontinuity is a shock if it satisfies the entropy condition

$$\sigma\big(\eta(\mathbf{u}) - \eta(\mathbf{u}_0)\big) - \big(q(\mathbf{u}) - q(\mathbf{u}_0)\big) \geqslant 0 \tag{7.17}$$

for any convex entropy–entropy flux pair (η, q).

For the Riemann problems (7.14) and (7.15) for system (1.16), the Riemann solutions generally contain rarefaction waves and shocks satisfying the following facts.

FACT (i). There exists a unique piecewise smooth entropy solution $(\rho, m)(t, x)$ containing vacuum states on the quarter $t \geqslant 0, x \geqslant 0$ for each problem of (7.14) and (7.15), at least locally in time.

FACT (ii). The regions

$$\Sigma = \big\{(\rho, m)\colon\ w_1 \leqslant w_0,\, w_2 \geqslant z_0,\, w_1 - w_2 \geqslant 0\big\}$$

and

$$\Sigma = \big\{(\rho, m)\colon\ w_1 \leqslant w_0,\, w_2 \geqslant z_0,\, w_1 - w_2 \geqslant 0\big\}, \quad z_0 \leqslant 0 \leqslant \frac{w_0 + z_0}{2},$$

are invariant for the Riemann problems (7.14) and (7.15) for system (1.16), respectively. More precisely, if the Riemann data lies in Σ, the corresponding Riemann solution lies in Σ and its corresponding integral average in x over $[a, b]$ also lies in Σ. □

With these properties of the Riemann solutions to (1.16), we can now establish estimate (7.13).

First we have from assumption (7.5) that

$$0 \leqslant \int_0^{\rho_0^h(x)} \frac{\sqrt{p'(s)}}{s}\,\mathrm{d}s \leqslant \frac{m_0^h(x)}{\rho_0^h(x)} \leqslant C_0 < \infty.$$

This means that there exists $w_0 = w_0(C_0) > 0$ such that

$$\begin{cases} w_1\big(\rho_0^h(x), m_0^h(x)\big) \leqslant w_0, \qquad w_2\big(\rho_0^h(x), m_0^h(x)\big) \geqslant 0, \\ w_1\big(\rho_0^h(x), m_0^h(x)\big) - w_2\big(\rho_0^h(x), m_0^h(x)\big) \geqslant 0. \end{cases}$$

Fact (ii) indicates that, for $0 \leqslant t \leqslant h$, $(\rho_0^h, m_0^h)(t,x)$ satisfy

$$\begin{cases} w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) - w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0, \end{cases}$$

which means that there exists $\widehat{C} = \widehat{C}(C_0)$ such that

$$\rho_0^h(t,x) \geqslant 0, \qquad 0 \leqslant \frac{m_0^h(t,x)}{\rho_0^h(t,x)} \leqslant \widehat{C} < \infty.$$

Choose $N = N(C_0)$ such that

$$\frac{(d-1)\widehat{C}(C_0)}{4\Lambda(C_0)N} = 1, \quad \text{that is,} \quad N = N(C_0) \equiv \frac{(d-1)\widehat{C}(C_0)}{4\Lambda(C_0)}. \tag{7.18}$$

Then, for $t \in [0,h)$, we have

$$\begin{cases} \rho^h(t,x) = \rho_0^h(t,x)\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}t\big) \geqslant 0, \\ \frac{m^h(t,x)}{\rho^h(t,x)} = \frac{m_0^h(t,x)}{\rho_0^h(t,x)} \geqslant 0. \end{cases}$$

For $0 \leqslant t < h$, we have

$$\begin{cases} w_1\big(\rho^h(t,x), m^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho^h(t,x), m^h(t,x)\big) - w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0. \end{cases}$$

Suppose that the above inequality holds for $t < kh$. Then, at $t = kh$, we similarly have from Fact (ii) that

$$\begin{cases} w_1\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) \leqslant w_0, \\ w_2\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) \geqslant 0, \\ w_1\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) - w_2\big(\rho^h(kh+0,x), m^h(kh+0,x)\big) \geqslant 0. \end{cases}$$

It follows from Fact (ii) that, for $kh \leqslant t < (k+1)h$,

$$\begin{cases} w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho_0^h(t,x), m_0^h(t,x)\big) - w_2\big(\rho_0^h(t,x), m_0^h(t,x)\big) \geqslant 0. \end{cases}$$

Therefore, we have

$$\begin{cases} w_1\big(\rho^h(t,x), m^h(t,x)\big) \leqslant w_0, \qquad w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0, \\ w_1\big(\rho^h(t,x), m^h(t,x)\big) - w_2\big(\rho^h(t,x), m^h(t,x)\big) \geqslant 0 \end{cases}$$

from the fact

$$\begin{cases} \rho^h(t,x) = \rho_0^h(t,x)\big(1 - \frac{d-1}{x}\frac{m_0^h(t,x)}{\rho_0^h(t,x)}(t-kh)\big)_+, \\ \frac{m^h(t,x)}{\rho^h(t,x)} = \frac{m_0^h(t,x)}{\rho_0^h(t,x)}. \end{cases}$$

Then we have again

$$0 \leqslant \rho^h(t,x) \leqslant \widehat{C}, \qquad 0 \leqslant m^h(t,x) \leqslant \widehat{C}\rho^h(t,x),$$

where $\widehat{C} = \widehat{C}(C_0)$ is solely determined by the initial data.

Step 3. H^{-1}-compactness of entropy dissipation measures for the approximate solutions. The measure sequence

$$\eta(\mathbf{u}^h)_t + q(\mathbf{u}^h)_x \quad \text{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+) \tag{7.19}$$

for any weak entropy–entropy flux pair (η, q).

Without loss of generality, we assume that the initial data has compact support because of the finiteness of propagation speed of approximate solutions, hence one can assume

$$\int_0^\infty \rho_0(x)\,\mathrm{d}x + \int_0^\infty m_0(x)\,\mathrm{d}x + \int_0^\infty \eta_*\big(\rho_0(x), m_0(x)\big)\,\mathrm{d}x < \infty,$$

where

$$\eta_* = \frac{1}{2}\frac{m^2}{\rho} + \frac{\kappa}{\gamma-1}\rho^\gamma$$

with corresponding entropy flux

$$q_* = m\left(\frac{m^2}{2\rho^2} + \frac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1}\right).$$

For any function $\phi \in C^\infty(\Pi_T)$ with $\Pi_T = [0,T] \times \mathbb{R}_+$, the entropy dissipation measures can be calculated in the form

$$\iint_{\Pi_T} \big(\eta(\mathbf{u}^h)\phi_t + q(\mathbf{u}^h)\phi_x\big)\,\mathrm{d}x\,\mathrm{d}t \\ = M^h(\phi) + N^h(\phi) + L^h(\phi) + \Sigma^h(\phi), \tag{7.20}$$

where

$$M^h(\phi) = \int_0^\infty \phi(T,x)\eta\big(\mathbf{u}_0^h(T,x)\big)\,\mathrm{d}x - \int_0^\infty \phi(0,x)\eta\big(\mathbf{u}_0^h(0,x)\big)\,\mathrm{d}x,$$

$$N^h(\phi) = \iint_{\Pi_T} \big(\big(\eta(\mathbf{u}^h) - \eta(\mathbf{u}_0^h)\big)\phi_t + \big(q(\mathbf{u}^h) - q(\mathbf{u}_0^h)\big)\phi_x\big)\,\mathrm{d}x\,\mathrm{d}t,$$

$$L^h(\phi) = \sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \big(\eta(\mathbf{u}_{0-}^{hk}) - \eta(\mathbf{u}_j^k)\big)\phi(kh,x)\,\mathrm{d}x,$$

$$\Sigma^h(\phi) = \int_0^T \sum \big\{\sigma[\eta]_0 - [q]_0\big\}\phi\big(t,x(t)\big)\,\mathrm{d}t,$$

where $\mathbf{u}_-^{hk} = \mathbf{u}^h(kh-0,x)$, $\phi_j^k = \phi(kh,jl)$, the summation is taken over all the shocks in $\mathbf{u}_0^h$ at a fixed time t, σ is the propagating speed of the shock, and $[\eta]_0$ and $[q]_0$ denote the jumps of $\eta(\mathbf{u}_0^h(t,x))$ and $q(\mathbf{u}_0^h(t,x))$ across the shock in $\mathbf{u}_0^h(t,x)$ from the left to right, respectively.

Noting that (ρ^h, m^h) have compact support in Π_T, one can substitute

$$(\eta, q, \phi) = (\rho, m, 1) \quad \text{and} \quad \left(m, \frac{m^2}{\rho} + p(\rho), 1\right)$$

in (7.20). We conclude

$$\sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} m^h(kh-0,x)\,\mathrm{d}x\,h \leqslant C < \infty, \tag{7.21}$$

$$\sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} \frac{(m^h(kh-0,x))^2}{\rho^h(kh-0,x)}\,\mathrm{d}x\,h \leqslant C < \infty, \tag{7.22}$$

using the Rankine–Hugoniot condition (7.16) and noting that

$$\begin{cases} \sum_{k\geqslant 1} \int_0^\infty \big(\rho_{0-}^{hk} - \rho_j^k\big)\,\mathrm{d}x = \sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} m^h(kh-0,x)\,\mathrm{d}x\,h, \\ \sum_{k\geqslant 1} \int_0^\infty \big(m_{0-}^{hk} - m_j^k\big)\,\mathrm{d}x = \sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \frac{d-1}{x} \frac{(m^h(kh-0,x))^2}{\rho^h(kh-0,x)}\,\mathrm{d}x\,h. \end{cases}$$

Then we choose $(\eta, q) = (\eta_*, q_*)$ and $\phi = 1$ in (7.20) and use estimates (7.21) and (7.22) to obtain

$$\int_0^T \sum \{\sigma[\eta_*]_0 - [q_*]_0\} \, dt \leqslant C, \tag{7.23}$$

$$\sum_{j,k} \int_{(j-1/2)l}^{(j+1/2)l} \int_0^1 (1-\theta)\big(\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\big)^\top \times \nabla^2 \eta_*\big(\mathbf{u}_j^k + \theta\big(\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\big)\big)\big(\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\big) \, d\theta \, dx \leqslant C. \tag{7.24}$$

In particular, since $\nabla^2 \eta_* \geqslant c_0 > 0$, we obtain

$$\sum_{j,k,0 \leqslant jl \leqslant L} \int_{(j-1/2)l}^{(j+1/2)l} \big|\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k\big|^2 \, dx \leqslant C(L). \tag{7.25}$$

Noting that $\mathbf{u}_0^h(t,x)$ are of the form $V(\frac{x-jl}{t-kh})$, then there exists $C(L) < \infty$ such that

$$\sum_{j,k,0 \leqslant jl \leqslant L} \int_{(k-1)h}^{kh} \int_{(j-1/2)l}^{(j+1/2)l} \big|\mathbf{u}_0^h(t,x) - \mathbf{u}_0^h(kh-0,x)\big|^2 \, dx \, dt \leqslant C(L)h. \tag{7.26}$$

Then, similarly to the proof of Ding, Chen and Luo [116], we use (7.21)–(7.26) to conclude (7.19).

Step 4. *Convergence and consistency.* Applying Theorem 3.1 with (7.13) and (7.19), we see that there exist a subsequence (still denoted by) $\mathbf{u}^h(t,x)$ and an L^∞ function $\mathbf{u}(t,x) \equiv (\rho, m)(t,x)$ such that

$$\mathbf{u}^h(t,x) \to \mathbf{u}(t,x) \quad \text{a.e. when } h \to 0,$$

and

$$0 \leqslant \rho(t,x) \leqslant C, \qquad \big|m(t,x)\big| \leqslant C\rho(t,x) \quad \text{a.e.} \tag{7.27}$$

It now suffices to check the consistency of the limit function $(\rho, m)(t,x)$ with (7.4). For any nonnegative function $\psi(t,x) \in C_0^\infty(\mathbb{R}_+^2)$, set $\phi(t,x) = x^{d-1}\psi(t,x)$ and $a(x) = (d-1)/x$. Then we have

$$\begin{aligned} &\iint_{\mathbb{R}_+^2} \big(\eta\big(\mathbf{u}^h\big)\phi_t + q\big(\mathbf{u}^h\big)\phi_x - a(x)\nabla\eta\big(\mathbf{u}^h\big)g\big(\mathbf{u}^h\big)\phi\big) \, dx \, dt \\ &\quad + \int_0^\infty \eta\big(\mathbf{u}^h(0,x)\big)\phi(0,x) \, dx \\ &\equiv I_1^h + I_2^h, \end{aligned} \tag{7.28}$$

where

$$I_1^h = \iint_{\mathbb{R}^2_+} \big(\eta(\mathbf{u}_0^h)\phi_t + q(\mathbf{u}_0^h)\phi_x + a(x)\nabla\eta(\mathbf{u}_0^h)g(\mathbf{u}_0^h)\phi\big)\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty \eta\big(\mathbf{u}_0^h(0,x)\big)\phi(0,x)\,\mathrm{d}x.$$

Notice that $|\mathbf{u}^h - \mathbf{u}_0^h| \leqslant a(x)|g(\mathbf{u}_0^h)|h$ and $\mathbf{u}^h$ are uniformly bounded. We have

$$|I_2^h| \leqslant C\Big(h + \iint_{\operatorname{supp}\phi} \big|g(\mathbf{u}^h) - g(\mathbf{u}_0^h)\big|\,\mathrm{d}x\,\mathrm{d}t + \iint_{\operatorname{supp}\phi} \big|\nabla\eta(\mathbf{u}^h) - \nabla\eta(\mathbf{u}_0^h)\big|\,\mathrm{d}x\,\mathrm{d}t\Big) \to 0 \tag{7.29}$$

when $h \to 0$. Furthermore,

$$I_1^h = \sum_{k\geqslant 1}\int_0^\infty \big(\eta(\mathbf{u}_{0-}^{hk}) - \eta(\mathbf{u}_j^k)\big)\phi(kh,x)\,\mathrm{d}x - \iint_{\mathbb{R}^2_+} a(x)\nabla\eta(\mathbf{u}_0^h)g(\mathbf{u}_0^h)\phi\,\mathrm{d}x\,\mathrm{d}t \equiv I_{11}^h + I_{12}^h. \tag{7.30}$$

Notice that

$$|I_{11}^h| = \Big|\sum_{j,k}\int_{(j-1/2)l}^{(j+1/2)l} (\phi - \phi_j^k)\big(\eta(\mathbf{u}_{0-}^{hk}) - \eta(\mathbf{u}_j^k)\big)\,\mathrm{d}x\Big| \leqslant C\sqrt{h}\|\phi\|_{C_0^1}\Big(\sum_{j,k,0\leqslant jl\leqslant L}\int_{(j-1/2)l}^{(j+1/2)l} |\mathbf{u}_{0-}^{hk} - \mathbf{u}_j^k|^2\,\mathrm{d}x\Big)^{1/2} \leqslant C\sqrt{h} \to 0 \tag{7.31}$$

when $h \to 0$ and

$$I_{12}^h \geqslant -\Big|\sum_{j,k}\int_{(k-1)h}^{kh}\mathrm{d}t\int_{(j-1/2)l}^{(j+1/2)l} a(x)\big(\phi\nabla\eta(\mathbf{u}_0^h)g(\mathbf{u}_0^h) - \phi_j^k\nabla\eta(\mathbf{u}_{0-}^{hk})g(\mathbf{u}_{0-}^{hk})\big)\,\mathrm{d}x\Big| \geqslant -\big(J_1^h + J_2^h\big), \tag{7.32}$$

where

$$J_1^h = \left| \sum_{j,k} \int_{(k-1)h}^{kh} \mathrm{d}t \int_{(j-1/2)l}^{(j+1/2)l} a(x)(\phi - \phi_j^k) \nabla\eta(\mathbf{u}_{0-}^{hk}) g(\mathbf{u}_{0-}^{hk}) \,\mathrm{d}x \right|$$
$$\leqslant Ch\|\phi\|_{C^1} \to 0 \tag{7.33}$$

when $h \to 0$ and

$$J_2^h \leqslant \iint_{\{\mathrm{supp}\,\phi\}\cap\Omega_h} a(x)\left|\nabla\eta(\mathbf{u}_{0-}^{hk}) g(\mathbf{u}_{0-}^{hk}) - \nabla\eta(\mathbf{u}_0^h) g(\mathbf{u}_0^h)\right| \mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{\{\mathrm{supp}\,\phi\}\cap\Omega_h^c} a(x)\left|\nabla\eta(\mathbf{u}_{0-}^{hk}) g(\mathbf{u}_{0-}^{hk}) - \nabla\eta(\mathbf{u}_0^h) g(\mathbf{u}_0^h)\right| \mathrm{d}x\,\mathrm{d}t$$

with $\Omega_h = \{(t,x)\colon\ \rho_0(t,x) \geqslant h^{1/4}\}$, and thus

$$J_2^h \leqslant C\left(h^{-1/4}\left(\iint_{\mathrm{supp}\,\phi} |\mathbf{u}_0^h - \mathbf{u}_{0-}^{hk}|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} + h^{1/4}\right) \leqslant Ch^{1/4} \to 0 \tag{7.34}$$

when $h \to 0$, by using the Hölder inequality.

Since $(\rho^h, m^h) \to (\rho, m)$ a.e., then it is routine to use the dominated convergence theorem to show that the function $(\rho, \mathbf{m})(t, \mathbf{x})$ determined by the function $(\rho, m)(t, x)$,

$$(\rho, \mathbf{m})(t, \mathbf{x}) = \left(\rho(t, |\mathbf{x}|), m(t, |\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}\right), \tag{7.35}$$

satisfies the standard notion of entropy solutions.

Finally, notice that the pairs $\pm(\rho, m)$ and $\pm(m, m^2/\rho + p(\rho))$ are all convex entropy–entropy flux pairs. It follows that (ρ, m) satisfies (7.4) in the sense of distributions. For (7.4), we take the test function $\phi(t,x) = \alpha(t)X^k(x)$ with

$$\alpha(t) \in C_0^\infty(0,\infty), \quad \alpha(t) \geqslant 0,$$

and

$$\begin{cases} X^k(x) \in C_0^\infty(0,\infty), \quad X^k\big|_{[0,x_0/2]} \equiv 1,\ 0 \leqslant X^k(x) \leqslant 1, \\ X^k(x) \to \chi_{[0,x_0]}(x) \quad \text{as } k \to \infty \end{cases}$$

for Lebesgue points $x_0 \in (0,\infty)$, $x_0 \to 0$, of the function

$$\int_0^\infty \left(m(t,x) + \frac{m(t,x)^2}{\rho(t,x)} + p(\rho(t,x))\right)\alpha(t)\,\mathrm{d}t.$$

Then, adding the two identities and setting $k \to \infty$, we have

$$\int_0^\infty \left(m(t, x_0) + \frac{m(t, x_0)^2}{\rho(t, x_0)} + p\big(\rho(t, x_0)\big) \right) \alpha(t)\, \mathrm{d}t \\ \leqslant C\left(TV\big(\alpha(\cdot)\big) x_0 + \int_0^\infty \int_0^{x_0} \frac{d-1}{x} \left(m(t, x) + \frac{m(t, x)^2}{\rho(t, x)} \right) \alpha(t)\, \mathrm{d}x\, \mathrm{d}t \right)$$

since $\int_0^\infty (\rho(t, x) + m(t, x))\, \mathrm{d}x \leqslant C < \infty$.

Therefore, we have

$$\lim_{x_0 \to 0} \int_0^\infty \left(m(t, x_0) + \frac{m(t, x_0)^2}{\rho(t, x_0)} + p\big(\rho(t, x_0)\big) \right) \alpha(t)\, \mathrm{d}t = 0$$

for any $\alpha(t) \in C_0^\infty(0, \infty)$ by using

$$\int_0^\infty \int_0^\infty \frac{d-1}{x} m(t, x)\, \mathrm{d}x\, \mathrm{d}t + \int_0^\infty \int_0^\infty \frac{d-1}{x} \frac{m(t, x)^2}{\rho(t, x)}\, \mathrm{d}x\, \mathrm{d}t \leqslant C < \infty.$$

Similarly, we take $(\eta, q) = (\rho, m)$ and $(m, \frac{m^2}{\rho} + p(\rho))$ in (5.3), respectively, and take $\phi(t, x) = \alpha^l(t) X^k(x)$ with

$$\begin{cases} \alpha^l(t) \in C_0^\infty(0, \infty), & \alpha^l\big|_{[0, T-\varepsilon]} \equiv 1,\ 0 \leqslant \alpha^l(t) \leqslant 1, \\ \alpha^l(t) \to \chi_{[0,T]}(t) & \text{as } l \to \infty, \\ TV\big(\alpha^l(\cdot)\big) \leqslant C, & C \text{ independent of } l, \end{cases}$$

and

$$\begin{cases} X^k(y) \in C_0^\infty(0, \infty), & X^k\big|_{[x_0/2, x/2]} \equiv 1,\ 0 \leqslant X^k(y) \leqslant 1, \\ X^k(y) \to \chi_{[x_0, x]}(y) & \text{as } k \to \infty \ \text{ for } x \in [x_0, \infty). \end{cases}$$

Adding the two identities and letting $k \to \infty$, we have

$$\frac{1}{T} \int_0^T \left(m(t, x) + \frac{m(t, x)^2}{\rho(t, x)} + p\big(\rho(t, x)\big) \right) \alpha^l(t)\, \mathrm{d}t \\ \leqslant \frac{1}{T} TV\big(\alpha^l(\cdot)\big) \sup_{0 \leqslant t \leqslant T} \int_0^\infty \big(\rho(t, y) + m(t, y)\big)\, \mathrm{d}y \\ + \frac{1}{T} \int_0^\infty \int_{x_0}^x \frac{d-1}{x} \left(m(t, y) + \frac{m(t, y)^2}{\rho(t, y)} \right) \mathrm{d}y\, \mathrm{d}t$$

$$+\frac{1}{T}\int_0^T\left(m(t,x_0)+\frac{m(t,x_0)^2}{\rho(t,x_0)}+p\big(\rho(t,x_0)\big)\right)\mathrm{d}t$$
$$\leqslant C\left(\frac{1}{T}+\frac{1}{T}\int_0^T\left(m(t,x_0)+\frac{m(t,x_0)^2}{\rho(t,x_0)}+p\big(\rho(t,x_0)\big)\right)\mathrm{d}t\right).$$

Set $x_0\to 0$ and then $l\to\infty$. We obtain

$$\frac{1}{T}\int_0^T\left(m(t,x)+\frac{m(t,x)^2}{\rho(t,x)}+p\big(\rho(t,x)\big)\right)\mathrm{d}t\leqslant\frac{C}{T}\quad\text{for a.e. } x\in\mathbb{R}_+.$$

This implies that

$$\frac{1}{T}\int_0^T(\rho,m)(t,x)\,\mathrm{d}t\to(0,0)\quad\text{for a.e. } x\in\mathbb{R}_+ \text{ as } T\to\infty,$$

which arrives at (7.7). This completes the proof. □

7.2. *Self-similar solutions*

The second type of geometric structure is self-similarity. One of the most challenging problems is to study solutions with data that give rise to self-similar solutions (such solutions especially include Riemann solutions) and to develop a unifying framework to treat hyperbolic–elliptic mixed problems with mixed boundary conditions that are derived from compressible flows.

Compressible flow equations in two space dimensions with one or more linearly degenerate modes of wave propagation have additional difficulties. In that case, the global flow is governed by a reduced (self-similar) system which is of both (hyperbolic–elliptic) mixed and composite type in the subsonic region. The linearly degenerate waves give rise to one or more families of degenerate characteristics which remain real in the subsonic region. The reduced equations typically couple a hyperbolic–elliptic mixed problem for the density and/or the pressure with a hyperbolic (transport) equation for the vorticity.

For the Euler equations (1.4) for $\mathbf{x}\in\mathbb{R}^2$, self-similar solutions

$$(\rho,u_1,u_2,p)=(\rho,u_1,u_2,p)(\xi,\eta),\qquad(\xi,\eta)=\frac{\mathbf{x}}{t},$$

are determined by

$$\begin{cases}\partial_\xi(\rho U)+\partial_\eta(\rho V)=-2\rho,\\ \partial_\xi\big(\rho U^2+p\big)+\partial_\eta(\rho UV)=-3\rho U,\\ \partial_\xi(\rho UV)+\partial_\eta\big(\rho V^2+p\big)=-3\rho V,\\ \partial_\xi\big(U(E+p)\big)+\partial_\eta\big(V(E+p)\big)=-2(E+p),\end{cases}\tag{7.36}$$

where $(U,V)=(u_1-\xi,u_2-\eta)$ is the pseudovelocity and $E=\rho(e+(U^2+V^2)/2)$.

It is straightforward to calculate and obtain four eigenvalues

$$\lambda_0 = \frac{V}{U} \quad \text{(two multiplicity)}$$

and

$$\lambda_\pm = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2},$$

where c is the sonic speed.

When $U^2 + V^2 > c^2$, system (7.36) is hyperbolic with four real eigenvalues and the flow is called pseudosupersonic.

When $U^2 + V^2 < c^2$, system (7.36) is hyperbolic–elliptic composite type (two repeated eigenvalues are real and the other two are complex): two equations are hyperbolic and the other two are elliptic.

The region $U^2 + V^2 = c^2$ in the (ξ, η) plane is called the pseudosonic region in the flow.

In general, system (7.36) is both hyperbolic–elliptic mixed and composite type, and the flow is pseudotransonic.

For a bounded solution (ρ, u_1, u_2, p), the flow must be pseudosupersonic when $\xi^2 + \eta^2 \to \infty$.

An important prototype problem for both practical applications and the theory of multidimensional complex wave patterns is the problem of diffraction of a shock wave which is incident along an inclined ramp. When a plane shock hits a wedge head on, a self-similar reflected shock moves outward as the original shock moves forward (e.g., [15,43,100,150, 252,291,328]). The computational and asymptotic analysis shows that various patterns of reflected shocks may occur, including regular and Mach reflections. The reflected shock is a transonic shock in the self-similar coordinates, for which the corresponding equation changes its type from hyperbolic to elliptic across the shock. There has been no rigorous mathematical result on the *global* existence and structure of shock reflections for the potential flow equation and the full Euler equations. Some results were recently obtained for simplified models. The transonic small-disturbance (TSD) equation in Section 4.3 was derived and used in [173,177,187,252] and the references cited therein for asymptotic analysis of shock reflections; and some steps of this analysis have been justified in [40]. Zheng [357] made an effort on the existence of a regular reflection solution for the pressure gradient equation when the wedge is close to a flat wall.

It is important to establish the existence and stability of shock reflection solutions and clarify the transition among regular reflection, simple Mach reflection, double Mach reflection, and complex Mach reflection.

A good starting point is the potential flow equation (4.4) for this problem. A self-similar solution is a solution of the form

$$\Psi = t\phi(\xi, \eta), \qquad (\xi, \eta) = \frac{\mathbf{x}}{t}.$$

By introducing the function

$$\varphi(\xi,\eta) = -\frac{\xi^2+\eta^2}{2} + \phi(\xi,\eta),$$

the system can be rewritten in the form of a second-order equation of mixed hyperbolic–elliptic type

$$\operatorname{div}_{(\xi,\eta)}\big(\rho\big(|\nabla\varphi|^2,\varphi\big)\nabla\varphi\big) + 2\rho\big(|\nabla\varphi|^2,\varphi\big) = 0 \tag{7.37}$$

with

$$\rho(q^2,z) = \left(1 - \frac{q^2+2z}{2}\right)^{1/(\gamma-1)}.$$

Similar to (6.1), equation (7.37) at $|\nabla\varphi| = q$ is hyperbolic (pseudosupersonic) if

$$\rho\big(q^2,z\big) + q\rho_q\big(q^2,z\big) < 0$$

and elliptic (pseudosubsonic) if

$$\rho\big(q^2,z\big) + q\rho_q\big(q^2,z\big) > 0.$$

The nature of the shock reflection pattern has been explored in [252] for weak incident shocks (strength b) and small wedge angles $2\theta_w$ by a number of different scalings, a study of mixed equations, and matching asymptotics for the different scalings, where the parameter $\beta = c_1\theta_w^2/b(\gamma+1)$ ranges from 0 to ∞ and c_1 is the sound speed behind the incident shock. It was shown that, for $\beta > 2$, regular reflection of both strong and weak kinds is possible as well as Mach reflection; for $\beta < 1/2$, Mach reflection occurs and the flow behind the reflection is subsonic and can be constructed in principle (with an open elliptic problem) and matched; and for $1/2 < \beta < 2$, the flow behind a Mach reflection may be transonic and the corresponding nonlinear boundary value problem of mixed type has been discussed. The basic pattern of reflection was shown to be an almost semicircular shock, for regular reflection, emanating from the reflection point on the wedge and, for Mach reflection, matched with a local interaction flow. It is important to establish some rigorous proofs for this problem with the aid of free boundary approaches as discussed in Section 6.3. Such a rigorous proof for the existence of shock reflection solutions has successfully been established in [63] when the wedge angle is large.

7.3. *Global solutions with special Cauchy data*

Several cases of initial data for the Cauchy problem may be solved for constructing global solutions for the compressible Euler equations (1.11) or (1.4).

CASE 1. Initial data of the form

$$(\rho, u_1, u_2)|_{t=0} = \begin{cases} (\rho_-, u_{1-}, u_{2-}) & \text{if } L(\mathbf{x}) < 0, \mathbf{x} \in \mathbb{R}^2, \\ (\rho_+, u_{1+}, u_{2+}) & \text{if } L(\mathbf{x}) > 0, \end{cases} \tag{7.38}$$

for (1.11). The initial discontinuity $L(\mathbf{x}) = 0$ is a smooth curve which separates the $\mathbf{x}$-plane into two unbounded parts, and $\nabla_{\mathbf{x}} L$ is continuous. This Cauchy problem (1.11) and (7.38) can be considered as a multidimensional generalization of the one-dimensional Riemann problem. It is also a natural problem from the viewpoint of physics. Conventional self-similarity transformations or symmetric transformations are not available to such a problem.

Certain preliminary observations have shown in the case where the global solutions are connected by two-dimensional rarefaction waves, with the discontinuity $L(\mathbf{x}) = 0$ being convex or concave, and two initial constants (ρ_-, u_{1-}, u_{2-}) and (ρ_+, u_{1+}, u_{2+}) satisfying a natural relation. A natural strategy is to develop the so-called envelope method and some particular implicit functions which may enable the construction of the two-dimensional rarefaction waves to be possible. It has also been observed that the state functions inside the rarefaction waves and the intermediate state functions between the two rarefaction waves must be smooth. It is interesting to obtain a complete global solution. For the pressureless Euler equations, some results have been obtained by Yang and Huang [343].

CASE 2. Initial data of the form

$$(\rho, u_1, u_2)|_{t=0} = \begin{cases} (\rho_-, u_{1-}, u_{2-}) & \text{if } L(\mathbf{x}) < 0, \\ (\rho_-, u_{1+}, u_{2+})(\mathbf{x}) & \text{if } L(\mathbf{x}) > 0, \end{cases}$$

where (ρ_-, u_{1-}, u_{2-}) is a constant state and $(\rho_+, u_{1+}, u_{2+})(\mathbf{x})$ is a smooth initial function. It is important to determine the class of initial functions $(\rho_+, u_{1+}, u_{2+})(\mathbf{x})$ which leads to the existence of two-dimensional global solutions that have only a single shock for such special initial data. In this regard, see [159] and [160].

CASE 3. Initial data consists of four different constant states $\mathbf{u}_i = (\rho_i, u_1^i, u_2^i)$, $i = 1, 2, 3, 4$, corresponding to four quadrants with a special relationship among the states, so that the unfolding solution at infinity consists of only one rarefaction wave along the direction of each semiaxis. Chang, Chen and Yang [44,45] and Lax and Liu [203] have similar numerical results for this case. The contour curve of the density ρ is simple: the two groups of planar rarefaction waves, R_{12} (along the η^+ axis) and R_{41} (along the ξ^+ axis), R_{34} (along the η^- axis) and R_{23} (along the ξ^- axis), are connected by a family of straight lines $\xi + \eta = \alpha$, where α is a constant parameter. Hence, ρ is symmetric about $\xi - \eta = \tilde{\alpha}$ for a particular $\tilde{\alpha}$, while the contour curve of the self-Mach number is relatively complex but follows some rules.

It is interesting to construct two-dimensional global solutions to the Euler equations (1.11) with this type of initial data. The idea is first to estimate the solution of ρ from its contour curve, then to plug ρ into equations (1.11), according to the symmetry of ρ, so as to construct u_1 and u_2.

8. Divergence-measure fields and hyperbolic conservation laws

Naturally, we want to approach the questions of existence, stability, uniqueness, and long-time behavior of entropy solutions for multidimensional compressible flows for fluids (as represented by the Euler equations of inviscid flows such as system (1.4) and system (1.11)) and solids (as represented by the equations of nonlinear elastodynamics such as system (4.15), (4.17) and (4.18)) with as much generality as possible. In this section, we discuss some recent efforts in developing a theory of divergence-measure fields to construct a global framework for studying solutions of multidimensional compressible flows and, more generally, hyperbolic systems of conservation laws.

8.1. *Connections*

Consider a system of hyperbolic conservation laws in d space dimensions in (1.1). As mentioned earlier, the main feature of nonlinear hyperbolic conservation laws (1.1), especially (1.4) or (1.11), is that, no matter how smooth the initial data is, solutions may develop singularities and form shock waves and vorticity waves, among others, in finite time. For the one-dimensional problem of (1.1), in particular, for the one-dimensional version of the Euler equations (1.15) or (1.17) in Lagrangian coordinates, one may expect solutions in BV; this is indeed the case by Glimm's theorem [145] which indicates that there exists a global entropy solution in BV when the initial data has sufficiently small total variation and stays away from vacuum. On the other hand, when the initial data is large, even away from vacuum, solutions may develop vacuum instantaneously as $t > 0$ or approach the vacuum states indefinitely. In this case, the specific volume $\tau = 1/\rho$ may become a Radon measure or an L^1 function, rather than a BV function (cf. [332]).

In particular, we emphasize again that, as discussed in Section 2.6, the BV bound generically fails for multidimensional hyperbolic conservation laws. In general, for multidimensional conservation laws, especially the Euler equations, because of complex interactions among shocks, rarefaction waves, vortex sheets, and vorticity waves, solutions of (1.1) are expected to be in the following class of entropy solutions:

(i) $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$ or $L^p(\mathbb{R}^{d+1}_+)$, $1 \leqslant p \leqslant \infty$,

(ii) $\mathbf{u}(t,\mathbf{x})$ satisfies the Lax entropy inequality

$$\mu_\eta := \partial_t \eta\big(\mathbf{u}(t,\mathbf{x})\big) + \nabla_{\mathbf{x}} \cdot \mathbf{q}\big(\mathbf{u}(t,\mathbf{x})\big) \leqslant 0 \quad \text{in the sense of distributions,} \tag{8.1}$$

for any convex entropy–entropy flux pair $(\eta, \mathbf{q}) : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^d$ so that $\eta(\mathbf{u}(t,\mathbf{x}))$ and $\mathbf{q}(\mathbf{u}(t,\mathbf{x}))$ are distributions.

One of the main issues in conservation laws is to study the behavior of solutions in this class to explore all possible information on solutions, including large-time behavior, uniqueness, stability, and existence of traces, *with neither specific reference to any particular method for constructing the solutions nor additional regularity assumptions.*

The Schwartz lemma infers from (8.1) that the distribution μ_η is in fact a Radon measure,

$$\operatorname{div}_{(t,\mathbf{x})}\big(\eta\big(\mathbf{u}(t,\mathbf{x})\big), \mathbf{q}\big(\mathbf{u}(t,\mathbf{x})\big)\big) \in \mathcal{M}\big(\mathbb{R}^{d+1}_+\big).$$

Furthermore, when $\mathbf{u} \in L^\infty$, this is also true for any C^2 entropy–entropy flux pair $(\eta, \mathbf{q})$ (η not necessarily convex) if (1.1) has a strictly convex entropy, which was first observed in [51].

More generally, we have the following definition.

DEFINITION. Let $\mathcal{D} \subset \mathbb{R}^N$ be open. For $1 \leqslant p \leqslant \infty$, $\mathbf{F}$ is called a $\mathcal{D}M^p(\mathcal{D})$ field if $\mathbf{F} \in L^p(\mathcal{D}; \mathbb{R}^N)$ and

$$\|\mathbf{F}\|_{\mathcal{D}M^p(\mathcal{D})} := \|\mathbf{F}\|_{L^p(\mathcal{D};\mathbb{R}^N)} + \|\operatorname{div}\mathbf{F}\|_{\mathcal{M}(\mathcal{D})} < \infty, \tag{8.2}$$

and the field $\mathbf{F}$ is called a $\mathcal{D}M^{\mathrm{ext}}(\mathcal{D})$-field if $\mathbf{F} \in \mathcal{M}(\mathcal{D}; \mathbb{R}^N)$ and

$$\|\mathbf{F}\|_{\mathcal{D}M^{\mathrm{ext}}(\mathcal{D})} := \big\|(\mathbf{F}, \operatorname{div}\mathbf{F})\big\|_{\mathcal{M}(\mathcal{D})} < \infty. \tag{8.3}$$

Furthermore, for any bounded open set $\mathcal{D} \subset \mathbb{R}^N$, $\mathbf{F}$ is called a $\mathcal{D}M^p_{\mathrm{loc}}(\mathbb{R}^N)$ field if $\mathbf{F} \in \mathcal{D}M^p(\mathcal{D})$, and $\mathbf{F}$ is called a $\mathcal{D}M^{\mathrm{ext}}_{\mathrm{loc}}(\mathbb{R}^N)$ if $\mathbf{F} \in \mathcal{D}M^{\mathrm{ext}}(\mathcal{D})$. A field $\mathbf{F}$ is simply called a $\mathcal{D}M$ field in $\mathcal{D}$ if $\mathbf{F} \in \mathcal{D}M^p(\mathcal{D})$, $1 \leqslant p \leqslant \infty$, or $\mathbf{F} \in \mathcal{D}M^{\mathrm{ext}}(\mathcal{D})$.

It is easy to check that these spaces, under the respective norms $\|\mathbf{F}\|_{\mathcal{D}M^p(\mathcal{D})}$ and $\|\mathbf{F}\|_{\mathcal{D}M^{\mathrm{ext}}(\mathcal{D})}$ are Banach spaces. These spaces are larger than the space of BV fields. The establishment of the Gauss–Green theorem, traces, and other properties of BV functions in the 1950s (cf. [133]; also [8,144,330]) has significantly advanced our understanding of solutions of nonlinear partial differential equations and related problems in the calculus of variations, differential geometry and other areas, especially for the one-dimensional theory of hyperbolic conservation laws. A natural question is whether the $\mathcal{D}M$ fields have similar properties, especially the normal traces and the Gauss–Green formula to deal with entropy solutions for multidimensional conservation laws. At a first glance, it seems impossible due to the Whitney paradox [338].

EXAMPLE 8.1 (Whitney paradox [338]). The field

$$\mathbf{F}(y_1, y_2) = \left(\frac{-y_2}{y_1^2 + y_2^2}, \frac{y_1}{y_1^2 + y_2^2}\right)$$

belongs to $\mathcal{D}M^1_{\mathrm{loc}}(\mathbb{R}^2)$; however, for $\Omega = (0,1) \times (0,1)$,

$$\int_\Omega \operatorname{div}\mathbf{F} = 0 \neq \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}\mathcal{H}^1 = \frac{\pi}{2},$$

if one understands $\mathbf{F} \cdot \mathbf{n}$ in the classical sense. This implies that the classical Gauss–Green theorem fails.

EXAMPLE 8.2. For any $\mu_i \in \mathcal{M}(\mathbb{R})$, $i = 1, 2$, with finite total variation,

$$\mathbf{F}(y_1, y_2) = \big(\mu_1(y_2), \mu_2(y_1)\big) \in \mathcal{D}M^{\mathrm{ext}}\big(\mathbb{R}^2\big).$$

A nontrivial example of such fields is provided by the Riemann solutions of the one-dimensional Euler equations in Lagrangian coordinates for which vacuum generally develops (see [69]).

On the other hand, motivated by various nonlinear problems from conservation laws, as well as for rigorous derivation of systems of balance laws with measure source terms from the physical principle of balance law and the recovery of Cauchy entropy flux through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation, a suitable notion of normal traces and corresponding Gauss–Green formula for divergence-measure fields are required.

Some earlier efforts were made on generalizing the Gauss–Green theorem for some special situations, and relevant results can be found in [9] for an abstract formulation for $\mathbf{F} \in L^\infty$, Rodrigues [283] for $\mathbf{F} \in L^2$, and Ziemer [358] for a related problem for $\operatorname{div}\mathbf{F} \in L^1$; also see [12,36] and [359]. In [67], an explicit way to calculate the suitable normal traces was first observed for $\mathbf{F} \in \mathcal{D}M^\infty$, under which a generalized Gauss–Green theorem was shown to hold, which has motivated the establishment of a theory of divergence-measure fields in [67,69,83,84].

8.2. *Basic properties of divergence-measure fields*

Now we list some basic properties of divergence-measure fields.

PROPOSITION 8.1. (i) *Let* $\{\mathbf{F}_j\}$ *be a sequence in* $\mathcal{D}M^p(\mathcal{D})$ *such that*

$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \text{in } L^p_{\mathrm{loc}}\big(\mathcal{D};\mathbb{R}^N\big) \text{ for } 1 \leqslant p < \infty, \tag{8.4}$$

$$\mathbf{F}_j \overset{*}{\rightharpoonup} \mathbf{F} \quad \text{in } L^\infty_{\mathrm{loc}}\big(\mathcal{D};\mathbb{R}^N\big) \text{ for } p = \infty. \tag{8.5}$$

Then

$$\|\mathbf{F}\|_{L^p(\mathcal{D})} \leqslant \liminf_{j\to\infty} \|\mathbf{F}_j\|_{L^p(\mathcal{D})}, \qquad |\operatorname{div}\mathbf{F}|(\mathcal{D}) \leqslant \liminf_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\mathcal{D}).$$

(ii) *Let* $\{\mathbf{F}_j\}$ *be a sequence in* $\mathcal{D}M^{\mathrm{ext}}(\mathcal{D})$ *such that*

$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \text{in } \mathcal{M}_{\mathrm{loc}}\big(\mathcal{D};\mathbb{R}^N\big).$$

Then

$$|\mathbf{F}|(\mathcal{D}) \leqslant \liminf_{j\to\infty} |\mathbf{F}_j|(\mathcal{D}), \qquad |\operatorname{div}\mathbf{F}|(\mathcal{D}) \leqslant \liminf_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\mathcal{D}).$$

In particular, if $\mathbf{F}$ *has compact support in* $\mathcal{D}$, *then*

$$|\operatorname{div}\mathbf{F}_j|(\mathcal{D}) \to |\operatorname{div}\mathbf{F}|(\mathcal{D}) \quad \text{as } j \to \infty.$$

This proposition immediately implies that spaces $\mathcal{DM}^p(\mathcal{D})$, $1 \leqslant p \leqslant \infty$, and $\mathcal{DM}^{\text{ext}}(\mathcal{D})$ are Banach spaces under norms (8.2) and (8.3), respectively.

PROPOSITION 8.2. *Let* $\{\mathbf{F}_j\}$ *be a sequence in* $\mathcal{DM}(\mathcal{D})$ *satisfying*

$$\lim_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\mathcal{D}) = |\operatorname{div}\mathbf{F}|(\mathcal{D})$$

and one of the following three conditions

$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \text{in } L^p_{\text{loc}}(\mathcal{D};\mathbb{R}^N) \text{ for } 1 \leqslant p < \infty,$$

$$\mathbf{F}_j \overset{*}{\rightharpoonup} \mathbf{F} \quad \text{in } L^\infty_{\text{loc}}(\mathcal{D};\mathbb{R}^N) \text{ for } p = \infty,$$

$$\mathbf{F}_j \rightharpoonup \mathbf{F} \quad \text{in } \mathcal{M}_{\text{loc}}(\mathcal{D};\mathbb{R}^N).$$

Then, for every open set $\Omega \subset \mathcal{D}$,

$$|\operatorname{div}\mathbf{F}|\big(\bar{\Omega}\cap\mathcal{D}\big) \geqslant \lim\sup_{j\to\infty} |\operatorname{div}\mathbf{F}_j|\big(\bar{\Omega}\cap\mathcal{D}\big). \tag{8.6}$$

In particular, if $|\operatorname{div}\mathbf{F}|(\partial\Omega\cap\mathcal{D}) = 0$, *then*

$$|\operatorname{div}\mathbf{F}|(\Omega) = \lim_{j\to\infty} |\operatorname{div}\mathbf{F}_j|(\Omega). \tag{8.7}$$

We now use the standard positive symmetric mollifiers $\omega:\mathbb{R}^N \to \mathbb{R}$ satisfying

$$\omega(\mathbf{y}) \in C_0^\infty(\mathbb{R}^N), \quad \omega(\mathbf{y}) \geqslant 0,\ \omega(\mathbf{y}) = \omega(|\mathbf{y}|),\ \int_{\mathbb{R}^N} \omega(\mathbf{y})\,d\mathbf{y} = 1,$$

$$\operatorname{supp}\omega(\mathbf{y}) \subset B_1 \equiv \{\mathbf{y}\in\mathbb{R}^N:\ |\mathbf{y}| < 1\}.$$

We denote

$$\omega^\varepsilon(\mathbf{y}) = \varepsilon^{-N}\omega\Big(\frac{\mathbf{y}}{\varepsilon}\Big), \qquad \mathbf{F}^\varepsilon = \mathbf{F} * \omega^\varepsilon, \tag{8.8}$$

that is,

$$\mathbf{F}^\varepsilon(\mathbf{y}) = \varepsilon^{-N}\int_{\mathbb{R}^N} \mathbf{F}(\mathbf{x})\omega\Big(\frac{\mathbf{y}-\mathbf{x}}{\varepsilon}\Big)\,d\mathbf{x} = \int_{\mathbb{R}^N} \mathbf{F}(\mathbf{y}+\varepsilon\mathbf{x})\omega(\mathbf{x})\,d\mathbf{x}. \tag{8.9}$$

Then $\mathbf{F}^\varepsilon \in C^\infty(\Omega;\mathbb{R}^N)$ for any $\Omega \Subset \mathcal{D}$ when ε is sufficiently small. We recall that, for any $f, g \in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f^\varepsilon g\,d\mathbf{x} = \int_{\mathbb{R}^N} f g^\varepsilon\,d\mathbf{x}. \tag{8.10}$$

The following fact for $\mathcal{DM}$ fields is analogous to a well-known property of BV functions.

PROPOSITION 8.3. *Let* $\mathbf{F} \in \mathcal{DM}(\mathcal{D})$. *Let* $\Omega \Subset \mathcal{D}$ *be open and* $|\operatorname{div}\mathbf{F}|(\partial\Omega) = 0$. *Then, for any* $\varphi \in C(\mathcal{D}; \mathbb{R})$,

$$\lim_{\varepsilon\to 0}\langle \operatorname{div}\mathbf{F}^\varepsilon, \varphi\chi_\Omega\rangle = \langle \operatorname{div}\mathbf{F}, \varphi\chi_\Omega\rangle.$$

Furthermore, if $\mathbf{F} \in \mathcal{DM}^{\text{ext}}(\mathcal{D})$ *and* $|\mathbf{F}|(\partial\Omega) = 0$, *then, for any* $\varphi \in C(\mathcal{D}; \mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0}\langle \mathbf{F}^\varepsilon, \varphi\chi_\Omega\rangle = \langle \mathbf{F}, \varphi\chi_\Omega\rangle.$$

Now we discuss some product rules for divergence-measure fields.

PROPOSITION 8.4. *Let* $\mathbf{F} = (F_1, \dots, F_N) \in \mathcal{DM}(\mathcal{D})$. *Let* $g \in BV \cap L^\infty(\mathcal{D})$ *be such that*
(i) $\partial_{y_j} g(\mathbf{y})$ *is* $|F_j|$*-integrable, for each* $j = 1, \dots, N$,
(ii) *the set of non-Lebesgue points of* $\partial_{y_j} g(\mathbf{y})$ *has* $|F_j|$*-measure zero,*
(iii) $g(\mathbf{y})$ *is* $(|\mathbf{F}| + |\operatorname{div}\mathbf{F}|)$*-integrable,*
(iv) *the set of non-Lebesgue points of* $g(\mathbf{y})$ *has* $(|\mathbf{F}| + |\operatorname{div}\mathbf{F}|)$*-measure zero.*
Then $g\mathbf{F} \in \mathcal{DM}(\mathcal{D})$ *and*

$$\operatorname{div}(g\mathbf{F}) = g \operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla g. \tag{8.11}$$

In particular, if $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$, *then* $g\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$ *for any* $g \in BV \cap L^\infty(\mathcal{D})$; *moreover, if* g *is also Lipschitz over any compact set in* $\mathcal{D}$, *then*

$$\operatorname{div}(g\mathbf{F}) = g \operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla g. \tag{8.12}$$

In fact, for $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$, one may refine the above result to yield that (8.12) holds a.e. in a more general case, not only for local Lipschitz functions. In this case, we must take the absolutely continuous part of ∇g. For $g \in BV$, let $(\nabla g)_{\text{ac}}$ and $(\nabla g)_{\text{sing}}$ denote the absolutely continuous part and the singular part of the Radon measure ∇g, respectively. Then we have the proposition.

PROPOSITION 8.5. *Given* $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$ *and* $g \in BV \cap L^\infty(\mathcal{D})$, *the identity*

$$\operatorname{div}(g\mathbf{F}) = \bar{g}\operatorname{div}\mathbf{F} + \overline{\mathbf{F}\cdot\nabla g}$$

holds in the sense of Radon measures in $\mathcal{D}$, *where* $\bar{g}$ *is the limit of a mollified sequence for* g *through a positive symmetric mollifier, and* $\overline{\mathbf{F}\cdot\nabla g}$ *is a Radon measure, absolutely continuous with respect to* $|\nabla g|$, *whose absolutely continuous part with respect to the Lebesgue measure in* $\mathcal{D}$ *coincides with* $\mathbf{F}\cdot(\nabla g)_{\text{ac}}$ *almost everywhere in* $\mathcal{D}$.

8.3. *Normal traces and the Gauss–Green formula*

We now discuss the Gauss–Green formula for $\mathcal{DM}$ fields over $\Omega \subset \mathcal{D}$ by introducing a suitable notion of normal traces over the boundary $\partial\Omega$ of a bounded open set with Lipschitz deformable boundary, established in [67,69].

DEFINITION 8.1. Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset. We say that $\partial\Omega$ is a deformable Lipschitz boundary, provided that

(i) for any $\mathbf{x} \in \partial\Omega$, there exist $r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap Q(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N : \gamma(y_1, \ldots, y_{N-1}) < y_N\} \cap Q(\mathbf{x}, r),$$

where $Q(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N : |y_i - x_i| \leqslant r,\ i = 1, \ldots, N\}$,

(ii) there exists $\Psi : \partial\Omega \times [0, 1] \to \overline{\Omega}$ such that Ψ is a homeomorphism, bi-Lipschitz over its image, and $\Psi(\omega, 0) = \omega$ for any $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Denote $\partial\Omega_s \equiv \Psi(\partial\Omega \times \{s\})$, $s \in [0, 1]$, and denote Ω_s the open subset of Ω whose boundary is $\partial\Omega_s$.

REMARK 8.1. The domains with deformable Lipschitz boundaries clearly include bounded domains with Lipschitz boundaries, the star-shaped domains and the domains whose boundaries satisfy the cone property. It is also clear that, if Ω is the image through a bi-Lipschitz map of a domain $\overline{\Omega}$ with a Lipschitz deformable boundary, then Ω itself possesses a Lipschitz deformable boundary.

For $\mathcal{DM}^p$ fields with $1 < p \leqslant \infty$, we have the following theorem.

THEOREM 8.1. *Let* $\mathbf{F} \in \mathcal{DM}^p(\mathcal{D})$, $1 < p \leqslant \infty$. *Let* $\Omega \subset \mathcal{D}$ *be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional* $\mathbf{F} \cdot \mathbf{n}$ *over* $\mathrm{Lip}(\partial\Omega)$ *such that, for any* $\phi \in \mathrm{Lip}(\mathbb{R}^N)$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \phi \rangle_{\partial\Omega} = \langle \operatorname{div} \mathbf{F}, \phi \rangle_{\Omega} + \int_{\Omega} \nabla\phi \cdot \mathbf{F}\, dx. \tag{8.13}$$

Moreover, let $\mathbf{n} : \Psi(\partial\Omega \times [0, 1]) \to \mathbb{R}^N$ *be such that* $\mathbf{n}(\mathbf{x})$ *is the unit outer normal to* $\partial\Omega_s$ *at* $\mathbf{x} \in \partial\Omega_s$, *defined for a.e.* $\mathbf{x} \in \Psi(\partial\Omega \times [0, 1])$. *Let* $h : \mathbb{R}^N \to \mathbb{R}$ *be the level set function of* $\partial\Omega_s$, *that is,*

$$h(\mathbf{y}) := \begin{cases} 0 & \textit{for } \mathbf{y} \in \mathbb{R}^N - \overline{\Omega}, \\ 1 & \textit{for } \mathbf{y} \in \Omega - \Psi\big(\partial\Omega \times [0, 1]\big), \\ s & \textit{for } \mathbf{y} \in \partial\Omega_s, 0 \leqslant s \leqslant 1. \end{cases}$$

Then, for any $\psi \in \mathrm{Lip}(\partial\Omega)$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \psi \rangle_{\partial\Omega} = -\lim_{s \to 0} \frac{1}{s} \int_{\Psi(\partial\Omega \times (0,s))} \mathcal{E}(\psi) \nabla h \cdot \mathbf{F}\, d\mathbf{x}, \tag{8.14}$$

where $\mathcal{E}(\psi)$ *is any Lipschitz extension of* ψ *to the whole space* $\mathbb{R}^N$.

In the case $p=\infty$, the normal trace $\mathbf{F}\cdot\mathbf{n}$ is a function in $L^\infty(\partial\Omega)$ satisfying

$$\|\mathbf{F}\cdot\mathbf{n}\|_{L^\infty(\partial\Omega)} \leqslant C\|\mathbf{F}\|_{L^\infty(\Omega)}$$

for some constant C independent of $\mathbf{F}$. Furthermore, for any field $\mathbf{F}\in\mathcal{D}M^\infty(\Omega)$,

$$\langle\mathbf{F}\cdot\mathbf{n},\psi\rangle_{\partial\Omega} = \operatorname{ess}\lim_{s\to 0}\int_{\partial\Omega_s}(\mathbf{F}\cdot\mathbf{n})\big(\psi\circ\Psi_s^{-1}\big)\,\mathrm{d}\mathcal{H}^{N-1} \quad \textit{for any } \psi\in L^1(\Omega). \tag{8.15}$$

Finally, for $\mathbf{F}\in\mathcal{D}M^p(\Omega)$ with $1<p<\infty$, $\mathbf{F}\cdot\mathbf{n}$ can be extended to a continuous linear functional over $W^{1-1/p,p}\cap L^\infty(\partial\Omega)$.

EXAMPLE 8.3. The field

$$\mathbf{F}(y_1,y_2) = \left(\sin\left(\frac{1}{y_1-y_2}\right), \sin\left(\frac{1}{y_1-y_2}\right)\right)$$

belongs to $\mathcal{D}M^\infty(\mathbb{R}^2)$. It is impossible to define any reasonable notion of traces over the line $y_1=y_2$ for the component $\sin(1/(y_1-y_2))$. Nevertheless, the unit normal $\mathbf{n}_\tau$ to the line $y_1-y_2=\tau$ is the vector $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ so that the scalar product $\mathbf{F}(y_1,y_1-\tau)\cdot\mathbf{n}_\tau$ is identically zero over this line. Hence, we find that

$$\mathbf{F}\cdot\mathbf{n}\equiv 0 \quad \text{over the line } y_1=y_2$$

and the Gauss–Green formula implies that, for any $\phi\in C_0^1(\mathbb{R}^2)$,

$$0 = \langle \operatorname{div}\mathbf{F}|_{y_1>y_2},\phi\rangle = -\int_{y_1>y_2}\mathbf{F}\cdot\nabla\phi\,\mathrm{d}\mathbf{y}.$$

This identity could also be directly obtained by applying the dominated convergence theorem to the analogous identity obtained from the classical Gauss–Green formula.

As indicated by Examples 8.1 and 8.2, it is more delicate for fields in $\mathcal{D}M^1$ and $\mathcal{D}M^{\mathrm{ext}}$. Then we have to define the normal traces as functionals over the spaces $\mathrm{Lip}(\gamma,\partial\Omega)$ with $\gamma>1$ (see [309]).

For $1<\gamma\leqslant 2$, the elements of $\mathrm{Lip}(\gamma,\partial\Omega)$ are $(N+1)$-components vectors, where the first component is the function itself, and the other N components are its "first-order partial derivatives". In particular, as a functional over $\mathrm{Lip}(\gamma,\partial\Omega)$, the values of the normal trace of a field in $\mathcal{D}M^1$ or $\mathcal{D}M^{\mathrm{ext}}$ on $\partial\Omega$ depend on not only the values of the respective functions over $\partial\Omega$ but also the values of their first-order derivatives over $\partial\Omega$. To define the normal traces for $\mathbf{F}\in\mathcal{D}M^1(\Omega)$ or $\mathcal{D}M^{\mathrm{ext}}(\Omega)$, we resort to the properties of the Whitney extensions of functions in $\mathrm{Lip}(\gamma,\partial\Omega)$ to $\mathrm{Lip}(\gamma,\mathbb{R}^N)$.

We have the following analogue of Theorem 8.1 which covers fields in $\mathcal{D}M^1$ and $\mathcal{D}M^{\mathrm{ext}}$.

THEOREM 8.2. *Let* $\mathbf{F} \in \mathcal{DM}^1(\mathcal{D})$ *or* $\mathcal{DM}^{\text{ext}}(\mathcal{D})$. *Let* $\Omega \subset \mathcal{D}$ *be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional* $\mathbf{F} \cdot \mathbf{n}$ *over* $\text{Lip}(\gamma, \partial\Omega)$ *for any* $\gamma > 1$ *such that, for any* $\phi \in \text{Lip}(\gamma, \mathbb{R}^N)$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \phi \rangle_{\partial\Omega} = \langle \operatorname{div} \mathbf{F}, \phi \rangle_{\Omega} + \langle \mathbf{F}, \nabla\phi \rangle_{\Omega}. \tag{8.16}$$

Moreover, let $h : \mathbb{R}^N \to \mathbb{R}$ *be the level set function as defined in Theorem* 8.1; *and in the case that* $\mathbf{F} \in \mathcal{DM}^{\text{ext}}(\mathcal{D})$, *we also assume that* $\partial_{x_i} h$ *is* $|F_i|$*-measurable and its set of non-Lebesgue points has* $|F_i|$*-measure zero,* $i = 1, \dots, N$. *Then, for any* $\psi \in \text{Lip}(\gamma, \partial\Omega)$, $\gamma > 1$,

$$\langle \mathbf{F} \cdot \mathbf{n}, \psi \rangle_{\partial\Omega} = -\lim_{s \to 0} \frac{1}{s} \langle \mathbf{F}, \mathcal{E}(\psi) \nabla h \rangle_{\Psi(\partial\Omega \times (0,s))}, \tag{8.17}$$

where $\mathcal{E}(\psi) \in \text{Lip}(\gamma, \mathbb{R}^N)$ *is the Whitney extension of* ψ *on* $\partial\Omega$ *to* $\mathbb{R}^N$.

REMARK 8.2. In general, for $\mathbf{F} \in \mathcal{DM}^1(\mathcal{D})$ or $\mathcal{DM}^{\text{ext}}(\mathcal{D})$, the normal trace $\mathbf{F} \cdot \mathbf{n}$ may be no longer a function. This can be seen in Example 8.1 for $\mathbf{F} \in \mathcal{DM}^1_{\text{loc}}(\mathbb{R}^2)$ with $\Omega = \{\mathbf{y}\colon y_1^2 + y_2^2 < 1, y_2 > 0\}$, for which $\mathbf{F} \cdot \mathbf{n}$ is a measure over $\partial\Omega$.

As a corollary of the Gauss–Green formula for $\mathcal{DM}^\infty$ fields, we have the following proposition.

PROPOSITION 8.6. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set with Lipschitz boundary and*

$$\mathbf{F}_1 \in \mathcal{DM}^\infty(\Omega), \qquad \mathbf{F}_2 \in \mathcal{DM}^\infty\big(\mathbb{R}^N - \overline{\Omega}\big).$$

Then

$$\mathbf{F}(y) = \begin{cases} \mathbf{F}_1(\mathbf{y}), & \mathbf{y} \in \Omega, \\ \mathbf{F}_2(\mathbf{y}), & \mathbf{y} \in \mathbb{R}^N - \overline{\Omega}, \end{cases} \tag{8.18}$$

belongs to $\mathcal{DM}^\infty(\mathbb{R}^N)$, *and*

$$\begin{aligned} \|\mathbf{F}\|_{\mathcal{DM}^\infty(\mathbb{R}^N)} \leqslant{}& \|\mathbf{F}_1\|_{\mathcal{DM}^\infty(\Omega)} + \|\mathbf{F}_2\|_{\mathcal{DM}^\infty(\mathbb{R}^N - \overline{\Omega})} \\ &+ \|\mathbf{F}_1 \cdot \mathbf{n} - \mathbf{F}_2 \cdot \mathbf{n}\|_{L^\infty(\partial\Omega)} \mathcal{H}^{N-1}(\partial\Omega). \end{aligned}$$

The analysis above over sets with Lipschitz boundary has been extended to the analysis over sets of finite perimeter for $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$.

DEFINITION 8.2. Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For any open set $\mathcal{D} \subset \mathbb{R}^N$, we say that E is a set of finite perimeter in $\mathcal{D}$ if the characteristic function of E, χ_E, belongs to $BV(\mathcal{D})$. We refer to a set of finite perimeter in $\mathbb{R}^N$ simply as a set of finite perimeter.

REMARK 8.3. If E is a set of finite perimeter in $\mathcal{D} \subset \mathbb{R}^N$, then $\nabla\chi_E$ (the gradient of χ_E in the sense of distributions) is a vector-valued Radon measure in $\mathcal{D}$. We denote the total variation of $\nabla\chi_E$ as $|\nabla\chi_E|$. It can be shown (cf. [8,132]) that

$$\nabla\chi_E = \mathbf{n}_E|\nabla\chi_E|,$$

where $\mathbf{n}_E$ is the measure-theoretic inward unit normal to E.

DEFINITION 8.3. Let E be a set of finite perimeter in $\mathcal{D} \subset \mathbb{R}^N$. The *reduced boundary* of E, denoted as $\partial^* E$, is the set of all points $\mathbf{y} \in \operatorname{supp}(|\nabla\chi_E|) \cap \mathcal{D}$ such that

(i) $\int_{B(\mathbf{y},r)} |\nabla\chi_E| > 0$ for all $r > 0$;

(ii) $\lim_{r\to 0}(\int_{B(\mathbf{y},r)} \nabla\chi_E / \int_{B(\mathbf{y},r)} |\nabla\chi_E|) = \mathbf{n}_E(\mathbf{y})$;

(iii) $|\mathbf{n}_E(\mathbf{y})| = 1$.

We recall that the space of functions of bounded variation, BV, in fact represents an equivalence class of functions so that changing the value of a function in this class on a set of $\mathcal{L}^N$-measure zero does not change the function itself. From Definition 8.2, it follows that the same is true for sets of finite perimeter. Since we are concerned with only equivalence classes of sets, *we assume here that a set of finite perimeter E is the representative given by the following proposition*, which can be found in [144].

PROPOSITION 8.7. *If $E \subset \mathbb{R}^N$ is a Borel set, then there exists a Borel set $\widetilde{E}$ equivalent to E, which differs only by a set of $\mathcal{L}^N$-measure zero, such that*

$$0 < \left|\widetilde{E} \cap B(\mathbf{y}, r)\right| < \omega_N r^N \tag{8.19}$$

for all $\mathbf{y} \in \partial\widetilde{E}$ and all small $r > 0$, where ω_N is the measure of the unit ball in $\mathbb{R}^N$.

DEFINITION 8.4. For every $\alpha \in [0, 1]$ and every $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$, define

$$E^\alpha := \left\{\mathbf{y} \in \mathbb{R}^N \colon \lim_{r\to 0} \frac{|E \cap B(\mathbf{y}, r)|}{|B(\mathbf{y}, r)|} = \alpha\right\}, \tag{8.20}$$

the set of all points with density $\alpha \in [0, 1]$. We now define the *essential boundary* of E, $\partial^s E$, as

$$\partial^s E = \mathbb{R}^N \setminus \left(E^0 \cup E^1\right). \tag{8.21}$$

The sets E^0 and E^1 may be considered as the measure-theoretic exterior and interior of E, which motivate the definition of essential boundaries.

REMARK 8.4. If E is a set of finite perimeter in $\mathcal{D} \subset \mathbb{R}^N$, it has been shown (cf. [8]) that

$$\partial^* E \subset E^{1/2} \subset \partial^s E, \tag{8.22}$$

$$\mathcal{H}^{N-1}\left(\partial^s E \setminus \partial^* E\right) = 0, \tag{8.23}$$

and

$$|\nabla \chi_E| = \mathcal{H}^{N-1} \lfloor \partial^* E. \tag{8.24}$$

DEFINITION 8.5. Let $f \in L^1(\mathcal{D})$ and $\mathbf{a} \in \mathbb{R}^N$. We say that $f_{\mathbf{a}}(\mathbf{y}_0)$ is the approximate limit of f at $\mathbf{y}_0 \in \mathcal{D}$ restricted to $\Pi_{\mathbf{a}} := \{\mathbf{y} \in \mathbb{R}^N \colon \mathbf{y} \cdot \mathbf{a} \geqslant 0\}$ if, for any $\delta > 0$,

$$\lim_{r \to 0} \frac{|\{\mathbf{y} \in \mathbb{R}^N \colon |f(\mathbf{y}) - f_{\mathbf{a}}(\mathbf{y}_0)| < \delta\} \cap B(\mathbf{y}_0, r) \cap \Pi_{\mathbf{a}}|}{|B(\mathbf{y}_0, r) \cap \Pi_{\mathbf{a}}|} = 1. \tag{8.25}$$

DEFINITION 8.6. We say that $\mathbf{y}_0 \in \mathcal{D}$ is a regular point of a function $f \in BV(\mathcal{D})$ if there exists a vector $\mathbf{a} \in \mathbb{R}^N$ such that the approximate limits $f_{\mathbf{a}}(\mathbf{y}_0)$ and $f_{-\mathbf{a}}(\mathbf{y}_0)$ exist. The vector $\mathbf{a}$ is called a *defining vector.*

If $\mathbf{y}_0$ is a regular point of $f \in BV(\mathcal{D})$, then there are two possibilities

$$\text{either} \quad f_{\mathbf{a}}(\mathbf{y}_0) = f_{-\mathbf{a}}(\mathbf{y}_0) \quad \text{or} \quad f_{\mathbf{a}}(\mathbf{y}_0) \neq f_{-\mathbf{a}}(\mathbf{y}_0).$$

It can be proved (cf. [330]) that, in the first case, any $\mathbf{b} \in \mathbb{R}^N$ is a defining vector and $f_{\mathbf{b}}(\mathbf{y}_0) = f_{\mathbf{a}}(\mathbf{y}_0)$; in the second case, $\mathbf{a}$ is unique up to the sign, i.e., the only defining vectors are $\mathbf{a}$ and $-\mathbf{a}$.

REMARK 8.5. A classical result in the BV theory says that $\mathcal{H}^{N-1}$ almost every $\mathbf{y} \in \mathcal{D}$ is a regular point of $f \in BV(\mathcal{D})$; see [8,132,330].

DEFINITION 8.7. Given $f \in L^1_{\mathrm{loc}}(\mathcal{D})$, we define

$$\bar{f}(\mathbf{y}) := \lim_{\varepsilon \to 0} f^{\varepsilon}(\mathbf{y}), \tag{8.26}$$

where $f^{\varepsilon} := f * \omega^{\varepsilon}$ with $\omega^{\varepsilon}(\mathbf{y}) = \varepsilon^{-N} \omega(\mathbf{y}/\varepsilon)$ for the standard positive symmetric mollifier ω defined in (8.8).

REMARK 8.6. It can be proved that, if $f \in BV(\mathcal{D})$, then $\bar{f}$ is defined at each regular point. Moreover, if $\mathbf{y}_0$ is a regular point of f, then

$$\bar{f}(\mathbf{y}_0) = \frac{1}{2}\big(f_{\mathbf{a}}(\mathbf{y}_0) + f_{-\mathbf{a}}(\mathbf{y}_0)\big),$$

where $\mathbf{a}$ is a defining vector (cf. [330]).

If E is a set of finite perimeter in $\mathcal{D}$, we have from Remark 8.6 that $\bar{\chi}_E$ is defined $\mathcal{H}^{N-1}$-almost everywhere. In fact, we have

$$\bar{\chi}_E(\mathbf{y}) = \begin{cases} \frac{1}{2} & \text{if } \mathbf{y} \in \partial^* E, \\ 1 & \text{if } \mathbf{y} \in E^1, \\ 0 & \text{if } \mathbf{y} \in E^0. \end{cases} \tag{8.27}$$

We recall here that $\mathcal{H}^{N-1}(\partial^s E \setminus \partial^* E) = 0$.

As Proposition 8.8 indicates,

$$\operatorname{div} \mathbf{F} \ll \mathcal{H}^{N-1} \quad \text{for } \mathbf{F} \in \mathcal{D}M^\infty(\mathcal{D}).$$

Thus, the values of $\bar{\chi}_E$ on the set $\partial^s E \setminus \partial^* E$ can be ignored. This fact is essential in the proof of the Gauss–Green formula for $\mathcal{D}M^\infty$ fields over sets of finite perimeter.

PROPOSITION 8.8. *Let* $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$. *Then the Radon measure* $\operatorname{div} \mathbf{F}$ *in* $\mathcal{D}$ *is absolutely continuous with respect to the* $(N-1)$*-Hausdorff measure* $\mathcal{H}^{N-1}$. *That is, if* $A \subset \mathcal{D}$ *be a Borel measurable set such that* $\mathcal{H}^{N-1}(A) = 0$, *then* $|\operatorname{div} \mathbf{F}|(A) = 0$.

PROOF. Since there are Borel measurable sets $\mathcal{D}_+$ and $\mathcal{D}_-$, $\mathcal{D}_+ \cup \mathcal{D}_- = \mathcal{D}$, such that $\operatorname{div} \mathbf{F}$ is a nonnegative measure over $\mathcal{D}_+$ and a nonpositive measure over $\mathcal{D}_-$, one may assume $A \subset \mathcal{D}_+$ and hence $|\operatorname{div} \mathbf{F}|(A) = (\operatorname{div} \mathbf{F})_+(A) = \operatorname{div} \mathbf{F}(A)$. Also, since $(\operatorname{div} \mathbf{F})_+$ is a Radon measure, it suffices to prove the assertion for any compact A.

Now, for any $\delta > 0$, we can find a finite number J of balls of radius less than δ such that

$$A \subset \bigcup_{i=1}^{J} B(\mathbf{y}_i; r_i), \qquad \sum_{i=1}^{J} r_i^{N-1} < \delta,$$

since $\mathcal{H}^{N-1}(A) = 0$. Then we may apply the Gauss–Green formula for $\mathcal{D}M^\infty$ fields over the set

$$\Omega = \Omega_\delta \equiv \bigcup_{i=1}^{J} B(\mathbf{y}_i; r_i)$$

with Lipschitz deformable boundary and any function $\phi \in C_0^1(\mathbb{R}^N)$ that is identically equal to one over $\overline{\Omega_\delta}$. Then

$$\left|\operatorname{div} \mathbf{F}(\Omega_\delta)\right| \leqslant \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(\partial \Omega_\delta) \leqslant C\|\mathbf{F}\|_\infty \sum_{i=1}^{J} r_i^{N-1} \leqslant \delta C \|\mathbf{F}\|_\infty.$$

Now, since $\chi_{\Omega_\delta} \to \chi_A$ pointwise in $\mathcal{D}$ as $\delta \to 0$ (recall that A is compact), one has

$$|\operatorname{div} \mathbf{F}|(A) = \operatorname{div} \mathbf{F}(A) = 0.$$

This completes the proof. □

Now, Proposition 8.5 immediately implies the following proposition.

PROPOSITION 8.9. *Let* $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$. *If* $E \Subset \mathcal{D}$ *is a set of finite perimeter in* $\mathcal{D}$, *then*

$$\operatorname{div}(\chi_E \mathbf{F}) = \bar{\chi}_E \operatorname{div} \mathbf{F} + \overline{\mathbf{F} \cdot \nabla \chi_E}, \tag{8.28}$$

where $\overline{\mathbf{F}\cdot\nabla\chi_E} = w - \lim_{\varepsilon\to 0}\mathbf{F}\cdot\nabla(\chi_E)^\varepsilon$ *for* $(\chi_E)^\varepsilon = \chi_E * \omega^\varepsilon$. *Furthermore, the measure* $\overline{\mathbf{F}\cdot\nabla\chi_E}$ *is absolutely continuous with respect to the measure* $|\nabla\chi_E|$.

THEOREM 8.3. *Let* $\mathbf{F}\in\mathcal{D}M^\infty(\mathcal{D})$. *If* $E \Subset \mathcal{D}$ *is a bounded set of finite perimeter, then there exists an* $\mathcal{H}^{N-1}$*-integrable function (denoted by)* $\mathbf{F}\cdot\mathbf{n}\in L^\infty(\partial^s E;\mathcal{H}^{N-1})$ *such that*

$$\int_{E^1}\operatorname{div}\mathbf{F} = -\int_{\partial^s E}\overline{\mathbf{F}\cdot\nabla\chi} = -\int_{\partial^s E}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{N-1}. \tag{8.29}$$

Then we have the following Gauss–Green formula.

THEOREM 8.4 (Gauss–Green formula). *Let* $\mathbf{F}\in\mathcal{D}\mathcal{M}^\infty(\mathcal{D})$. *Let* $E\Subset\mathcal{D}$ *be a bounded set of finite perimeter. Then there exists an* $\mathcal{H}^{N-1}$*-integrable function*

$$\mathbf{F}\cdot\mathbf{n}\quad \textit{on } \partial^s E$$

such that, for any $\phi\in C_0^1(\mathbb{R}^N)$,

$$\int_{E^1}\phi\operatorname{div}\mathbf{F} = -\int_{\partial^s E}\mathbf{F}\cdot\mathbf{n}\phi\,\mathrm{d}\mathcal{H}^{N-1} - \int_{E^1}\mathbf{F}\cdot\nabla\phi\,\mathrm{d}\mathbf{y}.$$

THEOREM 8.5. *Let* $\Omega\Subset E\Subset\mathcal{D}$ *be bounded open sets where* E *is a set of finite perimeter in* $\mathbb{R}^N$. *Let* $\mathbf{F}_1\in\mathcal{D}M^\infty(\mathcal{D})$ *and* $\mathbf{F}_2\in\mathcal{D}M^\infty(\mathbb{R}^N-\overline{\Omega})$. *Then*

$$\mathbf{F}(\mathbf{y}) = \begin{cases}\mathbf{F}_1(\mathbf{y}), & \mathbf{y}\in E,\\ \mathbf{F}_2(\mathbf{y}), & \mathbf{y}\in\mathbb{R}^N-\overline{E},\end{cases} \tag{8.30}$$

belongs to $\mathcal{D}M^\infty(\mathbb{R}^N)$, *and*

$$\|\mathbf{F}\|_{\mathcal{D}M^\infty(\mathbb{R}^N)} \leqslant \|\mathbf{F}_1\|_{\mathcal{D}M^\infty(E)} + \|\mathbf{F}_2\|_{\mathcal{D}M^\infty(\mathbb{R}^N-\overline{E})} + \|\mathbf{F}_1\cdot\mathbf{n} - \mathbf{F}_2\cdot\mathbf{n}\|_{L^1(\partial^s E;\mathcal{H}^{N-1})}.$$

The normal trace over a surface of finite perimeter, introduced by Chen and Torres [83], can be understood as the weak-star limit of the normal traces in Theorem 8.1 by Chen and Frid [67] over the Lipschitz deformation surfaces of the surface, which implies their consistency.

Some entropy methods based on the theory of divergence-measure fields presented above have been developed and applied for solving various nonlinear problems for conservation laws and related nonlinear equations. These problems especially include

(1) stability of Riemann solutions, which may contain rarefaction waves, contact discontinuities, and/or vacuum states, in the class of entropy solutions of the Euler equations for gas dynamics in [69,70,86],

(2) decay of periodic entropy solutions for hyperbolic conservation laws in [65],

(3) initial and boundary layer problems for hyperbolic conservation laws in [67,82, 83,329],

(4) rigorous derivation of systems of balance laws from the physical principle of balance law and the recovery of Cauchy entropy flux through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation in [84],

(5) nonlinear degenerate parabolic–hyperbolic equations in [37,73,81,248].

One of the entropy methods is to identify Lyapunov-type functionals and employ the Gauss–Green formula to establish the uniqueness and stability of entropy solutions; see [69,70,86]. In this regard, some related Lyapunov-type functionals have been identified for small BV solutions obtained by the Glimm scheme, the wave-front tracking scheme, and the vanishing viscosity method; see [20,33,111,167,204,210] and the references cited therein for the details. It would be interesting to apply the theory of divergence-measure fields to developing more efficient entropy methods for solving more various problems in partial differential equations and related areas whose solutions are only measures or L^p functions.

For more details, see [67,69,83,84].

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References

[1] R.K. Agarwal and D.W. Halt, *A modified CUSP scheme in wave/particle split form for unstructured grid Euler flows*, Frontiers of Computational Fluid Dynamics, D.A. Caughey and M.M. Hafez, eds, Wiley, New York (1994), 155–168.

[2] S. Alinhac, *Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux* [*Life spans of the classical solutions of two-dimensional axisymmetric compressible Euler equations*], Invent. Math. **111** (1993), 627–670.

[3] S. Alinhac, *Blowup for Nonlinear Hyperbolic Equations*, Birkhäuser, Boston (1995).

[4] H.W. Alt and L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144.

[5] H.W. Alt, L.A. Caffarelli and A. Friedman, *A free-boundary problem for quasilinear elliptic equations*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 1–44.

[6] H.W. Alt, L.A. Caffarelli and A. Friedman, *Compressible flows of jets and cavities*, J. Differential Equations **56** (1985), 82–141.

[7] L. Ambrosio and C. De Lellis, *Existence of solutions for a class of hyperbolic systems of conservation laws in several space dimensions*, Int. Math. Res. Not. **41** (2003), 2205–2220.

[8] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Clarendon Press, Oxford University Press, New York (2000).

[9] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4) **135** (1983), 293–318.

[10] V.I. Arnold, *Mathematical Methods in Classical Mechanics*, 2nd Edition, Springer-Verlag, New York (1989).

[11] M. Artola and A. Majda, *Nonlinear development of instabilities in supersonic vortex sheets, I: The basic kink modes*, Phys. D. **28** (1987), 253–281; *II: Resonant interaction among kink modes*, SIAM J. Appl. Math. **49** (1989), 1310–1349; *Nonlinear kink modes for supersonic vortex sheets*, Phys. Fluids A **1** (1989), 583–596.

[12] C. Baiocchi and A. Capelo, *Variational and Quasi-Variational Inequalities, Applications to Free-Boundary Problems, Vols 1 and 2*, Wiley, Chichester–New York (1984).

[13] J.M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal. **63** (1977), 337–403.

[14] J.M. Ball, *A version of the fundamental theorem for Young measures*, PDEs and Continuum Models of Phase Transitions, Nice, 1988, Lecture Notes in Phys., Vol. 344, Springer-Verlag, Berlin–New York (1989), 207–215.

[15] G. Ben-Dor, *Shock Wave Reflection Phenomena*, Springer-Verlag, New York (1991).

[16] H. Berestycki, L. Caffarelli and L. Nirenberg, *Uniform estimates for regularization of free boundary problems*, Analysis and Partial Differential Equations, Lecture Notes in Pure and Appl. Math., Vol. 122, Dekker, New York (1990), 567–619.

[17] H. Berestycki, L. Caffarelli and L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), 69–94; *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), 1089–1111; *Inequalities for second-order elliptic equations with applications to unbounded domains I*, Duke Math. J. **81** (1996), 467–494.

[18] L. Bers, *Existence and uniqueness of subsonic flows past a given profile*, Comm. Pure Appl. Math. **7** (1954), 441–504.

[19] A.V. Betsadze, *Equations of the Mixed Type*, Macmillan, New York (1964).

[20] S. Bianchini and A. Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Preprint (2001).

[21] A. Blokhin and Y. Trakhinin, *Stability of strong discontinuities in fluids and MHD*, Handbook of Mathematical Fluid Dynamics, Vol. 1, S. Friedlander and D. Serre, eds, Elsevier, Amsterdam, The Netherlands (2002), 545–652.

[22] G. Boillat, *La propagation des ondes*, Gauthier-Villars, Paris (1965).

[23] G. Boillat, *Chocs caractéristiques*, C. R. Math. Acad. Sci. Paris **274** (1972), 1018–1021.

[24] G. Boillat, *Involutions des systèmes conservatifs*, C. R. Math. Acad. Sci. Paris **307** (1988), 891–894.

[25] G. Bouchitté and G. Buttazzo, *Characterization of optimal shapes and masses through Monge–Kantorovich equation*, J. Eur. Math. Soc. **3** (2001), 139–168.

[26] F. Bouchut, *On zero pressure gas dynamics*, Advances in Kinetic Theory and Computing, Ser. Adv. Math. Appl. Sci., Vol. 22, World Scientific, River Edge, NJ (1994), 171–190.

[27] F. Bouchut and F. James, *Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness*, Comm. Partial Differential Equations **24** (1999), 2173–2189.

[28] U. Brauer, *Breakdown of smooth solutions of the three-dimensional Euler–Poisson system*, J. Math. Phys. **39** (1998), 1050–1074.

[29] Y. Brenier, *Hydrodynamic structure of the augmented Born–Infeld equations*, Arch. Ration. Mech. Anal. **172** (2004), 65–91.

[30] Y. Brenier and E. Grenier, *Sticky particles and scalar conservation laws*, SIAM J. Numer. Anal. **35** (1998), 2317–2328.

[31] P. Brenner, *The Cauchy problem for the symmetric hyperbolic systems in L_p*, Math. Scand. **19** (1966), 27–37.

[32] A. Bressan, *The unique limit of the Glimm scheme*, Arch. Ration. Mech. Anal. **130** (1995), 205–230.

[33] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*, Oxford University Press, Oxford (2000).

[34] A. Bressan, *An ill-posed Cauchy problem for a hyperbolic system in two space dimensions*, Rend. Sem. Mat. Univ. Padova **110** (2003), 103–117.

[35] A. Bressan, T.-P. Liu and T. Yang, *L^1 stability estimates for $n \times n$ conservation laws*, Arch. Ration. Mech. Anal. **149** (1999), 1–22.

[36] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York (1991).

[37] R. Bürger, H. Frid and K.H. Karlsen, *On a free boundary problem for a strongly degenerate quasilinear parabolic equation arising in a model for pressure filtration*, SIAM J. Math. Anal. **34** (2002), 611–635.

[38] L.A. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries, I. Lipschitz free boundaries are $C^{1,\alpha}$*, Rev. Mat. Iberoamericana **3** (1987), 139–162; *II. Flat free boundaries are Lipschitz,*

Comm. Pure Appl. Math. **42** (1989), 55–78; *III. Existence theory, compactness, and dependence on X*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **15** (1989), 583–602.

[39] L.A. Caffarelli and A. Friedman, *Continuity of the density of a gas flow in a porous medium*, Trans. Amer. Math. Soc. **252** (1979), 99–113.

[40] S. Canić, B.L. Keyfitz and E.H. Kim, *Free boundary problems for the unsteady transonic small disturbance equation: Transonic regular reflection*, Methods Appl. Anal. **7** (2000), 313–335; *Free boundary problems for a quasilinear degenerate elliptic equation: Regular reflection of weak shocks*, Comm. Pure Appl. Math. **55** (2002), 71–92.

[41] S. Canić, B.L. Keyfitz and G. Lieberman, *A proof of existence of perturbed steady transonic shocks via a free boundary problem*, Comm. Pure Appl. Math. **53** (2000), 484–511.

[42] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer-Verlag, New York (1988).

[43] T. Chang and G.-Q. Chen, *Diffraction of planar shock along the compressive corner*, Acta Math. Sci. **6** (1986), 241–257.

[44] T. Chang, G.-Q. Chen and S. Yang, *2-D Riemann problem in gas dynamics and formation of spiral*, Nonlinear Problems in Engineering and Science – Numerical and Analytical Approach, Beijing, 1991, Science Press, Beijing (1992), 167–179.

[45] T. Chang, G.-Q. Chen and S. Yang, *On the Riemann problem for two-dimensional Euler equations I: Interaction of shocks and rarefaction waves*, Discrete Contin. Dyn. Syst. **1** (1995), 555–584; *II: Interaction of contact discontinuities*, Discrete Contin. Dyn. Syst. **6** (2000), 419–430.

[46] T. Chang and L. Hsiao, *The Riemann Problem and Interaction of Waves in Gas Dynamics*, Longman/Wiley, Harlow–New York (1989).

[47] J.S. Chapman and T. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Cambridge University Press, Cambridge (1970).

[48] J.-Y. Chemin, *Dynamique des gaz à masse totale finie*, Asymptot. Anal. **3** (1990), 215–220.

[49] J.Y. Chemin, *Remarques sur l'apparition de singularités dans les écoulements eulériens compressibles*, Comm. Math. Phys. **133** (1990), 323–329.

[50] G.-Q. Chen, *Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics (III)*, Acta Math. Sci. **6** (1986), 75–120 (in English); **8** (1988), 243–276 (in Chinese).

[51] G.-Q. Chen, *Hyperbolic systems of conservation laws with a symmetry*, Comm. Partial Differential Equations **16** (1991), 1461–1487.

[52] G.-Q. Chen, *The method of quasidecoupling for discontinuous solutions to conservation laws*, Arch. Ration. Mech. Anal. **121** (1992), 131–185.

[53] G.-Q. Chen, *Remarks on global solutions to the compressible Euler equations with spherical symmetry*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 243–259.

[54] G.-Q. Chen, *Compactness methods and nonlinear hyperbolic conservation laws*, Some Current Topics on Nonlinear Conservation Laws, Stud. Adv. Math., Vol. 15, Amer. Math. Soc., Providence, RI (2000), 33–75.

[55] G.-Q. Chen, *Vacuum states and stability of rarefaction waves for compressible flow*, Methods Appl. Anal. **7** (2000), 337–362.

[56] G.-Q. Chen and J. Chen, *Vacuum states and global stability of rarefaction waves for the Euler equations for compressible fluids*, Preprint, Northwestern University (2005).

[57] G.-Q. Chen, J. Chen and K. Song, *Existence and stability of two-dimensional transonic flows through a bounded nozzle for the full Euler equations*, Preprint, Northwestern University (2005).

[58] G.-Q. Chen and C.M. Dafermos, *The vanishing viscosity method in one-dimensional thermoelasticity*, Trans. Amer. Math. Soc. **347** (1995), 531–541.

[59] G.-Q. Chen and C.M. Dafermos, *Global solutions in L^∞ for a system of conservation laws of viscoelastic materials with memory*, J. Partial Differential Equations **10** (1997), 369–383.

[60] G.-Q. Chen and M. Feldman, *Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type*, J. Amer. Math. Soc. **16** (2003), 461–494; *Steady transonic shocks and free boundary problems in infinite cylinders for the Euler equations*, Comm. Pure Appl. Math. **57** (2004), 310–356.

[61] G.-Q. Chen and M. Feldman, *Existence and stability of multidimensional transonic flows through an infinite nozzle of arbitrary cross-sections*, Preprint, December (2003); Preprint 2004-069, available at http://www.math.ntnu.no/conservation/2004.

[62] G.-Q. Chen and M. Feldman, *Free boundary problems and transonic shocks for the Euler equations in unbounded domains*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) **3** (2004), 827–869.

[63] G.-Q. Chen and M. Feldman, *Global solutions to shock reflection by blunt wedges for potential flow*, Preprint, Northwestern University (2005).

[64] G.-Q. Chen and H. Frid, *Large-time behavior of entropy solutions in L^∞ for multidimensional conservation laws*, Advances in Nonlinear Partial Differential Equations and Related Areas, G.-Q. Chen et al., eds, World Scientific, Singapore (1998), 28–44.

[65] G.-Q. Chen and H. Frid, *Decay of entropy solutions of nonlinear conservation laws*, Arch. Ration. Mech. Anal. **146** (1999), 95–127.

[66] G.-Q. Chen and H. Frid, *Large-time behavior of entropy solutions of conservation laws*, J. Differential Equations **152** (1999), 308–357.

[67] G.-Q. Chen and H. Frid, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **147** (1999), 89–118.

[68] G.-Q. Chen and H. Frid, *Vanishing viscosity limit for initial–boundary value problems for conservation laws*, Nonlinear Partial Differential Equations, Evanston, IL, 1998, G.-Q. Chen and E. DiBenedetto, eds, Contemp. Math., Vol. 238, Amer. Math. Soc., Providence, RI (1999), 35–51.

[69] G.-Q. Chen and H. Frid, *Extended divergence-measure fields and the Euler equations for gas dynamics*, Comm. Math. Phys. **236** (2003), 251–280.

[70] G.-Q. Chen, H. Frid and Y. Li, *Uniqueness and stability of Riemann solutions with large oscillation in gas dynamics*, Comm. Math. Phys. **228** (2002), 201–217.

[71] G.-Q. Chen and J. Glimm, *Global solution to the compressible Euler equations with geometrical structure*, Comm. Math. Phys. **179** (1996), 153–193.

[72] G.-Q. Chen and P.-T. Kan, *Hyperbolic conservation laws with umbilic degeneracy (I), (II)*, Arch. Ration. Mech. Anal. **130** (1995), 231–276; **160** (2001), 325–354.

[73] G.-Q. Chen and K.H. Karlsen, *L^1-framework for continuous dependence and error estimates for quasilinear anisotropic degenerate parabolic equations*, Trans. Amer. Math. Soc. **357** (2005), to appear; *Quasilinear anisotropic degenerate parabolic equations with time–space dependent diffusion coefficients*, Comm. Pure Appl. Anal. **4** (2005), 241–266.

[74] G.-Q. Chen and P. LeFloch, *Entropies and flux-splittings for the isentropic Euler equations*, Chinese Ann. Math. Ser. B **22** (2001), 145–158.

[75] G.-Q. Chen and P. LeFloch, *Compressible Euler equations with general pressure law*, Arch. Ration. Mech. Anal. **153** (2000), 221–259; *Existence theory for the isentropic Euler equations*, Arch. Ration. Mech. Anal. **166** (2003), 81–98.

[76] G.-Q. Chen, C.D. Levermore and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. **47** (1994), 787–830.

[77] G.-Q. Chen and T.-H. Li, *Global entropy solutions in L^∞ to the Euler equations and Euler–Poisson equations for isothermal fluids with spherical symmetry*, Methods Appl. Anal. **10** (2003), 215–243.

[78] G.-Q. Chen, B.-H. Li and T.-H. Li, *Entropy solutions in L^∞ for the Euler equations in nonlinear elastodynamics and related equations*, Arch. Ration. Mech. Anal. **170** (2003), 331–357.

[79] G.-Q. Chen and H. Liu, *Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids*, SIAM J. Math. Anal. **34** (2003), 925–938; *Concentration and cavitation in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids*, Phys. D **189** (2004), 141–165.

[80] G.-Q. Chen and T.-P. Liu, *Zero relaxation and dissipation limits for hyperbolic conservation laws*, Comm. Pure Appl. Math. **46** (1993), 255–281.

[81] G.-Q. Chen and B. Perthame, *Well-posedness for non-isotropic degenerate parabolic–hyperbolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), 645–668.

[82] G.-Q. Chen and M. Rascle, *Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **153** (2000), 205–220.

[83] G.-Q. Chen and M. Torres, *Divergence-measure fields, sets of finite perimeter, and conservation laws*, Arch. Ration. Mech. Anal. **175** (2005), 245–267.

[84] G.-Q. Chen, M. Torres and W. Ziemer, *Cauchy flux, balance laws, and divergence-measure fields*, Preprint, Northwestern University (2005).

[85] G.-Q. Chen and D. Wagner, *Global entropy solutions to exothermically reacting, compressible Euler equations*, J. Differential Equations **191** (2003), 277–322.
[86] G.-Q. Chen and D. Wang, *The Cauchy problem for the Euler equations for compressible fluids*, Handbook of Mathematical Fluid Dynamics, Vol. 1, S. Friedlander and D. Serre, eds, Elsevier, Amsterdam, The Netherlands (2002), 421–543.
[87] G.-Q. Chen, Y. Zhang and D. Zhu, *Stability of supersonic vortex sheets in steady Euler flows past Lipschitz walls*, Preprint, Northwestern University (2005).
[88] G.-Q. Chen, Y. Zhang and D. Zhu, *Existence and stability of supersonic Euler flows past Lipschitz wedges*, Arch. Ration. Mech. Anal. (2005), submitted.
[89] S.-X. Chen, *Asymptotic behavior of supersonic flow past a convex combined wedge*, Chinese Ann. Math. Ser. B **19** (1998), 255–264.
[90] S.-X. Chen, *Supersonic flow past a concave double wedge*, Sci. China Ser. A **41** (1998), 39–47.
[91] S.-X. Chen, *Global existence of supersonic flow past a curved convex wedge*, J. Partial Differential Equations **11** (1998), 73–82.
[92] S.-X. Chen, *Existence of stationary supersonic flows past a point body*, Arch. Ration. Mech. Anal. **156** (2001), 141–181.
[93] S.-X. Chen, P. Xin and H. Yin, *Global shock waves for the supersonic flow past a perturbed cone*, Comm. Math. Phys. **228** (2002), 47–84.
[94] I.-L. Chern, J. Glimm, O. McBryan, B. Plohr and S. Yaniv, *Front tracking for gas dynamics*, J. Comput. Phys. **62** (1986), 83–110.
[95] A.J. Chorin, *Vorticity and Turbulence*, Springer-Verlag, New York (1994).
[96] A.J. Chorin and J.E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag, Berlin–Heidelberg–New York (1979).
[97] J.D. Cole and L.P. Cook, *Transonic Aerodynamics*, Elsevier, Amsterdam, The Netherlands (1986).
[98] P. Constantin, *Some mathematical problems of fluid mechanics*, Proc. Internat. Congress of Mathematicians, Zürich, 1994, Vols 1 and 2, Birkhäuser, Basel (1995), 1086–1095.
[99] F. Coquel and B. Perthame, *Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics*, SIAM J. Numer. Anal. **35** (1998), 2223–2249.
[100] R. Courant and K.O. Friedrichs, *Supersonic Flow and Shock Waves*, Springer-Verlag, New York (1962).
[101] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York (1953).
[102] M.S. Cramer and A.R. Seebass, *Focusing of a weak shock at an arâte*, J. Fluid Mech. **88** (1978), 209–222.
[103] B. Dacorogna, *Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals*, Lecture Notes in Math., Vol. 922, Springer-Verlag, New York (1982).
[104] C.M. Dafermos, *Generalized characteristics and the structure of solutions of hyperbolic conservation laws*, Indiana Univ. Math. J. **26** (1977), 1097–1119.
[105] C.M. Dafermos, *The second law of thermodynamics and stability*, Arch. Ration. Mech. Anal. **70** (1979), 167–179.
[106] C.M. Dafermos, *Quasilinear hyperbolic systems with involutions*, Arch. Ration. Mech. Anal. **94** (1986), 373–389.
[107] C.M. Dafermos, *Solutions in L^∞ for a conservation law with memory*, Analyse Mathématique et Applications, Gauthier-Villars, Paris (1988), 117–128.
[108] C.M. Dafermos, *Generalized characteristics in hyperbolic systems of conservation laws*, Arch. Ration. Mech. Anal. **107** (1989), 127–155.
[109] C.M. Dafermos, *Large time behavior of periodic solutions of hyperbolic systems of conservation laws*, J. Differential Equations **121** (1995), 183–202.
[110] C.M. Dafermos, *Entropy and the stability of classical solutions of hyperbolic systems of conservation laws*, Recent Mathematical Methods in Nonlinear Wave Propagation, Montecatini Terme, 1994, Lecture Notes in Math., Vol. 1640, Springer-Verlag, Berlin (1996), 48–69.
[111] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Berlin (2000).
[112] C. De Lellis, F. Otto and M. Westdickenberg, *Structure of entropy solutions for multi-dimensional scalar conservation laws*, Arch. Ration. Mech. Anal. **170** (2003), 137–184.
[113] S. Demoulini, D.M. Stuart and A.E. Tzavaras, *A variational approximation scheme for three-dimensional elastodynamics with polyconvex energy*, Arch. Ration. Mech. Anal. **157** (2001), 325–344.

[114] R.L. Deschambault and I.I. Glass, *An update on nonstationary oblique shock-wave reflections, actual isopicnics and numerical experiments*, J. Fluid Mech. **131** (1983), 27–57.

[115] X. Ding, G.-Q. Chen and P. Luo, *Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics (I), (II)*, Acta Math. Sci. **5** (1985), 483-500, 501-540 (in English); **7** (1987), 467–480, **8** (1988), 61–94 (in Chinese).

[116] X. Ding, G.-Q. Chen and P. Luo, *Convergence of the fractional step Lax–Friedrichs scheme and Godunov scheme for isentropic gas dynamics*, Comm. Math. Phys. **121** (1989), 63–84.

[117] X. Ding, T. Zhang, C.-H. Wang, L. Hsiao and T.-C. Li, *A study of the global solutions for quasilinear hyperbolic systems of conservation laws*, Scientica Sinica **16** (1973), 317–335.

[118] R. DiPerna, *Existence in the large for quasilinear hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **52** (1973), 244–257.

[119] R. DiPerna, *Decay and asymptotic behavior of solutions to nonlinear hyperbolic systems of conservation laws*, Indiana Univ. Math. J. **24** (1975), 1047–1071.

[120] R. DiPerna, *Global existence of solutions to nonlinear hyperbolic systems of conservation laws*, J. Differential Equations **20** (1976), 187–212.

[121] R. DiPerna, *Decay of solutions of hyperbolic systems of conservation laws with a convex extension*, Arch. Ration. Mech. Anal. **64** (1977), 1–46.

[122] R. DiPerna, *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J. **28** (1979), 137–188.

[123] R. DiPerna, *Convergence of viscosity method for isentropic gas dynamics*, Comm. Math. Phys. **91** (1983), 1–30.

[124] R. DiPerna, *Convergence of approximate solutions to conservation laws*, Arch. Ration. Mech. Anal. **88** (1985), 223–270.

[125] G. Dong, *Nonlinear Partial Differential Equations of Second Order*, Transl. Math. Monogr., Vol. 95, Amer. Math. Soc., Providence, RI (1991).

[126] W. E, K. Khanin, A. Mazel and Y. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Ann. of Math. (2) **151** (2000), 877–960.

[127] W. E, Y. Rykov and Y. Sinai, *Generalized variational principles, global weak solutions, and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics*, Comm. Math. Phys. **177** (1996), 349–380.

[128] B. Engquist and W. E, *Large time behavior and homogenization of solutions of two-dimensional conservation laws*, Comm. Pure Appl. Math. **46** (1993), 1–26.

[129] L. Euler, *Institutiones Calculi Integralis, III*, Petropoli (1790).

[130] L.C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. A **111** (1989), 141–172.

[131] L.C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, CBMS Regional Conf. Ser. in Math., Vol. 72, Amer. Math. Soc., Providence, RI (1990).

[132] L.C. Evans and R.F. Gariepy, *Lecture Notes on Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL (1992).

[133] H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin–Heidelberg–New York (1969).

[134] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford Lecture Ser. Math. Appl., Vol. 26, Oxford University Press, Oxford (2004).

[135] A.F. Filippov, *Differential equations with discontinuous right-hand side*, Mat. Sb. **51** (1960), 99–128.

[136] R. Finn and D. Gilbarg, *Three-dimensional subsonic flows, and asymptotic estimates for elliptic partial differential equations*, Acta Math. **98** (1957), 265–296.

[137] J. Francheteau and G. Métivier, *Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels*, Astérisque, Vol. 268, Soc. Math. France, Paris (2000).

[138] S. Friedland, J.W. Robbin and J. Sylvester, *On the crossing rule*, Comm. Pure Appl. Math. **37** (1984), 19–38.

[139] K.O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. **7** (1954), 345–392.

[140] K. Friedrichs and P. Lax, *Systems of conservation laws with a convex extension*, Proc. Natl. Acad. Sci. USA **68** (1971), 1686–1688.

[141] I.M. Gamba and C.S. Morawetz, *A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: Existence theorem for potential flow*, Comm. Pure Appl. Math. **49** (1996), 999–1049.
[142] I.M. Gamba, R. Rosales and E. Tabak, *Constraints on possible singularities for the unsteady transonic small disturbance (UTSD) equations*, Comm. Pure Appl. Math. **52** (1999), 763–779.
[143] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, Berlin (1983).
[144] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Basel (1984).
[145] J. Glimm, *Solutions in the large for nonlinear hyperbolic system of equations*, Comm. Pure Appl. Math. **18** (1965), 95–105.
[146] J. Glimm, *The interaction of nonlinear hyperbolic waves*, Comm. Pure Appl. Math. **41** (1988), 569–590.
[147] J. Glimm, *Nonlinear waves: Overview and problems*, Multidimensional Hyperbolic Problems and Computations, Minneapolis, MN, 1989, IMA Vol. Math. Appl., Vol. 29, Springer-Verlag, New York (1991), 89–106.
[148] J. Glimm, C. Klingenberg, O. McBryan, B. Plohr, D. Sharp and S. Yaniv, *Front tracking and two-dimensional Riemann problems*, Adv. Appl. Math. **6** (1985), 259–290.
[149] J. Glimm and P.D. Lax, *Decay of Solutions of Nonlinear Hyperbolic Conservation Laws*, Mem. Amer. Math. Soc., Vol. 101, Amer. Math. Soc., Providence, RI (1970).
[150] J. Glimm and A. Majda, *Multidimensional Hyperbolic Problems and Computations*, IMA Vol. Math. Appl., Vol. 29, Springer-Verlag, New York (1991).
[151] J. Glimm, G. Marshall and B. Plohr, *A generalized Riemann problem for quasi-one dimensional gas flow*, Adv. Appl. Math. **5** (1984), 1–30.
[152] J. Glimm, H.C. Kranzer, D. Tan and F.M. Tangerman, *Wave fronts for Hamilton–Jacobi equations: The general theory for Riemann solutions in* $\mathbb{R}^n$, Comm. Math. Phys. **187** (1997), 647–677.
[153] E. Godlewski and P.-A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Springer-Verlag, New York (1996).
[154] S. Godunov, *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*, Mat. Sb. **47** (1959), 271–360.
[155] S. Godunov, *An interesting class of quasilinear systems*, Dokl. Akad. Nauk SSSR **139** (1961), 521–523; English transl.: Soviet Math. **2** (1961), 947–949.
[156] S. Godunov, *Elements of Continuum Mechanics*, Nauka, Moscow (1978).
[157] S. Godunov, *Lois de conservation et integrals d'énergie des équations hyperboliques*, Nonlinear Hyperbolic Problems, St. Etienne, 1986, Lecture Notes in Math., Vol. 1270, Springer-Verlag, Berlin (1987), 135–149.
[158] L. Gosse and A. Tzavaras, *Convergence of relaxation schemes to the equations of elastodynamics*, Math. Comp. **70** (2001), 555–577.
[159] M. Grassin, *Global smooth solutions to Euler equations for a perfect gas*, Indiana Univ. Math. J. **47** (1998), 1397–1432.
[160] M. Grassin and D. Serre, *Existence de solutions globales et régulières aux équations d'Euler pour un gaz parfait isentropique*, C. R. Math. Acad. Sci. Paris **325** (1997), 721–726.
[161] E. Grenier, *Existence globale pour le système des gaz sans pression*, C. R. Math. Acad. Sci. Paris **321** (1995), 171–174.
[162] C.-H. Gu, *A method for solving the supersonic flow past a curved wedge*, Fudan Univ. J. **7** (1962), 11–14 (in Chinese).
[163] C. Gu, D. Li, W. Yu and Z. Hou, *Discontinuous initial-value problem for hyperbolic systems of quasilinear equations (I)–(III)*, Acta Math. Sinica **11** (1961), 314–323, 324–327; **12** (1962), 132–143 (in Chinese).
[164] M.F. Hamilton, *Fundamentals and applications of nonlinear acoustics*, Nonlinear Wave Propagation in Mechanics, T.W. Wright, ed., AMD-77, Amer. Soc. Mech. Engrs, New York (1986), 1–28.
[165] E. Harabetian, *Diffraction of a weak shock by a wedge*, Comm. Pure Appl. Math. **40** (1987), 849–863.
[166] D. Hoff, *The zero-Mach limit of compressible flows*, Comm. Math. Phys. **192** (1998), 543–554.
[167] H. Holden and N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, Springer-Verlag, New York (2002).
[168] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag, Berlin (1997).
[169] W.J. Hrusa and S.A. Messaoudi, *On formation of singularities in one-dimensional nonlinear thermoelasticity*, Arch. Ration. Mech. Anal. **111** (1990), 135–151.

[170] L. Hsiao, *Quasilinear Hyperbolic Systems and Dissipative Mechanisms*, World Scientific, Singapore (1997).
[171] J. Hu and P.G. LeFloch, *L^1 continuous dependence property for systems conservation laws*, Arch. Ration. Mech. Anal. **151** (2000), 45–93.
[172] F. Huang and Z. Wang, *Well posedness for pressureless flow*, Comm. Math. Phys. **222** (2001), 117–146.
[173] J. Hunter, *Transverse diffraction of nonlinear waves and singular rays*, SIAM J. Appl. Math. **48** (1988), 1–37.
[174] J. Hunter, *Hyperbolic waves and nonlinear geometrical acoustics*, Trans. 6th Army Conference on Appl. Math. and Computing, Vol. 2 (1989), 527–569.
[175] J. Hunter, *Nonlinear wave diffraction*, Geometrical Optics and Related Topics, Cortona 1996, Progr. Nonlinear Differential Equations Appl., Vol. 32, Birkhäuser, Boston, MA (1997), 221–243.
[176] J. Hunter and M. Brio, *Weak shock reflection*, J. Fluid Mech. **410** (2000), 235–261.
[177] J. Hunter and J. Keller, *Weak shock diffraction*, Wave Motion **6** (1984), 79–89, 321.
[178] H.K. Jenssen, *Blowup for systems of conservation laws*, SIAM J. Math. Anal. **31** (2000), 894–908.
[179] F. John, *Formation of singularities in one-dimensional nonlinear wave propagation*, Comm. Pure Appl. Math. **27** (1974), 377–405.
[180] F. John, *Restrictions on the coefficients of hyperbolic systems of partial differential equations*, Proc. Nat. Acad. Sci. USA **74** (1977), 4150–4151.
[181] F. John, *Blow-up for quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34** (1981), 29–51.
[182] F. John, *Nonlinear Wave Equations, Formation of Singularities*, Univ. Lecture Ser., Vol. 2, Amer. Math. Soc., Providence, RI (1990).
[183] T. Kato, *Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^2$*, J. Funct. Anal. **9** (1972), 296–305.
[184] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Ration. Mech. Anal. **58** (1975), 181–205.
[185] T. Kato, *Quasi-linear equations of evolution with applications to partial differential equations*, Lecture Notes in Math., Vol. 448, Springer-Verlag, Berlin (1975), 25–70.
[186] S. Kawashima, A. Matsumura and T. Nishida, *On the fluid dynamical approximation to the Boltzmann equation at the level of the Navier–Stokes equations*, Comm. Math. Phys. **70** (1979), 97–124.
[187] J.B. Keller and A. Blank, *Diffraction and reflection of pulses by wedges and corners*, Comm. Pure Appl. Math. **4** (1951), 75–94.
[188] J.B. Keller and L. Ting, *Periodic vibrations of systems governed by nonlinear partial differential equations*, Comm. Pure Appl. Math. **19** (1966), 371–420.
[189] M. Kikuchi and Y. Shibata, *On the mixed problem for some quasilinear hyperbolic system with fully nonlinear boundary condition*, J. Differential Equations **80** (1989), 154–197.
[190] S. Klainerman and A. Majda, *Formation of singularities for wave equations including the nonlinear vibrating string*, Comm. Pure Appl. Math. **33** (1980), 241–263.
[191] H. Koch, *Mixed problems for fully nonlinear hyperbolic equations*, Math. Z. **214** (1993), 9–42.
[192] W. Kosinski, *Gradient catastrophe in the solutions of nonconservative hyperbolic systems*, J. Math. Anal. Appl. **61** (1977), 672–688.
[193] S. Kruzhkov, *First-order quasilinear equations with several space variables*, Mat. Sb. **123** (1970), 228–255.
[194] A. Kurganov and E. Tadmor, *Solution of two-dimensional Riemann problems for gas dynamics without Riemann problem solvers*, Numer. Methods Partial Differential Equations **18** (2002), 584–608.
[195] P.D. Lax, *Nonlinear hyperbolic equations*, Comm. Pure Appl. Math. **6** (1953), 231–258.
[196] P.D. Lax, *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure Appl. Math. **7** (1954), 159–193.
[197] P.D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. **10** (1957), 537–566.
[198] P.D. Lax, *Development of singularities in solutions of nonlinear hyperbolic partial differential equations*, J. Math. Phys. **5** (1964), 611–613.
[199] P.D. Lax, *Shock waves and entropy*, Contributions to Nonlinear Functional Analysis, E.A. Zarantonello, ed., Academic Press, New York (1971), 603–634.
[200] P.D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, Philadelphia, PA (1973).

[201] P.D. Lax, *The multiplicity of eigenvalues*, Bull. Amer. Math. Soc. **6** (1982), 213–214.
[202] P.D. Lax, *Hyperbolic systems of conservation laws in several space variables*, Current Topics in Partial Differential Equations, Kinokuniya, Tokyo (1986), 327–341.
[203] P.D. Lax and X.-D. Liu, *Solutions of two dimensional Riemann problem of gas dynamics by positive schemes*, SIAM J. Sci. Comput. **19** (1998), 319–340.
[204] P.G. LeFloch, *Hyperbolic Systems of Conservation Laws: The theory of classical and nonclassical shock waves*, Lectures Math. ETH Zürich, Birkhäuser, Basel (2002).
[205] P.G. LeFloch and V. Shelukhin, *Symmetries and global solvability of the isothermal gas dynamics equations*, Arch. Ration. Mech. Anal. **175** (2005), 389–430.
[206] R.J. LeVeque, *Numerical Methods for Conservation Laws*, 2nd Edition, Birkhäuser, Basel (1992).
[207] M. Lewicka, *Well-posedness for hyperbolic systems of conservation laws with large BV data*, Arch. Ration. Mech. Anal. **173** (2004), 415–445.
[208] M. Lewicka and K. Trivisa, *On the L^1 well-posedness of systems of conservation laws near solutions containing two large shocks*, J. Differential Equations **179** (2002), 133–177.
[209] J. Li, *Note on the compressible Euler equations with zero temperature*, Appl. Math. Lett. **14** (2001), 519–523.
[210] J. Li and H. Yang, *Delta-shocks as limits of vanishing viscosity for multidimensional zero-pressure gas dynamics*, Quart. Appl. Math. **59** (2001), 315–342.
[211] J. Li and T. Zhang, *Generalized Rankine–Hugoniot relations of delta-shocks in solutions of transportation equations*, Advances in Nonlinear Partial Differential Equations and Related Areas, Beijing, 1997, G.-Q. Chen et al., eds, World Scientific, River Edge, NJ (1998), 219–232.
[212] J. Li and T. Zhang, *On the initial-value problem for zero-pressure gas dynamics*, Hyperbolic Problems: Theory, Numerics, Applications, Zürich, 1998, Vol. 2, Birkhäuser, Basel (1999), 629–640.
[213] J. Li, T. Zhang and S. Yang, *The Two-Dimensional Riemann Problem in Gas Dynamics*, Pitman Monographs Surveys Pure Appl. Math., Vol. 98, Longman, Essex (1998).
[214] T.-T. Li, *On a free boundary problem*, Chinese Ann. Math. **1** (1980), 351–358.
[215] T.-T. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*, Wiley, Chichester (1994).
[216] T.-T. Li and W.-C. Yu, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke University Press, Durham (1985).
[217] W.-C. Lien and T.-P. Liu, *Nonlinear stability of a self-similar 3-dimensional gas flow*, Comm. Math. Phys. **204** (1999), 525–549.
[218] M.J. Lighthill, *The diffraction of a blast I*, Proc. Roy. Soc. London **198** (1949), 454–470.
[219] L.W. Lin, *On the vacuum state for the equations of isentropic gas dynamics*, J. Math. Anal. Appl. **121** (1987), 406–425.
[220] P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Vols 1 and 2*, Oxford University Press, Oxford (1998).
[221] P.-L. Lions and N. Masmoudi, *Incompressible limit for a viscous compressible fluid*, J. Math. Pures Appl. (9) **77** (1998), 585–627.
[222] P.-L. Lions, B. Perthame and P. Souganidis, *Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates*, Comm. Pure Appl. Math. **49** (1996), 599–638.
[223] P.-L. Lions, B. Perthame and E. Tadmor, *Kinetic formulation of the isentropic gas dynamics and p-systems*, Comm. Math. Phys. **163** (1994), 169–172.
[224] P.-L. Lions, B. Perthame and E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc. **7** (1994), 169–191.
[225] T.-P. Liu, *Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math. **30** (1977), 585–610.
[226] T.-P. Liu, *Linear and nonlinear large-time behavior of solutions of hyperbolic conservation laws*, Comm. Pure Appl. Math. **30** (1977), 767–796.
[227] T.-P. Liu, *The deterministic version of the Glimm scheme*, Comm. Math. Phys. **57** (1977), 135–148.
[228] T.-P. Liu, *Solutions in the large for the equations of nonisentropic gas dynamics*, Indiana Univ. Math. J. **26** (1977), 147–177.
[229] T.-P. Liu, *Initial-boundary value problems for gas dynamics*, Arch. Ration. Mech. Anal. **64** (1977), 137–168.

[230] T.-P. Liu, *The development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations*, J. Differential Equations **33** (1979), 92–111.
[231] T.-P. Liu, *Quasilinear hyperbolic systems*, Comm. Math. Phys. **68** (1979), 141–172.
[232] T.-P. Liu, *Nonlinear stability and instability of transonic gas flow through a nozzle*, Comm. Math. Phys. **83** (1983), 243–260.
[233] T.-P. Liu, *Nonlinear resonance for quasilinear hyperbolic equation*, J. Math. Phys. **28** (1987), 2593–2602.
[234] T.-P. Liu, *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. **108** (1987), 153–175.
[235] T.-P. Liu, *Hyperbolic and Viscous Conservation Laws*, CBMS–NSF Regional Conf. Ser. in Appl. Math., Vol. 72, SIAM, Philadelphia, PA (2000).
[236] T.-P. Liu and T. Yang, *L^1 stability for systems of hyperbolic conservation laws*, Nonlinear Partial Differential Equations and Applications, G.-Q. Chen and E. DiBenedetto, eds, Contemp. Math., Vol. 238, Amer. Math. Soc., Providence, RI (1999), 183–192.
[237] T.-P. Liu and T. Yang, *L^1 stability of weak solutions for 2×2 systems of hyperbolic conservation laws*, J. Amer. Math. Soc. **12** (1999), 729–774.
[238] T.-P. Liu and T. Yang, *Well-posedness theory for hyperbolic conservation laws*, Comm. Pure Appl. Math. **52** (1999), 1553–1586.
[239] A. Majda, *The Stability of Multidimensional Shock Fronts*, Mem. Amer. Math. Soc., Vol. 275, Amer. Math. Soc., Providence, RI (1983).
[240] A. Majda, *The Existence of Multidimensional Shock Fronts*, Mem. Amer. Math. Soc., Vol. 281, Amer. Math. Soc., Providence, RI (1983).
[241] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York (1984).
[242] A. Majda, *One perspective on open problems in multi-dimensional conservation laws*, Multidimensional Hyperbolic Problems and Computations, Minneapolis, MN, 1989, IMA Vol. Math. Appl., Vol. 29, Springer-Verlag, New York (1991), 217–238.
[243] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge (2002).
[244] T. Makino, K. Mizohata and S. Ukai, *Global weak solutions of the compressible Euler equations with spherical symmetry (I), (II)*, Japan J. Indust. Appl. Math. **9** (1992), 431–449; **11** (1994), 417–426.
[245] T. Makino and S. Takeno, *Initial boundary value problem for the spherically symmetric motion of isentropic gas*, Japan J. Indust. Appl. Math. **11** (1994), 171–183.
[246] T. Makino, S. Ukai and S. Kawashima, *Sur la solution à support compact de l'iquations d'Euler compressible*, Japan J. Indust. Appl. Math. **3** (1986), 249–257.
[247] J. Málek, J. Nečas, M. Rokyta and M. Růžička, *Weak and Measure-Valued Solutions to Evolutionary PDEs*, Chapman and Hall, London (1996).
[248] C. Mascia, A. Porretta and A. Terracina, *Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations*, Arch. Ration. Mech. Anal. **163** (2002), 87–124.
[249] G. Métivier, *Stability of multi-dimensional weak shocks*, Comm. Partial Differential Equations **15** (1990), 983–1028.
[250] C.S. Morawetz, *On the non-existence of continuous transonic flows past profiles I, II, III*, Comm. Pure Appl. Math. **9** (1956), 45–68; **10** (1957), 107–131; **11** (1958), 129–144.
[251] C.S. Morawetz, *On a weak solution for a transonic flow problem*, Comm. Pure Appl. Math. **38** (1985), 797–818.
[252] C.S. Morawetz, *Potential theory for regular and Mach reflection of a shock at a wedge*, Comm. Pure Appl. Math. **47** (1994), 593–624.
[253] C.S. Morawetz, *On steady transonic flow by compensated compactness*, Methods Appl. Anal. **2** (1995), 257–268.
[254] C.B. Morrey, *Quasiconvexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.
[255] C.B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin (1966).
[256] J. Moser, *A rapidly convergent iteration method and non-linear partial differential equations I*, Ann. Sc. Norm. Sup. Pisa (3) **20** (1966), 265–315.
[257] I. Müller and Y. Ruggeri, *Extended Thermodynamics*, Springer-Verlag, New York (1993).
[258] F. Murat, *Compacité par compensation*, Ann. Sc. Norm. Pisa Cl. Sci. (4) **5** (1978), 489–507.

[259] F. Murat, *The injection of the positive cone of H^{-1} in $W_{-1,q}$ is completely continuous for all $q < 2$*, J. Math. Pures Appl. (9) **60** (1981), 309–322.

[260] F. Murat, *Compacité par compensation, condition nécessaire et suffisante de continuité faible sous une hypothèse de range constant*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), 69–102.

[261] F. Murat, *A survey on compensated compactness*, Contributions to Modern Calculus of Variations, Bologna, 1985, Pitman Res. Notes Math. Ser., Vol. 148, Longman, Harlow (1987), 145–183.

[262] T. Nishida, *Global solution for an initial-boundary value problem of a quasilinear hyperbolic systems*, Proc. Japan Acad. **44** (1968), 642–646.

[263] T. Nishida and J. Smoller, *Solutions in the large for some nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math. **26** (1973), 183–200.

[264] T. Nishida and J. Smoller, *Mixed problems for nonlinear conservation laws*, J. Differential Equations **23** (1977), 244–269.

[265] M. Okada and T. Makino, *Free boundary problem for the equation of spherically symmetrical motion of viscous gas*, Japan J. Appl. Math. **10** (1993), 219–235.

[266] O. Oleinik, *Some Asymptotic Problems in the Theory of Partial Differential Equations*, Lezioni Lincee [Lincei Lectures], Cambridge University Press, Cambridge (1996).

[267] S. Osher, *Shock modelling in transonic and supersonic flow*, Advances in Computational Transonics, Recent Adv. Numer. Methods Fluids, Vol. 4, Pineridge, Swansea (1985), 607–643.

[268] Y.-J. Peng, *Solutions faibles globales pour l'èquation d'Euler d'un fluide compressible avec de grandes données initiales*, Comm. Partial Differential Equations **17** (1992), 161–187.

[269] B. Perthame, *Time decays: An analogy between kinetic transport, Schrödinger and gas dynamics equations*, Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, Vol. XIII, Paris, 1994/1996, Pitman Res. Notes Math. Ser., Vol. 391, Longman, Harlow (1998), 281–293.

[270] B. Perthame, *Kinetic Formations of Conservation Laws*, Oxford Lecture Ser. Math. Appl., Vol. 21, Oxford University Press, Oxford (2002).

[271] W.F. Pfeffer, *Derivation and Integration*, Cambridge Tracts in Math., Vol. 140, Cambridge University Press, Cambridge (2001).

[272] S.D. Poisson, *Mémoire sur l'intégration des equations linéaires aux derivées partielles*, J. l'Ecole Polytechnique **12** (1823), 19.

[273] F. Poupaud and M. Rascle, *Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients*, Comm. Partial Differential Equations **22** (1997), 337–358.

[274] F. Poupaud, M. Rascle and J.-P. Vila, *Global solutions to the isothermal Euler–Poisson system with arbitrarily large data*, J. Differential Equations **123** (1995), 93–121.

[275] T. Qin, *Symmetrizing the nonlinear elastodynamic system*, J. Elasticity **50** (1998), 245–252.

[276] M.A. Rammaha, *Formation of singularities in compressible fluids in two-space dimensions*, Proc. Amer. Math. Soc. **107** (1989), 705–714.

[277] M.A. Rammaha, *On the formation of singularities in magnetohydrodynamic waves*, J. Math. Anal. Appl. **188** (1994), 940–955.

[278] J. Rauch, *BV estimates fail for most quasilinear hyperbolic systems in dimension great than one*, Comm. Math. Phys. **106** (1986), 481–484.

[279] A.D. Rendall, *The initial value problem for self-gravitating fluid bodies*, Mathematical Physics X, Leipzig, 1991, Springer-Verlag, Berlin (1992), 470–474.

[280] H.K. Rhee, R. Aris and N.R. Amundsen, *First-Order Differential Equations, Theory and Application of Hyperbolic Systems of Quasilinear Equations, I, II*, Prentice Hall Internat. Ser., Prentice Hall, Englewood Cliffs, NJ (1986); (1989).

[281] B. Riemann, *Uber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Abh. Koenig. Gesell. Wiss. Goettingen **8** (1860), 43–65.

[282] N.H. Risebro, *A front-tracking alternative to the random choice method*, Proc. Amer. Math. Soc. **117** (1993), 1125–1139.

[283] J.-F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland Math. Stud., Vol. 134, Elsevier, Amsterdam, The Netherlands (1987).

[284] T. Ruggeri and A. Strumia, *Main field and convex covariant density for quasilinear hyperbolic systems*, Ann. Inst. H. Poincaré **34** (1981), 65–84.

[285] D.G. Schaeffer, *Supersonic flow past a nearly straight wedge*, Duke Math. J. **43** (1976), 637–670.

[286] D. Schaeffer and M. Shearer, *The classification of* 2×2 *systems of nonstrictly hyperbolic conservation laws, with application to oil recovery*, Comm. Pure Appl. Math. **40** (1987), 141–178; *Riemann problems for nonstrictly hyperbolic* 2×2 *systems of conservation laws*, Trans. Amer. Math. Soc. **304** (1987), 267–306.

[287] S. Schochet, *Sufficient conditions for local existence via Glimm's scheme for large BV data*, J. Differential Equations **89** (1991), 317–354.

[288] C.W. Schulz-Rinne, J.P. Collins and H.M. Glaz, *Numerical solution of the Riemann problem for two-dimensional gas dynamics*, SIAM J. Sci. Comput. **14** (1993), 1394–1414.

[289] L. Schwartz, *Théorie des distributions*, Actualites scientifiques et industrielles 1091, 1122, Herman, Paris (1950–1951).

[290] D. Serre, *La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d'espace*, J. Math. Pures Appl. (9) **65** (1986), 423–468.

[291] D. Serre, *Écoulements de fluides parfaits en deux variables indépendantes de type espace. Rèflexion d'un choc plan par un dièdre compressif*, Arch. Ration. Mech. Anal. **132** (1995), 15–36.

[292] D. Serre, *Spiral waves for the two-dimensional Riemann problem for a compressible fluid*, Ann. Fac. Sci. Toulouse Math. (6) **5** (1996), 125–135.

[293] D. Serre, *Solutions classiques globales des équations d'Euler pour un fluide parfait compressible*, Ann. Inst. Fourier (Grenoble) **47** (1997), 139–153.

[294] D. Serre, *Systems of Conservation Laws I, II*, Cambridge University Press, Cambridge (1999).

[295] D. Serre, *Hyperbolicity of the nonlinear models of Maxwell's equations*, Arch. Ration. Mech. Anal. **172** (2004), 309–331.

[296] W.-C. Sheng and T. Zhang, *The Riemann Problem for the Transportation Equations in Gas Dynamics*, Mem. Amer. Math. Soc., Vol. 654, Amer. Math. Soc., Providence, RI (1999).

[297] M. Shiffman, *On the existence of subsonic flows of a compressible fluid*, J. Ration. Mech. Anal. **1** (1952), 605–652.

[298] T.C. Sideris, *Global behavior of solutions to nonlinear wave equations in three dimensions*, Comm. Partial Differential Equations **8** (1983), 1291–1323.

[299] T.C. Sideris, *Formation of singularities in solutions to nonlinear hyperbolic equations*, Arch. Ration. Mech. Anal. **86** (1984), 369–381.

[300] T.C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985), 475–485.

[301] T.C. Sideris, *The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit*, Indiana Univ. Math. J. **40** (1991), 535–550.

[302] T.C. Sideris, *Delayed singularity formation in 2D compressible flow*, Amer. J. Math. **119** (1997), 371–422.

[303] T.C. Sideris, B. Thomases and D. Wang, *Long time behavior of solutions to the 3D compressible Euler equations with damping*, Comm. Partial Differential Equations **28** (2003), 795–816.

[304] M. Slemrod, *Resolution of the spherical piston problem for compressible isentropic gas dynamics via a self-similar viscous limit*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 1309–1340.

[305] J. Smith, *The Riemann problem in gas dynamics*, Trans. Amer. Math. Soc. **249** (1979), 1–50.

[306] J. Smoller, *Shock Waves and Reaction–Diffusion Equations*, 2nd Edition, Springer-Verlag, New York (1994).

[307] J. Smoller and B. Temple, *Global solutions of the relativistic Euler equations*, Comm. Math. Phys. **156** (1993), 67–99.

[308] K. Song, *The pressure-gradient system on non-smooth domains*, Comm. Partial. Differential Equations **28** (2003), 199–221.

[309] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ (1970).

[310] B. Sturtevant and V.A. Kulkarny, *The focusing of weak shock waves*, J. Fluid Mech. **73** (1976), 1086–1118.

[311] E. Tabak and R. Rosales, *Focusing of weak shock waves and the von Neumann paradox of oblique shock reflection*, Phys. Fluids **6** (1994), 1874–1892.

[312] E. Tadmor, *A minimum entropy principle in the gas dynamics equations*, Appl. Numer. Math. **2** (1986), 211–219.

[313] E. Tadmor, *Approximate solutions of nonlinear conservation laws and related equations*, Recent Advances in Partial Differential Equations, Venice, 1996, Proc. Sympos. Appl. Math., Vol. 54, Amer. Math. Soc., Providence, RI (1998).

[314] E. Tadmor, M. Rascle and P. Bagnerini, *Compensated compactness for 2D conservation laws*, Preprint (2005).

[315] S. Takeno, *Initial-boundary value problems for isentropic gas dynamics*, Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), 1–23; *Free piston problem for isentropic gas dynamics*, Japan J. Indust. Appl. Math. **12** (1995), 163–194.

[316] D. Tan and T. Zhang, *Two dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws I: Four-J cases*, J. Differential Equations **111** (1994), 203–254; *II: Initial data involving some rarefaction waves*, J. Differential Equations **111** (1994), 255–283.

[317] D. Tan, T. Zhang and Y. Zheng, *Delta-shock waves as limits of vanishing viscosity for hyperbolic system of conservation laws*, J. Differential Equations **112** (1994), 1–32.

[318] L. Tartar, *Une nouvelle méthode de resolution d'equations aux derivées partielles nonlinéaires*, Lecture Notes in Math., Vol. 665, Springer-Verlag, Berlin (1977), 228–241.

[319] L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear Analysis and Mechanics, Heriot–Watt Symposium IV, Res. Notes in Math., Vol. 39, Pitman, Boston–London (1979), 136–212.

[320] L. Tartar, *The compensated compactness method applied to systems of conservation laws*, Systems of Nonlinear P.D.E., J.M. Ball, ed., NATO Series, Reidel, Dordrecht (1983), 263–285.

[321] B. Temple, *Solutions in the large for the nonlinear hyperbolic conservation laws of gas dynamics*, J. Differential Equations **41** (1981), 96–161.

[322] B. Temple and R. Young, *The large time existence of periodic solutions for the compressible Euler equations*, Mat. Contemp. **11** (1996), 171–190.

[323] B. Temple and R. Young, *The large time stability of sound waves*, Comm. Math. Phys. **179** (1996), 417–466.

[324] A. Tesdall and J.K. Hunter, *Self-similar solutions for weak shock reflection*, SIAM J. Appl. Math. **63** (2002), 42–61.

[325] R. Timman, *Unsteady motion in transonic flow*, Symposium Transsonicum, IUTAM, Aachen, September 1962, K. Oswatitsch, ed., Springer-Verlag, Berlin (1964), 394–401.

[326] C. Truesdell, *Rational Thermodynamics*, 2nd Edition, Springer-Verlag, New York (1984).

[327] A. Tzavaras, *Materials with internal variables and relaxation to conservation laws*, Arch. Ration. Mech. Anal. **146** (1999), 129–155.

[328] M. Van Dyke, *An Album of Fluid Motion*, The Parabolic Press, Stanford (1982).

[329] A. Vasseur, *Strong traces for solutions to multidimensional scalar conservation laws*, Arch. Ration. Mech. Anal. **160** (2001), 181–193.

[330] A. Volpert, *The space BV and quasilinear equations*, Mat. Sb. **73** (1967), 255–302 (in Russian); Math. USSR-Sb. **2** (1967), 225–267.

[331] V. Volterra, *Sulle vibrazioni luminose nei mezzi isotropi*, Rend. Accad. Naz. Lincel **1** (1892), 161–170.

[332] D.H. Wagner, *Equivalence of Euler and Lagrangian equations of gas dynamics for weak solutions*, J. Differential Equations **68** (1987), 118–136.

[333] D. Wang, *Global solutions and stability for self-gravitating isentropic gases*, J. Math. Anal. Appl. **229** (1999), 530–542.

[334] Z. Wang and X. Ding, *Uniqueness of generalized solutions for the Cauchy problem of transportation equations*, Acta Math. Sci. (English Ed.) **17** (1997), 341–352.

[335] Z. Wang, F. Huang and X. Ding, *On the Cauchy problem of transportation equations*, Acta Math. Appl. Sinica (English Ser.) **13** (1997), 113–122.

[336] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York (1973).

[337] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.

[338] H. Whitney, *Geometric Integration Theory*, Princeton University Press, Princeton, NJ (1957).

[339] P. Woodward, *Simulation of the Kelvin–Helmholtz instability of a supersonic slip surface with the piecewise-parabolic method (PPM)*, Numerical Methods for the Euler Equations of Fluid Dynamics, Rocquencourt, 1983, SIAM, Philadelphia, PA (1985), 493–508.

[340] P. Woodward and P. Colella, *The numerical simulation of two-dimensional fluid flow with strong shocks*, J. Comput. Phys. **54** (1984), 115–173.

[341] X.-M. Wu, *Equations of Mathematical Physics*, Chinese Advanced Education Publishing Co., Beijing (1956) (in Chinese).

[342] G.-J. Yang, *The Euler–Poisson–Darboux Equations*, Yuannan University Press, Yuannan (1989) (in Chinese).

[343] X. Yang and F. Huang, *Two-dimensional Riemann problems of simplified Euler equation*, Chinese Sci. Bull. **43** (1998), 441–444.

[344] L. Ying and Z. Teng, *Hyperbolic Systems of Conservation Laws and Difference Methods*, Chinese Scientific Publishing Co., Beijing (1991) (in Chinese).

[345] L.C. Young, *Lectures on the Calculus of Variations and Optimal Control Theories*, Saunders, Philadelphia, PA (1969).

[346] G.I. Zabolotskaya and R.V. Khokhlov, *Quasi-plane waves in the nonlinear acoustics of confined beams*, Soviet Phys. Acoustics **15** (1969), 35–40.

[347] A.R. Zakharian, M. Brio, J.K. Hunter and G.M. Webb, *The von Neumann paradox in weak shock reflection*, J. Fluid Mech. **422** (2000), 193–205.

[348] Ya.B. Zeldovich, *Gravitational instability: An approximate theory for large density perturbations*, Astronaut. Astrophys. **5** (1970), 84–89.

[349] T. Zhang and Y.-F. Guo, *A class of initial-value problem for systems of aerodynamic equations*, Acta Math. Sinica (N.S.) **15** (1965), 386–396.

[350] T. Zhang and Y. Zheng, *Conjecture on the structure of solutions of the Riemann problem for two-dimensional gas dynamics*, SIAM J. Math. Anal. **21** (1990), 593–630.

[351] T. Zhang and Y. Zheng, *Exact spiral solutions of the two-dimensional Euler equations*, Discrete Contin. Dyn. Syst. **3** (1997), 117–133.

[352] T. Zhang and Y. Zheng, *Axisymmetric solutions of the Euler equations for polytropic gases*, Arch. Ration. Mech. Anal. **142** (1998), 253–279.

[353] Y.-Q. Zhang, *Global existence of steady supersonic potential flow past a curved wedge with a piecewise smooth boundary*, SIAM J. Math. Anal. **31** (1999), 166–183.

[354] Y.-Q. Zhang, *Steady supersonic flow past an almost straight wedge with large vertex angle*, J. Differential Equations **192** (2003), 1–46.

[355] Y. Zheng, *Systems of Conservation Laws: Two-Dimensional Riemann Problems*, Birkhäuser, Boston (2001).

[356] Y. Zheng, *A global solution to a two-dimensional Riemann problem involving shocks as free boundaries*, Acta Math. Appl. Sinica (English Ser.) **19** (2003), 559–572.

[357] Y. Zheng, *Two-dimensional regular shock reflection for the pressure gradient system of conservation laws*, Preprint (2004).

[358] W.P. Ziemer, *Cauchy flux and sets of finite perimeter*, Arch. Ration. Mech. Anal. **84** (1983), 189–201.

[359] W.P. Ziemer, *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*, Springer-Verlag, New York (1989).