Sheared-flow Generalization of the Harris Sheet

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Abstract

A novel, exact class of solutions to the Vlasov–Maxwell system, with self–generated magnetic fields and nonuniform plasma flows, are constructed. It is shown that a gyrotropic distribution function (independent of gyrophase) does not allow equilibrium shear flow; introduction of agyrotropy is essential for the maintenance of spatially nonuniform velocity fields. The new self–consistent sheared–flow solutions include the shearless Harris Sheet [E.G. Harris, Nuovo Cimento **23**, 117 (1962)] solution as a special case. These equilibria are likely to be relevant to a variety of astrophysical flows (most natural flows are sheared) and to a better understanding of the laboratory phenomena observed, for example, in the device MRX (Magnetic Reconnection Experiment, M. Yamada, H. Ji, S. Hsu, T. Carter, R. Kulsrud, N. Bretz, F. Jodes, Y. Ono, and F. Perkins, Phys. Plasma **4**, 1936 (1997)) designed to study magnetic reconnection.

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I. INTRODUCTION

In 1962 Harris [1] displayed an exact, one–dimensional solution to the Vlasov equation with a localized plasma current $J(x)$, and therefore a near discontinuity in the self–consistent magnetic field $B(x)$. Such a configuration, generally called the Harris Sheet, is relevant to various magnetospheric, solar and astrophysical phenomena, as well as to certain laboratory experiments, such as the MRX [2,3] device at Princeton. Indeed there is evidence that the profiles in MRX are qualitatively similar to those predicted by Harris.

However, in order to model either the laboratory experiments or the space and astrophysical phenomena, several modifications of the original theory are necessary. The Harris solution assumes a one–dimensional Cartesian geometry, while cylindrical symmetry is often pertinent—and we shall find that cylindrical effects are significant. More importantly, the flow speeds of both species in the (quasineutral) Harris solution are spatially uniform; indeed, for the form of the distribution function assumed, velocity shear is not permitted. The inclusion of velocity shear,therefore, is essential to give a widespread applicability to the theory. Most flows of experimental and observational interest are sheared.

Notice that for spatially uniform flows, the self–consistent plasma current profile is constrained to follow the density profile, $n(x)$:

$$J(x) \propto n(x), \tag{1}$$

restricting the class and nature of equilibrium configurations. In the present work, we derive exact Vlasov solutions, in both Cartesian and cylindrical geometry, with arbitrary amounts of shear in the flow speeds of both species. Thus the current profile and the density profile are fully independent. While these solutions preserve the sheet–current nature of the Harris solution, their generality (in profiles and geometry) should allow more useful comparison to observation and experiment.

The Harris solution, derived by assuming a uniform–velocity, drifting–Maxwellian distribution function, is, by construction, free of velocity shear. If we were to naively generalize this very distribution to assign spatial dependence to the drift speed, we would find that this distribution function no longer satisfies the Vlasov equation; a simple drifting Maxwellian with sheared flow is just not an equilibrium distribution. Although this interesting fact can be understood in a variety of ways, the explanation is most transparent in the case of a magnetized plasma, where the gyroradius is smaller than gradient scale lengths. (Note, however, that the solutions we consider can have arbitrary magnetic field strength and need not be "magnetized.") It is well known [4] that a magnetized plasma is nearly *gytrotropic*, that is, nearly isotropic in velocity space in the two directions transverse to the magnetic field:

$$f(\boldsymbol{x}, \boldsymbol{v}) \approx \overline{f}(\boldsymbol{x}, v_\parallel, |\boldsymbol{v}_\perp|). \tag{2}$$

Here $v_\parallel = \boldsymbol{b} \cdot \boldsymbol{v}$ and $\boldsymbol{v}_\perp = \boldsymbol{b} \times (\boldsymbol{v} \times \boldsymbol{b})$, with $\boldsymbol{b} \equiv \boldsymbol{B}/B$. We show in Sec. II that a gyrotropic distribution cannot reach equilibrium in the presence of velocity shear. On the other hand, in the magnetized case even a very small agyrotropy can allow for a strongly sheared flow.

The Harris distribution is isotropic (and therefore gyrotropic) in the moving frame, ruling out velocity shear. In Sec. III we generalize it to derive the Cartesian equilibrium with velocity shear. The result is closely related to a family of equilibria derived previously by Mahajan [5]. The cylindrical case is developed in Sec. IV, using an analogous distribution but simplifying the argument by means of dynamical constants. The scope of our results and their possible bearing on the MRX program are discussed in the concluding section.

II. VELOCITY SHEAR AND AGYROTROPY

Our purpose in this section is to show that the plasma distribution functions cannot be gyrotropic in the presence of (perpendicular) velocity shear. The reason is simple: shear drives gyroviscosity, and the gyroviscous tensor is not gyrotropic. The following demonstration, however, does not assume familiarity with gyroviscosity.

All that is needed is the second moment of the kinetic equation. Let $\boldsymbol{u}$ denote the flow velocity of some plasma species,

$$n\boldsymbol{u} \equiv \int d^3v f(\boldsymbol{x}, \boldsymbol{v}, t)\boldsymbol{v}$$

where f is the distribution function. Denoting the velocity in the moving frame by $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{u}$, we define the stress tensor,

$$\boldsymbol{p} = \int d^3v f(\boldsymbol{x}, \boldsymbol{v}, t) m\boldsymbol{w}\boldsymbol{w}.$$

Its trace is of course the scalar pressure, denoted by

$$p \equiv (1/3)Tr(\boldsymbol{p}).$$

We denote the traceless part of $\boldsymbol{p}$ by $\boldsymbol{\pi}$:

$$\boldsymbol{\pi} \equiv \boldsymbol{p} - \boldsymbol{I}p$$

where $\boldsymbol{I}$ is the unit tensor. We will also need the third order moment

$$\boldsymbol{\Gamma}_3 \equiv \int d^3v f(\boldsymbol{x}, \boldsymbol{v}, t) m\boldsymbol{w}\boldsymbol{w}\boldsymbol{w} \tag{3}$$

whose trace, with respect to any two indices, is the heat–flow vector,

$$\boldsymbol{q} = \frac{1}{2}\mathrm{Tr}(\boldsymbol{\Gamma}_3).$$

For generality we include collisions, denoting the second (tensor) moment of the collision operator by $\boldsymbol{C}_2$:

$$\boldsymbol{C}_2 \equiv \int d^3v m\boldsymbol{w}\boldsymbol{w},$$

although our attention in this work is on the collisionless, Vlasov systems. With these definitions we compute the exact $m\boldsymbol{w}\boldsymbol{w}$–moment of the Boltzmann equation to obtain the evolution law

$$\frac{\partial \boldsymbol{p}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma}_3 + (\boldsymbol{p} \cdot \nabla)\boldsymbol{u} + [(\boldsymbol{p} \cdot \nabla)\boldsymbol{u}]^T + \boldsymbol{p}\nabla \cdot \boldsymbol{u} + \Omega[\boldsymbol{b} \times \boldsymbol{\pi} + (\boldsymbol{b} \times \boldsymbol{\pi})^T] = \boldsymbol{C}_2. \tag{4}$$

Here we have introduced the gyrofrequency

$$\Omega = eB/mc$$

and the superscript T refers to the transpose tensor.

The trace of (4) is the familiar law (see, for example, [6]) for plasma energy conservation,

$$\frac{\partial p}{\partial t} + \frac{2}{3}\nabla \cdot \boldsymbol{q} + \frac{2}{3}\boldsymbol{\pi}\colon \nabla \boldsymbol{u} + \frac{5}{3}p\nabla \cdot \boldsymbol{u} = \frac{1}{3}\mathrm{Tr}(\boldsymbol{C}_2),$$

but our attention here is reserved for the traceless part.

We express the traceless version in Cartesian coordinates, with the z–axis along the direction of the magnetic field. For simplicity we assume that spatial variation occurs only in the x–direction, and that the drift or flow velocity is in the y–direction,

$$\boldsymbol{u} = \widehat{y}u(x).$$

Then the traceless part of (4) leads to the equation

$$\partial_x \left(\Gamma_{3x\alpha\beta} - \frac{2}{3}\delta_{\alpha\beta}q_x \right) + pW_{\alpha\beta} + \pi_{\alpha x}\partial_x u_\beta + \pi_{\beta x}\partial_x u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\pi_{\gamma x}\partial_x u_\gamma$$
$$+\Omega(\epsilon_{\alpha z\gamma}\pi_{\gamma\beta} + \epsilon_{\beta z\gamma}\pi_{\gamma\alpha}) = \widehat{C}_{2\alpha\beta}. \qquad (5)$$

Here $\widehat{C}$ refers to the traceless part of the collisional tensor, we use the abbreviation

$$\partial_\alpha \equiv \frac{\partial}{\partial\alpha}$$

and

$$W_{\alpha\beta} \equiv \partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\nabla \cdot \boldsymbol{u}$$

is the rate of strain tensor. For a magnetized plasma, (5) can be solved for π in order to derive both gyroviscosity (from the stress tensor) and collisional viscosity (from the collisional term) [6]. For present purposes we can ignore collisions and consider the simple case when $\alpha = x$, $\beta = y$:

$$\partial_x\Gamma_{3xxy} + (p + \pi_{xx})\partial_x u + \Omega(\pi_{xx} - \pi_{yy}) = 0. \qquad (6)$$

It is easily seen from (2) that both Γ_{3xxy} and $(\pi_{xx} - \pi_{yy})$ vanish for a gyrotropic plasma, in which, therefore, velocity shear cannot be supported. This explains the absence of shear in the conventional Harris sheet.

In a magnetized plasma, velocity shear is mainly supported by the difference $\pi_{xx} - \pi_{yy}$, the contribution from Γ_3 being of higher order in the gyroradius expansion. Indeed, we show next that agyrotropy with vanishing Γ_3 occurs quite naturally in distribution functions similar to that of Harris: drifting Maxwellians.

The distribution given in the following section is a natural extension of the Harris distribution and probably its simplest generalization. Its Maxwellian form could result from persistent collisional diffusion, which is effective even when the collision frequency is small enough to justify Vlasov theory. However, collisional friction would act to eliminate flow shear. Therefore our analysis makes physical sense in the presence of weak collisions when some (unspecified) external agent drives the sheared flow.

III. GENERALIZED SHEET IN CARTESIAN GEOMETRY

A. Nonlinear equations

A simple distribution with the properties necessary to support velocity shear, as found in the previous section, is given by

$$f(x, v_x, v_y, v_z) = \frac{n(x)\sqrt{1+\alpha}}{\pi^{3/2} v_t^3} \exp\left[-s_x^2 - s_z^2 - (1+\alpha)(s_y - \widehat{u}(x))^2\right]. \tag{7}$$

Here $v_t = \sqrt{2T/m}$ is a thermal speed, $\boldsymbol{s} \equiv \boldsymbol{v}/v_t$, $\widehat{u} \equiv u/v_t$ and $\alpha = \mathit{constant}$ measures the temperature anisotropy between the x–y directions:

$$T_x - T_y = \frac{\alpha T_x}{1+\alpha}.$$

The temperature anisotropy is the source of agyrotropy in this distribution. Note that spatial variation enters f only through the flow $u(x)$ and the density $n(x)$. We substitute this distribution into the Vlasov equation

$$\boldsymbol{v} \cdot \nabla f + \frac{e}{m}(-\nabla\Phi + c^{-1}\boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{8}$$

where $\Phi(x)$ is the electrostatic potential, and find that it provides an exact solution only if the density and flow satisfy

$$\frac{n'}{n} + \frac{e\Phi'}{T} = \frac{mu\Omega}{T} \tag{9}$$

$$u' = -\frac{\alpha\Omega}{1+\alpha}. \tag{10}$$

Here the primes denote x–derivatives. Note that, as was anticipated, the velocity shear vanishes at $\alpha = 0$. It should be further noted that the velocity shear has a very specific form, proportional to the magnetic field. In fact this is a general feature of all kinetic equilibria with a direction of symmetry. Since the equilibrium distribution must be constructed from the constants of motion, the only source for inhomogeneity in physical variables (density and velocity) are the electromagnetic potentials. For the velocity field, therefore, the spatial variation must come from the canonical momentum, and hence from the component of the vector potential in the ignorable direction. This feature becomes explicit in the next section.

In order to combine these results with Maxwell's equations we need to distinguish the ion and electron distributions. We distinguish electron parameters with an e–subscript, leaving ion parameters unsubscripted. Thus the quasineutrality condition $n = n_e$ yields the electric field in terms of the magnetic and the velocity fields,

$$\frac{e\Phi'}{T}(1+\tau) = \frac{m}{T}\Omega(u + \tau u_e) \tag{11}$$

where

$$\tau \equiv T/T_e$$

is the temperature ratio. Substituting this result into (9) we find

$$\frac{n'}{n} = \frac{m}{T}\frac{\tau}{1+\tau}\Omega(u - u_e). \tag{12}$$

Regarding the electron flow, we note that the electron version of (10) will contain the electron gyrofrequency, which is larger than Ω by a factor of the mass ratio m/m_e. For a maximal ordering (in which u and u_e contribute comparably to the current) we take $(m/m_e)\alpha_e \cong \alpha$, writing

$$\frac{m}{m_e}\alpha_e = \zeta\alpha.$$

Thus, neglecting m_e/m compared to unity, we can write the electron version of (10) as

$$u_e' = -\zeta\alpha\Omega.$$

That is,

$$u' - u_e' = \left(\frac{\alpha}{1+\alpha} - \zeta\alpha\right)\Omega. \tag{13}$$

Finally we consider Amperé's law. Only its y–component is of interest:

$$B' = -\frac{4\pi e}{c} n(u - u_e).$$

Measuring B in terms of the gyrofrequency, we express this relation as

$$\delta^2\Omega' = -\frac{n(x)}{n_0}(u - u_e) \tag{14}$$

where n_0 is a constant reference value of the density and δ is the ion collisionless skin depth

$$\delta \equiv \frac{c}{\omega_{pi}} = \frac{c m_i^{1/2}}{e\sqrt{4\pi n_0}}.$$

Equations (12), (13), and (14) constitute a closed set for the three unknown functions $n(x)$, $u(x) - u_e(x)$ and $\Omega(x)$.

B. Structure of the Cartesian sheet

It is convenient to introduce the following dimensionless variables

$$X = \sqrt{\frac{\tau}{1+\tau}}\frac{x}{\delta},$$
$$F(X) = \sqrt{\frac{\tau}{1+\tau}}\frac{\delta\Omega}{v_t},$$
$$N(X) = n/n_0,$$
$$U(X) = (u - u_e)/v_t.$$

Then our three equations become

$$F' = -NU, \tag{15}$$

$$N' = 2FNU, \tag{16}$$

$$U' = \alpha_* F, \tag{17}$$

where

$$\alpha_* = \frac{\alpha(1+\tau)}{\tau(1+\alpha)}[1 - \zeta(1+\alpha)]$$

conveniently measures the agyrotropy of the 2–species plasma. Here of course the primes indicate derivatives with respect to the dimensionless variable X.

The nonlinear equations (15)–(7) can be analyzed using standard methods. Although it is not possible to obtain a general closed-form solution, it is possible to reduce the system to a quadrature. But before working out explicit solutions, we point out that the generalization has added a qualitatively new feature, in addition to the expected modification of the standard Harris sheet. We note from Eq. (17) that for an even (in x) density function, the product FU must be odd in x. Therefore for the standard Harris sheet with a constant velocity U (even function), the magnetic field must necessarily be odd. The picture changes completely with sheared velocity fields: the system now allows even parity magnetic fields in conjunction with odd parity velocity fields. The existence of velocity fields can lead to equilibria with topologically distinct magnetic structures.

Let us first recover the modified Harris sheet. The appropriate boundary conditions, consistent with the presence of a field–reversing current layer at $x = 0$, are:

$$\Omega(0) = 0, \tag{18}$$

$$n(0) = n_0, \tag{19}$$

$$u(0) - u_e(0) = J_0/(en_0), \tag{20}$$

where J_0 is a specified constant. For the normalized variables, these translate into

$$F(0) = 0, \tag{21}$$

$$N(0) = 1, \tag{22}$$

$$U(0) = U_0 \equiv J_0/(en_0 v_t). \tag{23}$$

Combining (15) and (16) yields the relation

$$(F^2)' + N' = 0, \tag{24}$$

which expresses plasma diamagnetism in a conventional way. (In this regard, note that F^2 is essentially the inverse of the plasma beta, $\beta = 4\pi nT/B^2$.) In view of (21) and (22) the solution is

$$N = 1 - F^2. \tag{25}$$

Similarly from (16), (17), and (23) we find

$$U = \sqrt{U_0^2 + \alpha_* \log N} = \sqrt{U_0^2 + \alpha_* \log(1 - F^2)}, \tag{26}$$

or equivalently

$$N = \exp\left[\frac{U^2 - U_0^2}{\alpha_*}\right]. \tag{27}$$

Combining (18), (25), and (26), we finally obtain the first–order, separable, differential equation

$$F' = (1 - F^2) * \sqrt{U_0^2 + \alpha_* \log(1 - F^2)} \tag{28}$$

describing the magnetic field in the modified sheet. Notice that for $\alpha_* = 0$, the shearless Harris sheet ($U_0 = 1$),

$$F = \tanh X, \tag{29}$$

$$N = \operatorname{sech}^2 X, \tag{30}$$

is automatically recovered.

We have not been able to express the solution to (28) in terms of known elementary functions. It is straightforward, however, to find the effects of the velocity shear (controlled by the agyrotropy parameter α_*) on the sheet structure by numerically solving (28). In Figures (1)–(3), we plot (as a function of x) the set of normalized self–consistent physical quantities, the magnetic and velocity fields and the density and current profiles, for several values of α_*.

Notice that for moderate values of α_*, the fundamental character of the sheet is maintained even in the presence of a respectable shear flow. It is this quality of ruggedness which makes the Harris sheet such an interesting and important structure; it is recognizable even in plasmas that differ considerably from the simple ideal plasma investigated by Harris. However, the presence of the shear does bring about essential and important changes. The density and the current profile are no longer the same and the sheet–width also changes due to the sheared field. Furthermore the form of solution changes dramatically as the shear parameter changes sign.

The origin of this bifurcation at $\alpha_* = 0$ may be seen by a re–examination of the defining equations. Differentiating (17) and, using (15), we find

$$U'' + \alpha_* N\, U = 0, \tag{31}$$

and since the density profile factor N is always positive (and choosing its maximum value to be $N = 1$), Eq. (31) yields oscillatory solutions for the velocity field for positive α_*. Since N is a sharp function of x, U, though periodic will be quite a complicated function.

Several other qualitative features of the solution can be inferred:

(1) Since $\log N \leq 0$ (therefore $N \leq 1$), Eq. (26) reveals that $U > (<) U_0$ for $\alpha_* < (>) 0$; the flow speed is bounded for positive α_*.

(2) For any finite positive α_*, the density N cannot even reach zero; in fact, the lowest value attained is

$$N = e^{-\frac{U_0^2}{\alpha_*}}. \tag{32}$$

(3) The oscillation frequency is proportional to $\sqrt{\alpha_*}$.

Although a detailed discussion of the oscillatory solution is beyond the scope of this paper, a few explanatory remarks may be in order. The sheared velocity field tends to increase (decrease) the effective strength of the self–magnetic field for negative (positive) value of α_*. The self–organizing Harris sheet requires a minimum strength of the magnetic field for confining pressure (density in our case because the temperature is a constant) — this minimum strength corresponds to $\alpha_* = 0$, the original shearless configuration. For negative α_*, the confining ability of the magnetic field is fortified by the shear–generated effective field, and the basic localized character of the Harris sheet is maintained with appropriate modifications; the density localization is sharper, and the current and density profile are no more the same.

For positive α_*, however, the 'effective' magnetic field goes below the minimum required, and the plasma is no longer confined. For relatively small α_*, the density shows a periodic but peaked structure. As α_* increases, the density tends to be unity with a very small periodic part superimposed upon it. This situation idealizes a finite structure with nearly periodic striation, similar to the structures sometimes seen in non-neutral plasmas [8]. A more detailed assessment of the physics and realizability of the positive α_* structures will be given in a later publication.

As pointed out earlier, the sheared velocity field allows the possibility of magnetic configurations which are topologically distinct from the Harris sheet. The new configurations have an even parity magnetic field (a magnetic well rather than a 'discontinuity' of the velocity field. These configurations, shown in Fig. 5, are quite similar to the diamagnetic structures discussed by Mahajan and Yoshida [7].

IV. CYLINDRICAL GEOMETRY

A. Distribution function

In the case of cylindrical symmetry, we use polar coordinates (r, θ, z), allowing variation only in the radial direction. The magnetic field is taken to be axial, $\boldsymbol{B} = \widehat{z}B(r)$ as before, and the flow is assumed to be azimuthal,

$$\boldsymbol{u} = \widehat{\theta}u(r).$$

It is convenient to express

$$B(r) = A'(r) + A(r)/r,$$

where A is the θ–component of the vector potential. As in the previous section we measure B in terms of the gyrofrequency, and therefore introduce the normalized vector potential

$$a(r) \equiv \frac{eA(r)}{mc}. \tag{33}$$

We could now march through the cylindrical version of our previous development, but it is quicker to notice that the energy $\mathcal{E}$ and canonical angular momentum p, given respectively by

$$\mathcal{E} \equiv \frac{1}{2}mv^2 + e\Phi(r), \tag{34}$$

$$p \equiv mr(v_\theta + a), \tag{35}$$

are constants of the motion. (A third constant, v_z, does not enter our analysis.) Therefore the Vlasov equation is solved by any function of $\mathcal{E}$ and p; in the spirit of Harris and of Sec. III we choose the distribution

$$f(r, v_r, v_\theta, v_z) = \frac{n_0}{\pi^{3/2} v_t^3} \exp\left(-\frac{\mathcal{E} + \omega p + (1/2)\kappa^2 p^2}{T}\right). \tag{36}$$

Here the constants n_0 and $T = (1/2)mv_t^2$ have the same meaning as in Sec. III, while ω, representing the rotation frequency in the rigid–body limit, and κ, representing an inverse shear–length, are constants characterizing the azimuthal flow.

To include the case of rapid (thermal speed) rotation we allow

$$\omega r \cong v_t; \tag{37}$$

to treat flows with arbitrary shear we assume

$$\kappa r \cong 1. \tag{38}$$

B. Maxwell equations

We now require our cylindrical distribution to satisfy quasineutrality and Ampere's law. Thus we compute the two moments

$$n(r) = \int d^3v f(r, v_r, v_\theta, v_z), \tag{39}$$

$$n(r)u(r) = \int d^3v v_\theta f(r, v_r, v_\theta, v_z). \tag{40}$$

Evaluation of either integral is a straightforward exercise in completing the square in the exponent. For the density we find

$$n(r) = \frac{n_0}{\sqrt{1+\kappa^2 r^2}} \exp\left[-\frac{e\Phi(r)}{T} + \xi(r)\right]$$

where

$$\xi(r) \equiv \frac{\omega^2 r^2 + 2a(\omega r - a\kappa^2 r^2)}{v_t^2(1+\kappa^2 r^2)}. \tag{41}$$

Note here that the gauge choice

$$\Phi(0) = 0,$$

implies $n_0 = n(0)$.

In equating the ion and electron densities, we use the same convention as in the previous Section, omitting the ion subscript, and adopt a similarly maximal ordering:

$$\kappa_e^2 a_e \cong \kappa^2 a.$$

Indeed we introduce the parameter

$$\zeta \equiv \frac{m\kappa_e^2}{m_e \kappa^2} \tag{42}$$

which is analogous to, although distinct from, the ζ of Sec. III. We also allow for comparable ion and electron rotation frequencies,

$$\omega_e \cong \omega.$$

Since (33) shows that

$$a_e = -\frac{m}{m_e}a$$

we see that $\kappa_e^2 r^2 \cong m_e/m$ can be neglected compared to unity, and that $\omega_e r$ is negligible compared to a_e. Thus the electron version of (41) is

$$\xi_e(r) = -\tau v_t^{-2} a(r)(\zeta\kappa^2 r^2 a(r) + 2\omega_e r). \tag{43}$$

With these remarks, it is straightforward to solve the quasineutrality condition $n = n_e$ for $\Phi(r)$; one finds

$$(1+\tau)\frac{e\Phi(r)}{T} = \xi(r) - \xi_e(r) - \frac{1}{2}\log(1+\kappa^2 r^2). \tag{44}$$

The density is therefore given by

$$n(r) = n_0(1+\kappa^2 r^2)^{-\frac{\tau}{2(1+\tau)}} \exp\left(\frac{\xi_e + \tau\xi}{1+\tau}\right). \tag{45}$$

Notice that the vector potential appears quadratically in the exponent; we explicate this dependence by using (41) and (43) to write

$$\frac{\xi_e + \tau\xi}{1+\tau} = \xi_0(r) + \xi_1(r)a(r) + \xi_2(r)a^2(r) \tag{46}$$

with

$$\xi_0 = \frac{m\omega^2 r^2}{2(T+T_e)(1+\kappa^2 r^2)}$$

$$\xi_1 = \frac{mr[(1+\kappa^2 r^2)\omega_e - \omega]}{(T+T_e)(1+\kappa^2 r^2)}$$

$$\xi_2 = \frac{m\kappa^2 r^2(1+\zeta+\zeta\kappa^2 r^2)}{2(T+T_e)(1+\kappa^2 r^2)}.$$

Next we turn our attention to the v_θ–moment. Straightforward evaluation of the integral in (40) provides the flow speeds

$$(1+\kappa^2 r^2)u = \omega r - a(r)\kappa^2 r^2, \tag{47}$$

$$u_e = \omega_e r + \zeta a(r)\kappa^2 r^2. \tag{48}$$

Notice that ω measures the rigid–body rotation frequency in the Harris limit, $\kappa \to 0$, as anticipated. On the other hand comparison with (10) shows that the cylindrical case is considerably more complicated than slab geometry.

Thus we have explicit expressions for the density and for the flow speeds of both species. All that is needed to close the system is a single equation for the vector potential, $a(r)$. This is provided by Ampere's law, in the form

$$\widehat{\theta} \cdot \nabla \times (\nabla \times \boldsymbol{A}) = (4\pi/c) en(u - u_e).$$

Combining (45), (47), and (48) and normalizing in the usual way we find that

$$\begin{aligned} \delta^2 \left[\frac{(ra)'}{r} \right]' &= (1 + \kappa^2 r^2)^{-\frac{2+3\tau}{2(1+\tau)}} \exp\left(\xi_0 + \xi_1 a + \xi_2 a^2\right) \\ &\times \left\{ \omega_e r (1 + \kappa^2 r^2) - \omega r + a \kappa^2 r^2 [\zeta(1 + \kappa^2 r^2) - 1] \right\}. \end{aligned} \tag{49}$$

Equation (49), complicated as it is, can be readily solved using Mathematica. Once we know the vector potential a, all quantities of physical interest can be readily computed.

It turns out that the rigid–rotor cylindrical equilibrium is analytically solvable like its Cartesian counterpart, the constant flow speed equilibrium. The simplified system can be cast in the following suggestion form

$$\frac{1}{N} \frac{1}{r} \frac{dN}{dr} = -b + h \tag{50}$$

$$\frac{1}{r} \frac{db}{dr} = N, \tag{51}$$

where N and b are the appropriately normalized density and the z component of the magnetic field. The constant h is a measure of the centrifugal force ($\sim \delta^2 \omega^2 / v_t^2$), a purely cylindrical effect. For the boundary conditions that at $r = r_0$, $N = 1$ and $b = 0$ (the equivalent Cartesian sheet located at $r = r_0$), the exact solution is

$$\begin{aligned} b &= h + (2 + h^2)^{1/2} \tanh \left[\frac{(2 + h^2)^{1/2}}{4} (r^2 - r_0^2) - \tanh^{-1} \frac{h}{(2 + h^2)^{1/2}} \right] \\ N &= \frac{1}{2} (2 + h^2) \mathrm{sech}^2 \left[\frac{(2 + h^2)^{1/2}}{4} (r^2 - r_0^2) - \tan^{-1} \frac{h}{(2 + h^2)^{1/2}} \right], \end{aligned}$$

reminding us of the Cartesian analogue. Notice that the centrifugal force term modifies the effective field and distorts the symmetric character of the equivalent Cartesian sheet. For $r_0 = 0$, the physical fields vary as $\tanh r^2$ and $\mathrm{sech}^2\, r^2$ implying that the cylindrical geometry forces a stronger density fall–off in the radial direction. For more general velocity fields, Eq. (49) has to be solved.

Extensive numerical calculations show that the cylindrical effects, though quantitatively very strong, leave the qualitative character of the various Cartesian configurations unchanged. Their importance in detailed comparisons with the experiments like MRX, cannot be, however, underestimated. Some typical solutions are displayed in Figs. 5–6.

The primary aim of this paper was to investigate the properties of a Harris–sheet–like solution for a near–Maxwellian sheared plasma in both the Cartesian and cylindrical geometries. The results of our investigation can be summarized as follows:

(1) Sheared flows are possible if and only if the distribution function is agyrotropic. A temperature anisotropy, for example, can lead to a sheared flow.

(2) In addition to the extended Harris–sheet structures, the sheared velocity fields can generate a totally distinct magnetic configuration in which the magnetic field provides a localized well, instead of a discontinuity associated with the Harris sheet. In the Cartesian geometry this pertains when the velocity field has odd parity.

(3) The effect of the sheared field has a 'direction' — if for a given sign of the anisotropy parameter, the sheared field tends to fortify the confining capabilities of the magnetic field, it tends to decrease them for the opposite sign. In fact, for the examples given in the paper, for positive α_*, the solutions become periodic, and the confinement is lost. This case needs further investigation.

(4) The effects of the cylindrical geometry can be quantitatively quite strong. They must be taken into consideration for a detailed comparison with the experiment.

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REFERENCES

[1] E.G. Harris, Nuovo Cimento **23**, 117 (1962).

[2] M. Yamada, H. Ji, S. Hsu, T. Carter, R. Kulsrud, N. Bretz, F. Jodes, Y. Ono, and F. Perkins, Phys. Plasma **4**, 1936 (1997).

[3] M. Yamada, J. Geophys. Res. **104**, 14,529 (1999).

[4] G.F. Chew, M.L. Goldberger, and F.E. Low, Proc. Roy. Soc. A (London) **236**, 112 (1956).

[5] S.M. Mahajan, Phys. Fluids B **1**(1), 43 (1989).

[6] R.D. Hazeltine and F.L. Waelbroeck, *The Framework of Plasma Physics*, (Perseus Books, Reading, Mass, 1998).

[7] S.M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **71**, 4863 (1998).

[8] Daniel H. E. Dubin and T. M. O'Neil, Rev. Mod. Phys. **71**, 87 (1999).