BATALIN-VILKOVISKY STRUCTURES ON Ext AND Tor

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ABSTRACT. This article studies the algebraic structure of homology theories defined by a left Hopf algebroid U over a possibly noncommutative base algebra A, such as for example Hochschild, Lie algebroid (in particular Lie algebra and Poisson), or group and étale groupoid (co)homology. Explicit formulae for the canonical Gerstenhaber algebra structure on $\mathrm{Ext}_U(A, A)$ are given. The main technical result constructs a Lie derivative satisfying a generalised Cartan-Rinehart homotopy formula whose essence is that $\mathrm{Tor}^U(M, A)$ becomes for suitable right U-modules M a Batalin-Vilkovisky module over $\mathrm{Ext}_U(A, A)$, or in the words of Nest, Tamarkin, Tsygan and others, that $\mathrm{Ext}_U(A, A)$ and $\mathrm{Tor}^U(M, A)$ form a differential calculus. As an illustration, we show how the well-known operators from differential geometry in the classical Cartan homotopy formula can be obtained. Another application consists in generalising Ginzburg's result that the cohomology ring of a Calabi-Yau algebra is a Batalin-Vilkovisky algebra to twisted Calabi-Yau algebras.

CONTENTS

1. Introduction

1.1. **Differential calculi.** By its definition in terms of (co)chain complexes or derived functors, the cohomology or homology of a mathematical object is typically only a graded module over some base ring. Thus an obvious task is to exhibit its full algebraic structure, and to understand which features of the original object this structure reflects.

For the (co)homology of associative algebras, this has been studied, amongst others, by Rinehart [Ri], Gerstenhaber [Ge], Goodwillie [Go], Getzler [Get] and Nest, Tamarkin and Tsygan, see e.g. [NTs3, TaTs1, TaTs2, Ts]. The ultimate answer is that Hochschild cohomology and homology form what Nest, Tamarkin and Tsygan call a differential calculus:

Definition 1.1. Let k be a commutative ring.

(*i*) A *Gerstenhaber algebra* over k is a graded commutative k-algebra $(V, \smile)$

$$V = \bigoplus_{p \in \mathbb{Z}} V^p, \quad \alpha \smile \beta = (-1)^{pq} \beta \smile \alpha \in V^{p+q}, \quad \alpha \in V^p, \beta \in V^q,$$

with a graded Lie bracket $\{\cdot,\cdot\} : V^{p+1} \otimes_k V^{q+1} \to V^{p+q+1}$ on the *desuspension*

$$V[1] := \bigoplus_{p \in \mathbb{Z}} V^{p+1}$$

of V for which all operators $\{\gamma, \cdot\}$ satisfy the graded Leibniz rule

$$\{\gamma, \alpha \smile \beta\} = \{\gamma, \alpha\} \smile \beta + (-1)^{pq} \alpha \smile \{\gamma, \beta\}, \quad \gamma \in V^{p+1}, \alpha \in V^q.$$

(*ii*) A *Gerstenhaber module* over V is a graded $(V, \smile)$-module $(\Omega, \frown)$,

$$\Omega = \bigoplus_{n \in \mathbb{Z}} \Omega_n, \quad \alpha \frown x \in \Omega_{n-p}, \quad \alpha \in V^p, x \in \Omega_n,$$

with a representation of the graded Lie algebra $(V[1], \{\cdot,\cdot\})$

$$\mathcal{L} : V^{p+1} \otimes_k \Omega_n \to \Omega_{n-p}, \quad \alpha \otimes_k x \mapsto \mathcal{L}_\alpha(x),$$

which satisfies for $\alpha \in V^{p+1}, \beta \in V^q, x \in \Omega$ the mixed Leibniz rule

$$\beta \frown \mathcal{L}_\alpha(x) = \{\beta, \alpha\} \frown x + (-1)^{pq} \mathcal{L}_\alpha(\beta \frown x).$$

(*iii*) Such a module is *Batalin-Vilkovisky* if there is a k-linear differential

$$\mathsf{B} : \Omega_n \to \Omega_{n+1}, \quad \mathsf{BB} = 0,$$

such that $\mathcal{L}_\alpha$ is for $\alpha \in V^p$ given by the homotopy formula

$$\mathcal{L}_\alpha(x) = \mathsf{B}(\alpha \frown x) - (-1)^p \alpha \frown \mathsf{B}(x).$$

(*iv*) A pair (V, Ω) of a Gerstenhaber algebra and of a Batalin-Vilkovisky module over it is also called a *differential calculus*.

Be aware that the term "Gerstenhaber module" is used in several different ways in the literature. The above one is based on the requirement that the operators

$$\iota_\alpha := \alpha \frown \cdot : \Omega \to \Omega$$

form a Gerstenhaber algebra quotient of V with bracket

$$\{\iota_\alpha, \iota_\beta\} := [\iota_\alpha, \mathcal{L}_\beta]$$

and agrees (up to slightly different sign conventions) with the one used in [DeHeKa]. One will often additionally find that the mixed Leibniz rule

$$\mathcal{L}_{\alpha \smile \beta} = \mathcal{L}_\alpha \iota_\beta + (-1)^i \iota_\alpha \mathcal{L}_\beta, \quad \alpha \in V^i, \beta \in V, \tag{1.1}$$

is demanded. This is necessary for $V \oplus \Omega$ to become naturally a (square zero) extension of V as a Gerstenhaber algebra. For Batalin-Vilkovisky modules, Equation (1.1) is satisfied automatically, so the definition of these is essentially unequivocal.

The definition of a Gerstenhaber algebra itself also admits a modification in which the operators $\{\cdot, \gamma\}$, rather than $\{\gamma, \cdot\}$, are assumed to satisfy the graded Leibniz rule. This had been the convention in Gerstenhaber's original paper [Ge], cf. Remark 3.19 below.

1.2. **Aims and objectives.** The main aim of this paper is to further highlight the ubiquity of such Batalin-Vilkovisky structures by giving conditions for

$$V := \mathrm{Ext}_U(A, A), \quad \Omega := \mathrm{Tor}^U(M, A)$$

to form a differential calculus when U is a left Hopf algebroid (a $\times_A$-Hopf algebra) over a possibly noncommutative k-algebra A; we will recall some background on left Hopf algebroids in §2 below. Here we only remind the reader that the rings governing most parts of classical homological algebra all carry this structure, so that our results apply for example to Hochschild and Lie-Rinehart (in particular Lie algebra, de Rham, Lie algebroid and Poisson) (co)homology as well as to that of any Hopf algebra (e.g. group (co)homology).

Besides for the case of Hochschild (co)homology with canonical coefficients $M = A$ that has been referred to above, our results are also already known for Lie-Rinehart (co)homology due to the work of Rinehart and of Huebschmann [Ri, Hue1]. However, the Hopf algebroid generalisation is, in our opinion, not only interesting because of new special cases to which it applies, but also leads to conceptually clearer statements and proofs, for instance because of the manifest distinction of homology and cohomology coefficients (right respectively left U-modules). Hence we hope that the paper is of interest also to people working in different but analogous settings in algebra, geometry and topology, see e.g. [BeFa, GiTr, Me1, Me2, DoShV] and the references therein.

1.3. **The Gerstenhaber algebra.** The Gerstenhaber algebra structure that we consider can be viewed as a special case of Menichi's operadic construction [Me1] that, in turn, closely follows Gerstenhaber's original work on Hochschild cohomology [Ge], or of Shoikhet's generalisation [Sho] of Schwede's homotopy theory approach to the Hochschild case [Schw]. Both imply that the derived endomorphisms $\mathrm{Ext}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ of the unit object $\mathbb{1}$ of a mildly restricted abelian monoidal category $\mathcal{C}$ carry a natural Gerstenhaber algebra structure. So, morally speaking, it is a monoidal structure on the category of coefficients that is reflected by a Gerstenhaber algebra structure on cohomology.

The aim of §3 below is to compute for $\mathcal{C} = U\text{-}\mathbf{Mod}$ (where U is any bialgebroid) explicit formulae for $\smile$ and $\{\cdot, \cdot\}$ in terms of the canonical cochain complex

$$\delta : C^\bullet(U, A) := \mathrm{Hom}_{A^{\mathrm{op}}}(U^{\otimes_{A^{\mathrm{op}}} \bullet}{}_{\triangleleft}, A) \to C^{\bullet+1}(U, A)$$

that arises from the bar resolution of A. We refer to the main text for the notation used here and below, but decided to copy all formulae into the introduction.

On the level of cochains $\varphi \in C^p(U, A), \psi \in C^q(U, A)$ the cup product turns out to be

$$(\varphi \smile \psi)(u^1, \dots, u^{p+q}) = \varphi\big(u^1, \dots, u^{p-1}, \psi(u^{p+1}, \dots, u^{p+q}) \blacktriangleright u^p\big). \tag{1.2}$$

We then define along the classical lines *Gerstenhaber products* $\circ_i$ by

$$\begin{aligned}&(\varphi \circ_i \psi)(u^1, \dots, u^{p+q-1}) \\ &\quad := \varphi(u^1, \dots, u^{i-1}, \mathsf{D}_\psi(u^i, \dots, u^{i+q-1}), u^{i+q}, \dots, u^{p+q-1}),\end{aligned}$$

for $i = 1, \dots, p$, where the operator

$$\mathsf{D}_\varphi : U^{\otimes_{A^{\mathrm{op}}} p} \to U, \quad (u^1, \dots, u^p) \mapsto \varphi(u^1_{(1)}, \dots, u^p_{(1)}) \triangleright u^1_{(2)} \cdots u^p_{(2)}$$

replaces the classical insertion operations used by Gerstenhaber. The $\circ_i$ are used to construct the Gerstenhaber bracket as usual as

$$\{\varphi, \psi\} := \varphi \bar{\circ} \psi - (-1)^{|p||q|} \psi \bar{\circ} \varphi \tag{1.3}$$

with

$$\varphi \bar{\circ} \psi := (-1)^{|p||q|} \sum_{i=1}^{p} (-1)^{|q||i|} \varphi \circ_i \psi, \quad |n| := n - 1.$$

In §3 we will prove:

Theorem 1.2. *If U is a bialgebroid over A, then the maps* (*1.2*) *and* (*1.3*) *induce a Gerstenhaber algebra structure on $H^\bullet(U, A) := H^\bullet(C^\bullet(U, A), \delta)$.*

When U is a left Hopf algebroid and $U_\triangleleft \in A^{\mathrm{op}}$-$\mathbf{Mod}$ is projective, the bar resolution is a projective resolution, so $H^\bullet(U, A) \simeq \mathrm{Ext}_U(A, A)$ and the above result yields Gerstenhaber brackets on various Ext-algebras. Even for Hopf algebras (i.e., for $A = k$) this has been discussed still fairly recently, see e.g. [FSo, Tai, Me2].

1.4. **The Gerstenhaber module.** In [KoKr2] we have studied the fact that for a left Hopf algebroid U a left U-comodule structure on a right U-module M induces a para-cyclic k-module structure on the canonical chain complex

$$C_\bullet(U, M) := M \otimes_{A^{\mathrm{op}}} ({}_\blacktriangleright U_\triangleleft)^{\otimes_{A^{\mathrm{op}}} \bullet}$$

that computes $\mathrm{Tor}^U(M, A)$ when U is a right A-projective.

The question whether this leads to a Batalin-Vilkovisky module structure on the simplicial homology $H_\bullet(U, M)$ of this para-cyclic object hinges on the compatibility between the left U-comodule and the right U-module structure on M. In full generality, we define for $\varphi \in C^p(U, A)$ the *cap product*

$$\iota_\varphi(m, u^1, \dots, u^n) = (m, u^1, \dots, u^{n-p-1}, \varphi(u^{n-|p|}, \dots, u^n) \blacktriangleright u^{n-p}), \tag{1.4}$$

and the *Lie derivative* (see the main text for all necessary details)

$$\mathcal{L}_\varphi := \sum_{i=1}^{n-|p|} (-1)^{\theta_i^{n,p}} \mathsf{t}^{n-|p|-i} \, \mathsf{D}'_\varphi \, \mathsf{t}^{i+p} + \sum_{i=1}^{p} (-1)^{\xi_i^{n,p}} \mathsf{t}^{n-|p|} \, \mathsf{D}'_\varphi \, \mathsf{t}^i, \tag{1.5}$$

where θ and ξ are sign functions, D'_φ is D_φ applied on the last p components of an element in $C_n(U, M)$, and t is the cyclic operator of the para-cyclic module $C_\bullet(U, M)$ as in (2.15).

In general, these do not induce a Gerstenhaber module structure on $H_\bullet(U, M)$, but only on the homology $H^M_\bullet(U)$ of the universal cyclic quotient $C^{\mathrm{cyc}}_\bullet(U, M)$, see §2.4. A sufficient condition for the two to coincide is that M is a stable anti Yetter-Drinfel'd module in which case the para-cyclic k-module is cyclic, see again §2.4 and §4.2 below. However, a more general case that is ubiquitous in examples is the following:

Definition 1.3. A para-cyclic k-module $(C_\bullet, \mathsf{d}_\bullet, \mathsf{s}_\bullet, \mathsf{t}_\bullet)$ is *quasi-cyclic* if we have

$$C_\bullet = \ker(\mathrm{id} - \mathsf{t}_\bullet^{\bullet+1}) \oplus \mathrm{im}(\mathrm{id} - \mathsf{t}_\bullet^{\bullet+1}).$$

We refer to §2.4 for the detailed explanation of this condition and of its consequences. In complete generality, we introduce for any module-comodule M (see Definition 2.3) the set $C^\bullet_M(U) \subseteq C^\bullet(U, A)$ consisting of those cochains for which the operators ι_φ and $\mathcal{L}_\varphi$ descend to $C^{\rm cyc}_\bullet(U, M)$. This turns out to be a subcomplex whose cohomology will be denoted by $H^\bullet_M(U)$. Then we prove:

Theorem 1.4. *For all modules-comodules M over a left Hopf algebroid U, (1.2) and (1.3) induce a Gerstenhaber algebra structure on $H^\bullet_M(U)$, and (1.4) and (1.5) induce a $H^\bullet_M(U)$-Gerstenhaber module structure on $H^M_\bullet(U)$.*

1.5. **The Batalin-Vilkovisky module.** Once this is established, we introduce the operator

$$\mathsf{S}_\varphi := \sum_{j=0}^{n-p} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}} \mathsf{t}\mathsf{s}_{n-|p|}\, \mathsf{t}^{n-p-i}\, \mathsf{D}'_\varphi\, \mathsf{t}^{n+i-|j|},$$

where η is again a sign function, and prove that for $\varphi \in C^\bullet_M(U)$ the *Cartan-Rinehart homotopy formula*

$$\mathcal{L}_\varphi = [\mathsf{B} + \mathsf{b}, \mathsf{S}_\varphi + \iota_\varphi] - \iota_{\delta\varphi} - \mathsf{S}_{\delta\varphi}$$

is satisfied. Here b and B are the simplicial resp. cyclic differentials on $C^{\rm cyc}_\bullet(U, M)$ and δ is the cosimplicial differential on $C^\bullet_M(U)$. This implies our main result:

Theorem 1.5. *For all module-comodules M over a left Hopf algebroid U, the pair $(H^\bullet_M(U), H^M_\bullet(U))$ carries a canonical structure of a differential calculus.*

In the simplest case where M is an SaYD module, we already mentioned that $C^{\rm cyc}_\bullet(U, M)$ coincides with $C_\bullet(U, M)$, and therefore we obtain:

Corollary 1.6. *If M is a stable anti Yetter-Drinfel'd module over a left Hopf algebroid U and if $U_\triangleleft \in A^{\rm op}$-$\mathbf{Mod}$ is projective, then the pair $\big(\mathrm{Ext}_U(A, A), \mathrm{Tor}^U(M, A)\big)$ carries a canonical structure of a differential calculus.*

For the special case of commutative associative algebras, the earliest account of the set of operators $\mathsf{b}, \mathsf{B}, \iota, \mathcal{L}$, and S is due to Rinehart [Ri], where these operators are called (in the same order) Δ, $\bar{d}$, c, θ, and f, respectively. About twenty years later, the commutativity assumption was dropped and the Lie derivative appeared for 1-cocycles in [Co, p. 124], where it is denoted by δ^*, and in [Go], where additionally the operators ι and S are introduced, denoted by e and E, respectively. Finally, these operators were generalised from 1-cocycles to arbitrary cochains both in [Get], where they are denoted by $\mathbf{b}$ and $\mathbf{B}$, as well as in [GDTs, NTs3, NTs2, Ts], the notation of which we take over.

1.6. **Applications.** A prominent example that forces one to go beyond SaYD modules is that of the Hochschild homology of an algebra A with coefficients in $M = A_\sigma$ for some automorphism σ of A, that is, M is A as a k-module with A-bimodule structure given by $a \blacktriangleright b \blacktriangleleft c := ab\sigma(c)$. Whenever σ is semisimple, the resulting para-cyclic k-module is quasi-cyclic, and in the final section of the paper we prove that this implies the following generalisation of a result of Ginzburg [Gi] from Calabi-Yau algebras (which form the case in which σ is inner) to twisted Calabi-Yau algebras (see Definition 7.5), such as the standard quantum groups [BrZh], Koszul algebras whose Koszul dual is Frobenius as, for example, Manin's quantum plane [VdB1], or the Podleś quantum 2-sphere [Kr]:

Theorem 1.7. *If A is a twisted Calabi-Yau algebra with semisimple modular automorphism, then the Hochschild cohomology $H^\bullet(A, A)$ of A is a Batalin-Vilkovisky algebra.*

Besides this application, we also explain in the penultimate section of the paper how one can use our formulae to obtain the classical operators in Cartan's *magic* formula in differential geometry, i.e., the Lie derivative, the insertion operator, and the de Rham differential in the setting of Lie-Rinehart algebras (or Lie algebroids, and in particular the tangent bundle of a smooth manifold) by taking for U the *jet space* JL, which is the dual of the universal enveloping algebra VL of a Lie-Rinehart algebra (A, L).

Acknowledgements. It is our pleasure to thank Ryszard Nest, Boris Shoikhet, and Boris Tsygan for inspiring discussions and explaining to us some aspects of their work. N.K. acknowledges funding by the Excellence Network of the University of Granada (GENIL) and would like to thank the University of Glasgow for hospitality and support. U.K. furthermore acknowledges funding by the Polish Government Grant N201 1770 33, the Marie Curie PIRSES-GA-2008-230836 network and the Royal Society - Russian Foundation for Basic Research Grant of Misha Feigin and Dima Vasiliev, and thanks ITEP Moscow for hospitality.

2. Preliminaries

In this section we recall preliminaries on bialgebroids, Hopf algebroids, and cyclic homology, mainly from our two papers [KoKr1, KoKr2] as we use therein the same notation and conventions as here. For more detailed information on bialgebroids and Hopf algebroids and references to the original sources, we recommend Böhm's survey [B].

2.1. **Bialgebroids.** Throughout this paper, A and U are (unital associative) k-algebras, and we assume that there is a fixed k-algebra map

$$\eta : A^{\mathrm{e}} := A \otimes_k A^{\mathrm{op}} \to U.$$

This induces forgetful functors

$$U\text{-}\mathbf{Mod} \to A^{\mathrm{e}}\text{-}\mathbf{Mod}, \quad U^{\mathrm{op}}\text{-}\mathbf{Mod} \to A^{\mathrm{e}}\text{-}\mathbf{Mod}$$

that turn left U-modules N respectively right U-modules M into A-bimodules with actions

$$a \triangleright n \triangleleft b := \eta(a \otimes_k b)n, \quad a \blacktriangleright m \blacktriangleleft b := m\eta(b \otimes_k a), \quad a, b \in A, n \in N, m \in M.$$

In particular, left and right multiplication in U defines A-bimodule structures of both these types on U itself. Unless explicitly stated otherwise, we a priori consider U as an A-bimodule using the actions $\triangleright, \triangleleft$ arising from left multiplication in U. For example, in (2.1) below the actions $\triangleright, \triangleleft$ are used to define $U \otimes_A U$, and later we will require U to be right A-projective meaning that $U_\triangleleft \in A^{\mathrm{op}}\text{-}\mathbf{Mod}$ is projective.

Generalising the standard result for bialgebras (which is the case $A = k$), Schauenburg has proved [Sch] that the monoidal structures on U-$\mathbf{Mod}$ for which the forgetful functor to A^{e}-$\mathbf{Mod}$ is strictly monoidal (where A^{e}-$\mathbf{Mod}$ is monoidal via $\otimes_A$) correspond to what is known as *(left) bialgebroid* (or $\times_A$-bialgebra) structures on U. We refer, e.g., to our earlier paper [KoKr1] for a detailed definition (which is due to Takeuchi [Tak]). Let us only recall that a bialgebroid has a coproduct and a counit

$$\Delta : U \to U \otimes_A U, \quad \varepsilon : U \to A, \tag{2.1}$$

which turn U into a coalgebra in A^{e}-$\mathbf{Mod}$. One of the subtleties to keep in mind is that unlike for $A = k$ the counit ε is not necessarily a ring homomorphism but only yields a left U-module structure on A with action of $u \in U$ on $a \in A$ given by $ua := \varepsilon(u \blacktriangleleft a)$. Furthermore, Δ is required to corestrict to a map from U to the Sweedler-Takeuchi product $U \times_A U$, which is the A^{e}-submodule of $U \otimes_A U$ whose elements $\sum_i u_i \otimes_A v_i$ fulfil

$$\textstyle\sum_i a \blacktriangleright u_i \otimes_A v_i = \sum_i u_i \otimes_A v_i \blacktriangleleft a, \ \forall a \in A. \tag{2.2}$$

In the sequel, we will freely use Sweedler's notation $\Delta(u) =: u_{(1)} \otimes_A u_{(2)}$.

2.2. **Hopf algebroids.** In the same paper [Sch], Schauenburg generalised the notion of a Hopf algebra to the bialgebroid setting. What he called $\times_A$-Hopf algebras will be called *left Hopf algebroids* here. Again, we refer to [KoKr1] for the definition, examples and more background information, and only recall that the crucial piece of structure (in addition to a bialgebroid one) is the so-called *translation map*

$$U \to {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, \tag{2.3}$$

for which we use the Sweedler-type notation

$$u \mapsto u_+ \otimes_{A^{\mathrm{op}}} u_-.$$

Example 2.1. For a Hopf algebra over $A = k$, the translation map is given by

$$u \mapsto u_{(1)} \otimes_k S(u_{(2)}),$$

where S is the antipode, and its relevance is already discussed in great detail by Cartan and Eilenberg [CE].

We will make permanent use of the following identities that hold for the map (2.3), see [Sch, Proposition 3.7]:

Proposition 2.2. *Let U be a left Hopf algebroid over A. For all $u, v \in U$, $a, b \in A$ one has*

$$\begin{aligned}
u_+ \otimes_{A^{\mathrm{op}}} u_- &\in U \times_{A^{\mathrm{op}}} U, && (2.4)\\
u_{+(1)} \otimes_A u_{+(2)} u_- &= u \otimes_A 1 \in U_{\triangleleft} \otimes_A {}_{\triangleright}U, && (2.5)\\
u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} u_{(2)} &= u \otimes_{A^{\mathrm{op}}} 1 \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, && (2.6)\\
u_{+(1)} \otimes_A u_{+(2)} \otimes_{A^{\mathrm{op}}} u_- &= u_{(1)} \otimes_A u_{(2)+} \otimes_{A^{\mathrm{op}}} u_{(2)-}, && (2.7)\\
u_+ \otimes_{A^{\mathrm{op}}} u_{-(1)} \otimes_A u_{-(2)} &= u_{++} \otimes_{A^{\mathrm{op}}} u_- \otimes_A u_{+-}, && (2.8)\\
(uv)_+ \otimes_{A^{\mathrm{op}}} (uv)_- &= u_+ v_+ \otimes_{A^{\mathrm{op}}} v_- u_-, && (2.9)\\
u_+ u_- &= s(\varepsilon(u)), && (2.10)\\
\varepsilon(u_-) \blacktriangleright u_+ &= u, && (2.11)\\
(s(a)t(b))_+ \otimes_{A^{\mathrm{op}}} (s(a)t(b))_- &= s(a) \otimes_{A^{\mathrm{op}}} s(b), && (2.12)
\end{aligned}$$

where in (2.4) we mean the Sweedler-Takeuchi product

$$U \times_{A^{\mathrm{op}}} U := \left\{ \textstyle\sum_i u_i \otimes_{A^{\mathrm{op}}} v_i \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \mid \sum_i u_i \triangleleft a \otimes_{A^{\mathrm{op}}} v_i = \sum_i u_i \otimes_{A^{\mathrm{op}}} a \blacktriangleright v_i \right\},$$

which is an algebra by factorwise multiplication, but with opposite multiplication on the second factor, and where in (2.10) and (2.12) we use the source *and* target *maps*

$$s, t : A \to U, \quad s(a) := \eta(a \otimes_k 1), \quad t(b) := \eta(1 \otimes_k b). \tag{2.13}$$

For us, the relevance of the translation map stems mostly from the fact that it turns the category U^{op}-$\mathbf{Mod}$ of right U-modules into a module category over the monoidal category U-$\mathbf{Mod}$. Explicitly, the product of $N \in U$-$\mathbf{Mod}$ with $M \in U^{\mathrm{op}}$-$\mathbf{Mod}$ is the tensor product of the underlying A-bimodules with right action given by

$$(n \otimes_A m)u := u_- n \otimes_A m u_+, \quad u \in U, m \in M, n \in N.$$

2.3. **Module-comodules and anti Yetter-Drinfel'd modules.** Throughout this paper, M will denote a right U-module, and in fact one which is simultaneously a comodule:

Definition 2.3. By a *module-comodule* (with compatible induced left A-action) over a bialgebroid U we shall mean a right U-module $M \in U^{\mathrm{op}}$-$\mathbf{Mod}$ for which the underlying left A-module ${}_{\blacktriangleright}M$ is also equipped with a left U-coaction

$$\Delta_M : M \to U_{\triangleleft} \otimes_A {}_{\blacktriangleright}M, \quad m \mapsto m_{(-1)} \otimes_A m_{(0)}.$$

Recall, e.g. from [B], that Δ_M is then an A^e-module morphism $M \to U_{\triangleleft} \times_A {}_{\blacktriangleright}M$, where $U_{\triangleleft} \times_A {}_{\blacktriangleright}M$ is the A^e-submodule of $U_{\triangleleft} \otimes_A {}_{\blacktriangleright}M$ whose elements $\sum_i u_i \otimes_A m_i$ fulfil

$$\textstyle\sum_i a \blacktriangleright u_i \otimes_A m_i = \sum_i u_i \otimes_A m_i \blacktriangleleft a, \ \forall a \in A. \tag{2.14}$$

The following particular class of module-comodules was introduced in [HKhRS] for Hopf algebras and in [BŞ] for left Hopf algebroids:

Definition 2.4. A module-comodule over a left Hopf algebroid is called an *anti Yetter-Drinfel'd module (aYD)* if the full A^e-module structure ${}_{\blacktriangleright}M_{\blacktriangleleft}$ of the module coincides with that underlying the comodule, and if one has

$$(mu)_{(-1)} \otimes_A (mu)_{(0)} = u_- m_{(-1)} u_{+(1)} \otimes_A m_{(0)} u_{+(2)}$$

for all $m \in M, u \in U$. A module-comodule is called *stable (SaYD)* if one has

$$m_{(0)} m_{(-1)} = m.$$

2.4. **The (para-)cyclic k-modules $C_\bullet(U, M)$ and $C^{\mathrm{cyc}}_\bullet(U, M)$.** The Batalin-Vilkovisky modules that we are going to study in this paper are obtained as the simplicial homology of para-cyclic k-modules of the following form [KoKr2]:

Proposition 2.5. *For every right module M over a bialgebroid U there is a well-defined simplicial k-module structure on*

$$C_\bullet(U, M) := M \otimes_{A^{\mathrm{op}}} ({}_{\blacktriangleright}U_{\triangleleft})^{\otimes_{A^{\mathrm{op}}} \bullet}$$

whose face and degeneracy maps are given by

$$\begin{aligned} \mathsf{d}_i(m, x) &= \begin{cases} (m, u^1, \ldots, \varepsilon(u^n) \blacktriangleright u^{n-1}), & \text{if } i = 0, \\ (m, \ldots, u^{n-i} u^{n-i+1}, \ldots, u^n) & \text{if } 1 \leqslant i \leqslant n-1, \\ (mu^1, u^2, \ldots, u^n) & \text{if } i = n, \end{cases} \\ \mathsf{s}_j(m, x) &= \begin{cases} (m, u^1, \ldots, u^n, 1) & \text{if } j = 0, \\ (m, \ldots, u^{n-j}, 1, u^{n-j+1}, \ldots, u^n) & \text{if } 1 \leqslant j \leqslant n-1, \\ (m, 1, u^1, \ldots, u^n) & \text{if } j = n, \end{cases} \end{aligned} \tag{2.15}$$

Here and in what follows, we denote the elementary tensors in $C_\bullet(U, M)$ by

$$(m, x) := (m, u^1, \ldots, u^n), \quad m \in M, u^1, \ldots, u^n \in U.$$

For a module-comodule M over a left Hopf algebroid U, the k-module $C_\bullet(U, M)$ becomes a para-cyclic k-module via

$$\mathsf{t}_n(m, x) = (m_{(0)} u^1_+, u^2_+, \ldots, u^n_+, u^n_- \cdots u^1_- m_{(-1)}). \tag{2.16}$$

This para-cyclic k-module is cyclic if M is a stable anti Yetter-Drinfel'd module.

Recall that this means that the operators $(\mathsf{d}_i, \mathsf{s}_j, \mathsf{t}_n)$ satisfy all the defining relations of a cyclic k-module in the sense of Connes (see e.g. [Co] or [L] for the definition of a cyclic k-module), except for the one that requires that

$$\mathsf{T}_n := \mathsf{t}_n^{n+1}$$

equals the identity (we do not even require it to be an isomorphism) which, as mentioned in the proposition, is only satisfied when M is an SaYD module.

The relations between the operators $(\mathsf{d}_i, \mathsf{s}_j, \mathsf{t}_n)$ imply that T_n commutes with all of them, so they descend to well-defined operators on the coinvariants

$$C^{\mathrm{cyc}}_\bullet(U, M) := C_\bullet(U, M)/\mathrm{im}\,(\mathrm{id} - \mathsf{T}_\bullet),$$

and hence this becomes a cyclic k-module.

In this paper, we will not study the cyclic homology of this object, but rather the simplicial homology of both $C_\bullet(U, M)$ and $C^{\mathrm{cyc}}_\bullet(U, M)$:

Definition 2.6. For any bialgebroid U and any $M \in U^{\mathrm{op}}$-**Mod**, we denote the simplicial homology of $C_\bullet(U, M)$, that is, the homology with respect to the boundary map

$$\mathsf{b} := \sum_{i=0}^{n} (-1)^i \mathsf{d}_i, \tag{2.17}$$

by $H_\bullet(U, M)$ and call it the *homology of U with coefficients in M*. For a module-comodule over a left Hopf algebroid, we denote the simplicial homology of $C_\bullet^{\mathrm{cyc}}(U, M)$ by $H_\bullet^M(U)$.

In general, $H_\bullet(U, M)$ differs from $H_\bullet^M(U)$, see [HaKr2] for an example. However, if $C_\bullet(U, M)$ is quasi-cyclic in the sense of Definition 1.3, we can apply [HaKr1, Proposition 2.1]:

Proposition 2.7. *If $C_\bullet$ is a quasi-cyclic k-module, then the canonical quotient map*

$$C_\bullet \to C_\bullet/\mathrm{im}(\mathrm{id} - \mathsf{t}_\bullet^{\bullet+1})$$

is a quasi-isomorphism of the chain complexes that are defined by the underlying simplicial k-module structures of $C_\bullet$ and $C_\bullet/\mathrm{im}(\mathrm{id} - \mathsf{t}_\bullet^{\bullet+1})$, respectively.

This means that if $C_\bullet(U, M)$ happens to be quasi-cyclic, then classes in $H_\bullet^M(U)$ can be represented by cycles in $C_\bullet(U, M)$ that are invariant under $\mathsf{T}_\bullet$.

Mostly, we will now work on the reduced (normalised) complexes of $C_\bullet(U, M)$ resp. of $C_\bullet^{\mathrm{cyc}}(U, M)$ by the subcomplex spanned by the images of the degeneracy maps of these simplicial k-modules. Being slightly sloppy, we will denote operators that descend from the original complexes to these quotients by the same symbols if no confusion can arise. Furthermore, we shall drop in all what follows the subscript on t and T indicating the degree of the element on which it acts.

2.5. **The operators $\mathsf{N}, \mathsf{s}_{-1}$ and B.** On every para-cyclic k-module, one defines the *norm operator*, the *extra degeneracy*, and the *cyclic differential*

$$\mathsf{N} := \sum_{i=0}^{n} (-1)^{in} \mathsf{t}^i, \qquad \mathsf{s}_{-1} := \mathsf{t}\,\mathsf{s}_n, \qquad \mathsf{B} = (\mathrm{id} - \mathsf{t})\,\mathsf{s}_{-1}\,\mathsf{N}. \tag{2.18}$$

Recall that B coincides on the reduced complex $\bar{C}_\bullet(U, M)$ with the map (induced by) $\mathsf{s}_{-1}\,\mathsf{N}$, so we are also slightly sloppy about this and denote the latter by B as well, as we, in fact, will only consider the induced map on the reduced complex.

It follows from the para-cyclic relations that one has

$$\mathsf{B}^2 = (\mathrm{id} - \mathsf{T})(\mathrm{id} - \mathsf{t})\mathsf{s}_{-1}\mathsf{s}_{-1}\mathsf{N}, \quad \mathsf{b}\mathsf{B} + \mathsf{B}\mathsf{b} = \mathrm{id} - \mathsf{T}, \tag{2.19}$$

so in general B does not turn $H_\bullet(U, M)$, but only $H_\bullet^M(U)$, into a cochain complex.

In the case of an SaYD module M one can give a compact expression for B: one first computes directly with the help of (2.5), (2.6), (2.7), and (2.8) the powers of t:

Lemma 2.8. *If M is an SaYD module, the i^{th} power for $1 \leqslant i \leqslant n$ of the cyclic operator t can be expressed as*

$$\mathsf{t}^i(m, x) = (m_{(0)} u^1_{+(2)} \cdots u^{i-1}_{+(2)} u^i_+, u^{i+1}_+, \ldots, u^n_+, u^n_- \cdots u^1_- m_{(-1)}, u^1_{+(1)}, \ldots, u^{i-1}_{+(1)}),$$

where we abbreviated here, as elsewhere, $(m, x) = (m, u^1, \ldots, u^n)$.

Then a further direct computation gives:

Lemma 2.9. *If M is an SaYD module, the action of $\mathsf{B} = \mathsf{s}_{-1}\mathsf{N}$ on $\bar{C}_\bullet(U, M)$ can be expressed as*

$$\begin{aligned} \mathsf{s}_{-1}\mathsf{N}(m, x) = \sum_{i=0}^{n} (-1)^{in} (m_{(0)} u^1_{+(2)} \cdots u^i_{+(2)}, u^{i+1}_+, \\ \ldots, u^n_+, u^n_- \cdots u^1_- m_{(-1)}, u^1_{+(1)}, \ldots, u^i_{+(1)}). \end{aligned} \tag{2.20}$$

Example 2.10. For $n = 1$, the above expression reduces to

$$\mathsf{s}_{-1}\mathsf{N}(m,u) = (m_{(0)}, u_+, u_- m_{(-1)}) - (m_{(0)}u_{+(2)}, u_- m_{(-1)}, u_{+(1)}).$$

In particular, for a Hopf algebra over $A = k$ this reads

$$\mathsf{s}_{-1}\mathsf{N}(m,u) = (m_{(0)}, u_{(1)}, S(u_{(2)})m_{(-1)}) - (m_{(0)}u_{(2)}, S(u_{(3)})m_{(-1)}, u_{(1)}).$$

3. The Gerstenhaber algebra

Unless stated explicitly otherwise, U is throughout this section an arbitrary left A-bialgebroid. We will first give explicit formulae for a canonical DG coalgebra structure Δ^P on the chain complex (P, b') that is obtained when applying the bar construction for the comonad $U \otimes_{A^{\mathrm{op}}} \cdot$ to the unit object $A \in U$-$\mathbf{Mod}$. Applying $\mathrm{Hom}_U(\cdot, A)$ to P yields a cochain complex $(C^\bullet(U,A), \delta)$. On the underlying graded vector space we define the structure of a (nonsymmetric) operad with multiplication. This, in particular, defines a DG algebra structure $(C^\bullet(U,A), \smile, \delta)$ and a Gerstenhaber algebra structure on its cohomology $H^\bullet(U,A)$. The fact that this DG algebra coincides with the one obtained by dualising the DG coalgebra structure on P proves that as long as U is a right A-projective left Hopf algebroid, $H^\bullet(U,A)$ is the cohomology ring $\mathrm{Ext}_U(A,A)$ that we studied in [KoKr1].

We will throughout use the convention in which DG algebras are cochain complexes while DG coalgebras and DG modules over DG algebras are chain complexes.

3.1. **The bar resolution P.** The bar construction for $U \otimes_{A^{\mathrm{op}}} \cdot$ applied to $A \in U$-$\mathbf{Mod}$ yields the chain complex $(P_\bullet, \mathsf{b}')$ of left U-modules, where

$$P_n := ({}_\blacktriangleright U_\triangleleft)^{\otimes_{A^{\mathrm{op}}} n+1}$$

is a U-module via left multiplication in the first tensor component, and b' is given by

$$\begin{aligned}\mathsf{b}'(u^0, \ldots, u^n) := &\sum_{i=0}^{n-1} (-1)^i (u^0, \ldots, u^i u^{i+1}, \ldots, u^n)\\ &+ (-1)^n (u^0, \ldots, u^{n-2}, \varepsilon(u^n) \blacktriangleright u^{n-1}).\end{aligned}$$

Note that the tensor product over A^{op} is chosen in such a way that

$$(u^0, \ldots, a \blacktriangleright u^i, u^{i+1}, \ldots, u^n) = (u^0, \ldots, u^i, u^{i+1} \triangleleft a, \ldots, u^n)$$

holds, which is necessary for b' to be well-defined. We recall [KoKr1, Lemma 2]:

Lemma 3.1. *If U is a left Hopf algebroid and $U_\triangleleft \in A^{\mathrm{op}}$-$\mathbf{Mod}$ is projective, then $(P_\bullet, \mathsf{b}')$ is a projective resolution of $A \in U$-$\mathbf{Mod}$.*

3.2. **The DG coalgebra structure on P.** As U-$\mathbf{Mod}$ is monoidal, so is the category of chain complexes of U-modules and our aim is to turn P into a coalgebra in this category.

Definition 3.2. We define

$$\Delta^P : P \to P \otimes_A P, \quad \Delta^P(u^0, \ldots, u^n) := \sum_{i=0}^{n} \Delta^P_{ni}(u^0, \ldots, u^n),$$

where for $i = 0, \ldots, n$ the maps $\Delta^P_{ni} : P_n \to P_i \otimes_A P_{n-i}$ are given by

$$(u^0, \ldots, u^n) \mapsto (u^0_{(1)}, \ldots, u^i_{(1)}) \otimes_A (u^0_{(2)} \cdots u^i_{(2)}, u^{i+1}, \ldots, u^n).$$

We verify by direct computation:

Lemma 3.3. *Δ^P is coassociative.*

Proof. For $j = 0, \ldots, i$, we have

$$
\begin{aligned}
&((\Delta^P_{ij} \otimes_A \mathrm{id}_{P_{n-i}})\, \Delta^P_{ni})(u^0, \ldots, u^n) \\
&= (u^0_{(1)}, \ldots, u^j_{(1)}) \otimes_A (u^0_{(2)} \cdots u^j_{(2)}, u^{j+1}_{(1)}, \ldots, u^i_{(1)}) \\
&\quad \otimes_A (u^0_{(3)} \cdots u^j_{(3)} u^{j+1}_{(2)} \cdots u^i_{(2)}, u^{i+1}, \ldots, u^n),
\end{aligned}
$$

and for $j = 0, \ldots, n-i$, we have

$$
\begin{aligned}
&((\mathrm{id}_{P_i} \otimes_A \Delta^P_{n-ij})\, \Delta^P_{ni})(u^0, \ldots, u^n) \\
&= (u^0_{(1)}, \ldots, u^i_{(1)}) \otimes_A (u^0_{(2)} \cdots u^i_{(2)}, u^{i+1}_{(1)}, \ldots, u^{i+j}_{(1)}) \\
&\quad \otimes_A (u^0_{(3)} \cdots u^i_{(3)} u^{i+1}_{(2)} \cdots u^{i+j}_{(2)}, u^{i+j+1}, \ldots, u^n).
\end{aligned}
$$

So for Δ^P to be coassociative, we need

$$
\begin{aligned}
&\sum_{i=0}^{n} \sum_{j=0}^{i} (u^0_{(1)}, \ldots, u^j_{(1)}) \otimes_A (u^0_{(2)} \cdots u^j_{(2)}, u^{j+1}_{(1)}, \ldots, u^i_{(1)}) \\
&\qquad\qquad \otimes_A (u^0_{(3)} \cdots u^j_{(3)} u^{j+1}_{(2)} \cdots u^i_{(2)}, u^{i+1}, \ldots, u^n) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n-r} (u^0_{(1)}, \ldots, u^r_{(1)}) \otimes_A (u^0_{(2)} \cdots u^r_{(2)}, u^{r+1}_{(1)}, \ldots, u^{r+s}_{(1)}) \\
&\qquad\qquad \otimes_A (u^0_{(3)} \cdots u^r_{(3)} u^{r+1}_{(2)} \cdots u^{r+s}_{(2)}, u^{r+s+1}, \ldots, u^n),
\end{aligned}
$$

which is seen to be correct by some basic substitution in the indices, writing first

$$\sum_{i=0}^{n} \sum_{j=0}^{i} = \sum_{j=0}^{n} \sum_{i=j}^{n},$$

and then substituting j by r and i by $s = i - j$. □

Proposition 3.4. *If we define*

$$\varepsilon^P := \varepsilon : P_0 = U \to A$$

and $\varepsilon^P|_{P_n} = 0$ *for* $n > 0$*, then* $(P, \mathsf{b}', \Delta^P, \varepsilon^P)$ *is a differential graded coalgebra.*

Proof. Both the counit property and the Leibniz rule

$$\Delta^P\, \mathsf{b}' = (\mathsf{b}' \otimes_A \mathrm{id}_P + \mathrm{id}_P \otimes_A \mathsf{b}')\, \Delta^P \tag{3.1}$$

are easily verified. We only remark that the above Equation (3.1) is meant to be interpreted using the Koszul sign convention, meaning that we have for all $c \in P_p, d \in P_q$

$$(\mathrm{id}_P \otimes_A \mathsf{b}')(c \otimes_A d) = (-1)^p c \otimes_A \mathsf{b}'(d),$$

but $(\mathsf{b}' \otimes_A \mathrm{id}_P)(c \otimes_A d) = \mathsf{b}'(c) \otimes_A d$, as id_P is of degree 0. □

3.3. **Comparison of P and $P \otimes_A P$.** Recall that so far it is sufficient to assume U to be a left A-bialgebroid which is the algebraic underpinning of the fact that U-**Mod** is monoidal with unit object A. Using, for example, the standard spectral sequence of the bicomplex $P_\bullet \otimes_A P_\bullet$, one easily verifies that the tensor product $P \otimes_A P$ has homology $A \otimes_A A \simeq A$; so it is, like P, a resolution of A. However, only when U is a left Hopf algebroid, P and $P \otimes_A P$ are necessarily quasi-isomorphic since in this case the tensor product of two projectives in U-**Mod** is projective [KoKr1, Theorem 5]. Proposition 3.4 tells us that

$$\Delta^P : P \to P \otimes_A P, \quad \mathrm{id}_P \otimes_A \varepsilon^P : P \otimes_A P \to P$$

are morphisms of chain complexes that are one-sided inverses of each other. In the left Hopf algebroid case the following proposition provides a homotopy that shows that the maps become in this situation quasi-inverse to each other. Note that this proposition is true for all left Hopf algebroids, assuming no projectivity of U over A (although, of course, without that P is not necessarily a projective resolution).

Proposition 3.5. *If U is a left Hopf algebroid over A, then the maps*

$$\mathsf{h}_n : \bigoplus_{i+j=n} P_i \otimes_A P_j \to \bigoplus_{k+l=n+1} P_k \otimes_A P_l$$

given by

$$\begin{aligned}&(u^0,\ldots,u^i) \otimes_A (v^0,\ldots,v^j)\\ \mapsto &\sum_{r=0}^{i} (-1)^i (u^0_{+(1)},\ldots,u^r_{+(1)}) \otimes_A (u^0_{+(2)}\cdots u^r_{+(2)}, u^{r+1}_+,\ldots,u^i_+, u^i_- \cdots u^0_- v^0, v^1,\ldots,v^j)\end{aligned}$$

define a homotopy equivalence

$$\Delta^P\,(\mathrm{id}_P \otimes_A \varepsilon^P) \sim \mathrm{id}_{P\otimes_A P},$$

so Δ^P and $\mathrm{id}_P \otimes_A \varepsilon^P$ are mutual quasi-inverses and we have $P \simeq P \otimes_A P$ as objects in the derived category $\mathcal{D}^-(U)$.

Proof. In degree $n = 0$, the homotopy is

$$\mathsf{h}_0 : u \otimes_A v \mapsto u_{+(1)} \otimes_A (u_{+(2)}, u_- v) = u_{(1)} \otimes_A (u_{(2)+}, u_{(2)-} v)$$

and using the bialgebroid axioms as well as (2.4)–(2.12), we get

$$\begin{aligned}((\mathrm{id}_U \otimes_A \mathsf{b}')\,\mathsf{h}_0)(u \otimes_A v) &= u_{(1)} \otimes_A (u_{(2)+}u_{(2)-}v - \varepsilon(u_{(2)-}v) \blacktriangleright u_{(2)+})\\ &= u_{(1)} \otimes_A \varepsilon(u_{(2)}) \triangleright v - u_{(1)} \otimes_A \varepsilon(\varepsilon(v) \blacktriangleright u_{(2)-}) \blacktriangleright u_{(2)+}\\ &= u_{(1)} \triangleleft \varepsilon(u_{(2)}) \otimes_A v - u_{(1)} \otimes_A \varepsilon(u_{(2)-}) \blacktriangleright u_{(2)+} \triangleleft \varepsilon(v)\\ &= u \otimes_A v - u_{(1)} \otimes_A u_{(2)} \triangleleft \varepsilon(v)\\ &= (\mathrm{id}_{U\otimes_A U} - \Delta^P\,(\mathrm{id}_U \otimes_A \varepsilon^P))(u \otimes_A v).\end{aligned}$$

Analogously, one computes that one has also for $n > 0$

$$\begin{aligned}&\mathsf{h}_{n-1}\,(\mathsf{b}' \otimes_A \mathrm{id}_P + \mathrm{id}_P \otimes_A \mathsf{b}') + (\mathsf{b}' \otimes_A \mathrm{id}_P + \mathrm{id}_P \otimes_A \mathsf{b}')\,\mathsf{h}_n\\ = \;&\mathrm{id}_P - \Delta^P\,(\mathrm{id}_P \otimes_A \varepsilon^P).\end{aligned}$$

□

This fact demonstrates, on the one hand, the homological difference between the bialgebroid and the left Hopf algebroid case, and it also illustrates, on the other hand, that the cup and cap products we define below are indeed the derived versions of the composition and contraction product that we dealt with abstractly in [KoKr1].

3.4. $C^\bullet(U,N)$ **and the cup product.** We retain the assumption that U is an A-bialgebroid and further denote by P the DG coalgebra defined in the previous sections.

Definition 3.6. We define for all $N \in U$-$\mathbf{Mod}$ the cochain complex

$$\hat{C}^\bullet(U,N) := \mathrm{Hom}_U(P_\bullet, N)$$

with coboundary map $\hat{\delta} := \mathrm{Hom}_U(\mathsf{b}', N)$, that is,

$$\hat{\delta} : \hat{C}^p(U,N) \to \hat{C}^{p+1}(U,N), \quad \hat{\delta}\hat{\varphi} := \hat{\varphi}\mathsf{b}'.$$

Furthermore, we define the *cup product* $\smile : \hat{C}^\bullet(U,A) \otimes_k \hat{C}^\bullet(U,N) \to \hat{C}^\bullet(U,N)$ by

$$(\hat{\varphi} \smile \hat{\psi})(c) := \hat{\psi}(\hat{\varphi}(c_{(1)}) \triangleright c_{(2)}) = \hat{\varphi}(c_{(1)}) \triangleright \hat{\psi}(c_{(2)}),$$

where $c_{(1)} \otimes_A c_{(2)}$ is $\Delta^P(c)$ in Sweedler notation.

Note that for $N = A$ the cup product becomes simply the convolution product

$$(\hat{\varphi} \smile \hat{\psi})(c) = \hat{\varphi}(c_{(1)})\hat{\psi}(c_{(2)}), \tag{3.2}$$

and that Proposition 3.4 implies:

Corollary 3.7. *$(\hat{C}^\bullet(U,A), \hat{\delta}, \smile)$ is a differential graded algebra and $(\hat{C}^\bullet(U,N), \hat{\delta}, \smile)$ is a differential graded left module over $\hat{C}^\bullet(U,A)$.*

By U-linearity of $\hat{\psi} \in \hat{C}^\bullet(U,A)$ we obtain in a standard fashion the isomorphism

$$\hat{C}^p(U,N) \xrightarrow{\simeq} C^p(U,N) := \mathrm{Hom}_{A^{\mathrm{op}}}(U^{\otimes_{A^{\mathrm{op}}} p}{}_{\triangleleft}, N), \quad \hat{\psi} \mapsto \psi := \hat{\psi}(1, \cdot). \tag{3.3}$$

The inverse map is given by

$$\varphi \mapsto \{\hat{\varphi} : (u^0, \ldots, u^p) \mapsto u^0 \varphi(u^1, \ldots, u^p)\}.$$

Under this isomorphism, the differential $\hat{\delta}$ is transformed into

$$\delta : C^\bullet(U,N) \to C^{\bullet+1}(U,N)$$

given by

$$\begin{aligned} \delta\varphi(u^1, \ldots, u^{p+1}) := \ & u^1 \varphi(u^2, \ldots, u^{p+1}) \\ & + \sum_{i=1}^{p} (-1)^i \varphi(u^1, \ldots, u^i u^{i+1}, \ldots, u^{p+1}) \\ & + (-1)^{p+1} \varphi(u^1, \ldots, \varepsilon(u^{p+1}) \blacktriangleright u^p). \end{aligned} \tag{3.4}$$

Observe that by duality, $C^\bullet(U,A)$ carries the structure of a cosimplicial k-module. This will be used in Definition 5.5 when defining the associated reduced complex $\bar{C}^\bullet(U,A)$.

Finally, the cup product can be expressed on $C^\bullet(U,A)$ as follows:

Lemma 3.8. *The cup product (3.2) assumes on $\varphi \in C^p(U,A), \psi \in C^q(U,A)$ the form*

$$(\varphi \smile \psi)(u^1, \ldots, u^{p+q}) = \varphi\big(u^1, \ldots, u^{p-1}, \psi(u^{p+1}, \ldots, u^{p+q}) \blacktriangleright u^p\big). \tag{3.5}$$

Proof. For U-linear $\hat{\varphi} : P_p \to A$ and $\hat{\psi} : P_q \to A$, the explicit meaning of (3.2) is on an element $P_n \ni c := (u^0, \ldots, u^n)$

$$(\hat{\varphi} \smile \hat{\psi})(c) = \begin{cases} \hat{\varphi}(u^0_{(1)}, \ldots, u^p_{(1)}) \hat{\psi}(u^0_{(2)} \cdots u^p_{(2)}, u^{p+1}, \ldots, u^n) & \text{if } p+q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Using the U-linearity of the cochains, the Sweedler-Takeuchi property (2.2), the fact that all A-actions on U commute, and the property of the tensor product in question, we obtain

$$\begin{aligned} & \hat{\varphi}(u^0_{(1)}, \ldots, u^p_{(1)}) \hat{\psi}(u^0_{(2)} \cdots u^p_{(2)}, u^{p+1}, \ldots, u^n) \\ & = \hat{\varphi}(u^0_{(1)}, \ldots, u^p_{(1)}) \varepsilon\big(u^0_{(2)} \cdots u^p_{(2)} \blacktriangleleft \hat{\psi}(1, u^{p+1}, \ldots, u^n)\big) \\ & = \hat{\varphi}\big(u^0_{(1)} \triangleleft \varepsilon\big(u^0_{(2)} \blacktriangleleft \varepsilon(u^1_{(2)} \blacktriangleleft \cdots \blacktriangleleft \varepsilon(u^p_{(2)}) \ldots)\big), u^1_{(1)}, \ldots, \hat{\psi}(1, u^{p+1}, \ldots, u^{p+q}) \blacktriangleright u^p_{(1)}\big) \\ & = \hat{\varphi}\big(u^0, u^1, \ldots, u^{p-1}, \hat{\psi}(1, u^{p+1}, \ldots, u^{p+q}) \blacktriangleright u^p\big). \end{aligned}$$

Applying now the isomorphism (3.3) yields the claim. $\square$

In the following, we will mostly be working with this alternative complex $(C^\bullet(U,A), \delta)$ and small Greek letters will usually denote cochains therein.

3.5. **The comp algebra structure on $C^\bullet(U,A)$.** For the construction of the Gerstenhaber bracket, we associate to any p-cochain $\varphi \in C^p(U,A)$ the operator

$$\mathsf{D}_\varphi : U^{\otimes_{A^{\mathrm{op}}} p} \to U, \qquad (u^1, \ldots, u^p) \mapsto \varphi(u^1_{(1)}, \ldots, u^p_{(1)}) \triangleright u^1_{(2)} \cdots u^p_{(2)}. \tag{3.6}$$

For zero cochains, i.e., elements in A, this becomes the map $A \to U, a \mapsto s(a)$, where s is the source map in (2.13).

These operators provide the correct substitute of the insertion operations used by Gerstenhaber to define what he called a *pre-Lie system* in [Ge] and a *(right) comp algebra* in [GeSch]. Indeed, we can now define, in analogy with [Ge], the *Gerstenhaber products*

$$\circ_i : C^p(U,A) \otimes_k C^q(U,A) \to C^{p+q-1}(U,A), \quad i = 1, \ldots, p,$$

by

$$\begin{aligned} &(\varphi \circ_i \psi)(u^1, \ldots, u^{p+q-1}) \\ &\quad := \varphi(u^1, \ldots, u^{i-1}, \mathsf{D}_\psi(u^i, \ldots, u^{i+q-1}), u^{i+q}, \ldots, u^{p+q-1}), \end{aligned} \tag{3.7}$$

and for zero cochains we define $a \circ_i \psi = 0$ for all i and all ψ, whereas

$$\varphi \circ_i a := \varphi(u^1, \ldots, u^{i-1}, s(a), u^i, \ldots, u^{p-1}).$$

These Gerstenhaber products satisfy the following associativity relations:

Lemma 3.9. *For $\varphi \in C^p(U, A)$, $\psi \in C^q(U, A)$, and $\chi \in C^r(U, A)$ we have*

$$(\varphi \circ_i \psi) \circ_j \chi = \begin{cases} (\varphi \circ_j \chi) \circ_{i+r-1} \psi & \text{if } j < i, \\ \varphi \circ_i (\psi \circ_{j-i+1} \chi) & \text{if } i \leqslant j < q+i, \\ (\varphi \circ_{j-q+1} \chi) \circ_i \psi & \text{if } j \geqslant q+i. \end{cases}$$

.

Proof. Straightforward computation. □

The structure of a right comp algebra is completed by adding the *distinguished element* (analogously to [GeSch, p. 62])

$$\mu := \varepsilon\, m_U \in C^2(U, A), \tag{3.8}$$

where m_U is the multiplication map of U.

Remark 3.10. The associativity of m_U implies $\mu \circ_1 \mu = \mu \circ_2 \mu$. Furthermore, one has

$$\mathsf{D}_\mu = m_U, \tag{3.9}$$

as will be used later.

Remark 3.11. Equivalently, this structure turns $O(n) := C^n(U, A)$ into a nonsymmetric operad in the category of k-modules, see e.g. [LV, §5.8.13] or [MaShnSt, Me1], with composition

$$O(n) \otimes_k O(i_1) \otimes_k \cdots \otimes_k O(i_n) \to O(i_1 + \cdots + i_n)$$

given by

$$\varphi \otimes_k \psi_1 \otimes_k \cdots \otimes_k \psi_n \mapsto \varphi\big(\mathsf{D}_{\psi_1}(\cdot), \mathsf{D}_{\psi_2}(\cdot), \ldots, \mathsf{D}_{\psi_n}(\cdot)\big).$$

Together with μ, the operad O becomes an operad with multiplication whose unit is $\mathrm{id}_A \in C^0(U, A)$.

3.6. **The Gerstenhaber algebra $H^\bullet(U, A)$.** Recall that $|n| = n - 1$.

Definition 3.12. For two cochains $\varphi \in C^p(U, A), \psi \in C^q(U, A)$ we define

$$\varphi \bar{\circ} \psi := (-1)^{|p||q|} \sum_{i=1}^{p} (-1)^{|q||i|} \varphi \circ_i \psi \in C^{|p+q|}(U, A)$$

and their *Gerstenhaber bracket* by

$$\{\varphi, \psi\} := \varphi \bar{\circ} \psi - (-1)^{|p||q|} \psi \bar{\circ} \varphi. \tag{3.10}$$

Furthermore, one verifies by straightforward computation:

Lemma 3.13. *For $\varphi \in C^p(U, A)$ and $\psi \in C^q(U, A)$, we have*

$$\varphi \smile \psi = (\mu \circ_1 \varphi) \circ_{p+1} \psi = (\mu \circ_2 \psi) \circ_1 \varphi$$

and

$$\delta\varphi = \{\mu, \varphi\}. \tag{3.11}$$

We can now state the main theorem of this section (cf. Theorem 1.2), which follows from Gerstenhaber's results. First, let us agree about notation:

Definition 3.14. For a bialgebroid U and every $N \in U$-$\mathbf{Mod}$ we denote the cohomology of $C^\bullet(U,N)$ by $H^\bullet(U,N)$ and call this the *cohomology of U with coefficients in N*.

Remark 3.15. If U is a right A-projective left Hopf algebroid so that P is, in view of Lemma 3.1, a projective resolution of $A \in U$-$\mathbf{Mod}$, then we have $H^\bullet(U,N) \simeq \mathrm{Ext}_U(A,N)$, but in general we use the symbol $H^\bullet(U,N)$ for the cohomology of the explicit cochain complex $C^\bullet(U,N)$.

Theorem 3.16. *If U is a bialgebroid over A, then the maps (3.5) and (3.10) induce a Gerstenhaber algebra structure on $H^\bullet(U,A)$.*

Proof. It is a general fact that by using the above formulae for $\delta, \smile$ as definitions, any right comp algebra becomes a DG algebra on whose cohomology $\{\cdot,\cdot\}$ induces a Gerstenhaber algebra structure, see e.g. [GeSch, McCSm] and the references therein. $\square$

Remark 3.17. The fact that the cup product is graded commutative up to homotopy follows abstractly using the "Hilton-Eckmann trick", see, e.g., [Su] or [KoKr1, Theorem 3] for the concrete bialgebroid incarnation. In Gerstenhaber's approach it follows from

$$(-1)^{|q|}\varphi\bar{\circ}\delta\psi - (-1)^{|q|}\delta(\varphi\bar{\circ}\psi) + \delta\varphi\bar{\circ}\psi = \psi \smile \varphi - (-1)^{pq}\varphi \smile \psi,$$

which means that $\delta(\varphi\bar{\circ}\psi) = (-1)^q\big(\psi \smile \varphi - (-1)^{pq}\varphi \smile \psi\big)$ if φ and ψ are cocycles, so their graded commutator is a coboundary.

Remark 3.18. If A is commutative and η factorises through the multiplication map of A, that is, if the source and target maps of U coincide so that $a \triangleright u = u \triangleleft a$ holds for all $a \in A, u \in U$, then the tensor flip

$$\tau : U \otimes_A U \to U \otimes_A U, \quad u \otimes_A v \mapsto v \otimes_A u$$

is well defined. Consequently, it makes sense to then speak about cocommutative left Hopf algebroids, meaning that $\tau \circ \Delta = \Delta$. For example, this holds for the example of the universal enveloping algebra of a Lie-Rinehart algebra, see §6. In this case an explicit computation shows that the Gerstenhaber bracket $\{\cdot,\cdot\}$ vanishes which is clear also for abstract reasons, see [Tai].

Remark 3.19. Before moving on we also quickly remark that the reader may find formulae for Gerstenhaber brackets in the literature that use a slightly different sign convention. Some confusion that arises from this can be avoided by using the notion of the *opposite* $(V, \smile_{\mathrm{op}}, \{\cdot,\cdot\}_{\mathrm{op}})$ of a Gerstenhaber algebra $(V, \smile, \{\cdot,\cdot\})$: this is defined by

$$u \smile_{\mathrm{op}} v := v \smile u, \quad \{u,v\}_{\mathrm{op}} := -\{v,u\},$$

and it is verified straightforwardly that this indeed is a Gerstenhaber algebra again. When defining a Gerstenhaber algebra from a right comp algebra, the same changes can be made on the level of the comp algebra itself. The differential then has to be rescaled on degree p by a factor of $(-1)^p$ in order to obtain a DG algebra again.

4. The Gerstenhaber module

This section introduces the structures on homology that correspond to the cup product and the Gerstenhaber bracket on $H^\bullet(U,A)$: the cap product between $H^\bullet(U,A)$ and $H_\bullet(U,M)$ and then a Hopf algebroid generalisation of the Lie derivative that has been defined by Rinehart on Lie-Rinehart and Hochschild (co)homology. This, for module-comodules M over a left Hopf algebroid U, will be defined only on $H^M_\bullet(U)$ rather than on $H_\bullet(U,M)$, and dually it will be necessary to replace $H^\bullet(U,A)$ by a Gerstenhaber algebra $H^\bullet_M(U)$ that is the cohomology of a suitable comp subalgebra $C^\bullet_M(U) \subseteq C^\bullet(U,A)$.

4.1. $C_\bullet(U, M)$ **and the cap product.** The first steps in this section are completely dual to those in the previous one. First of all, we define the homology of a bialgebroid with coefficients in a right module. The following is the counterpart of Definition 3.6:

Definition 4.1. For any bialgebroid U and any $M \in U^{\mathrm{op}}$-$\mathbf{Mod}$ we define

$$\hat{C}_\bullet(U, M) := M \otimes_U P_\bullet,$$

which becomes a chain complex of k-modules with boundary map $\hat{\mathsf{b}} := \mathrm{id}_M \otimes_U \mathsf{b}'$. Using the coalgebra structure Δ^P of P, we furthermore introduce the *cap product*

$$\frown : \hat{C}^p(U, A) \otimes_k \hat{C}_n(U, M) \to \hat{C}_{n-p}(U, M)$$

by

$$\hat{\varphi} \frown (m \otimes_U c) := m \otimes_U c_{(1)} \triangleleft \hat{\varphi}(c_{(2)}). \tag{4.1}$$

Analogously to (3.3), we have an isomorphism of k-modules

$$\hat{C}_n(U, M) \xrightarrow{\simeq} C_n(U, M) = M \otimes_{A^{\mathrm{op}}} U^{\otimes_{A^{\mathrm{op}}} n}, \tag{4.2}$$

given by

$$m \otimes_U (u^0, \cdots, u^n) \mapsto (mu^0, u^1, \ldots, u^n).$$

Here and in what follows, we are again using the notation

$$(m, u^1, \ldots, u^n) := m \otimes_{A^{\mathrm{op}}} u^1 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u^n$$

to better distinguish the tensor product over A^{op} from that one over A.

Remark 4.2. As a straightforward computation shows, the simplicial differential b from (2.17) differs from the one induced by $\hat{\mathsf{b}}$ only by a sign factor: if we suppress the isomorphism (4.2), then we have on $C_n(U, M)$

$$\mathsf{b} = (-1)^n \hat{\mathsf{b}},$$

so the two boundary maps yield the same homology $H_\bullet(U, M)$.

Remark 4.3. In analogy with Remark 3.15, if U is a right A-projective Hopf algebroid, then we have $H_\bullet(U, M) \simeq \mathrm{Tor}^U(M, A)$.

Let us compute what happens to the cap product under the isomorphisms (3.3) and (4.2):

Lemma 4.4. *The cap product of $\varphi \in C^p(U, A)$ with $(m, x) \in C_n(U, M)$ is given by*

$$\varphi \frown (m, x) = (m, u^1, \ldots, u^{n-p-1}, \varphi(u^{n-|p|}, \ldots, u^n) \blacktriangleright u^{n-p}), \tag{4.3}$$

where we again use the abbreviation $(m, x) = (m, u^1, \ldots, u^n)$ as in Proposition 2.5.

Proof. For $\hat{\varphi} \in \hat{C}^p(U, A)$ (recall that these are the U-linear cochains), we have by a computation similar to that in the proof of Lemma 3.8

$$\begin{aligned}
&\hat{\varphi} \frown \big(m \otimes_U (u^0, \ldots, u^n)\big) \\
&= \hat{\varphi}(u^0_{(2)} \cdots u^{n-p}_{(2)}, u^{n-|p|}, \ldots, u^n) m \otimes_U (u^0_{(1)}, \ldots, u^{n-p}_{(1)}) \\
&= \varepsilon\big(u^0_{(2)} \cdots u^{n-p}_{(2)} \blacktriangleleft \hat{\varphi}(1, u^{n-|p|}, \ldots, u^n)\big) m \otimes_U \big(u^0_{(1)}, \ldots, u^{n-p}_{(1)}\big) \\
&= m \otimes_U \big(u^0_{(1)} \triangleleft \varepsilon(u^0_{(2)} \cdots u^{n-p}_{(2)}), \ldots, u^{n-p-1}_{(1)}, \hat{\varphi}(1, u^{n-|p|}, \ldots, u^n) \blacktriangleright u^{n-p}_{(1)}\big) \\
&= m \otimes_U \big(u^0, \ldots, u^{n-p-1}, \hat{\varphi}(1, u^{n-|p|}, \ldots, u^n) \blacktriangleright u^{n-p}\big).
\end{aligned} \tag{4.4}$$

The claim follows by applying the isomorphisms (3.3) and (4.2). $\square$

In the sequel we will carry out extensive computations concerning algebraic relations satisfied by the operators

$$\iota_\varphi := \varphi \frown \cdot : C_n(U, M) \to C_{n-p}(U, M).$$

As a first illustration, we formulate the following analogue of Corollary 3.7 in this notation. This could still be nicely written out using $\frown$, but the computations in the subsequent sections will be too complex for that.

Proposition 4.5. *$(C_\bullet(U,M), \mathsf{b}, \frown)$ is a left DG module over $(C^\bullet(U,A), \delta, \smile)$, i.e.,*

$$\iota_\varphi \, \iota_\psi = \iota_{\varphi \smile \psi}, \tag{4.5}$$

$$[\mathsf{b}, \iota_\varphi] = \iota_{\delta\varphi}, \tag{4.6}$$

where $[\cdot,\cdot]$ denotes the graded commutator, that is, we explicitly have for $\varphi \in C^p(U,A)$

$$[\mathsf{b}, \iota_\varphi] = \mathsf{b}\, \iota_\varphi - (-1)^p \iota_\varphi \, \mathsf{b},$$

as ι_φ is of degree p while b is of degree 1.

Proof. This follows instantly from the DG coalgebra axioms when using the original presentation (4.1) for the cap product. □

Consequently, $(H_\bullet(U,M), \frown)$ is a left module over the ring $(H^\bullet(U,A), \smile)$.

4.2. **The comp module structure on $C_\bullet(U,M)$.** A finer analysis, parallel to the one carried out for $C^\bullet(U,A)$ in §3.5, shows that $C_\bullet(U,M)$ carries a structure that we will refer to as that of a *comp module* over $C^\bullet(U,A)$: for $i = 1, \ldots, n - |p|$ we define

$$\bullet_i : C^p(U,A) \otimes_k C_n(U,M) \to C_{n-|p|}(U,M)$$

by

$$\varphi \bullet_i (m,x) := (m, u^1, \ldots, u^{i-1}, \mathsf{D}_\varphi(u^i, \ldots, u^{i+|p|}), u^{i+p}, \ldots, u^n).$$

Observe that for zero cochains, i.e., for elements in A, this means that

$$a \bullet_i (m,x) := (m, u^1, \ldots, u^{i-1}, s(a), u^i, \ldots, u^n), \quad i = 1, \ldots, n+1,$$

where s is the source map from (2.13).

These maps satisfy the following associativity conditions, as is verified by straightforward computation:

Lemma 4.6. *Let $\varphi \in C^p(U,A)$, $\psi \in C^q(U,A)$, and $(m,x) \in C_n(U,M)$. Then, for $j = 1, \ldots n - |q|$, one has*

$$\varphi \bullet_i \big(\psi \bullet_j (m,x)\big) = \begin{cases} \psi \bullet_j \big(\varphi \bullet_{i+|q|} (m,x)\big) & \text{if } j < i \leqslant n - |p| - |q|, \\ (\varphi \circ_{j-i+1} \psi) \bullet_i (m,x) & \text{if } j - |p| \leqslant i \leqslant j, \\ \psi \bullet_{j-|p|} \big(\varphi \bullet_i (m,x)\big) & \text{if } 1 \leqslant i < j - |p|. \end{cases} \tag{4.7}$$

Of course, the middle line in (4.7) can also be read from right to left so as to get an idea how an element $\varphi \circ_i \psi$ acts on $C_\bullet(U,M)$ via $\bullet_i$.

Remark 4.7. Despite the similarity, the above associativity relations are quite different from those that hold for the $\circ_i$ in a comp algebra. For example, there seems to be no way to express the cap product $\frown$ in terms of μ and $\bullet_i$ by a formula analogous to the one given in Lemma 3.13 for the cup product $\smile$. However, Lemma 4.17 below will provide a counterpart of the second part of Lemma 3.13.

For later use, let us also note down the following relations:

Lemma 4.8. *Let $\varphi \in C^p(U,A)$, $\psi \in C^q(U,A)$, and $(m,x) \in C_n(U,M)$. For $i = 1, \ldots, n - |p+q|$ one has*

$$(\varphi \smile \psi) \bullet_i (m,x) = \mu \bullet_i \big(\varphi \bullet_i (\psi \bullet_{i+p} (m,x))\big), \tag{4.8}$$

$$\varphi \bullet_i \big(\psi \frown (m,x)\big) = \psi \frown \big(\varphi \bullet_i (m,x)\big). \tag{4.9}$$

Proof. Eq. (4.8) is easily proven by means of the Sweedler-Takeuchi property (2.2) and (3.9). Eq. (4.9) follows from the fact that the coproduct of U is an A^{e}-module homomorphism. □

Similar as for the cap product with a fixed cochain, we introduce a new notation for the operator $\varphi \bullet_i \cdot$, where $\varphi \in C^p(U, A)$, in order to keep the presentation of our computations below as compact as possible: whenever $p \leqslant n$ and for $i = 1, \dots, n - |p|$, we define

$$\mathsf{D}_\varphi^{i\text{th}} : C_n(U, M) \to C_{n-|p|}(U, M), \quad (m, x) \mapsto \varphi \bullet_i (m, x).$$

In particular, we will make frequent use of the short hand notation

$$\mathsf{D}'_\varphi := \mathsf{D}_\varphi^{(n-|p|)\text{th}}.$$

For example, in this notation we have:

Lemma 4.9. *For any $\varphi \in C^p(U, A)$, for $0 \leqslant p < n$ we have on $C_n(U, M)$*

$$\begin{aligned} \mathsf{d}_0 \, \mathsf{D}'_\varphi &= \iota_\varphi, && && (4.10)\\ \mathsf{d}_i \, \mathsf{D}'_\varphi &= \mathsf{D}'_\varphi \, \mathsf{d}_{i+|p|}, && \textit{for} \quad i = 2, \dots, n - |p|, && (4.11)\\ \mathsf{s}_j \, \mathsf{D}'_\varphi &= \mathsf{D}'_\varphi \, \mathsf{s}_{j+|p|}, && \textit{for} \quad j = 1, \dots, n - |p|. && (4.12) \end{aligned}$$

Proof. Using (2.15), (4.3), and with D_φ as in (3.6), Eq. (4.10) follows directly from the identity

$$\varepsilon \mathsf{D}_\varphi = \varphi,$$

which we prove now: one verifies in a straightforward manner that

$$\bar{\Delta} : U^{\otimes_{A^{\mathrm{op}}} p} \to (U^{\otimes_{A^{\mathrm{op}}} p})_{\triangleleft} \otimes_{A} {}_{\triangleright}U, \quad (u^1, \dots, u^p) \mapsto (u^1_{(1)}, \dots, u^p_{(1)}) \otimes_A u^1_{(2)} \cdots u^p_{(2)}$$

defines a right U-comodule structure on $(U^{\otimes_{A^{\mathrm{op}}} p})_{\triangleleft}$. Using source and target maps from (2.13) and denoting by m_U the multiplication in U, we can then write

$$\varepsilon \mathsf{D}_\varphi = \varepsilon m_U\big(s\varphi \otimes \mathrm{id}\big)\bar{\Delta} = \varepsilon m_U\big(s\varphi \otimes s\varepsilon\big)\bar{\Delta} = (\varphi \otimes \varepsilon)\bar{\Delta} = \varphi m_{U^{\mathrm{op}}}(\mathrm{id} \otimes t\varepsilon)\bar{\Delta} = \varphi,$$

which holds by A-linearity of a bialgebroid counit, the right A-linearity of φ and the fact that $\bar{\Delta}$ is a coaction.

Eqs. (4.11) and (4.12) follow by straightforward computation, using the fact that the involved face and degeneracy maps can be written as

$$\begin{aligned} \mathsf{d}_i(m, x) &= \mu \bullet_{n-i} (m, x), \\ \mathsf{s}_j(m, x) &= (\varepsilon 1_U) \bullet_{n-|i|} (m, x), \end{aligned} \qquad \text{for } i, j = 1, \dots, n-1,$$

where $(m, x) \in C_\bullet(U, M)$, and then applying the properties (4.7). ☐

4.3. **The comp algebra $C^\bullet_M(U)$.** When U is a left Hopf algebroid (not just a bialgebroid as before) and M is a module-comodule, the para-cyclic structure on $C_\bullet(U, M)$ given in Proposition 2.5 relates the products $\bullet_i$ to each other:

Lemma 4.10. *For any $\varphi \in C^p(U, A)$, we have for $0 \leqslant p \leqslant n$ and $(m, x) \in C_\bullet(U, M)$*

$$\varphi \bullet_i \big(\mathsf{t}(m, x)\big) = \begin{cases} \mathsf{t}\big(\varphi \bullet_{i+1} (m, x)\big) & \textit{for } i = 1, \dots, n - p, \\ \mathsf{t}\big(\iota_\varphi \, \mathsf{s}_{-1}(m, x)\big) & \textit{for } i = n - |p|. \end{cases} \qquad (4.13)$$

Proof. The case for $1 \leqslant i \leqslant n - p$ is a simple computation using (2.7) and (2.12):

$$\begin{aligned} &\varphi \bullet_i \big(\mathsf{t}(m, u^1, \dots, u^n)\big) \\ &= \big(m_{(0)} u^1_+, u^2_+, \dots, u^i_+, \mathsf{D}_\varphi(u^{i+1}_+, \dots, u^{i+p}), u^{i+p+1}_+, \dots, u^n_+, u^n_- \cdots u^1_- m_{(-1)}\big) \\ &= \big(m_{(0)} u^1_+, u^2_+, \dots, \big(\mathsf{D}_\varphi(u^{i+1}, \dots, u^{i+p})\big)_+, \dots \\ &\quad \dots, u^n_+, u^n_- \cdots \big(\mathsf{D}_\varphi(u^{i+1}, \dots, u^{i+p})\big)_- \cdots u^1_- m_{(-1)}\big) \\ &= \mathsf{t}\big(\varphi \bullet_{i+1} (m, u^1, \dots, u^n)\big). \end{aligned}$$

As for the case $i = n - |p|$, one first observes that no aYD condition (i.e., compatibility of U-action and U-coaction) is needed for the explicit computation, which we leave to the reader. ☐

The comp module structure of $C_\bullet(U,M)$ does not descend, for general module-comodules M over left Hopf algebroids, to the universal cyclic quotient $C^{\mathrm{cyc}}_\bullet(U,M)$. Since we will have to work from some point on on the latter, we define:

Definition 4.11. If U is a left Hopf algebroid and M is a module-comodule, we define

$$C^\bullet_M(U) := \big\{\varphi \in C^\bullet(U,A) \mid \mathsf{D}^{i\text{th}}_\varphi(\mathrm{im}(\mathrm{id}-\mathsf{T})) \subseteq \mathrm{im}(\mathrm{id}-\mathsf{T}) \forall i\big\}.$$

Obviously, one has $C^\bullet_M(U) = C^\bullet(U,A)$ whenever M is an SaYD module. Observe furthermore that the middle relation in (4.7) immediately implies:

Lemma 4.12. *$C^\bullet_M(U) \subseteq C^\bullet(U,A)$ is a comp subalgebra.*

In particular, it is a DG subalgebra, so it makes sense to talk about its cohomology:

Definition 4.13. The cohomology of $C^\bullet_M(U)$ will be denoted by $H^\bullet_M(U)$.

Applying Eq. (4.13) repeatedly, one obtains that on $C^{\mathrm{cyc}}_\bullet(U,M)$ all operators $\mathsf{D}^{i\text{th}}_\varphi$ can be expressed in terms of D'_φ and the cyclic operator. More precisely, Lemma 4.6 respectively Eq. (4.9) imply:

Lemma 4.14. *If M is a module-comodule over a left Hopf algebroid U, then for any $\varphi \in C^p_M(U)$ and $\psi \in C^q_M(U)$ we have*

$$\mathsf{D}^{i\text{th}}_\varphi = \mathsf{t}^{n-|p|-i}\mathsf{D}'_\varphi \mathsf{t}^{i+p}, \qquad i = 1,\ldots,n-|p|, \tag{4.14}$$

and

$$\mathsf{t}^{n-|p+q|-i}\mathsf{D}'_\varphi \mathsf{t}^{i+p}\iota_\psi = \iota_\psi \mathsf{t}^{n-|p|-i}\mathsf{D}'_\varphi \mathsf{t}^{i+p} \tag{4.15}$$

as operators on $C^{\mathrm{cyc}}_\bullet(U,M)$.

We conclude this subsection with another technical lemma:

Lemma 4.15. *Let M be a module-comodule over a left Hopf algebroid U and $\varphi \in C^p_M(U)$ as well as $\psi \in C^q_M(U)$.*

(*i*) *If ψ is a cocycle, then the equation*

$$\mathsf{d}_1\mathsf{D}'_\psi = \sum_{i=1}^{q}(-1)^{i+q}\mathsf{D}'_\psi \mathsf{d}_i + (-1)^q \mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi \mathsf{t}^n \tag{4.16}$$

holds for $0 < q < n$ on $C^{\mathrm{cyc}}_n(U,M)$.

(*ii*) *For $0 \leqslant p \leqslant n$, the identities*

$$\mathsf{D}'_\varphi = \mathsf{t}\,\iota_\varphi\,\mathsf{s}_{-1}\,\mathsf{t}^n \tag{4.17}$$

and

$$\iota_\varphi\,\mathsf{s}_{-1} = \mathsf{t}^{n-|p|}\,\mathsf{D}'_\varphi\,\mathsf{t} \tag{4.18}$$

hold on $C^{\mathrm{cyc}}_\bullet(U,M)$.

Proof. All statements are either obvious or follow by a straightforward computation. For example, (4.16) is proven with the help of (4.14) and (3.4). Eqs. (4.17) and (4.18) follow directly from (4.13) as we have $\mathrm{id}-\mathsf{T} = 0$ on $C^{\mathrm{cyc}}_\bullet(U,M)$. ☐

4.4. **The Lie derivative.** Now we define a Hopf algebroid generalisation of the *Lie derivative* that will subsequently be shown to define a Gerstenhaber module structure on $H^M_\bullet(U)$. Throughout, U is a left Hopf algebroid and M is a module-comodule.

Definition 4.16. For $\varphi \in C^p(U,A)$, we define

$$\mathcal{L}_\varphi : C_n(U,M) \to C_{n-|p|}(U,M)$$

in degree n with $p < n+1$ to be

$$\mathcal{L}_\varphi := \sum_{i=1}^{n-|p|}(-1)^{\theta^{n,p}_i}\mathsf{t}^{n-|p|-i}\,\mathsf{D}'_\varphi\,\mathsf{t}^{i+p} + \sum_{i=1}^{p}(-1)^{\xi^{n,p}_i}\mathsf{t}^{n-|p|}\,\mathsf{D}'_\varphi\,\mathsf{t}^i, \tag{4.19}$$

where the signs are given by

$$\theta_i^{n,p} := |p|(n - |i|), \qquad \xi_i^{n,p} := n|i| + |p|.$$

In case $p = n + 1$, we set

$$\mathcal{L}_\varphi := (-1)^{|p|} \iota_\varphi \, \mathsf{B},$$

and for $p > n + 1$, we define $\mathcal{L}_\varphi := 0$.

We will speak of the first block in the Lie derivative as of the *untwisted part* and of the second block as of the *twisted part*, a terminology which will become vivid in §4.5.

Clearly, $\mathcal{L}_\varphi$ descends for $\varphi \in C^\bullet_M(U)$ to a well-defined operator on $C^{\mathrm{cyc}}_\bullet(U, M)$. In particular, this applies to the distinguished element μ from (3.8). For this specific cochain, we obtain the following counterpart to the second half of Lemma 3.13:

Lemma 4.17. *The differential of $C^{\mathrm{cyc}}_\bullet(U, M)$ is given by*

$$\mathsf{b} = -\mathcal{L}_\mu. \tag{4.20}$$

Proof. Using (3.9), one obtains $\mathsf{D}'_\mu = \mathsf{d}_1$ and correspondingly for the Lie derivative by the relations of a para-cyclic module:

$$\begin{aligned}
\mathcal{L}_\mu &= \sum_{i=1}^{n-1} (-1)^{n-i+1} \mathsf{t}^{n-1-i} \, \mathsf{d}_1 \, \mathsf{t}^{i+2} + \sum_{i=1}^{2} (-1)^{n(i-1)+1} \mathsf{t}^{n-1} \, \mathsf{d}_1 \, \mathsf{t}^i, \\
&= \sum_{i=1}^{n-1} (-1)^{n-i+1} \mathsf{d}_{n-i} \, \mathsf{t}^{n+1} - \mathsf{d}_n \, \mathsf{t}^n + (-1)^{n+1} \mathsf{d}_n \mathsf{t}^{n+1} \\
&= \sum_{j=1}^{n-1} (-1)^{j+1} \mathsf{d}_j \, \mathsf{t}^{n+1} - \mathsf{d}_0 \, \mathsf{t}^{n+1} + (-1)^{n+1} \mathsf{d}_n \mathsf{t}^{n+1} = -\mathsf{b}
\end{aligned}$$

on the quotient $C^{\mathrm{cyc}}_\bullet(U, M)$. □

4.5. **The case of 1-cochains.** For the reader's convenience, we treat some special cases in detail that will help understanding the general formula for $\mathcal{L}_\varphi$ and how it has been derived.

First of all, consider a 1-cochain $\varphi \in C^1(U, A)$. By extending scalars from k to the ring $k[\![r]\!]$ of formal power series in an indeterminate r, we define for any $k[\![r]\!]$-linear map

$$\mathsf{D} : C_n(U, M)[\![r]\!] \to C_n(U, M)[\![r]\!]$$

the operators

$$\mathsf{t}^{\mathsf{D}} := \mathsf{D} \, \mathsf{t}, \quad \mathsf{T}^{\mathsf{D}} := (\mathsf{t}^{\mathsf{D}})^{n+1}.$$

We apply this with D being the exponential series

$$\exp(r\varphi) := \sum_{i \geqslant 0} \frac{1}{i!} (r \mathsf{D}'_\varphi)^i.$$

Thinking of a 1-cocycle φ as of a generalised vector field, of $\exp(r\varphi)$ as of its flow, and of

$$\Omega_\varphi := \mathrm{id} - \mathsf{T}^{\exp(r\varphi)}$$

as of a curvature along an integral curve motivates the fact that a short computation yields

$$\mathcal{L}_\varphi = \tfrac{d}{dr} \Omega_\varphi|_{r=0}$$

for $n > 0$, which in this case is explicitly given by

$$\mathcal{L}_\varphi = \sum_{i=0}^{n} \mathsf{t}^{n-i} \, \mathsf{D}'_\varphi \, \mathsf{t}^{i+1} = \sum_{i=1}^{n} \mathsf{t}^{n-i} \, \mathsf{D}'_\varphi \, \mathsf{t}^{i+1} + \mathsf{t}^n \, \mathsf{D}'_\varphi \, \mathsf{t}.$$

Next, let us study $\mathcal{L}_\varphi$ in greater detail on $C^{\mathrm{cyc}}_\bullet(U,M)$. Note first that, when descending to the quotient $C^{\mathrm{cyc}}_n(U,M)$, the untwisted part in (4.19) can be written as follows:

$$\begin{aligned}\sum_{i=1}^{n-|p|}(-1)^{\theta_i^{n,p}}\mathsf{t}^{n-|p|-i}\,\mathsf{D}'_\varphi\,\mathsf{t}^{i+p}(m,x) &= \sum_{i=1}^{n-|p|}(-1)^{\theta_i^{n,p}}\mathsf{D}^{i\mathrm{th}}_\varphi(m,x)\\ &= \sum_{i=1}^{n-|p|}(-1)^{\theta_i^{n,p}}\varphi\bullet_i(m,x).\end{aligned}$$

If we now introduce the operator

$$\mathsf{E}_\varphi : U\to U,\quad u\mapsto \varphi(u_-)\blacktriangleright u_+,$$

then $\mathcal{L}_\varphi$ can be further rewritten as follows:

Proposition 4.18. *For every module-comodule M over a left Hopf algebroid U, the Lie derivative $\mathcal{L}_\varphi$ for $\varphi\in C^1_M(U)$ assumes on $C^{\mathrm{cyc}}_n(U,M)$ the form*

$$\begin{aligned}\mathcal{L}_\varphi(m,x) &= \sum_{i=1}^{n}\big(m,u^1,\ldots,\mathsf{D}_\varphi(u^i),\ldots,u^n\big)\\ &\quad+\big(m_{(0)},u^1_+,\ldots,u^{n-1}_+,\varphi(u^n_-\cdots u^1_- m_{(-1)})\blacktriangleright u^n_+\big).\end{aligned}\tag{4.21}$$

This can be alternatively written as

$$\begin{aligned}\mathcal{L}_\varphi(m,x) &= \big(\varphi(m_{(-1)})m_{(0)},u^1,\ldots,u^n\big)+\sum_{i=1}^{n}\big(m,u^1,\ldots,\mathsf{D}_\varphi(u^i),\ldots,u^n\big)\\ &\quad+\sum_{j=1}^{n}\big(m,u^1,\ldots,\mathsf{E}_\varphi(u^j),\ldots,u^n\big)\\ &\quad-\sum_{k=1}^{n}\big(m_{(0)},u^1_+,\ldots,u^{k-1}_+,\delta\varphi(u^k_-,u^{k-1}_-\cdots u^1_- m_{(-1)})\blacktriangleright u^k_+,u^{k+1},\ldots,u^n\big).\end{aligned}\tag{4.22}$$

Proof. The explicit form for the untwisted part of $\mathcal{L}$, i.e., the first summand in (4.21) was explained above, whereas the twisted part follows by a straightforward computation using the powers of t in Lemma 2.8. Eq. (4.22) follows by using Eq. (3.4) for $p=1$ as well as (2.4) and (2.11). ☐

Example 4.19. In degree $n=1$, the above reads

$$\begin{aligned}\mathcal{L}_\varphi(m,u) &= \big(\varphi(m_{(-1)})m_{(0)},u\big)+\big(m,\varphi(u_{(1)})\triangleright u_{(2)}\big)+\big(m,\varphi(u_-)\blacktriangleright u_+\big)\\ &\quad-\big(m_{(0)},\delta\varphi(u_-,m_{(-1)})\blacktriangleright u_+\big),\end{aligned}$$

and in degree $n=2$ it becomes

$$\begin{aligned}\mathcal{L}_\varphi(m,u,v) &= \big(\varphi(m_{(-1)})m_{(0)},u,v\big)+\big(m,\varphi(u_{(1)})\triangleright u_{(2)},v\big)+\big(m,u,\varphi(v_{(1)})\triangleright v_{(2)}\big)\\ &\quad+\big(m,\varphi(u_-)\blacktriangleright u_+,v\big)+\big(m,u,\varphi(v_-)\blacktriangleright v_+\big)\\ &\quad-\big(m_{(0)},\delta\varphi(u_-,m_{(-1)})\blacktriangleright u_+,v\big)-\big(m_{(0)},u_+,\delta\varphi(v_-,u_-m_{(-1)})\blacktriangleright v_+\big).\end{aligned}$$

Example 4.20. In case φ is a 1-co*cycle*, one has the cocycle condition

$$\varphi(uv)=\varepsilon\big(\varphi(v)\blacktriangleright u\big)+\varphi\big(\varepsilon(v)\blacktriangleright u\big),\tag{4.23}$$

which implies $\varphi(1)=0$. The Lie derivative in degree zero then reads, as before

$$\mathcal{L}_\varphi(m)=\varphi(m_{(-1)})m_{(0)}=\varphi(m_{(-1)})\blacktriangleright m_{(0)},$$

whereas in degree n reduces to

$$\begin{aligned}\mathcal{L}_\varphi(m,x) &= \big(\varphi(m_{(-1)})m_{(0)}, u^1, \ldots, u^n\big)\\ &\quad + \sum_{i=1}^{n} \big(m, u^1, \ldots, \mathsf{D}_\varphi(u^i), \ldots, u^n\big)\\ &\quad + \sum_{j=1}^{n} \big(m, u^1, \ldots, \mathsf{E}_\varphi(u^j), \ldots, u^n\big).\end{aligned} \tag{4.24}$$

In particular, in degree one this reads

$$\mathcal{L}_\varphi(m,u) = \big(\varphi(m_{(-1)})m_{(0)}, u\big) + \big(m, \varphi(u_{(1)}) \triangleright u_{(2)}\big) + \big(m, \varphi(u_-) \blacktriangleright u_+\big).$$

Observe that in (4.24) the single summands where the E_φ appear are not well-defined but only their sum is (a similar comment applies to (4.22)). To exemplify this, consider in degree 2 the map

$$(u,v) \mapsto \big(\mathsf{E}_\varphi(u), v\big) + \big(u, \mathsf{E}_\varphi(v)\big).$$

Using (4.23) and (2.12), one has

$$(a \blacktriangleright u, v) \mapsto \big(\mathsf{E}_\varphi(u), v \triangleleft a\big) + \big(u, v \triangleleft \varphi(s(a))\big) + \big(a \blacktriangleright u, \mathsf{E}_\varphi(v)\big),$$

and it is easy to see that $(u, v \triangleleft a)$ has the same image.

4.6. **The case of an SaYD module.** In the case of stable anti Yetter-Drinfel'd modules, one can find an expression for $\mathcal{L}_\varphi$ on $C^{\mathrm{cyc}}_\bullet(U,M)$ analogous to the one given in (4.21) for the special case of 1-cochains. This is achieved by the following result:

Proposition 4.21. *If M is an SaYD module and $\varphi \in C^p_M(U)$, one has on $C^{\mathrm{cyc}}_\bullet(U,M)$*

$$\begin{aligned}&\mathcal{L}_\varphi(m,x) =\\ &\sum_{i=1}^{n-|p|} (-1)^{\theta^{n,p}_i} \big(m, u^1, \ldots, \mathsf{D}_\varphi(u^i, \ldots, u^{i+|p|}), \ldots, u^n\big)\\ &+ \sum_{i=0}^{|p|} (-1)^{\xi^{n,p}_{i+1}} \big(m_{(0)}u^1_{+(2)} \cdots u^i_{+(2)}, u^{i+1}_+, \ldots, u^{n-p+i}_+,\\ &\qquad \varphi(u^{n-|p|+i+1}_+, \ldots, u^n_+, u^n_- \cdots u^1_- m_{(-1)}, u^1_{+(1)}, \ldots, u^i_{+(1)}) \blacktriangleright u^{n-|p|+i}_+\big).\end{aligned}$$

Proof. Straightforward computation using Lemma 2.8 as well as Schauenburg's relations (2.4)–(2.12), the fact that the two A^{e}-module structures originating from the U-action and U-coaction coincide for SaYD modules, and the Sweedler-Takeuchi condition (2.14) for comodules. $\square$

Example 4.22. For $p = 2$ and $n = 3$, this reads:

$$\begin{aligned}\mathcal{L}_\varphi(m,u,v,w) &= -\big(m, \mathsf{D}_\varphi(u,v), w\big) + \big(m, u, \mathsf{D}_\varphi(v,w)\big)\\ &\quad - \big(m_{(0)}, u_+, \varphi(w_+, w_- v_- u_- m_{(-1)}) \blacktriangleright v_+\big)\\ &\quad + \big(m_{(0)}u_{+(2)}, v_+, \varphi(w_- v_- u_- m_{(-1)}, u_{+(1)}) \blacktriangleright w_+\big).\end{aligned}$$

4.7. **The DG Lie algebra module structure.** We now prove that the Lie derivative $\mathcal{L}$ defines a DG Lie algebra representation of $(C^\bullet_M(U)[1], \{.,.\})$:

Theorem 4.23. *For any two cochains $\varphi \in C^p_M(U)$ and $\psi \in C^q_M(U)$, we have on the quotient $C^{\mathrm{cyc}}_\bullet(U,M)$*

$$[\mathcal{L}_\varphi, \mathcal{L}_\psi] = \mathcal{L}_{\{\varphi,\psi\}}, \tag{4.25}$$

where the bracket on the right hand side is the Gerstenhaber bracket (3.10). Furthermore, we have

$$[\mathsf{b}, \mathcal{L}_\varphi] + \mathcal{L}_{\delta\varphi} = 0. \tag{4.26}$$

Proof. The proof relies on Eqs. (4.12), (4.18), and (4.14): assume that $1 \leqslant q \leqslant p$ and $p+q \leqslant n+1$, as the proof for zero cochains and the case $q=0, p=n+1$ can be carried out by similar, but easier computations. Recall that throughout we consider the operators induced on $C^{\mathrm{cyc}}_\bullet(U,M)$ and hence may identify T and id.

Using (4.19), we explicitly compute the expressions for $\mathcal{L}_\varphi\mathcal{L}_\psi$ and $\mathcal{L}_\psi\mathcal{L}_\varphi$. The underbraced terms will afterwards be computed and compared one by one. One has

$$
\begin{aligned}
\mathcal{L}_\varphi\mathcal{L}_\psi = {} & \underbrace{\sum_{i=1}^{n-|p|-|q|}\ \sum_{j=1}^{n-|q|}(-1)^{\theta_i^{n-|q|,p}+\theta_j^{n,q}}\mathsf{t}^{n-|p|-|q|-i}\mathsf{D}'_\varphi\mathsf{t}^{n-|q|+p+i-j}\mathsf{D}'_\psi\mathsf{t}^{j+q}}_{(1)} \\
& + \underbrace{\sum_{i=1}^{p}\ \sum_{j=1}^{n-|q|}(-1)^{\xi_i^{n-|q|,p}+\theta_j^{n,q}}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^{n-|q|+i-j}\mathsf{D}'_\psi\mathsf{t}^{j+q}}_{(2)} \\
& + \underbrace{\sum_{i=1}^{n-|p|-|q|}\ \sum_{j=1}^{q}(-1)^{\theta_i^{n-|q|,p}+\xi_j^{n,q}}\mathsf{t}^{n-|p|-|q|-i}\mathsf{D}'_\varphi\mathsf{t}^{n-|q|+p+i}\mathsf{D}'_\psi\mathsf{t}^{j}}_{(3)} \\
& + \underbrace{\sum_{i=1}^{p}\ \sum_{j=1}^{q}(-1)^{\xi_i^{n-|q|,p}+\xi_j^{n,q}}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^{n-|q|+i}\mathsf{D}'_\psi\mathsf{t}^{j}}_{(4)},
\end{aligned}
$$

along with

$$
\begin{aligned}
-(-1)^{|p||q|}\mathcal{L}_\psi\mathcal{L}_\varphi = {} & \underbrace{\sum_{j=1}^{n-|p|-|q|}\ \sum_{i=1}^{n-|p|}(-1)^{\theta_j^{n,q}+\theta_i^{n,p}+1}\mathsf{t}^{n-|q|-|p|-j}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+q+j-i}\mathsf{D}'_\varphi\mathsf{t}^{i+p}}_{(5)} \\
& + \underbrace{\sum_{j=1}^{q}\ \sum_{i=1}^{n-|p|}(-1)^{\xi_j^{n-|p|,q}+\theta_i^{n-|q|,p}+1}\mathsf{t}^{n-|q|-|p|}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+j-i}\mathsf{D}'_\varphi\mathsf{t}^{i+p}}_{(6)} \\
& + \underbrace{\sum_{j=1}^{n-|q|-|p|}\ \sum_{i=1}^{p}(-1)^{\theta_j^{n,q}+\xi_i^{n,p}+1}\mathsf{t}^{n-|q|-|p|-j}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+q+j}\mathsf{D}'_\varphi\mathsf{t}^{i}}_{(7)} \\
& + \underbrace{\sum_{j=1}^{q}\ \sum_{i=1}^{p}(-1)^{\xi_j^{n-|p|,q}+\xi_i^{n,p}+1+|p||q|}\mathsf{t}^{n-|q|-|p|}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+j}\mathsf{D}'_\varphi\mathsf{t}^{i}}_{(8)}.
\end{aligned}
$$

Furthermore, it follows from (4.7) and (4.15) that, for $i=1,\ldots,p$, we have the identities

$$
\mathsf{D}'_{\varphi\circ_i\psi} = \mathsf{D}'_\varphi\mathsf{D}^{k\mathrm{th}}_\psi = \mathsf{D}'_\varphi\mathsf{t}^{n-|q|-k}\mathsf{D}'_\psi\mathsf{t}^{k+q}, \quad \text{where} \quad k = n-|p|-|q|,\ldots,n-|q|.
$$

Hence

$$
\begin{aligned}
\mathcal{L}_{\varphi\bar{\circ}\psi} &= (-1)^{|p||q|} \sum_{i=1}^{n-|p|-|q|} \sum_{k=n-|p|-|q|}^{n-|q|} (-1)^{\theta_i^{n,|p+q|} + |q||k-n+|p+q||} \mathsf{t}^{n-|p|-|q|-i} \mathsf{D}'_\varphi \mathsf{D}_\psi^{k\text{th}} \mathsf{t}^{i+|p+q|} \\
&\quad + (-1)^{|p||q|} \sum_{i=1}^{|p+q|} \sum_{k=n-|p|-|q|}^{n-|q|} (-1)^{\xi_i^{n,|p+q|} + |q||k-n+|p+q||} \mathsf{t}^{n-|p|-|q|} \mathsf{D}'_\varphi \mathsf{D}_\psi^{k\text{th}} \mathsf{t}^{i+|p+q|} \\
&= \sum_{i=1}^{n-|p|-|q|} \sum_{k=n-|p|-|q|}^{n-|q|} (-1)^{\theta_i^{n,p} + \theta_i^{n,q} + |q|(|k|-n)} \mathsf{t}^{n-|p|-|q|-i} \mathsf{D}'_\varphi \mathsf{t}^{n-|q|-k} \mathsf{D}'_\psi \mathsf{t}^{k+q+i+|p+q|} \\
&\quad + \sum_{i=1}^{|p+q|} \sum_{k=n-|p|-|q|}^{n-|q|} (-1)^{\xi_i^{n,|p+q|} + |q|(|k|-n)} \mathsf{t}^{n-|p|-|q|} \mathsf{D}'_\varphi \mathsf{t}^{n-|q|-k} \mathsf{D}'_\psi \mathsf{t}^{k+q+i} \\
&= \underbrace{\sum_{i=1}^{n-|p|-|q|} \sum_{l=i}^{|p|+i} (-1)^{\theta_i^{n,p} + \theta_i^{n,q} + |q|(l+i+|p|)} \mathsf{t}^{n-|p|-|q|-i} \mathsf{D}'_\varphi \mathsf{t}^{|p|+i-l} t^{n-|q|+1} \mathsf{D}'_\psi \mathsf{t}^{l+q}}_{(9)} \\
&\quad + \underbrace{\sum_{i=1}^{|p+q|} \sum_{l=n-|p|+1}^{n+1} (-1)^{\xi_i^{n,|p+q|} + \theta_l^{n,q}} \mathsf{t}^{n-|p|-|q|} \mathsf{D}'_\varphi \mathsf{t}^{n-|l|} \mathsf{D}'_\psi \mathsf{t}^{l+i}}_{(10)},
\end{aligned}
$$

where we substituted $l := k - n + |p| + |q| + i$ in the first summand of the last equation, $l := k + q$ in the second summand, and used the fact that we descend to the quotient $C_\bullet^{\mathrm{cyc}}(U, M)$. Now it is easy to see that

$$
(9) = \sum_{i=1}^{n-|p|-|q|} \sum_{l=i}^{|p|+i} (-1)^{\theta_i^{n,p} + \theta_l^{n,q} + |q||p|} \mathsf{D}_\varphi^{i\text{th}} \mathsf{D}_\psi^{l\text{th}}.
$$

Likewise,

$$
\begin{aligned}
-(-1)^{|p||q|} \mathcal{L}_{\psi\bar{\circ}\varphi} &= \underbrace{\sum_{j=1}^{n-|q|-|p|} \sum_{l=i}^{|q|+i} (-1)^{\theta_j^{n,q} + \theta_l^{n,p} + 1} \mathsf{D}_\psi^{j\text{th}} \mathsf{D}_\varphi^{l\text{th}}}_{(11)} \\
&\quad + \underbrace{\sum_{j=1}^{|q+p|} \sum_{l=n-|q|+1}^{n+1} (-1)^{\xi_j^{n,|q+p|} + \theta_l^{n,p} + |p||q| + 1} \mathsf{t}^{n-|q|-|p|} \mathsf{D}'_\psi \mathsf{t}^{n-|l|} \mathsf{D}'_\varphi \mathsf{t}^{l+j}}_{(12)}.
\end{aligned}
$$

We can now write on the quotient $C_\bullet^{\mathrm{cyc}}(U, M)$

$$
\begin{aligned}
(1) &= \sum_{i=1}^{n-|p|-|q|} \sum_{j=1}^{n-|q|} (-1)^{\theta_i^{n-|q|,p} + \theta_j^{n,q}} \mathsf{D}_\varphi^{i\text{th}} \mathsf{D}_\psi^{j\text{th}} \\
&= \underbrace{\sum_{j=1}^{n-p-|q|} \sum_{i=j+1}^{n-|p|-|q|} (-1)^{\theta_i^{n-|q|,p} + \theta_j^{n,q}} \mathsf{D}_\varphi^{i\text{th}} \mathsf{D}_\psi^{j\text{th}}}_{(13)} + \underbrace{\sum_{j=p+1}^{n-|q|} \sum_{i=1}^{j-p} (-1)^{\theta_i^{n-|q|,p} + \theta_j^{n,q}} \mathsf{D}_\varphi^{i\text{th}} \mathsf{D}_\psi^{j\text{th}}}_{(14)} \\
&\quad + \underbrace{\sum_{i=1}^{n-|p|-|q|} \sum_{l=i}^{|p|+i} (-1)^{\theta_i^{n-|q|,p} + \theta_l^{n,q}} \mathsf{D}_\varphi^{i\text{th}} \mathsf{D}_\psi^{l\text{th}}}_{(15)}
\end{aligned}
$$

and

$$
\begin{aligned}
(5) &= \sum_{j=1}^{n-|q|-|p|}\sum_{i=1}^{n-|p|}(-1)^{\theta_j^{n,q}+\theta_i^{n,p}+1}\mathsf{D}_\psi^{j\mathrm{th}}\mathsf{D}_\varphi^{i\mathrm{th}} \\
&= \underbrace{\sum_{i=1}^{n-q-|p|}\sum_{j=i+1}^{n-|q|-|p|}(-1)^{\theta_j^{n,q}+\theta_i^{n,p}+1}\mathsf{D}_\psi^{j\mathrm{th}}\mathsf{D}_\varphi^{i\mathrm{th}}}_{(16)} + \underbrace{\sum_{i=q+1}^{n-|p|}\sum_{j=1}^{i-q}(-1)^{\theta_j^{n,q}+\theta_i^{n,p}+1}\mathsf{D}_\psi^{j\mathrm{th}}\mathsf{D}_\varphi^{i\mathrm{th}}}_{(17)} \\
&\quad + \underbrace{\sum_{j=1}^{n-|q|-|p|}\sum_{l=j}^{|q|+j}(-1)^{\theta_j^{n,q}+\theta_l^{n,p}+1}\mathsf{D}_\psi^{j\mathrm{th}}\mathsf{D}_\varphi^{l\mathrm{th}}}_{(18)}.
\end{aligned}
$$

We directly see that $(9) = (15)$, along with $(11) = (18)$. Furthermore, by a simple observation one sees that

$$
\begin{aligned}
(13) &= \sum_{j=1}^{n-q-|p|}\sum_{i=j+1}^{n-|q|-|p|}(-1)^{\theta_i^{n-|q|,p}+\theta_j^{n,q}}\mathsf{D}_\psi^{j\mathrm{th}}\mathsf{D}_\varphi^{(i+|q|)\mathrm{th}} \\
&= \sum_{j=1}^{n-q-|p|}\sum_{k=j+q}^{n-|p|}(-1)^{\theta_{k-|q|}^{n-|q|,p}+\theta_j^{n,q}}\mathsf{D}_\psi^{j\mathrm{th}}\mathsf{D}_\varphi^{k\mathrm{th}} =: (19),
\end{aligned}
$$

where in the second step we substituted $k := i + |q|$. Reordering the double sums in (19),

$$
\sum_{j=1}^{n-q-|p|}\sum_{k=j+q}^{n-|p|} = \sum_{k=q+1}^{n-|p|}\sum_{j=1}^{k-q},
$$

and by $\theta_{k-|q|}^{n-|q|,p} = \theta_k^{n,p}$, we conclude that $(13) = (19) = -(17)$. Analogously, one proves that $(14) = (16)$.

After a tedious, but straightforward re-ordering of summands one furthermore has

$$
\begin{aligned}
(2) &= \sum_{i=0}^{p-2}\sum_{k=q+1}^{p-|q|-i}(-1)^{\xi_{k+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^k \\
&\quad + \sum_{i=1}^{p}\sum_{k=0}^{i-1}(-1)^{\xi_{k+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^{n-|k|} \\
&\quad + \sum_{i=p+1}^{n-|q|}\sum_{k=i-p-1}^{i-2}(-1)^{\xi_{|k|+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^{n-k},
\end{aligned}
$$

whereas

$$
\begin{aligned}
(10) &= \sum_{i=1}^{|p|}\sum_{k=0}^{p+|q|-i}(-1)^{\xi_{k+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^k + \sum_{k=1}^{p+|q|}(-1)^{\xi_k^{n,p}}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{D}'_\psi\mathsf{t}^k \\
&\quad + \sum_{i=2}^{|p|}\sum_{k=0}^{i-2}(-1)^{\xi_{|k|+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^{n-k}.
\end{aligned}
$$

From these expressions one obtains after equally tedious but straightforward computations

$$
\begin{aligned}
(2) - (10) &= \underbrace{\sum_{i=0}^{|p|}\sum_{k=1}^{q}(-1)^{\xi_{k+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^k}_{(20)} \\
&\quad + \underbrace{\sum_{i=p}^{n-|q|}\sum_{k=i-p}^{i-1}(-1)^{\xi_{k+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^{n-|k|}}_{(21)},
\end{aligned}
$$

and one verifies directly that $(20) = (4)$.

So far all terms in the expressions for $\mathcal{L}_\varphi\mathcal{L}_\psi$ and $\mathcal{L}_{\varphi\bar{\circ}\psi}$ cancelled, except

$$(3) \quad = \quad \sum_{i=1}^{n-|p|-|q|}\sum_{j=1}^{q}(-1)^{\theta_i^{n-|q|,p}+\xi_j^{n,q}}\mathsf{t}^{n-|p|-|q|-i}\mathsf{D}'_\varphi\mathsf{t}^{n+p-|q|+i}\mathsf{D}'_\psi\mathsf{t}^j,$$

$$(21) \quad = \quad \sum_{i=p}^{n-|q|}\sum_{k=i-p}^{i-1}(-1)^{\xi_{k+i}^{n,p}+|q||i|}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\varphi\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^{n-|k|}.$$

Repeating the same type of arguments that led to (21) analogously cancels all terms in $-(-1)^{|p||q|}\mathcal{L}_\psi\mathcal{L}_\varphi$ and $-(-1)^{|p||q|}\mathcal{L}_{\psi\bar{\circ}\varphi}$, except

$$(7) \quad = \quad \sum_{j=1}^{n-|q|-|p|}\sum_{i=1}^{p}(-1)^{\theta_j^{n,q}+\xi_i^{n,p}+1}\mathsf{t}^{n-|q|-|p|-j}\mathsf{D}'_\psi\mathsf{t}^{n+q-|p|+j}\mathsf{D}'_\varphi\mathsf{t}^i,$$

$$(22) \quad := \quad \sum_{i=q}^{n-|p|}\sum_{k=i-q}^{i-1}(-1)^{\xi_{k+i}^{n,q}+|p|(|i|+|q|)}\mathsf{t}^{n-|q|-|p|}\mathsf{D}'_\psi\mathsf{t}^i\mathsf{D}'_\varphi\mathsf{t}^{n-|k|}.$$

Using (4.15), (4.18), and (4.12), and the relations of a cyclic k-module we see that

$$\begin{aligned}
(3) &= \sum_{i=1}^{n-|p|-|q|}\sum_{j=1}^{q}(-1)^{\theta_i^{n-|q|,p}+\xi_j^{n,q}}\mathsf{t}^{n-|p|-|q|-i}\mathsf{D}'_\varphi\mathsf{t}^{p+i}\iota_\psi\mathsf{t}\mathsf{s}_n\mathsf{t}^{j-1}\\
&= \sum_{i=1}^{n-|p|-|q|}\sum_{j=1}^{q}(-1)^{\theta_i^{n-|q|,p}+\xi_j^{n,q}}\iota_\psi\mathsf{t}^{n-|p|-|i|}\mathsf{D}'_\varphi\mathsf{t}^{|p|+i}\mathsf{s}_0\mathsf{t}^j\\
&= \sum_{i=1}^{n-|p|-|q|}\sum_{j=1}^{q}(-1)^{\theta_i^{n-|q|,p}+\xi_j^{n,q}}\iota_\psi\mathsf{t}\mathsf{s}_{n-p+1}\mathsf{t}^{n-|p|-|i|}\mathsf{D}'_\varphi\mathsf{t}^{|p|+i+j}\\
&= \sum_{i=1}^{n-|p|-|q|}\sum_{j=1}^{q}(-1)^{\theta_i^{n-|q|,p}+\xi_j^{n,q}}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\psi\mathsf{t}^{n-|p|-|i|}\mathsf{D}'_\varphi\mathsf{t}^{|p|+i+j} =: (23).
\end{aligned}$$

Substitution of $l := n-|p|-|i|$ and subsequently of $k := l-j$ produces

$$\begin{aligned}
(23) &= \sum_{l=q}^{n-|p|}\sum_{j=1}^{q}(-1)^{\xi_j^{n,q}+|p|(|l|+|q|)}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\psi\mathsf{t}^l\mathsf{D}'_\varphi\mathsf{t}^{n-|l|+j}\\
&= \sum_{l=q}^{n-|p|}\sum_{k=l-q}^{l-1}(-1)^{\xi_{l+k}^{n,q}+|p|(|l|+|q|)}\mathsf{t}^{n-|p|-|q|}\mathsf{D}'_\psi\mathsf{t}^l\mathsf{D}'_\varphi\mathsf{t}^{n-|k|},
\end{aligned}$$

and this is directly seen to be (22). Likewise, one shows that $(7) = (21)$.

For Eq. (4.26), simply use (4.20) to express b, then apply (4.25) to the case where $\varphi := \mu$ and finally make use of (3.11):

$$\{\mathsf{b}, \mathcal{L}_\varphi\} = -\{\mathcal{L}_\mu, \mathcal{L}_\varphi\} = -\mathcal{L}_{\{\mu,\varphi\}} = -\mathcal{L}_{\delta\varphi}. \qquad \square$$

4.8. **The Gerstenhaber module** $H_\bullet^M(U)$**.** By the identities (4.6) and (4.26), both operators ι_φ and $\mathcal{L}_\varphi$ descend to well defined operators on the Hochschild homology $H_\bullet^M(U)$, provided that φ is a cocycle. In this case, the following theorem together with Proposition 4.23 proves that ι and $\mathcal{L}$ turn $H_\bullet^M(U)$ into a module over the Gerstenhaber algebra $H^\bullet_M(U)$, cf. Def. 1.1 (ii):

Theorem 4.24. *If M is a module-comodule over a left Hopf algebroid U, then for any two cocycles $\varphi \in C^p_M(U)$, $\psi \in C^q_M(U)$, the induced maps*

$$\begin{array}{rccc}
\mathcal{L}_\varphi: & H_\bullet^M(U) & \to & H^M_{\bullet-|p|}(U),\\
\iota_\psi: & H_\bullet^M(U) & \to & H^M_{\bullet-q}(U)
\end{array}$$

satisfy

$$[\iota_\psi, \mathcal{L}_\varphi] = \iota_{\{\psi,\varphi\}}. \tag{4.27}$$

Proof. Throughout we use relations that we have shown above to hold for operators on $C^{\text{cyc}}_\bullet(U, M)$, but as we now consider the induced operators on homology, we will also assume tacitly that the operators only act on cycles and that we compute modulo boundaries.

Assume $p + q \leqslant n + 1$ (otherwise both sides in (4.27) are zero). Without restriction we may assume that $0 < q < p$, the case of $p = q$ and that of zero cochains being skipped as the proof is similar, but somewhat simpler. We now have

$$\begin{aligned}
\iota_\psi \mathcal{L}_\varphi = & \sum_{i=1}^{n-|p|-q} (-1)^{\theta_i^{n,p}} \iota_\psi \mathsf{t}^{n-|p|-i} \mathsf{D}'_\varphi \mathsf{t}^{i+p} + \sum_{i=n-|p|-|q|}^{n-|p|} (-1)^{\theta_i^{n,p}} \iota_\psi \mathsf{t}^{n-|p|-i} \mathsf{D}'_\varphi \mathsf{t}^{i+p} \\
& + \sum_{i=1}^{p} (-1)^{\xi_i^{n,p}} \iota_\psi \mathsf{t}^{n-|p|} \mathsf{D}'_\varphi \mathsf{t}^{i} \\
= & \underbrace{\sum_{i=1}^{n-|p+q|} (-1)^{\theta_i^{n,p}} \iota_\psi \mathsf{t}^{n-|p|-i} \mathsf{D}'_\varphi \mathsf{t}^{i+p}}_{(1)} + \underbrace{\sum_{k=1}^{q} (-1)^{|p|(|q|+|k|)} \iota_{\psi \circ_k \varphi}}_{(2)} + \underbrace{\sum_{i=1}^{p} (-1)^{\xi_i^{n,p}} \iota_\psi \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{i-1}}_{(3)},
\end{aligned}$$

using (3.7) and (4.14) for the second term and (4.18) for the third term. Observe that already

$$(2) = \iota_{\psi \bar{\circ} \varphi}.$$

On the other hand, we see that

$$-(-1)^{q|p|} \mathcal{L}_\varphi \iota_\psi = \underbrace{\sum_{i=1}^{n-q-|p|} (-1)^{\theta_i^{n,p}+1} \mathsf{t}^{n-q-|p|-i} \mathsf{D}'_\varphi \mathsf{t}^{i+p} \iota_\psi}_{(4)} + \underbrace{\sum_{i=1}^{p} (-1)^{\xi_i^{n-q,|q||p|}} \mathsf{t}^{n-q-|p|} \mathsf{D}'_\varphi \mathsf{t}^{i} \iota_\psi}_{(5)}.$$

By Equation (4.9), one immediately observes that $(1) = -(4)$, hence we are left to prove that

$$(3) + (5) = -(-1)^{|q||p|} \iota_{\varphi \bar{\circ} \psi} = -\sum_{i=1}^{p} (-1)^{|q||i|} \iota_{\varphi \circ_i \psi}, \tag{4.28}$$

or, in our former terminology, only the "twisted" parts in the Lie derivative still matter.

By (4.18), we see that

$$(5) = \sum_{i=1}^{p} (-1)^{\xi_i^{n-q,|q||p|}} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{i-1} \iota_\psi = \underbrace{\sum_{i=1}^{p-1} (-1)^{\xi_i^{n-q,|q||p|}} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{i-1} \iota_\psi}_{(6)} + \underbrace{(-1)^{\xi_p^{n-q,|q||p|}} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-1} \iota_\psi}_{(7)},$$

and we continue with

$$\begin{aligned}
(6) &= \sum_{i=1}^{p-1} (-1)^{\xi_i^{n-q,|q||p|}} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{i-1} \mathsf{d}_0 \mathsf{D}'_\psi = \sum_{i=1}^{p-1} (-1)^{\xi_i^{n-q,|q||p|}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{i-1} \mathsf{D}'_\psi \\
&= \sum_{i=1}^{p-1} \sum_{\substack{j=0 \\ j \neq i}}^{n-q+2} (-1)^{\xi_i^{n-q,|q||p|} + |j-i|} \iota_\varphi \mathsf{d}_j \mathsf{s}_{-1} \mathsf{t}^{i-1} \mathsf{D}'_\psi \\
&= \underbrace{\sum_{i=1}^{p-1} \sum_{\substack{j=1 \\ j \neq i}}^{n-|q|} (-1)^{\xi_i^{|n|-q,|q||p|} + j} \iota_\varphi \mathsf{d}_j \mathsf{s}_{-1} \mathsf{t}^{i-1} \mathsf{D}'_\psi}_{(8)} \\
&\quad \underbrace{+ \sum_{i=1}^{p-1} (-1)^{\xi_i^{|n|-q,|q||p|}} \iota_\varphi \mathsf{t}^{i-1} \mathsf{D}'_\psi + \sum_{i=1}^{p-1} (-1)^{\xi_{|i|}^{|n|-q,|q||p|} + 1} \iota_\varphi \mathsf{t}^{i} \mathsf{D}'_\psi}_{(9)},
\end{aligned}$$

where in the third line we used (4.6) together with the fact that φ is a cocycle, and that we deal here with the induced maps on $H^M_\bullet(U)$, i.e., $\mathsf{b}\iota_\varphi = 0 = \iota_\varphi \mathsf{b}$. Observe now that

$$(9) = (-1)^{|q||p|+1} \iota_\varphi \mathsf{D}'_\psi + (-1)^{n|p|} \iota_\varphi \mathsf{t}^{|p|} \mathsf{D}'_\psi = \underbrace{(-1)^{|q||p|+1} \iota_{\varphi \circ_p \psi}}_{(10)} + \underbrace{(-1)^{n|p|} \iota_\varphi \mathsf{t}^{|p|} \mathsf{D}'_\psi}_{(11)}.$$

Furthermore,

$$(8) = \underbrace{\sum_{i=0}^{p-3}\sum_{j=1}^{n-|q|}(-1)^{\xi_{|i|}^{n-q,|q||p|}+|j|}\iota_\varphi \mathsf{s}_{-1}\mathsf{t}^i\mathsf{d}_j\mathsf{D}'_\psi}_{(12)} + \underbrace{\sum_{j=1}^{n-|q|-|p|}(-1)^{\xi_{|p|}^{|n|,q}+j}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{d}_j\mathsf{D}'_\psi}_{(13)},$$

where by (4.16) and (4.11) we have

$$\begin{aligned}
(12) &= \sum_{i=0}^{p-3}\sum_{j=q+1}^{n}(-1)^{\xi_{|i|}^{n-q,qp}+|j+p|}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{D}'_\psi\mathsf{d}_j + \sum_{i=0}^{p-3}\sum_{j=1}^{q}(-1)^{\xi_{|i|}^{n-q,qp}+|j+p|}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{D}'_\psi\mathsf{d}_j \\
&\quad + \sum_{i=0}^{p-3}(-1))^{\xi_{|i|}^{n-q,|q||p|}+|q|}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^n \\
&= \underbrace{\sum_{i=0}^{p-3}(-1)^{\xi_{|i|}^{n-q,qp}+p}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{D}'_\psi\mathsf{d}_0}_{(14)} + \underbrace{\sum_{i=0}^{p-3}(-1)^{\xi_{|i|}^{n-q,qp}+|p|}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^n}_{(15)},
\end{aligned}$$

where in the second line we used that the representatives in $H^M_\bullet(U)$ are cycles. By a similar argument we get, still with (4.16),

$$\begin{aligned}
(13) = &\sum_{j=2}^{n-|q|-|p|}(-1)^{\xi_{|p|}^{|n|,q}+j}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{d}_j\mathsf{D}'_\psi + (-1)^{|n|p}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^n \\
&+ \sum_{j=1}^{q}(-1)^{\xi_{|p|}^{|n|,|j|}}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{d}_j \\
= &\underbrace{(-1)^{|n|p}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^n}_{(16)} + \underbrace{\sum_{j=n-|p|+1}^{n}(-1)^{\xi_{|p|}^{|n|,j}}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{d}_j}_{(17)} \\
&+ \underbrace{(-1)^{|n|p+1}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{d}_0}_{(18)}.
\end{aligned}$$

We now see that

$$\begin{aligned}
&(14) + (18) + (15) + (16) \\
&\quad= \sum_{i=0}^{p-2}(-1)^{\xi_{|i|}^{n-q,qp}+p}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{D}'_\psi\mathsf{d}_0 + \sum_{i=0}^{p-2}(-1)^{\xi_{|i|}^{n-q,qp}+|p|}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^n \\
&\quad= \sum_{i=0}^{p-2}(-1)^{\xi_{|i|}^{n-q,qp}+p}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{d}_{n-|q|}\mathsf{D}'_\psi\mathsf{t}^n + \sum_{i=0}^{p-2}(-1)^{\xi_{|i|}^{n-q,qp}+|p|}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i\mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^n \\
&\quad= \sum_{i=0}^{p-2}(-1)^{\xi_{|i|}^{n-q,qp}+p}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^i(\mathsf{d}_0-\mathsf{d}_1)\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^n =: (19).
\end{aligned}$$

Let us come back to the other half and compute (3): to this end, consider first

$$\begin{aligned}
&\iota_\psi\iota_\varphi(m,u^1,\ldots,u^n) \\
&\quad= \big(m,u^1,\ldots,\psi(u^{n-|p+q|},\ldots,\varphi(u^{n-|p|},\ldots,u^n)\blacktriangleright u^{n-p})\blacktriangleright u^{n-p-q}\big) \\
&\quad= \big(m,u^1,\ldots,\varepsilon\big(\varphi(u^{n-|p|},\ldots,u^n)\blacktriangleright \mathsf{D}_\psi(u^{n-|p+q|},\ldots,u^{n-p})\big)\blacktriangleright u^{n-p-q}\big) \\
&\quad= \big(m,u^1,\ldots,\varphi(\mathsf{D}_\psi(u^{n-|p+q|},\ldots,u^{n-p})u^{n-|p|},\ldots,u^n)\blacktriangleright u^{n-p-q}\big) \\
&\qquad+\sum_{i=n-|p|}^{n-1}(-1)^{i-n+p}\Big(m,u^1,\ldots,\varphi\big(\mathsf{D}_\psi(u^{n-|p+q|},\ldots,u^{n-p}),\ldots,u^iu^{i+1},\ldots,u^n\big)\blacktriangleright u^{n-p-q}\Big) \\
&\qquad+(-1)^p\Big(m,u^1,\ldots,\varphi\big(\mathsf{D}_\psi(u^{n-|p+q|},\ldots,u^{n-p}),\ldots,\varepsilon(u^n)\blacktriangleright u^{n-1}\big)\blacktriangleright u^{n-p-q}\Big),
\end{aligned}$$

which is true since φ is a cocycle; that is, with the help of (4.14),

$$\iota_\psi\iota_\varphi = \sum_{i=0}^{p}(-1)^{i+p}\iota_\varphi\mathsf{d}_i\mathsf{t}^p\mathsf{D}'_\psi t^{n-|p|}.$$

Hence, by (4.11) and (4.12),

$$\begin{aligned}
(3) &= \sum_{j=1}^{p} (-1)^{\xi_j^{n,p}} \iota_\psi \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{j-1} \\
&= \sum_{j=1}^{p} \sum_{i=0}^{p} (-1)^{\xi_j^{n,i}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{|p|} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j} \\
&= \underbrace{\sum_{j=1}^{p-1} \sum_{i=0}^{p} (-1)^{\xi_j^{n,i}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{|p|} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}}_{(20)} + \underbrace{\sum_{i=0}^{p} (-1)^{\xi_p^{n,i}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{|p|} \mathsf{D}'_\psi}_{(21)},
\end{aligned}$$

where we continue with

$$(21) = (-1)^{n|p|+1} \iota_\varphi \mathsf{t}^{|p|} \mathsf{D}'_\psi + \sum_{k=n-|q|-|p|+1}^{n-|q|} (-1)^{|n|p-|q|+k} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{d}_k \mathsf{D}'_\psi + (-1)^{|n||p|} \iota_\varphi \mathsf{d}_p \mathsf{s}_{-1} \mathsf{t}^{|p|} \mathsf{D}'_\psi,$$

and these three terms are precisely, by (4.11) and (4.10) again, the terms $-(11)$, $-(16)$, and $-(7)$, respectively. We furthermore have

$$(20) = \underbrace{\sum_{j=1}^{p-1} \sum_{i=1}^{p} (-1)^{\xi_j^{n,i}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}}_{(22)} + \underbrace{\sum_{j=1}^{p-1} (-1)^{n|j|+1} \iota_\varphi \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}}_{(23)},$$

where

$$(23) = \underbrace{\sum_{j=2}^{p-1} (-1)^{n|j|+1} \iota_\varphi \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}}_{(24)} - \underbrace{\iota_\varphi \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-p+2}}_{(25)},$$

and we observe that $(25) = \iota_{\varphi \circ_1 \psi}$.

For better orientation let us state were we are at this point: we are left with the equations

$$(19) \quad = \quad \sum_{i=1}^{p-1} (-1)^{\xi_i^{n-q,qp}+p} \iota_\varphi (\mathsf{d}_i - \mathsf{d}_{i+1}) \mathsf{s}_{-1} \mathsf{t}^i \mathsf{D}'_\psi \mathsf{t}^n, \tag{4.29}$$

$$(22) \quad = \quad \sum_{j=1}^{p-1} \sum_{i=1}^{p} (-1)^{\xi_j^{n,i}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}, \tag{4.30}$$

$$(24) \quad = \quad \sum_{j=2}^{p-1} (-1)^{n|j|+1} \iota_\varphi \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}, \tag{4.31}$$

and we are also missing the terms, cf. (4.28),

$$-\sum_{i=2}^{p-1} (-1)^{|q||i|} \iota_{\varphi \circ_i \psi}.$$

The proof proceeds now in recursive steps, which at each step reproduce formally the Equations (4.29)–(4.31), but with lower degrees, and one of the $\iota_{\varphi \circ_i \psi}$. We only give the next step: start with

$$(22) = \underbrace{\sum_{j=1}^{p-2} \sum_{i=1}^{p} (-1)^{\xi_j^{n,i}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}}_{(26)} + \underbrace{\sum_{i=1}^{p} (-1)^{\xi_{|p|}^{n,i}} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^n}_{(27)},$$

where

$$
\begin{aligned}
(26) = {} & \underbrace{\sum_{j=1}^{p-2} \sum_{i=n-|q|-|p|+1}^{n-|q|} (-1)^{\xi^{n,i}_{|j|}+q+p} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{d}_i \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}}_{(28)} \\
& + \underbrace{\sum_{j=1}^{p-3} (-1)^{\xi^{n,p}_j} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{d}_1 \mathsf{t} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j}}_{(29)} + \underbrace{(-1)^{|n||p|} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{d}_1 \mathsf{t} \mathsf{D}'_\psi \mathsf{t}^{n-1}}_{(30)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(28) = {} & \underbrace{\sum_{j=n-p+2}^{n-2} \sum_{i=0}^{n-j} (-1)^{\xi^{n,|i|}_{|p+j|}} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^j \mathsf{d}_i}_{(31a)} \\
& + \underbrace{\sum_{j=n-p+2}^{n-2} \sum_{i=0}^{j-n+p-2} (-1)^{\xi^{n,|i|}_{p+j}} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^j \mathsf{d}_{n-i}}_{(31b)} \\
& + \underbrace{(-1)^{|n||p|} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-1} (\mathsf{d}_0 - \mathsf{d}_1)}_{(32)}.
\end{aligned}
$$

Since the representatives of the elements we consider are in $\ker \mathsf{b}$, we conclude

$$
\begin{aligned}
(31a) + (31b) &= \sum_{j=1}^{p-3} \sum_{i=p-j}^{n-j} (-1)^{\xi^{n,i}_{|j+p|}} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-|p|+j} \mathsf{d}_i \\
&= \sum_{j=1}^{p-3} \sum_{i=1}^{n-|p|} (-1)^{\xi^{n,|i+p|}_j} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{d}_i \mathsf{t}^{n-|p|+1+j} =: (33).
\end{aligned}
$$

Now, again by (4.16), we have

$$
\begin{aligned}
(33) + (29) &= \sum_{j=1}^{p-3} \sum_{i=1}^{n-|q|-|p|} (-1)^{\xi^{n,|i+p+q|}_j} \iota_\varphi \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{d}_i \mathsf{D}'_\psi \mathsf{t}^{n-p+2+j} \\
&= \sum_{j=1}^{p-3} \sum_{i=p}^{n-|q|} (-1)^{\xi^{n,i+q}_j} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-p+2+j} \\
&= \sum_{j=1}^{p-3} \sum_{i=0}^{p-1} (-1)^{\xi^{n,|i+q|}_j} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-p+2+j} \\
&\quad + \sum_{j=1}^{p-3} (-1)^{nj} \iota_\varphi \mathsf{d}_{n-q+2} \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-p+2+j} \\
&= \underbrace{\sum_{j=1}^{p-3} \sum_{i=1}^{p-1} (-1)^{\xi^{n,|i+q|}_j} \iota_\varphi \mathsf{d}_i \mathsf{s}_{-1} \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-p+2+j}}_{(34)} + \underbrace{\sum_{j=1}^{p-3} (-1)^{\xi^{n,|q|}_j} \iota_\varphi \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-p+2+j}}_{(35)} \\
&\quad + \underbrace{\sum_{j=1}^{p-3} (-1)^{nj} \iota_\varphi \mathsf{t}^{p-1} \mathsf{D}'_\psi \mathsf{t}^{n-p+2+j}}_{(36)},
\end{aligned}
$$

where in the third equation we used one more time $\mathsf{b}\iota_\varphi = 0 = \iota_\varphi \mathsf{b}$, which holds in our situation. One furthermore has

$$
(35) = \underbrace{\sum_{j=3}^{p-2} (-1)^{nj+q} \iota_\varphi \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-p+1+j}}_{(37)} + \underbrace{(-1)^q \iota_\varphi \mathsf{t}^{p-2} \mathsf{D}'_\psi \mathsf{t}^{n-p+3}}_{(38)},
$$

and we see that $(38) = -(-1)^{|q|}\iota_{\varphi\circ_2\psi}$, that is, the second summand in (4.28). Moreover,

$$(27) + (19) = \underbrace{\sum_{i=1}^{p-2}(-1)^{\xi_i^{n-q,qp}+p}\iota_\varphi(\mathsf{d}_i - \mathsf{d}_{i+1})\mathsf{s}_{-1}\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^n}_{(39)} + \underbrace{\sum_{i=1}^{p-2}(-1)^{np+|i|}\iota_\varphi\mathsf{d}_i\mathsf{s}_{-1}\mathsf{t}^{p-1}\mathsf{D}'_\psi\mathsf{t}^n}_{(40)},$$

where

$$(40) = \sum_{i=n-|q|-|p|+1}^{n-q}(-1)^{\xi_p^{|n|,|i+q|}}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{d}_i\mathsf{D}'_\psi\mathsf{t}^n = \underbrace{\sum_{i=n-p+3}^{n}(-1)^{|n||p|+i}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^{n-1}\mathsf{d}_i}_{(41)}.$$

Furthermore, we obtain

$$\begin{aligned}
(41) + (32) &= \sum_{i=2}^{n-p+2}(-1)^{\xi_p^{|n|,i}}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^{n-1}\mathsf{d}_i \\
&= \sum_{i=1}^{q}(-1)^{|n||p|+i}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{d}_i\mathsf{t}^n + \sum_{i=q+1}^{n-|p|}(-1)^{|n||p|+i}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{d}_i\mathsf{t}^n \\
&= \underbrace{(-1)^{|n||p|+1}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{d}_1\mathsf{t}\mathsf{D}'_\psi\mathsf{t}^{n-1}}_{(42)} + \underbrace{\sum_{i=2}^{n-|q|-|p|}(-1)^{|n||p|+|q+i|}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{d}_i\mathsf{D}'_\psi\mathsf{t}^n}_{(43)},
\end{aligned}$$

where for the first term in the last line we used (4.16). By $\mathsf{b}\iota_\varphi = 0 = \iota_\varphi\mathsf{b}$ again, one has

$$\begin{aligned}
(43) &= \sum_{i=p}^{n-|q|}(-1)^{|n||p|+i+p+q}\iota_\varphi\mathsf{d}_i\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^n \\
&= \underbrace{\sum_{i=1}^{p-1}(-1)^{n|p|+i+q}\iota_\varphi\mathsf{d}_i\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^n}_{(44)} + \underbrace{(-1)^{n|p|+q}\iota_\varphi\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^n}_{(45)} + \underbrace{(-1)^{np}\iota_\varphi\mathsf{t}^{p-1}\mathsf{D}'_\psi\mathsf{t}^n}_{(46)}.
\end{aligned}$$

Finally, we see that $(42) = -(30)$, that $(36) + (46) = -(24)$, and that

$$(34) + (44) = \sum_{j=2}^{p-1}\sum_{i=1}^{p-1}(-1)^{nj+i+q}\iota_\varphi\mathsf{d}_i\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+j} =: (47),$$

as well as

$$(37) + (45) = \sum_{j=3}^{p-1}(-1)^{nj+q}\iota_\varphi\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+j} =: (48).$$

We are now left with the three terms

$$(39) = \sum_{i=1}^{p-2}(-1)^{\xi_i^{n-q,qp}+p}\iota_\varphi(\mathsf{d}_i - \mathsf{d}_{i+1})\mathsf{s}_{-1}\mathsf{t}^i\mathsf{D}'_\psi\mathsf{t}^n, \tag{4.32}$$

$$(47) = \sum_{j=2}^{p-1}\sum_{i=1}^{p-1}(-1)^{\xi_{|j|}^{n,|i|}+q}\iota_\varphi\mathsf{d}_i\mathsf{s}_{-1}\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+j}, \tag{4.33}$$

$$(48) = \sum_{j=3}^{p-1}(-1)^{nj+q}\iota_\varphi\mathsf{t}^{p-2}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+j}, \tag{4.34}$$

and these correspond (with alternating signs) to the Eqs. (4.29)–(4.31), but with one summand less and p lowered by one, respectively. Also, we obtained $\iota_{\varphi\circ_2\psi}$, see (38), on the way. Repeating the same steps as above another $p-3$ times yields the missing terms

$$-\sum_{i=3}^{p-1}(-1)^{|q||i|}\iota_{\varphi\circ_i\psi} = -\sum_{i=3}^{p-1}(-1)^{|q||i|}\iota_\varphi\mathsf{t}^{p-i}\mathsf{D}'_\psi\mathsf{t}^{n-|p|+i},$$

in (4.28), and cancels the rest. Observe that in (4.33) and (4.34) the factor $(-1)^q$ appears in contrast to (4.30) and (4.31), but in correspondence to the sign rule in (4.28). $\square$

5. The Batalin-Vilkovisky module

This section contains the both conceptually and computationally most involved aspect of our paper, which is a Hopf algebroid generalisation of the Cartan-Rinehart homotopy formula. This is a relation on the (co)chain level which implies on (co)homology the Batalin-Vilkovisky relation that expresses $\mathcal{L}_\varphi$ as the graded commutator of B and ι_φ. In other words, establishing this formula will complete the proof that $H^\bullet_M(U)$ and $H^M_\bullet(U)$ form a differential calculus.

5.1. **The operators S_φ.** We begin by defining the generalisation of the operator denoted by S in the work Nest, Tsygan and Tamarkin [NTs3, Ts, TaTs1, TaTs2], by $\mathbf{B}$ in Getzler's work [Get], and by f in Rinehart's paper [Ri]. This operator may be considered as a generalisation of the cap product for the cyclic bicomplex. Throughout this section, U is assumed to be a left Hopf algebroid and M is a module-comodule (not necessarily an SaYD module).

Definition 5.1. Given $\varphi \in C^p(U, A)$, we define

$$\mathsf{S}_\varphi : C_n(U, M) \to C_{n-p+2}(U, M)$$

for $p \leqslant n$ by

$$\mathsf{S}_\varphi := \sum_{j=0}^{n-p} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}} \mathsf{s}_{-1}\, \mathsf{t}^{n-p-i}\, \mathsf{D}'_\varphi\, \mathsf{t}^{n+i-|j|},$$

where the sign is given by

$$\eta^{n,p}_{j,i} := nj + |p|i.$$

For $p > n$, we put

$$\mathsf{S}_\varphi := 0.$$

Remark 5.2. Observe that the extra degeneracy (2.18) is given here as $\mathsf{s}_{-1} = \mathsf{t}\,\mathsf{s}_{n-|p|}$.

In general, inserting the explicit formula for t, D'_φ and s_{-1} results in truly unpleasant expressions. However, in case M is an SaYD module and hence $C_\bullet(U, M)$ a cyclic module, these can be at least somewhat simplified:

Proposition 5.3. *If M is an SaYD module over a left Hopf algebroid U, then S_φ, for $\varphi \in C^p(U, A)$, $p \leqslant n$, assumes the following form:*

$$\begin{aligned}\mathsf{S}_\varphi(m, x) = \sum_{i=0}^{n-p} \sum_{j=i+1}^{n-|p|} &(-1)^{n(i+|p|)+|p|(j+i+1)} \big(m_{(0)} u^1_{+(2)} \cdots u^i_{+(2)}, u^{i+1}_+, \ldots, \\ &\mathsf{D}_\varphi(u^j_+, \ldots, u^{j+|p|}_+), \ldots, u^n_+, u^n_- \cdots u^1_- m_{(-1)}, u^1_{+(1)}, \ldots, u^i_{+(1)}\big).\end{aligned}$$

Proof. Direct computation. ☐

Example 5.4. For $n = 1$, $p = 1$, the above means:

$$\mathsf{S}_\varphi(m, u) = \big(m_{(0)}, \varphi(u_{+(1)}) \triangleright u_{+(2)}, u_- m_{(-1)}\big),$$

while it becomes for $n = 2, p = 1$:

$$\begin{aligned}\mathsf{S}_\varphi(m, u, v) &= \big(m_{(0)}, \varphi(u_{+(1)}) \triangleright u_{+(2)}, v_+, v_- u_- m_{(-1)}\big) \\ &\quad + \big(m_{(0)}, u_+, \varphi(v_{+(1)}) \triangleright v_{+(2)}, v_- u_- m_{(-1)}\big) \\ &\quad + \big(m_{(0)} u_{+(2)}, \varphi(v_{+(1)}) \triangleright v_{+(2)}, v_- u_- m_{(-1)}, u_{+(1)}\big).\end{aligned}$$

For $n = 3$ and $p = 2$, we get

$$\begin{aligned}\mathsf{S}_\varphi(m, u, v, w) &= -\big(m_{(0)}, \varphi(u_{+(1)}, v_{+(1)}) \triangleright u_{+(2)} v_{+(2)}, w_+, w_- v_- u_- m_{(-1)}\big) \\ &\quad + \big(m_{(0)}, u_+, \varphi(v_{+(1)}, w_{+(1)}) \triangleright v_{+(2)} w_{+(2)}, w_- v_- u_- m_{(-1)}\big) \\ &\quad + \big(m_{(0)} u_{+(2)}, \varphi(v_{+(1)}, w_{+(1)}) \triangleright v_{+(2)} w_{+(2)}, w_- v_- u_- m_{(-1)}, u_{+(1)}\big).\end{aligned}$$

5.2. **The relation** $[\mathsf{B}, \mathsf{S}_\varphi] = 0$**.** Our first result is that S_φ commutes with B. As this simplifies the formula for B, we will from now on be working on the reduced chain complex $\bar{C}_\bullet(U, M)$ resp. $\bar{C}^{\mathrm{cyc}}_\bullet(U, M)$, which dually requires passing also to the reduced cochain complex:

Definition 5.5. We denote by $\bar{C}^\bullet(U, A)$ respectively $\bar{C}^\bullet_M(U)$ the intersection of the kernels of the codegeneracies in the cosimplicial k-modules $C^\bullet(U, A)$ respectively $C^\bullet_M(U)$.

Proposition 5.6. *For any $\varphi \in \bar{C}^p(U, A)$ the identity*

$$[\mathsf{B}, \mathsf{S}_\varphi] = 0 \tag{5.1}$$

holds on the reduced chain complex $\bar{C}_\bullet(U, M)$.

Proof. Explicitly, the graded commutator reads on the reduced complex

$$[\mathsf{B}, \mathsf{S}_\varphi] = \mathsf{t}\, \mathsf{s}_{n-p+2}\, \mathsf{N}\, \mathsf{S}_\varphi - (-1)^{p-2}\, \mathsf{S}_\varphi\, \mathsf{t}\, \mathsf{s}_n\, \mathsf{N}.$$

If $p > n+1$, the entire expression is already zero. Hence assume that $p \leqslant n+1$ and first consider the second summand: it suffices to show that the image of $\mathsf{S}_\varphi\, \mathsf{t}\, \mathsf{s}_n$ on elements of degree n is degenerate, and this can be seen as follows:

$$\begin{aligned}
\mathsf{S}_\varphi\, \mathsf{t}\, \mathsf{s}_n &= \sum_{j=0}^{n-p+1} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}}\, \mathsf{t}\, \mathsf{s}_{n-p+2}\, \mathsf{t}^{n-p-i+1}\, \mathsf{D}'_\varphi\, \mathsf{t}^{n+i-j+2}\, \mathsf{t}\, \mathsf{s}_n \\
&= \sum_{j=0}^{n-p+1} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}}\, \mathsf{t}\, \mathsf{s}_{n-p+2}\, \mathsf{t}^{n-p-i+1}\, \mathsf{D}'_\varphi\, \mathsf{t}^{n+i-j+1}\, \mathsf{s}_0\, \mathsf{t} \\
&= \sum_{j=0}^{n-p+1} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}}\, \mathsf{t}\, \mathsf{s}_{n-p+2}\, \mathsf{t}^{n-p-i+1}\, \mathsf{D}'_\varphi\, \mathsf{s}_{n-(j-i)+1}\, \mathsf{t}^{n+i-j+2} \\
&= \sum_{j=0}^{n-p+1} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}}\, \mathsf{t}\, \mathsf{s}_{n-p+2}\, \mathsf{t}^{n-p-i+1} \mathsf{s}_{n-(j-i)-p+2}\, \mathsf{D}'_\varphi\, \mathsf{t}^{n+i-j+2} \\
&= \sum_{j=0}^{n-p+1} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}}\, \mathsf{t}\, \mathsf{s}_{n-p+2}\, \mathsf{t}^{n-p-j+2}\, \mathsf{s}_{n-p+1}\, \mathsf{t}^{j-i-1}\, \mathsf{D}'_\varphi\, \mathsf{t}^{n+i-j+2} \\
&= \sum_{j=0}^{n-p+1} \sum_{i=0}^{j} (-1)^{\eta^{n,p}_{j,i}}\, \mathsf{t}\, \mathsf{s}_{n-p+2}\, \mathsf{t}^{n-p-j}\, \mathsf{s}_0\, \mathsf{t}^{j-i}\, \mathsf{D}'_\varphi\, \mathsf{t}^{n+i-j+2},
\end{aligned}$$

using the simplicial and cyclic relations as well as (4.12) in the third line, along with the fact that $j - i = 0, \ldots, n-p+1$. Now we distinguish the following cases: we have on $\bar{C}^{\mathrm{cyc}}_\bullet(U, M)$

$$\mathsf{t}\mathsf{s}_{n-p+2}\mathsf{t}^{n-p-j}\mathsf{s}_0 = \begin{cases} \mathsf{t}\mathsf{s}_{n-p+2}\mathsf{t}^{n-p+3}\mathsf{s}_0 = \mathsf{t}\mathsf{s}_{n-p+2}\mathsf{s}_{n-p+3}\mathsf{t}^{n-p+3} & \text{if } \ j = n-p+1, \\ \mathsf{t}\mathsf{s}_{n-p+2}\mathsf{s}_0 & \text{if } \ j = n-p, \\ \mathsf{t}\mathsf{s}_{n-p+2}\mathsf{t}\mathsf{s}_0 & \text{if } \ j = n-p-1, \\ \mathsf{t}\mathsf{s}_{n-p+2}\mathsf{s}_{n-p-j}\mathsf{t}^{n-p-j} & \text{if } \ j \leqslant n-p-2, \end{cases}$$

and a quick computation reveals that in all these cases one produces degenerate elements.

That the first summand $\mathsf{t}\mathsf{s}_{n-p+2}\mathsf{N}\mathsf{S}_\varphi$ is also degenerate follows by a similar argument, and this finishes the proof. □

5.3. **The Cartan-Rinehart homotopy formula.** We are now in a position to state:

Theorem 5.7. *If M is a module-comodule over a left Hopf algebroid U, then for any cochain $\varphi \in \bar{C}^\bullet_M(U)$ the homotopy formula*

$$\mathcal{L}_\varphi = [\mathsf{B} + \mathsf{b}, \mathsf{S}_\varphi + \iota_\varphi] - \iota_{\delta\varphi} - \mathsf{S}_{\delta\varphi} \tag{5.2}$$

holds on $\bar{C}^{\mathrm{cyc}}_\bullet(U, M)$.

Remark 5.8. Observe that using (5.1) and (4.3), this can be rewritten as

$$\mathcal{L}_\varphi = [\mathsf{B}, \iota_\varphi] + [\mathsf{b}, \mathsf{S}_\varphi] - \mathsf{S}_{\delta\varphi}. \tag{5.3}$$

Remark 5.9. Apart from the obvious classical Cartan homotopy [C], this formula has been given in the context of associative algebras, i.e., in the classical cyclic homology of algebras, in [Ri] for the commutative case, in [NTs3, Get] for the noncommutative situation, and in more restricted settings such as for 1-cocycles in [Go, Co, X2].

Proof of Theorem 5.7. We stress that throughout we work on $\bar{C}^{\rm cyc}_\bullet(U, M)$. Rewrite first

$$\begin{aligned}[\mathsf{B}, \iota_\varphi] + [\mathsf{b}, \mathsf{S}_\varphi] - \mathsf{S}_{\delta\varphi} &= \mathsf{B}\iota_\varphi - (-1)^p \iota_\varphi \mathsf{B} + \mathsf{b}\mathsf{S}_\varphi - (-1)^{p-2}\mathsf{S}_\varphi \mathsf{b} - \mathsf{S}_{\delta\varphi} \\ &= \mathsf{B}\iota_\varphi + (-1)^{|p|}\iota_\varphi \mathsf{B} + \mathsf{b}\mathsf{S}_\varphi + (-1)^{|p|}\mathsf{S}_\varphi \mathsf{b} - \mathsf{S}_{\delta\varphi}.\end{aligned}$$

Observe then that the statement in the cases $p > n+1$ and $p = n+1$ follows by definition. For $p < n+1$, let us write down (4.19):

$$\mathcal{L}_\varphi = \underbrace{\sum_{i=1}^{n-|p|} (-1)^{\theta_i^{n,p}} \mathsf{t}^{n-|p|-i} \mathsf{D}'_\varphi \mathsf{t}^{i+p}}_{(1)} + \underbrace{\sum_{i=1}^{p} (-1)^{\xi_i^{n,p}} \mathsf{t}^{n-|p|} \mathsf{D}'_\varphi \mathsf{t}^{i}}_{(2)},$$

and also write with (4.10) and (4.18) on $\bar{C}^{\rm cyc}_\bullet(U, M)$

$$\begin{aligned}\mathsf{B}\iota_\varphi &= \sum_{k=0}^{n-p} (-1)^{k(n-p)} \mathsf{s}_{-1}\mathsf{t}^k \mathsf{d}_0 \mathsf{D}'_\varphi =: (3), \\ (-1)^{|p|}\iota_\varphi \mathsf{B} &= \sum_{k=0}^{n} (-1)^{|p|+nk} \mathsf{t}^{n-|p|} \mathsf{D}'_\varphi \mathsf{t}^{k+1} \\ &= \underbrace{\sum_{k=1}^{n} (-1)^{|p|+n|k|} \mathsf{t}^{n-|p|} \mathsf{D}'_\varphi \mathsf{t}^{k}}_{(4)} + \underbrace{(-1)^{|p|} \mathsf{t}^{n-|p|} \mathsf{D}'_\varphi}_{(5)}.\end{aligned}$$

A lengthy computation using the simplicial and cyclic relations yields

$$\begin{aligned}\mathsf{b}\mathsf{S}_\varphi = {} & \sum_{j=0}^{n-p}\sum_{i=0}^{j} (-1)^{\eta_{j,i}^{n,p}} \mathsf{t}^{n-p-i} \mathsf{D}'_\varphi \mathsf{t}^{n+i-|j|} + \sum_{j=0}^{n-p}\sum_{i=0}^{j} (-1)^{\eta_{|j|,|i|}^{n,|p|}+1} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_0\mathsf{D}'_\varphi \mathsf{t}^{n+i-|j|} \\ & + \underbrace{\sum_{k=2}^{n-|p|}\sum_{i=1}^{k-1}\sum_{j=i}^{n-|p|} (-1)^{\eta_{|j|,|i|}^{n,p}+k-i} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_k\mathsf{D}'_\varphi \mathsf{t}^{n+i-|j|}}_{(6)} \\ & + \underbrace{\sum_{k=1}^{n-p}\sum_{j=k}^{n-p}\sum_{i=k}^{j} (-1)^{\eta_{j,i}^{n,p}+k+n-|p|-i} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_k\mathsf{D}'_\varphi \mathsf{t}^{n+i-|j|}}_{(7)} \\ & + \underbrace{\sum_{j=0}^{n-p} (-1)^{\eta_{j,0}^{n,p}+n-p} \mathsf{t}^{n-|p|}\mathsf{D}'_\varphi \mathsf{t}^{n-|j|}}_{(8)} + \underbrace{\sum_{j=0}^{n-p-1}\sum_{i=0}^{j} (-1)^{\eta_{|j|,|i|}^{n,p}+n-p} \mathsf{t}^{n-p-i}\mathsf{D}'_\varphi \mathsf{t}^{n+i-|j|}}_{(9)} \\ = {} & \underbrace{\sum_{j=0}^{n-p-1}\sum_{i=0}^{j} (-1)^{\eta_{j,i}^{n,p}} \mathsf{t}^{n-p-i}\mathsf{D}'_\varphi \mathsf{t}^{n+i-|j|}}_{(10)} + \underbrace{\sum_{i=1}^{n-|p|} (-1)^{\eta_{n-p,|i|}^{n,p}} \mathsf{t}^{n-|p|-i}\mathsf{D}'_\varphi \mathsf{t}^{p+i}}_{(11)} \\ & + \underbrace{\sum_{j=1}^{n-p}\sum_{i=0}^{j-1} (-1)^{\eta_{|j|,|i|}^{n,|p|}+1} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_0\mathsf{D}'_\varphi \mathsf{t}^{n+i-|j|}}_{(12)} + \underbrace{\sum_{i=0}^{n-p} (-1)^{\eta_{|i|,|i|}^{n,|p|}+1} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_0\mathsf{D}'_\varphi}_{(13)} \\ & + (6) + (7) + (8) + (9).\end{aligned}$$

Observe that by $(-1)^{\eta^{n,p}_{|j|,|i|}+n-p} = (-1)^{\eta^{n,p}_{j,i}+1}$ one has $(9) = -(10)$. Likewise, by $(-1)^{\eta^{n,p}_{n-p,|i|}} = (-1)^{\theta^{n,p}_i}$, we see that $(11) = (1)$. By substitution $k := n-p-i$, one obtains $(-1)^{k(n-p)} = (-1)^{\eta^{n,|p|}_{|i|,|i|}}$, and hence $(13) = -(3)$. Finally, $(2) = (4) + (5) + (8)$ by substitution of $i := n - |j|$ in (8). We continue computing

$$
\begin{aligned}
(6) = &\sum_{k=2}^{n-|p|}\sum_{i=1}^{k-1}\sum_{j=i}^{n-p}(-1)^{\eta^{n,|p|}_{|j|,|i|}+k+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_k\mathsf{D}'_\varphi\mathsf{t}^{n+i-|j|}\\
&+\sum_{k=2}^{n-|p|}\sum_{i=1}^{k-1}(-1)^{\eta^{n,|p|}_{n-p,|i|}+k+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_k\mathsf{D}'_\varphi\mathsf{t}^{p+i}\\
= &\underbrace{\sum_{k=2}^{n-p}\sum_{i=1}^{k-1}\sum_{j=i}^{n-p}(-1)^{\eta^{n,|p|}_{|j|,|i|}+k+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_k\mathsf{D}'_\varphi\mathsf{t}^{n+i-|j|}}_{(14)}\\
&+\underbrace{\sum_{k=2}^{n-p}\sum_{i=1}^{k-1}(-1)^{\eta^{n,|p|}_{n-p,|i|}+k+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_k\mathsf{D}'_\varphi\mathsf{t}^{p+i}}_{(15)}\\
&+\underbrace{\sum_{j=1}^{n-p}\sum_{i=1}^{j}(-1)^{\eta^{n,|p|}_{|j|,|i|}+n+p}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_{n-|p|}\mathsf{D}'_\varphi\mathsf{t}^{n+i-|j|}}_{(16)}\\
&+\underbrace{\sum_{i=1}^{n-p}(-1)^{\eta^{n,|p|}_{n-p,|i|}+n+p}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_{n-|p|}\mathsf{D}'_\varphi\mathsf{t}^{p+i}}_{(17)}.
\end{aligned}
$$

With (4.11) one sees

$$
(15) = \sum_{k=p+1}^{n-1}\sum_{i=1}^{k-p}(-1)^{\eta^{n,|p|}_{|p|,i}+k}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{t}^{p+i} =: (18),
$$

and we also simplify

$$
(16) = \underbrace{\sum_{j=1}^{n-p-1}\sum_{i=1}^{j}(-1)^{\eta^{n,|p|}_{|j|,i}}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_{n-|p|}\mathsf{D}'_\varphi\mathsf{t}^{n+i-j}}_{(19)} + \underbrace{\sum_{i=1}^{n-p}(-1)^{\eta^{n,|p|}_{i,i}}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_{n-|p|}\mathsf{D}'_\varphi}_{(20)}.
$$

Furthermore,

$$
\begin{aligned}
(7) = &\underbrace{\sum_{k=2}^{n-p}\sum_{j=k}^{n-p}\sum_{i=k}^{j}(-1)^{\eta^{n,|p|}_{|j|,|i|}+k+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_k\mathsf{D}'_\varphi\mathsf{t}^{n+i-|j|}}_{(21)}\\
&+\underbrace{\sum_{j=1}^{n-p}\sum_{i=1}^{j}(-1)^{\eta^{n,|p|}_{|j|,|i|}}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_1\mathsf{D}'_\varphi\mathsf{t}^{n+i-|j|}}_{(22)}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(-1)^{|p|}\mathsf{S}_\varphi\mathsf{b} = & \underbrace{\sum_{i=1}^{n-p}(-1)^{\eta^{|n|,p}_{|i|,|i|}+n+|p|}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_n}_{(23)} \\
& + \underbrace{\sum_{k=0}^{n-1}\sum_{j=1}^{n-p}\sum_{i=1}^{j}(-1)^{\eta^{|n|,p}_{|j|,|i|}+k+i-j+|p|}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{t}^{n+i-|j|}}_{(24)} \\
& + \underbrace{\sum_{i=1}^{n-p-1}\sum_{k=p+i}^{n-1}(-1)^{\eta^{|n|,p}_{|n-p|,|i|}+k-|i|}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{t}^{p+i}}_{(25)},
\end{aligned}
$$

and we directly observe that $(23) = -(20)$ and $(25) = -(18)$, whereas

$$
\begin{aligned}
(24) = & \underbrace{\sum_{k=1}^{p}\sum_{j=1}^{n-p}\sum_{i=1}^{j}(-1)^{\eta^{n,|p|}_{|j|,i}+k+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{t}^{n+i-|j|}}_{(26)} \\
& + \underbrace{\sum_{k=p+1}^{n-1}\sum_{j=1}^{n-p}\sum_{i=1}^{j}(-1)^{\eta^{n,|p|}_{|j|,i}+k+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{t}^{n+i-|j|}}_{(27)} \\
& + \underbrace{\sum_{j=1}^{n-p}\sum_{i=1}^{j}(-1)^{\eta^{n,|p|}_{|j|,i}+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_0\mathsf{t}^{n+i-|j|}}_{(28)},
\end{aligned}
$$

where by the cyclic relations

$$
(28) = \underbrace{\sum_{j=1}^{n-p-1}\sum_{i=1}^{j}(-1)^{\eta^{n,|p|}_{|j|,i}+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_n\mathsf{t}^{n+i-j}}_{(29)} + \underbrace{\sum_{i=1}^{n-p}(-1)^{\eta^{n,|p|}_{p,i}+1}\mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_n\mathsf{t}^{p+i}}_{(30)}.
$$

By means of (4.11), one now sees that $(14)+(21) = -(27)$ and that $(29) = -(19)$, along with $(30) = -(17)$.

To conclude the proof, we need to show that $\mathsf{S}_{\delta\varphi}$ equals the only remaining terms (12), (22), and (26). Note first that from (4.6), (4.11), (4.10), as well as from the cyclic and simplicial relations follows for the $(p+1)$-cochain $\delta\varphi$:

$$
\begin{aligned}
\mathsf{D}'_{\delta\varphi} &= \mathsf{t}\iota_{\delta\varphi}\mathsf{s}_{-1}\mathsf{t}^n = \mathsf{t}\mathsf{b}\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^n + (-1)^{|p|}\mathsf{t}\iota_\varphi\mathsf{b}\mathsf{s}_{-1}\mathsf{t}^n \\
&= \sum_{k=1}^{n-|p|+1}(-1)^{|k|}\mathsf{t}\mathsf{d}_0\mathsf{d}_k\mathsf{D}'_\varphi\mathsf{s}_{-1}\mathsf{t}^n + \sum_{k=0}^{n+1}(-1)^{k+|p|}\mathsf{t}\mathsf{d}_0\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{s}_{-1}\mathsf{t}^n \\
&= \mathsf{t}\mathsf{d}_0\mathsf{d}_1\mathsf{D}'_\varphi\mathsf{s}_{-1}\mathsf{t}^n + \sum_{k=0}^{p}(-1)^{k+|p|}\mathsf{t}\mathsf{d}_0\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{s}_{-1}\mathsf{t}^n \\
&= \mathsf{t}\mathsf{d}_0\iota_\varphi\mathsf{s}_{-1}\mathsf{t}^n + (-1)^{|p|}\mathsf{t}\mathsf{d}_0\mathsf{D}'_\varphi\mathsf{t}^n + \sum_{k=1}^{p}(-1)^{k+|p|}\mathsf{t}\iota_\varphi\mathsf{s}_{-1}\mathsf{d}_{k-1}\mathsf{t}^n \\
&= \mathsf{t}^{n-p+1}\mathsf{d}_1\mathsf{D}'_\varphi + (-1)^{|p|}\mathsf{t}\mathsf{d}_0\mathsf{D}'_\varphi\mathsf{t}^n + \sum_{k=1}^{p}(-1)^{k+|p|}\mathsf{t}^{n-p+1}\mathsf{D}'_\varphi\mathsf{d}_k.
\end{aligned}
$$

Hence we have for the $(p+1)$-cochain $\delta\varphi$:

$$
\begin{aligned}
\mathsf{S}_{\delta\varphi} = & \sum_{j=0}^{n-(p+1)} \sum_{i=0}^{j} (-1)^{\eta_{j,i}^{n,|p|}} \mathsf{s}_{-1}\mathsf{t}^{n-p-(i+1)}\mathsf{d}_1\mathsf{D}'_\varphi\mathsf{t}^{n+i-|j|} \\
& + \sum_{k=1}^{p} \sum_{j=0}^{n-(p+1)} \sum_{i=0}^{j} (-1)^{\eta_{j,i}^{n,|p|}+k+|p|} \mathsf{s}_{-1}\mathsf{t}^{n-p-(i+1)}\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{t}^{n+i-|j|} \\
& + \sum_{j=0}^{n-(p+1)} \sum_{i=0}^{j} (-1)^{\eta_{j,i}^{n,|p|}+|p|} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_0\mathsf{D}'_\varphi\mathsf{t}^{n+i-j} \\
= & \sum_{j=1}^{n-p} \sum_{i=1}^{j} (-1)^{\eta_{|j|,|i|}^{n,|p|}} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_1\mathsf{D}'_\varphi\mathsf{t}^{n+i-|j|} \\
& + \sum_{k=1}^{p} \sum_{j=1}^{n-p} \sum_{i=1}^{j} (-1)^{\eta_{|j|,|i|}^{n,|p|}+k+|p|} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{D}'_\varphi\mathsf{d}_k\mathsf{t}^{n+i-|j|} \\
& + \sum_{j=0}^{n-(p+1)} \sum_{i=0}^{j} (-1)^{\eta_{j,|i|}^{n,|p|}+1} \mathsf{s}_{-1}\mathsf{t}^{n-p-i}\mathsf{d}_0\mathsf{D}'_\varphi\mathsf{t}^{n+i-j},
\end{aligned}
$$

and these summands are exactly the terms (22), (26), and (12), which concludes the proof of (5.3) and hence of (5.2). ☐

With the help of the homotopy formula, we can easily prove:

Corollary 5.10. *For any cochain $\varphi \in \bar{C}^\bullet_M(U)$, we have on $\bar{C}^{\rm cyc}_\bullet(U, M)$*

$$[\mathcal{L}_\varphi, \mathsf{B}] = 0. \tag{5.4}$$

Proof. Using (5.3), (4.3), and (2.19), we see by the graded Jacobi identity that

$$
\begin{aligned}
[\mathcal{L}_\varphi, \mathsf{B}] &= [[\mathsf{B}, \iota_\varphi], \mathsf{B}] + [[\mathsf{b}, \mathsf{S}_\varphi], \mathsf{B}] - [\mathsf{S}_{\delta\varphi}, \mathsf{B}] \\
&= [\mathsf{B}, [\iota_\varphi, \mathsf{B}]] - (-1)^p[\iota_\varphi, [\mathsf{B}, \mathsf{B}]] + [\mathsf{b}, [\mathsf{S}_\varphi, \mathsf{B}]] - (-1)^{p-2}[\mathsf{S}_\varphi, [\mathsf{b}, \mathsf{B}]] \\
&= 0,
\end{aligned}
$$

where the fact that $[\mathsf{B}, [\iota_\varphi, \mathsf{B}]] = 0$ directly follows from the graded Jacobi identity. ☐

Remark 5.11. With some more effort, it can be shown that (5.4) even holds on the non-reduced complex, but we do not need this.

5.4. **Proof of Theorem 1.5.** If $\varphi \in \bar{C}^\bullet_M(U)$ is a cocycle, then for the induced maps

$$\mathcal{L}_\varphi : H^M_\bullet(U) \to H^M_{\bullet-|p|}(U), \qquad \iota_\varphi : H^M_\bullet(U) \to H^M_{\bullet-p}(U),$$

the Rinehart homotopy formula (5.2) simplifies to

$$\mathcal{L}_\varphi = [\mathsf{B}, \iota_\varphi].$$

Using this and (4.5) one has

Corollary 5.12. *For cocycles $\varphi, \psi \in \bar{C}^\bullet_M(U)$, the induced maps on $H^M_\bullet(U)$ obey*

$$\mathcal{L}_{\varphi\smile\psi} = \mathcal{L}_\varphi\iota_\psi + (-1)^{\deg\varphi}\iota_\varphi\mathcal{L}_\psi.$$

Proof. This is now only one line:

$$\mathcal{L}_{\varphi\smile\psi} = [\mathsf{B}, \iota_{\varphi\smile\psi}] = [\mathsf{B}, \iota_\varphi]\iota_\psi + (-1)^{\deg\varphi}\iota_\varphi[\mathsf{B}, \iota_\psi] = \mathcal{L}_\varphi\iota_\psi + (-1)^{\deg\varphi}\iota_\varphi\mathcal{L}_\psi. \quad ☐$$

We now sum up the results of Theorems 4.23, 4.24, and 5.7, and state the main theorem (cf. Theorem 1.5) of this paper:

Theorem 5.13. *If U is a left Hopf algebroid over A, and M is a module-comodule, then ι given in (4.3) and the Lie derivative $\mathcal{L}$ given in (4.19) turn $H^M_\bullet(U)$ into a Batalin-Vilkovisky module over the Gerstenhaber algebra $H^\bullet_M(U)$ defined by Theorem 3.16.*

6. Lie-Rinehart algebras and jet spaces

This section contains a brief sketch of how to generalise the above results to complete left Hopf algebroids (the Hopf algebroid generalisation of complete Hopf algebras, see e.g. [Q]), and how this allows one to obtain the well-known calculus for Lie-Rinehart algebras (Lie algebroids) given by the Lie derivative, insertion operator, and the de Rham differential (cf. the original reference [Ri] and also, for example, [Hue1, GrUr, Hue2, Kos, X1]), and in particular the classical Cartan calculus from differential geometry that arises as the special case of the tangent Lie algebroid (see [C]).

In § 6.1 we introduce the jet space JL of a Lie-Rinehart algebra ([KoP], see also [CaRoVdB]), and explain its complete Hopf algebroid structure. Then we sketch in § 6.2 how to adapt the constructions of this paper to this setting. Finally, in the last two sections we recall the definition of the generalised Hochschild-Kostant-Rosenberg morphisms and use them to relate the differential calculus of Theorem 1.5 to the standard one on the exterior algebras of L respectively L^* that gives rise to Lie-Rinehart cohomology.

6.1. **Universal enveloping algebras and jet spaces.** Let (A, L) be a Lie-Rinehart algebra over a commutative k-algebra A with anchor map $L \to \mathrm{Der}_k(A), X \mapsto \{a \mapsto X(a)\}$, and VL be its universal enveloping algebra (see [Ri] for details). This is naturally a left Hopf algebroid, see e.g. [KoKr1]; as therein, we denote by the same symbols elements $a \in A$ and $X \in L$ and the corresponding generators in VL. The source and target maps $s = t$ are equal to the canonical injection $A \to VL$. The coproduct and the counit are given by

$$\begin{array}{rclrcl} \Delta(X) & := & X \otimes_A 1 + 1 \otimes_A X, & \varepsilon(X) & := & 0, \\ \Delta(a) & := & a \otimes_A 1, & \varepsilon(a) & := & a, \end{array} \tag{6.1}$$

whereas the inverse of the Hopf-Galois map is

$$X_+ \otimes_{A^{\mathrm{op}}} X_- := X \otimes_{A^{\mathrm{op}}} 1 - 1 \otimes_{A^{\mathrm{op}}} X, \qquad a_+ \otimes_{A^{\mathrm{op}}} a_- := a \otimes_{A^{\mathrm{op}}} 1, \tag{6.2}$$

where we retain the notation $\otimes_{A^{\mathrm{op}}}$ for the tensor product ${}_\blacktriangleright VL \otimes_{A^{\mathrm{op}}} VL_\triangleleft$ although A is commutative. By universality, these maps can be extended to VL.

Definition 6.1. The A-linear dual $JL := \mathrm{Hom}_A(VL, A)$ is called the *jet space* of (A, L).

By duality, JL carries a commutative A^{e}-algebra structure with product

$$(fg)(u) = f(u_{(1)})g(u_{(2)}), \qquad f, g \in JL,\ u \in VL, \tag{6.3}$$

unit given by the counit ε of VL, and source and target maps given by

$$s(a)(u) := a\varepsilon(u) = \varepsilon(au), \qquad t(a)(u) := \varepsilon(ua), \qquad a \in A, u \in VL. \tag{6.4}$$

Observe that these do not coincide although A is commutative.

The A^{e}-algebra JL is complete with respect to the (topology defined by the) decreasing filtration whose degree p part consists of those functionals that vanish on the A-linear span $(VL)_{\leqslant p} \subseteq VL$ of all monomials in up to p elements of L. For finitely generated projective L, Rinehart's generalised PBW theorem [Ri] identifies JL with the completed symmetric algebra of the A-module $L^* = \mathrm{Hom}_A(L, A)$.

Example 6.2. The simplest example beyond Lie algebras is $A = k[x], L = \mathrm{Der}_k(A)$, in which case L is generated as an A-module by $p := \frac{d}{dx}$. Then VL is isomorphic to the first Weyl algebra. In particular, there is an A-algebra isomorphism $JL \simeq A[\![h]\!]$ under which h^i corresponds to the A-linear functional on $A[p]$ that maps p^j to $\delta_{ij} \in A$. Here JL is considered as A-algebra via the source map s which becomes under the isomorphism the standard unit map of $A[\![h]\!]$. However, the target map t maps a polynomial $a \in A$ to the power series given by its jet

$$t(a) = a + \frac{da}{dx}h + \frac{d^2a}{dx^2}h^2 + \cdots .$$

The filtration of JL induces one of $JL \otimes_A JL$ and if we denote by $JL\hat{\otimes}_A JL$ the completion, then the product of VL yields a coproduct $\Delta : JL \to JL\hat{\otimes}_A JL$ determined by

$$f(uv) =: \Delta(f)(u \otimes_{A^{\mathrm{op}}} v) = f_{(1)}(uf_{(2)}(v)), \tag{6.5}$$

see Lemma 3.16 in [KoP, §3.4]. This is part of a *complete Hopf algebroid* structure on JL. We refer to [Q, Appendix A] for complete Hopf algebras, the Hopf algebroid generalisation is straightforward. The counit of JL is given by $f \mapsto f(1_{VL})$, and the antipode is

$$(Sf)(u) := \varepsilon(u_+ f(u_-)), \qquad u \in VL, f \in JL, \tag{6.6}$$

which for $u \in L \subseteq VL$ is known under the name *Grothendieck connection*. A short computation gives $S^2 = \mathrm{id}$. The translation map (2.3) is

$$f_+ \hat{\otimes}_{A^{\mathrm{op}}} f_- := f_{(1)} \hat{\otimes}_{A^{\mathrm{op}}} S(f_{(2)}). \tag{6.7}$$

Note that JL is not only a left but a *full* complete Hopf algebroid in the sense of Böhm and Szlachányi [B]. Over noncommutative base algebras this would generally require *two* bialgebroid structures that coincide here. In particular, JL is also a Hopf algebroid over a commutative base ring in the more narrow sense studied already for decades [Hov, Ra].

6.2. $C^\bullet(JL, A)$ **and** $C_\bullet(JL, A)$**.** For complete Hopf algebroids such as JL, the theory developed in this paper needs to be modified as follows, in order for the structure maps (e.g. the cyclic operator t) to be well-defined: in $P_\bullet$ and in the chain complex $C_\bullet(JL, M)$, the completed tensor products have to be used. Similarly, in the definition of a module-comodule and of an SaYD module the coaction might be given by maps $M \to JL\hat{\otimes}_A M$.

Dually, $C^\bullet(JL, A)$ has to be defined as $\mathrm{Hom}^{\mathrm{cont}}_{A^{\mathrm{op}}}(JL^{\hat{\otimes}_{A^{\mathrm{op}}} \bullet}{}_{\triangleleft}, A)$, where cont means that the cochains have to be continuous (A being discrete), as only the operators assigned to these cochains will be well-defined on the completed tensor products.

Unlike for general left Hopf algebroids, we have for JL canonical homology coefficients: using that JL is commutative, one easily verifies that A carries a natural structure of an SaYD module over JL whose action and coaction are given by

$$\begin{array}{rclcrcl} A \otimes JL & \to & A, & \qquad & (a, f) & \mapsto & a\varepsilon(f), \\ A & \to & JL \otimes_A A, & & a & \mapsto & s(a) \otimes_A 1_A, \end{array} \tag{6.8}$$

where s is the source map from (6.4). Hence Theorem 1.5 yields a canonical differential calculus $(H^\bullet(JL, A), H_\bullet(JL, A))$ associated to any Lie-Rinehart algebra (A, L) that we want to discuss in more detail as an illustration of the abstract theory.

6.3. **Lie-Rinehart (co)homology.** In order to do so, recall that the space $\mathrm{Hom}_A(\bigwedge^\bullet_A L, A)$ of alternating A-multilinear forms is a cochain complex of k-modules with respect to

$$\mathsf{d} : \mathrm{Hom}_A(\textstyle\bigwedge^n_A L, A) \to \mathrm{Hom}_A(\textstyle\bigwedge^{n+1}_A L, A)$$

given by (where the terms $\hat{X}^i$ are omitted)

$$\begin{aligned} \mathsf{d}\omega(X^0, \ldots, X^n) := & \sum_{i=0}^{n} (-1)^i X^i\big(\omega(X^0, \ldots, \hat{X}^i, \ldots, X^n)\big) \\ & + \sum_{i<j} (-1)^{i+j} \omega([X^i, X^j], X^0, \ldots, \hat{X}^i, \ldots, \hat{X}^j, \ldots, X^n). \end{aligned} \tag{6.9}$$

In case (A, L) arises from a Lie algebroid E, the above is the complex of E-differential forms (see, for example, [CanWe]), and in case E is the tangent bundle of a smooth manifold, these are the conventional differential forms that appear in differential geometry.

Definition 6.3. $H^\bullet(\mathrm{Hom}_A(\bigwedge_A L, A), \mathsf{d})$ is called the *Lie-Rinehart cohomology* of L.

From [KoP, Theorem 3.21] we gather that there is a morphism of chain complexes

$$F : \big(\bar{C}_\bullet(JL, A), \mathsf{b}\big) \to \big(\mathrm{Hom}_A(\textstyle\bigwedge_A^\bullet L, A), 0\big) \tag{6.10}$$

given in degree n by

$$F(f^1, \ldots, f^n)(X^1 \wedge \cdots \wedge X^n) := (-1)^n \big(Sf^1 \wedge \cdots \wedge Sf^n\big)(X^1, \ldots, X^n).$$

Here $Sf^1 \wedge \cdots \wedge Sf^n$ is the wedge product of alternating multilinear forms. As $C_\bullet(JL, A)$ is defined via completed tensor products, we have

$$C_n(JL, A) \simeq \varprojlim \mathrm{Hom}_A\left((VL^{\otimes_A n})_{\leqslant p}, A\right), \tag{6.11}$$

where $(VL^{\otimes_A n})_{\leqslant p}$ is the degree p part of the filtration induced by that of VL. The antipodes appear above as this isomorphism (6.11) is given by

$$(f^1, \ldots, f^n)(u^1, \ldots, u^n) := Sf^1(u^1) \cdots Sf^n(u^n). \tag{6.12}$$

That F is well-defined on the reduced complex $\bar{C}_\bullet(JL, A)$ follows since degenerate chains vanish under F as (2.11) gives for $X \in L$

$$\varepsilon(X_+ 1_{JL}(X_-)) = \varepsilon(X_+ \varepsilon(X_-)) = \varepsilon(X) = 0. \tag{6.13}$$

When L is finitely generated projective over A, the wedge product of multilinear forms provides an isomorphism

$$\textstyle\bigwedge_A^\bullet L^* \to \mathrm{Hom}_A(\bigwedge_A^\bullet L, A)$$

that we suppress in the sequel. Furthermore, the pairing (6.12) yields an isomorphism (cf. [CaRoVdB, Eq. (4.10)])

$$C^n(JL, A) \simeq VL^{\otimes_A \bullet}. \tag{6.14}$$

Finally, if we denote by $\mathrm{pr} : VL \to L$ the projection on L resulting from Rinehart's PBW theorem, we have:

Proposition 6.4. *Assume that L is finitely generated projective over A and define*

$$F'(\alpha^1 \wedge \cdots \wedge \alpha^n) := \sum_{\sigma \in S_n} (-1)^\sigma \big(\mathrm{pr}^* \alpha^{\sigma(1)}, \ldots, \mathrm{pr}^* \alpha^{\sigma(n)}\big)$$

for $\alpha^1, \ldots, \alpha^n \in L^*$. Then we have

$$FF' = n! \, \mathrm{id}_{\bigwedge_A^n L^*}.$$

In particular, if $\mathbb{Q} \subseteq k$, then the morphism F has a right inverse.

Proof. This follows by straightforward computation, using that (6.2) yields

$$S(\mathrm{pr}^* \alpha) = -\mathrm{pr}^* \alpha \tag{6.15}$$

for every 1-form $\alpha \in L^*$. ☐

Dual to (6.10), one has a morphism

$$F^* : \big(\textstyle\bigwedge_A^\bullet L, 0\big) \to (\bar{C}^\bullet(JL, A), \delta) \tag{6.16}$$

of cochain complexes explicitly given as

$$X^1 \wedge \cdots \wedge X^n \mapsto \big\{(f^1, \ldots, f^n) \mapsto (-1)^n \sum_{\sigma \in S_n} (-1)^\sigma (Sf^1)(X^{\sigma(1)}) \cdots (Sf^n)(X^{\sigma(n)})\big\}.$$

6.4. **The calculus structure for Lie-Rinehart algebras.** Our main aim is to use now F, F^*, and F' to compare the calculus structure on $\big(H^\bullet(JL,A), H_\bullet(JL,A)\big)$ resulting from (the topological version of) Theorem 1.5 with the well-known calculus on $\big(\bigwedge^\bullet_A L, \bigwedge^\bullet_A L^*\big)$ given by the exterior differential, the insertion operator, the Lie derivative for differential forms, along with the classical Cartan homotopy formula (see [Ri, Hue1, Hue2], or [CanWe, Kos, X1] for the case of Lie algebroids and in particular the original reference [C] for the tangent bundle of a smooth manifold). First, recall that these operators, besides d from (6.9), are given by

$$
\begin{aligned}
\mathsf{i}_X : \textstyle\bigwedge^n_A L^* &\to \textstyle\bigwedge^{n-1}_A L^*, \quad \omega \mapsto \omega(\cdot, \ldots, X),\\
\mathsf{L}_X : \textstyle\bigwedge^n_A L^* &\to \textstyle\bigwedge^{n}_A L^*, \quad \mathsf{L}_X\omega(Y^1, \ldots, Y^n) := X\big(\omega(Y^1, \ldots, \hat{Y}^i, \ldots, Y^n)\big)\\
&\qquad\qquad - \sum_{i=1}^n \omega(Y^1, \ldots, [X, Y^i], \ldots, Y^n).
\end{aligned}
$$

where $Y^1, \ldots, Y^n \in L$.

Let us then consider the Gerstenhaber bracket on $C^\bullet(JL, A) \simeq VL^{\otimes_A \bullet}$. Now, $VL^{\otimes_A n}$ carries a canonical comp algebra structure given by

$$
\begin{aligned}
&(u^1 \otimes_A \cdots \otimes_A u^p) \circ^{\mathrm{tens}}_i (v^1 \otimes_A \cdots \otimes_A v^q)\\
&:= (u^1 \otimes_A \cdots \otimes_A u^{i-1} \otimes_A u^i_{(1)} v^1 \otimes_A \cdots \otimes_A u^i_{(q)} v^q \otimes_A u^{i+1} \otimes_A \cdots \otimes_A u^p,
\end{aligned}
\tag{6.17}
$$

for $i = 1, \ldots, p$, and where $\Delta^q(u) = u_{(1)} \otimes_A \cdots \otimes_A u_{(q)}$ is the iterated coproduct (where $\Delta^0 := \varepsilon$ and $\Delta^1 := \mathrm{id}$). This is a slight generalisation to bialgebroids from a statement in [GeSch, p. 65], and the expression is well defined with (2.2).

In the first part of the following proposition we state that (6.17) corresponds to our general expression (3.7) of the Gerstenhaber products by means of the isomorphism (6.14), and in particular that the resulting Gerstenhaber bracket corresponds to the classical Schouten-Nijenhuis bracket on the exterior algebra $\bigwedge^\bullet_A L$. In the second part, we show how the relevant operators from the two mentioned calculi are connected to each other; for the sake of simplicity we restrict to the case where one acts with an element $X \in L = \bigwedge^1_A L$:

Proposition 6.5. *If L is finitely generated projective over A, then for $1 \leqslant i \leqslant p$ one has*

$$
(u^1 \otimes_A \cdots \otimes_A u^p) \circ_i (v^1 \otimes_A \cdots \otimes_A v^q) = (u^1 \otimes_A \cdots \otimes_A u^p) \circ^{\mathrm{tens}}_i (v^1 \otimes_A \cdots \otimes_A v^q),
$$

where the left hand side is the Gerstenhaber product from (3.7). In particular, if $\mathbb{Q} \subseteq k$, then the Gerstenhaber bracket from (3.10) corresponds to the classical Schouten-Nijenhuis bracket by means of the map $\frac{1}{n!}F^*$ from (6.16).

Furthermore, for the operations d, i_X, and L_X of differential, insertion, and Lie derivative of (generalised) forms along a (generalised) vector field $X \in L$, one has on $\bigwedge^n_A L^*$

$$
\begin{aligned}
(n+1)\,\mathsf{d} &= F\mathsf{B}F', &(6.18)\\
(n-1)\,\mathsf{i}_X &= F\iota_{F^*X}F', &(6.19)\\
n\,\mathsf{L}_X &= F\mathcal{L}_{F^*X}F'. &(6.20)
\end{aligned}
$$

Proof. For the general Gerstenhaber product (3.7) one computes with the commutativity of JL, (6.3)–(6.5), (2.4), and using the isomorphism (6.14),

$$
\begin{aligned}
&\big((u^1 \otimes_A \cdots \otimes_A u^p) \circ_i (v^1 \otimes_A \cdots \otimes_A v^q)\big)\big(f^1, \ldots, f^{p+|q|}\big) \\
&= \big(u^1 \otimes_A \cdots \otimes_A u^p\big)\big(f^1, \ldots, f^{i-1}, \mathsf{D}_{v^1 \otimes_A \cdots \otimes_A v^q}(f^i, \ldots, f^{i+|q|}), f^{i+q}, \ldots, f^{p+|q|}\big) \\
&= Sf^1(u^1) \cdots Sf^{i-1}(u^{i-1})\big(S\big(s(Sf^i_{(1)}(v^1) \cdots Sf^{i+|q|}_{(1)}(v^q))f^i_{(2)} \cdots f^{i+|q|}_{(2)}\big)\big)(u^i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Sf^{i+q}(u^{i+1}) \cdots Sf^{p+|q|}(u^p) \\
&= Sf^1(u^1) \cdots Sf^{i-1}(u^{i-1})\, \varepsilon\big(u^i_{(1)+}\varepsilon(v^1_+ f^i_{(1)}(v^1_-))f^i_{(2)}(u^i_{(1)-})\big) \cdots \\
&\qquad\qquad \varepsilon\big(u^i_{(q)+}\varepsilon(v^q_+ f^{i+|q|}_{(1)}(v^q_-))f^{i+|q|}_{(2)}(u^i_{(q)-})\big) Sf^{i+q}(u^{i+1}) \cdots Sf^{p+|q|}(u^p) \\
&= Sf^1(u^1) \cdots Sf^{i-1}(u^{i-1})\, \varepsilon\big(u^i_{(1)+}v^1_+ f^i(v^1_- u^i_{(1)-})\big) \cdots \\
&\qquad\qquad \varepsilon\big(u^i_{(q)+}v^q_+ f^{i+|q|}(v^q_- u^i_{(q)-})\big) Sf^{i+q}(u^{i+1}) \cdots Sf^{p+|q|}(u^p) \\
&= Sf^1(u^1) \cdots Sf^{i-1}(u^{i-1})\, Sf^i(u^i_{(1)}v^1) \cdots Sf^{i+|q|}(u^i_{(q)}v^q) Sf^{i+q}(u^{i+1}) \cdots Sf^{p+|q|}(u^p) \\
&= \big((u^1 \otimes_A \cdots \otimes_A u^p) \circ_i^{\mathrm{tens}} (v^1 \otimes_A \cdots \otimes_A v^q)\big)\big(f^1, \ldots, f^{p+|q|}\big)
\end{aligned}
$$

for $f^i \in JL$ and $u^j, v^k \in VL$. The fact that the Gerstenhaber bracket resulting from (6.17) corresponds to the (generalised) Schouten-Nijenhuis bracket on $\bigwedge_A^\bullet L$ by means of the (generalised) Hochschild-Kostant-Rosenberg map was already shown in [Ca, Theorem 1.4]. Hence, observing that the map $\frac{1}{n!}F^*$ is the mentioned HKR morphism followed by (6.14), the first claim is proven.

Concerning the identity (6.18), as stated in (6.13), the degenerate elements of B vanish under F, whereas the operator (2.20) assumes the form

$$
\mathsf{s}_{-1}\mathsf{N}(f^1, \ldots, f^n) = \sum_{i=0}^{n} (-1)^{in}(f^{i+1}_+, \ldots, f^n_+, f^n_- \cdots f^1_-, f^1_+, \ldots, f^i_+)
$$

for an element $(f^1, \ldots, f^n) \in C_n(JL, A)$, as is quickly revealed by a direct computation using (6.8), (2.2), and the commutativity of JL. Hence, since S is an involution and with (6.7), (2.4), (6.1), and (6.3)–(6.6) one has

$$
\begin{aligned}
&\big(F\mathsf{B}F'(\alpha^1, \ldots, \alpha^n)\big)\big(X^0 \wedge \cdots \wedge X^n\big) \\
&= F\Big(\sum_{i=0}^{n}(-1)^{in}\sum_{\sigma \in S_n}(-1)^\sigma\big((\alpha^{\sigma(i+1)}\mathrm{pr}\,)_+, \ldots, (\alpha^{\sigma(n)}\mathrm{pr}\,)_+, \\
&\qquad\qquad (\alpha^{\sigma(n)}\mathrm{pr}\,)_- \cdots (\alpha^{\sigma(1)}\mathrm{pr}\,)_-, (\alpha^{\sigma(1)}\mathrm{pr}\,)_+, \ldots, (\alpha^{\sigma(i)}\mathrm{pr}\,)_+\big)\Big)\Big(X^0 \wedge \cdots \wedge X^n\Big) \\
&= (n+1)\sum_{\sigma \in S_n}(-1)^\sigma S\big((\alpha^1\mathrm{pr}\,)_{(1)}\big)\big(X^{\sigma(1)}\big) \cdots S\big((\alpha^n\mathrm{pr}\,)_{(1)}\big)\big(X^{\sigma(n)}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\big((\alpha^n\mathrm{pr}\,)_{(2)} \cdots (\alpha^1\mathrm{pr}\,)_{(2)}\big)\big(X^{\sigma(0)}\big) \\
&= (n+1)\sum_{\sigma \in S_n}(-1)^\sigma \varepsilon\big(X_+^{\sigma(1)}(\alpha^1\mathrm{pr}\,)(X_-^{\sigma(1)}X_{(1)}^{\sigma(0)})\big) \cdots \varepsilon\big(X_+^{\sigma(n)}(\alpha^n\mathrm{pr}\,)(X_-^{\sigma(n)}X_{(n)}^{\sigma(0)})\big) \\
&= (n+1)\sum_{i=1}^{n}\sum_{\sigma \in S_n}(-1)^\sigma \varepsilon\big(X_+^{\sigma(1)}(\alpha^1\mathrm{pr}\,)(X_-^{\sigma(1)})\big) \cdots \varepsilon\big(X_+^{\sigma(i)}(\alpha^i\mathrm{pr}\,)(X_-^{\sigma(i)}X^{\sigma(0)})\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \varepsilon\big(X_+^{\sigma(n)}(\alpha^n\mathrm{pr}\,)(X_-^{\sigma(n)})\big) \\
&= (n+1)\sum_{i=1}^{n}\Big[(-1)^{n-1}\sum_{\sigma \in S_n}(-1)^\sigma \alpha^1(X^{\sigma(1)}) \cdots X^{\sigma(i)}\big(\alpha^i(X^{\sigma(0)})\big) \cdots \alpha^n(X^{\sigma(n)}) \\
&\quad + (-1)^n\sum_{\sigma \in S_n}(-1)^\sigma \alpha^1(X^{\sigma(1)}) \cdots (\alpha^i\mathrm{pr}\,)(X^{\sigma(i)}X^{\sigma(0)}) \cdots \alpha^n(X^{\sigma(n)})\Big] \\
&= (n+1)\,\mathsf{d}(\alpha^1 \wedge \cdots \alpha^n)(X_0, \ldots, X_n),
\end{aligned}
$$

where the last line follows from the fact that the vector fields are derivations on A and that $\mathrm{pr}\,(XY - YX) = \mathrm{pr}\,([X, Y]) = [X, Y]$.

As for the insertion operator, we compute with (6.15), (6.3)–(6.6), and $St = s$:

$$\begin{aligned}
&\big(F\iota_{F^*X^n}F'(\alpha^1,\dots,\alpha^n)\big)\big(X^1\wedge\cdots\wedge X^{n-1}\big)\\
&= \sum_{\sigma\in S_n}(-1)^\sigma F\Big(\mathrm{pr}^*\alpha^{\sigma(1)},\dots,\mathrm{pr}^*\alpha^{\sigma(n-2)},\big((\mathrm{pr}^*\alpha^{\sigma(n)})(F^*X^n)\big)\blacktriangleright\mathrm{pr}^*\alpha^{\sigma(n-1)}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big(X^1\wedge\cdots\wedge X^{n-1}\Big)\\
&= (-1)^{n-1}(n-1)\sum_{\sigma\in S_n}(-1)^\sigma(S(\alpha^1\mathrm{pr}))(X^{\sigma(1)})\cdots(S(\alpha^{n-2}\mathrm{pr}))(X^{\sigma(n-2)})\\
&\qquad\qquad\qquad\qquad\Big(S\big((\alpha^{n-1}\mathrm{pr})t(\alpha^n\mathrm{pr}(F^*X^{\sigma(n)}))\big)\Big)\Big(X^{\sigma(n-1)}\Big)\\
&= (n-1)\sum_{\sigma\in S_n}(-1)^\sigma\alpha^1(X^{\sigma(1)})\cdots\alpha^{n-2}(X^{\sigma(n-2)})s(\alpha^n(X^{\sigma(n)}))(X^{\sigma(n-1)}_{(1)})(\alpha^{n-1}\mathrm{pr})(X^{\sigma(n-1)}_{(2)})\\
&= (n-1)\sum_{\sigma\in S_n}(-1)^\sigma\alpha^1(X^{\sigma(1)})\cdots\alpha^{n-1}(X^{\sigma(n-1)})\alpha^n(X^{\sigma(n)})\\
&= (n-1)\big(\mathrm{i}_{X^n}(\alpha^1\wedge\cdots\wedge\alpha^n)\big)\big(X^1,\dots,X^{n-1}\big),
\end{aligned}$$

hence (6.19) is proven.

In a similar way, one proves (6.20) the details of which we omit since the computation is similar to those of the two preceding identities. □

7. HOCHSCHILD (CO)HOMOLOGY AND TWISTED CALABI-YAU ALGEBRAS

In this final section we discuss as an example the action of the Hochschild cohomology $H^\bullet(A,A)$ of an associative algebra A on the Hochschild homology $H_\bullet(A,M)$ with coefficients in suitable A-bimodules M. In particular, the differential calculus discussed in [NTs3] is generalised towards nontrivial coefficients which are not even SaYD modules, and this is used to prove Theorem 1.7.

7.1. **The Hopf algebroid A^{e} and the coefficients A_σ.** As said in the introduction, all the main results of this paper were historically first obtained for the Hochschild cohomology $H^\bullet(A,A)$ and homology $H_\bullet(A,A)$ of an associative k-algebra A. This arises as the special case in which U is the enveloping algebra A^{e} of A, with $\eta=\mathrm{id}_{A^{\mathrm{e}}}$ and coproduct and counit given by

$$\Delta: U\to U\otimes_A U,\ a\otimes_k b\mapsto(a\otimes_k 1)\otimes_A(1\otimes_k b),\quad \varepsilon: U\to A,\ a\otimes_k b\mapsto ab.$$

One then has

$$_\blacktriangleright U\otimes_{A^{\mathrm{op}}}U_\triangleleft = U\otimes_k U/\mathrm{span}_k\{(a\otimes_k cb)\otimes_k(a'\otimes_k b')-(a\otimes_k b)\otimes_k(a'\otimes_k b'c)\},$$

where cb and $b'c$ is understood to be the product in A, and one easily verifies that

$$(a\otimes_k b)_+\otimes_{A^{\mathrm{op}}}(a\otimes_k b)_- := (a\otimes_k 1)\otimes_{A^{\mathrm{op}}}(b\otimes_k 1)$$

yields an inverse of the Galois map as was originally pointed out by Schauenburg. For simplicity, we shall assume throughout this section that k is a field which implies in particular that $U=A^{\mathrm{e}}$ is A-projective (in fact free) with respect to all four actions $\triangleright,\triangleleft,\blacktriangleright,\blacktriangleleft$.

Like JL in the previous section, $U=A^{\mathrm{e}}$ is an example of a full Hopf algebroid in the sense of Böhm and Szlachányi whose antipode $S(a\otimes_k b):=b\otimes_k a$ is an involution. We use this to identify left and right U-modules. Obviously, U-modules can also be identified with A-bimodules with symmetric action of k, and in the sequel M is such a bimodule that will be viewed freely as left or right U-module as necessary.

In particular, any algebra endomorphism $\sigma: A\to A$ defines an A-bimodule A_σ which is A as k-vector space with the A-bimodule respectively right A^{e}-module structure

$$b\blacktriangleright m\blacktriangleleft a = m(a\otimes_k b) := bx\sigma(a),\qquad a,m\in A, b\in A^{\mathrm{op}}.$$

These bimodules are prototypical examples of the homology coefficients we are interested in. They carry a left A^{e}-comodule structure given by

$$A_\sigma\to A^{\mathrm{e}}\otimes_A A_\sigma,\quad m\mapsto(m\otimes_k 1)\otimes_A 1,$$

for which the induced left A-module structure is ${}_{\blacktriangleright}A$. However, in general A_σ is not a stable anti Yetter-Drinfel'd module, see [KoKr2] for a discussion of this fact.

Up to isomorphism, A_σ only depends on the class of σ in the outer automorphism group $\mathrm{Out}(A)$ of A, and $\sigma \mapsto A_\sigma$ yields an embedding of the latter into the Picard group of U-$\mathbf{Mod}$ that appears to have been considered in detail for the first time by Fröhlich [Fr]. The study of the (co)homology of A with coefficients in these bimodules has many motivations. Nest and Tsygan suggested to view the Hochschild cohomology groups $H^\bullet(A, A_\sigma)$ as defining a quantum analogue of the Fukaya category [NTs3, NTs2] while Kustermans, Murphy and Tuset related $H_\bullet(A, A_\sigma)$ to Woronowicz's concept of covariant differential calculi over compact quantum groups [KuMuTu]. Moreover, they arise naturally in the description of the Hochschild (co)homology of the crossed product $A \rtimes_\sigma \mathbb{Z}$, see [GetJ].

7.2. **The Hochschild (co)chain complex.** In this situation, the chain complex $C_\bullet(U, M) = M \otimes_{A^{\mathrm{op}}} U^{\otimes_{A^{\mathrm{op}}} \bullet}$ is isomorphic to the standard Hochschild chain complex

$$C_\bullet(A, M) := M \otimes_k A^{\otimes_k \bullet}$$

by means of the map

$$m \otimes_{A^{\mathrm{op}}} (a_1 \otimes_k b_1) \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} (a_n \otimes_k b_n) \mapsto b_n \cdots b_1 m \otimes_k a_1 \otimes_k \cdots \otimes_k a_n.$$

For $M = A_\sigma$, the para-cyclic structure on $C_\bullet(U, A_\sigma)$ from Proposition 2.5 becomes under this isomorphism

$$\mathsf{d}_i(m \otimes_k y) = \begin{cases} a_n m \otimes_k a_1 \otimes_k \cdots \otimes_k a_{n-1} & \text{if } i = 0, \\ m \otimes_k \cdots \otimes_k a_{n-i} a_{n-i+1} \otimes_k \cdots & \text{if } 1 \leqslant i \leqslant n-1, \\ m\sigma(a_1) \otimes_k a_2 \otimes_k \cdots \otimes_k a_n & \text{if } i = n, \end{cases}$$

$$\mathsf{s}_i(m \otimes_k y) = \begin{cases} m \otimes_k a_1 \otimes_k \cdots \otimes_k a_n \otimes_k 1 & \text{if } i = 0, \\ m \otimes_k \cdots \otimes_k a_{n-i} \otimes_k 1 \otimes_k a_{n-i+1} \otimes_k \cdots & \text{if } 1 \leqslant i \leqslant n-1, \\ m \otimes_k 1 \otimes_k a_1 \otimes_k \cdots \otimes_k a_n & \text{if } i = n, \end{cases}$$

$$\mathsf{t}_n(m \otimes_k y) = \sigma(a_1) \otimes_k a_2 \otimes_k \cdots \otimes_k a_n \otimes_k m,$$

where $m \in A$ and where we abbreviate $y := a_1 \otimes_k \cdots \otimes_k a_n$. In particular, one has

$$\mathsf{T} = \sigma \otimes_k \cdots \otimes_k \sigma,$$

so $C_\bullet(A, A_\sigma)$ is cyclic if and only if $\sigma = \mathrm{id}$ (in which case A_σ is an SaYD module).

Likewise, there is an isomorphism of cochain complexes of k-vector spaces

$$C^\bullet(U, A) \to C^\bullet(A, A) := \mathrm{Hom}_k(A^{\otimes_k \bullet}, A), \quad \varphi \mapsto \tilde{\varphi},$$

where the latter is the standard Hochschild cochain complex [Ho] and $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}(a_1 \otimes_k \cdots \otimes_k a_n) := \varphi\big((a_1 \otimes_k 1) \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} (a_n \otimes_k 1)\big)$$

so that

$$\varphi\big((a_1 \otimes_k b_1) \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} (a_n \otimes_k b_n)\big) = \tilde{\varphi}(a_1 \otimes_k \cdots \otimes_k a_n) b_n \cdots b_1.$$

The resulting operators involved in the calculus structure are given by

$$
\begin{aligned}
\mathsf{B}(m \otimes_k y) &= \sum_{i=0}^{n} (-1)^{in} 1 \otimes_k a_{i+1} \otimes_k \cdots \otimes_k a_n \otimes_k m \otimes_k \sigma(a_1) \otimes_k \cdots \otimes_k \sigma(a_i),\\
\iota_{\tilde\varphi}(m \otimes_k y) &= \tilde\varphi(a_{n-|p|}, \ldots, a_n) m \otimes_k a_1 \otimes_k \cdots \otimes_k a_{n-p},\\
\mathsf{S}_{\tilde\varphi}(m \otimes_k y) &= \sum_{j=0}^{n-p} \sum_{i=0}^{j} (-1)^{\eta_{j,i}^{n,p}} 1 \otimes_k \sigma(a_{n-|p|-j}) \otimes_k \cdots \otimes_k \tilde\varphi\big(\sigma(a_{n-|p|+i-j}) \otimes_k \cdots \otimes_k \sigma(a_{n+i-j})\big)\\
&\qquad \otimes_k \cdots \otimes_k \sigma(a_n) \otimes_k \sigma(m) \otimes_k \sigma^2(a_1) \otimes_k \cdots \otimes_k \sigma^2(a_{n-p-j})\\
\mathcal{L}_{\tilde\varphi}(m \otimes_k y) &= \sum_{i=1}^{n-|p|} (-1)^{\theta_i^{n,p}} \sigma(m) \otimes_k \cdots \otimes_k \tilde\varphi\big(\sigma(a_i) \otimes_k \cdots \otimes_k \sigma(a_{i+|p|})\big) \otimes_k \cdots \otimes_k \sigma(a_n)\\
&\quad + \sum_{i=1}^{p} (-1)^{\xi_i^{n,p}} \sigma\Big(\tilde\varphi\big(a_{n-|p|+i} \otimes_k \cdots \otimes_k a_n \otimes_k m \otimes_k \sigma(a_1) \otimes_k \cdots \otimes_k \sigma(a_{i-1})\big)\Big)\\
&\qquad \otimes_k \sigma(a_i) \otimes_k \cdots \otimes_k \sigma(a_{n-p+i}),
\end{aligned}
$$

Here we again work with the reduced complexes, so $\tilde\varphi \in \bar C^p(A,A)$ and $(m \otimes_k y)$ represents a class in $\bar C_\bullet(A, A_\sigma)$. For $\sigma = \mathrm{id}$ these operators appeared in [Ri, NTs3, Get].

7.3. **The case of semisimple** σ**.** A particularly well-behaved case is when the automorphism σ is semisimple (diagonalisable), that is, if there is a subset $\Sigma \subseteq k\backslash\{0\}$ and a decomposition of k-vector spaces

$$A = \bigoplus_{\lambda\in\Sigma} A_\lambda, \quad A_\lambda = \{a \in A \mid \sigma(a) = \lambda a\}.$$

Note that we have $1 \in \Sigma$ because $\sigma(1) = 1$, and also that an algebra A equipped with such an automorphism is exactly the same as a G-graded algebra, where G is a submonoid of the multiplicative group $k\backslash\{0\}$, as $\sigma(ab) = \sigma(a)\sigma(b)$ implies $A_\lambda A_\mu \subseteq A_{\lambda\mu}$ (thus the monoid $G \subseteq k\backslash\{0\}$ resulting from $\sigma \in \mathrm{Aut}(A)$ is the one generated by Σ).

This grading yields decompositions of $C^\bullet(A,A)$ and $C_\bullet(A, A_\sigma)$. The chain complex $C_\bullet(A, A_\sigma)$ becomes G-graded by the total degree of a tensor,

$$C_\bullet(A, A_\sigma) = \bigoplus_{\lambda\in G} C_\bullet(A, A_\sigma)_\lambda, \quad C_n(A, A_\sigma)_\lambda = \bigoplus_{\substack{\lambda_0,\ldots,\lambda_n\in G\\ \lambda_0\cdots\lambda_n=\lambda}} A_{\lambda_0} \otimes_k \cdots \otimes_k A_{\lambda_n},$$

which is a decomposition of chain complexes of k-vector spaces. This coincides with the decomposition into eigenspaces of T, and in particular we have

$$\ker(\mathrm{id} - \mathsf{T}) = C_\bullet(A, A_\sigma)_1, \quad \mathrm{im}(\mathrm{id} - \mathsf{T}) = \bigoplus_{\lambda\in G\backslash\{1\}} C_\bullet(A, A_\sigma)_\lambda.$$

It is also immediately seen that this decomposition is in fact one of para-cyclic k-vector spaces, so we have:

Lemma 7.1. *If A is an algebra over a field k and $\sigma \in \mathrm{Aut}(A)$ is a semisimple automorphism, then the para-cyclic k-vector space $C_\bullet(A, A_\sigma)$ is quasi-cyclic.*

Unless G is finite, the decomposition of the cochain complex $C^\bullet(A,A)$ is slightly more subtle. Given a cochain $\tilde\varphi \in C^p(A,A)$, we denote by $\tilde\varphi_\lambda$ its homogeneous component of degree $\lambda \in k\backslash\{0\}$. That is, $\tilde\varphi_\lambda : A^{\otimes_k p} \to A$ is given on the homogeneous component

$$(A^{\otimes_k p})_\mu := \bigoplus_{\substack{\mu_1,\ldots,\mu_p\in G\\ \mu_1\cdots\mu_p=\mu}} A_{\mu_1} \otimes_k \cdots \otimes_k A_{\mu_p}$$

of elements of $A^{\otimes_k p}$ of total degree $\mu \in G$ by

$$\tilde\varphi_\lambda := \pi_{\lambda\mu} \circ \tilde\varphi : (A^{\otimes_k p})_\mu \to A_{\lambda\mu},$$

where $\pi_\nu : A \to A_\nu$ is the projection onto the degree ν part of A. If we denote by

$$C^p(A,A)_\lambda := \{\tilde\varphi \in C^p(A,A) \mid \tilde\varphi((A^{\otimes_k p})_\mu) \subseteq A_{\lambda\mu}\}$$

the set of all λ-homogeneous p-cochains, then $\tilde{\varphi} \mapsto \{\tilde{\varphi}_\lambda\}_{\lambda \in k\backslash\{0\}}$ defines an embedding

$$C^\bullet(A, A) \to \prod_{\lambda \in k\backslash\{0\}} C^\bullet(A, A)_\lambda$$

of cochain complexes of k-vector spaces which is, however, not a quasi-isomorphism in general. Still, we can split off the homogeneous part of degree 1,

$$C^\bullet(A, A) \simeq C^\bullet(A, A)_1 \oplus \Big(C^\bullet(A, A) \cap \prod_{\lambda \in k\backslash\{0,1\}} C^\bullet(A, A)_\lambda\Big),$$

and $C^\bullet(A, A)_1$ consists precisely of those cochains $\tilde{\varphi}$ for which $\mathsf{D}'_{\tilde{\varphi}}$ commutes with T.

Note that $C^\bullet(A, A)_1$ is not equal to $C^\bullet_{A_\sigma}(A, A)$ in general. We rather have:

Lemma 7.2. *With the assumptions and notation as above, we have*

$$C^p_{A_\sigma}(A, A) = \big\{\tilde{\varphi} \in C^p(A, A) \mid \forall \lambda \in k\backslash\{0, 1\} \forall \mu \in G : \tilde{\varphi}_\lambda|_{(A^{\otimes_k p})_{\lambda^{-1}\mu^{-1}}} = 0\big\}.$$

Proof. This follows from the fact that the operator $\mathsf{D}'_{\tilde{\varphi}_\lambda}$ maps a chain $x \otimes_k y \in C_{n+p}(A, A_\sigma)_{\lambda^{-1}} \subseteq \mathrm{im}(\mathrm{id} - \mathsf{T})$ to $x \otimes_k \tilde{\varphi}_\lambda(y) \in C_{n+1}(A, A_\sigma)_1 \subseteq \ker(\mathrm{id} - \mathsf{T})$. ☐

From this it is clear that the projections onto the homogeneous parts leave $C^\bullet_{A_\sigma}(A, A) \subseteq C^\bullet(A, A)$ invariant, so $C^\bullet_{A_\sigma}(A, A)$ splits as well as a direct sum of cochain complexes into $C^\bullet(A, A)_1$ and $C^\bullet_{A_\sigma}(A, A) \cap \prod_{\lambda \neq 1} C^\bullet(A, A)_\lambda$. We therefore obtain:

Lemma 7.3. *If A is an algebra over a field k and $\sigma \in \mathrm{Aut}(A)$ is a semisimple automorphism, then $C^\bullet(A, A)_1$ is a comp subalgebra of $C^\bullet_{A_\sigma}(A, A)$, and the induced morphisms*

$$H^\bullet(C(A, A)_1) \to H^\bullet_{A_\sigma}(A, A), \quad H^\bullet(C(A, A)_1) \to H^\bullet(A, A)$$

are injective and split as maps of $H^\bullet(C(A, A)_1)$-modules.

Example 7.4. Let k be any field, A be the polynomial ring $k[x]$, and σ be specified by $\sigma(x) = qx$ for some fixed $q \in k\backslash\{0\}$ which is assumed to be not a root of unity. Then we have $\Sigma = \{q^n \mid n \in \mathbb{N}\} = G \simeq \mathbb{N}$, and $\ker(\mathrm{id} - \mathsf{T})$ consists only of the (degenerate) multiples of $1 \otimes_k \cdots \otimes_k 1$. Then $C^p(A, A)_1 \simeq k$ for all p while $C^\bullet_{A_\sigma}(A, A)$ consists of all cochains that do not decrease the degree (where "decrease" refers to the ordering of $G \simeq \mathbb{N}$). In particular, $C^0(A, A)_1 \simeq k$ while $C^0_{A_\sigma}(A, A) \simeq A$, and as A is commutative, we also have $H^0_{A_\sigma}(A, A) \simeq A$ while $H^0(C(A, A)_1) \simeq k$.

7.4. **Twisted Calabi-Yau algebras.** More recently, the Hochschild homology groups with coefficients in A_σ have been studied intensively for the fact that large classes of algebras have been recognised to be what is nowadays called a twisted Calabi-Yau algebra:

Definition 7.5. An algebra A is a *twisted Calabi-Yau algebra* with *modular automorphism* $\sigma \in \mathrm{Aut}(A)$ if the A^{e}-module A has (as an A^{e}-module) a finitely generated projective resolution of finite length and there exists $d \in \mathbb{N}$ and isomorphisms of right A^{e}-modules

$$\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \simeq \begin{cases} 0 & i \neq d, \\ A_\sigma & i = d. \end{cases}$$

The number d is then necessarily the *dimension* of A in the sense of [CE], that is, the projective dimension of $A \in A^{\mathrm{e}}$-$\mathbf{Mod}$, and the Ischebeck spectral sequence [I] leads to a Poincaré-type duality

$$H^\bullet(A, A) \simeq H_{d-\bullet}(A, A_\sigma). \tag{7.1}$$

We refer to [BerSo, BrZh, Gi, Ke, Kr, LiW, VdB1, VdB2, VdBdTdV] and the references therein for more information and background, and in particular plenty of examples.

It had been our aim in [KoKr1] to understand the duality (7.1) in the wider context of Hopf algebroids and to observe that (7.1) is an isomorphism of graded $H^\bullet(A, A)$-modules. From that point of view, the essence of the present paper is that (7.1) is even compatible with the Gerstenhaber structure which implies Theorem 1.7. For $\sigma = \mathrm{id}$ this theorem has been proven by Ginzburg in [Gi] and just as therein, the fact is more or less immediate once the full differential calculus structure is established:

Proof of Theorem 1.7. First we need to observe that in the case of a twisted Calabi-Yau algebra, we have $H^\bullet(A, A) \simeq H^\bullet(C(A, A)_1)$. Indeed, we know already that the duality isomorphism (7.1) is an isomorphism of $H^\bullet(A, A)$-modules, see, for instance, Theorem 1 in [KoKr1]. By Lemma 7.1 we know that the homology is in fact concentrated in degree 1 with respect to the G-grading. Hence the cohomology is also concentrated in degree 1, that is, the embedding $C^\bullet(A, A)_1 \to C^\bullet(A, A)$ is a quasi-isomorphism.

Now Theorem 1.5 states in combination with Theorem 1 in [KoKr1] precisely that $H^\bullet(A, A)$ and $H_\bullet(A, A_\sigma)$ form for a twisted Calabi-Yau algebra with semisimple modular automorphism σ what Lambre calls a differential calculus with duality [La, Définition 1.2]. Hence [La, Corollaire 1.6] directly implies Theorem 1.7. $\square$

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