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Employment Higher Degree B-Spline Function for Solving Higher Order Differential Equations

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Abstract As the B-spline method was developed for solving higher order differential equations, we present a brief survey to construct a higher degree B-spline. The new technique has been given in this field, accordingly a numerical illustration used to solve boundary value problems by employ quintic B-spline function. An example has been given for calculating maximum absolute error through *n* nodes.

Keywords: *B-spline, boundary value problems, approximate solution, absolute error*

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1. Introduction

B-spline is a spline function that has minimal support with respect to given degree, smoothness, and domain partition [8], and named B-splines because they formed a basis for all splines [3]. Quartic spline solution of third order singularly perturbed B V P has been studied by [2]. Theoretical background for stable computation by using B-splines with their derivatives studied by [7]. [4] employ quartic B-spline collocation method for solving one-dimensional hyperbolic telegraph equation and exploitation. Quintic B-spline for the numerical solution of fourth order parabolic partial differential equations to find maximum error given by [5] while [1] discussed quartic B-spline differential quadrature method, and [6] employs quartic B-spline method to solve the self-adjoint boundary value problems. [9] in his paper approximate errors calculated by using cubic B-spline function.

As for us in this paper we construct a higher degree B-spline by two different method for solving self adjoint boundary value problems, in the following section we display deriving methods. Section 3 as example define a quintic B-spline. Section 4 describes the definition of Quintic B-spline. Finally Section 5 consists of a computer procedure to compute maximum error for several nodes.

2. Construction of B-Spline

If $(c_i)^n_{i=1}$is a sequence of control points and x=$\{x_i\}_{i=2}^{n+d}$ is (n+d-1) knots for spline of degree d; we have seen that a typical spline can be written as

$$f(x)=\sum_{i=d+1}^{n} p_{i,d}\mathrm{B}_{i,0}(x), x\in[x_{d+1},x_{n+1}],$$

where $B_{I,0}$ is written as

$$B_{i,0}=\begin{cases}1, & for \quad x_i\le x<x_{i+1}\\ 0, & for\ otherwise\end{cases} \tag{1}$$

Any spline of degree 0 can be expressed as a linear combination of the B-spline $B_{i,0}$.

And f(x)=$\sum_{i=d+1}^{n} p_{i,d}(x)$ is piecewise constant function and $x_{d+1}<x<\ldots<x_{n+1}$ although the end knots allowed to coincide.

So higher B-spline is generate fromlower degree of B-splines by

$$\begin{aligned}B_{i,k}(x)&=\left(\frac{x-x_i}{x_{i+k}-x_i}\right)B_{i,k-1}(x)\\&+\left(\frac{x_{i+k+1}-x}{x_{i+k+1}-x_{i+1}}\right)B_{i+1,k-1}(x);k\ge1,i\in Z.\end{aligned} \tag{2}$$

The $B_{i,k}$ functions as defined in (2) are called B-spline of degree *k*.

Another method to generate higher degree B-spline and it is valid only equidistant points:

The B-spline of order m is defined as follows:

$$\begin{aligned}B_{i,m}(t)&=\frac{1}{h^m}\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^{m+1-j}(x_{i-2+j}-t)^m\\&=\frac{1}{h^m}\Delta^{m+1}(x_{i-2}-t)^m.\end{aligned}$$

Where

$$\Delta f(x_0)=f(x_1)-f(x_0)$$
$$\Delta^{k+1}f(x_0)=\Delta^k f(x_1)-\Delta^k f(x_0).$$

3. Quintic B-spline

Let π be a uniform partition of the interval [0,1] such that $a=x_0<x_1<\ldots<x_n=b$ where $h=x_{i+1}-x_i$ or $h=\frac{1}{n}$,then

$$B_{i,5}(x) = \frac{1}{h^5}\begin{cases} (x_{i+4}-x)^5-6(x_{i+3}-x)^5+15(x_{i+2}-x)^5 \\ -20(x_{i+1}-x)^5+15(x_i-x)^5-6(x_{i-1}-x)^5, \\ x_{i-2}<x\le x_{i-1} \\ (x_{i+4}-x)^5-6(x_{i+3}-x)^5+15(x_{i+2}-x)^5 \\ -20(x_{i+1}-x)^5+15(x_i-x)^5, x_{i-1}<x\le x_i \\ (x_{i+4}-x)^5-6(x_{i+3}-x)^5+15(x_{i+2}-x)^5 \\ -20(x_{i+1}-x)^5, x_i<x\le x_{i+1} \\ (x_{i+5}-x)^5-6(x_{i+3}-x)^5+15(x_{i+2}-x)^5, \\ x_{i+1}<x\le x_{i+2} \\ (x_{i+4}-x)^5-6(x_{i+3}-x)^5, x_{i+2}<x\le x_{i+3} \\ (x_{i+4}-x)^5, x_{i+3}<x\le x_{i+4} \\ 0, otherwise \end{cases}$$

$B_{i,5}(x)$ is the B-spline basis function of 5^{th} degree which also called quintic B-spline vanish outside interval. Each quintic B-spline cover five elements. The basis function is non-zero on five knot spans. The set of quintic B-splines $\{B_{-3}, B_{-2}, B_{-1}, \ldots, B_N, B_{N+1}, B_{N+2}\}$ form a basis for the functions over interval [0, 1].

Now let s(x) be the B-spline interpolating function at the nodal points. Then s(x) can be written as $s(x)=\sum_{j=-3}^{n+2} c_j B_j(x)$ where c_j's are unknown coefficients and $B_i(x)$'s are quintic B-spline functions. The value of B_i^5 at the nodal points can be obtained and its differentiating with respect to x, which are summarized in Table 2.

Table 4.1. We found the coefficients of quintic B-spline and its derivative at nodal points from the definition of our B-spline

X	X_{i-2}	X_{i-1}	x_i	X_{i+1}	X_{i+2}	X_{i+3}	X_{i+4}
Bi	0	1	26	66	26	1	0
B_i'	0	$5/h$	$50/h$	0	$-50/h$	$-5/h$	0
B_i''	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0
B_i'''	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0
$B_i^{(4)}$	0	$120/h^3$	$-480/h^4$	$720/h^4$	$-480/h^4$	$120/h^4$	0

4. Description of the Method

Consider the self-adjoin fourth-order singularly perturbed boundary value problem of the form:

$$Lu(x)=-u^{(4)}(x)+a(x)u(x)=f(x), a(x)\ge 0 \quad (3)$$

$$u(0)=\alpha, u(1)=\beta, u'(0)=\gamma, u'(1)=\delta \quad (4)$$

Where α, β, γ and $\delta$are constants and $\in$is a small positive parameter ($0<\in\le 1$),a(x), and f(x) are sufficiently smooth functions. In this survey, we take a(x)=a= constant. Let u(x)= $s(x)=\sum_{j=-3}^{n+2} c_j B_j$ be the approximate solution of boundary value problem (3). Then let $x_0, x_1, \ldots, x_n$ be n+1 grid points in the interval [0,1]. So that we have, $x_i=x_0+ih$, $x_0=0$, $x_n=1$, i=1, 2, …, n; $h=\frac{1}{n}$ at the knots, we get

$$S(x_i)=\sum_{j=-3}^{n+2} c_j B_j(x_i) \quad (5)$$

$$S'(x_i)=\sum_{j=-3}^{n+2} c_j B'_j(x_i) \quad (6)$$

$$S''(x_i)=\sum_{j=-3}^{n+2} c_j B''_j(x_i) \quad (7)$$

$$S'''(x_i)=\sum_{j=-3}^{n+2} c_j B'''_j(x_i) \quad (8)$$

$$S^{(4)}(x_i)=\sum_{j=-3}^{n+2} c_j B^{(4)}_j(x_i) \quad (9)$$

Putting the value of equations (5)-(9) in equation (3), we get

$$-\in\sum_{j=-3}^{n+2} c_j B^{(4)}_j(x_i)+a(x_i)\sum_{j=-3}^{n+2} c_j B_j(x) = f(x_i), i=0, 1, 2, \ldots, n. \quad (10)$$

And the boundary condition becomes,

$$\sum_{j=-3}^{n+2} c_j B_j(x_0)=\alpha \quad (11)$$

$$\sum_{j=-3}^{n+2} c_j B_j(x_n)=\beta \quad (12)$$

$$\sum_{j=-3}^{n+2} c_j B'_j(x_0)=\gamma \quad (13)$$

$$\sum_{j=-3}^{n+2} c_j B'_j(x_n)=\delta \quad (14)$$

The values of the spline function at the knots are determined using table (4.1) and substituting in equations (10)-(14) a system of (n+4) equations with (n+4) unknown. Now, we can write the above system of equations in the following form

$$S(X_n)=I_n,$$

where $X_n=(c_{-3},c_{-2},c_{-1},\ldots,c_0,c_1,\ldots,c_{n+2})^T$ are unknowns,

$$I_n=(\alpha, h\gamma, h^4 f(x_0),\ldots,h^4 f(x_{n-1}), h^4 f(x_n), \beta)^T.$$

From equation (10):

$$-\in\sum_{j=-3}^{n+2} C_j B''''_j(x_i)+a(x_i)\sum_{j=-3}^{n+2} C_j b_j(x_i) = f(x_i), i=0, 1, 2, \ldots, n$$

and boundary condition(11-14),

$$\sum_{j=-3}^{n+2} C_j B_j(x_0)=\alpha,$$

$$\sum_{j=-3}^{n+2} C_j B_j(x_0)=\beta,$$

$$\sum_{j=-3}^{n+2} C_j B'_j(x_0)=\gamma.$$

$\sum_{j=-3}^{n+2} C_j B'_j(x_n)=\delta$, we get the following:

If i=0,then

$$-\in(\frac{120}{h^4}c_{-3}-\frac{480}{h^4}c_{-2}+\frac{720}{h^4}c_{-1}-\frac{480}{h^4}c_0+\frac{120}{h^4}c_1) +a(x_0)(1c_{-3}+26c_{-2}+66c_{-1}+26c_0+1c_1)=f(x_0). \quad (15)$$

For i=1, we obtain

$$-\in(\frac{120}{h^4}c_{-2}-\frac{480}{h^4}c_{-1}+\frac{720}{h^4}c_0-\frac{480}{h^4}c_1+\frac{120}{h^4}c_2) +a(x_1)(1c_{-2}+26c_{-1}+66c_0+26c_1+1c_2)=f(x_1). \quad (16)$$

For i=2, then we have

$$-\in(\frac{120}{h^4}-\frac{480}{h^4}c_0+\frac{720}{h^4}c_1-\frac{480}{h^4}c_2)$$
$$+a(x_2)(1c_{-1}+26c_0+66c_1+26c_2)=f(x_2) \quad (17)$$

For i=3, then

$$-\in(\frac{120}{h^4}c_0-\frac{480}{h^4}c_1+\frac{720}{h^4}c_2)$$
$$+a(x_3)(1c_0+26c_1+66c_2)=f(x_3). \quad (18)$$

If i=4, thus

$$-\in(\frac{120}{h^4}c_1-\frac{480}{h^4}c_2)+a(x_4)(1c_1+26c_2)=f(x_4) \quad (19)$$

If i=5, then

$$-\in\left(\frac{120}{h^4}c_2\right)+a(x_5)(1c_2)=f(x_5) \quad (20)$$

For i=6 ,

$$-\in(0+\ldots+0)+a(x_6)(0+\ldots+0)=f(x_6) \quad (21)$$

For i-n-4, then

$$-\in\left(\frac{120}{h^4}c_{n-3}\right)+a(x_{n-4})c_{n-3}=f(x_{n-4}) \quad (22)$$

For i=n-3, then

$$-\in\left(-\frac{480}{h^4}c_{n-3}+\frac{120}{h^4}c_{n-2}\right)$$
$$+\mathrm{a}(x_{n-3})(26c_{n-3}+1c_{n-2})=f(x_{n-3}) \quad (23)$$

For i-n-2, then

$$-\in\left(\frac{720}{h^4}c_{n-3}-\frac{480}{h^4}c_{n-2}+\frac{120}{h^4}c_{n-1}\right)$$
$$+\mathrm{a}(x_{n-2})(66c_{n-3}+26c_{n-2}+1c_{n-1})=f(x_{n-2}) \quad (24)$$

For i=n-1, then

$$-\in(-\frac{480}{h^4}c_{n-3}+\frac{720}{h^4}c_{n-2}-\frac{480}{h^4}c_{n-1}+\frac{120}{h^4}c_n)$$
$$+\mathrm{a}(x_{n-1})(26c_{n-3}+66c_{n-2}+26c_{n-1}+1c_n)=f(x_{n-1}) \quad (25)$$

Finally for i-n, we obtain that

$$-\in\left(\begin{array}{l}\frac{120}{h^4}c_{n-3}-\frac{480}{h^4}c_{n-2}\\+\frac{720}{h^4}c_{n-1}-\frac{480}{h^4}c_n+\frac{120}{h^4}c_{n+1}\end{array}\right)$$
$$+\mathrm{a}(x_n)\left(\begin{array}{l}1c_{n-3}+26c_{n-2}+66c_{n-1}\\+26c_n+1c_{n+1}\end{array}\right)=f(x_n) \quad (26)$$

And boundary conditions(11)-(14) gives:

$$c_{-3}+26c_{-2}+66c_{-1}+26c_0+1c_1=\alpha, \quad (27)$$
$$c_{n-3}+26c_{n-2}+66c_{n-1}+26c_n+1c_{n+1}=\beta\ , \quad (28)$$
$$\frac{1}{n}(-5c_{-3}-50c_{-2}+0+50c_0+5c_1)=\gamma, \quad (29)$$
$$\frac{1}{n}(-5c_{n-3}-50c_{n-2}+0+50c_n+5c_{n+1})=\delta. \quad (30)$$

5. Numerical Result

In this section we solve higher order B. V. Ps. By using quintic B-spline interpolation as follows:

For order four B. V. Ps. Take the following

Example 1: Consider the fourth order boundary value problem:

$$-\in y^{(4)}+4y=x, \text{with BC}: \mathrm{y}(0)=0, \mathrm{y}(1)=0,$$
$$\mathrm{y}'(0)=0, \mathrm{y}'(1)=0, x\in[0,1].$$

The maximum error bound gives by the following table:

Table 1. Absolute maximum errors at given N=10, 20, 40 and∈ =10^{-1}, 10^{-2}, 10^{-3}

N \ ∈	10^{-1}	10^{-2}	10^{-3}
10	2.823061491×10^{-2}	3.667433027×10^{-1}	1.416508233×10^{-2}
20	1.273861219×10^{-2}	6.491690733×10^{-1}	1.443154357×10^{-1}
40	1.414624235×10^{-2}	6.536330990×10^{-1}	1.582789743

For order three B. V. Ps. Take the following

Example 2: Consider the following third order singular perturbation problem :

$$-\in y^{(4)}+4y=x, \text{with BC}: \mathrm{y}(0)=0, \mathrm{y}(1)=0,$$
$$\mathrm{y}'(0)=0, \mathrm{y}'(1)=0, x\in[0,1].$$

The maximum error bound gives by the following table:

Table 2. Absolute maximum errors at given N=10, 20, 40 and∈ =10^{-1}, 10^{-2}, 10^{-3}

N \ ∈	10^{-1}	10^{-2}	10^{-3}
10	8.989325582×10^{-3}	8.412423306×10^{-2}	6.933135145×10^{-1}
20	8.125154045×10^{-3}	7.401708169×10^{-2}	4.398766060×10^{-1}
40	6.676666933×10^{-3}	6.198015561×10^{-2}	2.930445177×10^{-1}

For order two B. V. Ps. Take the following

Example 3: Consider the second order boundary value problem with singular perturbation form: $-\in y''+\frac{1}{9}y=\frac{1}{8}x^2$,and subject to the boundary conditions

$$\mathrm{y}(0)=0, \mathrm{y}(1)=0, \textit{ for } x\in[0,1].$$

The maximum error bound gives by the following table:

Table 3. Absolute maximum errors at given N=10, 20, 40 and∈ $=10^{-1}, 10^{-2}, 10^{-3}$

N \ ∈	10^{-1}	10^{-2}	10^{-3}
10	$3.139421929\times 10^{-10}$	$1.650288161\times 10^{-8}$	$1.598924686\times 10^{-4}$
20	$2.362429354\times 10^{-10}$	$7.323553680\times 10^{-10}$	$4.292993654\times 10^{-9}$
40	$2.377712504\times 10^{-10}$	$7.553306635\times 10^{-10}$	$1.380656857\times 10^{-9}$

6. Conclusion

In this paper, we design higher order B –Spline to solve second, third, and fourth order singular perturbed boundary value problems. Also there examples are presented with different values of n and∈ and they showed the efficiency and of our design .

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