ON NON-HOLOMORPHIC FUNCTIONAL CALCULUS FOR COMMUTING OPERATORS

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ABSTRACT. We provide a general scheme to extend Taylor's holomorphic functional calculus for several commuting operators to classes of non-holomorphic functions. These classes of functions will depend on the growth of the operator valued forms that define the resolvent cohomology class. The proofs are based on a generalisation of the so-called resolvent identity to several commuting operators. We give a concrete interpretation of the general result in the case when the spectrum is contained in a convex set in $\mathbb{C}^n$.

1. INTRODUCTION

Let X, Y be two Banach spaces. We denote by $L(X,Y)$ the Banach space of all continuous linear operators from X to Y and we let $L(X) = L(X,X)$. We denote by e the identity operator of $L(X)$. For a subset $A \subset L(X)$ we let A'' denote the bicommutant, that is the Banach algebra of all operators in $L(X)$ which commute with every operator $b \in L(X)$ such that $ab = ba$ for all $a \in A$.

Suppose that $a \in L(X)$. The spectrum of a is then defined as

$$\sigma(a) = \{z \in \mathbb{C} : z - a \text{ is invertible}\},$$

where $z - a$ is the operator $ze - a$. If f is a holomorphic function in a neighbourhood of $\sigma(a)$ then one can define the operator $f(a)$ by the integral

$$f(a) = \frac{1}{2\pi i}\int_{\partial D} f(z)(z-a)^{-1}dz, \tag{1.1}$$

where D is an appropriate neighbourhood of $\sigma(a)$. This expression defines a continuous algebra homomorphism

$$f \mapsto f(a) : \mathcal{O}(\sigma(a)) \to (a)'',$$

such that $1(a) = e$ and $z(a) = a$, called the Riesz functional calculus. We want to extend this algebra homomorphism to functions not necessarily holomorphic in a neighbourhood of the spectrum. In order to

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do this, we define

$$f(a) = -\frac{1}{2\pi i}\int \bar{\partial} f(z) \wedge (z-a)^{-1} dz \tag{1.2}$$

for all $f \in S_a$, where S_a is defined by

$$S_a = \left\{ f \in C_c^1(\mathbb{C}) : \|f\|_a := \left\|\bar{\partial} f(z) \wedge (z-a)^{-1} dz\right\|_\infty < \infty \right\}.$$

It is evident that $f(a)$ is a bounded linear operator on X which commutes with each operator that commutes with a, that is $f(a) \in (a)''$. By Stokes theorem the definition of $f(a)$ only depends on the behaviour of f near $\sigma(a)$. Suppose that D is an open set such that $\sigma(a) \subset D$ and that $f \in \mathcal{O}(D)$. Then if $\phi \in C_c^1(D)$ is equal to 1 in a neighbourhood of $\sigma(a)$, we have that $\phi f \in S_a$ and $\phi f(a)$ defined by (1.2) equals $f(a)$ defined by (1.1).

We now prove the basic theorem of this non-holomorphic functional calculus, that is it is an algebra homomorphism and the spectral mapping theorem holds.

Theorem 1.1. ***(Dynkin)*** *The mapping*

$$f \mapsto f(a) : S_a \to (a)'',$$

where $a \in L(X)$, is a continuous algebra homomorphism that continuously extends the holomorphic functional calculus. Moreover, if $f \in S_a$ then $\sigma(f(a)) = f(\sigma(a))$.

Proof. The map $f \mapsto f(a)$ is obviously linear and continuous. We have the so-called resolvent identity,

$$(w-z)(z-a)^{-1}(w-a)^{-1} = (z-a)^{-1} - (w-a)^{-1} \tag{1.3}$$

where $z, w \in \mathbb{C}$. The multiplicative property then follows,

$$f(a)g(a) = \frac{1}{(2\pi i)^2}\int_z\int_w \bar{\partial} f(z) \wedge (z-a)^{-1} dz \wedge \bar{\partial} g(w) \wedge (w-a)^{-1} dw$$

$$= \frac{1}{(2\pi i)^2}\int_z\int_w \bar{\partial} f(z) \wedge (z-a)^{-1} dz \wedge \bar{\partial} g(w) \wedge (w-z)^{-1} dw$$

$$+\frac{1}{(2\pi i)^2}\int_z\int_w \bar{\partial} f(z) \wedge (z-w)^{-1} dz \wedge \bar{\partial} g(w) \wedge (w-a)^{-1} dw$$

$$= -\frac{1}{2\pi i}\int_z g(z)\bar{\partial} f(z) \wedge (z-a)^{-1} dz$$

$$-\frac{1}{2\pi i}\int_w f(w)\bar{\partial} g(w) \wedge (w-a)^{-1} dw = fg(a),$$

by Fubini-Tonelli's theorem.

Suppose that D is an open neighbourhood of $\sigma(a)$ and that $f_n \in \mathcal{O}(D)$ is a sequence such that $f_n \to 0$ uniformly on compacts. Then if $\phi \in C_c^1(D)$ is a function equal to 1 in a neighbourhood of $\sigma(a)$ we have that $\|f_n\phi\|_a \to 0$. Thus the mapping $f \mapsto f(a)$ continuously extends the holomorphic functional calculus.

If $w \notin f(\sigma(a))$ and $\phi \in C_c^1(\mathbb{C})$ is equal to 1 in an appropriate neighbourhood of $g(\sigma(a))$, then

$$\frac{\phi}{w-f} \in S_a,$$

and hence $w - f(a)$ is invertible and thus $w \notin \sigma(f(a))$. Therefore we have the inclusion $\sigma(f(a)) \subset f(\sigma(a))$. Suppose that $w \in f(\sigma(a))$ and assume that $w = 0$. Then $0 = f(\zeta)$ for some $\zeta \in \sigma(a)$. Let

$$g(z) = \frac{f(z)}{z-\zeta}.$$

Then

$$f(a) = -\frac{1}{2\pi i}\int_z (z-\zeta)\bar{\partial}g(z) \wedge (z-a)^{-1}dz$$

$$= (\zeta - a)\frac{1}{2\pi i}\int_z \bar{\partial}g(z) \wedge (z-a)^{-1}dz$$

$$-\frac{1}{2\pi i}\int_z (z-a)\bar{\partial}g(z) \wedge (z-a)^{-1}dz.$$

The last integral equals $f(\zeta)$, which is 0, and hence $0 \in \sigma(f(a))$ since otherwise $\zeta - a$ would be invertible. Therefore $f(\sigma(a)) \subset \sigma(f(a))$, and hence the theorem is proved. □

Furthermore, we have a rule of composition for this functional calculus.

Theorem 1.2. ***(Rule of composition)*** *If $g \in S_a$ and f is a holomorphic function in a neighbourhood of $\sigma(a)$, then $\phi\,(f \circ g) \in S_a$ and $f(g(a)) = \phi\,(f \circ g)\,(a)$, if $\phi \in C_c^1(\mathbb{C})$ is equal to 1 in a neighbourhood of $\sigma(a)$.*

Proof. Suppose that $\psi \in C_c^1(\mathbb{C})$ is equal to 1 in a neighbourhood of $\sigma(g(a))$. There is a function $\phi \in C_c^1(\mathbb{C})$ such that ϕ is equal to 1 in a neighbourhood of $\sigma(a)$ and

$$h = \frac{\phi}{w-g} \in S_a$$

for each fixed $w \in \operatorname{supp}\left|\bar{\partial}\psi\right|$. The function $\phi(f \circ g)$ is in S_a since

$$\frac{\partial\left(\phi\left(f \circ g\right)\right)}{\partial \bar{z}} = f \circ g\frac{\partial \phi}{\partial \bar{z}} + \phi\frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar{z}}.$$

We have that

$$f(g(a)) = -\frac{1}{2\pi i}\int_w f(w)\bar{\partial}_w\psi(w) \wedge (w-g(a))^{-1}dw$$

$$= \frac{1}{(2\pi i)^2} \int_w \int_z f(w) \bar{\partial}_w \psi(w) \wedge dw \wedge \bar{\partial}_z h(z) \wedge (z-a)^{-1} dz$$

$$= \frac{1}{(2\pi i)^2} \int_z \bar{\partial}_z \int_w f(w) \bar{\partial}_w \psi(w) \wedge \frac{\phi(z) dw}{w - g(z)} \wedge (z-a)^{-1} dz$$

$$= -\frac{1}{2\pi i} \int_z \bar{\partial}_z (\phi f \circ g) \wedge (z-a)^{-1} dz = \phi(f \circ g)(a),$$

and hence the theorem is proved. □

For further results regarding this functional calculus, see Dynkin [6].

Now to the notion of spectrum of a commuting tuple of operators. Suppose that $a = (a_1, \ldots, a_n) \in L(X)^n$ is a commuting tuple of operators, that is $a_i a_j = a_j a_i$ for all i and j. Denote by

$$\Lambda = \oplus_{p=0}^n \Lambda^p$$

the exterior algebra of $\mathbb{C}^n$ over $\mathbb{C}$. If $s_1, \cdots, s_n$ is a basis of $\mathbb{C}^n$ then Λ has the basis

$$s_\emptyset = 1, \quad s_I = s_{i_1} \wedge \cdots \wedge s_{i_p}, \quad I = \{i_1, \cdots, i_p\},$$

where $i_1 < \cdots < i_p$ and $1 \le p \le n$, and we denote $\Lambda = \Lambda(s)$ in this case. We let $K_\bullet(a, X)$ be the Koszul complex induced by a,

$$\cdots \to K_{p+1}(a, X) \xrightarrow{\delta_{p+1}} K_p(a, X) \xrightarrow{\delta_p} K_{p-1}(a, X) \to \cdots,$$

where

$$K_p(a, X) = \Lambda^p(s, X) = X \otimes_{\mathbb{C}} \Lambda^p(s)$$

and

$$\delta_p(x s_I) = 2\pi i \sum_{k=1}^p (-1)^{k-1} a_{i_k} x s_{i_1} \wedge \cdots \wedge \widehat{s}_{i_k} \wedge \cdots \wedge s_{i_p}.$$

If $K_\bullet(a, X)$ is exact then a is called non-singular, otherwise singular. Then the spectrum is defined as

$$\sigma(a) = \{z \in \mathbb{C}^n : z - a \text{ is singular}\}.$$

One also defines the split spectrum as

$$sp(a) = \{z \in \mathbb{C}^n : K_\bullet(z - a, X) \text{ is not split}\},$$

where split means that for every integer p there are operators h and k such that $e = \delta_{p+1} h + k \delta_p$. If X is a Hilbert space or $n = 1$ then $sp(a) = \sigma(a)$. In general we have that $\sigma(a) \subset sp(a)$, but not the reverse inclusion, see Müller [11].

We will consider operators parametrized by a variable z, such as $z \mapsto z - a$. In that case the boundary map δ_p depends on z and we will henceforth suppress the index p and write δ_p as δ_{z-a} for every p. We also let $s_i = dz_i$.

Now suppose that $T \in L(X,Y)$ has closed range and let $k(T)$ be the norm of the inverse of T considered as a map from $X/\operatorname{Ker} T$ to $\operatorname{Im} T$. The next lemma is Lemma 2.1.3 of [7], and it implies that if a_0 is a non-singular tuple then a is non-singular if $\|a_0 - a\|$ is small enough.

Lemma 1.3. *Suppose that X, Y, Z are Banach spaces, $\alpha_0 \in L(X,Y)$, $\beta_0 \in L(Y,Z)$, $\operatorname{Im} \beta_0$ closed and $\operatorname{Ker} \beta_0 = \operatorname{Im} \alpha_0$, that is*

$$X \xrightarrow{\alpha_0} Y \xrightarrow{\beta_0} Z$$

is exact. Let r be a number such that $r > \max\{k(\alpha_0), k(\beta_0)\}$. If $\alpha \in L(X,Y)$, $\beta \in L(Y,Z)$, $\operatorname{Im} \alpha \subset \operatorname{Ker} \beta$ and $\|\alpha - \alpha_0\|, \|\beta - \beta_0\| < 1/6r$ then $\operatorname{Im} \alpha = \operatorname{Ker} \beta$ and $k(\alpha) \leq 4r$.

Hence $\sigma(a)$ is closed. Furthermore, the spectrum has the projection property, see Theorem 2.5.4 of [7].

Theorem 1.4. *If $a \in L(X)^n$ and $a' = (a, a_{n+1}) \in L(X)^{n+1}$ are commuting and $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is defined by $\pi(z, z_{n+1}) = z$ then $\pi(\sigma(a')) = \sigma(a)$.*

It follows that

$$\sigma(a) \subset \sigma(a_1) \times \cdots \times \sigma(a_n)$$

and hence $\sigma(a)$ is bounded. Thus $\sigma(a)$ is a compact subset of $\mathbb{C}^n$. Conversely, any compact set K in $\mathbb{C}^n$ can arise as the spectrum of a commuting tuple of operators. This one sees by letting the operators a_k to be multiplication by z_k on the Banach space $C(K)$ of continuous functions on $K \subset \mathbb{C}^n$.

The next theorem says that pointwise exactness is equivalent to continuous exactness, see Corollary 2.1.4 of [7].

Theorem 1.5. *Suppose that X, Y, Z are Banach spaces and that Ω is a paracompact topological space. Furthermore suppose that $\alpha \in C(\Omega, L(X,Y))$ and $\beta \in C(\Omega, L(Y,Z))$ such that $\operatorname{Im} \beta(\lambda)$ is closed and $\operatorname{Ker} \beta(\lambda) = \operatorname{Im} \alpha(\lambda)$ for all $\lambda \in \Omega$. Then*

$$\operatorname{Ker} \left(C(\Omega, Y) \xrightarrow{\beta} C(\Omega, Z) \right) = \operatorname{Im} \left(C(\Omega, X) \xrightarrow{\alpha} C(\Omega, Y) \right).$$

Moreover for each point $\lambda \in \Omega$ and vector $x \in \operatorname{Ker} \alpha(\lambda)$ there is a function $f \in C(\Omega, X)$ with $\alpha f = 0$ and $f(\lambda) = x$.

Thus the complex

$$K_\bullet(a, C(\mathbb{C}^n \setminus \sigma(a), X))$$

is exact. The next theorem is more complicated to prove, see Taylor [14], Theorem 2.16 and Eschmeier and Putinar [7], Section 6.4.

Theorem 1.6. *Suppose that U is an open subset of $\mathbb{C}^n$, Y_p are Banach spaces, $\alpha_p \in \mathcal{O}(U, L(Y_p, Y_{p-1}))$ and that*

$$\cdots \to Y_{p+1} \xrightarrow{\alpha_{p+1}(z)} Y_p \xrightarrow{\alpha_p(z)} Y_{p-1} \to \cdots$$

is exact for all $z \in U$. Then the complex

$$\cdots \to C^\infty(U, Y_{p+1}) \xrightarrow{\alpha_{p+1}} C^\infty(U, Y_p) \xrightarrow{\alpha_p} C^\infty(U, Y_{p-1}) \to \cdots$$

is exact.

Hence the complex

$$K_\bullet(a, C^\infty(\mathbb{C}^n \setminus \sigma(a), X))$$

is exact.

This notion of joint spectrum for a commuting tuple of operator was introduced by Taylor, [13], in 1970. Furthermore, he proved the holomorphic functional calculus and the spectral mapping theorem for this spectrum in [14]. His first proof of the holomorphic functional calculus was based on the Cauchy-Weil integral. Using homological algebra he generalized the construction to not necessarily commuting tuples of operators in [16]. See Kisil and Ramirez de Arellano [9] for more recent developments of non-commuting functional calculus. In [1, 2] Andersson proved the holomorphic functional calculus for commuting operators using Cauchy-Fantappie-Leray formulas.

The purpose of this paper is to study generalisations of Theorem 1.1 to the case of several commuting operators. In [5] and [3] results of this kind are obtained in the case when the spectrum is contained in $\mathbb{R}^n$, or more generally, in a totally real submanifold $\mathbb{C}^n$. Our main results are contained in Section 3. The basic tool in the proof of these results is a generalisation of the resolvent identity (1.3) to several commuting operators, this is proved Section 2. In order to explain the ideas of the proof in Section 3 we apply the resolvent identity to give a simple proof of the multiplicative property for Taylor's holomorphic functional calculus. Our construction of the holomorphic functional calculus follows the ideas in [1, 2] and in Section 2 we recollect the basic ideas.

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2. Holomorphic functional calculus

Remember that X is a Banach space, $a \in L(X)^n$ is a tuple of commuting operators on X, and $z \in \mathbb{C}^n$ is a variable. Remember also the fact that if the complex $K_\bullet(z - a, X)$ is exact for every z in an open set U then there is a smooth solution u in U to the equation $\delta_{z-a} u = f$ if f is a closed and smooth X-valued form in U.

We now construct the resolvent on $\mathbb{C}^n \setminus \sigma(a)$. We have that

$$\delta_{z-a}\bar\partial \sum_k f_k dz_k = -2\pi i \sum_{k,l}(z_k - a_k)\frac{\partial f_k}{\partial \bar z_l} d\bar z_l = -\bar\partial \delta_{z-a} \sum_k f_k dz_k,$$

and therefore $\delta_{z-a}\bar\partial = -\bar\partial\delta_{z-a}$ for 1-forms and hence for all forms since δ_{z-a} and $\bar\partial$ are anti-derivations. Suppose that $K_\bullet(z-a, X)$ is exact and $x \in X$. Then we can define a sequence u_i in $\mathbb{C}^n \setminus \sigma(a)$ by

$$\delta_{z-a}u_1 = x, \quad \delta_{z-a}u_{i+1} = \bar\partial u_i, \tag{2.1}$$

since $\bar\partial$ and δ_{z-a} anti-commute. If this sequence starts with $x = 0$ then there is a form w_n such that $u_n = \bar\partial w_n$, this follows from the fact that we successively can find w_i such that

$$w_1 = 0, \quad \delta_{z-a}w_{i+1} = \bar\partial w_i - u_i. \tag{2.2}$$

Thus if one has two sequences u_i and u_i' as in (2.1) then the difference $u_n - u_n'$ is exact. Hence u_n defines a Dolbeault cohomology class $\omega_{z-a}x$ of bidegree $(n, n-1)$, which is called the resolvent cohomology class.

Suppose we have two cohomology classes, $\omega_{z-a}x$ and $\omega_{w-b}x$, where $z, w \in \mathbb{C}^n$, $a, b \in L(X)^n$, corresponding to sequences u_i and v_i, respectively. Then one defines the X-valued cohomology class $\omega_{z-a} \wedge \omega_{w-b}x$ as the class of c_{2n}, where c_i solve

$$c_1 = 0, \quad \delta_{z-a,w-b}c_{i+1} = \bar\partial c_i + v_i - u_i. \tag{2.3}$$

To see that this really is a well defined cohomology class, let u_i', v_i' and c_i' be other choices of sequences. Let w_i^u and w_i^v be the sequences given by (2.2) for the sequences $u_i - u_i'$ and $v_i - v_i'$ respectively. Then we have that

$$c_1 - c_1' + w_1^v - w_1^u = 0$$

and

$$\delta_{z-a,w-b}\left(c_{i+1} - c_{i+1}' + w_{i+1}^v - w_{i+1}^u\right) = \bar\partial\left(c_i - c_i' + w_i^v - w_i^u\right).$$

Hence, by (2.2) again, there exists a sequence w_i^c such that $c_{2n} - c_{2n}' = \bar\partial w_{2n}^c$.

Now suppose that we instead have operator valued forms, u_i, such that

$$\delta_{z-a}u_1 = e, \quad \delta_{z-a}u_{i+1} = \bar\partial u_i, \tag{2.4}$$

so that u_n represents the operator valued cohomology class ω_{z-a}. Then we have that $\omega_{z-a}\wedge\omega_{w-b}x$ is the class of $u_n \wedge v_n$, where v_i is an X-valued sequence defining $\omega_{w-b}x$. This follows from the fact

$$\delta_{z-a}\left(u_1 \wedge v_n\right) = v_n, \quad \delta_{z-a}\left(u_{i+1} \wedge v_n\right) = \bar\partial\left(u_i \wedge v_n\right)$$

and the following proposition.

Proposition 2.1. *If v_i is a sequence defining $\omega_{w-b}x$ and*

$$\delta_{z-a}f_1 = v_n, \quad \delta_{z-a}f_{i+1} = \bar{\partial} f_i,$$

then f_n represents $\omega_{z-a} \wedge \omega_{w-b}x$.

Proof. Let c_i be any sequence that defines $\omega_{z-a} \wedge \omega_{w-b}x$, so that c_i satisfies (2.3). Denote by $c_i^{k,l}$ the component of c_i which is of degree k in dz and degree l in dw. We have that $\delta_{z-a}c_i^{0,i} = 0$, so there is a form f such that $c_i^{0,1} = \delta_{z-a}f$. This gives

$$\delta_{z-a,w-b}c_i = \delta_{z-a,w-b}\left(c_i - c_i^{0,1} - \delta_{w-b}f\right),$$

and hence we can assume that the component $c_i^{0,i}$ vanishes. We have that

$$\delta_{z-a}c_{n+1}^{1,n} = v_n, \quad \delta_{z-a}c_{n+i+1}^{i+1,n} = \bar{\partial} c_{n+i}^{i,n},$$

and therefore there is a form w_n such that

$$f_n - c_{2n}^{n,n} + \bar{\partial} w_n = 0.$$

Since $c_{2n} = c_{2n}^{n,n}$ the proposition is proved. □

In one variable there is only one possible representative for $\omega_{z-a}x$, $a \in L(X)$,

$$\omega_{z-a}x = \frac{1}{2\pi i}(z-a)^{-1}dz\, x,$$

and we have that ω_{z-a} is operator valued. The key part of the proof of the holomorphic functional calculus in one variable is the resolvent identity (1.3), which we can reformulate as

$$\omega_{z-a} \wedge \omega_{w-a} + \omega_{w-a} \wedge \omega_{z-w} + \omega_{w-z} \wedge \omega_{z-a} = 0.$$

We will now generalize this equality to several commuting operators. Let $\Delta = \{(z,w) \in \mathbb{C}^{2n} : z = w\}$ be the diagonal in what follows.

Lemma 2.2. *For every $x \in X$, we have the equality*

$$\omega_{z-a} \wedge \omega_{w-a}x + \omega_{w-a} \wedge \omega_{z-w}x + \omega_{w-z} \wedge \omega_{z-a}x = 0, \tag{2.5}$$

on $((\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))) \setminus \Delta$.

Proof. Define the sequence m_k by

$$m_k = \frac{1}{(2\pi i)^k}\frac{\partial |z-w|^2}{|z-w|^2} \wedge \left(\bar{\partial}\frac{\partial |z-w|^2}{|z-w|^2}\right)^{k-1}. \tag{2.6}$$

The equalities,

$$\delta_{z-a,w-a}m_1 = \frac{1}{2\pi i\,|z-w|^2}\delta_{z-a,w-a}\partial |z-w|^2 = 1, \tag{2.7}$$

$$\delta_{z-a,w-a}m_{k+1} = \frac{1}{(2\pi i)^k}\left(\bar{\partial}\frac{\partial |z-w|^2}{|z-w|^2}\right)^k = \bar{\partial} m_k, \tag{2.8}$$

for all $k \le n$, and $m_k = 0$ for all $k > n$, holds on $\mathbb{C}^{2n} \setminus \Delta$. Let u_i be a sequence as in (2.1) that defines $\omega_{z-a}x$. Define u_i^1 and u_i^2 by $u_i^1 = \pi_1^* u_i$ and $u_i^2 = \pi_2^* u_i$, where $\pi_1(z,w) = z$ and $\pi_2(z,w) = w$ are the projections. Let c_i be a sequence that satisfies the equalities

$$c_1 = 0, \quad \delta_{z-a,w-a} c_{l+1} = \bar\partial c_l + u_l^2 - u_l^1. \tag{2.9}$$

Using the equalities (2.7), (2.8) and (2.9) (for $l \ge n$), we get that

$$-\bar\partial \sum_{k+l=2n} m_k \wedge c_l = \delta_{z-a,w-a} \sum_{k+l=2n+1} m_k \wedge c_l - \bar\partial \sum_{k+l=2n} m_k \wedge c_l$$

$$= \sum_{l=n+1}^{2n} \delta_{z-a,w-a} m_{2n+1-l} \wedge c_l - \sum_{l=n}^{2n-1} \bar\partial m_{2n-l} \wedge c_l$$

$$+ \sum_{k=1}^{n} m_k \wedge \left(\bar\partial c_{2n-k} - \delta_{z-a,w-a} c_{2n+1-k}\right)$$

$$= -\bar\partial m_n \wedge c_n + c_{2n} + m_n \wedge \left(u_n^1 - u_n^2\right).$$

Thus

$$-\bar\partial \sum_{k+l=2n} m_k \wedge c_l = c_{2n} + u_n^2 \wedge m_n + m_n \wedge u_n^1 \tag{2.10}$$

outside the diagonal. We have that the component of m_n without dw and $d\bar w$ represents ω_{z-w} and that the component of m_n without dz or $d\bar z$ repesents ω_{w-z}. Since c_{2n} represents $\omega_{z-a} \wedge \omega_{w-a} x$, the lemma follows from (2.10). $\square$

Choose representatives $\widetilde\omega_{z-a}x$, $\widetilde\omega_{w-a}x$ and $\widetilde\omega_{z-a} \wedge \widetilde\omega_{w-a}x$ for $\omega_{z-a}x$, $\omega_{w-a}x$ and $\omega_{z-a} \wedge \omega_{w-a}x$ respectively on $(\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))$. Let $\widetilde\omega_{z-w} = m_n$. Then (2.5) says that the form defined on $((\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))) \setminus \Delta$

$$\widetilde\omega_{z-a} \wedge \widetilde\omega_{w-a}x + \widetilde\omega_{w-a} \wedge \widetilde\omega_{z-w}x + \widetilde\omega_{z-w} \wedge \widetilde\omega_{z-a}x \tag{2.11}$$

is exact. We want this expression to be an exact current over Δ as well. Suppose that (2.11) holds on $(\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma(a))$. We have that $[\Delta] = \bar\partial \widetilde\omega_{z-w}$, where $[\Delta]$ denotes the current of integration over $[\Delta]$. If we apply $\bar\partial$ to (2.11), interpreted as a current, we get

$$0 = -\widetilde\omega_{w-a} \wedge [\Delta] + [\Delta] \wedge \widetilde\omega_{z-a} = [\Delta] \wedge (\widetilde\omega_{z-a} - \widetilde\omega_{w-a}).$$

Hence $i^*(\widetilde\omega_{z-a} - \widetilde\omega_{w-a}) = 0$, where i is the function defined by $i(\tau) = (\tau, \tau)$. The next theorem gives the desired equality in the case where we have $i^*\widetilde\omega_{z-a} = i^*\widetilde\omega_{w-a}$.

Theorem 2.3. ***(Resolvent identity)*** *Suppose that $\widetilde{\omega}_{z-a}x$, $\widetilde{\omega}_{w-a}x$ and $\widetilde{\omega}_{z-a}\wedge\widetilde{\omega}_{w-a}x$ are representatives for $\omega_{z-a}x$, $\omega_{w-a}x$ and $\omega_{z-a}\wedge\omega_{w-a}x$, respectively. Let $\widetilde{\omega}_{z-w}=m_n$, where m_n is defined in (2.6). Then the current*

$$\widetilde{\omega}_{z-a}\wedge\widetilde{\omega}_{w-a}x+\widetilde{\omega}_{w-a}\wedge\widetilde{\omega}_{z-w}x+\widetilde{\omega}_{z-w}\wedge\widetilde{\omega}_{z-a}x$$

defined on $(\mathbb{C}^n\setminus\sigma(a))\times\mathbb{C}^n\cap\mathbb{C}^n\times(\mathbb{C}^n\setminus\sigma(a))$ is exact if and only if $i^*\widetilde{\omega}_{z-a}=i^*\widetilde{\omega}_{w-a}$, where $i:\mathbb{C}^n\to\mathbb{C}^{2n}$ is defined by $i(\tau)=(\tau,\tau)$.

Proof. The necessity of having $i^*\left(\widetilde{\omega}_{z-a}-\widetilde{\omega}_{w-a}\right)=0$ has already been proved. Now suppose that $i^*\left(\widetilde{\omega}_{z-a}-\widetilde{\omega}_{w-a}\right)=0$. Let u_i^1, u_i^2, m_i and c_i be the sequences in the proof of Lemma 2.2. Let $\delta=\delta_{z-a,w-a}$. Then we have that $i^*\delta=\delta_{\tau-a}i^*$ by induction, since

$$i^*\delta\left(f\,dz_k+g\,dw_l\right)=\left(\tau_k-a_k\right)f(\tau,\tau)+\left(\tau_l-a_l\right)g(\tau,\tau)$$

$$=\delta_{\tau-a}i^*\left(f\,dz_k+g\,dw_l\right)$$

and

$$i^*\delta(u\wedge v)=i^*\delta u\wedge i^*v-i^*u\wedge i^*\delta v=\delta_{\tau-a}i^*\left(u\wedge v\right),$$

if u is a 1-form. Thus

$$i^*c_1=0,\quad \delta_{\tau-a}i^*c_{i+1}=\bar\partial i^*c_i$$

and hence, by (2.2), there is a form w_n of τ such that $i^*c_n=\bar\partial w_n$. For all test forms f we have the identity

$$\bar\partial m_n\wedge c_n.f=\int_\Delta i^*\left(c_n\wedge f\right)=\int_\Delta\bar\partial w_n\wedge i^*f=\int_\Delta w_n\wedge i^*\bar\partial f.$$

Therefore the calculation in the proof of Lemma 2.2 gives the equality

$$-\bar\partial\left([\Delta]\wedge w_n+\sum_{k=1}^n m_k\wedge c_{2n-k}\right)=c_{2n}+u_n^2\wedge m_n+m_n\wedge u_n^1. \tag{2.12}$$

Since $\widetilde{\omega}_{\tau-a}x$ and u_n represent the same cohomology class, there is a form q such that $\widetilde{\omega}_{\tau-a}x-u_n=\bar\partial q$. Let $q^1=\pi_1^*q$ and $q^2=\pi_2^*q$. Then

$$\widetilde{\omega}_{z-w}\wedge\left(\widetilde{\omega}_{z-a}x-\widetilde{\omega}_{w-a}x-\left(u_n^1-u_n^2\right)\right)$$

$$=\widetilde{\omega}_{z-w}\wedge\left(\bar\partial q^1-\bar\partial q^2\right)=[\Delta]\wedge\left(q^1-q^2\right)-\bar\partial\left(\widetilde{\omega}_{z-w}\wedge\left(q^1-q^2\right)\right)$$

$$=-\bar\partial\left(\widetilde{\omega}_{z-w}\wedge\left(q^1-q^2\right)\right).$$

Thus, since $\widetilde{\omega}_{z-a}\wedge\widetilde{\omega}_{w-a}x-c_{2n}$ is an exact current, the theorem is proved. □

Now we give the definition of $f(a)$. If f is a holomorphic function in a neighbourhood of $\sigma(a)$ then we define $f(a)$ by the formula

$$f(a)x=-\int f\bar\partial\phi\wedge\omega_{z-a}x\quad\text{for all } x\in X, \tag{2.13}$$

where $\phi \in C_c^\infty$ is equal to 1 in a neighbourhood of $\sigma(a)$. This definition is independent of the choice of ϕ. To see this, suppose that $\varphi \in C_c^\infty$ is equal to 0 in a neighbourhood of the spectrum. Then we have that

$$\int \bar{\partial}\varphi \wedge \omega_{z-a}x = \int \bar{\partial}\varphi \wedge u_n = \int \bar{\partial}\left(\varphi \wedge u_n\right) = 0,$$

if u_n is a smooth form in $\mathbb{C}^n \setminus \sigma(a)$ representing $\omega_{z-a}x$. Note also that, by Stokes theorem, we have the equality

$$-\int f\bar{\partial}\phi \wedge \omega_{z-a}x = \int_{\partial D} f\omega_{z-a}x,$$

where D is a small enough neighbourhood of $\sigma(a)$. We now prove that $f(a) \in (a)''$.

Lemma 2.4. *If $f(a)$ is defined by the formula (2.13), then $f(a) \in (a)''$.*

Proof. Suppose that $x, y \in X$ and $c, d \in \mathbb{C}$. Denote by u_i^x the sequence (2.1). Then

$$\delta_{z-a}\left(u_1^{cx+dy} - cu_1^x - du_1^y\right) = 0$$

and

$$\delta_{z-a}\left(u_{i+1}^{cx+dy} - cu_{i+1}^x - du_{i+1}^y\right) = \bar{\partial}\left(u_i^{cx+dy} - cu_i^x - du_i^y\right),$$

so u_n^{cx+dy} and $cu_n^x + du_n^y$ define the same cohomology class. Therefore the resolvent is linear, i.e.,

$$\omega_{z-a}\left(cx + dy\right) = c\omega_{z-a}x + d\omega_{z-a}y,$$

and hence $f(a)$ is a linear operator.

The map δ_{z-a} is linear, continuous and surjective between the Frechet space of all $C_{p+1,q}^\infty(U, X)$ forms to the Frechet space of all δ_{z-a}-closed $C_{p,q}^\infty(U, X)$ forms, where $U = \mathbb{C}^n \setminus \sigma(a)$. Let $K_1 \subset \mathbb{C}^n \setminus \sigma(a)$ be a given compact set and let $t_1 = 0$. Then the open mapping theorem gives the existence of a sequence of compact sets $K_i \subset \mathbb{C}^n \setminus \sigma(a)$ and natural numbers t_i such that the equation $\delta_{z-a}u = v$ has a solution u, which satisfies

$$\|u\|_{K_i, t_i+1} \leq C\|v\|_{K_{i+1}, t_{i+1}}$$

for all closed v. Thus we can choose the sequence (2.1) so that

$$\|u_1\|_{K_n, t_n+1} \leq C\|x\|_{K_{n+1}, t_{n+1}} = C\|x\|$$

and

$$\|u_{i+1}\|_{K_{n-i}, t_{n-i}+1} \leq C\left\|\bar{\partial}u_i\right\|_{K_{n-i+1}, t_{n-i+1}} \leq C\|u_i\|_{K_{n-i+1}, t_{n-i+1}+1}.$$

Hence

$$\|f(a)x\| \leq \int \left\|f\bar{\partial}\phi \wedge u_n\right\| \leq C\,|f|_{\text{supp }\phi}\,\|x\| \tag{2.14}$$

and thus the operator $f(a)$ is bounded.

Suppose that $b \in L(X)$ is an operator which commutes with the tuple a. Then

$$\delta_{z-a} b u_1^x = bx, \quad \delta_{z-a} b u_{i+1}^x = \bar\partial b u_i^x,$$

so bu_n^x and u_n^{bx} defines the same cohomology class. Therefore

$$b\omega_{z-a} x = \omega_{z-a} bx$$

and thus $f(a) \in (a)''$. □

We can now prove Taylor's theorem.

Theorem 2.5. ***(Taylor)*** *The mapping*

$$f \mapsto f(a) : \mathcal{O}(\sigma(a)) \to (a)'' \tag{2.15}$$

is a continuous algebra homomorphism such that $1(a) = e$ *and* $z_k(a) = a_k$.

Proof. The map $f \mapsto f(a)$ is continuous by (2.14). We now prove that $f(a)g(a) = fg(a)$. Let u_i, u_i^1, u_i^2 and c_i be as in Lemma 2.2. By the proof of Proposition 2.1 we can assume that the component $c_i^{0,i}$ vanishes. Since

$$\delta_{z-a} c_{n+1}^{1,n} = u_n(w), \quad \delta_{z-a} c_{n+i+1}^{i+1,n} = \bar\partial c_{n+i}^{i,n},$$

we have that c_{2n} represents $\omega_{z-a} u_n(w)$ and thus we have that

$$f(a)u_n(w) = -\int_z f(z)\bar\partial\phi_1(z) \wedge \omega_{z-a} u_n(w) = -\int_z f(z)\bar\partial\phi_1(z) \wedge c_{2n}.$$

Multiplying this equality by $g(w)\bar\partial\phi_2(w)$ and integrating with respect to w we get

$$f(a)g(a)x = \int_w \int_z f(z)g(w)\bar\partial\phi_2(w) \wedge \bar\partial\phi_1(z) \wedge c_{2n}.$$

The resolvent identity (2.12) then gives that the right hand side is equal to

$$\iint fg\bar\partial\phi_1 \wedge \bar\partial\phi_2 \wedge m_n \wedge u_n^1 + \iint fg\bar\partial\phi_1 \wedge \bar\partial\phi_2 \wedge u_n^2 \wedge m_n,$$

and hence we get, by the Bochner-Martinelli integral formula,

$$-\int \left(fg\phi_2\bar\partial\phi_1 + f\phi_1 g\bar\partial\phi_2\right) \wedge u_n = -\int fg\bar\partial\left(\phi_1\phi_2\right) \wedge u_n = fg(a)x,$$

since $u_n^1 = \pi_1^* u_n$ and $u_n^2 = \pi_2^* u_n$. Since the map (2.15) obviously is linear, it is an algebra homomorphism.

It remains to prove that $1(a) = e$ and $z_k(a) = a_k$. The first equality follows by representing ω_{z-a} by

$$\frac{1}{(2\pi i)^n}\left(|z|^2 e - \bar z a\right)^{-n} \partial|z|^2 \wedge \left(\bar\partial\partial|z|^2\right)^{n-1},$$

cf. [1], and integrating against $\bar{\partial}\phi$, where ϕ is a radial cutoff function which is equal to 1 in a neighbourhood of $\sigma(a)$. The second equality follows from the first equality and

$$(z_k - a_k)\, u_n = \frac{1}{2\pi i}\left(\delta_{z-a} u_n\right) \wedge dz_k = \frac{1}{2\pi i}\bar{\partial}\left(u_{n-1} \wedge dz_k\right),$$

where u_i is a sequence that satisfies (2.1). $\square$

The next theorem says what happens when one has a norm convergent sequence in $L(X)^n$. Note first that if $a_k \to a_0$ in operator norm and D is an open set such that $\sigma(a_0) \subset D$ then

$$\sigma(a_k) \subset D$$

for all but a finite number of k. Suppose that is not the case. Theorem 1.4 gives then that all the sets $\sigma(a_k)$ are supported in a fixed bounded set, hence we would have a convergent sequence $z_k \in \mathbb{C}^n \setminus D$ such that $z_k - a_k$ is singular. Therefore, by Lemma 1.3, this would contradict the assumption that $\sigma(a_0) \subset D$. Notice also that if $\sigma(a) = sp(a)$ then the conclusion in the following theorem would be that $f(a_k) \to f(a_0)$ in operator norm.

Theorem 2.6. *Suppose that $a_k \in L(X)^n$ are commuting for every $k \geq 0$ and that $\|a_k - a_0\| \to 0$ as $k \to \infty$. If f is holomorphic in a neighbourhood of $\cup_{k\geq 0}\sigma(a_k)$, then $f(a_k)x \to f(a_0)x$ for every $x \in X$.*

Proof. Consider the Banach space

$$c(X) = \left\{(x_k)_{k=0}^{\infty} : \lim_{k\to\infty} \|x_k - x_0\| = 0\right\}$$

with norm $\|(x_k)_{k=0}^{\infty}\|_{\infty} = \sup_{k\leq 0} \|x_k\|$ and the tuple of n operators $a' \in L(c(X))^n$ defined by $a'(x_k)_{k=0}^{\infty} = (a_k x_k)_{k=0}^{\infty}$. Suppose that a_k is a non-singular tuple for every $k \geq 0$ and that f is a closed $c(X)$-form, that is $\delta_{a'} f = 0$. Then $\delta_{a_k} f_k = 0$ for every $k \geq 0$. Hence there is a solution u_0 of the equation $\delta_{a_0} u_0 = f_0$ since a_0 is non-singular. Lemma 1.3 gives a uniform constant C and v_k such that $\delta_{a_k} v_k = \delta_{a_k} u_0 - f_k$ and

$$\|v_k\| \leq C\|\delta_{a_k} u_0 - f_k\| \leq C\|\delta_{a_k} - \delta_{a_0}\|\|u_0\| + C\|f_0 - f_k\|.$$

Thus $u_k = u_0 - v_k$ solve the equations $\delta_{a_k} u_k = f_k$ and $u_k \to u_0$ if $k \to \infty$. Hence $u = (u_k)_{k=0}^{\infty}$ is a solution of $\delta_{a'} u = f$ and the complex $K_\bullet(a', c(X))$ is exact, and thus a' is non-singular. That is, we have proved the inclusion

$$\sigma(a') \subset \bigcup_{k\geq 0} \sigma(a_k).$$

Let u_i be smooth $c(X)$-forms defined on $\mathbb{C}^n \setminus \sigma(a')$ by the equations

$$\delta_{z-a'} u_1 = x, \quad \delta_{z-a'} u_{i+1} = \bar{\partial} u_i.$$

Thus $(u_n)_k$ represent $\omega_{z-a_k}x$ for all $k > 0$ and $(u_n)_0 = \lim_{k\to\infty}(u_n)_k$ represents $\omega_{z-a_0}x$. Suppose that $\phi \in C_c^\infty$ is equal to 1 in a neighbourhood the union of $\sigma(a_k)$. Then

$$\lim_{k\to\infty} f(a_k)x = -\lim_{k\to\infty}\int f\bar{\partial}\phi\wedge(u_n)_k = -\int f\bar{\partial}\phi\wedge(u_n)_0 = f(a_0)x$$

for all $x \in X$, and hence the theorem is proved. □

3. Non-holomorphic functional calculus

In this section we will extend the holomorphic functional calculus of Section 2 to functions such that $\left|\bar{\partial}f(z)\right|$ tends to zero when z approches the spectrum. If f is a C^1-function with compact support, we define whenever possible

$$f(a)x = -\int \bar{\partial}f\wedge u_n^x,$$

where u_n^x is a form that represents $\omega_{z-a}x$.

Several problems occur. There is a problem with the possible dependence of the choice of representative u_n^x of the class $\omega_{z-a}x$. Other problems are to investigate whether

$$f(a)\in(a)'',\quad f(a)g(a)=fg(a),\quad \sigma(f(a))=f(\sigma(a)),$$

$$g(f(a)) = g\circ f(a)$$

and whether $f(a) = 0$ if $f = 0$ on $\sigma(a)$. We will prove that $f(a)g(a) = fg(a)$, $f(a)\in(a)''$ and $\sigma(f(a)) = f(\sigma(a))$ for a certain algebra S_a (3.7) of functions. In order to do this, we will need a slightly stronger condition on $\bar{\partial}f$ than in the case $n = 1$. To begin with, we will see what is needed for the muliplicative property to hold.

Suppose that $E \supset \sigma(a)$ is a compact set such that there exists a sequence u_i on $\mathbb{C}^n\setminus E$ satisfying (2.4). Then we have that u_n is operator valued and represents ω_{z-a} in $\mathbb{C}^n\setminus E$. The definition of $f(a)$ in this case is

$$f(a) = -\int\bar{\partial}f\wedge u_n.$$

Define a sequence c_l by

$$c_1 = 0,\quad \delta_{z-a,w-a}c_{l+1} = \bar{\partial}c_l + u_l^2 - u_l^1, \tag{3.1}$$

where $u_l^1 = \pi_1^* u_l$ and $u_l^2 = \pi_2^* u_l$. Then we have that c_{2n} represents $\omega_{z-a}\wedge\omega_{w-a}$. We now prove the multiplicative property.

Proposition 3.1. *Let u_i be a sequence defined on $\mathbb{C}^n\setminus E$, where $E\supset\sigma(a)$ is a compact set, as in (2.4), and suppose that c_l, $n\le l\le 2n$ are forms that satisfies the condition,*

$$i^*c_n = 0,\quad \delta_{z-a,w-a}c_{l+1} = \bar{\partial}c_l + u_l^2 - u_l^1,\quad c_{2n} = u_n^1\wedge u_n^2, \tag{3.2}$$

where $i(\tau) = (\tau, \tau)$. *Moreover suppose that* $f, g \in C^2_c$ *such that*

$$\int \left\| \bar\partial f \wedge u_n \right\| < \infty, \quad \int \left\| \bar\partial g \wedge u_n \right\| < \infty$$

and

$$\int_z \int_w \frac{\left\| \bar\partial f(z) \wedge \bar\partial g(w) \wedge c_l \right\|}{d\,(z, E)\, d\,(w, E)\, |z-w|^{2(2n-l)-1}} < \infty, \tag{3.3}$$

for all l *such that* $n \le l < 2n$. *Then* $f(a)g(a) = fg(a)$.

Proof. First note that

$$f(a)g(a) = -\int_z \int_w \bar\partial f(z) \wedge \bar\partial g(w) \wedge u^1_n \wedge u^2_n$$

and that, by the Bochner-Martinelli integral formula,

$$fg(a) = -\int \left(g \bar\partial f + f \bar\partial g \right) \wedge u_n$$

$$= \int_z \int_w \bar\partial f(z) \wedge \bar\partial g(w) \wedge m_n \wedge u^1_n - \int_z \int_w \bar\partial f(z) \wedge \bar\partial g(w) \wedge m_n \wedge u^2_n.$$

Let χ_ε be the convolution of the characteristic function of the set

$$\{(z, w) : d((z, w), E \times \mathbb{C}^n \cup \mathbb{C}^n \times E) \ge 2\varepsilon\}$$

and the function $\varepsilon^{-4n} \rho(\cdot/\varepsilon)$, where ρ is a non-negative smooth function with compact support in the unit ball of $\mathbb{C}^{2n}$ such that its integral is equal to 1. Since

$$\left\| \bar\partial f(z) \wedge \bar\partial g(w) \wedge \left(u^1_n \wedge u^2_n + m_n \wedge u^1_n - m_n \wedge u^2_n \right) \right\|$$

is integrable, we must prove that

$$\lim_{\varepsilon \to 0} \int_z \int_w \chi_\varepsilon \bar\partial f(z) \wedge \bar\partial g(w) \wedge \left(u^1_n \wedge u^2_n + m_n \wedge u^1_n - m_n \wedge u^2_n \right) = 0.$$

The resolvent identity (2.10) gives that

$$-\bar\partial \sum_{k+l=2n} m_k \wedge c_l + [\Delta] \wedge c_n = u^1_n \wedge u^2_n + m_n \wedge u^1_n - m_n \wedge u^2_n$$

in the sense of currents (note that the proof of this formula only made use of the forms c_l for $l \ge n$). Hence, since $i^* c_n = 0$, we must prove that

$$\lim_{\varepsilon \to 0} \int_z \int_w \chi_\varepsilon \bar\partial f(z) \wedge \bar\partial g(w) \wedge \bar\partial \sum_{k+l=2n} m_k \wedge c_l = 0. \tag{3.4}$$

Integration by parts gives that (3.4) is equivalent to

$$\lim_{\varepsilon \to 0} \int_z \int_w \bar\partial \chi_\varepsilon \wedge \bar\partial f(z) \wedge \bar\partial g(w) \wedge \sum_{k+l=n} m_k \wedge c_l = 0. \tag{3.5}$$

Note that $\left| \bar\partial \chi_\varepsilon \right| \le C\varepsilon^{-1}$ and that $\left| \bar\partial \chi_\varepsilon \right|$ has support in

$$\varepsilon \le d((z, w), E \times \mathbb{C}^n \cup \mathbb{C}^n \times E) \le 3\varepsilon.$$

We also have that

$$d\left((z,w), E\times\mathbb{C}^n \cup \mathbb{C}^n \times E\right) \geq \min\left\{d(z,E), d(w,E)\right\}$$

$$\geq C d\left(z,E\right) d\left(w,E\right)$$

on a bounded set, where $C > 0$ is a constant (depending on the bound). Thus (3.5) follows since

$$\int_z\int_w \frac{\left\|\bar\partial f(z)\wedge\bar\partial g(w)\wedge\sum_{k+l=2n} m_k\wedge c_l\right\|}{d((z,w), E\times\mathbb{C}^n\cup\mathbb{C}^n\times E)} < \infty$$

by (3.3). Hence the proposition is proved. □

To be able to separate the condition (3.3) we will assume that u_i commute with a. We can then choose the sequence c_i in the following way.

Proposition 3.2. *Suppose that u_i is a sequence as in (2.4) and that $au_i = u_i a$. Then*

$$c_i = \sum_{k+l=i} u_k^1 \wedge u_l^2$$

satisfies (3.1).

Proof. We have that $c_1 = 0$, and since a and u_i commute,

$$\delta c_{i+1} - \bar\partial c_i = \sum_{k+l=i+1}\left(\delta u_k^1\wedge u_l^2 - u_k^1\wedge\delta u_l^2\right)$$

$$-\sum_{k+l=i}\left(\delta u_{k+1}^1\wedge u_l^2 - u_k^1\wedge\delta u_{l+1}^2\right) = u_i^2 - u_i^1,$$

where $\delta = \delta_{z-a,w-a}$. Thus c_i satisfies (3.1). □

Unfortunately, the sequence c_i in Proposition 3.2 does not necessarily satisfy $i^* c_n = 0$. However, by the proof of Theorem 2.3 we have that $i^* c_n$ is exact.

We have an explicit choice of sequence that satisfies (2.4). Suppose that s satisfies the equalities $\delta_{z-a}s = e$ and $as = sa$. Then

$$\delta_{z-a}s = e, \quad \delta_{z-a}\left(s\wedge\left(\bar\partial s\right)^i\right) = \left(\bar\partial s\right)^i = \bar\partial\left(s\wedge\left(\bar\partial s\right)^{i-1}\right)$$

and hence $u_i = s\wedge\left(\bar\partial s\right)^{i-1}$ satisfies (2.4). The sequence c_i of Proposition 3.2 is then

$$c_i = \sum_{k+l=i} s^1\wedge\left(\bar\partial s^1\right)^{k-1}\wedge s^2\wedge\left(\bar\partial s^2\right)^{l-1}, \tag{3.6}$$

where $s^1 = \pi_1^* s$ and $s^2 = \pi_2^* s$. Note that if $s\wedge s = 0$ then $s\wedge\left(\bar\partial s\right) = \left(\bar\partial s\right)\wedge s$ and hence $i^* c_n = 0$.

Let $E \supset \sigma(a)$ be a compact set and let s be a given form such that s is defined on $\mathbb{C}^n \setminus E$, $\delta_{z-a} s = e$ and $as = sa$. Define the class S_a by

$$S_a = \left\{ f \in C_c^2 \left(\mathbb{C}^n\right) : \|f\|_a < \infty \right\}, \tag{3.7}$$

where

$$\|f\|_a = \sum_{i=1}^{n} \left\| \frac{\bar{\partial} f \wedge s \wedge \left(\bar{\partial} s\right)^{i-1}}{d(z,E)} \right\|_\infty$$

$$+ \sum_{k+l=n} \left\| \frac{\bar{\partial} f \wedge s \wedge \left(\bar{\partial} s\right)^{k-1} \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}}{d(z,E)} \right\|_\infty .$$

Note that the second sum vanishes if $s \wedge s = 0$. This is always the case if $n = 2$ since then $\delta_{z-a}\left(s \wedge s\right) = s - s = 0$ and δ_{z-a} injective. If $n = 1$ then S_a defined by (3.7) is a slightly smaller class than S_a defined in the introduction. This is because the left hand side in the resolvent identity (2.10) is 0 if $n = 1$. If $f \in S_a$ then $f(a)$ is defined by

$$f(a) = - \int \bar{\partial} f \wedge s \wedge \left(\bar{\partial} s\right)^{n-1}.$$

Of course we have that $f(a) \in L(X)$ if $f \in S_a$. Note that S_a is an algebra. In the next lemma we will use Proposition 3.1 to prove that $f(a)g(a) = fg(a)$ if $f, g \in S_a$.

Lemma 3.3. *If $f, g \in S_a$ then $f(a)g(a) = fg(a)$.*

Proof. Let c_i be the sequence defined by (3.6) and let

$$d_i = \sum_{k+l=i} s^2 \wedge \left(\bar{\partial} s^2\right)^{k-1} \wedge s^2 \wedge \left(\bar{\partial} s^2\right)^{l-1}.$$

By a computation similar to the proof of Proposition 3.2, we see that the sequence d_i satisfies the relation

$$\delta_{z-a,w-a} d_{i+1} = \bar{\partial} d_i,$$

and hence that $\bar{\partial} d_n = 0$. For every $l > n$ define c_l' by $c_l' = c_l$ and define c_n' by $c_n' = c_n - d_n$. Then c_l' satisfies the condition (3.2) since $\bar{\partial} d_n = 0$ and $i^* c_n = i^* d_n$. We have that $|z-w|^{-2n+1}$ is a locally integrable function on $\mathbb{C}^{2n}$ and hence

$$\int_z \int_w \frac{\left\|\bar{\partial} f(z) \wedge \bar{\partial} g(w) \wedge c_i\right\|}{d\left(z,E\right) d\left(w,E\right) \left|z-w\right|^{2n-1}}$$

$$\leq \sum_{k+l=i} \int_z \int_w \frac{\left\|\bar{\partial} f(z) \wedge s^1 \wedge \left(\bar{\partial} s^1\right)^{k-1}\right\| \left\|\bar{\partial} g(w) \wedge s^2 \wedge \left(\bar{\partial} s^2\right)^{l-1}\right\|}{d\left(z,E\right) d\left(w,E\right) \left|z-w\right|^{2n-1}} < \infty.$$

Similarly, we have that

$$\int_z \int_w \frac{\left\|\bar{\partial} f(z) \wedge \bar{\partial} g(w) \wedge d_n\right\|}{d\left(z,E\right) d\left(w,E\right) \left|z-w\right|^{2n-1}} < \infty,$$

since $\|g\|_a < \infty$. Thus the statement follows from Proposition 3.1. □

In order to prove that $f(a) \in (a)''$ we construct the resolvent $\omega_{z-a,w-b}$ and use the multiplicative property of the functional calculus of the tuple (a,b), where $b \in L(X)$ commutes with a.

Lemma 3.4. *If $f \in S_a$ then $f(a) \in (a)''$.*

Proof. Suppose that $b \in L(X)$ is an operator such that $ab = ba$. Define the form

$$v(w) = \frac{1}{2\pi i}(w-b)^{-1}\,dw$$

Define the sequence c_k by

$$c_1 = 0, \quad c_k = v \wedge s \wedge \left(\bar{\partial} s\right)^{k-2}.$$

Then we have the equations

$$c_1 = 0, \quad \delta_{z-a,w-b} c_2 = s - v$$

and

$$\delta_{z-a,w-b} c_{k+1} = s \wedge \left(\bar{\partial} s\right)^{k-1} - v \wedge \left(\bar{\partial} s\right)^{k-1} = \bar{\partial} c_k + s \wedge \left(\bar{\partial} s\right)^{k-1}.$$

Let χ be a smooth cutoff function such that $\{\chi, 1-\chi\}$ is a partition of unity subordinate the cover

$$\left\{\{(z,w) : z \notin E, |w| < 3\|b\|\}, \{(z,w) : |w| > 2\|b\|\}\right\}$$

of $\mathbb{C}^n \times \mathbb{C} \setminus E \times \{w : |w| \leq 2\|b\|\}$. This is a special choice of function χ used in Lemma 3.2 of [1] which enables us to avoid an integration by parts. Define the sequence a_k outside $E \times \{w : |w| \leq 2\|b\|\}$ by

$$a_1 = \chi s + (1-\chi)\,v, \quad a_k = \chi s \wedge \left(\bar{\partial} s\right)^{k-1} - \bar{\partial}\chi \wedge c_k.$$

We then have that

$$\delta_{z-a,w-b} a_1 = e, \quad \delta_{z-a,w-b} a_2 = \chi \bar{\partial} s + \bar{\partial}\chi \wedge (s - v) = \bar{\partial} a_1$$

and that

$$\delta_{z-a,w-b} a_{k+1} = \chi\left(\bar{\partial} s\right)^k + \bar{\partial}\chi \wedge \left(\bar{\partial} c_k + s \wedge \left(\bar{\partial} s\right)^{k-1}\right) = \bar{\partial} a_k,$$

and thus

$$a_{n+1} = -\bar{\partial}\chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{n-1}$$

represents $\omega_{z-a,w-b}$. Choose $\phi \in C_c^\infty(\mathbb{C})$ which is 1 in a neighbourhood of $\{w \in \mathbb{C} : |w| < 3\|b\|\}$. Then we have that

$$(\phi f)(a,b) = -\int_w \int_z \bar{\partial}\left(\phi(w) f(z)\right) \wedge a_{n+1}(z,w)$$

$$= \iint f \bar{\partial}_w \phi \wedge \bar{\partial}_z \chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{n-1}$$

$$+ \iint \phi \bar{\partial}_z f \wedge \bar{\partial}_w \chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{n-1} = -\int \bar{\partial} f \wedge s \wedge \left(\bar{\partial} s\right)^{n-1} = f(a).$$

Let $a_k^1 = \pi_1^* a_k$ and $a_k^2 = \pi_2^* a_k$, where

$$\pi_1(z_1, w_1, z_2, w_2) = (z_1, w_1) \text{ and } \pi_2(z_1, w_1, z_2, w_2) = (z_2, w_2).$$

Define the sequence c_i' by

$$c_1' = 0, \quad c_i' = \sum_{k+l=i} a_k^1 \wedge a_l^2$$

so that by Proposition 3.2,

$$c_1 = 0, \quad \delta_{z_1-a, w_1-b, z_2-a, w_2-b} c_{i+1}' = \bar{\partial} c_i' + a_i^2 - a_i^1.$$

Let $F = E \times \{w : |w| \le 2\|b\|\}$. Define the function g by $g(z,w) = w\psi(z,w)$ where $\psi \in C_c^\infty$ is equal to 1 in a neighbourhood of F. We have that

$$\left\| \frac{\bar{\partial}(\phi f) \wedge a_k}{d\left((z,w), F\right)} \right\|_\infty$$

$$\le \left\| \frac{\chi \bar{\partial}(\phi f) \wedge s \wedge \left(\bar{\partial} s\right)^{k-1}}{d(z,E)} \right\|_\infty$$

$$+ \left\| \frac{\bar{\partial}(\phi f) \wedge \bar{\partial}\chi \wedge v \wedge s \wedge \left(\bar{\partial} s\right)^{k-2}}{d(z,E)} \right\|_\infty < \infty$$

since $f \in S_a$. Hence we have that

$$\iint \frac{\left\| \bar{\partial}(\phi(w_1) f(z_1)) \wedge \bar{\partial} g(z_2, w_2) \wedge c_l' \right\|}{d\left((z_1, w_1), F\right) d\left((z_2, w_2), F\right) \left|(z_1, w_1) - (z_2, w_2)\right|^{2n+1}} < \infty$$

for all l. Define the forms c_l'' by the equations $c_l'' = c_l'$ if $l > n+1$ and

$$c_{n+1}'' = c_{n+1}' - \sum_{k+l=n+1} a_k^2 \wedge a_l^2.$$

Then we have that c_{n+1}'' satisfies $i^* c_{n+1}'' = 0$ and hence by Proposition 3.1 we have that $(\phi f)(a,b) g(a,b) = g(a,b)(\phi f)(a,b)$ since

$$\iint \frac{\left\| \bar{\partial}(\phi(w_1) f(z_1)) \wedge \bar{\partial} g(z_2, w_2) \wedge \sum_{k+l=n+1} a_k^2 \wedge a_l^2 \right\|}{d\left((z_1, w_1), F\right) d\left((z_2, w_2), F\right) \left|(z_1, w_1) - (z_2, w_2)\right|^{2n+1}} < \infty.$$

Thus $f(a)b = bf(a)$ since $g(a,b) = b$ by the holomorphic functional calculus. □

We can now prove a generalisation of the holomorphic functional calculus.

Theorem 3.5. ***(Non-holomorphic functional calculus)*** *Suppose that a is an n-tuple of commuting operators and that $E \supset \sigma(a)$ is compact such that it exists a smooth form s defined on $\mathbb{C}^n \setminus E$ with*

$\delta_{z-a}s = e$ and $as = sa$. Let S_a be the class defined by (3.7) and let $f(a)$, $f \in S_a$, be the operator defined by

$$f(a) = -\int \bar{\partial} f \wedge s \wedge \left(\bar{\partial} s\right)^{n-1}.$$

Then we have that the map $f \mapsto f(a) : S_a \to (a)''$ is a continuous algebra homomorphism that continuously extends the map $f \mapsto f(a) : \mathcal{O}(E) \to (a)''$.

Proof. By Lemma 3.4 the map $f \mapsto f(a) : S_a \to (a)''$ is well defined. The map is continuous and linear. Lemma 3.3 gives that the map is multiplicative, and thus the map is an algebra homomorphism. To see that it continuously extends the map $f \mapsto f(a) : \mathcal{O}(E) \to (a)''$, suppose that we have a sequence $f_n \in \mathcal{O}(U)$, where U is an open neighbourhood of E, and that $f_n \to 0$ uniformly on compacts. Then

$$\|f_n \phi\|_a \to 0,$$

where $\phi \in C_c^\infty(U)$ is a function equal to 1 in a neighbourhood of E. □

We now go on and prove the spectral mapping theorem for this functional calculus. To do this, we need the following lemma which shows that $f(w)$ acts as $f(a)$ on $H_p(w-a, c, X)$.

Lemma 3.6. *Suppose that there is an operator valued form s outside E such that $\delta_{z-a}s = e$ and $sa = as$. Furthermore, suppose that $c \in ((a)'')^m$, $w \in \sigma(a)$ and $k \in K_p(w-a, c, X)$ (with respect to a basis $dw_1, \ldots, dw_n, e_{n+1}, \ldots, e_{n+m}$ of $\mathbb{C}^{n+m}$) such that $\delta_{w-a,c}k = 0$. If $f \in S_a$, then*

$$\left(f(a) - f(w)\right) k = \delta_{w-a,c} \int_z \bar{\partial} f(z) \wedge \sum_{l=1}^{n} m''_{n+1-l} \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} \wedge k,$$

where m''_i is defined in the proof.

Proof. We have that

$$\delta_{z-a,w-a} m_1 = e, \quad \delta_{z-a,w-a} m_{i+1} = \bar{\partial} m_i,$$

by (2.7) and (2.8), where m_i is defined by (2.6). We also have that

$$\delta_{z-a,w-a} s = e, \quad \delta_{z-a,w-a}\left(s \wedge \left(\bar{\partial} s\right)^i\right) = \bar{\partial}_z \left(s \wedge \left(\bar{\partial} s\right)^{i-1}\right),$$

where s only depends on z. Therefore the same calculation as in the proof of Proposition 3.2 shows that

$$\delta_{z-a,w-a} \sum_{k+l=i+1} m_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} - \bar{\partial} \sum_{k+l=i} m_k \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= s \wedge \left(\bar{\partial} s\right)^{i-1} - m_i.$$

Let $i = n$ and identify the component without any dw and $d\bar{w}$ in this expression to get,

$$\delta_{w-a} \sum_{k+l=n+1} m_k'' \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= s \wedge (\bar{\partial} s)^{n-1} - m_n' + \bar{\partial}_z \sum_{k+l=n} m_k' \wedge s \wedge \left(\bar{\partial} s\right)^{l-1},$$

where

$$m_k' = \frac{1}{(2\pi i)^k} \frac{\partial_z |z-w|^2}{|z-w|^2} \wedge \left(\bar{\partial}_z \frac{\partial_z |z-w|^2}{|z-w|^2}\right)^{k-1}$$

and m_k'' is the component of m_k with one dw and no $d\bar{w}$. Let χ_ε be the convolution of the characteristic function of the set

$$\{z : d(z, E) \geq 2\varepsilon\}$$

and the function $\varepsilon^{-2n}\rho(\cdot/\varepsilon)$, where ρ is a non-negative smooth function with compact support in the unit ball of $\mathbb{C}^n$ such that its integral is equal to 1. We have that

$$\int_z \bar{\partial}_z f(z) \wedge \bar{\partial}_z \sum_{k+l=n} m_k' \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= \lim_{\varepsilon \to 0} \int_z \chi_\varepsilon \bar{\partial}_z f(z) \wedge \bar{\partial}_z \sum_{k+l=n} m_k' \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}$$

$$= \lim_{\varepsilon \to 0} \int_z \bar{\partial}_z \chi_\varepsilon \wedge \bar{\partial}_z f(z) \wedge \sum_{k+l=n} m_k' \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} = 0$$

since $\left|\bar{\partial}\chi_\varepsilon\right| \leq C\varepsilon^{-1}$ and $\left|\bar{\partial}\chi_\varepsilon\right|$ has support in $\varepsilon \leq d(z, E) \leq 3\varepsilon$. Hence we have that

$$f(a) - f(w) = \int_z \bar{\partial} f(z) \wedge \left(s \wedge \left(\bar{\partial} s\right)^{n-1} - m_n'\right)$$

$$= \delta_{w-a} \int_z \bar{\partial} f(z) \wedge \sum_{k+l=n+1} m_k'' \wedge s \wedge \left(\bar{\partial} s\right)^{l-1}.$$

Therefore,

$$(f(a) - f(w))\, k = \delta_{w-a,c} \int_z \bar{\partial} f(z) \wedge \sum_{l=1}^{n} m_{n+1-l}'' \wedge s \wedge \left(\bar{\partial} s\right)^{l-1} \wedge k,$$

since $(w-a, c)$ and s commute. □

We can now prove the spectral mapping theorem.

Theorem 3.7. ***(Spectral mapping theorem)*** *If f is tuple of functions in S_a, where S_a is defined by (3.7), then $\sigma(f(a)) = f(\sigma(a))$.*

Proof. Suppose that we can prove the statement; if $z \in \sigma(a)$ then $(z-a, f(a))$ is non-singular if and only if $f(z) \neq 0$. In that case $(z-a, w-f(a))$ is non-singular if and only if $w - f(z) \neq 0$ and hence

$$\sigma(f(a)) = \pi_2 \sigma(a, f(a)) = \pi_2 \{(z,w) : w = f(z), z \in \sigma(a)\} = f(\sigma(a))$$

by Theorem 1.4.

Suppose that $z \in \sigma(a)$. We have the induction hypothesis that if m is a natural number then the tuple $(z-a, f(a))$ is non-singular if and only if $f(z) \neq 0$ for all m-tuples f of functions in S_a. The case $m = 0$ follows from Lemma 3.6. Assume that the hypothesis has been proved for m. Given $f' = (f_1, \ldots, f_{m+1})$ let $f = (f_1, \ldots, f_m)$. Then there is a long exact sequence

$$\ldots \to H_p(z-a, f(a), X) \to H_p(z-a, f'(a), X)$$

$$\to H_{p-1}(z-a, f(a), X) \xrightarrow{f_{m+1}(a)} H_{p-1}(z-a, f(a), X) \to \ldots,$$

for this see Taylor [13], Lemma 1.3. Lemma 3.6 gives that the last homomorphism is equal to $f_{m+1}(z)$. Hence

$$H_p(z-a, f'(a), X) = 0$$

if $f_{m+1}(z) \neq 0$ and

$$\operatorname{Im} H_p(z-a, f'(a), X) = H_{p-1}(z-a, f(a), X)$$

if $f_{m+1}(z) = 0$. Therefore the induction hypothesis hold for $m+1$ and hence the theorem follows. $\square$

We will now consider a concrete situation where we can give an answer to all the questions we set up in the beginning of this section. Suppose that $E \supset \sigma(a)$ is a compact and convex set with C^2 boundary and that we are given a form u_n representing ω_{z-a} on $\mathbb{C}^n \setminus E$. Then we can use the holomorphic functional calculus to construct a form s, $\delta_{z-a} s = e$, such that for each i, $s \wedge (\bar{\partial} s)$ admits estimates controled by the growth of u_n.

Theorem 3.8. *Suppose that $\sigma(a) \subset E$, where E is a compact and convex set with C^2-boundary. Let u^x be a differential form representing $\omega_{z-a} x$ outside E such that $\|u^x(z)\| \leq \|x\| e^{q(r(z))}$ where $r(z) = d(z, E)$, q is a decreasing function and $q(0) = \infty$. Then the map*

$$f \mapsto f(a) : S_u \to (a),$$

where S_u is

$$\left\{ f \in C_c^2 : \left|\bar{\partial} f(z)\right| \leq C r(z) \sup_{\varepsilon > 0} \left\{\varepsilon^n e^{-q(r(z)-\varepsilon)}\right\} \text{ for all } z \in \mathbb{C}^n \right\},$$

is a continuous algebra homomorphism.

Proof. We have that $r \in C^2$ (see [10], Exercise 4 page 136) and $dr \neq 0$ in $U \setminus \sigma(a)$ where U is a neighbourhood of $\partial\sigma(a)$. Therefore

$$r(z) - r(\zeta) \leq 2\mathrm{Re}\ \sum_{k=1}^{n}(z_k - \zeta_k)\frac{\partial r}{\partial z_k}(z) \leq 2\left|\delta_{z-\zeta}\partial r(z)\right|$$

for all $z \in U \setminus \sigma(a)$ and $\zeta \in \mathbb{C}^n$ since r is convex. Thus the form s defined by

$$s(z,\zeta) = \frac{\partial r(z)}{\delta_{z-\zeta}\partial r(z)},$$

is well-defined for all (z,ζ) such that $r(z) > r(\zeta)$. We get

$$s(z,\zeta) \wedge \left(\bar{\partial} s(z,\zeta)\right)^{j-1} = \frac{\partial r(z) \wedge \left(\bar{\partial}\partial r(z)\right)^{j-1}}{\left(\delta_{z-\zeta}\partial r(z)\right)^{j}}.$$

Integrating over ζ, we get by the holomorphic functional calculus that

$$s(z,a) \wedge \left(\bar{\partial} s(z,a)\right)^{j-1}$$

$$= \int_{r(\zeta)=r(z)-\varepsilon} \left(\delta_{z-\zeta}\partial r(z)\right)^{-j} u^{x}(\zeta)\partial r(z) \wedge \left(\bar{\partial}\partial r(z)\right)^{j-1},$$

for $0 < \varepsilon < r(z)$. Therefore,

$$\left\| s(z,a) \wedge \left(\bar{\partial} s(z,a)\right)^{j-1} \right\| \leq Ce^{g(r(z)-\varepsilon)}\left(r(z) - r(\zeta)\right)^{-j} \leq Ce^{q(r(z)-\varepsilon)}\varepsilon^{-n}.$$

An application of Theorem 3.5 finishes our proof. □

Theorem 3.9. *If $g \in S_u$ and f is holomorphic in a neighbourhood of $\sigma(g(a))$, then*

$$\sigma(g(a)) = g(\sigma(a)), \quad \phi(f \circ g)(a) = f(g(a)),$$

where $\phi \in C_c^\infty$ is equal to 1 in a neighbourhood of $\sigma(a)$, and $g(a) = 0$ if $g(z) = 0$ for all $z \in E$.

Proof. The equality $\sigma(g(a)) = g(\sigma(a))$ follows from Theorem 3.7. Assume that $0 \in E$ and let

$$h(z) = r(z)\sup_{\varepsilon > 0}\left\{\varepsilon^n e^{-q(r(z)-\varepsilon)}\right\}.$$

Then $|\bar{\partial} g_r| \leq h_r \leq h$ where $g_r(z) = g(rz)$ and $h_r(z) = h(rz)$ and $r \leq 1$. Hence $g_r(a) \to g(a)$ when $r \to 1$, by dominated convergence, and thus $g(a) = 0$ if $g(z) = 0$ for all $z \in \sigma(a)$. The rule of composition is true for holomorphic functions, and hence we have that $(f \circ g_r)(a) = f(g_r(a))$. Since $\phi(f \circ g) \in S_u$ we can let $r \to 1$ in this equality to get $\phi(f \circ g)(a) = f(g(a))$, by Theorem 2.6. □

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