Weber's Law and Weberized TV Restoration

Jianhong Shen

School of Mathematics
University of Minnesota
Minneapolis, MN 55455, USA *

Abstract

Most conventional image processors consider little the influence of human vision psychology. *Weber's Law* in psychology and psychophysics claims that human's perception and response to the intensity fluctuation δu of visual signals are weighted by the background stimulus u, instead of being plainly uniform. This paper attempts to integrate this well known perceptual law into the classical total variation (TV) image restoration model of Rudin, Osher, and Fatemi [*Physica D*, **60**:259-268, 1992]. We study the issues of existence and uniqueness for the proposed Weberized nonlinear TV restoration model, making use of the direct method in the space of functions with bounded variations. We also propose an iterative algorithm based on the linearization technique for the associated nonlinear Euler-Lagrange equation.

PACS (2001): 87.19.Dd (information processing in vision and hearing); 87.57.Nk (image analysis).

Keywords: Weber's Law; Vision; Psychophysics; Image restoration.

1 Introduction: Weber's Law

As information carriers, all images are eventually perceived and interpreted by the human visual system. As a result, human vision psychology and psychophysics play an important role in the successful communication of image information. From the imaging science point of view, this fact implies that any ideal image processor should take into account the consequences of vision psychology and psychophysics.

The current paper makes an attempt in this direction. We develop an image restoration model that intends to incorporate one of the most well known and influential psychological results - *Weber's Law* for sound and light perception. We study its major mathematical properties (e.g., existence and uniqueness), as well as the computational strategy for the associated nonlinear PDE.

* Email: jhshen@math.umn.edu. Tel: (612) 625-3570. Fax: (612) 626-2017.

Weber's Law was first described in 1834 by German physiologist E. H. Weber [Web34], and was later formulated quantitatively by the great experimental psychologist Gustav Fechner [Fec58], founder of the modern psychophysics. The law reveals the universal influence of the background stimulus u on human's sensitivity to the intensity increment δu, or the so called JND (just-noticeable-difference), in the perception of both sound and light. It claims that the so-called Weber's fraction is a constant:

$$\delta u / u = \text{const.} \tag{1}$$

Many experiments have demonstrated that in a large range of stimuli u, Weber's Law indeed provides a good approximation.

Empirical evidence is easy to gather from daily life for a qualitative understanding of Weber's Law. In a fully packed stadium where the background sound intensity u is high, one has to speak close to shouting in order to be effectively heard by the other folks. The same observation is true for visual communication. The stars can be clearly spotted in a dark night without a bright full moon, and away from the urban neon lights. But otherwise our naked vision has much difficulty in finding them. In the current paper, we apply Weber's Law in the context of visual perception. Therefore, u stands for the background light intensity and δu the intensity fluctuation.

Since almost all images are eventually to be observed and interpreted by humans, an ideal digital image processor has to take into account the effects of human psychology and psychophysics, such as that of Weber's Law. This is an important new area that needs to be further explored. The current paper makes the first infantile attempt of integrating Weber's Law into image restoration schemes. We demonstrate our main idea through "Weberizing" the well known classical model of total variation (TV) denoising and enhancement by Rudin, Osher, and Fatemi [ROF92, RO94].

The organization goes as follows. In Section 2, we quickly review the TV restoration model in image processing, and explain the idea behind its "Weberization." In Section 3, we first rigorously interpret the Weberized TV restoration energy and its admissible space, and then apply the direct method to study the existence and uniqueness of the minimizers. The computational approach to the minimization of the Weberized TV energy is addressed in Section 4, accompanied by some typical numerical results. The conclusion goes into Section 5.

2 Weberized TV for Image Restoration

Let u_0 denote the observed raw image data, which is assumed to be a degraded version of the original good image u. Distortions in u_0 are typically modeled by blurring and noising:

$$u_0 = Ku + n, \tag{2}$$

where K is a linear blurring operator, or a lowpass filter with $K1 \equiv 1$, and n denotes white noise. The goal of image restoration is to recover the original good image u from one *single* observation of u_0 (since strictly speaking, u_0 is a random field). In this paper, we shall assume that the noise is spatially homogeneous, and can be well approximated by Gaussian. We shall also focus on the pure denoising case when there is no severe blurring.

The TV restoration model first proposed by Rudin, Osher, and Fatemi [RO94, ROF92] is to minimize the following Bayesian type energy [GG84, Mum94, CS02]:

$$E[u|u_0] = \int_\Omega |\nabla u| dx + \frac{\lambda}{2} \int_\Omega (u - u_0)^2 dx, \qquad (3)$$

in the space of functions with bounded variations $\mathrm{BV}(\Omega)$. (Here, imitating the conditional expectation in probability theory, the vertical bar defines two domains for the known variables and the unknowns.) The first regularity term is understood beyond the conventional Sobolev space $W^{1,1}(\Omega)$, instead, as the TV Radon measure [Giu84]. $\mathrm{BV}(\Omega)$ has been proven a sufficiently good image space for most images without much *texture* [CS02]. The main characteristic of the BV image model is that it legalizes 1-dimensional singularities, or popularly referred to as "edges," an important visual cue in human and computer vision [MH80].

Ever since their first introduction into digital image processing in [RO94, ROF92], the BV image model and TV restoration model (3) have witnessed many successful new developments during the past decade. We refer to our recent survey paper [CS02] and the references therein for more detail. In particular, with his collaborators, the author of the present paper has been able to extend and generlize the models onto digital graph domains [COS01], onto the so-called *nonflat* image features that live on Riemannian manifolds [CS00], and onto the novel area of image inpainting and geometric image interpolations [CS01]. Figure 1 shows one application of the TV restoration model for the error concealment of a blurry image transmitted through a wireless network with randomly lost packets.

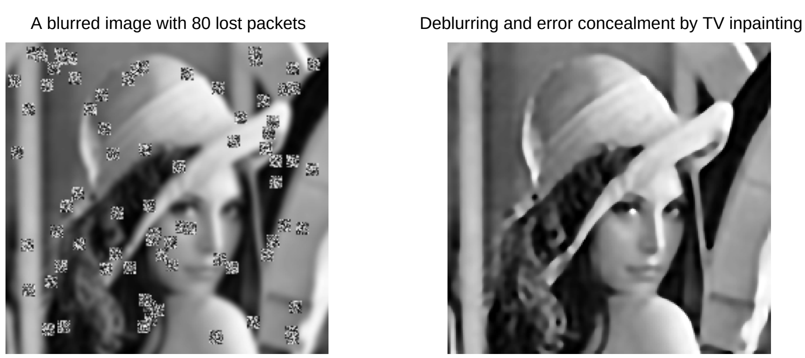

Figure 1: TV restoration of a blurry image with simulated random packet loss.

Most conventional restoration models do not take into account that our visual sensitivity to the regularity or local fluctuation δu depends on the ambient intensity level u. That is, models such as (3) assume that a local variation, $\delta u = 0.02$ say, should be treated equally independent of the background intensity level u, no matter whether it is $u = 0.1$ or $u = 0.8$. But this exactly violates Weber's Law, according to which, a fluctuation level of $\delta u = 0.02$ against a background intensity $u = 0.1$ is much more significant than the same amount against $u = 0.8$. In fact it is approximately equivalent to a level of

$$\delta u = \frac{0.8}{0.1} \times 0.02 = 0.16$$

in the latter situation.

It is out of this consideration that we propose to "Weberize" the classical TV restoration model (3). The key is to replace the uniform local variation

$$|\nabla u| = \sqrt{u_x^2 + u_y^2} = \frac{\partial u}{\partial \vec{n}}, \qquad \vec{n} = \frac{\nabla u}{|\nabla u|}$$

by the Weberized local variation $|\nabla u|_w$,

$$|\nabla u|_w = \frac{|\nabla u|}{u} = \frac{1}{u}\frac{\partial u}{\partial \vec{n}},$$

which encodes the influence of the background intensity u according to Weber's Law (1).

Therefore, instead of (3), we propose the *Weberized* TV restoration model

$$\min_u E_w[u|u_0] = \int_\Omega |\nabla u|_w dx + \frac{\lambda}{2}\int_\Omega (u - u_0)^2 dx. \qquad (4)$$

The current paper is devoted to the study of the mathematical properties of this new model, including issues related to the existence and uniqueness of the minimizers, and its computational approach.

3 Existence and Uniqueness

We first work out some natural assumptions on the data model, and then investigate the existence and uniqueness of the minimizers to the Weberized TV restoration model (4).

3.1 Assumptions and the admissible space

In Weber's fraction $\delta u/u$, u denotes the intensity value. Thus $u \geq 0$. We shall call $u = 0$ the "blackhole" since physically it means no photons are emitted or reflected. The blackhole is the singularity of both Weber's fraction and the Weberized local variation $|\nabla u|_w = |\nabla u|/u$. Therefore, technically we should stay away from the blackhole and assume that $u > 0$.

The blackhole similarly imposes some natural restrictions on the noise model

$$u_0 = u + n.$$

Since u_0 also represents the intensity value, we must have $u_0 \geq 0$, which implies that $u \geq -n$. Now that in both the TV restoration model (3) and its Weberized version (4), noise reduction has been controlled by the least square energy, we are implicitly assuming that the noise can be well approximated by some Gaussian $N(0, \sigma^2)$ [Str93]. In particular, n (or its probability density function) should be almost symmetric, and the condition $u \geq -n$ is equivalent to $u \geq |n|$, which can be roughly translated to that the signal-to-noise ratio SNR ≥ 1. Therefore,

$$u_0 = u + n \leq u + u = 2u,$$

and we obtain a natural technical constraint for the Weberized TV restoration model (4):

$$u \geq \frac{u_0}{2}. \tag{5}$$

Before investigating the existence of the Weberized TV retoration (E:Ew)

$$\min_u E_w[u|u_0] = \int_\Omega |\nabla u|_w dx + \frac{\lambda}{2}\int_\Omega (u - u_0)^2 dx, \tag{6}$$

we need to explain the exact meaning of the Weberized TV, and the admissible space for the restoration energy E_w.

First, due to the least square energy control in E_w, we assume that $u_0 \in L^2(\Omega)$. As a result, $u \in L^2(\Omega)$. Throughout the paper, we also assume that Ω is a Lipschitz open domain with a finite Lebesgue measure $|\Omega| < \infty$.

Second, the Weberized TV energy

$$\int_\Omega |\nabla u|_w dx = \int_\Omega \frac{|\nabla u|}{u} dx$$

is understood in the sense of the *coarea* formula [Giu84]. More generally, let $\phi(u) : (0, \infty) \to (0, \infty)$ be a continuous function. Then, for any $u \in \mathrm{BV}(\Omega)$ and $u(x) > 0$, we define

$$\int_\Omega \phi(u)|\nabla u| dx := \int_0^\infty \mathrm{Per}(u < \lambda)\phi(\lambda) d\lambda. \tag{7}$$

Here for any $\lambda > 0$, the perimeter of the set $F_\lambda = \{x | u < \lambda\}$ is defined as [Giu84]:

$$\mathrm{Per}(u < \lambda) = \mathrm{Per}(F_\lambda, \Omega) = \int_\Omega |\nabla 1_{F_\lambda}| dx. \tag{8}$$

(Note: in this paper we shall always use the conventional notation $\int_\Omega |\nabla f| dx$ to denote the TV Radon measure $\int_\Omega |Df|$ [Giu84].) When $\phi \equiv 1$, (7) is precisely the classical coarea formula.

Another equivalent way is to introduce the integral Φ of ϕ: $\Phi'(u) = \phi(u)$. Then the definition (7) is identical to

$$\int_\Omega \phi(u)|\nabla u| dx := \int_\Omega |\nabla \Phi(u)| dx = \mathrm{TV}(\Phi(u)). \tag{9}$$

For instance, for the Weberized TV, $\phi(u) = 1/u$, $v = \Phi(u) = \ln u$. Therefore, the Weberized TV restoration can be rewritten as

$$E_w[u|u_0] = \int_\Omega |\nabla v| dx + \frac{\lambda}{2}\int_\Omega (u - u_0)^2 dx. \tag{10}$$

The combination of all the three elements discussed above leads to the following natural admissible space for the Weberized TV restoration (10):

$$\mathcal{D} = \{ u > 0 \mid u \in L^2(\Omega),\ \mathrm{TV}(\ln u) < \infty,\ u \geq u_0/2 \ \}. \tag{11}$$

This is the space that we shall work with from now on.

3.2 Existence of Weberized TV restoration

First we prove a *Maximum Principle* type result for the Weberized energy form (10). The technique is characteristic to total variation related energies, but becomes difficult for conventional Sobolev type regularity energies.

Lemma 1 *Suppose that* $0 < u_0(x) \leq A$ *for all* $x \in \Omega$*, and* $u \in \mathcal{D}$ *is a minimizer of the Weberized TV restoration energy* E_w *restricted in the admissible space* $\mathcal{D}$ *as defined in (11). Then* $u \leq A$ *(in the Lebesgue a. e. sense). In particular,* $\|u\|_\infty \leq \|u_0\|_\infty$.

Proof. Define $[u]_A = u \wedge A = \min(u, A)$. Then

$$\begin{aligned}
\int_\Omega (u-u_0)^2 dx &= \int_{u \leq A} (u-u_0)^2 dx + \int_{u > A} (u-u_0)^2 dx \\
&\geq \int_{u \leq A} (u-u_0)^2 dx + \int_{u > A} (A-u_0)^2 dx \\
&= \int_\Omega ([u]_A - u_0)^2 dx.
\end{aligned}$$

The equality in the inequality line holds if and only if $u > A$ has Lebesgue measure 0. Meanwhile, by the coarea formula (7),

$$\begin{aligned}
\int_\Omega |\nabla [u]_A|_w dx &= \int_0^\infty \mathrm{Per}([u]_A < \lambda) \frac{d\lambda}{\lambda} = \int_0^A \mathrm{Per}([u]_A < \lambda) \frac{d\lambda}{\lambda} \\
&= \int_0^A \mathrm{Per}\big(\ ([u]_A < \lambda | u \leq A) \cup ([u]_A < \lambda | u > A)\ \big) \frac{d\lambda}{\lambda} \\
&= \int_0^A \mathrm{Per}\big(\ [u]_A < \lambda | u \leq A\ \big) \frac{d\lambda}{\lambda} = \int_0^A \mathrm{Per}(u < \lambda) \frac{d\lambda}{\lambda} \\
&\leq \int_0^\infty \mathrm{Per}(u < \lambda) \frac{d\lambda}{\lambda} = \int_\Omega |\nabla u|_w dx,
\end{aligned}$$

where for the second equality, we have applied $\mathrm{Per}(\Omega, \Omega) = 0$. Together, we have established directly that

$$E_w\big[\ [u]_A\ \big] \leq E_w[\ u\ ],$$

and the equality holds if and only if $u \leq A, a.e.$ Since u is a minimizer in $\mathcal{D}$ and $[u]_A \in \mathcal{D}$, the equality must hold and thus $u \leq A, a.e.$ □

Notice that such direct method is almost unique to TV related energies. The ceiling operator $u \to [u]_A = u \wedge A$ (or the flooring operator $u \to u \vee A$) is readily compatible with the TV measure. The same technique is valid even for more general nonnegative functions $\phi(u)$ as in (7).

Theorem 1 (Existence) *Assume that the observation u_0 satisfies*

$$0 < u_0 \le A \qquad \textit{and} \qquad v_0 = \ln u_0 \in L^1_{loc}(\Omega).$$

Then the Weberized TV restoration model (6) or (10) has at least one minimizer when restricted in the admissible space $\mathcal{D}$ (see (11)).

Proof. Notice that the admissible space $\mathcal{D}$ is nonempty since $u \equiv A \in \mathcal{D}$. Let $u_n \in \mathcal{D}, n = 1, 2, \cdots$ be a minimizing sequence of the Weberized TV restoration energy E_w restricted in $\mathcal{D}$. In the spirit of Lemma 1, we can assume that $u_n \le A, n = 1, 2, \cdots$. Then

$$\int_\Omega |\nabla u_n|_w dx = \int_\Omega \frac{|\nabla u_n|}{u_n} dx \ge \frac{1}{A} \int_\Omega |\nabla u_n| dx.$$

Therefore, (u_n) is a bounded sequence in the Banach space $\mathrm{BV}(\Omega)$ endowed with the BV norm:

$$\|f\|_{\mathrm{BV}} = \mathrm{TV}(f) + \|f\|_{L^1}.$$

By the weak compactness, (u_n) has a subsequence, still denoted by (u_n) for convenience, that converges strongly in $L^1(\Omega)$ to some u_∞: $u_n \to u_\infty$. Furthermore, after a refinement of the subsequence if necessary, we can assume that

$$u_n(x) \to u_\infty(x), \qquad a.e. \quad x \in \Omega.$$

Then by the Lebesgue Dominated Convergence Theorem,

$$\int_\Omega (u_\infty - u_0)^2 dx = \lim_{n \to \infty} \int_\Omega (u_n - u_0)^2 dx. \tag{12}$$

For the control over the Weberized TV, define $v_n = \ln u_n, n = 1, 2, \cdots$ and $v_\infty = \ln u_\infty$. Then

$$v_n(x) \to v_\infty(x), \qquad a.e. \quad x \in \Omega,$$

and since $u_n \in \mathcal{D}$, we also have

$$v_n, \ v_\infty \ge \ln(u_0/2), \quad n = 1, 2, \cdots \tag{13}$$

For any compactly supported vectorial test function

$$\vec{g} = (g_1, g_2) \in C_0^\infty(\Omega, R^2), \quad \text{with} \quad \|\vec{g}\| = \sqrt{g_1^2 + g_2^2} \le 1,$$

we have

$$v_n \nabla \cdot \vec{g} \to v_\infty \nabla \cdot \vec{g}, \quad a.e. \quad x \in \Omega,$$

and

$$|v_n \nabla \cdot \vec{g}| \le \big(\, |\ln(u_0/2)| \vee |\ln A| \, \big) |\nabla \cdot \vec{g}|. \tag{14}$$

Now that the observation $v_0 = \ln u_0 \in L^1_{\text{loc}}(\Omega)$ and that $\vec{g}$ is compactly supported, the right hand side of (14) must belong to $L^1(\Omega)$. Hence, again by the Lebesgue Dominated Convergence Theorem,

$$\int_\Omega v_\infty \nabla \cdot \vec{g}\, dx = \lim_{n\to\infty} \int_\Omega v_n \nabla \cdot \vec{g}\, dx.$$

Consequently, for each of such test functions $\vec{g}$,

$$\int_\Omega v_\infty \nabla \cdot \vec{g}\, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n| dx.$$

Therefore, by definition [Giu84], we must have

$$\int_\Omega |\nabla v_\infty| dx \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n| dx. \tag{15}$$

Finally, the combination of (12) and (15) gives

$$E_w[u_\infty] \le \liminf_{n\to\infty} E_w[u_n].$$

It is easy to see that $u_\infty \in \mathcal{D}$. Since u_n is a minimizing sequence, we therefore have shown that u_∞ is in fact a minimizer. This completes the proof. □

We close this subsection with a remark on the conditions of the existence theorem. In numerous digital applications, the conditions in Theorem 1 are naturally satisfied since intensity values are always scaled to a positive interval $[m,\ M]$. For instance, $M = 256$ and $m = 1$ in most 8-bit display systems. When $m = 0$, one level of elementary shifting, $u \to u + \epsilon$ say, resolves the *blackhole* problem. Such practice is equivalent to what some vision psychologists have called the *modified Weber fraction* $\delta u/(u+\epsilon)$, in which the small intensity value ϵ is called the *activation level*.

3.3 Uniqueness of Weberized TV Restoration

Unlike the classical TV restoration model (3), the Weberized energy

$$E_w[u|u_0] = \int_\Omega |\nabla u|_w dx + \frac{\lambda}{2} \int_\Omega (u - u_0)^2 dx$$

is not convex. As a result, uniqueness is no longer a direct product of convexity.

We start with a computational lemma that is again unique to total variation related energies.

Lemma 2 *Let $\phi(u) : (0,\infty) \to (0,\infty)$ be a C^1 function, and*

$$J[u] = \int_\Omega \phi(u) |\nabla u| dx.$$

then the formal Euler-Lagrange differential of $J[u]$ is

$$\frac{\partial J}{\partial u} = -\phi(u)\nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right]\Big|_\Omega + \frac{\phi(u)}{|\nabla u|}\frac{\partial u}{\partial \vec{n}}\Big|_{\partial\Omega}. \tag{16}$$

Proof. We witness an intrinsic cancellation mechanism characteristic to the TV energy during the standard computation of Calculus of Variation $J \to J + \delta J$:

$$\begin{aligned}\delta J &= \int_\Omega \left(\phi'(u)|\nabla u|\delta u + \phi(u)\frac{\nabla u}{|\nabla u|}\delta(\nabla u) \right) dx \\ &= \int_\Omega \left(\phi'(u)|\nabla u| - \nabla \cdot \left[\phi(u)\frac{\nabla u}{|\nabla u|} \right] \right) \delta u \; dx + \int_{\partial\Omega} \frac{\phi(u)}{|\nabla u|}\frac{\partial u}{\partial \vec{n}} \; \delta u \; ds \\ &= \int_\Omega \left(-\phi(u)\nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right] \right) \delta u \; dx + \int_{\partial\Omega} \frac{\phi(u)}{|\nabla u|}\frac{\partial u}{\partial \vec{n}} \; \delta u \; ds,\end{aligned}$$

where ds denotes the arc-length element of the boundary. The cancellation occurs in the first integral of the second line. This completes the proof. □

Applying the lemma to the Weberized TV restoration energy E_w, we obtain the formal equilibrium Euler-Lagrange equation:

$$-\frac{1}{u}\nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right] + \lambda(u - u_0) = 0, \qquad \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{along } \partial\Omega. \tag{17}$$

Or, applying the time marching scheme along the gradient descent direction,

$$\frac{\partial u}{\partial t} = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right] + \lambda u(u_0 - u), \tag{18}$$

with the same Neumann adiabatic boundary condition, and an appropriate initial guess. Notice that (18) is a nonlinear diffusion-reaction type equation. At each fixed pixel $x \in \Omega$, set $a = u_0(x)$. Then the pure reaction mechanism is given by the ODE

$$\frac{du}{dt} = f(u) = \lambda u(a - u), \qquad u > 0, \tag{19}$$

which has an unstable repelling boundary $u = 0$, and the unique globally stable attractor $u = a$. It is this property that hints that the Weberized TV restoration model may have a unique solution.

The following uniqueness theorem is at a *formal* level in the sense that our proof relies on the formal Euler-Lagrange equations (17). A mathematically more rigorous proof, or an appropriate reformulation of the uniqueness issue, still remains an open problem for our readers.

Like the Existence Theorem 1, the following Uniqueness Theorem is again established in the natural admissible space $\mathcal{D}$ in Section 3.1.

Theorem 2 (Uniqueness) *Assume that $u = z(x) \in \mathcal{D}$ is a minimizer of the Weberized TV restoration energy E_w restricted in $\mathcal{D}$. If $z(x) > u_0(x)/2$ for all $x \in \Omega$, then $u = z(x)$ is unique.*

Proof. Since $z(x) > u_0(x)/2$, $u = z(x)$ is away from the boundary of the admissible space $\mathcal{D}$ and Calculus of Variation is valid, which leads to the Euler-Lagrange equation for $u = z(x)$:

$$-\frac{1}{u}\nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right] + \lambda(u - u_0) = 0, \qquad \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{along } \partial\Omega, \tag{20}$$

or equivalently,

$$-\nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right] + \lambda u(u-u_0) = 0, \qquad \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{along } \partial\Omega. \tag{21}$$

Define a new reference energy $E_r[u|u_0]$ for the Weberized TV restoration E_w as follows:

$$E_r[u|u_0] = \int_\Omega |\nabla u| dx + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 \left(\frac{2u+u_0}{3}\right) dx. \tag{22}$$

It is easy to derive that (21) is exactly the Euler-Lagrange equilibrium equation for $E_r[u|u_0]$. At each fixed pixel $x \in \Omega$, set $a = u_0(x)$, and define a cubic potential of u by

$$F(u) = \frac{\lambda}{2}(u-a)^2 \ \frac{2u+a}{3}. \tag{23}$$

Then

$$F'(u) = \lambda u(u-a) \quad \text{and} \quad F''(u) = 2\lambda(u-a/2).$$

In particular, $F''(u) > 0$, for all $u > a/2$. As a result,

$$J[u] = \frac{\lambda}{2}\int_\Omega (u-u_0)^2 \left(\frac{2u+u_0}{3}\right) dx$$

is *strictly* convex when restricted on $u : u(x) > u_0(x)/2, \ x \in \Omega$. Now that the TV Radon measure is semi-convex, together we conclude that the reference energy $E_r[u|u_0]$ is strictly convex:

$$E_r\left[\frac{u_1+u_2}{2}\Big|u_0\right] \le \frac{E_r[u_1|u_0] + E_r[u_2|u_0]}{2},$$

for any $u_1, u_2 \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ and $u_1, u_2 > u_0/2$. The equality holds if and only if when $u_1 = u_2$. Consequently, its equilibrium Euler-Lagrange equation (21) (even unnecessarily being a global minimum) has at most one solution that satisfies $u > u_0/2$, which implies that $u = z(x)$ is indeed unique as claimed. □

The lower bound $u = u_0/2$ has been initially motivated by the *blackhole* constraint on Weber's fraction, as discussed in Section 3.1. It is perhaps more than a coincidence that it also naturally appears as the inflection point of the reference energy E_r in the proof of uniqueness.

4 The Computational Approach and Examples

As for the classical TV restoration, there are many computational tools available for digitally implementing the energy minimization (see, for example, [CGM99, ROF92, MO00, VO96]). In this paper, as in [COS01, VO96], we propose to apply the linearization technique to iteratively solve the Euler-Lagrange equation (17).

Define $\tilde{\lambda} = \tilde{\lambda}(u) = \lambda u$. Then Eq. (17) can be re-written as

$$-\nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right] + \tilde{\lambda}(u - u_0) = 0, \quad x \in \Omega, \tag{24}$$

with the Neumann adiabatic condition along the boundary of the image domain. It is formally identical to the classical TV denoising equation [ROF92, VO96], except that the fitting constant λ now depends on u. Notice that $\tilde{\lambda} > 0$ since $u > 0$.

To numerically solve (24), we apply the linearization technique. Eq. (24) is to be solved iteratively ($u^{(n)} \to u^{(n+1)}$) based on the linearization

$$L^{(n)} u^{(n+1)} = \tilde{\lambda}^{(n)} u_0, \tag{25}$$

where $\tilde{\lambda}^{(n)} = \lambda u^{(n)}$, and $L^{(n)}$ stands for the linear elliptic operator

$$-\nabla \cdot \left[\frac{1}{|\nabla u^{(n)}|} \nabla \right] + \tilde{\lambda}^{(n)}.$$

As well practiced in the TV restoration literature [CL97, VO96], $L^{(n)}$ is computationally better conditioned to

$$L_\epsilon^{(n)} = -\nabla \cdot \left[\frac{1}{|\nabla u^{(n)}|_\epsilon} \nabla \right] + \tilde{\lambda}^{(n)}, \tag{26}$$

where the notation $|a|_\epsilon$ stands for $\sqrt{a^2 + \epsilon^2}$ for some positive parameter $\epsilon \ll 1$. Such conditioning gets rid of the singularity of $L^{(n)}$ on the homogeneous regions of the current guess $u^{(n)}$ where $|\nabla u^{(n)}|$ is close to zero. Notice that in terms of the energy formulation, (25) and (26) are equivalent to tracking down the unique minimizer $u^{(n+1)}$ of the quadratic energy

$$E[u|u_0, u^{(n)}] = \frac{1}{2} \int_\Omega \frac{1}{|\nabla u^{(n)}|_\epsilon} |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega \tilde{\lambda}^{(n)} (u - u_0)^2 \, dx.$$

Figures 2 and 3 have been generated by this algorithm, and the central-difference based finite difference schemes for the linearized equation (25). Figure 2 displays the noisy test image (the left panel) and its Weberized TV restoration (the right panel). Figure 3 shows a typical 1-dimensional horizontal slice taken from both the noisy image and its Weberized TV restoration. One clearly observes that unlike the conventional TV restoration, the Weberized version is able to distribute the minimum amount of irregularity adaptively over the image domain according to Weber's Law. Therefore, in the restored image, the minimum fluctuation δu is allowed to be larger on regions where the background intensity u is higher and human's visual sensitivity is weaker.

5 Conclusion

Most conventional image processors consider little how human subjects "feel" about the outputs. Weber's Law claims that human's perception and response to the intensity fluctuation δu of both aural and visual signals are not simply uniform, instead,

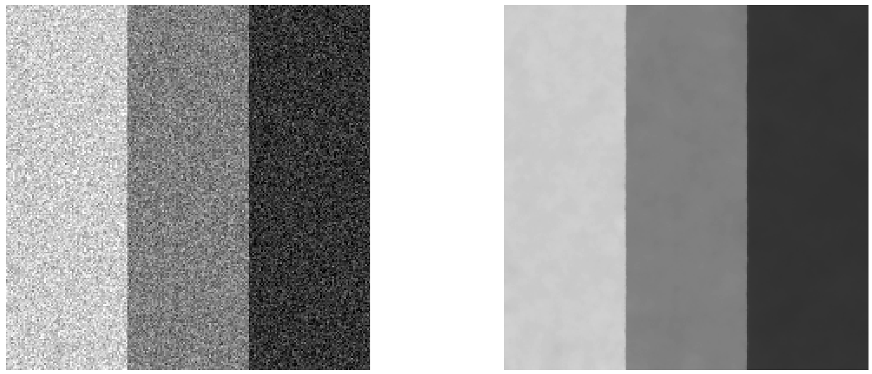

Figure 2: Weberized TV restoration of a test image with homogeneous noise.

should be weighted by the ambient stimulus u. The current paper has attempted to integrate this famous psychological effect into the classical TV image restoration model of Rudin, Osher, and Fatemi [ROF92].

We have studied the issues of existence and uniqueness for the proposed Weberized TV restoration model, based on the direct method in the space of functions with bounded variations $\mathrm{BV}(\Omega)$. We have also proposed an iterative algorithm based on the linearization technique for the nonlinear Euler-Lagrange equation.

We consider the present work as an infantile step in the big blueprint of integrating important psychological and psychophysical results (either empirical or statistical) into the contemporary imaging science and technology. Our long-term goal has been set on the exploration of all possible important interactions.

Acknowledgments

I deeply thank my great teachers and friends Professors Dan Kersten and Paul Schrater at the Department of Psychology, University of Minnesota, for patiently teaching me both vision psychology and psychophysics.

References

[CGM99] T. F. Chan, G. H. Golub, and P. Mulet. A nonlinear primal-dual method for total variation-based image restoration. *SIAM J. Sci. Comput.*, 20(6):1964–1977, 1999.

[CL97] A. Chambolle and P. L. Lions. Image recovery via Total Variational minimization and related problems. *Numer. Math.*, 76:167–188, 1997.

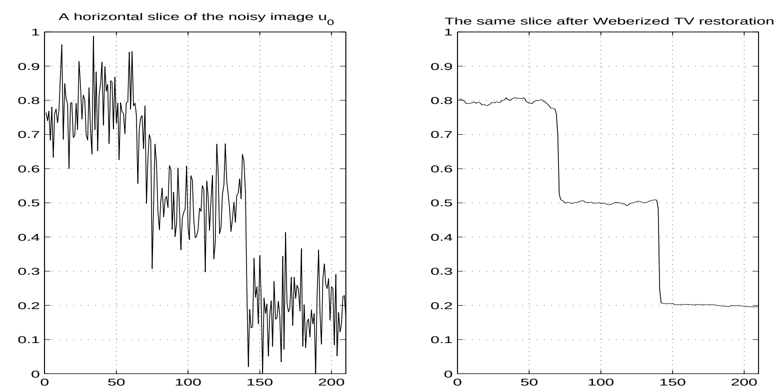

Figure 3: A typical horizontal slice from the noisy image (left) and its Weberized TV restoration (right). One is able to clearly observe two main features of the Weberized TV restoration model: (1) like the classical TV model, it is able to maintain the sharpness of edges in the noisy image; (2) but unlike the direct TV restoration model, it is able to distribute the minimum amount of irregularity adaptively over the image domain according to Weber's Law: the minimum amount of fluctuation (in the minimized energy) is allowed to be larger on regions where the background intensity is higher and human's visual sensitivity gets weaker.

[COS01] T. F. Chan, S. Osher, and J. Shen. The digital TV filter and nonlinear denoising. *IEEE Trans. Image Process.*, 10(2):231–241, 2001.

[CS00] T. F. Chan and J. Shen. Variational restoration of non-flat image features: models and algorithms. *SIAM J. Appl. Math.*, 61(4):1338–1361, 2000.

[CS01] T. F. Chan and J. Shen. Mathematical models for local non-texture inpaintings. *SIAM J. Appl. Math.*, 62(3):1019–1043, 2001.

[CS02] T. F. Chan and J. Shen. On the role of the BV image model in image restoration. UCLA's Mathematics Department CAM Report 02-14 available at: www.math.ucla.edu/˜imagers, 2002.

[Fec58] G. T. Fechner. Über ein wichtiges psychophysiches Grundgesetz und dessen Beziehung zur Schäzung der Sterngrössen. Abk. k. Ges. Wissensch. *Math.-Phys.*, K1, 4, 1858.

[GG84] S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Trans. Pattern Anal. Machine Intell.*, 6:721–741, 1984.

[Giu84] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser, Boston, 1984.

[MH80] D. Marr and E. Hildreth. Theory of edge detection. *Proc. Royal Soc. London*, B:207: 187–217, 1980.

[MO00] A. Marquina and S. Osher. Explicit algorithms for a new time dependent model based on level set motion for nonlinear deblurring and noise removal. *Siam. J. Sci. Comput.*, 22:387–405, 2000.

[Mum94] D. Mumford. *Geometry Driven Diffusion in Computer Vision*, chapter “The Bayesian rationale for energy functionals”, pages 141–153. Kluwer Academic, 1994.

[RO94] L. Rudin and S. Osher. Total variation based image restoration with free local constraints. *Proc. 1st IEEE ICIP*, 1:31–35, 1994.

[ROF92] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[Str93] G. Strang. *Introduction to Applied Mathematics*. Wellesley-Cambridge Press, MA, 1993.

[VO96] C.R. Vogel and M. E. Oman. Iterative methods for total variation denoising. *SIAM J. Sci. Comput.*, 17(1):227–238, 1996.

[Web34] E. H. Weber. De pulsu, resorptione, audita et tactu. *Annotationes anatomicae et physiologicae*, Leipzig: Koehler, 1834.