Microlocal Regularity of an Inverse Problem for the Multidimensional Wave Equation

Gang Bao

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by

Gang Bao

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APPROVED, THESIS COMMITTEE:

William W. Symes, Chairman
Professor of Mathematical Sciences

Barbara L. Keyfitz
Professor of Mathematics
University of Houston

John C. Polking
Professor of Mathematics

Chao-Cheng Wang
Noah Harding Professor of Mathematical Sciences

Mary F. Wheeler
Noah Harding Professor of Mathematical Sciences

Houston, Texas
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Abstract

Many physical processes such as reflection seismology, oil exploration, and ground-penetrating radar may be modeled as inverse problems for the multidimensional acoustic wave equation with point energy sources. The inverse problem is to identify the coefficients from the knowledge of boundary measurements of the solution.

In this research we formulate an inverse problem for the wave equation with constant wave speed as a functional equation involving a *forward* map, which maps the coefficient (density) to the boundary value of the solution (excess pressure).

We begin by examining some fundamental results in nonsmooth microlocal analysis. Rauch's lemma on the algebraic property of microlocal Sobolev spaces and a Beals-Reed linear propagation of singularities theorem are extended. We then present a trace regularity theorem which indicates that with microlocal restrictions against tangential oscillations in the coefficient, the boundary value is just as regular as the solution itself. The trace theorem also gives the first hint of the appropriate domain and range for the forward map. However, compared to the one dimensional case,

much more overall smoothness has to be imposed to assure the optimal regularity of timelike traces.

The Hadamard theory on progressing wave expansions is employed to study the fundamental solution to the linear acoustic wave equation. To establish the regularity of the solution, the solutions of transport equations are investigated by applying the Rauch-type results.

The central result for the regularity of the inverse problem is an upper bound for the linearized forward map with nonsmooth reference density. In order to establish this regularity result, a dual technique is developed which dramatically reduces the difficulties of the inverse problem. Our method has the potential to obtain some regularity results even for the important nonsmooth reference velocity case. Similar analyses could result in a continuity result and a differentiability result for the forward map. These regularity properties are obviously crucial in the design and analysis of the algorithms for solving the inverse problem.

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For my parents and Linli

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Chapter 1

Preliminaries

1.1 Description of problem

A simplified model which governs many physical processes such as seismic and acoustic wave propagation is the following reduced linear acoustic wave equation:

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - \nabla\sigma \cdot \nabla)u = f \ , \tag{1.1}$$

where $\sigma = \sigma(x)$ is the logarithm of the density, $c = c(x)$ is the sound speed of the medium, and $f = f(x,t)$ is the source term which introduces the energy to the problem. If σ, c and f are given along with appropriate side conditions, the forward (or direct) problem is to determine $u = u(x,t)$, the excess pressure. For appropriate choices of σ, c, and f, u is determined uniquely by standard linear hyperbolic theory of partial differential equations (*p.d.e.*).

In this work we study the inverse problem which arises in reflection seismology, oil exploration, ground-penetrating radar, etc. To understand the problem further, let us look at a simple exploration seismology experiment explained in detail in Lailly [20]. Near the surface of the earth, a seismic source is fired at some point (point source). The seismic waves propagate into the earth. Since the earth's structure varies (as do its physical properties) part of the energy of the wave will be reflected back to the surface and can be measured. The inverse problem is to deduce the interior properties of the earth from the recorded data. Mathematically, the inverse problem is to determine the coefficients σ, c by knowing additional boundary value conditions of u.

A natural problem of mathematical and physical importance is to pursue the *right* models so that the reflected waves they generate carry sufficient information for determining the physical properties of the medium. By the theory of geometric optics, the models which are too smooth (*i.e.*, the coefficients σ, c are smooth) on the wavelength scale do not generate reflected waves. On the other hand, no energy penetrates extremely oscillatory media, hence models that are too rough generate no reflected waves.

Another important reason that one wants to work with nonsmooth models comes from a computational point of view. It is clear that to solve inverse problems numerically requires efficient minimization algorithms. By far, the most efficient minimization algorithms are Newton-type algorithms. According to the infinite dimensional optimization theory (see e.g. Kantorovich and Akilov [19]), in order to formulate any *effective* convergent Newton-type algorithm, one has to study the problem in a Banach space. Moreover, dealing with minimization problems, the best available results are perhaps those for Hilbert spaces. Note that even though C^∞-topology induces countable semi-norms, it is not a Banach space. At this point, we do not know any effective convergent Newton-type minimization algorithm in a non-Banach space. Besides, it is natural to use the weakest norm and the biggest possible space of models.

To fix the ideas, write $x \in \mathbb{R}^n$ as (x', x_n), where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ representing depth of the "flat" earth. We assume that the medium is the whole space $\mathbb{R}^n$ and $u = 0$ in the past. Take $f(x,t) = \delta(x,t)$ as an ideal point source. This assumption seems reasonable when the spatial extent of the source is much smaller than a typical wavelength and all frequency components to be measured are present in f. More explanations on the validity of these assumptions may be found in Symes [37]. Throughout this work we shall restrict ourselves to the special case of constant velocity c, though we believe that the ideas in this work may be extended to cover some more general cases. We then have the following simple model:

$$\Box u - \nabla\sigma \cdot \nabla u = \delta(x,t)\ , \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{1.2}$$

$$u = 0\ , \quad t < 0\ , \tag{1.3}$$

where $\Box$ is defined to be $\partial_t^2 - \Delta$, and Δ is the Laplacian. Thus the inverse problem is:

Recover $\sigma = \sigma(x)$ by the knowledge of $u = u(x,t)$ on the subset $\{x_n = 0\}$.

Define the forward map F as:

$$F : \sigma \rightarrow (\phi u)\mid_{x_n=0}\ , \tag{1.4}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$ is supported inside the conoid $\{t > |x|\}$ and near$\{x_n = 0\}$. The reason for introducing this cut-off function, $\phi(x,t)$ is that we want to make sure the restriction of distribution u to the hypersurface $\{x_n = 0\}$ is well defined even though the equation (1.2) has a singular right-hand side. Since the inverse problem is just to invert this functional relation F, we are naturally interested in all the properties of this forward map.

Because F is nonlinear, one wants to work with the formal linearization (or formal derivative) DF, with respect to the reference state (σ_0, u_0), defined by first order perturbation theory (Born-approximation). Let u_0 be the solution of (1.2), (1.3) corresponding to σ_0. Assume that $u_0 + \epsilon\delta u$ is the solution corresponding to $\sigma_0 + \epsilon\delta\sigma$. Substituting them into the equation and ignoring higher order terms in ϵ, we obtain the linearized problem

$$\Box\delta u - \nabla\sigma_0 \cdot \nabla\delta u = \nabla\delta\sigma \cdot \nabla u_0 \tag{1.5}$$

$$\delta u = 0\,, \quad t < 0\,. \tag{1.6}$$

DF is defined by

$$DF(\sigma_0)\delta\sigma = (\phi\delta u)\mid_{x_n=0}\,. \tag{1.7}$$

It is our main goal in this work to determine the appropriate spaces of the domain and range of F for which

the formal derivative DF is bounded.

We believe that similar analysis will lead to the continuity or even differentiability of F. These properties are obviously crucial in the design and analysis of algorithms for solving the inverse problems.

1.2 Previous research

When the spatial dimension is one or c and σ depend only on x_n (layered problem) there is a large literature available. For a similar problem in which the medium was assumed to be excited by an impulsive load on the surface $\{x_n = 0\}$ instead of point sources, the properties of the forward map have been studied fairly satisfactorily by Symes and others (see Symes [34] for references). It was shown by Symes that, for the constant wave speed case, the forward map is a C^1-*diffeomorphism* by applying the method of geometrical optics together with energy estimates.

When the spatial dimension $n > 1$ and c, σ depend on all space variables (non-layered problem), very little is known in mathematics. Symes [32, 33, 35], Sacks and Symes [29], Rakesh [25], and Sun [31] have some partial results. The difficulties are essentially due to the ill-posed nature of the timelike hyperbolic Cauchy problem and the presence of nonsmooth coefficients. For the one dimensional wave equation, both coordinate directions are spacelike, which indicates that the problem is hyperbolic with respect to both directions. Apparently, this is not the case when the spatial dimension is larger than one.

Rakesh in [25] looked at a related linearized velocity inversion problem with constant density and point sources. Assuming smooth background velocity, he obtained some results on both upper and lower bounds for the linearized forward map. The essential observation in Rakesh's work is that DF is a Fourier integral operator (see also Beylkin [7]). Unfortunately, the calculus of Fourier integral operators employed in Rakesh's work is not applicable to the nonsmooth reference velocity case since the linearized forward map is a Fourier integral operator only when the reference velocity is smooth.

In [32], Symes gave a pair of examples, based on the geometric optics construction, which show that both $DF(1)$ and $DF(1)^{-1}$ are unbounded for a slightly different problem. As the examples show, within the Sobolev scales no strengthening or weakening of topologies of the domain and range can make both DF and DF^{-1} bounded. This fact also implies a strategy of regularization: Change the topology in the domain so that DF becomes bounded, then ask for optimal regularization of DF^{-1} in the sense of best possible lower bound estimate for DF. In both examples of Symes, the unboundedness was caused by rapid oscillation of σ in the x'-direction or the tangential directions, hence the problem is actually "partially well-posed", i.e., only more smoothness of the coefficients in tangential directions (essentially grazing ray directions) will be required to cure the difficulty. This might be the main reason the anisotropic Sobolev spaces $H^{m,s}(\mathbb{R}^n)$ or Hörmander spaces, were introduced in [29], [35] and [31]. As defined originally in Hörmander [14],

$$H^{m,s}(\mathbb{R}^n) = \{f \in \mathcal{D}', D^{\alpha}_{x',\, x_n} f \in L^2(\mathbb{R}^n), \forall \alpha = (\alpha_1, ..., \alpha_n), \mid \alpha \mid \leq m + s, \alpha_n \leq m\}$$

where

$$D^{\alpha}_{x',\, x_n} = D^{\alpha_1, ..., \alpha_{n-1}}_{x'} D^{\alpha_n}_{x_n}.$$

Thus, in $H^{m,s}(\mathbb{R}^n)$, the distributions are in H^m with s additional orders of smoothness in the x'-variables.

In Theorem 4.1 of [29] Sacks and Symes showed by using the full strength of sideways energy estimates that for a linearized density determination problem with constant velocity and plane wave sources, DF is bounded from $H^{1,1}$ to H^1, provided the reference coefficient is in $H^{1,s}$ for some $s > n + 2$. They also proved the injectivity of DF. However, as they pointed out, the lower bound for DF was not that satisfactory. Our techniques and results are quite different from theirs. We intend to assure the optimal regularity of the timelike trace under weaker hypotheses.

There remains an extremely important issue to be addressed, namely,

What is an appropriate space for the domain of DF ?

In 1983, as one of his conjectures, Symes suggested that microlocal restrictions on the coefficients might regularize the inverse problem (see [33], [35], and [36]). In some sense, this was confirmed by recent joint work with Symes [2], where we were able to prove a trace theorem for the solutions of general linear *p.d.e.* with smooth coefficients. Roughly speaking, our theorem asserts that the solution will belong to H^s along a codimension one hypersurface if it belongs to H^s in a neighborhood of the hypersurface and to H^{s+1} microlocally in those directions where the *p.d.e.* is not microlocally strictly hyperbolic. Note that we gained back the half derivative from the standard trace theorem.

In this work, we relax the requirement of additional tangential smoothness to additional microlocal smoothness. The microlocal regularity of the forward map will be established. Various results on propagation of singularities and microlocal analysis will play essential roles.

Interested in applications to nonlinear *p.d.e.*, Beals, Bony, Rauch, Reed and many other mathematicians have over the past decade developed the theory of microlocal analysis concerning *p.d.e.* with nonsmooth coefficients. The fundamental fact is *Rauch's lemma* proven by Rauch in [26].

Lemma 1.1 (Rauch's Lemma) For some $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$, assume that $u, v \in H^s \cap H^r_{m\ell}(x_0, \xi_0)$ with $n/2 < s \leq r < 2s - n/2$. Then

$$uv \in H^s \cap H^r_{m\ell}(x_0, \xi_0) \ .$$

The propagation of singularities theorem for linear *p.d.e.* with smooth coefficients is due to Hörmander [15] and dates back to the early seventies. Hörmander's theorem basically says: The singularities propagate generally along the null bicharacteristics of the principal part of the operator. However, Hörmander's theorem itself does not say anything about how the singularities propagate when the *p.d.e.* has nonsmooth coefficients. Obviously the nonsmoothness of the coefficients will introduce new singularities to the solutions, so that only limited initial regularity can be propagated. In 1982, Beals and Reed [5] were able to prove a linear propagation of singularities theorem for strictly hyperbolic pseudodifferential operators with nonsmooth coefficients at lower order terms by following the general outline of the proof of Hörmander's theorem. Their theorem and its proof turn out to be the major tools in our work here.

It is known that in their applications to nonlinear wave equations, most of the results based on Rauch's lemma (or the method of Fourier analysis) can only deal with relatively weak singularities. This is very unfortunate, since in many situations strong singularities (like shocks) are present. In general, such strong singularities are much more difficult to work with. Nonetheless, this work exhibits that to some extent, strong singularities appearing in the linear wave equation (*e.g.* the fundamental solution) can also be tackled by this Fourier analysis method. The relation between the coefficients and solution with strong singularities remains to be fully understood, especially when the coefficients are less regular.

1.3 Plan of the work

The fundamental concepts of microlocal analysis and results of propagation of singularities are introduced in Chapter 2. We establish an extension of Rauch's lemma and a new commutator lemma. The main result of this chapter is an extended Beals-Reed theorem on propagation of singularities which allows relaxation of the smoothness requirements on the solution, and therefore on the coefficients. In the last part of this chapter, we obtain some better regularity results when the coefficients are independent of some variables.

In Chapter 3, we prove a trace regularity theorem by using the propagation of singularities theorem introduced in Chapter 2, along with an application of the pseudodifferential ($\psi.d.o.$) cut-off technique and standard hyperbolic energy estimates. Our trace theorem indicates that with microlocal restrictions against tangential oscillations in the coefficients, the trace is just as regular as the solution. In particular, it is as regular as the coefficients allow it to be. These properties of traces also indicate that the conclusion of our trace theorem is optimal. However, as compared to the layered problem, a much higher degree of overall smoothness has to be imposed.

The basic structure of fundamental solutions to second order multidimensional wave equations can be traced back to the works of Hadamard [13], where progressing wave expansions were introduced to construct approximate solutions. To explain the ideas of Hadamard's theory, we state a theorem due to Romanov which gives an explicit progressing wave expression of the fundamental solution when the spatial dimension is three. We point out that his proof actually works for other dimensions as well. The main result in Chapter 4 is a regularity theorem for the solution of the model problem (*i.e.* the fundamental solution to equations (1.2), (1.3)) by applying the method of progressing wave expansions inside the characteristic surface. A simple energy identity plays a crucial role in our analysis: It indicates that the regularity result can be established by analyzing the regularity of the transport equations. Based on a $\psi.d.o.$ cut-off technique and some properties of $\phi.d.o.$ as well as Rauch-type results, we are able to prove a regularity result for solutions of the transport equations which also turns out to be important in the next chapter.

Chapter 5 is devoted to a study of the regularity of the forward map. By analyzing the propagation of regularity for solution of a problem dual to the linearized problem, we obtain an upper bound for the formal derivative of the forward map at the nonsmooth reference density σ_0. In this process, a microlocal version of the classical trace theorem is introduced. An important step is deriving an estimate out of the result on propagation of singularities. It seems to us that the dual technique developed in this chapter may well be useful in other contexts.

Conclusions are presented in the last chapter. We also make a few comments on the effective computation of the multidimensional hyperbolic inverse problems. Some future interests will be briefly mentioned at the end of this chapter.

1.4 Notation

Throughout this thesis, the reader is assumed to be familiar with the basic calculus of *Pseudodifferential Operators* (from now, they will always be called " ψ.d.o. ") as stated in Taylor [39], Nirenberg [24] or Chazarain and Piriou [9]. A classical ψ.d.o. P of order m is denoted as $P \in OPS^m$ with its symbol $p \in S^m$. $ES(P)$ stands for *the essential support* of operator P. $WF(u)$ denotes *the wave front set* of a distribution u. H^s is the standard L^2-type Sobolev space and H^s_{loc} means a local Sobolev space. $\langle\xi\rangle$ means $(1+|\xi|^2)^{1/2}$. The Fourier transform of a distribution u is expressed as $\hat{u}$.

Usually, the constant from the Fourier Transform is assumed to be absorbed by the integral. For simplicity, C serves as a generalized positive constant the precise value of which is not needed. Finally, χ_Γ is the characteristic function of a set Γ.

1.5 Warning

When the reference density σ_0 is smooth, most of the regularity results for the forward map in this work will follow more easily from the calculus of *Fourier Integral Operators.* For a standard text on *F. I. O.* we refer to Duistermaat [11] or Hörmander [16]. However, this technique fails with the appearance of the nonsmooth reference density, an assumption important in this work.

Chapter 2

Microlocal Analysis and Propagation of Singularities

2.1 Introduction

In this chapter, the basic material of microlocal Sobolev spaces that will serve this work is introduced and discussed. The recent monograph of Beals [4] contains the most complete references to date in microlocal analysis and its applications to the study of nonlinear hyperbolic partial differential equations.

Our main result here is a linear propagation of singularities theorem which is an extension of the Beals-Reed theorem in [5]. The theorem assures that weaker regularity of the solution may also be propagated along the null bicharacteristics. The main ingredients in our proof are an extended Rauch's lemma and a commutator lemma.

We prove the theorem by following the general scheme of the proof of the Beals-Reed theorem in [5]. The pseudodifferential cutoff technique in their proof was analogous to a proof of Hörmander's theorem (see [15] for the original form) as described in Nirenberg [24]. Except for the use of Rauch's lemma since nonsmooth coefficients and right-hand side were present, the key step was a commutator lemma which allowed them to compute the action on $H^s \cap H^r_{m\ell}(\gamma)$ of a commutator of a $\psi.d.o.$ with a differential operator whose coefficient was nonsmooth. Then a local existence theorem with microlocal hypotheses completed their proof.

All the results in Sections 2.3-2.5 are improved in the last section under the additional hypothesis that the coefficients depend only on some of the variables (recall that the density in the model problem is time independent).

2.2 Some basic estimates

The following estimates will be used frequently in various contexts, essentially because it contains very useful information about certain kernels, other related kernel estimates may be found in Beals [3] and Beals and Reed [6].

We begin with Young's inequality (see Adams [1] for a proof).

Proposition 2.1 (Young's Inequality) Let $1 \leq p < +\infty$ and $u \in L^1(\mathbb{R}^n)$, $v \in L^p(\mathbb{R}^n)$. Then the convolution products

$$u * v = \int_{\mathbb{R}^n} u(x-y)v(y)dy \ , \quad v * u(x) = \int_{\mathbb{R}^n} v(x-y)u(y)dy$$

are well defined and are equal $a.e.$ in $\mathbb{R}^n$. Moreover,

$$u * v \in L^p(\mathbb{R}^n) \ , \quad \text{and} \quad \|u * v\|_p \leq \|u\|_1 \|v\|_p \ .$$

A combination of the Cauchy-Schwarz inequality and Young's inequality yields the kernel estimates Proposition 2.2 obtained originally by Rauch and Reed in [27].

Proposition 2.2 Define

$$T_{g,h}(\xi) = \int K(\xi,\eta)f(\eta)g(\xi-\eta)d\eta$$

where $f, g \in L^2$. Then the estimate

$$\|T_{g,h}(\xi)\|_{L^2} \leq C\|f\|_{L^2}\|g\|_{L^2}$$

holds if $K(\xi,\eta)$ can be decomposed into finitely many pieces, *i.e.* $K = \sum_i K_i(\xi,\eta)$ such that each of which satisfies one of the following conditions:

$$(1)\ \sup_{\xi}\int |K_i(\xi,\eta)|^2 d\eta \leq C_0 < +\infty,$$

$$(2)\ \sup_{\eta}\int |K_i(\xi,\eta)|^2 d\xi \leq C_0 < +\infty.$$

An immediate consequence of Proposition 2.2 leads to a key estimate in this chapter.

Corollary 2.1 Define

$$T_{g,h}(\xi) = \int \frac{f(\eta)g(\xi-\eta)}{\langle\eta\rangle^{\alpha}\langle\xi-\eta\rangle^{\beta}}d\eta$$

where $f, g \in L^2(\mathbb{R}^n)$, $\alpha+\beta > n/2$. Then

$$\|T_{g,h}(\xi)\|_{L^2} \leq C\|f\|_{L^2}\|g\|_{L^2}\ .$$

Finally, a distance argument is stated for completeness.

Proposition 2.3 Assume that K' is a closed cone which is strictly contained in an open cone K. If $\xi \in K'$, $\eta \in K^c$, then

(1) $|\xi - \eta| \geq C_1|\xi|$, $C_1 > 0$;

(2) if $|\xi| \geq C_0 > 0$, then $\langle \xi - \eta \rangle \geq C\langle \xi \rangle$.

Proof (1) is trivial. The statement (2) holds from the simple fact:

$$\langle \xi \rangle^2 = 1 + |\xi|^2 \leq (1 + 1/C_0)|\xi|^2 \leq C\langle \xi - \eta \rangle^2 .$$

□

2.3 Microlocal Sobolev spaces

We present some basic properties of microlocal Sobolev spaces. Details may be found in the references mentioned above. Only new results will be proved.

The standard *Schauder's lemma* asserts that $H^s(\mathbb{R}^n)$ is an algebra for $s > n/2$. Concerning the lower order Sobolev spaces, one can generalize Schauder's lemma in a number of ways.

Lemma 2.1 If $u \in H^{s_1}(\mathbb{R}^n)$ and $v \in H^{s_2}(\mathbb{R}^n)$, with $s_1 + s_2 \geq 0$, then

$$uv \in H^{\min(s_1, s_2, s_1+s_2-n/2-\delta)} \quad \text{for any} \quad \delta > 0 .$$

Proof The conclusion holds if either one of the following additional conditions is satisfied (see Beals [3] for a proof):

$$s_1, s_2 \geq 0 \quad \text{or} \quad s_1, s_2 \leq n/2 \text{ and } s_1 + s_2 \geq 0 .$$

Thus, in order to prove the lemma, it suffices to look at the case where $s_1 > n/2$ and $s_2 < 0$.

Observe that

$$\langle\xi\rangle^{s_2} \mid \widehat{uv}(\xi) \mid = \int K(\xi,\eta) \mid f(\xi-\eta)g(\eta) \mid d\eta,$$

where $\hat{u}(\xi-\eta) = \langle\xi-\eta\rangle^{s_1} f(\xi-\eta)$, $\hat{v}(\eta) = \langle\eta\rangle^{s_2} g(\eta)$, $f, g \in L^2(\mathbb{R}^n)$, and $K(\xi,\eta) = \dfrac{\langle\xi\rangle^{s_2}}{\langle\xi-\eta\rangle^{s_1}\langle\eta\rangle^{s_2}}$.

The conclusion then follows from Proposition 2.2, by considering three cases:

(1) If $\mid \eta \mid \leq 1/2 \mid \xi \mid$, then

$$K(\xi,\eta) \leq \frac{C}{\langle\xi-\eta\rangle^{s_1}};$$

(2) If $\mid \eta \mid \geq 1/2 \mid \xi \mid$ and $\mid \xi-\eta \mid \leq 1/2 \mid \xi \mid$, then $\mid \eta \mid \leq 3/2 \mid \xi \mid$ and

$$K(\xi,\eta) \leq \frac{C}{\langle\xi-\eta\rangle^{s_1}};$$

(3) If $\mid \eta \mid \geq 1/2 \mid \xi \mid$, $\mid \xi-\eta \mid \geq 1/2 \mid \xi \mid$, then since $\mid \eta \mid \leq \mid \xi-\eta \mid + \mid \xi \mid$,

$$K(\xi,\eta) \leq \frac{C}{\langle\xi\rangle^{s_1}}.$$

□

It is evident that the microlocal Sobolev spaces give a precise description about how regularity and singularities are propagated for solutions to linear strictly hyperbolic *p.d.e.*.

Definition 2.1 $u \in H^s \cap H^r_{m\ell}(x_0, \xi_0)$ if there exist $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$ and a conic neighborhood $\gamma \subset \mathbb{R}^n \backslash \{0\}$ of ξ_0 such that

$$\langle \xi \rangle^s (\phi u)^\wedge(\xi) \in L^2(\mathbb{R}^n) \textit{ and } \langle \xi \rangle^r \chi_\gamma(\xi)(\phi u)^\wedge(\xi) \in L^2(\mathbb{R}^n) .$$

In terms of classical ψ.*d.o.*, there is an equivalent way to characterize the microlocal Sobolev spaces. Recall that a classical ψ.*d.o.* $p(x, D)$ of order m is said to be microlocally elliptic at (x_0, ξ_0) if there is a constant C and a small conic neighborhood γ of ξ_0 such that its symbol satisfies

$$\mid p(x, \xi) \mid \geq C \langle \xi \rangle^m$$

on γ.

Proposition 2.4 $u \in H^r_{m\ell}(x_0, \xi_0)$ if and only if there is a ψ.*d.o.* $Q \in OPS^0$ such that Q is microlocally elliptic at (x_0, ξ_0) and $Qu \in H^r_{loc}$.

To work on microlocal Sobolev spaces, Rauch's lemma (whose standard form is stated in the introduction) is essential. It gives the algebraic property of this interesting class of spaces. Here, we prove a generalized Rauch's lemma whose statement was first presented in Symes [36].

Lemma 2.2 Suppose that for some $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$, the distributions u, v satisfy $u \in H^s \cap H^q_{m\ell}(x_0, \xi_0)$ and $v \in H^l \cap H^q_{m\ell}(x_0, \xi_0)$ with $n/2 < s$, $0 \leq l \leq s, q$, and $q < s + l - n/2$. Then

$$uv \in H^l \cap H^q_{m\ell}(x_0, \xi_0) .$$

In order to prove the lemma, the following Proposition 2.5 is needed.

Proposition 2.5 Assume that, for $n/2 < r$, $0 \leq l \leq q \leq r$, and $q < l + r - n/2$,

$$\tilde{w} \in H^r_{\text{loc}}(x_0) \text{ and } w \in H^l \cap H^q_{m\ell}(x_0, \xi_0) .$$

Then

$$\tilde{w}w \in H^l \cap H^q_{m\ell}(x_0, \xi_0) .$$

Proof Let K be a small conic neighborhood of ξ_0 such that $w \in H^l \cap H^q_{m\ell}(K)$. By Definition 2.1, we may represent w as

$$w = w_1 + w_2 \tag{2.1}$$

with

$$w_1 \in H^l_{loc} \cap H^\infty(K) , \quad w_2 \in H^q_{loc} .$$

Then

$$\tilde{w}w = \tilde{w}w_1 + \tilde{w}w_2 .$$

Since $n/2 < r$ and $q \leq r$, Lemma 2.1 gives

$$\tilde{w}w_2 \in H^q_{loc} . \tag{2.2}$$

Next using Lemma 2.1 one more time gives us $\tilde{w}w_1 \in H^l_{loc}$. Therefore from (2.1) and (2.2), it suffices to show that

$$\tilde{w}w_1 \in H^q_{m\ell}(x_0, \xi_0) . \tag{2.3}$$

Let K' be another conic neighborhood of ξ_0 which is strictly contained in K. Suppose that

$$f(\xi) = \langle\xi\rangle^l \hat{w}_1(\xi) \text{ and } g(\xi) = \langle\xi\rangle^r \hat{\tilde{w}}(\xi) \ ,$$

which means that $f, g \in L^2$. Thus in order to get (2.3), it is sufficient to show that

$$\langle\xi\rangle^q \chi_{K'}(\xi) \hat{\tilde{w} w_1}(\xi) = \int \frac{\chi_{K'}(\xi)\chi_{K^c}(\eta)\langle\xi\rangle^q f(\eta) g(\xi-\eta)}{\langle\xi-\eta\rangle^r \langle\eta\rangle^l} d\eta \qquad (2.4)$$

belongs to L^2. But this is simple from our previous results. Actually, since $\xi \in K'$ and $\eta \in K^c$, Proposition 2.3 implies $\langle\xi-\eta\rangle \geq C\langle\xi\rangle$, therefore Corollary 2.1 together with the hypothesis $q < l + r - n/2$ lead to the fact that (2.4) is in L^2 space. □

Proof of Lemma 2.2 If $q \leq s$ then $u \in H^s_{loc}$, the conclusion follows from Proposition 2.5. Therefore, we may assume that $s < q$.

Definition 2.1 allows us to rewrite

$$u = u_1 + u_2 \text{ and } \quad v = v_1 + v_2 \ ,$$

with

$$u_1 \in H^s_{\text{loc}} \cap H^\infty(x_0, \xi_0) \ , \quad u_2 \in H^q_{\text{loc}} \ ,$$

$$v_1 \in H^l_{\text{loc}} \cap H^\infty(x_0, \xi_0) \ , \quad v_2 \in H^q_{\text{loc}} \ ,$$

then

$$uv = u_1 v_1 + u_2 v + u v_2 - u_2 v_2 \ .$$

The fact $u_2 v_2 \in H^l \cap H^q_{m\ell}(x_0, \xi_0)$ is an immediate consequence of Lemma 2.1. From Proposition 2.5, we have

$$u_2 v \text{ and } u v_2 \in H^l \cap H^q_{m\ell}(x_0, \xi_0) \ .$$

Finally, Proposition 2.6 below and Lemma 2.1 yield

$$u_1 v_1 \in H^q_{m\ell}(x_0, \xi_0) \ , \quad \text{for } q \leq s + l - n/2 \ ,$$

$$u_1 v_1 \in H^l_{\text{loc}} \ .$$

□

Remark on Lemma 2.2. Note that for $l = s$, this lemma becomes the original Rauch's lemma. Since l can be any constant between 0 and s, q can be any number in the interval $[l, s+l-n/2)$, Lemma 2.2 is indeed an improvement of Rauch's lemma.

We claim that like Rauch's lemma, Lemma 2.2 cannot be strengthened. The following result and an example exhibit the extent to which the result is applicable.

Proposition 2.6 Let K_1, K_2, and K be cones in $\mathbb{R}^n \backslash 0$ and assume that $u_i \in H^{s_i}(\mathbb{R}^n)$ and $\tilde{\Pi} WF(w_i) \subset K_i$, $i = 1, 2$, where $\tilde{\Pi}$ denotes the projection on the second factor (or on the frequency space). If $K \subset\subset K_1^c \cap K_2^c$, then

$$\chi_K(D)(w_1 w_2) \in H^{\tilde{s}} \ , \ \text{ if } \tilde{s} \leq s_1 + s_2 - n/2 \ .$$

Remark. When $\tilde{s} < s_1 + s_2 - n/2$, the result is due to Rauch (Theorem 2.2 in [26]); the extreme case was first observed by Meyer in [23]. The proposition cannot be strengthened as is shown by the example described below (the example is illustrated in Figure 2.1). The idea for the example came from private conversation with Michael Beals at IMA. See also Beals [4].

Suppose that

$$\hat{w}_1(\xi) = \langle \xi \rangle^{-s_1 - n/2 - \delta} \chi_{K_1}(\xi) \text{ and } \hat{w}_2(\xi) = \langle \xi \rangle^{-s_2 - n/2 - \delta} \chi_{K_2}(\xi) \ ,$$

where $\delta > 0$ small, K_1 is a small conic neighborhood of $\{(\xi_1, 0, \cdots, 0) : \xi_1 > 0\}$, and K_2 is a small conic neighborhood of $\{(0, \xi_2, \cdots, 0) : \xi_2 > 0\}$ with $\hat{w}_1$, $\hat{w}_2 \geq 0$. Let K be a small conic neighborhood of $\{(\xi_1, \xi_2, \cdots, 0) : \xi_1 = \xi_2\}$. Then

$$\begin{aligned} \chi_K(\xi) w_1 \hat{w}_2(\xi) &= \int \frac{\chi_K(\xi)\chi_{K_1}(\xi - \eta)\chi_{K_2}(\eta)}{\langle \xi - \eta \rangle^{s_1 + n/2 + \delta} \langle \eta \rangle^{s_2 + n/2 + \delta}} d\eta \\ &\geq \int_{|\eta_2| \leq \epsilon |\eta_1|} \frac{1}{\langle \xi - \eta \rangle^{s_1 + n/2 + \delta} \langle \eta \rangle^{s_2 + n/2 + \delta}} d\eta \; . \end{aligned}$$

Now if $| \eta | \leq 1/2 | \xi |$, then $1/2 | \xi | \leq | \xi - \eta | \leq 3/2 | \xi |$. Hence

$$\begin{aligned} \chi_K(\xi) w_1 \hat{w}_2(\xi) &\geq C \int_{|\eta_2| \leq \epsilon |\eta_1|, \, |\eta| \leq 1/2|\xi|} \frac{1}{\langle \xi \rangle^{s_1 + n/2 + \delta} \langle \xi \rangle^{s_2 + n/2 + \delta}} d\eta \\ &\geq C \frac{\langle \xi \rangle^n}{\langle \xi \rangle^{s_1 + s_2 + n + 2\delta}} = C \langle \xi \rangle^{-s_1 - s_2 - 2\delta} \; . \end{aligned}$$

Therefore, $\chi_K(D) w_1 w_2$ is not in any space H^t for $t > s_1 + s_2 - n/2$.

For the sake of completeness, we end up this section with Hörmander's theorem on propagation of singularities for operators with smooth coefficients. The proof may be found in Hörmander [14] or Nirenberg [24]; the statement is taken from Taylor [39].

Let $p(x, \xi) \in S^m_{1,0}$ have a scalar principal symbol $p_m(x, \xi)$. Then the bicharacteristic strips of p_m are defined by the Hamiltonian system:

$$\frac{dx}{ds} = \nabla_\xi p_m(x, \xi) \; , \quad \frac{d\xi}{ds} = -\nabla_x p_m(x, \xi) \; .$$

The null bicharacteristic through (x_0, ξ_0) for $p_m(x_0, \xi_0)$ is the curve defined by the system above associated with the initial condition,

$$x(0) = x_0 \; , \quad \xi(0) = \xi_0 \; ,$$

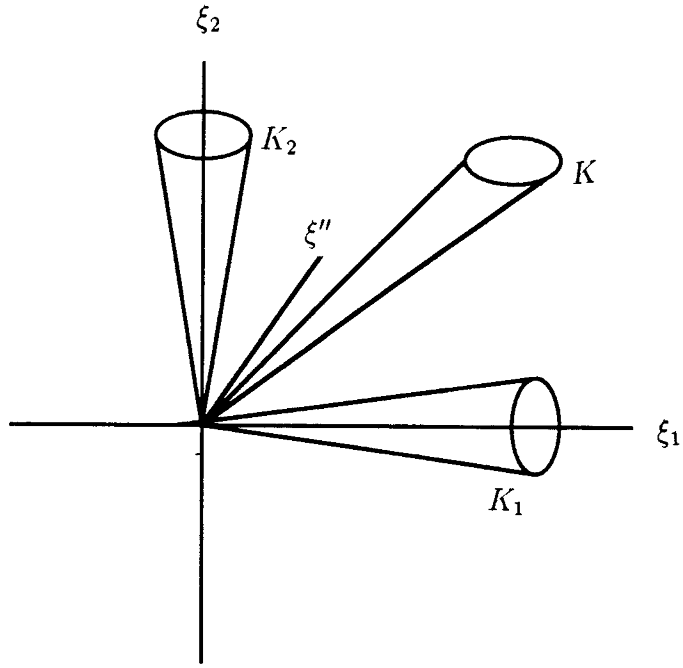

Figure 2.1 A counterexample to Proposition 2.6.

and $p_m(x,\xi) = 0$.

Theorem 2.1 (Hörmander) Let $p(x,D) \in OPS^m$ have a scalar principal symbol. Let $p_m(x_0,\xi_0) = 0$, let Γ be a null bicharacteristic through (x_0,ξ_0) for Rep_m, and assume $Imp_m \geq 0$ on a neighborhood of Γ. Suppose

$$P(x,D)u(x) = f(x) \in H^s_{m\ell}(\Gamma).$$

Then $u \in H^{s+m-1}_{m\ell}(\Gamma)$ if $u \in H^{s+m-1}_{m\ell}(x_0,\xi_0)$.

2.4 Commutator lemma

Having introduced the basic concepts of microlocal Sobolev spaces, we now present a commutator lemma which is necessary in order to prove any results on propagation of singularities for a *p.d.e.* with nonsmooth coefficients. As usual, a commutator $[A,B]$ represents $AB - BA$. Then the calculus of $\psi.d.o.$ indicates that if $p(x,D) \in OPS^{m_1}$ and $q(x,D) \in OPS^{m_2}$ then $[p(x,D), q(x,D)] \in OPS^{m_1+m_2-1}$. However, the situation becomes more complicated when one of the operators has nonsmooth coefficients.

Lemma 2.3 Let $p_1(x,\xi) \in S^1$ and $b_0(x,\xi) \in S^0$ be properly supported, and assume that for some $(x_0,\xi_0) \in T^*(\mathbb{R}^n)\backslash 0$, $a(x) \in H^s \cap H^r_{ml}(x_0,\xi_0)$ and $v(x) \in H^l \cap H^q_{ml}(x_0,\xi_0)$, with $1 + n/2 < s$, $0 \leq l \leq s$, $q \leq r$, and $q \leq l + s - (1 + n/2)$. Then

$$[b_0(x,D), a(x)p_1(x,D)]v(x) \in H^l \cap H^q_{ml}(x_0,\xi_0)\ .$$

We shall prove this lemma by the following two steps. Proposition 2.7 offers the local version of the commutator action, while the microlocal version is given as Proposition 2.8.

Proposition 2.7 Let $p_1(x,\xi) \in S^1$, $b_0(x,\xi) \in S^0$ be properly supported, let $1+n/2 < s$, and assume that $a(x) \in H^s$ and $v(x) \in H^l$ with $0 \le l \le s$. Then

$$[b_0(x,D), a(x)p_1(x,D)]v(x) \in H^l \ .$$

Proof Assume that b_0, p_1 depend on ξ only, and v, a are compactly supported (the general case requires some obvious modifications). We only consider $l \ge 1$ case. A slightly different analysis will lead to the conclusion for $0 \le l < 1$.

$$\begin{aligned}
&\widehat{[b_0(D), a(x)p_1(D)]v}(\xi) \\
&= b_0(\xi)\int \hat{a}(\eta)p_1(\xi-\eta)\hat{v}(\xi-\eta)d\eta - \int \hat{a}(\eta)p_1(\xi-\eta)b_0(\xi-\eta)\hat{v}(\xi-\eta)d\eta \\
&= \int \hat{a}(\eta)(b_0(\xi)-b_0(\xi-\eta))p_1(\xi-\eta)\hat{v}(\xi-\eta)d\eta \ .
\end{aligned}$$

Write $\hat{a}(\eta) = f(\eta)/\langle\eta\rangle^s$, $\hat{v}(\xi-\eta) = g(\xi-\eta)/\langle\xi-\eta\rangle^l$, then $f, g \in L^2$. Thus

$$\langle\xi\rangle^l\,\widehat{[b_0(D), a(x)p_1(D)]v}(\xi) = \int K(\xi,\eta)f(\eta)g(\xi-\eta)d\eta \ ,$$

where

$$K(\xi,\eta) \le \frac{C\langle\xi\rangle^l|b_0(\xi)-b_0(\xi-\eta)|}{\langle\eta\rangle^s\langle\xi-\eta\rangle^{l-1}} \ .$$

By Proposition 2.2, it suffices to divide K into finitely many pieces so that

$$\sup_\xi \int |K_i(\xi,\eta)|^2 d\eta < \infty \ , \quad \text{or} \quad \sup_\eta \int |K_i(\xi,\eta)|^2 d\xi < \infty \ .$$

But this is obvious from the given regularity assumptions and Proposition 2.3, together with the following facts:

(1) For $|\eta| > |\xi|/2$ and $|\xi - \eta| > |\xi|/2$,

$$|K| \leq \frac{C}{\langle \xi \rangle^{s-1}} .$$

(2) For $|\eta| > |\xi|/2$ and $|\xi - \eta| < |\xi|/2$,

$$|K| \leq \frac{C\langle \xi \rangle^{l}}{\langle \xi \rangle^{s} \langle \xi - \eta \rangle^{l-1}} \leq \frac{C}{\langle \xi - \eta \rangle^{s-1}} .$$

(3) For $|\eta| < |\xi|/2$, $\xi - \eta \approx \xi$, thus $|b_0(\xi) - b_0(\xi - \eta)| \leq C\langle \eta \rangle / \langle \xi \rangle$, and consequently,

$$|K| \leq \frac{C\langle \xi \rangle^{l} \langle \eta \rangle}{\langle \eta \rangle^{s} \langle \xi \rangle^{l-1} \langle \xi \rangle} = \frac{C}{\langle \eta \rangle^{s-1}} .$$

□

Proposition 2.8 Let $p_1(x,\xi) \in S^1$, $b_0(x,\xi) \in S^0$ be properly supported, and assume that for some $(x_0,\xi_0) \in T^*(\mathbb{R}^n)\backslash 0$, $a(x) \in H^s \cap H^q_{m\ell}(x_0,\xi_0)$ and $v(x) \in H^l \cap H^q_{m\ell}(x_0,\xi_0)$, with $1 + n/2 < s$, $0 \leq l \leq s$, q and $q < l + s - (1 + n/2)$. Then

$$[b_0(x,D), a(x)p_1(x,D)]v(x) \in H^q_{m\ell}(x_0,\xi_0) . \tag{2.5}$$

Proof As before, after making some simplifications, we have

$$\widehat{[b_0(D), a(x)p_1(D)]v}(\xi) = \int \hat{a}(\eta)(b_0(\xi) - b_0(\xi - \eta))p_1(\xi - \eta)\hat{v}(\xi - \eta)d\eta .$$

Let K be a small conic neighborhood of ξ_0 such that $a \in H^s \cap H^q_{m\ell}(K)$ and $v \in H^l \cap H^q_{m\ell}(K)$. Let K' be a strictly smaller conic neighborhood of ξ_0; then in order to prove (2.5) it suffices to show that

$$\chi_{K'}(\xi)\langle\xi\rangle^q \, \widehat{[b_0(D), a(x)p_1(D)]v}(\xi) \in L^2 \, . \tag{2.6}$$

Write

$$\hat{a}(\eta) = \frac{\chi_K(\eta)a_1(\eta)}{\langle\eta\rangle^r} + \frac{\chi_{K^c}(\eta)a_2(\eta)}{\langle\eta\rangle^s} \, , \tag{2.7}$$

$$\hat{v}\langle\xi-\eta\rangle = \frac{\chi_K(\xi-\eta)v_1(\xi-\eta)}{\langle\xi-\eta\rangle^q} + \frac{\chi_{K^c}(\xi-\eta)v_2(\xi-\eta)}{\langle\xi-\eta\rangle^l} \, , \tag{2.8}$$

where $a_i, v_i \in L^2$ $(i = 1, 2)$.

We only prove (2.5) for the case where $q > s$, since if $q \leq s$ then (2.7) will become $\hat{a}(\eta) = \dfrac{a_1(\eta)}{\langle\eta\rangle^s}$, and the same analysis will go through with much simpler arguments.

Substituting (2.7) and (2.8) into (2.6), we then have

$$\chi_{K'}\langle\xi\rangle^q \, \widehat{[b_0, aP_1]v}(\xi) = \sum_{i,j=1}^{2} \int K_{ij}(\xi,\eta)a_i(\xi)v_j(\xi-\eta)d\eta \, ,$$

where

(1) $K_{11}(\xi,\eta) = \dfrac{\chi_{K'}(\xi)\langle\xi\rangle^q\chi_K(\eta)\chi_K(\xi-\eta)(b_0(\xi)-b_0(\xi-\eta))\langle\xi-\eta\rangle}{\langle\eta\rangle^q\langle\xi-\eta\rangle^q}$,

(2) $K_{12}(\xi,\eta) = \dfrac{\chi_{K'}(\xi)\langle\xi\rangle^q\chi_K(\eta)\chi_{K^c}(\xi-\eta)(b_0(\xi)-b_0(\xi-\eta))\langle\xi-\eta\rangle}{\langle\eta\rangle^q\langle\xi-\eta\rangle^l}$,

(3) $K_{21}(\xi,\eta) = \dfrac{\chi_{K'}(\xi)\langle\xi\rangle^q\chi_{K^c}(\eta)\chi_K(\xi-\eta)(b_0(\xi)-b_0(\xi-\eta))\langle\xi-\eta\rangle}{\langle\eta\rangle^s\langle\xi-\eta\rangle^q}$,

(4) $K_{22}(\xi,\eta) = \dfrac{\chi_{K'}(\xi)\langle\xi\rangle^q\chi_{K^c}(\eta)\chi_{K^c}(\xi-\eta)(b_0(\xi)-b_0(\xi-\eta))\langle\xi-\eta\rangle}{\langle\eta\rangle^s\langle\xi-\eta\rangle^l}$.

It follows from Proposition 2.2 (essentially Cauchy-Schwarz inequality) that the corresponding estimates of the kernels will complete the proof:

(1) K_{11} will be handled exactly as in Proposition 2.7, knowing the fact that $1 + n/2 < q$.

(2) Since $q > s$, the hypothesis implies that $l - 1 > n/2$. On supp K_{12}, $\xi - \eta \in K^c$, $\xi \in K' \Rightarrow \langle\eta\rangle \geq C\langle\xi\rangle$; hence

$$\text{if } |\xi - \eta| \geq C|\xi|\,, \quad \text{then } |K_{12}| \leq \frac{C}{\langle\xi\rangle^{l-1}}\,,$$
$$\text{if } |\xi| \geq C|\xi - \eta|\,, \quad \text{then } |K_{12}| \leq \frac{C}{\langle\xi - \eta\rangle^{l-1}}\,.$$

(3) On supp K_{21}, $\eta \in K^c$, $\xi \in K' \Rightarrow \langle\xi - \eta\rangle \geq C\langle\xi\rangle$. Now

$$\text{if } |\eta| \geq |\xi|/2\,, \quad \text{then } |K_{21}| \leq \frac{C}{\langle\xi - \eta\rangle^{s-1}}\,, \quad s - 1 > n/2\,;$$
$$\text{if } |\eta| < |\xi|/2 \text{ then } \xi - \eta \approx \xi\,, \quad \text{thus } |b_0(\xi) - b_0(\xi - \eta)| \leq C\langle\eta\rangle/\langle\xi\rangle\,,$$

and it follows that $K_{21} \leq \dfrac{C\langle\xi\rangle^q\langle\eta\rangle\langle\xi - \eta\rangle}{\langle\eta\rangle^s\langle\xi - \eta\rangle^q\langle\xi\rangle} \leq \dfrac{C}{\langle\eta\rangle^{s-1}}$ for $s - 1 > n/2$.

(4) On supp K_{22}, $\eta \in K^c$, $\xi - \eta \in K^c$, $\xi \in K' \Rightarrow$ Proposition 2.6 can be applied to treat this term since $q \leq l + s - (1 + n/2)$.

□

2.5 Propagation of singularities theorem

We are now ready for a formal statement and a proof of the main result of this chapter: a linear propagation of singularities theorem for ψ.*d.o.* equations with nonsmooth coefficients at lower order terms.

Theorem 2.2 Let $p_m(x, D)$ be a strictly hyperbolic homogeneous ψ.*d.o.* of degree $m \geq 2$, $p_\alpha(x,\xi) \in S^{m-1}$, and $p_\beta(x,\xi) \in S^{m-2}$. Let Γ be a null bicharacteristic of p_m passing through $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$. Assume that

(i) $1 + n/2 < s$, $0 \leq l \leq s$, q, and $q < l + s - (1 + n/2)$;

(ii) $a_\alpha \in H^s \cap H^q_{m\ell}(\Gamma)$ and $a_\beta \in H^{s-1} \cap H^{q-1}_{m\ell}(\Gamma)$;

(iii) $v \in H^{l+m-2} \cap H^{q+m-2}_{m\ell}(\Gamma)$ and $f \in H^l \cap H^q_{m\ell}(\Gamma)$;

(iv) $v \in H^{q+m-2+\epsilon}_{m\ell}(x_0, \xi_0)$, for some $0 \leq \epsilon \leq 1$,

and that

$$[p_m(x, D) + \Sigma a_\alpha(x) p_\alpha(x, D) + \Sigma a_\beta(x) p_\beta(x, D)] v(x) = f(x) \ .$$

Then

$$v \in H^{q+m-2+\epsilon}_{m\ell}(\Gamma) \ .$$

Proof We shall basically follow the outline of the proof of the Beals-Reed theorem in [5], indicating the necessary modifications. Denote $x = (t, \bar{x})$, where t is a distinguished variable. W.L.O.G., we assume that p_m is homogeneous strictly hyperbolic with respect to the direction $(1,0)$ or

$$p_m(t,\bar{x},\tau,\bar{\xi}) = (\tau - k_1(t,\bar{x},\bar{\xi}))\cdots(\tau - k_m(t,\bar{x},\bar{\xi}))\ ,$$

with $k_i \in S^1(\mathbb{R}^{n-1}\backslash 0)$ real, homogeneous of degree one in $\bar{\xi}$, and distinct for $\xi \neq 0$.

Clearly, Lemma 2.2 reduces the equation to

$$[p_m + \Sigma a_\alpha p_\alpha]v \in H^l \cap H^q_{ml}(K)\ ,$$

where K is a conic nbhd of Γ and $\Gamma \subset\subset K$.

Let $b_0(x,\xi) \in S^0$ be chosen exactly as in Beals and Reed [5], so that supp $b_o \subset\subset K$, b_0 is elliptic near Γ, $[b_0, p_m]$ has order $m-2$, and $p_m b_0 = (D_t - k_1(t,\bar{x},D_{\bar{x}})q_{m-1}(x,D)b_0$ (mod C^∞) with $q_{m-1} \in S^{m-1}$ elliptic near Γ. Let

$$\Lambda = \langle D\rangle = (1 - (\frac{\partial^2}{\partial t^2} + \sum \frac{\partial^2}{\partial \bar{x}_i^2}))^{1/2}.$$

Then

$$\begin{aligned} b_0 a_\alpha p_\alpha v &= a_\alpha p_\alpha b_0 v + [b_0, a_\alpha p_\alpha]v\ , \\ [b_0, a_\alpha p_\alpha]v &= [b_0, a_\alpha\Lambda]\Lambda^{-1}p_\alpha v + a_\alpha\Lambda[b_0,\Lambda^{-1}p_\alpha]v\ . \end{aligned}$$

Observe that since $\Lambda^{-1}p_\alpha$ has order $m-2$, Lemma 2.3 leads to $[b_0, a_\alpha\Lambda]\Lambda^{-1}p_\alpha v \in H^l \cap H^q_{m\ell}(K)$. Since $\Lambda[b_0,\Lambda^{-1}p_\alpha]$ has order $m-2$, the last term will be in $H^l \cap H^q_{m\ell}(K)$ by Lemma 2.2. We conclude that

$$(p_m b_0 + \Sigma a_\alpha p_\alpha b_0)v \in H^l \cap H^q_{m\ell}(K)\ .$$

Let $\tilde{p}_\alpha = p_\alpha q_{-(m-1)}$ where $q_{-(m-1)}$ is the parametrix of q_{m-1} so that $q_{-(m-1)}q_{m-1} = 1$ on supp b_0. Set $w = q_{m-1}b_0 v$. We then have

$$(D_t - k_1(t, \bar{x}, D_{\bar{x}}))w + \Sigma a_\alpha \tilde{p}_\alpha w \in H^q_{m\ell}(K) \; ,$$

where $w \in H^{q-1}$ and $WF(w) \subset\subset K$. Therefore, we have obtained a well behaved hyperbolic Cauchy problem for w (assumption (iv) implies that the Cauchy data of $w \in H^{q-1+\epsilon}$ for t near 0). Our next proposition, which is an extension of the one in [5], yields that $w \in H^{q-1+\epsilon}$. Thus, $v \in H^{q+m-2+\epsilon}(\Gamma)$ will follow as a consequence of the fact $q_{m-1}b_0$ is microlocally elliptic of order $m-1$ near Γ. □

Proposition 2.9 Let $k_1(t, \bar{x}, \bar{\xi}) \in S^1(\mathbb{R}^{n-1})$, k_1 be real, and $p_0(x, \xi) \in S^0(\mathbb{R}^n)$. Assume that

(i) $n/2 < s$, $0 \leq q$;

(ii) $w \in H^{q-1}$ and $WF(w) \subset\subset K$;

(iii) $a \in H^s \cap H^q_{m\ell}(K)$;

(iv) $g \in H^{q-1+\epsilon}_{ml}(K)$, for some $0 \leq \epsilon \leq 1$;

(v) $w \in H^{q-1+\epsilon}$ for t near 0,

and that

$$(D_t - k_1(t, \bar{x}, D_{\bar{x}}))w = a(x)p_0(x, D)w + g(x) \; .$$

Then

$$w \in H^{q-1+\epsilon} \; .$$

An immediate consequence of Theorem 2.2 is a theorem on propagation of singularities due to Beals and Reed, Theorem 1 in [5].

2.6 Remark on Theorem 2.2

Notice that the Beals-Reed theorem as well as Rauch's Lemma are designed for the study of nonlinear propagation of singularities. In that case the coefficients or the right-hand side, roughly speaking, have same (or closely related) regularity as the solution to the problem or the transformed problem. Theorem 2.2 deals with much more general situations, since l could vary from 0 to s and $q \in [l, l+s-(1+n/2))$ (unlike in the Beals-Reed theorem).

We conjecture that Theorem 2.2 cannot be improved much concerning the regularity requirements for the coefficients and right-hand side, since the conclusions of Lemma 2.2 and Lemma 2.3 cannot be strengthened.

The most precise information about the propagation of singularities may be obtained in the case of one space dimension. Roughly speaking, the improved microlocal regularity is then propagated along null bicharacteristics with very few restriction on the order of smoothness. This certainly is not implied by Theorem 2.2. Note that the result itself is not too surprising if one observes that the one dimensional wave operator can be factored into products of differential operators. But it exhibits a substantial difference between the one dimension and multidimension for hyperbolic *p.d.e.*, is somehow remarkable.

2.7 Analysis on family of distributions

Since the coefficient $\sigma = \sigma(x)$ in the model equation is time-independent, one wants to obtain some better regularity results by taking advantage of this fact.

Once again, we begin with examining the algebraic properties of this class of distributions.

Lemma 2.4 (Generalized Schauder's Lemma) If $s_1, s_2 \geq 0$, $u(x) \in H^{s_1}(\mathbb{R}^{n_0})$, $v(x,y) \in H^{s_2}(\mathbb{R}^n)$, and $1 \leq n_0 \leq n$. Then

$$u(x)v(x,y) \in H^s(\mathbb{R}^n) \,,$$

with $s = min\{s_1, s_2, s_1 + s_2 - n_0/2 - \delta\}$, for any $\delta > 0$.

Proof Write

$$\hat{u}(\xi) = \langle\xi\rangle^{-s_1} f(\xi) \,, \quad \hat{v}(\xi,\eta) = \langle\xi,\eta\rangle^{-s_2} g(\xi,\eta)$$

with $f \in L^2(\mathbb{R}^{n_0})$, $g \in L^2(\mathbb{R}^n)$.

Then

$$\begin{aligned} \widehat{uv}(\xi,\eta) &= \hat{u}(\xi) * \hat{v}(\xi,\eta) \\ &= \int \langle\xi-\xi_1\rangle^{-s_1} f(\xi-\xi_1)\langle\xi_1,\eta\rangle^{-s_2} g(\xi_1,\eta) d\xi_1 \,. \end{aligned}$$

Since $|\xi - \xi_1| + |(\xi_1,\eta)| \geq |(\xi,\eta)|$, we have either $\xi_1 \in I_1$ so that $|\xi - \xi_1| \geq |(\xi,\eta)|/2$ or $\xi_1 \in I_2$ so that $|(\xi_1,\eta)| \geq |(\xi,\eta)|/2$. Thus

$\langle\xi,\eta\rangle^s |\widehat{uv}(\xi,\eta)|$

$$\leq \int_{I_1} \langle\xi_1,\eta\rangle^{-s_2} \langle\xi-\xi_1\rangle^{s-s_1} \quad | \quad f(\xi-\xi_1) g(\xi_1,\eta) \mid d\xi_1$$

$$+ \int_{I_2} \langle \xi - \xi_1 \rangle^{-s_1} \langle \xi_1, \eta \rangle^{s-s_2} \mid f(\xi - \xi_1) g(\xi_1, \eta) \mid d\xi_1 ,$$

or

$$\| \langle \xi, \eta \rangle^s | \widehat{uv}(\xi, \eta) | \|_{L^2(\xi,\eta)} \le max\{S_1, S_2\} ,$$

where

$$\begin{aligned} S_1 &\le C \| \int \langle \xi_1, \eta \rangle^{s-s_1-s_2} \mid f(\xi - \xi_1) g(\xi_1, \eta) \mid d\xi_1 \|_{L^2(\xi,\eta)} \\ &\le C \int d\eta \| \langle \xi_1, \eta \rangle^{s-s_1-s_2} \mid g(\xi_1, \eta) \mid \|_{L^1(\xi_1)} \| f \|_{L^2} \\ &\le C \| g \|_{L^2(\xi,\eta)} \| f \|_{L^2} , \end{aligned}$$

and

$$\begin{aligned} S_2 &= C \| \int \langle \xi - \xi_1 \rangle^{s-s_1-s_2} \mid f(\xi - \xi_1) g(\xi_1, \eta) \mid d\xi_1 \|_{L^2(\xi,\eta)} \\ &\le C \int d\eta \| \langle \xi \rangle^{s-s_1-s_2} \mid f(\xi) \mid \|_{L^1(\xi)} \cdot \| g(\xi_1, \eta) \|_{L^2(\xi_1)} \\ &\le C \| \langle \xi \rangle^{s-s_1-s_2} \|_{L^2(\xi)} \| f \|_{L^2(\xi)} \| g \|_{L^2(\xi,\eta)} \le C \| g \|_{L^2(\xi,\eta)} \| f \|_{L^2} . \end{aligned}$$

Note that here we have used Young's inequality and the assumption

$$s \le s_1 + s_2 - n_0/2 - \delta$$

so that the following inequalities

$$\| \langle \xi_1, \eta \rangle^{s-s_1-s_2} \|_{L^2(\xi_1)} \le \| \langle \xi_1 \rangle^{s-s_1-s_2} \|_{L^2(\mathbb{R}^{n_0})} < +\infty ,$$

and

$$\| \langle \xi \rangle^{s-s_1-s_2} \|_{L^2(\mathbb{R}^{n_0})} < +\infty$$

hold. □

We can also develop a corresponding generalized Rauch's type lemma that is similar to Lemma 2.2.

Lemma 2.5 Suppose that for some $(x_0, \xi_0) \in T^*(\mathbb{R}^{n_0})\backslash 0$ where $(x_0, y_0, \xi_0, \eta_0) \in T^*(\mathbb{R}^n)\backslash 0$ $(1 \leq n_0 \leq n)$, the distributions u, v satisfy $u(x) \in H^s \cap H^q_{m\ell}(x_0, \xi_0)$ and $v(x,y) \in H^l \cap H^q_{m\ell}(x_0, y_0, \xi_0, \eta_0)$, with $n_0/2 < s$, $0 \leq l \leq s$, q, and $q < l + s - n_0/2$. Then

$$u(x)v(x,y) \in H^l \cap H^q_{m\ell}(x_0, y_0, \xi_0, \eta_0) \ .$$

Proof The fact that $uv \in H^l_{loc}$ comes from Lemma 2.4. W.L.O.G., we may assume that u, v have compact supports in their own spaces. Moreover, we only prove the lemma for the case $q > s$; a natural modification of the proof will yield the conclusion for $q \leq s$. Let K be a conic neighborhood of (ξ_0, η_0) which is small enough so that $v \in H^l \cap H^q_{m\ell}(K)$ and $u \in H^s \cap H^q_{m\ell}(\Gamma)$, where Γ is the projection of Γ on the ξ-space. Let $K' \subset\subset K$, a strictly smaller conic neighborhood of (ξ_0, η_0). Γ' is the projection of K'. It suffices to show that

$$\widehat{uv}(\xi,\eta)\chi_{K'}(\xi,\eta)\langle \xi,\eta \rangle^q \in L^2(\mathbb{R}^n) \ .$$

Write $u = u_1 + u_2$, $v = v_1 + v_2$ such that

$$u_1 \in H^s \ , \quad u_2 \in H^q \text{ ,and } \ \text{supp } \hat{u}_1 \subseteq \Gamma^c \ , \quad \text{supp } \hat{u}_2 \subseteq \Gamma \ ,$$

$$v_1 \in H^l \ , \quad v_2 \in H^q \text{ ,and } \ \text{supp } \hat{v}_1 \subseteq K^c \ , \quad \text{supp } \hat{v}_2 \subseteq K \ .$$

Then

$$uv = u_1v_1 + u_1v_2 + u_2v_1 + u_2v_2 \ .$$

Let

$$\langle\xi\rangle^s\hat{u}_1(\xi) = f_1(\xi) \ , \qquad \langle\xi\rangle^q\hat{u}_2(\xi) = f_2(\xi) \ ,$$

$$\langle\xi,\eta\rangle^l\hat{v}_1(\xi,\eta) = g_1(\xi,\eta) \ , \qquad \langle\xi,\eta\rangle^q\hat{v}_2(\xi,\eta) = g_2(\xi,\eta) \ ,$$

then $f_i \in L^2(\mathbb{R}^{n_0})$ and $g_i \in L^2(\mathbb{R}^n)$, $i = 1, 2$. Thus we may decompose

$$\langle\xi,\eta\rangle^q\chi_{K'}(\xi,\eta)\widehat{uv}(\xi,\eta) = I_1 + I_2 + I_3 + I_4 \ ,$$

where

$$\begin{aligned}
I_1 &= \langle\xi,\eta\rangle^q \int \frac{\chi_{K'}(\xi,\eta)\chi_\Gamma(\xi_1)\chi_K(\xi-\xi_1,\eta)f_2(\xi_1)g_2(\xi-\xi_1,\eta)d\xi_1}{\langle\xi_1\rangle^q\langle\xi-\xi_1,\eta\rangle^q} \ , \\
I_2 &= \langle\xi,\eta\rangle^q \int \frac{\chi_{K'}(\xi,\eta)\chi_\Gamma(\xi_1)\chi_{K^c}(\xi-\xi_1,\eta)f_2(\xi_1)g_1(\xi-\xi_1,\eta)d\xi_1}{\langle\xi_1\rangle^q\langle\xi-\xi_1,\eta\rangle^l} \ , \\
I_3 &= \langle\xi,\eta\rangle^q \int \frac{\chi_{K'}(\xi,\eta)\chi_{\Gamma^c}(\xi_1)\chi_K(\xi-\xi_1,\eta)f_1(\xi_1)g_2(\xi-\xi_1,\eta)d\xi_1}{\langle\xi_1\rangle^s\langle\xi-\xi_1,\eta\rangle^q} \ , \\
I_4 &= \langle\xi,\eta\rangle^q \int \frac{\chi_{K'}(\xi,\eta)\chi_{\Gamma^c}(\xi_1)\chi_{K^c}(\xi-\xi_1,\eta)f_1(\xi_1)g_1(\xi-\xi_1,\eta)d\xi_1}{\langle\xi_1\rangle^s\langle\xi-\xi_1,\eta\rangle^l} \ .
\end{aligned}$$

Therefore, to accomplish the proof, we only need to show that $I_i \in L^2(\mathbb{R}^n)$ ($i = 1, \cdots, 4$).

(1) The fact $I_1 \in L^2(\xi,\eta)$ comes from Lemma 2.4.

(2) On support I_2, $(\xi-\xi_1,\eta) \in K^c$ and $(\xi,\eta) \in K' \Rightarrow \langle\xi_1, 0\rangle \geq C\langle\xi,\eta\rangle$. Hence

$$| I_2 | \leq \int \frac{| f_2(\xi_1)g_1(\xi-\xi_1,\eta) | d\xi_1}{\langle\xi-\xi_1,\eta\rangle^l} \ ,$$

therefore an extension of Corollary 2.1 and the hypotheses will yield

$$\|I_2\|_{L^2(\mathbb{R}^n)} \leq C\|g_2\|_{L^2}\|f_2\|_{L^2} \ .$$

(3) On support I_3, $\xi_1 \in \Gamma^c$, $(\xi, \eta) \in K' \Rightarrow \langle \xi - \xi_1, \eta \rangle \geq C\langle \xi, \eta \rangle$, Proposition 2.1 gives

$$\|I_3\|_{L^2(\xi)} \leq C \left\| \frac{| f_1(\xi) |}{\langle \xi \rangle^s} \right\|_{L^1(\xi)} \|g_2\|_{L^2(\mathbb{R}^n)} \ .$$

Thus, one gets from the hypothesis $s > n_0/2$ that

$$\|I_3\|_{L^2(\mathbb{R}^n)} \leq C\|f_1\|_{L^2(\mathbb{R}^{n_0})}\|g_2\|_{L^2(\mathbb{R}^n)} \ .$$

(4) On support I_4, $\xi_1 \in \Gamma^c$, $(\xi - \xi_1, \eta) \in K^c, (\xi, \eta) \in K$. Then, since $q \leq l + s - n_0/2$, one may apply an extension form of Proposition 2.6 to obtain

$$\|I_4\|_{L^2(\mathbb{R}^n)} \leq C\|f_1\|_{L^2}\|g_1\|_{L^2}.$$

□

Having established the corresponding generalized Schauder's lemma and the Rauch type lemma, one can proceed further to prove an extension of our commutator lemma (Lemma 2.3). We skip the proof because it is parallel to the one in Section 2.4.

Lemma 2.6 (Generalized Commutator Lemma) Let $p_1(x, y, \xi, \eta) \in S^1(\mathbb{R}^n)$ and $b_0(x, y, \xi, \eta) \in S^0(\mathbb{R}^n)$, and assume that for some $(x_0, \xi_0) \in T^*(\mathbb{R}^{n_0})$, $(x_0, y_0, \xi_0, \eta_0) \in T^*(\mathbb{R}^n)$ $(1 \leq n_0 \leq n)$, $a(x) \in H^s \cap H^q_{m\ell}(x_0, \xi_0)$ and

$v(x,y) \in H^l \cap H^q_{m\ell}(x_0, y_0, \xi_0, \eta_0)$, with $1 + n_0/2 < s$, $0 \le l \le s$, q, and $q < l + s - (1 + n_0/2)$. Then

$$[b_0, ap_1]v \in H^l \cap H^q_{m\ell}(x_0, y_0, \xi_0, \eta_0) \ .$$

Consequently, essentially the same arguments as in the proof of Theorem 2.2 lead to a generalized form of Theorem 2.2. Again, we omit the proof and leave it to the interested reader.

Theorem 2.3 Let $p_m(x,D)$ be a strictly hyperbolic homogeneous *ψ.d.o.* of degree $m \ge 2$, $p_\alpha(x,\xi) \in S^{m-1}$, $p_\beta(x,\xi) \in S^{m-2}$, with $x = (x_1, x_2) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0}$, $1 \le n_0 \le n$. Let Γ be a null bicharacteristic of p_m passing through $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$. Denote $K = \Pi(\Gamma)$, $\Pi : T^*(\mathbb{R}^n) \to T^*(\mathbb{R}^{n_0})$ the projection map. Assume that

(i) $1 + n_0/2 < s$, $0 \le l \le s$, q, and $q < l + s - (1 + n_0/2)$;

(ii) $a_\alpha \in H^s \cap H^q_{m\ell}(K)$ and $a_\beta \in H^{s-1} \cap H^{q-1}_{m\ell}(K)$;

(iii) $v \in H^{l+m-2} \cap H^{q+m-2}_{m\ell}(\Gamma)$ and $f \in H^l \cap H^q_{m\ell}(\Gamma)$;

(iv) $v \in H^{q+m-2+\epsilon}_{m\ell}(x_0, \xi_0)$, for some $0 \le \epsilon \le 1$,

and that

$$[p_m(x,D) + \Sigma a_\alpha(x_1) p_\alpha(x,D) + \Sigma a_\beta(x_1) p_\beta(x,D)]v(x) = f(x) \ .$$

Then

$$v \in H^{q+m-2+\epsilon}_{m\ell}(\Gamma) \ .$$

An interesting special case of Theorem 2.3 arises: For the layered problem, the coefficients (density) depend only on one space variable, so that we have the following degenerate form of Theorem 2.3.

Corollary 2.2 Let $p_m(x, D)$ be a strictly hyperbolic homogeneous $\psi.d.o.$ of degree $m \geq 2$, $p_\alpha(x, \xi) \in S^{m-1}$ and $p_\beta(x, \xi) \in S^{m-2}$, with $x = (x_1, x_2) \in \mathbb{R}^1 \times \mathbb{R}^{n-1}$. Let Γ be a null bicharacteristic of p_m passing through $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$. Let $K = \Pi(\Gamma)$, $\Pi : T^*(\mathbb{R}^n) \rightarrow T^*(\mathbb{R}^1)$ the projection map. Assume that

(i) $3/2 < s$, $0 \leq l \leq s$, q, and $q < l + s - 3/2$;

(ii) $a_\alpha \in H_{loc}^{min\{s,\, q\}}$ and $a_\beta \in H_{loc}^{min\{s,\, q\}-1}$;

(iii) $v \in H^{l+m-2} \cap H_{m\ell}^{q+m-2}(\Gamma)$ and $f \in H^l \cap H_{m\ell}^q(\Gamma)$;

(iv) $v \in H_{m\ell}^{q+m-2+\epsilon}(x_0, \xi_0)$, for some $0 \leq \epsilon \leq 1$,

and that

$$[p_m(x, D) + \Sigma a_\alpha(x_1) p_\alpha(x, D) + \Sigma a_\beta(x_1) p_\beta(x, D)]v(x) = f(x) .$$

Then

$$v \in H_{m\ell}^{q+m-2+\epsilon}(\Gamma) .$$

2.8 Microlocal norm

In this section, we introduce a notation which will be used frequently in the following chapters for simplifying the estimates. Since the microlocal Sobolev spaces are the main spaces we deal with in this work, it will be convenient to introduce a microlocal norm to measure them.

Definition 2.2 Let P be a standard $\psi.d.o.$ of order zero, then for any $q, r \in \mathbb{R}$, define

$$\|w\|^P_{q,r,\Omega} \stackrel{def.}{=} \|w\|_{q,\Omega} + \|Pw\|_{r,\Omega}$$

where Ω is a compact set in $\mathbb{R}^n$, or $\Omega = \mathbb{R}^n$ if P has compact support in spatial variables.

Having Definition 2.2, it is a simple exercise to interpret Rauch's lemma in terms of an estimate.

Proposition 2.10 Suppose w_1 and $w_2 \in H^q \cap H^r_{m\ell}(\gamma)$, $\gamma \in T^*(\mathbb{R}^n)$, and $n/2 < q \leq r < 2q - n/2$. Then there exist $\psi.d.o.$ P_1, P_2 of order zero such that for $i = 1, 2$,

$$ES(P_i) \subseteq \text{a sufficiently small conic neighborhood of } \gamma \ ,$$

and their principal symbols

$$P_i^0 = 1 \ \text{ on } \ \gamma \cap \{(x,\xi) : |\xi| > 1\} \ .$$

Furthermore, for $\phi \in C_0^\infty(\mathbb{R}^n)$, and $supp(\phi) \subset \Omega$ a compact set in $\mathbb{R}^n$,

$$\|\phi w_1 w_2\|^{P_1}_{q,r,\Omega} \leq C \|\phi w_1\|^{P_2}_{q,r,\Omega} \|\phi w_2\|^{P_2}_{q,r,\Omega} \ .$$

Proof The conclusion is slightly stronger than the original Rauch's lemma, though the proof is analogous. Therefore, we shall skip the proof by making the following observation. It is obvious to see that the proof of our generalized Rauch's lemma (Lemma 2.2) can be utilized to prove Rauch's lemma. Furthermore, the proposition will follow immediately if one writes down the corresponding estimates. □

Chapter 3

Some Trace Regularity Theorems

3.1 Introduction

The trace properties of solutions to the hyperbolic problem are obviously essential in understanding the forward map as well as the inverse problem itself. Therefore, a precise trace theorem is always the first thing one wants to demonstrate in working on this sort of inverse problem.

The classical trace theorem in Sobolev spaces asserts that the restriction map of a distribution to a codimension one hypersurface extends uniquely to a continuous linear operator from $H^s(\mathbb{R}^k)$ to $H^{s-1/2}(\mathbb{R}^{k-1})$, if $s > 1/2$. It is also well known that this result is sharp, see Lions and Magenes [22] or Taylor [39] for details. However, dealing with the solutions to hyperbolic *p.d.e.*, one may reasonably expect an improvement of their trace regularity. This is actually the case if the equation with smooth coefficients is strictly hyperbolic with respect to a codimension one trace hypersurface, since then standard energy estimates will yield that the trace map is from $H^s(\mathbb{R}^k)$ to $H^s(\mathbb{R}^{k-1})$ locally for any real s. Unfortunately, the same idea will not work if the trace surface is timelike, essentially because the presence of grazing rays prohibit the direct application of energy estimates. See Symes [32], Bao and Symes [2] for more comments on this aspect.

In [32], Symes proved a trace theorem for the solution of a second order multi-dimensional wave equation with constant coefficients: For finite energy initial data compactly supported away from the boundary (with the absence of the grazing rays), the trace is of class H^1_{loc} which is as regular as the solution in the interior. Some similar trace regularity results were obtained by Lasiecka and Triggiani [21] for the solutions of second order hyperbolic mixed problems based on the application of the Laplace-Fourier transform.

Recently in [2], we proved a trace theorem for general linear *p.d.e.* with smooth variable coefficients, applying the full strength of the Hörmander-Nirenberg pseudodifferential cutoff technique and the method of energy estimates. Our theorem shows that the difficulty above may be resolved by imposing more smoothness against grazing ray directions.

Roughly speaking, the analysis in this chapter is similar to that of Bao and Symes [2]. Two major differences are in order:

- Since in this work our attention is restricted to the second order equation, compared to the general case in [2] a much simpler $\psi.d.o.$ cut-off of the operator becomes possible.

- Note that the model problem is an initial value problem; therefore both of the propagation of singularity theorems, both Theorem 2.1 (Hörmander's theorem) and Theorem 2.3, have to be involved in the analysis in contrast to [2] where no side condition was introduced and discussed explicitly.

The main ingredient of this chapter is a *fattening* lemma which provides a convenient way to determine the regularity of the action of a smooth family of $\psi.d.o.$ on a distribution under some appropriate hypotheses. This lemma plays a dominant role in deriving our trace regularity theorems.

3.2 Properties of $\psi.d.o.$-like operators

It is easy to observe that by the definition of $\psi.d.o.$ a smooth family of $\psi.d.o.$ $P(x,y,D_x) \in OPS^m(\mathbb{R}^{k_0})$, for each $y \in \mathbb{R}^{k-k_0}$ with $k_0 < k$, is not necessarily a $\psi.d.o.$ in $\mathbb{R}^k$. For convenience, in the future, we shall denote the smooth family of $\psi.d.o.$ as $P \in C^\infty(\mathbb{R}^{k-k_0}, OPS^m(\mathbb{R}^{k_0}))$. The results in this section will conclude that a smooth family of $\psi.d.o.$ in fact behaves like a $\psi.d.o.$, hence will be called a $\psi.d.o.$-like operator in the future.

This section is devoted to the understanding of these $\psi.d.o.$-like operators.

We begin with our Proposition 3.1 which guarantees that similar Sobolev space continuous properties still hold.

Proposition 3.1 If $p(x,\xi) \in S^m(\mathbb{R}^{k_0})$, $1 \le k_0 \le k$, satisfies one of the following assumptions:

(1) $p(x,\xi) = p(\xi)$; that is p is independent of x;

(2) $p(x,\xi)$ has compact support in $|x| \leq c$,

then

$$p(x, D_x) \, : \, H^s(\mathbb{R}^k) \to H^{s-m}(\mathbb{R}^k)$$

continuously.

Proof For simplicity, we only prove the second statement here. The first one follows from the fact $\mathcal{F}_x[P(D_x)u(x,y)] = P(\xi)\hat{u}(\xi,y)$. It suffices to assume the $m = 0$ case and derive the appropriate norm estimates. Let $u \in \mathcal{S}$, the *Schwarz* space and write $p(x,\xi) = \int \mathcal{F}_x p(\eta,\xi)e^{ix\eta}d\eta$, with $\mathcal{F}_x p(\eta,\xi) = \int p(x_1,\xi)e^{-ix_1\eta}dx_1$. Assumption (2) on $p(x,\xi)$ implies that $\mid \mathcal{F}_x p(\eta,\xi) \mid \leq C_N(1+ \mid \eta \mid^2)^{-N/2}$, $\forall N > 0$. Since the *Fourier* transform of $P(x, D_x)u(x,y)$ has the form

$$\mathcal{F}(P(x,D_x)u)(\eta,\zeta) = \int \mathcal{F}_x p(\eta-\xi,\xi)\hat{u}(\xi,\zeta)d\xi \ ,$$

we have

$$\mid \mathcal{F}(P(x,D_x)u)(\eta,\zeta) \mid \leq C_N \int (1+\mid \eta-\xi \mid^2)^{-N/2} \mid \hat{u}(\xi,\zeta) \mid d\xi \ .$$

Therefore,

$$\|P(x,D_x)u\|^2_{H^s}$$

$$\leq C \int \| \int (1+\mid \eta-\xi \mid^2)^{-N_1/2}(1+\mid \xi \mid^2 + \mid \zeta \mid^2)^{s/2} \mid \hat{u}(\xi,\zeta) \mid d\xi \, \|^2_{L^2(\eta)} d\zeta$$

where $N_1 = N - s$. For large N, Young's inequality yields that

$$\|P(x,D_x)u\|^2_{H^s} \leq C \int \| \, (1+\mid \xi \mid^2 + \mid \zeta \mid^2)^{s/2} \mid \hat{u}(\xi,\zeta) \mid \|^2_{L^2(\xi)} d\zeta = C \| \, u \, \|^2_{H^s}.$$

□

Note that the only thing preventing $P(x,y,D_x)$ from being a $\psi.d.o.$ of order m is that its symbol $p(x,y,\xi)$ does not decrease in any directions other than the ξ-direction. This implies that *via* a pseudodifferential cutoff along those nondecay directions P may be regularized to be a $\psi.d.o.$, which leads to our next proposition.

From now on, $\tilde{\Pi}_2 : \ X \in T^*(\mathbb{R}^k) \to Y \in \mathbb{R}^k \times \mathbb{R}^{k_0}$ serves as a map for $k > k_0$,

$$\tilde{\Pi}_2(X) = \{(x,y,\xi) \in Y : \ (x,y,\xi,\eta) \in X\} .$$

Recall that the normal bundle of a foliation $\mathbb{R}^k = \mathbb{R}^{k-k_0} \times \mathbb{R}^{k_0}$ is the set

$$\mathcal{N} = \{(x,y,\xi,\eta) \in \mathbb{R}^{k_0} \times \mathbb{R}^{k-k_0} \times \mathbb{R}^{k_0} \times \mathbb{R}^{k-k_0}, \xi = 0\} .$$

Proposition 3.2 Assume that $P(x,y,D_x) \in C^\infty(\mathbb{R}^{k-k_0}, OPS^m(\mathbb{R}^{k_0}))$, $H(x,y,D_x,D_y) \in OPS^{m_0}(\mathbb{R}^k)$, $1 \le k_0 \le k$, $m, m_0 \in R$ and $\phi(x,y) \in C_0^\infty(\mathbb{R}^k)$. Furthermore, assume that

$$ES(H) \cap \mathcal{N} = \emptyset ,$$

where $\mathcal{N}$ is the normal bundle of $\mathbb{R}^{k_0} \times \mathbb{R}^{k-k_0}$. Then

$$P\phi H \in OPS^{m+m_0}(\mathbb{R}^k)$$

and

$$ES(P\phi H) \subset \tilde{\Pi}_2^{-1} ES(P(\cdot,y,\cdot)) \cap ES(H) .$$

Proof W.L.O.G., it is sufficient to consider the case with $m = m_0 = 0$ (some simple modifications lead to the general case). Thus it suffices to show that

$$Q = P(x, y, D_x)\phi(x, y)H(D_x, D_y)$$

is a $\psi.d.o.$ (in $OPS^0(\mathbb{R}^k)$). Observe that

$Qu(x, y)$

$$= \int\int [\int P(x, y, \xi)\hat{\phi}(\xi - \xi', y)e^{ix(\xi-\xi')}d\xi]H(\xi', \eta')\hat{u}(\xi', \eta')e^{ix\xi'+iy\eta'}d\xi' d\eta' \ ;$$

hence, the symbol of Q

$$Q_0(x, y, \xi', \eta') = \int P(x, y, \xi + \xi')\hat{\phi}(\xi, y)e^{ix\xi}d\xi H(\xi', \eta') \ .$$

The definition of $\psi.d.o.$ gives that

$$\begin{aligned} | \partial_{\xi'}^{\alpha_1} P(x, y, \xi + \xi') | &\leq C_{\alpha,K}(1+ | \xi + \xi' |)^{-\alpha_1} \\ &\leq C(1+ | \xi' |)^{-\alpha_1}(1+ | \xi |)^{\alpha_1}, \ \forall \alpha_1 > 0 \ . \end{aligned}$$

Therefore, for any $(\xi', \eta') \in \mathbb{R}^k$ and any (x, y) that is contained in a compact set in $\mathbb{R}^k$,

$| \partial_{\xi'}^{\alpha} Q_0(x, y, \xi', \eta') |$

$$\begin{aligned} &\leq C | \sum_{0\leq\alpha_1\leq\alpha} [\partial_{\xi'}^{\alpha_1} \int P(x, y, \xi + \xi')e^{ix\xi}\hat{\phi}(\xi, y)d\xi] \ \partial_{\xi'}^{\alpha-\alpha_1} H(\xi', \eta') | \\ &\leq \sum_{0\leq\alpha_1\leq\alpha} C(1+ | \xi' |)^{-\alpha_1}\partial_{\xi'}^{\alpha-\alpha_1} H(\xi', \eta') \\ &\leq C_{\alpha,K}(1+ | \xi' | + | \eta' |)^{-\alpha}. \end{aligned}$$

The last inequality comes from our construction of H; that is, $H(\xi', \eta')$ is nonzero only in the region $(1+ \mid \xi' \mid + \mid \eta' \mid) \leq C(1+ \mid \xi' \mid)$. Thus Q is a $\psi.d.o.$ in $\mathbb{R}^k$.

The fact

$$ES(P\phi H) \subset \tilde{\Pi}_2^{-1} ES(P(\cdot, y, \cdot))$$

is a simple exercise of the definitions of essential support as well as the map $\tilde{\Pi}_2$. From the above expression of the symbol of Q, it is obvious to see that

$$ES(P\phi H) \subset ES(H) \ .$$

□

Remark. In the appendix of [38], Taylor studied some properties of $\psi.d.o.$-like operators through two lemmas (Lemma A.1, Lemma A.2). While in Lemma A.1, for a smooth family of $\psi.d.o.$ $P_1 \in C^\infty(\mathbb{R}^1, OPS^1(\mathbb{R}^{n-1}))$, he obtained essentially the first conclusion of Proposition 3.2. He then showed in Lemma A.2 that if $(\partial/\partial_y - P_1)u \in C^\infty$, then $WF(u) \cap \mathcal{N} = \emptyset$.

We shall make an extensive use of the $\psi.d.o.$ cut-off technique behind Proposition 3.2 and examine further properties of these $\psi.d.o.$-like operators.

Our next proposition provides us with an important property of microlocal ellipticity.

Proposition 3.3 Suppose $P \in OPS^m(\mathbb{R}^k)$ and $Q \in OPS^0(\mathbb{R}^k)$, $s \in \mathbb{R}$, $q(x, \xi)$ has compact support in x, and P is microlocally elliptic on a conic neighborhood of $ES(Q)$. Then

$$||Qu||_{s,\Omega} \leq C_1||PQu||_{s-m,\Omega} + C_2||u||_{r,\Omega}$$

holds for any real r and $u \in C_0^\infty(\mathbb{R}^n)$, Ω is a compact set in $\mathbb{R}^n$ with $supp(u) \subset \Omega$, and the constants depending on r, s, m, and $supp(u)$.

Proof Since P is elliptic on a neighborhood of $ES(Q)$, there is another $\psi.d.o.$ R of order m such that

- $P + R$ is elliptic;
- $ES(R) \cap ES(Q) = \emptyset$.

According to Gårding's inequality, we then have for any real r

$$\begin{aligned} ||Qu||_{s,\Omega} &\leq C_1||(P+R)Qu||_{s-m,\Omega} + C_2'||Qu||_{r,\Omega} \\ &\leq C_1||PQu||_{s-m,\Omega} + C_2||u||_{r,\Omega} , \end{aligned}$$

where to derive the second inequality, we have used the fact that RQ is a smoothing operator. □

We are now ready to prove a lemma based on Propositions 3.1-3.3 and a $\psi.d.o.$ fattening technique. The usefulness of this lemma will become clear in the proofs of the coming trace theorems.

Lemma 3.1 (Fattening Lemma) Let $B(x, y, D_x) \in C^\infty(\mathbb{R}^{k-k_0}, OPS^m(\mathbb{R}^{k_0})$ and $A(x, y, D_x, D_y) \in OPS^{m_0}(\mathbb{R}^k)$, where $1 \leq k_0 \leq k$. Let

$$\mathcal{N} = \{(x, \xi) \in \mathbb{R}^k \times \mathbb{R}^k \ , \ \ (\xi_1, \cdots, \xi_{k_0}) = 0\}$$

be the normal bundle of $\mathbb{R}^{k_0} \times \mathbb{R}^{k-k_0}$. Also, assume that

(1) A is microlocal elliptic on a conic set $Ell(A)$, with $\mathcal{N} \subset\subset Ell(A)$;

(2) $u \in H^h \cap H^{h+1}_{m\ell}([T^*(\mathbb{R}^k)\backslash Ell(A)] \cap \tilde{\Pi}_2^{-1} ES(B(\cdot, y, \cdot)))$;

(3) $A\phi u \in H^{h-m_0+1}_{\text{loc}}(\mathbb{R}^k)$, where $\phi(x) \in C_0^\infty(\mathbb{R}^k)$.

Then

$$B\phi u \in H^{h-m+1}_{\text{loc}}(\mathbb{R}^k) ,$$

in addition, if B is either a convolutional operator or its symbol has compact support in spatial variables,

$$B\phi u \in H^{h-m+1}(\mathbb{R}^k) .$$

Remark on the lemma. Since the operator A plays a very important role here, a natural question arises: Given operator B and u, how can one find the appropriate operator A? It is obvious that the right candidate should satisfy assumptions (1) and (3); that is, its properties depend strongly on the properties of B and u. Fortunately, this is not a problem because the wave operator always satisfies two essential requirements: The wave operator $\Box$ is microlocally elliptic away from its characteristic variety, therefore we know exactly where (1) holds. According to the theorems on propagation of singularities, the improved regularity is propagated along the null bicharacteristics (the Hamiltonian flow) of $\Box$, which gives a first hint to find where the assumption (2) is satisfied.

Proof We only prove the first part of the conclusion, the second part can be shown by a simple application of Proposition 3.1.

By assumptions (1) and (2), we can find a conic set Ell_1, such that $\mathcal{N} \subset Ell_1 \subset\subset Ell(A)$, and $u \in H^h \cap H^{h+1}_{m\ell}([T^*(\mathbb{R}^k)\backslash Ell_1] \cap \tilde{\Pi}_2^{-1} ES(B(\cdot, y, \cdot)))$.

One can also construct a $\psi.d.o.$ $H_1 \in OPS^0(\mathbb{R}^k)$ which satisfies

- $ES(H_1) \subseteq Ell(A)$ and
- the symbol of H_1, $h_1 = 1$ on $Ell_1 \cap \{(x, \xi) : |\xi| > 1\}$.

Write $\phi = \phi\phi_1$ with $\phi_1 \in C_0^\infty(\mathbb{R}^k)$; we then have

$$B\phi u = B\phi_1 H\phi u + B\phi_1 H_1 \phi u$$

with $H = I - H_1$.

Since

$$AH_1\phi u = [A, H_1]\phi u + H_1 A\phi u$$

and $[A, H_1]$ has order $m_0 - 1$, we have

$$AH_1\phi u \in H^{h-m_0+1}_{loc} ,$$

which follows by assumptions (1) & (2) and Proposition 3.1.

From assumption (1) and the fact

$$ES(H_1) \subseteq Ell(A) ,$$

where $Ell(A)$ is the microlocal elliptic region of A, we see from Proposition 3.3 that

$$H_1\phi u \in H^{h+1}_{loc} .$$

Thus Proposition 3.1 gives

$$B\phi_1 H_1 \phi u \in H_{loc}^{h-m+1}.$$

On the other hand, from the construction of H, Proposition 3.2 implies that

$$B\phi_1 H \in OPS^m(\mathbb{R}^k) .$$

Moreover,

$$ES(B\phi_1 H) \subseteq \tilde{\Pi}_2^{-1} ES(B(\cdot, y, \cdot)) \cap ES(H) \subseteq [T^*(\mathbb{R}^k) \backslash Ell_1] \cap \tilde{\Pi}_2^{-1} ES(B(\cdot, y, \cdot)) .$$

Thus, a simple property of wavefront set yields

$$B\phi_1 H \phi u \in H_{loc}^{h-m+1}.$$

Eventually, combining the above arguments, we have

$$B\phi u = B\phi_1 H \phi u + B\phi_1 H_1 \phi u \in H_{loc}^{h-m+1} ,$$

which finishes the proof. □

Furthermore, an estimate can be obtained by carrying out all the corresponding estimates in the above proof.

Corollary 3.1 Under the assumptions of Lemma 3.1, the following estimate holds:

$$||\phi_0 B \phi u||_{h-m+1} \leq C||\phi u||_h + C||\phi_0 A \phi u||_{h-m_0+1} + C||\phi_0 P \phi u||_{h+1} ,$$

where $\phi_0(x) \in C_0^\infty(\mathbb{R}^n)$, $P \in OPS^0$ and $ES(P) \subseteq$ a sufficiently small conic neighborhood of $\Gamma = [T^*(\mathbb{R}^k) \backslash Ell_1] \cap \tilde{\Pi}_2^{-1} ES(B(\cdot, y, \cdot))$ and $p = 1$ on $\Gamma \cap \{(x, \xi) : |\xi| > 1\}$.

3.3 Propagation of singularities

From now on, the space variable is always denoted as $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and the Fourier variables dual to t, x are ω, ξ respectively. We shall focus mainly on the multidimensional case, $n \geq 2$.

For convenience, we state and prove a simple result on propagation of singularities by following closely Nirenberg's construction in [24]. Although the lemma is stated for the wave operator $\Box$, which is what we need in this work, it is clear from the proof that the corresponding result for general operators can be established with no further difficulty.

Let

$$\Box(x, t, \xi, \omega) = (1/2)(\omega^2 - \xi^2) ,$$

the bicharacteristic strips of $\Box$ are defined by the Hamiltonian System

$$\dot{x} = -\xi , \qquad \dot{\xi} = 0 ,$$
$$\dot{t} = \omega , \qquad \dot{\omega} = 0 .$$

The null bicharacteristics of $\Box$ are those that satisfy $\omega^2 = |\xi|^2$. For example, one can easily write down the characteristic through the point $(x_0, 0, \xi_0, \omega_0)$ with $\omega_0^2 = |\xi_0|^2$ as

$$\{(x, t, \xi, \omega) : \ x = x_0 - (\xi_0/\omega_0)t, \ \xi = \xi_0, \text{ and } \omega = \omega_0\} .$$

Lemma 3.2 Given a conic set γ, there exists a $B \in OPS^0$ such that

(1) $[B, \Box] \in OPS^0$;

(2) B is elliptic on the null bicharacteristics (Hamiltonian flow) generated by the wave operator $\Box$ out of γ.

Proof According to Nirenberg's proof of the theorem of propagation of singularities, we can find a $\psi.d.o.$ A of order zero for every null bicharacteristic of $\Box$ out of γ such that A is elliptic on a small conic neighborhood of the bicharacteristic and $[A, \Box] \in OPS^0$.

Now B may be constructed in the following way: $B = \sum A$ where A is defined as above. Then $B \in OPS^0$, it can be arranged to be elliptic on the Hamiltonian flow out of γ, and $[B, \Box] \in OPS^0$. Moreover, the local compactness of the unit sphere ensures that the summation is finite. $\Box$

Three remarks are in order:

(1) The same idea could lead to the existence of $\tilde{B} \in OPS^0$ with all the properties of B and, moreover, $[\tilde{B}, \Box] \in OPS^{-\infty}$. However, it is evident that with the presence of nonsmooth coefficients, the fact that $[\tilde{B}, \Box] \in OPS^{-\infty}$ will not benefit our analysis any further. The proofs of the Beals-Reed type theorems only rely on $[\tilde{B}, \Box] \in OPS^0$, anyway.

(2) With Lemma 3.2, both Hörmander's theorem and the Beals-Reed theorem enjoy an obvious but useful generalization. For example, consider problem

$$\Box u = f$$

$$u \in H^r_{m\ell}(\gamma)$$

where f is smooth. From Lemma 3.2, Hörmander's theorem will then yield

$$u \in H^r_{m\ell}(\Gamma) ,$$

where Γ is the Hamiltonian flow out of γ.

(3) By the definition of microlocal Sobolev spaces, $u \in H^s_{m\ell}(x_0, \xi_0)$ implies that there is a conic neighborhood of ξ_0 $\mathcal{K}$, such that $u \in H^s_{m\ell}(x_0, \mathcal{K})$. This fact has been implicitly used in Chapter 2. Therefore a combination of this fact and a simple compactness argument will imply that all the previous results on propagation of singularities hold in a small conic neighborhood (in the frequency space) of the null bicharacteristic as well.

3.4 Trace theorem I: Constant coefficients

In this section, we present a trace theorem for the hyperbolic *p.d.e.* with constant coefficients. In addition, we employ Hörmander's theorem (and Lemma 3.2) to describe how the singularities are propagated.

We first state a useful local regularity result for the solution to a first order hyperbolic Cauchy problem. We shall skip the proof because it is based on the method of energy estimates employed by Beals and Reed (Proposition on pages 176-177 in [5]). The difference is that a slightly different version of local regularity result is given in their paper with microlocal assumptions on the coefficients and the right-hand side. See also Proposition 2.9 in the preceding chapter.

Proposition 3.4 (Local Regularity) Let $P_1(x,t,D_x) \in C^\infty(\mathbb{R}, OPS^1(\mathbb{R}^{n-1}))$ and $P_0(x,t,D_x,D_t) \in OPS^0(\mathbb{R}^n)$, where $D_t - P_1$ is symmetric hyperbolic in the sense that $P_1 + P_1^* \in OPS^0$. Assume that

(i) $a \in H^r_{loc}$, $n/2 < r$ and $s \le r$,

(ii) $f \in H^s_{loc}$,

(iii) $v \in H^s_{loc}$ for t near 0,

and that

$$(D_t - P_1(x,t,D_x))v(x,t) = a(x,t)P_0(x,t,D_x,D_t)v(x,t) + f(x,t) .$$

Then

$$v \in H^s_{loc} .$$

We now proceed to show a trace regularity theorem.

Theorem 3.1 Suppose that u solves the hyperbolic problem

$$\begin{aligned} \Box u &= (\frac{\partial^2}{\partial t^2} - \Delta)u = 0 \\ u &\in H^s \cap H^{s+1}_{m\ell}(\gamma) , \quad \text{near } \{t=0\} , \end{aligned}$$

with

$$\gamma = \Omega \times \{(\xi,\omega) \in \mathbb{R}^{n+1} , \ \xi_n^2 + |\xi'|^2 = \omega^2 \text{ and } |\xi_n| \le \epsilon_0 |\xi'|\} ,$$

for small $\epsilon_0 > 0$, where Ω is a compact subset of $\{(x,t) \in \mathbb{R}^{n+1}\ ,\quad |t|, |x_n| \leq \epsilon_0\}$. Then

$$u|_{x_n=0} \in H^s_{loc}\ .$$

Idea of the proof. As we mentioned before, since the hypersurface $\{x_n = 0\}$ is a time-like surface, the method of energy estimates cannot be applied directly. To cure this difficulty, we shall alter the wave operator $\Box$ by a $\psi.d.o.$ cut-off technique so that $\{x_n = 0\}$ becomes a space-like surface. In other words, we shall construct a strictly hyperbolic $\psi.d.o.$ equation with respect to the trace $\{x_n = 0\}$. Since the operator in our construction is differential in x_n, the standard method of energy estimates (for example in John [18] or Taylor [39]) can be applied to get the basic estimate. Then, the microlocal hypotheses and Theorem 2.1 together with Lemmas 3.1, 3.2 will complete the proof.

Proof Let γ_0, γ_1 be two conic subsets of the set $\Omega_0 \times \{(\xi', \omega) \in \mathbb{R}^n\ ,\quad |\omega| \geq |\xi'|\}$ and let $\Pi_2\gamma_0 \subset \Pi_2\gamma_1$ (strictly), where Π_2 maps a set to its second factor or the frequency space (see Figure 3.1), and Ω_0 is generated by Ω in such a way that each point in Ω_0 may be traced back to Ω along the characteristics of the operator $\Box$. That is,

$$\Omega_0 = \{(x,t) \in \mathbb{R}^{n+1},\ \exists (x_0, t_0) \in \Omega,\ x = x_0 - \lambda t, |\lambda| = 1\}\ .$$

Then, we can find a convolutional operator $Q \in C^\infty(\mathbb{R}, OPS^0(\mathbb{R}^n))$, $q = q(\xi', \omega)$, that satisfies

- $ES(Q) \subset \gamma_1$ and $0 \leq Q_0 \leq 1$;

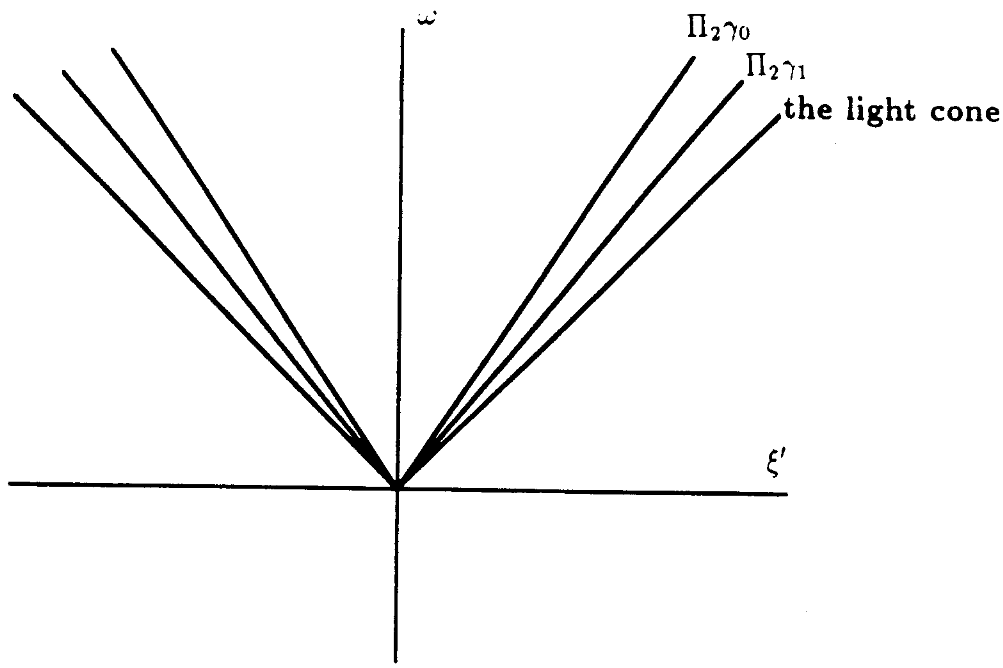

Figure 3.1 The projection of sets γ_0, γ_1 to the frequency space.

- $Q_0 = 1$ on $\gamma_0 \cap \{(x,t,\xi',\omega),\ |(\xi',\omega)| > 1\}$,

where $Q_0(\xi',\omega)$ is the principal symbol of Q. Define another operator E as

$$E \stackrel{def.}{=} Q\Box_{x',t} + (I-Q)\Delta_{x',t} \ ,$$

where $\Box_{x',t} = \partial_t^2 - \partial_{x'}^2$, $\Delta_{x',t} = \partial_t^2 + \partial_{x'}^2$.

Observe that the principal symbol of E

$$E_0 = Q_0(\omega^2 - \xi'^2) + (1-Q_0)(\omega^2 + \xi'^2) \geq C(\omega^2 + \xi'^2) \ ,$$

for $|(\omega,\xi')| \geq \delta$, with some positive constants C, δ. Hence, E is an elliptic $\psi.d.o.$ of order *two*.

Let $\phi = \phi(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ with supp $\phi \subseteq \{|x_n| \leq \epsilon_0\}$. We then have a strictly symmetric hyperbolic problem

$$(-\partial_{x_n}^2 + E)\phi u = [\Box, \phi]u + (I-Q)(\Delta_{x',t} - \Box_{x',t})\phi u \ .$$

Since ϕ is compactly supported, we actually have a symmetric hyperbolic Cauchy problem with zero Cauchy data. It follows from a hyperbolic energy estimate in Taylor [39] pages 73-75 or Proposition 3.4, by knowing that $[\Box,\phi]$ and $[\Delta_{x',t} - \Box_{x',t}]$ are operators of order one and two respectively, that

$$\begin{aligned} \|(\phi u)\,|_{x_n=0}\,\|_s &\leq C\|[\Box,\phi]u + (I-Q)(\Delta_{x',t} - \Box_{x',t})\phi u\|_{s-1} \\ &\leq C[\|\phi u\|_s + \|(I-Q)2\partial_{x'}^2\phi u\|_{s-1}] \ , \end{aligned}$$

where the second inequality makes sense because Q is a convolutional operator so that Proposition 3.1 is applicable.

Therefore, to obtain the desired conclusion it suffices to show that

$$(I-Q)\partial_{x'}^2\phi u \in H^{s-1} \ ,$$

which requires the use of Lemma 3.1. In order to apply Lemma 3.1, we choose $B=(I-Q)$ of order $m=0$, $A=\Box$ of order $m_0=2$. B is also a $\psi.d.o.$-like operator in $\mathbb{R}^n \times \mathbb{R}^1$ with $k_0=n$, $k=n+1$, and $h=s-2$.

Let us look at the assumption (1) of Lemma 3.1, $Ell(A)$ (the elliptic region of $A=\Box$) is easy to determine. Actually $\Box$ is elliptic away from the light cone $\{\omega^2 = |\xi|^2\}$.

Since $\Box u = 0$, the fact

$$\Box\partial_{x'}^2\phi u = [\Box, \partial_{x'}^2\phi]u \in H^{s-3}$$

verifies assumption (3). Hence, the only assumption that needs to be checked is that

$$u \in H^s \cap H_{m\ell}^{s+1}([T^*(\mathbb{R}^{n+1})\backslash Ell(\Box)] \cap \tilde{\Pi}_2^{-1}ES(I-Q)) \ ;$$

see Figure 3.2.

Since $u \in H^s_{\text{loc}}$ is a simple consequence of Proposition 3.3, we are left with verifying that $u \in H_{m\ell}^{s+1}([T^*(\mathbb{R}^{n+1})\backslash Ell(\Box)] \cap \tilde{\Pi}_2^{-1}ES(I-Q))$, and this demands Hörmander's theorem on propagation of singularities and Lemma 3.2.

Let γ_0, γ_1 approach the set $\Omega_0 \times \{(\xi,\omega) : \ |\omega| \geq |\xi'|\}$. The set

$$[T^*(\mathbb{R}^{n+1})\backslash Ell(\Box)] \cap \tilde{\Pi}_2^{-1}ES(I-Q)$$

is contained in a small (conic) neighborhood of the Hamiltonian flow out of γ. Hence Hörmander's theorem on propagation of singularities, Lemma 3.2 (in particular, remarks (2) and (3) there), and the microlocal initial hypotheses yield that

$$u \in H^{s+1}_{m\ell}([T^*(\mathbb{R}^{n+1})\backslash Ell(\Box)] \cap \tilde{\Pi}_2^{-1} ES(I-Q)) .$$

Thus we have proved the theorem. □

3.5 Trace theorem II: Variable coefficients

The ideas in the previous section can be generalized immediately to the case of nonsmooth coefficients in the lower order terms. With the presence of nonsmooth coefficients, the generalized Beals-Reed theorem (Theorem 2.3) are obvious necessary.

For the sake of simplicity, let us define

$$\mathcal{G} = [T^*(\mathbb{R}^{n+1})\backslash Ell(\Box)] \cap \tilde{\Pi}_2^{-1} ES(I-Q)$$

where $\mathcal{N} \subset\subset Ell(\Box)$, with $\mathcal{N}$ the normal bundle of $\mathbb{R}^n \times \mathbb{R}^1$. Q is a convolutional operator defined in the proof of Theorem 3.1 (see figure 3.2 for $\mathcal{G}$).

Theorem 3.2 Suppose that $s > 3+n/2$ and that u solves the hyperbolic initial value problem

$$\Box' u = [\Box - \nabla\sigma \cdot \nabla]u = f , \qquad (3.1)$$

$$u \in H^l \cap H^{l+1}_{m\ell}(\gamma) , \quad \text{near } \{t=0\} . \qquad (3.2)$$

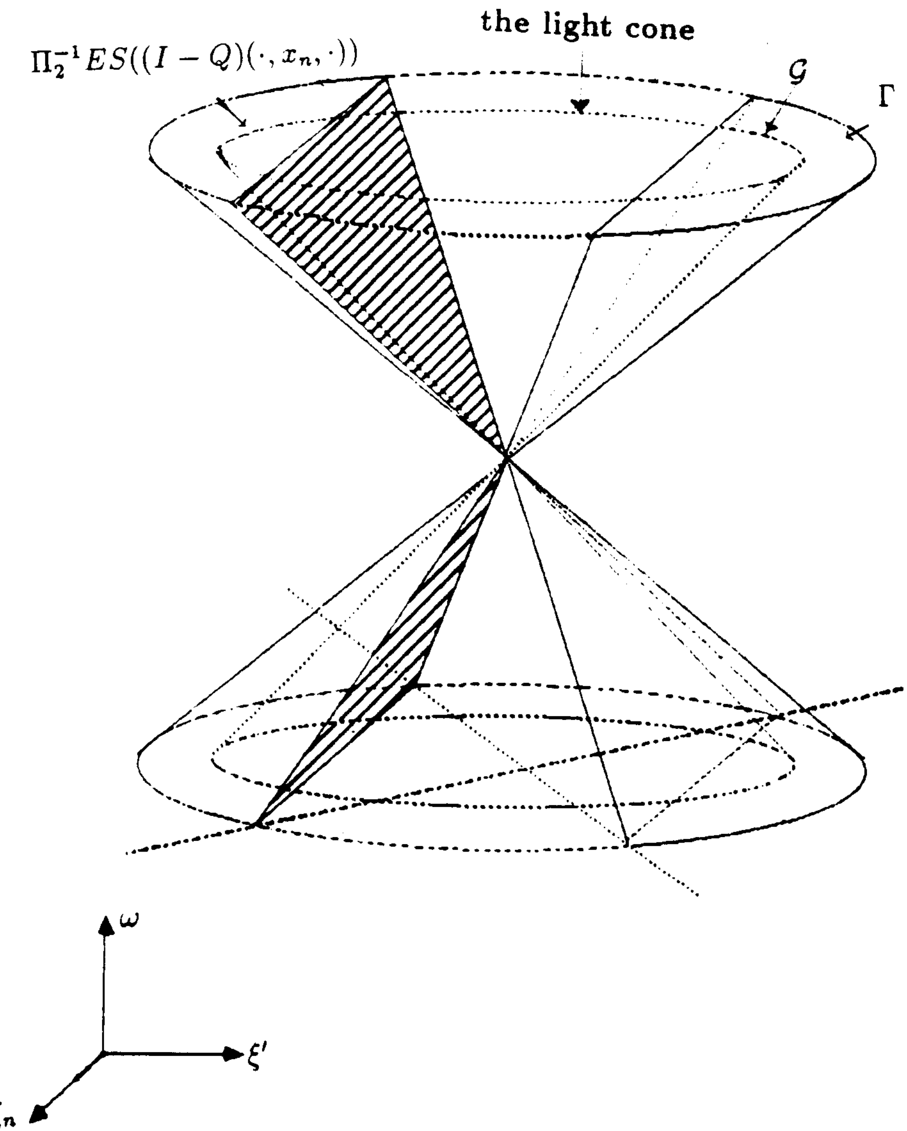

Figure 3.2 The projection of set
$\mathcal{G} = [T^*(\mathbb{R}^{n+1})\backslash Ell(\square)] \cap \Pi_2^{-1} ES((I-Q)(\cdot, x_n, \cdot))$ to the frequency space.

with

$$\gamma = \Omega \times \{(\xi, \omega) \in \mathbb{R}^{n+1} \ , \ \xi_n^2 + |\xi'|^2 = \omega^2 \text{ and } |\xi_n| \le \epsilon_0 |\xi'|\} \ ,$$

for small $\epsilon_0 > 0$. Ω is a compact subset of $\{(x,t) \in \mathbb{R}^{n+1} \ , \ \ |t|, |x_n| \le \epsilon_0\}$ and

$$\Gamma = \text{a small conic neighborhood of } \gamma \ .$$

Assume that

(i) $u \in H^{l-1} \cap H^l_{m\ell}(\Gamma)$, $1 \le l \le s$;

(ii) $\nabla\sigma(x) \in H^{s-1} \cap H^l_{m\ell}(K)$, $\Pi : \ \Gamma \subset T^*(\mathbb{R}^{n+1}) \to K \subset T^*(\mathbb{R}^n)$ is the projection map;

(iii) $f \in H^{l-1} \cap H^l_{m\ell}(\Gamma)$.

Then

$$u \mid_{x_n=0} \in H^l_{loc} \ .$$

Proof Proposition 3.3 implies that $u \in H^l_{\text{loc}}$. In the statement of Theorem 2.3 choose

$$(m, n_0, n, l, s, q, \epsilon) = (2, n, n+1, l-1, s-1, l, 1)$$

then the microlocal hypotheses and the fact $\mathcal{G} \subset \Gamma$ verify all the assumptions of Theorem 2.3. Hence similar to the proof of Theorem 3.1, we find $u \in H^{l+1}_{m\ell}(\mathcal{G})$, or

$$u \in H^l \cap H^{l+1}_{m\ell}(\mathcal{G}).$$

Notice that the main assumption, $s > 3 + n/2$, is required by the corresponding hypothesis (i) in Theorem 2.3.

Define E exactly as in the proof of Theorem 3.1. Similar arguments lead to

$$\begin{aligned} (-\partial_{x_n}^2 + E)\phi u &= \Box\phi u + (I-Q)(\Delta_{x',t} - \Box_{x',t})\phi u \\ &= [\Box, \phi]u + \phi f + \phi\nabla\sigma\cdot\nabla u + (I-Q)(\Delta_{x',t} - \Box_{x',t})\phi u \ , \end{aligned} \tag{3.3}$$

where again $\phi = \phi(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ with supp $\phi \subseteq \{|x_n| \le \epsilon_0\}$.

Hence

$$\|(\phi u)\ |_{x_n=0}\ \|_l \le C\|r.h.s.\text{ of }(3.1)\|_{l-1}$$

$$\le C[\|\phi_1 u\|_l + \|\phi f\|_{l-1} + \|\phi\nabla\sigma\cdot\nabla u\|_{l-1} + \|(I-Q)\partial_{x'}^2\phi u\|_{l-1}] \ , \tag{3.4}$$

where $\phi_1 \in C_0^\infty$ and $supp(\phi) \subseteq supp(\phi_0)$. But the generalized Schauder's lemma (Lemma 2.4) yields

$$\phi\nabla\sigma\cdot\nabla u \in H^{l-1} \ .$$

Therefore, to complete the proof it suffices to show that

$$(I-Q)\partial_{x'}^2\phi u \in H^{l-1} \ ,$$

which again requires Lemma 3.1.

Arguments similar to those in the proof of Theorem 3.1 are employed to verify all the assumptions of Lemma 3.1. Choose $B = I - Q \in C^\infty(\mathbb{R}^1, OPS^0(\mathbb{R}^n))$ of order $m = 0$, $A = \Box$ of order $m_0 = 2$, and $h = l - 2$ in the statement of Lemma 3.1. Then hypothesis (1) of Lemma 3.1 is obvious once again because $\Box$ is elliptic away from the light cone. Using the Beals-Reed type theorem on propagation of singularities (Theorem 2.3) as well as Lemma 3.2 and its remarks, we can verify hypothesis (2). To verify hypothesis (3), one only needs to look at

$$\Box\partial_{x'}^2\phi u = [\Box, \partial_{x'}^2\phi]u + \partial_{x'}^2\phi(\nabla\sigma \cdot \nabla u + f) \ ,$$

which is bounded by the first three terms in (3.2); hence the same arguments yield that $\Box\partial_{x'}^2\phi u \in H^{l-3}$.

The proof of Theorem 3.2 is then completed. □

Next we derive the corresponding trace estimate on the solution which will also be useful in Chapter 5. In this process and in the future, for the sake of simplicity it is convenient to introduce a useful notation.

Definition 3.1 A constant C is said to depend on the $H^s \cap H^r_{m\ell}(\mathcal{K})$-norm of $u \in C_0^\infty(\mathbb{R}^k)$ if the constant depends on $||u||_s + ||Qu||_{r,\Omega}$ for a $\psi.d.o.$ Q of order zero such that

- $ES(Q) \subseteq$ a sufficiently small conic neighborhood of $\mathcal{K}$ and
- $q = 1$ on $\mathcal{K} \cap \{(x,\xi) : \ |\xi| > 1\}$.

Ω is a compact set such that $supp(u) \subset \Omega$, or $\Omega = \mathbb{R}^k$ if $q(x,\xi)$ has compact support in x.

We also need a Gårding's type inequality concerning the microlocal ellipticity.

Lemma 3.3 Assume that $Q_1 \in OPS^{m_1}$, $Q_2 \in OPS^{m_2}$, with $m_1, m_2 \in \mathbb{R}$. Furthermore assume Q_2 is elliptic on $ES(Q_1)$. Then for any $r \in \mathbb{R}$ and $u \in C_0^\infty$,

$$||Q_1 u||_{s,\Omega} \leq C||Q_2 u||_{s+m_1-m_2,\Omega} + C||u||_{r,\Omega}$$

where Ω is a compact set, $supp(u) \subset \Omega$.

Proof W.L.O.G., it suffices to consider the case where $m_1 = m_2 = 0$. Let Q_2^{-1} be a parametrix of Q_2 on γ, a small conic neighborhood of $ES(Q_1)$. That is,

$$Q_2^{-1} Q_2 = I + K \quad \text{on } \gamma \; , \tag{3.5}$$

where K is a smoothing operator.

Construct a $\psi.d.o.$ P of order zero so that

- $ES(Q_1) \subset ES(P) \subset \gamma$;
- $p = 1$ on $ES(Q_1) \cap \{(x,\xi) : |\xi| > 1\}$,

where p is the symbol of P. Then Q_1 can be decomposed into two parts:

$$Q_1 = Q_1(I - P) + Q_1 P \; .$$

From (3.5) and the construction of P, we see that

$$Q_1 P = Q_1 Q_2^{-1} Q_2 P - Q_1 K P \; .$$

But since

$$ES(Q_1(I-P)) = ES(Q_1) \cap ES(I-P) = \emptyset \; ,$$

$Q_1(I-P)$ is a smoothing operator. Therefore, for any $r \in \mathbb{R}$,

$$\begin{aligned} ||Q_1 u||_{s,\Omega} &\leq C||Q_1 P u||_{s,\Omega} + C||u||_{r,\Omega} \\ &\leq C||Q_1 Q_2^{-1} Q_2 P u||_{s,\Omega} + C||u||_{r,\Omega} \\ &\leq C||Q_2 u||_{s,\Omega} + C||u||_{r,\Omega} . \end{aligned}$$

□

Remark. The lemma enjoys an obvious extension because in the proof we have only used the fact that Q_2 has a parametrix on γ.

Lemma 3.4 Under the same hypotheses as in the statement of Theorem 3.2, the estimate

$$||(\phi u)|_{x_n=0}||_l \leq C_1||\phi_0 u||_l + C||\phi_0 f||_{l-1} + C||P\phi_0 f||_{l,\Omega} \qquad (3.6)$$

holds for a ψ*.d.o.* P of order zero $ES(P) \subseteq \Gamma$, where $\phi(x,t)$ and $\phi_0 \in C_0^\infty(\mathbb{R}^{n+1})$ are supported near the trace hypersurface $\{x_n = 0\}$ and the constant C_1 depends on $H^{s-1} \cap H^{l_1}_{m\ell}(K)$-norm of $\psi\nabla\sigma$ with $\psi \in C_0^\infty(\mathbb{R}^n)$, Ω is a compact set in $\mathbb{R}^{n+1}$ such that $supp(\phi) \subset supp(\phi_0) \subset \Omega$.

Proof Clearly the conclusion of this lemma is slightly stronger than the previous theorem. Following the general outline of Theorem 3.2, we prove this lemma by getting all the necessary estimates.

Recall the estimate (3.4) in the proof of Theorem 3.2,

$$||(\phi u)|_{x_n=0}||_l \leq C[||\phi_1 u||_l + ||\phi f||_{l-1} + ||\phi\nabla\sigma\cdot\nabla u||_{l-1} + ||(I-Q)\partial^2_{x'}\phi u||_{l-1}] \quad (3.7)$$

where Q is again defined in the proof of Theorem 3.1. The generalized Schauder's lemma (Lemma 2.4) gives

$$||\phi\nabla\sigma\cdot\nabla u||_{l-1}\le C||\phi_1 u||_l$$

with constant C depending on $||\psi\nabla\sigma||_{s-1}$, for some $\psi\in C_0^\infty(\mathbb{R}^n)$.

Thus it suffices to estimate $||(I-Q)\phi u||_{l+1}$, which requires the $\psi.d.o.$ fattening technique developed in the proof of Lemma 3.1 (see also Corollary 3.1 for the estimate).

Recall the construction of the $\psi.d.o.$ H in the proof of Lemma 3.1. Similarly, one has

$$(I-Q)\partial_{x'}^2\phi u=(I-Q)\phi_1(I-H)\partial_{x'}^2\phi u+(I-Q)\phi_1 H\partial_{x'}^2\phi u\ . \qquad (3.8)$$

Choosing $A=\Box$, $B=I-Q$, $k=n+1$, $k_0=n$, and $h=l$, A is microlocally elliptic on $ES(I-H)$; hence, according to the proof of Lemma 3.1, one can easily write down the estimate

$$||(I-Q)\phi_1(I-H)\partial_{x'}^2\phi u||_{l-1}\le C||\phi u||_l\ , \qquad (3.9)$$

where C again depends on $||\psi\nabla\sigma||_{s-1}$.

Therefore to prove the lemma we only need to bound $||(I-Q)\phi_1 H\partial_{x'}^2\phi u||_{l+1}$.

As in the proof of Theorem 3.1 we can arrange ϵ to be sufficiently small and γ_0, γ_1 to approach the set $\Omega_0\times\{(\xi',\omega),|\omega|\ge|\xi'|\}$ so that $supp((I-Q)\phi_1 H)$ is near $\{x_n=0\}$ and $ES((I-Q)\phi_1 H)$ is contained in the flow out of γ (the set γ is defined in the statement of Theorem 3.1). It follows from Lemma 3.2 that there exists a $\psi.d.o.$ P of order zero such that

- P is elliptic on $ES((I-Q)\phi_1 H)$;
- $[\Box, P] \in OPS^0$;
- p is supported near $\{x_n = 0\}$.

From the ellipticity of P on $ES((I-Q)\phi_1 H)$, Gårding's type inequality Lemma 3.3 yields that

$$\begin{aligned} ||(I-Q)\phi_1 H \partial_{x'}^2 \phi u||_{l-1,\Omega} &\leq C||P\partial_{x'}^2 \phi u||_{l-1,\Omega} + C||\phi u||_{r,\Omega} \\ &\leq C||[P, \partial_{x'}^2 \phi]u||_{l-1,\Omega} + C||\phi P u||_{l+1,\Omega} + C||\phi u||_{r,\Omega} \\ &\leq C||\phi P u||_{l+1,\Omega} + C||\phi u||_{r,\Omega} \end{aligned}$$

for any $r \in \mathbb{R}$. Thus the proof has been reduced to bounding $||\phi P u||_{l+1,\Omega}$.

Acting P onto both sides of equation (3.1), one has

$$\begin{aligned} \Box P u &= [\Box, P]u + P\nabla\sigma \cdot \nabla u + Pf \\ u &= 0 \quad t < 0\,. \end{aligned}$$

Then the energy estimates together with a simple estimate implied by the commutator lemma (Lemma 2.6) yield

$$||\phi P u||_{l+1,\Omega} \leq C||\phi_0 u||_{l,\Omega} + C||\phi_0 f||_{l-1,\Omega} + C||P\phi_0 f||_{l,\Omega}\,, \tag{3.10}$$

where C depends on the $H^{s-1} \cap H^{l_1}_{m\ell}(K)$-norm of $\psi\nabla\sigma$.

Eventually, the proof is completed by combining (3.7) with (3.9) and (3.10). □

Chapter 4

Regularity of Fundamental Solution

4.1 Introduction

Since the excess pressure u in the model equation is in fact the fundamental solution, in order to study the regularity of the forward map, the regularity of the fundamental solution must be understood. It is evident that the real obstacle here is the singular right-hand side so that none of the propagation of singularity results discussed before could be applied to handle it directly. The goal in this chapter is to determine the regularity of the excess pressure by using the Hadamard theory of progressing wave expansion. Hadamard's construction (see *e.g.* Courant and Hilbert [10]) is nothing more than a singular decomposition in the sense that it decomposes the fundamental solution to the second order hyperbolic equation with smooth coefficients into two parts: A singular part which is the sum of a series of singular functions (singular only on the "wave front") and a regular part which is generally smoother than the singular part. The expansion exhibits precisely where the singularities take place and how singular they are. We refer the reader to Courant and Hilbert [10], Friedlander [12] and Romanov [28], for a detail study on the method of progressing wave expansions.

This chapter is started with a rigorous progressing wave expansion for general second order hyperbolic *p.d.e.*. We then specify the idea to work on our model problem. Clearly in order to get regularity of the fundamental solution it is crucial to study the transport equations. But what is more interesting is that by employing an energy identity given by Symes [36], the energy norm of the solution can be expressed in terms of an appropriate norm of the solution on the characteristic surface (solutions of the transport equations) without any remainder term. Notice that the regular part v_s vanishes at $t = \tau(x)$ when s is not too small. It is easy to show that the solution of the model problem with smooth coefficients is actually smooth inside the conoid $\{t = \tau(x)\}$. For more general cases (with nonsmooth coefficients), the previous Rauch-type results may be applied to obtaining an estimate.

The method involved in the regularity analysis for the solution of the model problem (the fundamental solution) is a natural one. What it shows, however, is encouraging: With the help of a simple energy identity, the previous Rauch-type results and the method of progressing wave expansions may well be applied to deal with certain strong singularities. Note that the right-hand side in the model equation is as singular as the Delta function.

4.2 Progressing wave expansion: General result

Consider a general second order hyperbolic equation

$$u_{tt} - Lu = f(x,t) \ ,$$

where L is a uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u \ ,$$

$$\mu \sum_{x=1}^{n} \alpha_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\alpha_i\alpha_j \le \frac{1}{\mu}\sum_{i=1}^{n}\alpha_i^2 \ , \quad 0 < \mu < +\infty \ .$$

Let D be an open simply connected domain of $\mathbb{R}^n$ containing the origin. The Riemann metric is considered as

$$d\tau = \left(\sum_{i,j=1}^{n} b_{i,j}(x)dx_i dx_j \right)^{1/2} \ , \quad B = (b_{ij})_{n\times n} = A^{-1} \ ,$$

and $\tau(x)$ is the distance between x and 0.

A point x in D can be specified using the Riemann coordinates $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, i.e., $x = f(\zeta)$. With respect to the variable ζ, the function $f(\zeta)$ has the inverse $\zeta = g(x)$. In this case,

$$\tau^2(x) = \sum_{i,j=1}^{n} b_{ij}(0)g_i(x)g_j(x) \ .$$

The conoid $\{t \ge \tau(x)\}$ is defined by the domain of influence of the point $(0,0)$.

We are now able to state a general regularity result

Lemma 4.1 (Romanov) For the spatial dimension $n = 3$, the fundamental solution which solves

$$\begin{aligned} &u_{tt} - Lu = \delta(x,t) \\ &u = 0 \ , \quad t < 0 \end{aligned} \tag{4.1}$$

is of the following structure:

$$u(x,t) = \frac{1}{2\pi} H(t) \sum_{k=0}^{s} \sigma_k(x) S_k(t^2 - \tau^2(x)) + v_s(x,t) \ , \tag{4.2}$$

where $H(t)$ is the Heaviside function, and

$$S_0(t) = \delta(t) \ , \ \ S_k(t) = \frac{t^{k-1}}{(k-1)!}H(t) \ , \quad \text{for } \ k \geq 1 \ .$$

The functions σ_k $(k = 0, \cdots, s)$ satisfy the recurrence relations:

$$\sigma_0(x) = \left(\frac{1}{|A(0)|} \left| \frac{\partial}{\partial x} g(x) \right| \right)^{1/2} \exp \left\{ -\frac{1}{2} \int_0^1 \sum_{i,k,s=1}^{n} \left[b_i(\xi) - \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j} a_{ij}(\xi) \right] \right.$$
$$\left. b_{ks}(0) g_k(x) \frac{\partial}{\partial \xi_i} g_s(\xi) \ |_{\xi = f(tg(x),0)} \ dt \right\} \ ,$$

$$\sigma_k(x) = \frac{1}{4} \sigma_0(x) \int_0^1 [\sigma_0(\xi)]^{-1} t^k L_\xi \sigma_{k-1}(\xi) \ |_{\xi = f(tg(x),0)} \ dt \ , \quad k > 0 \ .$$

Furthermore, v_s solves the Cauchy problem

$$\left(\frac{\partial^2}{\partial t^2} \quad - \quad L\right) v_s = [1/(2\pi)] H(t) S_s(t^2 - \tau^2(x)) L \sigma_s$$
$$v_s \quad = \quad 0 \ , \quad t < 0 \ .$$

Finally, if $a_{ij} \in C^{\ell+4}(D)$, $b_i \in C^{\ell+2}(D)$, $c \in C^{\ell}(D)$ then

$$\sigma_k \in C^{\ell-2k+2}(D \times D).$$

Remark. The lemma carries all the information about the singularities through expansion (4.2). The importance of this result is that it gives a natural way to determine the regularity of u in terms of the regularity of the coefficients σ_k.

A detailed proof may be found in Romanov [28], the calculation is a bit complicated there, but the idea is rather simple. The proof containes two major steps: using some known results on the distributions, one may arrange to match the Delta function on the right-hand side; the coefficients of the less singular parts all vanish, and moreover these relations may be deduced to a series of ordinary differential equations (transport equations). The standard way of solving the *o.d.e.* yields the recurrence relations of σ_k. However, to get the above construction, one has to take into full consideration the special structure of 3-D. Nonetheless, Theorem 4.1 below verifies that similar progressing wave expansions may be carried out for other spatial dimensions (though one might not be able to write down the explicit expansions as simply as in Lemma 4.1). Essentially the only difficulty is how to remove the Delta function on the right-hand side of equation (4.2), but it can be overcome by using the properties of special functions. The proof is fashioned after the proof of Lemma 4.1 in Romanov's book; hence we shall not hesitate to skip it. However, it should be pointed out that the behavior of the leading singularities for all spatial dimensions was well understood long ago (for example in Courant and Hilbert [10]).

Theorem 4.1 The fundamental solution which solves

$$\begin{aligned} u_{tt} &- Lu = \delta(x,t)\ , \quad (x,t) \in \mathbb{R}^{n+1} \\ u &= 0\ , \quad t < 0 \end{aligned}$$

is of the following structure:

$$u(x,t) = C_0 H(t) \sum_{k=0}^{s} \sigma_k(x) S_k(t^2 - \tau^2(x)) + v_s(x,t)\ ,$$

where C_0 is a constant depending on the dimension n, and

$$S_0(t) = \delta^{(\frac{n-3}{2})}(t) \text{ and } S_k'(t) = S_{k-1} , \quad \text{for } k \geq 1 ,$$
$$\delta^{(-1)}(t) = H(t) \text{ and } \delta^{(-1/2)}(t) = \frac{H(t)}{\sqrt{t}} .$$

The functions σ_k ($k = 0, \cdots, s$) satisfy the recurrence relations:

$$2\textstyle\sum_{i,j=1}^{n} a_{ij}\Gamma_{x_j}\dfrac{\partial \sigma_k}{\partial x_i} + \sigma_k L'\Gamma = -L\sigma_{k-1} , \quad k > -1 .$$

When $k = -1$, the right-hand side should be replaced by zero, where

$$L'\Gamma = \sum_{i,j=1}^{n} a_{ij}\Gamma_{x_i x_j} + \sum_{i=1}^{n} b_i\Gamma_{x_i} + C_1 ,$$

where C_1 is another fixed constant.

Furthermore v_s solves the Cauchy problem

$$\begin{aligned}(\frac{\partial^2}{\partial t^2} \quad &- \quad L)v_s = C_0 H(t) S_s(t^2 - \tau^2(x)) L\sigma_s \\ v_s \quad &= \quad 0 , \quad t < 0 .\end{aligned}$$

Once again, if $a_{ij} \in C^{\ell+4}(D)$, $b_i \in C^{\ell+2}(D)$, and $c \in C^{\ell}(D)$, then

$$\sigma_k \in C^{\ell-2k+2}(D \times D).$$

4.3 Energy identity

We now examine the regularity of the fundamental solution to the model problem quantitatively. Clearly Theorem 4.1 gives us a direct way to do so. We can represent the fundamental solution as the sum of the principal part and remainder and study the remainder by the Beals-Reed type propagation of singularity theorem. However, a great drawback of this idea is that additional regularity is needed to regularize the remainder term. In this section, taking the special structure of the model problem into account, we shall modify the above straightforward idea by introducing an energy identity. The advantage to this technique is that with the energy identity, we can essentially get rid of the remainder term in the expansion; therefore a refined regularity result should be expected.

To fix the ideas, let us consider a problem obtained from the model problem by integrating the problem in the time variable,

$$\begin{aligned} &(\Box - \nabla\sigma_0 \cdot \nabla)v_0 = \delta^{-\frac{n-1}{2}}(t)\delta(x)\,, \quad (x,t) \in \mathbb{R}^{n+1} \\ &v_0 = 0\,, \quad t < 0\,. \end{aligned} \tag{4.3}$$

Hadamard's construction leads to the progressing wave expansion for v_0,

$$v_0 = \sum_{k=0}^{s} b_k S_k(t - \tau(x)) + R_{v_0}(x,t) \tag{4.4}$$

where $\tau(x) = |x|$, S_0 is the Heaviside function, $S_k' = S_{k-1}$ $(k \geq 1)$, and R_{v_0} vanishes at $t = \tau(x)$. Moreover $\{b_k\}$ solve the transport equations, for $k = 1, \cdots, s$,

$$2\nabla\tau \cdot \nabla b_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0) b_0 \;=\; 0 \tag{4.5}$$

$$2\nabla\tau \cdot \nabla b_k + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0) b_k \;=\; \Delta b_{k-1} + \nabla\sigma_0 \cdot \nabla b_{k-1}\,. \tag{4.6}$$

Since the boundedness of the energy norm will naturally lead to the regularity, we attempt to bound the energy norm by recalling an energy identity stated in Symes [36].

Denote

$$B_T = \{x : \tau(x) \le T\}\ ,\ \ C_T = \{(x,t) : t = \tau(x) \le T\}\ .$$

We can then introduce an energy identity for the solution of the wave equation.

Proposition 4.1 (Energy Identity) Suppose w solves the inhomogeneous wave equation

$$\begin{aligned}(\Box - \nabla\sigma\nabla)w = f\ ,\quad (x,t) \in \mathbb{R}^{n+1}\\ w = 0\ ,\quad t < 0\ .\end{aligned} \tag{4.7}$$

Define

$$E_T(t) = \int_{B_T} dx e^{\sigma}(|\frac{\partial w}{\partial t}|^2 + |\nabla w|^2)\ .$$

Then the following identity holds

$$E_T(t) = \int_{C_T} dx e^{\sigma}|\frac{\partial w}{\partial t}\nabla\tau + \nabla w|^2 + \int\int_{B_T\times[0,t]} dxdte^{\sigma} f w_t\ . \tag{4.8}$$

Proof We shall assume that σ, f, w are smooth enough, and w has compact support in x for each t. The equation (4.7) may be rewritten as

$$e^{\sigma}\partial_t^2 w - \nabla\cdot(e^{\sigma}\nabla)w = e^{\sigma} f\ .$$

Multiply both sides by w_t and integrate over $B_T \times [0,t]$,

$$\int\int_{B_T\times[0,t]} dxdte^{\sigma} f w_t = \int\int_{B_T\times[0,t]} dxdt\{\frac{\partial}{2\partial_t} e^{\sigma}(|\frac{\partial w}{\partial t}|^2 + |\nabla w|^2) - \nabla\cdot(e^{\sigma}\nabla w w_t)\}\ .$$

Integration by parts (divergence theorem) yields

$$\int\int_{B_T\times[0,t]} dxdte^{\sigma} f w_t = E_T(t) - \int_{C_T} dx e^{\sigma}|\frac{\partial w}{\partial t}\nabla\tau + \nabla w|^2\ .$$

□

Remarks on the energy identity.

(1) Applying Proposition 4.1 to v_0, the remainder term is eliminated due to the fact that $R_{v_0} = 0$ on C_T. More interestingly, both the tangential and normal derivatives of v_0 are determined by the transport equations.

(2) After a simple calculation, we can deduce from (4.4) that

$$\begin{aligned} \nabla v_0|_{t\to\tau(x)+} &= \nabla b_0 - b_1 \nabla\tau \\ \frac{\partial v_0}{\partial t}|_{t\to\tau(x)+} &= b_1 \, . \end{aligned}$$

Therefore

$$(\frac{\partial v_0}{\partial t}\nabla\tau + \nabla v_0)|_{t\to\tau(x)+} = \nabla b_0 \, ,$$

where the term b_1 is killed due to a cancellation. In fact this is true in general: Given P a differential operator with constant coefficients of order k, it is easy to show that b_k does not appear in

$$(\frac{\partial P v_0}{\partial t}\nabla\tau + \nabla P v_0)|_{t\to\tau(x)+} \, ;$$

the leading term is ∇b_{k-1}.

With Proposition 4.1, one can then examine the regularity of v_0 in (4.3) in terms of the regularity of the solutions to the corresponding transport equations.

Corollary 4.1 The solution v_0 of (4.3) belongs to H^l inside $\{t = \tau(x)\}$ if and only if $b_k \in H^{l-k}$, where b_k solves the k-th transport equation of (4.5), (4.6) and $k = 0, \cdots, l-1$.

Proof From the above energy identity as well as the remarks it is obvious to show that the L_2-norm of $\partial_t^k v_0$ can be bounded by the H^{k-i}-norm of b_i for $i = 1, \cdots, k-1$. Hence it is sufficient to consider higher order x-derivatives. But this is not difficult either. As an example, let us look at the $\nabla^2 v_0$ term. Since ∇v_0 solves equation

$$(\Box - \nabla\sigma_0 \cdot \nabla)\nabla v_0 = \Delta\sigma_0 \nabla v_0 ,$$

which may be viewed as an inhomogeneous equation, the corresponding energy identity and a simple use of Gronwall's inequality will lead to the desired estimate. The general case followes by induction. □

4.4 Regularity of fundamental solution

In order to establish a regularity result with the presence of nonsmooth coefficients, we need the following results.

It is helpful to introduce an interesting invariant property of microlocal Sobolev spaces. The result was originally established by Bony [8] and was extended by Meyer [23]. See also Beals [3] for a different proof.

Proposition 4.2 Suppose that for some $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$, $u \in H^s \cap H^r_{m\ell}(x_0, \xi_0)$, $n/2 < s \leq r \leq 2s - n/2$, and $g \in C^\infty$, then

$$g(x, u) \in H^s \cap H^r_{m\ell}(x_0, \xi_0) .$$

Observe that all the transport equations in (4.5)-(4.6) have the same principal part $2\nabla\tau \cdot \nabla$ which is a smooth vector field. Therefore in order to understand the regularity of the solutions to (4.5)-(4.6) it is useful to study the properties of this smooth vector field.

It is evident that the equation

$$Vu = f$$

can be solved by the so called characteristic method (*i.e.* finding integral curves of the vector field V). Let $x = x(\lambda)$ be such a curve with λ as a parameter, then $x(\lambda)$ satisfies a system of ordinary differential equations

$$\frac{dx^j}{d\lambda} = \tau_j(x(\lambda)) \;\; j = 1, \cdots, n ,$$

therefore along such a curve

$$\frac{du}{d\lambda} = f \tag{4.9}$$

which may be viewed as a hyperbolic first order differential equation.

Lemma 4.2 Let V be the smooth vector field $\nabla\tau \cdot \nabla$. Consider

$$Vu = f \; , \;\; \text{or} \;\; \frac{du}{d\lambda} = f \; ,$$

where u is smooth and $supp(u) \subset \{\lambda > \delta\}$, for $\forall\delta > 0$ small. Then there exists a ψ.*d.o.* Q of order zero such that Q is elliptic on $Char(V)$ and $[V, Q] \in OPS^{-\infty}$. Moreover, for $s \in \mathbb{R}$, the inequalities

$$||\phi u||_{s,\Omega} \leq C||\tilde{\phi}QVu||_{s,\Omega} + C||\tilde{\phi}Vu||_{s-1,\Omega} + C||\tilde{\phi}u||_{r,\Omega} \quad (4.10)$$

$$||Qu||_{s,\Omega} \leq C||QVu||_{s,\Omega} + C||u||_{r,\Omega} \quad (4.11)$$

hold for any $r \in \mathbb{R}$, where ϕ and $\tilde{\phi} \in C_0^\infty$, $\Omega \subset K \times \{\lambda : \lambda > \delta^-\}$ with K a compact set, Ω is a sufficiently big compact set, and $supp(\phi) \subset supp(\tilde{\phi}) \subset \Omega$.

Proof The existence of operator Q follows from Nirenberg's construction which appeared in the proof of Theorem 6 in [24], together with a local compactness argument. The assumption on Q implies that Q is elliptic on γ, a small conic neighborhood of $Char(V)$. Thus, one may construct another ψ.*d.o.* R of order zero which has the properties:

- $R + Q$ is elliptic and
- $ES(R) \cap \gamma = \emptyset$.

Then Gårding's inequality gives

$$\begin{aligned} ||\phi u||_{s,\Omega} &\leq C||(R+Q)\phi u||_{s,\Omega} + C||\phi u||_{r,\Omega} \\ &\leq C||R\phi u||_{s,\Omega} + C||Q\phi u||_{s,\Omega} + C||\phi u||_{r,\Omega} \quad (4.12) \end{aligned}$$

for any $r \in \mathbb{R}$.

Since V is elliptic on $\gamma^c \supset ES(R)$, the Gårding's type result Lemma 3.3 yields that

$$||R\phi u||_{s,\Omega} \leq C||V\phi u||_{s-1,\Omega} + C||\phi u||_{r,\Omega} \; ; \tag{4.13}$$

hence

$$\begin{aligned} ||\phi u||_{s,\Omega} &\leq C||V\phi u||_{s-1,\Omega} + C||Q\phi u||_{s,\Omega} + C||\phi u||_{r,\Omega} \\ &\leq C||\phi V u||_{s-1,\Omega} + C||\phi Q u||_{s,\Omega} + C||\phi_1 u||_{s-1,\Omega} + C||\phi u||_{r,\Omega} \, , \end{aligned}$$

with $\phi_1 \in C_0^\infty(\mathbb{R}^n)$, and $supp(\phi) \subset supp(\phi_1) \subset \Omega$. Now we may apply a bootstrap argument. In fact, same analysis leads to

$$||\phi_i u||_{s-i,\Omega} \leq C||\phi_i V u||_{s-i-1,\Omega} + C||\phi_i Q u||_{s-i,\Omega} + C||\phi_{i+1} u||_{s-i-1,\Omega} + C||\phi_i u||_{r,\Omega} \, , \tag{4.14}$$

where $\phi_i \in C_0^\infty$, $supp(\phi) \subset supp(\phi_1) \subset \cdots \subset \Omega$, $i = 1, 2, \cdots$. Therefore, a simple calculation yields

$$||\phi u||_{s,\Omega} \leq C[||\tilde{\phi} V u||_{s-1,\Omega} + ||\tilde{\phi} Q u||_{s,\Omega} + ||\tilde{\phi} u||_{r,\Omega}] \, . \tag{4.15}$$

Thus it suffices to study the term $||Qu||_{s,\Omega}$. Observe that $Qu \in C^\infty(K)$ solves

$$VQu = QVu + [V, Q]u$$

which is a first order equation. Thus along the integral curve of the vector field V,

$$\frac{dQu}{d\lambda} = QVu + [V, Q]u \, , \tag{4.16}$$

which is a hyperbolic first order differential equation.

Moreover, the pseudolocal property of Q yields that

$$||(Qu)(\cdot,\delta)||_{s,K} \leq C||u||_{r,K} .$$

Hence the method of hyperbolic energy estimates in Taylor [39] pages 73-75 may be applied to (4.16) and leads to a simple estimate

$$||Qu(\cdot,\lambda)||_{s,K} \leq C\int_{\delta}^{\lambda}[||QVu||_{s,K} + ||[V,Q]u||_{s,K}]d\lambda$$

or

$$\begin{aligned} ||Qu||_{s,K} &\leq C||QVu||_{s,K} + ||[V,Q]u||_{s,K} \\ &\leq C||QVu||_{s,K} + ||u||_{r,K} . \end{aligned} \tag{4.17}$$

The second estimate uses the fact that $[V,Q]$ is a smoothing operator. The estimate (4.11) follows from differentiating the differential equations and above estimates.

Substituting estimates (4.11) and (4.13) to (4.12), we eventually obtain that

$$||\phi u||_{s,\Omega} \leq C||\tilde{\phi}Vu||_{s-1,\Omega} + C||\tilde{\phi}QVu||_{s,\Omega} + C||\tilde{\phi}u||_{r,\Omega} ,$$

which completes our proof. □

Until now, we have only considered the principal part of the transport equations. Fortunately, our next proposition implies that the lower order terms may actually be absorbed by the principal part, hence the whole analysis can go through.

Proposition 4.3 Assume that w, q solve

$$Vw = f \text{ and } Vq = a , \tag{4.18}$$

where again V denotes $\nabla\tau \cdot \nabla$. Then $\tilde{w} = we^{-q}$ solves the equation

$$V\tilde{w} + a\tilde{w} = fe^{-q} \,. \tag{4.19}$$

Proof Substituting $\tilde{w} = we^{-q}$ to the left-hand side of (4.19), one has by chain rule

$$V\tilde{w} + a\tilde{w} = Vwe^{-q} + (-Vq + a)we^{-q} \,.$$

Hence the assumptions in (4.18) verify the equation (4.19). □

Remark. We want to make the following observation: In the transport equations (4.5) and (4.6),

$$q = \sigma_0/2 + q_0$$

where q_0 solves equation $Vq_0 = \Delta\tau/2$. Thus q is nothing more than a smooth perturbation of $\sigma_0/2$.

With the above preparations, we are now ready to state and prove the main result of this chapter.

Theorem 4.2 Suppose that $\sigma_0 \in H^{l+\alpha} \cap H^{2l-1}_{m\ell}(Char(\nabla\tau \cdot \nabla))$ with one of the following assumptions: $\alpha > n/4$ and $l \geq 1 + \alpha$, or $\alpha = n/4$ and $l > 1 + n/4$. Let $Char(\nabla\tau \cdot \nabla) = \{(x,\xi) \in T^*(\mathbb{R}^n), \nabla\tau \cdot \xi = 0\}$. Then

$$v_0 \in H^l(U) \,,$$

where $U = \{(x,t) : \; x \in \Omega \;,\; t \in [0,T] \text{ and } t > \tau(x)\}$ is a compact set in $\mathbb{R}^{n+1}$.

Proof By Corollary 4.1, it suffices to show that $b_k \in H^{l-k}(\Omega)$, where b_k is the solution of the k-th transport equation of (4.5), (4.6), for $k = 0, \cdots, l-1$.

We once again introduce a function $q = \sigma_0/2 + q_0$ with $\nabla\tau \cdot \nabla q_0 = \Delta\tau/2$. Then according to Proposition 4.3, the transport equations (4.5), (4.6) may be transformed to equations

$$\nabla\tau \cdot \nabla b_0 e^q = 0 \qquad (4.20)$$

$$\nabla\tau \cdot \nabla b_k e^q = (\Delta b_{k-1}/2 + \nabla\sigma_0 \cdot \nabla b_{k-1}/2)e^q \ , \qquad (4.21)$$

for $k = 1, \cdots, l-1$.

From equation (4.20), the assumptions on σ_0, l, and α clearly indicate that $||b_0||_{l,\Omega} \leq C||\sigma_0||_{l,\Omega}$.

Since $\nabla\sigma_0 \in H^{l+\alpha-1} \cap H^{2l-2}_{m\ell}(\gamma)$, the assumptions imply that $l+\alpha-1$, $2l-2$ satisfy Rauch's condition, Proposition 4.3 and Lemma 2.2 (our generalized Rauch's lemma in Chapter 2) guarantee that all of the operations involving $\nabla\sigma_0$ may be performed.

To clarify the idea, we shall use Q_0 to represent all *ψ.d.o.* of order zero. Their essential supports are close to each other, and they all possess the properties of Q in Lemma 4.2. Therefore by using (4.10), (4.11) of Lemma 4.2 several times, after some similar simple calculations, one can write down the following inequalities,

$$||\phi b_1 e^q||_{l-1,\Omega} \leq C||\phi_1 Q_0 b_0||_{l+1,\Omega} + C||\phi_1 b_0||_{l,\Omega} + C||\phi_1\sigma_0||_{r,\Omega}$$

$$\cdots$$

$$||\phi b_k e^q||_{l-k,\Omega} \leq C||\phi_1 Q_0 b_0||_{l+k,\Omega} + C||\phi_1 b_0||_{l,\Omega} + C||\phi_1\sigma_0||_{r,\Omega} \ ,$$

where $k = 1, \cdots, l-1$, $\phi \in C_0^\infty(\Omega)$, $\phi_1 \in C_0^\infty$ has bigger support than ϕ, r is any real number, and C depends at most on $||\phi_1\sigma_0||^{Q_0}_{l+\alpha,2l-1,\Omega}$. Using the Gårding's type inequality of Lemma 3.3 one more time, and knowing the regularity of b_0, we can then complete the proof of this theorem. □

We want to make some comments on Theorem 4.2. It is unpleasant to have extra α-order derivatives on σ_0 in the statement of the theorem. This defect cannot be avoided because Rauch's condition is necessary to get the conclusion of Proposition 4.2. At this point, we do not know how to relax the hypothesis as long as the Rauch-type results are employed.

Chapter 5

Upper Bound for Linearized Forward Map

5.1 Introduction

We are now ready to establish a regularity result for the forward map F. It is well known that in the study of inverse problems the regularity of forward map is essential. To solve inverse problems numerically, the differentiability of the corresponding forward maps is necessary in order to access to any fast numerical schemes such as various types of Newton's methods. Also local properties of inverse problems may be understood through the study of the formal derivative of forward maps provided that the forward maps are differentiable. Moreover, the regularity results for forward maps are obviously crucial in the design and implementation of any numerical algorithms for solving inverse problems.

Our goal in this chapter is to determine the appropriate hypotheses under which $DF(\sigma_0)$, the linearization of F about a reference state σ_0, is bounded above. We believe that similar analyses could lead to a continuity result for F as well as the differentiability of F (these and other related issues will be addressed in a sequel to this work).

Recall the linearized problem corresponding to the reference state (u_0, σ_0), for $(t,x) \in \mathbb{R}^{n+1}$, $x = (x', x_n)$,

$$\begin{aligned} (\Box - \nabla\sigma_0\nabla)\delta u &= \nabla\delta\sigma\nabla u_0 \\ \delta u = 0\ , &\quad t < 0\ , \end{aligned} \tag{5.1}$$

where u_0 is the solution of the model problem corresponding to the reference density σ_0. The linearized forward map can be defined as

$$DF(\sigma_0)\delta\sigma = (\phi\delta u)\,|_{x_n=0}\ , \tag{5.2}$$

where $\phi(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ is supported inside the conoid $\{t > |x|\}$, and near $\{x_n = 0\}$.

Once again we consider a related problem,

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)\ v &= \nabla\delta\sigma \cdot \nabla v_0 \\ v = 0\ , &\quad t < 0\ , \end{aligned} \tag{5.3}$$

where $\delta u = \partial_t^{\frac{n-1}{2}} v$ and v_0 solves

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)v_0 &= \delta^{-\frac{n-1}{2}}(t)\delta(x) \\ v_0 = 0\ , &\quad t < 0\ , \end{aligned} \tag{5.4}$$

Observe that for $l \in \mathbb{R}$,

$$\begin{aligned} ||DF(\sigma_0)\delta\sigma||_l &= ||\ (\phi\delta u)\,|_{x_n=0}||_l \\ &\leq C||(\phi v)\,|_{x_n=0}\,||_{l_1}, \end{aligned} \tag{5.5}$$

where l_1 denotes $l + \dfrac{n-1}{2}$. Thus the real challenge here is to get an appropriate trace regularity estimate for v on a time-like hypersurface $\{x_n = 0\}$.

Throughout this chapter, we shall always assume that

(A) $supp(\delta\sigma) \subset \{x_n > \epsilon\}$,

for $\epsilon > 0$ small. After all, the most precise recovery of the density should take place near the trace surface.

5.2 Statement of theorem

We first state the main result of this chapter then give a brief description about the idea of its proof. The theorem will be proved in the sections which follow.

Recall Definition 3.1, by a constant C depending on the $H^s \cap H^r_{m\ell}(\mathcal{K})$-norm of u we mean that the constant C depends on $||u||^Q_{s,r} = ||u||_s + ||Qu||_{r,\Omega}$ for a $\psi.d.o.$ Q of order zero such that $ES(Q) \subseteq$ a sufficiently small conic neighborhood of $\mathcal{K}$ and $q = 1$ on $\mathcal{K} \cap \{(x,\xi) : \ |\xi| > 1\}$, where Ω is a compact set with $supp(u) \subset \Omega$.

Theorem 5.1 Assume that $\alpha \geq n/4$, $l \geq max\{\alpha + (3-n)/2, 3/2 + \delta_1\}$, $s = max\{3 + n/2 + \delta_2, l + n - 1 + \delta_2, l + \alpha + (n-1)/2\}$, $\forall \delta_1,\ \delta_2 > 0$, $\theta = Char(\nabla \cdot \tau) = \{(x,\xi) \in T^*(\mathbb{R}^n), \nabla\tau \cdot \xi = 0\}$, and $K = \{(x,\xi) \in T^*(\mathbb{R}^n), |\xi_n| \leq \epsilon|\xi|\}$. Then under the assumption (A), the following estimate holds

$$||DF(\sigma_0)\delta\sigma||_l \leq C||\delta\sigma||_{l+\frac{n-1}{2}} \tag{5.6}$$

where the constant C depends on the $H^s \cap H^{l+(n+1)/2}_{m\ell}(K) \cap H^{2l+n-2}_{m\ell}(\theta)$-norm of σ_0 but is independent of $\delta\sigma$.

An interesting special case of Theorem 5.1 is remarkable because the additional microlocal smoothness along the tangential direction will then be absorbed.

Corollary 5.1 In addition to the assumptions in the statement of Theorem 5.1, assume that the spatial dimension $n \geq 3$. Then under the assumption (A), the same estimate

$$||DF(\sigma_0)\delta\sigma||_l \leq C||\delta\sigma||_{l+\frac{n-1}{2}}$$

holds, where the constant C depends on the $H^s \cap H^{2l+n-2}_{m\ell}(\theta)$-norm of σ_0 but is independent of $\delta\sigma$.

However, it still remains to see whether or not the additional smoothness along the characteristic variety of transport equations can be removed.

Before getting into the details of the proof, let us first make the following general remarks on this theorem:

- The estimate (5.6) above has a similar form to a Rakesh's theorem (Theorem 2.5 in [25]). Actually a formal extension of our proof here could lead to an elementary proof of his theorem. On the contrary, notice that Rakesh's proof is based on the calculus of Fourier integral operators, therefore it breaks down when the reference density is nonsmooth (as we mentioned earlier the method of F. I. O. is only applicable to smooth reference density).
- Our approach here is based on the method of energy estimates associated with results on propagation of singularities and various trace regularity results. The

beauty of the method of energy estimates is that it possesses useful information on various parameters involved in the estimates. Most importantly, the idea of our approach has the potential to deal with much more difficult velocity inversion problems with nonsmooth reference velocity.

- To simplify the proof of Theorem 5.1, we shall first assume that σ_0 and $\delta\sigma$ are smooth functions with (sufficiently big) compact supports. We then derive the estimates. The precise smoothness requirement for σ_0 can be seen easily from the dependence of the constants on σ_0 in the estimates. It is also important to see that the coefficient is smooth is not a necessary assumption in order to use all the techniques involved in our proof.

In order to clarify the ideas, we shall split the proof in several steps:

- Applying our previous trace theorem, assumption (A), as well as results on propagation of singularities, the estimate of $||(\phi v)\ |_{x_n=0}\ ||_{l_1}$ may be reduced to the estimate of $||\phi v||_{l_1}$.

- We then decompose $\delta\sigma$ into two pieces: $Q_1\delta\sigma$ (good part) and $Q_2\delta\sigma$ (bad part), correspondingly decompose v into $v_1 + v_2$, so that they can be studied separately and then reassembled.

- We show that the good part actually leads to the desired estimate by the same technique used in the preceding chapter.

- The most difficult part is to show that the bad part has very little influence on the estimate therefore may be negligible. In order to do so, we introduce a dual problem. We show that it suffices to analyze how the singularities (regularity) of the solution to the dual problem propagate. The main ingredients in this step are an estimate derived from the propagation of singularities theorem and a microlocal version of the classical trace theorem.

Let $\phi \in C_0^\infty$ be supported inside the characteristic surface (also inside $\{x_n < \epsilon/2\}$). Multiplying ϕ to both sides of equation (5.3), we have

$$\begin{aligned} \Box \phi v &= \phi \nabla \sigma_0 \cdot \nabla v + [\Box, \phi] v \\ v &= 0 \ , \quad t < 0 \ . \end{aligned} \tag{5.7}$$

Here we have used the fact that according to the assumption (A), ϕ and $\delta\sigma$ have disjoint supports, so that $\phi \nabla \delta\sigma \nabla v_0 = 0$.

Once again with l_1 we denote $l + (n-1)/2$.

Lemma 5.1 Assume that $s > 3 + n/2$, $1 \leq l_1 \leq s$, and v solves problem (5.7) then there is a $\phi_0 \in C_0^\infty$ supported near $supp(\phi)$ such that the following estimate holds,

$$||(\phi v)|_{x_n=0}||_{l_1} \leq C||\phi_0 v||_{l_1} \ , \tag{5.8}$$

where C is a constant depending on the $H^{s-1} \cap H^{l_1}_{m\ell}(K)$-norm of $\nabla \sigma_0$, but is independent of $\delta\sigma$.

Proof This lemma is a direct application of Lemma 3.4 in Chapter 3, where the right-hand side f is chosen to be $\nabla\delta\sigma \cdot \nabla v_0$. Observe in the equation (5.7), by the assumption (A), ϕ and $\delta\sigma$ have disjoint supports; thus $\phi f = 0$. From the construction of the ψ*.d.o.* P in the proof of Lemma 3.3, its symbol p is supported near $\{x_n = 0\}$; therefore $Pf = 0$. □

5.3 Regularity of v_1

Construct two ψ*.d.o.* $Q_1, Q_2 \in OPS^0(\mathbb{R}^n)$, such that

- $Q_1 + Q_2 = I$;
- $ES(Q_2)$ is a small conic neighborhood of$\{\nabla\tau \cdot \xi = 0\}$;
- Q_2's symbol $q_2 = 1$ near $\{\nabla\tau \cdot \xi = 0\} \cap \{(x,\xi),\ |\xi| \geq 1\}$.

An immediate consequence of this construction is that for any ψ*.d.o.* $\tilde{Q}$ whose essential support is near $\{\nabla\tau \cdot \xi = 0\}$, the operator $\tilde{Q}Q_1$ is a smoothing operator. Accordingly, by linearity, the solution to (5.3) may also be decomposed into two pieces,

$$v = v_1 + v_2\ ,$$

where v_i (for $i = 1, 2$) satisfies

$$\begin{aligned} &(\Box - \nabla\sigma_0 \cdot \nabla)v_i = \nabla Q_i \delta\sigma \cdot \nabla v_0\ , \quad (x,t) \in \mathbb{R}^{n+1} \\ &v_i = 0\ , \quad t < 0\ . \end{aligned} \tag{5.9}$$

Therefore, in order to estimate $||\phi v||_{l_1}$, it suffices to estimate $||\phi v_i||_{l_1}$ for $i = 1, 2$. In the rest of this chapter, we shall proceed to estimate the two terms separately because of their different natures.

The analysis of v_1's regularity is parallel to that in Sections 4.2-4.3. From (5.9), Hadamard's construction again leads to the progressing wave expansion of v_1,

$$v_1 = \sum_{k=0}^{s} a_k S_k(t - \tau(x)) + R_{v_1}(x, t) \ , \tag{5.10}$$

where $\tau(x) = |x|$, S_0 is the Heaviside function, $S_k' = S_{k-1}$, R_{v_1} vanishes at $t = \tau(x)$, and $\{a_k\}$ solve the transport equations, for $k = 0, \cdots, s-1$,

$$2\nabla\tau \cdot \nabla a_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)a_0 = -b_0 \nabla\tau \cdot \nabla Q_1 \delta\sigma \tag{5.11}$$

$$2\nabla\tau\cdot\nabla a_{k+1} + (\Delta\tau + \nabla\tau\cdot\nabla\sigma_0)a_{k+1} = \Delta a_k + \nabla\sigma_0\cdot\nabla a_k + \nabla Q_1\delta\sigma(\nabla b_k - b_{k+1}\nabla\tau) \tag{5.12}$$

and b_k as defined in Section 4.3 solves the k-th transport equation (4.6) for v_0.

In order to get the regularity for v_1 we attempt to bound the energy norm by the energy identity Proposition 4.1 stated in Section 4.3. Since the whole process is so similar to the one in Sections 4.3-4.4, we think it is appropriate to only point out the major differences.

Now we can read out the regularity for v_1.

Lemma 5.2 Suppose that $\alpha > n/4$, $l \geq 1 + \alpha$, or $\alpha = n/4$, $l > 1 + \alpha$; then

$$||v_1||_{l_1} \leq C||\delta\sigma||_{l_1} \tag{5.13}$$

holds, where constant C depends on the $H^{l_1+\alpha} \cap H^{2l_1-1}_{m\ell}(\theta)$-norm of σ_0, and θ is a small conic neighborhood of $\{(x,\xi) \in T^*(\mathbb{R}^n), \nabla\tau \cdot \xi = 0\}$.

Proof Since the proof follows the same pattern as in Section 4.4, we shall only make the following observation: Applying the same ideas as in Section 4.4, since σ_0 is smooth, in general one should expect an estimate of the following form

$$\begin{aligned} ||v_1||_{l_1} &\leq C||Q_1\delta\sigma||_{l_1} + C||PQ_1\delta\sigma||_{2l_1} \\ &= C||Q_1\delta\sigma||^P_{l_1,2l_1} \end{aligned}$$

where C depends on σ_0, P is a $\psi.d.o.$ of order zero, and $ES(P)$ near $\{\nabla\tau \cdot \xi = 0\}$. However our construction of Q_1 implies that PQ_1 is a smoothing operator which is why we call v_1 the good part.

5.4 Microlocal version of trace theorem

In order to estimate the term $||v_2|_{x_n=0}||_{l_1}$, a microlocal version of the classical trace theorem is necessary.

The classical trace theorem in Sobolev spaces characterizes the regularity of a distribution restricted to a hypersurface. Dealing with inverse problems, one always has to face a difficult but crucial question: When does the restriction operator commute with another operator of interest? The result in this section indicates that a simple microlocal trace theorem, which not only works on the space restriction but also on the phase space restriction (*i.e.* a trace theorem on cotangent bundles), may lead to a way to cure the difficulty. Let $K \in \mathbb{R}^n$, $i : x \in \mathbb{R}^n \to (x,0) \in \mathbb{R}^{n+1}$. Define a semi-norm:

$$|u|_{K,s} = (\int_{\xi \in K} d\xi |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s})^{1/2} .$$

Then, a proof of the classical trace theorem (see *e.g.* in Taylor [39], pages 20-21) implies the following inequality.

Proposition 5.1 For $s > 1/2$, $u \in C_0^\infty$,

$$|i^* u|_{K,s-1/2} \leq C|u|_{K\times\mathbb{R},s} .$$

Thus the map i^* may be extended to be a bounded map from $H^s_{m\ell}(x \times \mathbb{R}, K \times \mathbb{R})$ to $H^{s-1/2}_{m\ell}(x, K)$, provided $s > 1/2$.

Once again, let Π_2 be the projection map to the frequency space (or the second factor). We may reformulate this result in terms of ψ*.d.o.*.

Proposition 5.2 If P_1 is a ψ*.d.o.* of order zero in $\mathbb{R}^n$, with $\Pi_2 ES(P_1) \subset K$, then there exists a ψ*.d.o.* P_2 of order zero in $\mathbb{R}^{n+1}$, and $\Pi_2 ES(P_2) \subset K \times \mathbb{R}$, such that for $s > 1/2$, $u \in C_0^\infty$,

$$||P_1 i^* u||_{s-1/2} \leq C||P_2 u||_s ,$$

where i^* again denotes a restriction operator to a codimension one hypersurface.

The above results together with our Gårding's type result Lemma 3.3 yield a microlocal version of trace theorem.

Lemma 5.3 Assume that E is an elliptic operator of order m in $\mathbb{R}^{n+1} \times K \times \mathbb{R}$, $P \in OPS^0(\mathbb{R}^n)$ and $\Pi_2 ES(P) \subset K$. Then for $s > 1/2$, $u \in C_0^\infty$,

$$||Pi^*u||_{s-1/2} \leq C||Eu||_{s-m} + C||u||_r$$

for any $r \in \mathbb{R}$.

5.5 Dual problem

According to Proposition 5.2, under some appropriate hypotheses, bounding $||v_2|_{x_n=0}||_{l_1}$ is equivalent to bounding $||v_2||_{l_1}$. Recall that v_2 solves

$$\begin{aligned} &\Box v_2 - \nabla\sigma_0 \cdot \nabla v_2 = \nabla Q_2 \delta\sigma \cdot \nabla v_0 \\ &v_2 = 0 \quad t < 0 \,. \end{aligned} \tag{5.14}$$

To simplify the arguments on its dual problem, we make use of the symmetric form of (5.14) by introducing $\rho(x) = e^{\sigma_0}$. Then (5.14) becomes

$$\begin{aligned} &\Box_1 v_2 = [\frac{1}{\rho}\partial_t^2 - \nabla \cdot (\frac{1}{\rho}\nabla)]v_2 = \frac{1}{\rho}\nabla Q_2 \delta\sigma \nabla v_0 \\ &v_2 = 0 \quad t < 0 \,. \end{aligned} \tag{5.15}$$

Now let us look at a dual problem to (5.15),

$$\begin{aligned} &\Box_1' w = [\frac{1}{\rho}\partial_t^2 - \nabla \cdot (\frac{1}{\rho}\nabla)]w = \psi \\ &w = 0 \quad t >> T_1 \,, \end{aligned} \tag{5.16}$$

where $\psi \in C_0^\infty$ and $supp(\psi) \subset \mathbb{R}^n \times [0, T_1] \cap \{|t| < |x|\}$.

Equivalently, we may reformulate (5.16) as

$$\begin{aligned} &\Box_1' w = \Box w + \nabla\sigma_0 \cdot \nabla w = e^{\sigma_0}\psi \\ &w = 0 \quad t >> T_1 \,. \end{aligned} \tag{5.17}$$

Thus if we can show that

$$|(\partial_t^{l_1} v_2, \psi)| \le C||\delta\sigma||_{l_1}||\psi||_0 \,, \tag{5.18}$$

then it can be concluded that

$$||\partial_t^{l_1} v_2||_0 \le C||\delta\sigma||_{l_1} \,. \tag{5.19}$$

Lemma 5.4 Suppose that $\alpha \ge n/4$, $l_1 \ge max\{1+\alpha, 1+n/2+\delta\}$, $l_1 + (n-1)/2 < \tilde{s}$ and θ is a small conic neighborhood of $\{(x,\xi) \in T^*(\mathbb{R}^n), \nabla\tau\cdot\xi = 0\}$. Then the estimate (5.19) holds where the constant C depends on the $H^{\tilde{s}} \cap H^{2l_1-1}_{m\ell}(\theta)$–norm of σ_0.

Proof Green's identity and integration by parts lead to

$$\begin{aligned}
&(\partial_t^{l_1} v_2, \psi) = (v_2, \Box_1' \partial_t^{l_1} w) \\
&= (\Box_1 v_2, \partial_t^{l_1} w) - \int_{t=\tau(x)} \frac{1}{\rho}[v_2 \frac{\partial}{\partial n}\partial_t^{l_1} w - \partial_t^{l_1} w \frac{\partial}{\partial n} v_2] ds \\
&= (\frac{1}{\rho}\nabla Q_2 \delta\sigma \cdot \nabla v_0, \partial_t^{l_1} w) - \int_{t=\tau(x)} ds \frac{1}{\rho}[v_2(\partial_t^{l_1+1} - \nabla\tau\nabla\partial_t^{l_1})w - \partial_t^{l_1} w(\partial_t - \nabla\tau\nabla)v_2] \,.
\end{aligned} \tag{5.20}$$

The first term in (5.20) is easy to handle. Actually, integration by parts and a simple use of Cauchy-Schwarz inequality lead to

$$\begin{aligned}
|(\frac{1}{\rho}\nabla Q_2\delta\sigma\nabla v_0, \partial_t^{l_1} w)| &= |(e^{-\sigma_0}\nabla Q_2 \delta\sigma \partial_t^{l_1-1}\nabla v_0, \partial_t w)| \\
&\le C||e^{-\sigma_0}\nabla Q_2\delta\sigma\partial_t^{l_1-1}\nabla v_0||_0 ||\partial_t w||_0 \,.
\end{aligned} \tag{5.21}$$

The energy estimate on w gives $||\partial_t w||_0 \le C||\psi||_0$. We may apply the generalized Schauder's lemma (Lemma 2.4) twice to obtain

$$||e^{-\sigma_0}\nabla Q_2\delta\sigma\partial_t^{l_1-1}\nabla v_0||_0 \;\le\; C||\nabla Q_2\delta\sigma||_{s_0}||e^{-\sigma_0}\partial_t^{l_1-1}\nabla v_0||_0$$

$$\leq \; C||\delta\sigma||_{s_0+1}||v_0||_{l_1} \;, \tag{5.22}$$

where $s_0 > n/2$, and $||v_0||_{l_1}$ can be handled by Theorem 4.2, provided that $\sigma_0 \in H^{l_1+\alpha} \cap H^{2l_1-1}_{m\ell}(\theta)$ and $\alpha > n/4$, $l_1 \geq 1+\alpha$, or $\alpha = n/4$, $l_1 > 1+\alpha$.

Thus it suffices to estimate the last two terms in (5.20). As usual, one may write down the progressing wave expansion for v_2. Actually, assuming that c_i solves the i-th transport equation ($i = 0, 1$), we have

$$2\nabla\tau \cdot \nabla c_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)c_0 = -b_0\nabla\tau \cdot \nabla Q_2\delta\sigma \tag{5.23}$$

$$2\nabla\tau \cdot \nabla c_1 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)c_1 = \Delta c_0 + \nabla\sigma_0 \cdot \nabla c_0 + \nabla Q_2\delta\sigma(\nabla b_0 - b_1\nabla\tau)\;. \tag{5.24}$$

Hence to control the last part of (5.20) we only need to analyze

$$I_0 = \int_{t=\tau(x)} c_0(\partial_t^{l_1+1} - \nabla\tau \cdot \nabla\partial_t^{l_1})w - c_1\partial_t^{l_1}w\;. \tag{5.25}$$

Since Q_2 is symmetric in the sense that $Q_2^* = Q_2$, the Cauchy-Schwarz inequality deduces

$$\begin{aligned} |I_0| \;&\leq\; C||\delta\sigma||_0[||Q_2 f(\sigma_0)Tr(\partial_t^{l_1}P_1 w)||_0 + ||Q_2 g(\sigma_0)Tr(\partial_t^{l_1}w)||_0] \\ &=\; C||\delta\sigma||_0 I_1\;, \end{aligned}$$

with P_1 a first order differential operator (or a linear combination of operators ∂_t and $\nabla\tau\nabla$). $Tr(u) = u|_{t=\tau(x)}$ is a restriction (trace) operator and f, g are smooth functions determined by (5.23), (5.24). It is not difficult to see that f only depends on σ_0, while g involves $\sigma_0, D\sigma_0$ and $\Delta\sigma_0$.

From Lemma 5.3, we know that there is a $\psi.d.o.$ $\tilde{Q}_2$ of order zero whose essential support is contained in a "cylindrical" conic neighborhood of $ES(Q_2)$ along ω-direction, such that $\Pi supp(\tilde{q}_2)$ is near the characteristic surface and $\Pi supp(q_2) \cap supp(\psi) = \emptyset$. That is,

$$|I_1| \leq C||\tilde{Q}_2 f(\sigma_0) P_1 w||_{l_1+1/2+} + C||\tilde{Q}_2 g(\sigma_0) w||_{l_1+1/2+} \,. \tag{5.26}$$

Thus Lemma 2.5 (the generalized Rauch's lemma) and the estimates involved in the proof imply that for $l_1 + 1/2 + n/2 < s_0$, $l_1 - 1/2 + n/2 < s_1$,

$$\begin{aligned} ||\tilde{Q}_2 f(\sigma_0) P_1 w||_{l_1} &\leq C_1(||w||_1 + ||Q_0 w||_{l_1+3/2+}) \\ ||\tilde{Q}_2 g(\sigma_0) w||_{l_1} &\leq C_2(||w||_1 + ||Q_0 w||_{l_1+1/2+}) \end{aligned}$$

with C_1, C_2 depending on $||\sigma_0||_{s_0}$ and $||\sigma_0||_{s_1}$ respectively, $Q_0 \in OPS^0$, $ES(Q_0)$ is near $ES(\tilde{Q}_2)$, and $\Pi supp(q_0) \cap supp(\psi) = \emptyset$.

Hence to finish the proof of Lemma 5.4 it is sufficient to show that

$$||Q_0 w||_{l_1+3/2+} \leq C||\psi||_0 \tag{5.27}$$

which can be proved by applying Lemma 5.5 below. The lemma (Lemma 5.5) will be proved in the remaining sections of this chapter. □

5.6 Regularity for solution of the dual problem

A result on propagation of singularities in Duistermaat's notes [11], Proposition 1.3.3 (see also Theorem 8.2.13 in Hörmander's book [17]) demonstrates the relation between the wavefront of the restriction of a distribution and the wavefront set of its own.

Applying this result and Hömander's theorem on propagation of singularities, it is easy to see that $Q_2 Tr(\partial_t^{l_1} P_1 v_2)$ is smooth. However the result does not directly lead to any explicit bound. In this section, we shall derive the necessary estimates by using a bootstrap argument. Our idea here is motivated by Nirenberg's proof of Hörmander's theorem on propagation of singularities. In fact, the main purpose of this section is to obtain a real estimate out of his proof.

Lemma 5.5 There exists an elliptic *ψ.d.o.* $\tilde{B}$ of order zero, such that $ES(\tilde{B})$ is contained in Cy, a "cylindrical" conic neighborhood of

$$\{(x,t,\xi,\omega) \in T^*\mathbb{R}^{n+1} \backslash 0,\ t^2 - |x|^2 = 0, \omega = \nabla\tau \cdot \xi\}$$

along ω direction, and the symbol of $\tilde{B}$, $\tilde{b}$ satisfies

$$\Pi supp(\tilde{b}) \cap supp(\psi) = \emptyset\ .$$

Then, for any $k \in \mathbb{R}$, $\phi \in C_0^\infty(\mathbb{R}^{n+1})$

$$||\phi \tilde{B} w||_k \leq C_k ||\psi||_0\ , \tag{5.28}$$

where the constant C depends on $||\sigma_0||_s$, $k - 2 + n/2 < s$.

The proof follows by showing two propositions below. Proposition 5.3 really gives an estimate based on Nirenberg's proof of Hörmander's theorem. It indicates that an estimate may be formed near any bicharacteristic, hence near the characteristic variety of operator $\Box = \partial_t^2 - \Delta$. We then proceed in Proposition 5.4 to argue that the remaining part of the cylindrical region, where the operator $\Box$ is elliptic, causes no trouble at all. With a concern about the nonsmooth σ_0, it should not be surprising that both propositions require the commutator lemma proved in Chapter 2.

Let β be a null bicharacteristic contained in Cy.

Proposition 5.3 There exists a *ψ.d.o.* B of order zero such that B is supported in a small conic neighborhood of β and B is elliptic near β, $\Pi supp(b) \cap supp(\psi) = \emptyset$. If, furthermore, $k - 2 + n/2 < s$, then the estimate

$$||Bw||_k \leq C_k ||\psi||_0$$

holds with C_k depending on $||\nabla\sigma_0||_s$.

Proof According to Nirenberg's construction, one can find a *ψ.d.o.* B_0 of order zero with

(1) b_0 supported in a small conic neighborhood of β, B_0 elliptic near β,

(2) $\Pi supp(b_0) \cap supp(\psi) = \emptyset$, and

(3) $[\Box, B_0] \in OPS^0$.

Since w solves (5.17), the method of energy estimates yields

$$||w||_1 \leq C||\psi||_0\ ,$$

where C is a constant depending on $||\nabla\sigma_0||_{\tilde{s}}$ for $\tilde{s} > n/2$.

Observe that from (5.17),

$$\Box_1' B_0 w = [\Box, B_0]w - [B_0, \nabla\sigma_0 \cdot \nabla]w + B_0 e^{\sigma_0}\psi \ .$$

Since ψ is supported inside the characteristic surface, $\Pi supp(b_0) \cap supp(\psi) = \emptyset$, we have

$$B_0 e^{\sigma_0}\psi = 0 \ .$$

Now energy estimates give

$$||B_0 w||_2 \leq C(||[\Box, B_0]w||_1 + ||[B_0, \nabla\sigma_0 \cdot \nabla]w||_1 \ . \tag{5.29}$$

Since $[\Box, B_0]$ is of order 0,

$$||[\Box, B_0]w||_1 \leq C||w||_1 \leq C||\psi||_0 \ .$$

The second term in (5.29) may be estimated by applying the generalized commutator lemma (Lemma 2.6 or more precisely Proposition 2.7) by choosing $q = 1, l = 1, 1 + n/2 < s_0$ in Proposition 2.7, so that

$$||[B_0, \nabla\sigma_0 \cdot \nabla w]||_1 \leq C||w||_1 \leq C||\psi||_0 \ ,$$

where C depends on $||\nabla\sigma_0||_{s_0}$.

Thus

$$||B_0 w||_2 \leq C_0||\psi_0|| \ , \tag{5.30}$$

with C_0 depending on $||\nabla\sigma_0||_{s_0}$.

Applying Nirenberg's construction once again, we can find a $\psi.d.o.$ B_1 such that $ES(B_1) \subset ES(B_0)$(strictly), B_1 also has properties (1) and (2) above; moreover $[\Box, B_1] \in OPS^{-1}$ and B_0 is elliptic near $ES(B_1)$. From (5.17) and $B_1 e^{\sigma_0}\psi = 0$,

$$\Box_1' B_1 w = [\Box, B_1]w - [B_1, \nabla\sigma_0 \cdot \nabla]w \ .$$

If we write down the energy estimates, after a simple $\psi.d.o.$ cut-off on B_1 we will find

$$||B_1 w||_3^2 \leq C||w||_1^2 + C||A_1[B_1, \nabla\sigma_0 \cdot \nabla]w||_2 ||B_1 w||_3 \ ,$$

where $A_1 \in OPS^0$, $ES(B_1) \subset ES(A_1) \subset ES(B_0)$, B_0 is elliptic on $ES(A_1)$, and $a_1 = 1$ on $ES(B_1) \cap \{(x,\xi), |\xi| \geq 1\}$.

Now since $w \in H^1 \cap H^2_{m\ell}(ES(B_0))$, Proposition 2.8 implies that $[B_1, \nabla\sigma_0 \cdot \nabla]w \in H^1 \cap H^2_{m\ell}(ES(A_1))$ and

$$||A_1[B_1, \nabla\sigma_0 \cdot \nabla]w||_2 \leq C(||w||_1 + ||A_1 w||_2) \ .$$

Here C depends on $||\nabla\sigma_0||_{s_1}$ for $2 + n/2 \leq s_1$.

Because of our construction, B_0 is elliptic on $ES(A_1)$; therefore Gårding's type inequality Lemma 3.3 leads to, for any real r,

$$||A_1 w||_2 \leq C||B_0 w||_2 + C||w||_r \leq C||\psi||_0$$

by (5.30).

Therefore we have shown that

$$||B_1 w||_3 \leq C_1||\psi||_0 \ ,$$

where C_1 depends on $||\nabla\sigma_0||_{s_1}$.

We can continue this process by constructing a sequence of ψ.*d.o.* B_i, A_1 ($i = 1, \cdots, k-2$), such that

- B_i has properties (1), (2), $[\Box, B_i] \in OPS^{-i}$,
- $ES(B_i) \subset ES(A_i) \subset ES(B_{i+1})$, and
- B_{i+1} is elliptic on $ES(A_i)$, $a_i = 1$ on $ES(B_i) \cap \{(x,\xi), |\xi| \geq 1\}$,
- Also

$$||B_i w||_{i+2} \leq C_i ||\psi||_0 ,$$

where C_i depends on $||\nabla\sigma_0||_{s_i}$ for $i + n/2 < s_i$.

Eventually we conclude by choosing $B = B_{k-2}$ so that, for $k - 2 + n/2 < s$,

$$||Bw||_k \leq C||\psi||_0$$

with C depending on $||\nabla\sigma_0||_s$. $\Box$

Proposition 5.4 Let P be a ψ.*d.o.* of order zero with the following properties: The wave operator $\Box$ is elliptic in a small conic neighborhood of $ES(P)$ and $\Pi supp(p) \cap supp(\psi) = \emptyset$. Then

$$||Pw||_k \leq C||\psi||_0 ,$$

where C depends on $||\nabla\sigma_0||_q$ for $k - 2 + n/2 < q$.

Proof The proof is based on the same type of bootstrap arguments as in the proof of last proposition.

Recall (5.17)

$$\Box w + \nabla\sigma_0 \cdot \nabla w = e^{\sigma_0}\psi \ . \tag{5.31}$$

From the support assumption on p, we see that $Pe^{\sigma_0}\psi = 0$. Hence, by applying P to both sides of (5.31), we find

$$\Box Pw = [\Box, P]w - [P, \nabla\sigma_0 \cdot \nabla]w - \nabla\sigma_0 \cdot \nabla Pw \ . \tag{5.32}$$

Now since $\Box$ is elliptic in a small conic neighborhood of $ES(P)$, there exists a $\psi.d.o.$ P_0 of order zero, such that $ES(P) \subset ES(P_0)$, P_0 is elliptic near $ES(P)$, and $\Box$ is elliptic in a small conic neighborhood of $ES(P_0)$. From the ellipticity of $P_0\Box$ on $ES(P)$, Proposition 3.3 gives, for any real number r,

$$||Pw||_k \leq C||P_0\Box Pw||_{k-2} + C||w||_r \ ,$$

or from (5.32)

$$||Pw||_k \leq C(||P_0[\Box, P]w||_{k-2} + ||P_0[P, \nabla\sigma_0 \cdot \nabla]w||_{k-2} + ||P_0\nabla\sigma_0 \cdot \nabla Pw||_{k-2}) \ .$$

Therefore an application of Proposition 2.8 and Lemma 2.5 yields

$$\begin{aligned} ||Pw||_k &\leq C_1||P_0w||_{k-1} + C_2(||w||_1 + ||P_0w||_{k-2}) + C_3(||w||_1 + ||P_0w||_{k-1}) \\ &\leq C||\psi||_0 + C||P_0w||_{k-1} \ . \end{aligned}$$

Here constants C_2 and C_3 depend on $||\nabla\sigma_0||_q$ for $k - 2 + n/2 < q$.

Thus the bootstrap arguments on P_0 will accomplish the proof. $\square$

A combination of Propositions 5.3, 5.4 and Gårding's type Lemma 3.3 assures the existence of an elliptic operator B with properties stated in Lemma 5.5. Near the characteristic variety of $\square$ in the cylindrical region Proposition 5.3 and an extension of Lemma 3.2 may be used, while away from its characteristic set operator $\square_1'$ is microlocally elliptic, hence Proposition 5.4 becomes applicable.

5.7 Proof of the claim

We conclude this chapter by proving an earlier claim.

Claim 5.1 Assume that v_2 solves equation (5.14), $l_1 \in \mathbb{R}$, $l_1 - 3/2 + n/2 < s$. Then the following estimate holds:

$$||v_2|_{x_n=0}||_{l_1} \leq C||\partial_t^{l_1} v_2|_{x_n=0}||_0 + C||\delta\sigma||_{l_1} \,, \tag{5.33}$$

with the constant depending on $||\sigma_0||_s$.

Proof We first construct a $\psi.d.o.$ $A \in OPS^0$ such that a, the symbol of A, is one on $|\omega| \geq \epsilon|\xi'|$, for $\xi = (\xi', \xi_n)$, and $ES(A) \subset \{|\omega| \geq \epsilon_0|\xi'|$, with $\epsilon > \epsilon_0\}$. Denote Tr as the restriction operator to $\{x_n = 0\}$; then we have

$$Tr(v_2) = v_2|_{x_n=0} = ATr(v_2) + (I - A)Tr(v_2)$$

or

$$||Tr(v_2)||_{l_1} \leq ||ATr(v_2)||_{l_1} + ||(I - A)Tr(v_2)||_{l_1} \,.$$

Since the operator $\partial_t^{l_1}$ is elliptic on $ES(A)$, a simple use of Lemma 3.3 leads to

$$||ATr(v_2)||_{l_1} \leq C||\partial_t^{l_1}Tr(v_2)||_0 + C||Tr(v_2)||_r$$

for any $r \in \mathbb{R}$.

On the other hand, the microlocal trace theorem implies that there exists a $\psi.d.o.$ $\tilde{A}$ of order zero such that $ES(\tilde{A}) \subset$ a cylindrical neighborhood of $\{|\omega| \leq \epsilon|\xi'|\}$ along the ξ_n-direction, and

$$||(I-A)Tr(v_2)||_{l_1} \leq C||\tilde{A}v_2||_{l_1+1/2} \, .$$

Therefore similar arguments as in the preceding section yield

$$||\tilde{A}v_2||_{l_1+1/2} \leq C||\delta\sigma||_{l_1}$$

with the constant C depending on $||\sigma_0||_s$, for $l_1 - 3/2 + n/2 < s$.

Combining the above discussions, we have proved the claim. □

Chapter 6

Concluding Remarks

6.1 Conclusion

In this thesis work, we introduce the methods of nonsmooth microlocal analysis to the study of multidimensional hyperbolic inverse problems. Through the study several trace regularity results are established, which turn out (as expected) to be crucial in the investigation of the inverse problem. In particular, we develop the first trace regularity result for the variable coefficients, see also [2]. A new regularity result is also established for the forward map. In this process, some fundamental results of linear microlocal analysis are examined. Rauch's lemma on the algebraic property of microlocal analysis and a Beals-Reed linear propagation of singularities theorem with nonsmooth coefficients at lower order terms are extended. We also determine the microlocal regularity of the fundamental solutions to second order hyperbolic *p.d.e.*. Our results show that the microlocal Sobolev spaces may be the right spaces to work with in the study of inverse problems for multidimensional wave propagation. The results in this study clearly exhibit the substantial differences between layered and nonlayered problems.

In the rest of this chapter, we shall comment on the computational aspect of the inverse problem and propose some related open problems.

6.2 Remark on computation

In this section, we briefly discuss the numerical indications of our regularity results. It is evident that the least-squares approach is relevant to solve the inverse problem numerically, *i.e.*,

$$\min_{\sigma} \|F(\sigma) - F_{data}\| \tag{6.1}$$

for some suitable norm " $\|\cdot\|$ ". It is, perhaps, not so evident that even for layered problems the least-squares approach leads to a very difficult optimization problem. As explained in detail by Santosa and Symes in [30], the difficulties are essentially caused by the nonconvexity of the objective function so that the problem often has many minima. To overcome this obstacle, a very delicate approach, "the coherency optimization method", was proposed by them to convexify the least-squares problem based on the known regularity results of the forward map. We refer to [30] for further discussions and results of solving layered problems.

Similarly, for nonlayered problems, an analogous method may be proposed to regularize this ill-posed least-squares problem. In stead of working on (6.1), we look at a perturbed optimization problem,

$$\min_{\sigma}(\|F(\sigma) - F_{data}\|^2 + \alpha\|\tilde{Q}\sigma\|^2) \tag{6.2}$$

where $\tilde{Q}$ is a $\psi.d.o.$ of order zero, and α is a small positive constant. We anticipate that in this way the problem will be regularized as for the layered problem, so that the convexity and stability will follow. However, we do not know any numerical experiments for nonlayered problems based on the ideas above. In any case, we believe that our regularity result holds some promise to the design and implementation of algorithms for solving the inverse problem.

6.3 Future work

As we mentioned at the beginning of this thesis very little is known in mathematics about nonlayered multidimensional inverse problems. There are a great deal of open problems to be addressed. It seems that our results as well as the techniques in this work have the potential to demonstrate some rather difficult situations such as nonlayered velocity inversion problems with nonsmooth background velocity and mixed forward problems. Most importantly, there is a real challenge, which is related to this work to some extent, concerning the development of a *generalized travel time transformation* to nonlayered media, see Symes [36]. Finally a natural question is how one can interpret the results from the geophysical point of view?

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