Conformal Mapping and Fluid Mechanics

Monday, December 02, 2013
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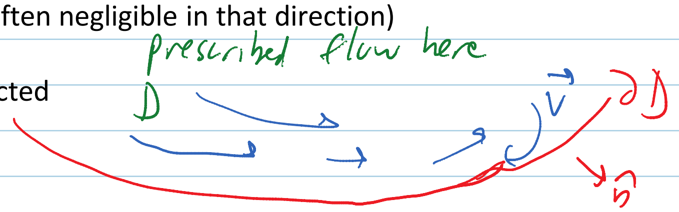
Homework 4 due (hard deadline) Wednesday, December 11 at 5 PM.

- more problems available
- rapid grading window: Friday, December 5 - Wednesday, December 11

Final exam on Thursday, December 12 at 6:30 PM.

Complex variable techniques can be used in clever ways to analyze problems in fluid mechanics in two-dimensional domains, when the flow is incompressible (subsonic) ($\nabla \cdot v = 0$), irrotational ($\nabla \times v = 0$), and steady (no time-dependence) where $v(\mathrm{x})$ is the fluid velocity.

Two-dimensional fluid mechanic problems are relevant when the fluid is thin in the third dimension (in which case the fluid velocity is often negligible in that direction) or otherwise uniform along the third dimension.
Can't get nontrivial flows in bounded, simply connected domains.



Also the velocity field has the property that on any rigid boundary of the domain D in which the fluid lies, we must have:

$$\vec{v}(\vec{x}) \cdot \hat{n}(\vec{x}) = 0 \qquad \text{for} \quad \vec{x} \in \partial D$$

unit normal to ∂D at $\vec{x}$

The key problem in such fluid mechanic problems is to describe the flow velocity field $v(x)$ given the shape of the domain and possibly some "far-field" boundary conditions if the domain is unbounded in some direction.

A few comments:

- If the domain D is simply connected then the irrotational (curl-free) condition implies the existence of what's known as a velocity potential: $v = \nabla\phi$.
- The incompressibility condition implies the existence of a stream function ψ

$$\vec{v} = \vec{\nabla}^{\perp} \psi = \begin{pmatrix} \partial\psi/\partial y \\ -\partial\psi/\partial x \end{pmatrix} = \vec{\nabla} \times (\psi \hat{e}_3)$$

- Notice that the equations for the fluid flow can be expressed in terms of the velocity potential and stream function as:

$$\Delta \phi = 0 \qquad \vec{\nabla}\phi(\vec{x}) \cdot \hat{n}(\vec{x}) = 0 \quad \text{for} \quad \vec{x} \in \partial D$$

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$$\Delta \psi = 0 \qquad \vec{\nabla}^{\perp}\psi(\vec{x}) \cdot \hat{n}(\vec{x}) = 0 \quad \text{for} \quad \vec{x} \in \partial D$$

$\Rightarrow$ ψ is constant over any smooth component of ∂D

$$\vec{\nabla}\psi \cdot \vec{\nabla}^{\perp}\psi = 0$$

Let's map these concepts into complex analysis, and see what it can do for us in helping to solve for these ideal two-dimensional fluid flows.

Define the complex velocity potential $\Omega(z) = \phi(x,y) + i\,\psi(x,y)$ where $z = x + iy$.

$$\vec{V}(\vec{x}) = \begin{pmatrix} V_1(x,y) \\ V_2(x,y) \end{pmatrix}$$

$$V_1(x,y) = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$V_2(x,y) = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

This is just the Cauchy-Riemann equations, meaning that the complex velocity potential $\Omega(z)$ for our ideal fluid flow is an analytic function.

The derivative of the complex velocity potential:

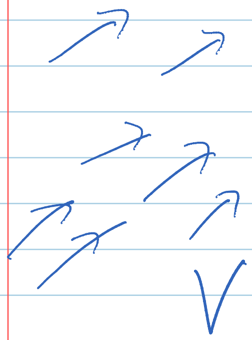
$$\frac{d\Omega(z)}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = V_1 + i(-V_2)$$

$$= V_1 - iV_2$$

$$V_1 + iV_2 = \overline{\frac{d\Omega}{dz}}$$

We see that the complex velocity potential must be an analytic function respecting the boundary conditions, and once we have it, we can easily obtain the flow field. Let's see how we can use this fact to solve some basic fluid mechanics problems.

Example A: Uniform free space flow: $\vec{V} = \vec{V}$: constant vector



In complex form:

$$V_1 + iV_2 = V_1 + iV_2 = \overline{\frac{d\Omega}{dz}}$$

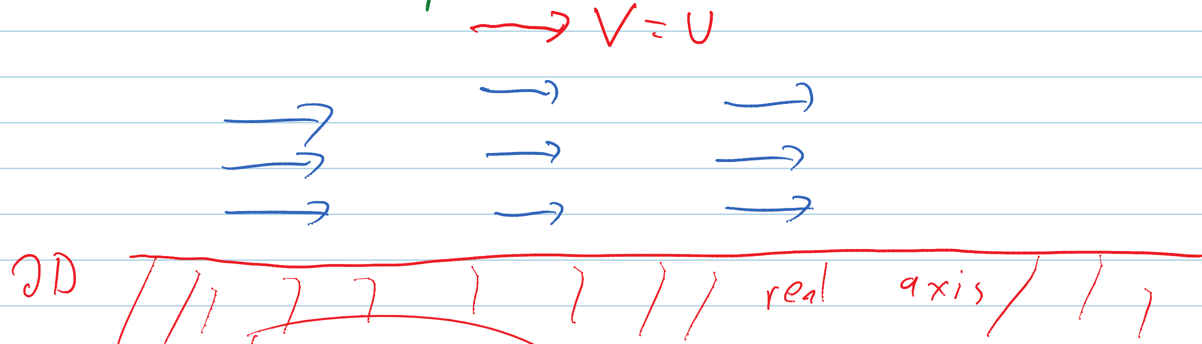
$$\| \quad V = \text{complex number}$$

$$\overline{\frac{d\Omega}{dz}} = V$$

$$\frac{d\Omega}{dz} = \overline{V}$$

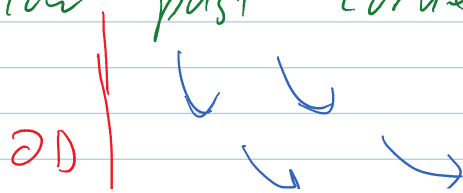
$$\boxed{\Omega = \overline{V} z}$$

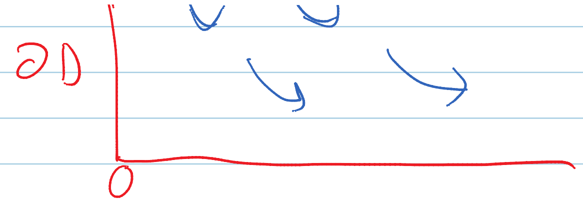
Example B: Flow past flat wall



$\boxed{\Omega = Uz}$ where U is real.

Example C: Flow past corner

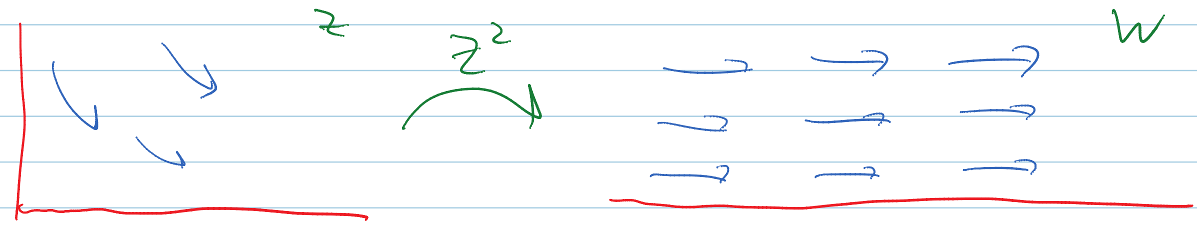




What is a consistent flow pattern past a corner according to the ideal fluid conditions?

$$\Omega = U z^2$$

Why does this work? This is a simple illustration of the principle of conformal mapping. One uses analytic functions to map a fluids problem (or more generally a Laplace equation problem) from a given domain to a domain on which the problem is solved.



$$\Omega(z) = \tilde{\Omega}(z^2)$$

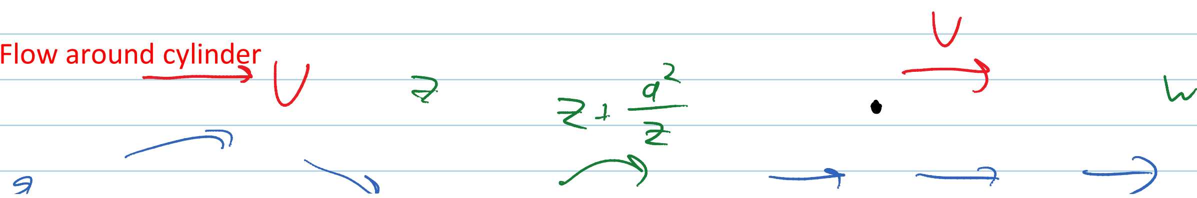
Known solution $\tilde{\Omega}(w)$

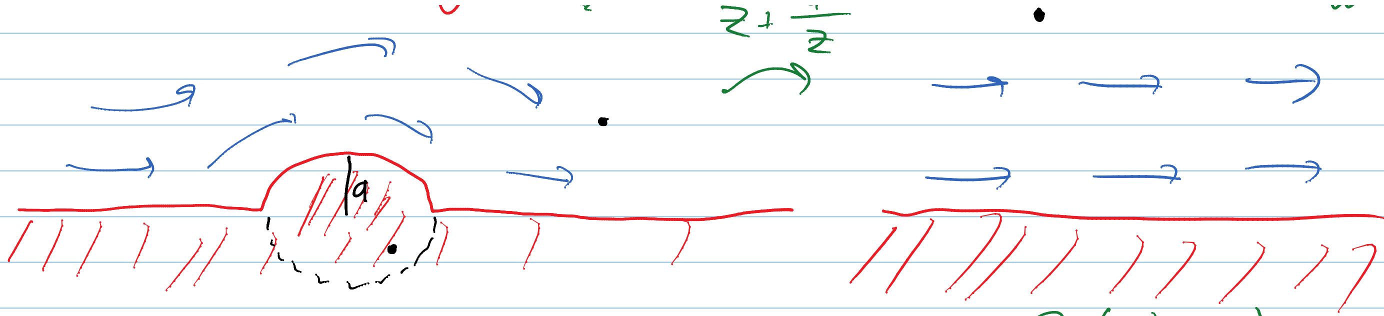
pull back to solution here

Why does this pullback of a solution through an analytic mapping solve the given problem:

- The pullback function is analytic on the prescribed domain because it's a composition of analytic functions.
- Boundary conditions are OK because analytic mappings are conformal, meaning they preserve relationships of angles.
- Conformal mapping works more broadly on problems involving Laplace's equation, because it is invariant under conformal mapping.

Example D: Flow around cylinder





$$\tilde{\Omega}(w) = Uw$$

Let's see why this mapping works. Easier to see by inversion:

$$w = z + \frac{a^2}{z}$$

$$z^2 - zw + a^2 = 0$$

$$z = \frac{w + (w^2 - 4a^2)^{1/2}}{2}$$

This gives us two roots, whose product is a^2. Therefore either both roots are on the circle of radius a about the origin, or one is inside the circle and one is outside.

Let's see where the boundary itself maps. Notice that if z is real, then its image under the mapping is real (i.e., on the boundary). As for the upper semicircle, we can parameterize it by $z = ae^{i\theta}$

$$w = z + \frac{a^2}{z} = ae^{i\theta} + \frac{a^2}{ae^{i\theta}} = ae^{i\theta} + ae^{-i\theta}$$

$$= \text{real}$$

So we see that the boundary in the given domain maps into the real axis (the boundary of the simpler domain in w plane), and one can check directly that this mapping is 1-1.

The only thing that's left to do is to check that the mapping actually takes the interior of the given domain to the interior of the target domain. To do this, note that the given domain can be defined:

$$D = \{z \in \mathbb{C} : |z| > a,\ \mathrm{Im}\, z > 0\}$$

$$= \{z = re^{i\theta} : r > a,\ 0 < \theta < \pi\}$$

Under the mapping $w = z + \frac{a^2}{\ }$

$$w = re^{i\theta} + \frac{a^2}{re^{i\theta}}$$
$$= re^{i\theta} + \frac{a^2}{r} e^{-i\theta}$$

$$\operatorname{Im} w = r \sin\theta - \frac{a^2}{r} \sin\theta$$
$$= \left(r - \frac{a^2}{r}\right) \sin\theta$$
$$> 0 \quad \text{because } r > a,\ 0 < \theta < \pi$$

And one can show that the mapping is surjective (onto) because every other domain in the wall in the given domain can be accounted for via the above observation of 2-1 mapping:

$$\{z = re^{i\theta} : r > a,\ 0 < \theta < \pi\} \to \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$$
$$\{z = re^{i\theta} : r < a,\ \pi < \theta < 2\pi\} \to \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$$
$$\{z = re^{i\theta} : r < a,\ 0 < \theta < \pi\} \to \{w \in \mathbb{C} : \operatorname{Im} w < 0\}$$
$$\{z = re^{i\theta} : r > a,\ \pi < \theta < 2\pi\} \to \{w \in \mathbb{C} : \operatorname{Im} w < 0\}$$

Now just need to check that the far field conditions (velocity = U for large $\Im z$). Let's do this after the fact.

Therefore we compose the solution on the simple flat wall domain with our analytic mapping to get:

$$\Omega(z) = \tilde{\Omega}\left(w = z + \frac{1}{z}\right) = U\left(z + \frac{1}{z}\right)$$

Check far-field conditions:

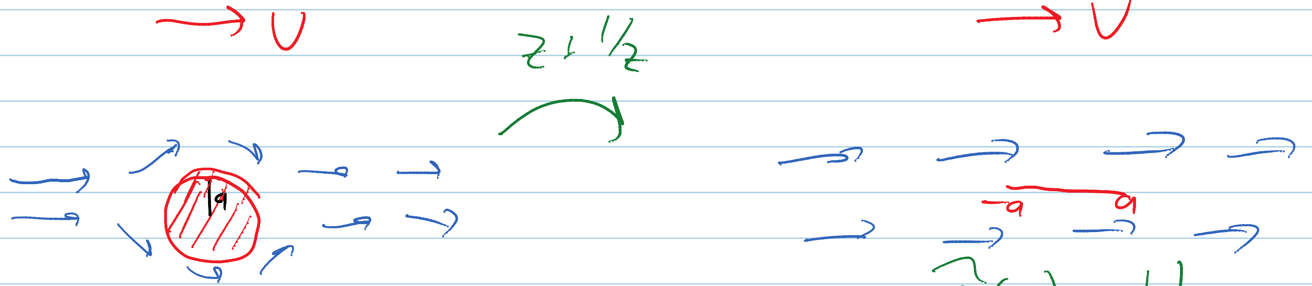
Check far-field conditions:

$$V = \overline{\Omega'(z)} = \overline{U(1 - 1/z^2)} = U\left(1 - \frac{1}{\bar{z}^2}\right)$$

$$\lim_{\substack{\operatorname{Im} z \to \infty}} V = U$$

So we have successfully solved this flow around a cylinder attached to a wall.

Closely related example: Flow past a free cylinder:

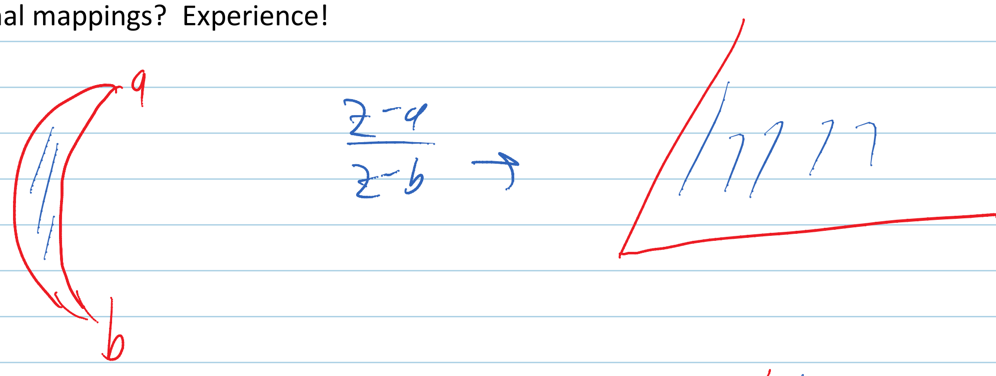


$$\tilde{\Omega}(w) = Uw$$

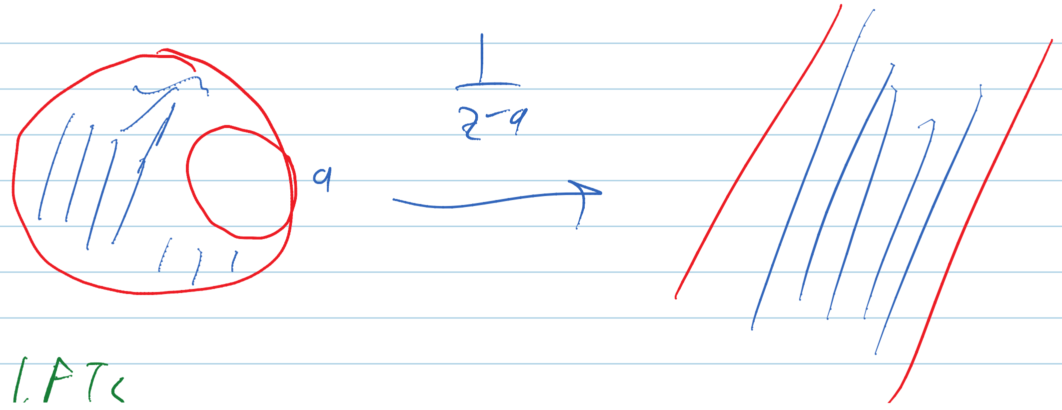
$$\Omega(z) = \tilde{\Omega}(z + 1/z) = U(z + 1/z)$$

How do we guess conformal mappings? Experience!

LFTs:



$$\frac{z-a}{z-b} \rightarrow$$



$$\frac{1}{z-a} \longrightarrow$$

Other non-LFTs

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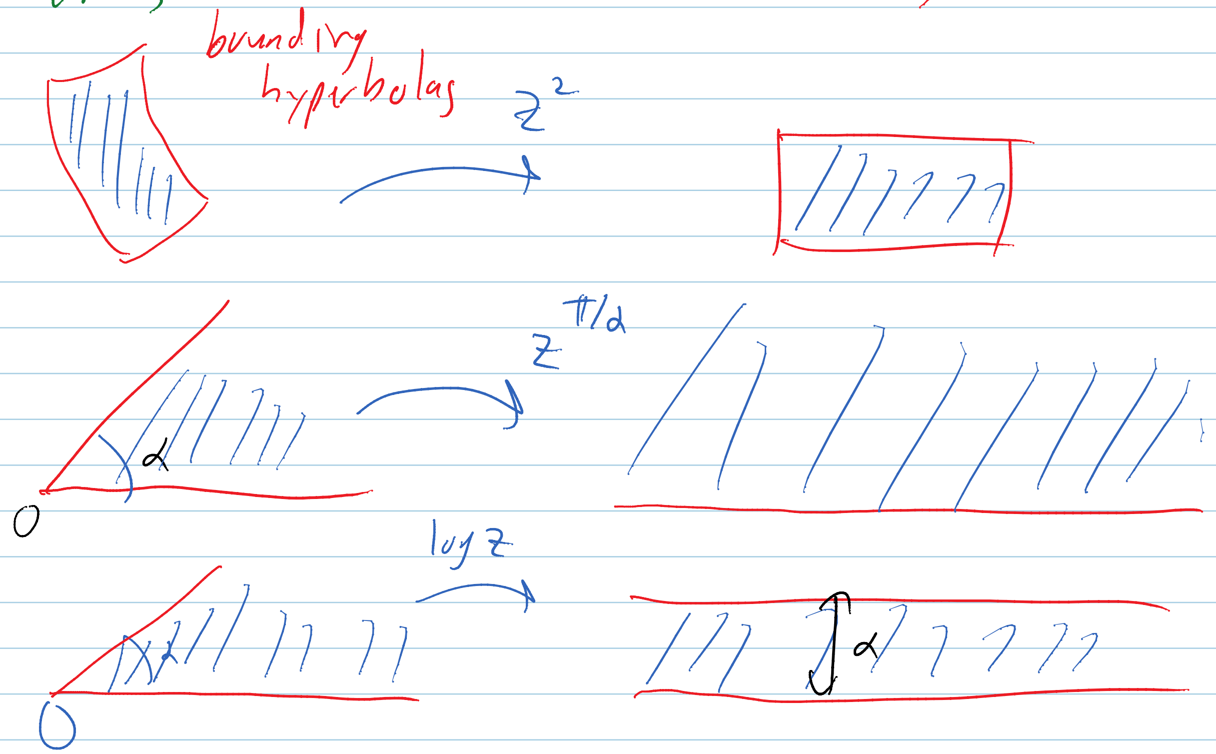
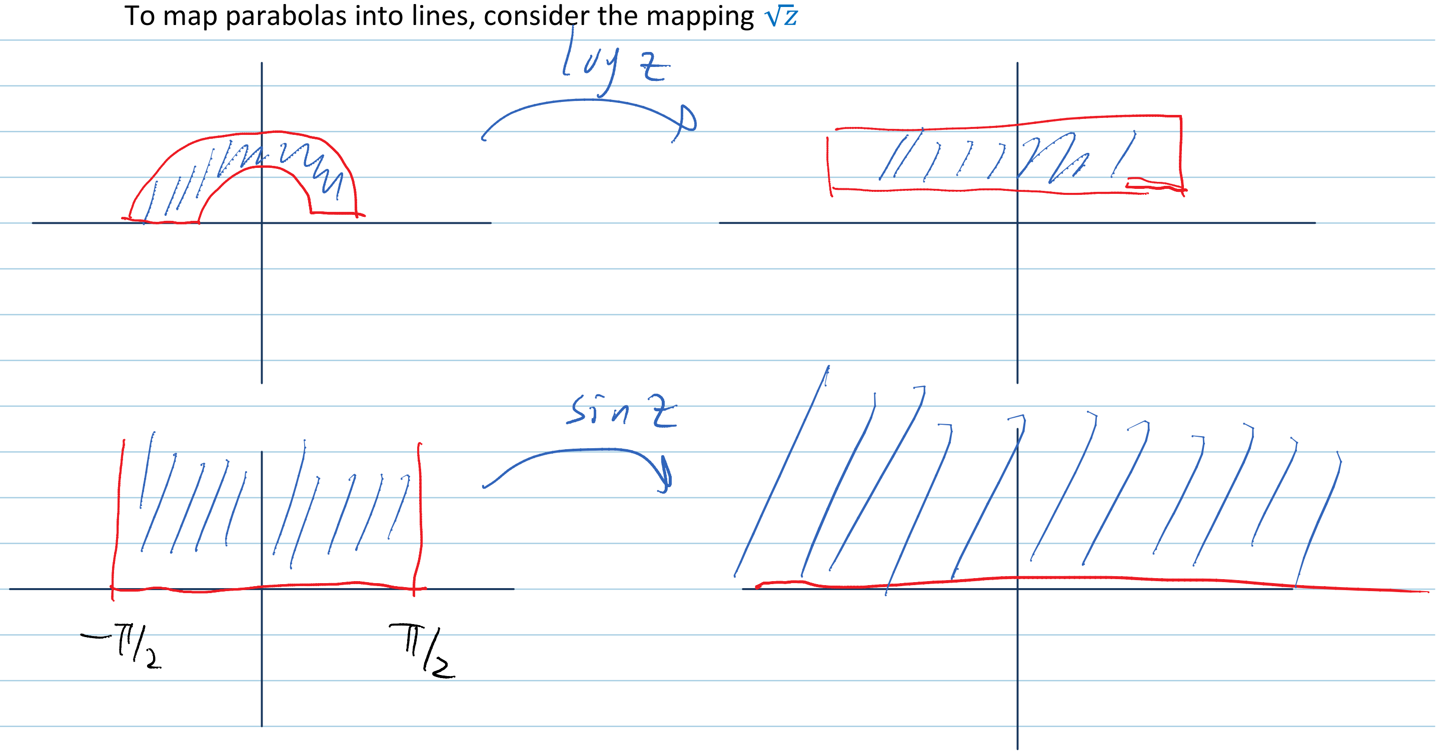


To map polygons into simpler shapes, use Schwarz-Christoffel formula.

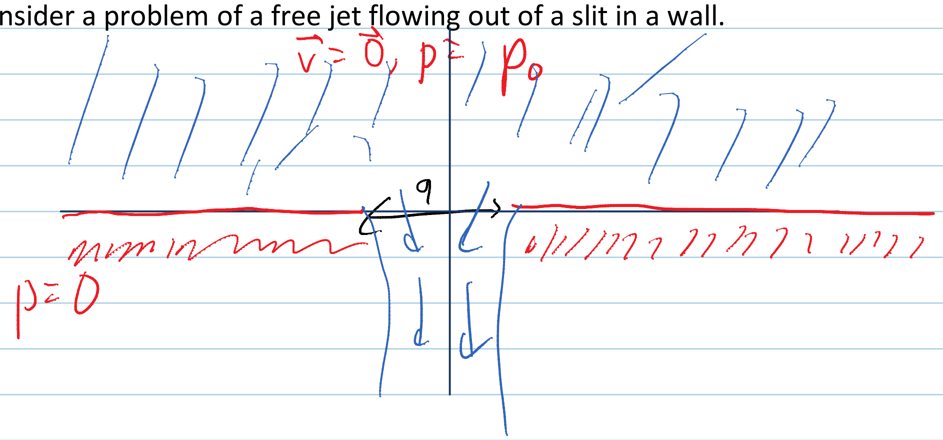
To handle ellipses and hyperbolas, think of using the mapping $\sin^{-1} z$

To map parabolas into lines, consider the mapping $\sqrt{z}$



The use of complex analysis on fluid problems can go much deeper than this conformal mapping. See for example the article D. Crowdy and M. Siegel, "Exact Solutions for the Evolution of a Bubble in Stokes Flow: A Cauchy Transform Approach," *SIAM J. Appl. Math.* **65** (3), 941-963 in the reading list.

Also, consider a problem of a free jet flowing out of a slit in a wall.



This has a challenging free boundary aspect to it; don't know the boundary of the jet; have to solve for it. How? Construct a complex velocity potential, and then use a sort of hodograph method. This is a technique in partial differential equations where you exchange the role of independent and dependent variables. Look at the problem not in the physical (x,y) plane, but in the image of this plane under the mapping by the complex velocity potential. Boundary conditions are more easily expressed in terms of rectangles in this complex velocity potential plane because stream function is constant along the boundary!