A PRIMER OF ANALYTIC NUMBER THEORY

This undergraduate introduction to analytic number theory develops analytic skills in the course of a study of ancient questions on polygonal numbers, perfect numbers, and amicable pairs. The question of how the primes are distributed among all integers is central in analytic number theory. This distribution is determined by the Riemann zeta function, and Riemann's work shows how it is connected to the zeros of his function and the significance of the Riemann Hypothesis.

Starting from a traditional calculus course and assuming no complex analysis, the author develops the basic ideas of elementary number theory. The text is supplemented by a series of exercises to further develop the concepts and includes brief sketches of more advanced ideas, to present contemporary research problems at a level suitable for undergraduates. In addition to proofs, both rigorous and heuristic, the book includes extensive graphics and tables to make analytic concepts as concrete as possible.

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From Pythagoras to Riemann

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This book is dedicated to all the former students
who let me practice on them.

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Preface

> Good evening. Now, I'm no mathematician but I'd like to talk about just a couple of numbers that have really been bothering me lately . . .
>
> *Laurie Anderson*

Number theory is a subject that is so old, no one can say when it started. That also makes it hard to describe what it is. More or less, it is the study of interesting properties of integers. Of course, what is interesting depends on your taste. This is a book about how analysis applies to the study of prime numbers. Some other goals are to introduce the rich history of the subject and to emphasize the active research that continues to go on.

History. In the study of right triangles in geometry, one encounters triples of integers x, y, z such that $x^2 + y^2 = z^2$. For example, $3^2 + 4^2 = 5^2$. These are called Pythagorean triples, but their study predates even Pythagoras. In fact, there is a Babylonian cuneiform tablet (designated Plimpton 322 in the archives of Columbia University) from the nineteenth century B.C. that lists fifteen very large Pythagorean triples; for example,

$$12709^2 + 13500^2 = 18541^2.$$

The Babylonians seem to have known the theorem that such triples can be generated as

$$x = 2st, \quad y = s^2 - t^2, \quad z = s^2 + t^2$$

for integers s, t. This, then, is the oldest theorem in mathematics. Pythagoras and his followers were fascinated by mystical properties of numbers, believing that numbers constitute the nature of all things. The Pythagorean school of mathematics also noted this interesting example with sums of cubes:

$$3^3 + 4^3 + 5^3 = 216 = 6^3.$$

This number, 216, is the Geometrical Number in Plato's *Republic*.[1]

The other important tradition in number theory is based on the *Arithmetica* of Diophantus. More or less, his subject was the study of *integer* solutions of equations. The story of how Diophantus' work was lost to the Western world for more than a thousand years is sketched in Section 12.2. The great French mathematician Pierre de Fermat was reading Diophantus' comments on the Pythagorean theorem, mentioned above, when he conjectured that for an exponent $n > 2$, the equation

$$x^n + y^n = z^n$$

has *no* integer solutions x, y, z (other than the trivial solution when one of the integers is zero). This was called "Fermat's Last Theorem," although he gave no proof; Fermat claimed that the margin of the book was too small for it to fit. For more than 350 years, Fermat's Last Theorem was considered the hardest open question in mathematics, until it was solved by Andrew Wiles in 1994. This, then, is the most recent major breakthrough in mathematics.

I have included some historical topics in number theory that I think are interesting, and that fit in well with the material I want to cover. But it's not within my abilities to give a complete history of the subject. As much as possible, I've chosen to let the players speak for themselves, through their own words. My point in including this material is to try to convey the vast timescale on which people have considered these questions.

The Pythagorean tradition of number theory was also the origin of numerology and much number mysticism that sounds strange today. It is my intention neither to endorse this mystical viewpoint nor to ridicule it, but merely to indicate how people thought about the subject. The true value of the subject is in the mathematics itself, not the mysticism. This is perhaps what Françoise Viète meant in dedicating his *Introduction to the Analytic Art* to his patron the princess Catherine de Parthenay in 1591. He wrote very colorfully:

> The metal I produced appears to be that class of gold others have desired for so long. It may be alchemist's gold and false, or dug out and true. If it is alchemist's gold, then it will evaporate into a cloud of smoke. But it certainly is true, ... with much vaunted labor drawn from those mines, inaccessible places, guarded by fire breathing dragons and noxious serpents ....

[1] If you watch the movie *Pi* closely, you will see that, in addition to $\pi = 3.14159\ldots$, the number 216 plays an important role, as a tribute to the Pythagoreans. Here's another trivia question: What theorem from this book is on the blackboard during John Nash's Harvard lecture in the movie *A Beautiful Mind*?

Analysis. There are quite a few number theory books already. However, they all cover more or less the same topics: the algebraic parts of the subject. The books that do cover the analytic aspects do so at a level far too high for the typical undergraduate. This is a shame. Students take number theory after a couple of semesters of calculus. They have the basic tools to understand some concepts of analytic number theory, if they are presented at the right level. The prerequisites for this book are two semesters of calculus: differentiation and integration. Complex analysis is specifically not required. We will gently review the ideas of calculus; at the same time, we can introduce some more sophisticated analysis in the context of specific applications. Joseph-Louis Lagrange wrote,

> I regard as quite useless the reading of large treatises of pure analysis: too large a number of methods pass at once before the eyes. It is in the works of applications that one must study them; one judges their ability there and one apprises the manner of making use of them.

(Among the areas Lagrange contributed to are the study of Pell's equation, Chapter 11, and the study of binary quadratic forms, Chapter 13.)

This is a good place to discuss what constitutes a proof. While some might call it heresy, a proof is an argument that is convincing. It, thus, depends on the context, on who is doing the proving and who is being convinced. Because advanced books on this subject already exist, I have chosen to emphasize readability and simplicity over absolute rigor. For example, many proofs require comparing a sum to an integral. A picture alone is often quite convincing. In this, it seems Lagrange disagreed, writing in the Preface to *Mécanique Analytique*,

> [T]he reader will find no figures in this work. The methods which I set forth do not require . . . geometrical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure.

In some places, I point out that the argument given is suggestive of the truth but has important details omitted. This is a trade-off that must be made in order to discuss, for example, Riemann's Explicit Formula at this level.

Research. In addition to having the deepest historical roots of all of mathematics, number theory is an active area of research. The Clay Mathematics Institute recently announced seven million-dollar "Millennium Prize Problems," see `http://www.claymath.org/prizeproblems/` Two of the seven problems concern number theory, namely the Riemann Hypothesis and the Birch Swinnerton-Dyer conjecture. Unfortunately, without introducing analysis, one can't understand what these problems are about. A couple of years ago, the National Academy of Sciences published a report on the current state of mathematical research. Two of the three important research areas in number theory they named were, again, the Riemann Hypothesis and the Beilinson conjectures (the Birch Swinnerton-Dyer conjecture is a small portion of the latter).

Very roughly speaking, the Riemann Hypothesis is an outgrowth of the Pythagorean tradition in number theory. It determines how the prime numbers are distributed among all the integers, raising the possibility that there is a hidden regularity amid the apparent randomness. The key question turns out to be the location of the zeros of a certain function, the Riemann zeta function. Do they all lie on a straight line? The middle third of the book is devoted to the significance of this. In fact, mathematicians have already identified the next interesting question after the Riemann Hypothesis is solved. What is the distribution of the spacing of the zeros along the line, and what is the (apparent) connection to quantum mechanics? These question are beyond the scope of this book, but see the expository articles Cipra, 1988; Cipra, 1996; Cipra, 1999; and Klarreich, 2000.

The Birch Swinnerton-Dyer conjecture is a natural extension of beautiful and mysterious infinite series identities, such as

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{\pi^2}{6},$$
$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\cdots=\frac{\pi}{4}.$$

Surprisingly, these are connected to the Diophantine tradition of number theory. The second identity above, Gregory's series for $\pi/4$, is connected to Fermat's observations that no prime that is one less than a multiple of four (e.g., 3, 7, and 11) is a hypotenuse of a right triangle. And every prime that is one more than a multiple of four *is* a hypotenuse, for example 5 in the (3, 4, 5) triangle, 13 in the (5, 12, 13), and 17 in the (8, 15, 17). The last third of the book is devoted to the arithmetic significance of such infinite series identities.

Advice. The Pythagoreans divided their followers into two groups. One group, the $\mu\alpha\theta\eta\mu\alpha\tau\iota\kappa\eta$, learned the subject completely and understood all the details. From them comes, our word "mathematician," as you can see for yourself if you know the Greek alphabet (mu, alpha, theta, eta, ...). The second group, the $\alpha\kappa o\upsilon\sigma\mu\alpha\tau\iota o\kappa\iota$, or "acusmatics," kept silent and merely memorized the master's words without understanding. The point I am making here is that if you want to be a mathematician, you have to participate, and that means doing the exercises. Most have solutions in the back, but you should at least make a serious attempt before reading the solution. Many sections later in the book refer back to earlier exercises. You will, therefore, want to keep them in a permanent notebook. The exercises offer lots of opportunity to do calculations, which can become tedious when done by hand. Calculators typically do arithmetic with floating point numbers, not integers. You will get a lot more out of the exercises if you have a computer package such as Maple, *Mathematica*, or PARI.

1. Maple is simpler to use and less expensive. In Maple, load the number theory package using the command `with(numtheory)`; Maple commands end with a semicolon.
2. *Mathematica* has more capabilities. Pay attention to capitalization in *Mathematica*, and if nothing seems to be happening, it is because you pressed the "return" key instead of "enter."
3. Another possible software package you can use is called PARI. Unlike the other two, it is specialized for doing number theory computations. It is free, but not the most user friendly. You can download it from `http://www.parigp-home.de/`

To see the movies and hear the sound files I created in *Mathematica* in the course of writing the book, or for links to more information, see my home page: `http://www.math.ucsb.edu/~stopple/`

Notation. The symbol $\exp(x)$ means the same as e^x. In this book, $\log(x)$ *always* means natural logarithm of x; you might be more used to seeing $\ln(x)$. If any other base of logarithms is used, it is specified as $\log_2(x)$ or $\log_{10}(x)$. For other notations, see the index.

Acknowledgments. I'd like to thank Jim Tattersall for information on Gerbert, Zack Leibhaber for the Viète translation, Lily Cockerill and David Farmer for reading the manuscript, Kim Spears for Chapter 13, and Lynne Walling for her enthusiastic support.

I still haven't said precisely what number theory – the subject – is. After a Ph.D. and fifteen further years of study, I think I'm only just beginning to figure it out myself.