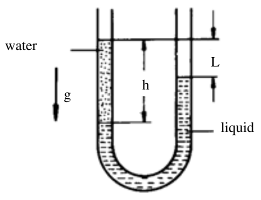
3. Exercises in Fluid Mechanics

3.1 Problems

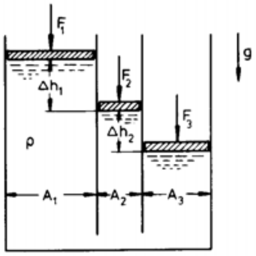
3.1.1 Hydrostatics

1.1 The density of a fluid ρ_f is to be determined with a U-tube. One stem is filled with water.



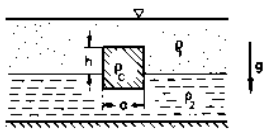
$h = 0.3\,\mathrm{m} \quad L = 0.2\,\mathrm{m} \quad \rho_w = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

1.2 In three communicating vessels pistons are exposed to the forces F_1, F_2 and F_3.



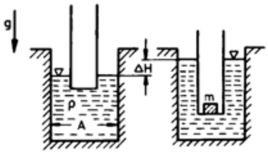
$F_1 = 1100\,\mathrm{N}$ $F_2 = 600\,\mathrm{N}$ $F_3 = 1000\,\mathrm{N}$
$A_1 = 0.04\,\mathrm{m}^2 \quad A_2 = 0.02\,\mathrm{m}^2$
$A_3 = 0.03\,\mathrm{m}^2 \quad \rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$
Determine the differences in height $\varDelta h_1$ and $\varDelta h_2$!

1.3 A cube floats in two laminated fluids, one on top of the other.



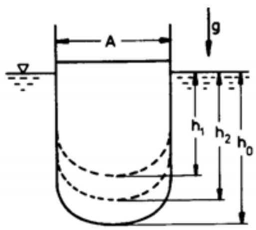
$\rho_1 = 850\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \rho_2 = 1000\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$
$\rho_C = 900\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad a = 0.1\,m$
Determine the height h!

1.4 A cylindrical vessel floats in another cylindrical vessel, filled with water. After adding a mass m the water surface is raised by $\varDelta H$.



Given: ρ, A, m
Determine the difference in height $\varDelta H$!

1.5 A boat with vertical side walls and a weight W_0 has a draught in sea water h_0 and displaces the volume τ_0. Before entering the mouth of a river the weight of the cargo is reduced by $\varDelta W$, in order to avoid the boat running aground. The draught is then h_1 and the volume is τ_1. The density of the sea water is ρ_S, and that of the water in the river ρ_R.



$\rho_S = 1.025 \cdot 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \rho_R = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$
$W_0 = 1.1 \cdot 10^9\,\mathrm{N} \quad \varDelta W = 10^8\,\mathrm{N}$
$h_0 = 11\,\mathrm{m} \quad h_1 = 10.5\,\mathrm{m} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine

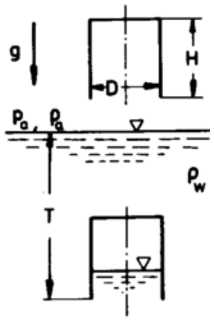
(a) the volume τ_0,

(b) the area of the deck A,

(c) the difference $\tau_2 - \tau_1$ of the displaced volumes in fresh water and sea water,

(d) the draught h_2 in fresh water!

1.6 A diving bell with the weight W is lowered into the sea.



$D = 3\,\mathrm{m}$ $\quad H = 3\,\mathrm{m}$ $\quad T = 22\,\mathrm{m}$

$\rho_a = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $\quad \rho_W = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

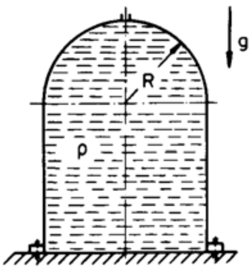
$p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$ $\quad W = 8\cdot 10^4\,\mathrm{N}$ $\quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

(a) How high does the water in the bell rise if the temperature remains constant?

(b) How large is the force (magnitude and direction) with which the bell must be held?

(c) At what depth of immersion is the force zero?

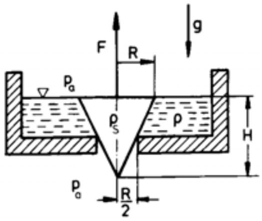
1.7 A container filled with water is fastened to a plate. It has a small opening in the top.



$R = 1\,\mathrm{m}$ $\quad \rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $\quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine the force in the screws under the assumption that the weight of the container can be neglected!

1.8 A conical plug with density ρ_s closes the outlet of a water basin. The base area of the cone levels with the surface of the fluid.

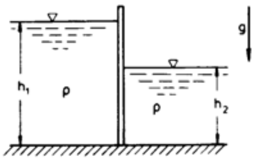


$R = 10^{-2}\,\mathrm{m}$ $\quad H = 10^{-2}\,\mathrm{m}$ $\quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

$\rho_s = 2\cdot 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $\quad \rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

How large must the force be to lift the plug?

1.9 A rectangular sluice gate with the width B separates two sluice chambers.



$B = 10\,\mathrm{m}$ $\quad h_1 = 5\,\mathrm{m}$ $\quad h_2 = 2\,\mathrm{m}$

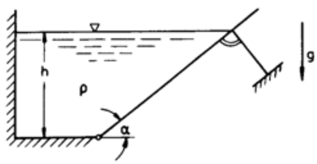
$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $\quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine

(a) the force acting on the sluice gate,

(b) the point of application of force!

1.10 A pivoted wall of a water container with width B is supported with a rod.

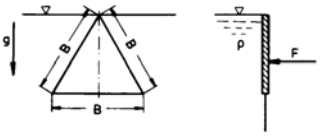


$h = 3\,\mathrm{m}$ $\quad B = 1\,\mathrm{m}$ $\quad \alpha = 30°$

$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $\quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine the force in the rod!

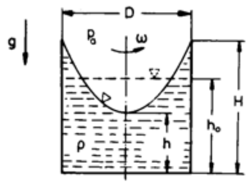
1.11 The triangular opening of a weir is closed with a plate.



$B = 1\,\mathrm{m} \quad \rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine
(a) the closing force,
(b) the point of application of force!

1.12 A fluid with a free surface rotates in an open circular cylindrical vessel with constant angular velocity, large enough, so that the fluid just reaches the edge of the vessel. When the fluid is at rest, it fills the vessel up to the height h_0.



$D = 0.5\,\mathrm{m} \quad h_0 = 0.7\,\mathrm{m} \quad H = 1\,\mathrm{m}$
$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine
(a) the height h and the angular velocity ω,
(b) the pressure distribution at the wall and on the bottom!

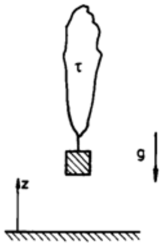
Hint:

$$\frac{\partial p}{\partial r} = \rho\,\omega^2\,r \quad \frac{\partial p}{\partial z} = -\rho\,g$$
$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}\,dz$$

1.13 Determine the pressure as a function of the height z
(a) for an isothermal atmosphere,
(b) for a linear temperature variation $T = T_0 - \alpha\,z$,
(c) for an isentropic atmosphere,
(d) for a height of 3000 m, 6000 m und 11000 m!

$z = 0:$
$R = 287\,\dfrac{\mathrm{Nm}}{\mathrm{kg\,K}} \quad T_0 = 287\,\mathrm{K} \quad \gamma = 1.4$
$p_0 = 10\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad \alpha = 6.5\cdot 10^3\,\dfrac{\mathrm{K}}{\mathrm{m}} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

1.14 A metereological balloon of mass m and initial volume τ_0 rises in an isothermal atmosphere. The envelope is slack until the maximum volume τ_1 is attained.



$p_0 = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad p_0 = 1.27\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$
$m = 2.5\,\mathrm{kg}$
$\tau_0 = 2.8\mathrm{m}^3 \quad \tau_1 = 10\mathrm{m}^3 \quad R = 287\dfrac{\mathrm{Nm}}{\mathrm{kg\,K}}$
$g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

(a) How large is the force the balloon must be held with before take off?
(b) At what altitude does the balloon attain the volume τ_1?
(c) How high does the balloon rise?

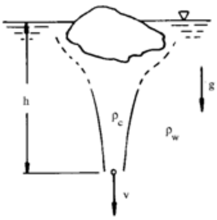
1.15 A balloon with an inelastic envelope has an opening at the bottom for equalization of the pressure with the surrounding air. The weight of the balloon without the gas filling is W. Before take off the balloon is held with the force F_s.
$W = 1000\,\mathrm{N} \quad F_s = 1720\,\mathrm{N}$
$R = 287\,\dfrac{\mathrm{Nm}}{\mathrm{kg\,K}}$
$T = 273\,\mathrm{K} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$
Determine the height of rise of the balloon in an isothermal atmosphere!

3.1.2 Hydrodynamics

If not mentioned otherwise, the flow is assumed to be loss-free in this chapter.

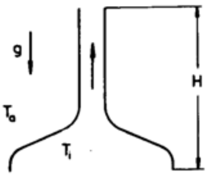
2.1 Given the velocity field
$u = u_0\,\cos\omega\,t \quad v = v\,\sin\omega\,t$
with $\frac{u_0}{w} = \frac{v_0}{w} = 1\,\mathrm{m}$.
Determine
(a) the streamlines for $\omega\,t = 0,\ \frac{\pi}{2},\ \frac{\pi}{4}$,
(b) the path lines,
(c) the path line of a particle, which at time $t = 0$ is in the point $x = 0$, $y = 1\,\mathrm{m}$!

2.2 Under an iceberg a steady downward flow is initiated by the cooling of the water by the ice. Determine the flow velocity v in the depth h under the assumption, that the cold water (density ρ_c) does not mix with the warm water (density ρ_w).



$$h = 50\,\text{m} \quad \frac{\rho_c - \rho_w}{\rho_c} = 0.01 \quad g = 10\,\frac{\text{m}}{\text{s}^2}$$

2.3 Hot exhaust air of temperature T_i flows through an open smokestack with a large suction scoop into the atmosphere. The external temperature is T_a.



$T_i = 450\,\text{K} \quad T_a = 300\,\text{K}$
$H = 100\,m \quad g = 10\,\frac{\text{m}}{\text{s}^2}$

Determine the discharge velocity, taking into account the influence of compressibility!
Hint: Use the Bernoulli equation in differential form:

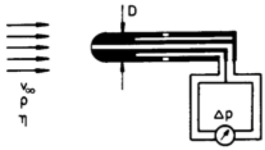
$$\frac{1}{\rho}\,dp + v\,dv + g\,dz = 0$$

2.4 Determine the free-stream velocity v_∞ of a Prandtl static pressure tube, taking into account the influence of the viscosity for

(a) $\mu = 10^{-3}\,\frac{\text{Ns}}{\text{m}^2}$,

(b) $\mu = 10^{-2}\,\frac{\text{Ns}}{\text{m}^2}$,

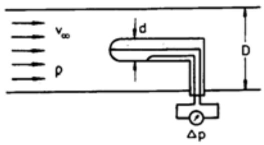
(c) $\mu = 10^{-1}\,\frac{\text{Ns}}{\text{m}^2}$.



$D = 6 \cdot 10^{-3}\,\text{m} \quad \Delta p = 125\,\frac{\text{N}}{\text{m}^2}$
$p = 10^3\,\frac{\text{kg}}{\text{m}^3}$

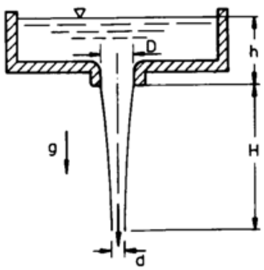
$$\begin{aligned} 2.5 \le Re \le 250: &\quad \beta = 1 + \tfrac{6}{Re} \\ 250 \le Re: &\quad \beta = 1 \end{aligned}$$

2.5 In order to determine the velocity in a pipe flow the pressure difference Δp is measured. The pressure difference deviates from the dynamic pressure of the undisturbed flow, if there is a large blockage in the pipe.



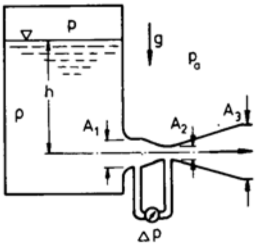
Plot $v_\infty / \sqrt{\frac{2\,\Delta p}{\rho}}$ as a function of $\frac{d}{D}$!

2.6 Water flows out of a large reservoir under the influence of gravity into the open air.



$h = 0.1\,\text{m} \quad H = 1.5\,\text{m} \quad D = 0.1\,\text{m}$
What is the diameter d of the water stream at the position H below the opening?

2.7 Water flows out of a large pressure tank into the open air. The pressure difference Δp is measured between the cross sections A_1 and A_2.



$A = 0.3\,\text{m}^2 \quad A_2 = 0.1\,\text{m}^2 \quad A_3 = 0.2\,\text{m}^2$

$h = 1\,\text{m} \quad \rho = 10^3\,\dfrac{\text{kg}}{\text{m}^3} \quad p_a = 10^5\,\dfrac{\text{N}}{\text{m}^2}$

$g = 10\,\dfrac{\text{m}}{\text{s}^2} \quad \Delta p = 0.64 \cdot 10^5\,\dfrac{\text{N}}{\text{m}^2}$

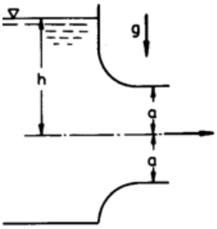
The outflow pipe was provided with a variable cross-section distribution in order to enable the measurement for the pressure. The pressure difference Δp is measured in the cross sections indicated in the sketch.

Determine

(a) the velocities v_1, v_2, v_3,

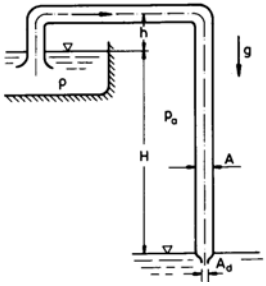
(b) the pressures p_1, p_2, p_3, and the pressure p above the water surface!

2.8 Water flows out of a large vessel through an opening of width B and height $2a$ into the open air.



For $\dfrac{a}{h} \to 0$ the volume flow per unit time is $\dot{Q}_0 = 2\,a\,B\,\sqrt{2\,g\,h}$. Determine the relative error $\dfrac{\dot{Q}_0 - \dot{Q}}{\dot{Q}}$ for $\dfrac{a}{h} = \dfrac{1}{4}, \dfrac{1}{2}, \dfrac{3}{4}$!

2.9 Two large reservoirs, one located above the other, are connected with a vertical pipe, with a nozzle attached to its end.



$A = 1\,\text{m}^2 \quad A_d = 0.1\,\text{m}^2 \quad h = 5\,\text{m}$

$H = 80\,\text{m} \quad p_a = 10^5\,\dfrac{\text{N}}{\text{m}^2} \quad \rho = 10^3\,\dfrac{\text{kg}}{\text{m}^3}$

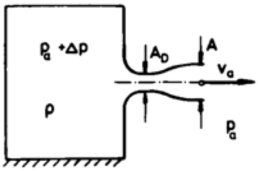
$g = 10\,\dfrac{\text{m}}{\text{s}^2}$

(a) How large is the volume flow per unit time?

(b) Sketch the curve of the static pressure in the pipe!

(c) At what size of the cross section of the exit will vapor bubbles be formed, if the vapor pressure is $p_v = 0.025 \cdot 10^5\,\dfrac{\text{N}}{\text{m}^2}$?

2.10 Air flows out of a large pressure tank through a well-rounded nozzle and a diffuser into the open air.



$\rho = 1.25\,\dfrac{\text{kg}}{\text{m}^3} \quad \Delta p = 10\,\dfrac{\text{N}}{\text{m}^2}$

Determine the velocity in the throat of the nozzle as a function of the ratio of the cross sections $\frac{A}{A_D}$

(a) for loss-free flow,

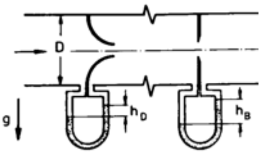
(b) for an efficiency of the diffuser of

$$\eta_D = \frac{p_a - p_D}{\frac{\varrho}{2}\,(v_D^2 - v_a^2)} = 0.84 \quad !$$

(c) What is the maximum velocity that can be attained for this efficiency of the diffuser?

2.11 Water flows through a nozzle mounted in a pipe (cross-sectional ratio m_D, discharge coefficient α_D) and an orifice

(m_D, α_B). The mercury gauges show differences in height of h_D and h_B.

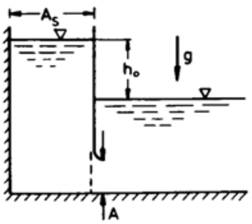


$m_D = 0.5 \quad \alpha_D = 1.08 \quad m_B = 0.6$
$h_D = 1\,\text{m} \quad h_B = 1.44\,\text{m} \quad D = 0.1\,\text{m}$
$\rho_W = 10^3\,\frac{\text{kg}}{\text{m}^3} \quad \rho_{Hg} = 13.6 \cdot 10^3\,\frac{\text{kg}}{\text{m}^3}$
$g = 10\,\frac{\text{m}}{\text{s}^2}$

Determine

1. the volume flow per unit time,
2. the discharge coefficient of the orifice!

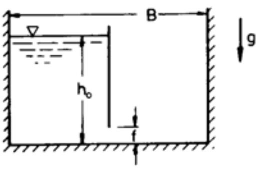
2.12 A sluice gate is suddenly opened.



$A_s = 3000\,\text{m}^2 \quad h(t=0) = h_0 = 5\,\text{m}$
$g = 10\,\frac{\text{m}}{\text{s}^2}$

How large must the cross section of the opening A be so that the water level of the bordering lake is attained within 10 minutes, if quasi-steady flow is assumed?

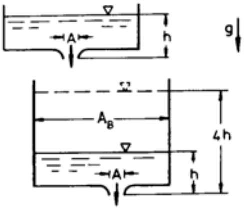
2.13 Two equally large reservoirs, one of which is filled with water, are separated from each other by a wall.



$B = 20\,\text{m} \quad h(t=0) = h_0 = 5\,\text{m}$
$f = 0.05\,\text{m} \quad g = 10\,\frac{\text{m}}{\text{s}^2}$

Determine the time necessary for equalizing the water levels, if the dividing wall is lifted by the amount $f \ll h_0$! Neglegt the contraction of the flow!

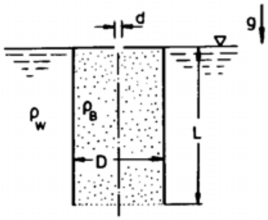
2.14 Water flows out of a large reservoir into a lower reservoir, the discharge opening of which is suddenly reduced to one third.



$A = 0.03\,\text{m}^2 \quad A_B = 1\,\text{m}^2$
$h = 5\,\text{m} \quad g = 10\,\frac{\text{m}}{\text{s}^2}$

Determine the time, in which the water level rises to the quadruple value of its initial height h!

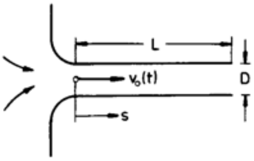
2.15 A pipe filled with gasoline is held vertically in water and closed at the upper end with a top. A small, well-rounded outlet in the top is opened.



$D = 0.1\,\text{m} \quad d = 0.01\,\text{m}m \quad L = 0.8\,\text{m}$
$\rho_B = 800\,\frac{\text{kg}}{\text{m}^3} \quad \rho_w = 10^3\,\frac{\text{kg}}{\text{m}^3} \quad g = 10\,\frac{\text{m}}{\text{s}^2}$

Since the gasoline is lighter than water, it will begin to flow upward through the small hole. The surface of the water surrounding the pipe is very large compared to the cross-sectional area of the pipe.
How long will it take, until the pipe is completely emptied from gasoline?

2.16 A fluid flows through a pipe with a well-rounded intake. Its velocity is $v_0(t)$.

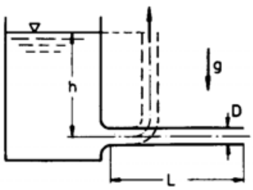


Show that the integral of the local acceleration can be approximated as follows:

$$\int_{-\infty}^{L} \frac{\partial v}{\partial t} ds = \left(\frac{D}{\sqrt{2}} + L \right) \frac{dv_0}{dt}$$

Hint: Assume that for $s < -\frac{D}{\sqrt{8}}$ the fluid flows radially towards the intake with the velocity $v = \frac{\dot{Q}}{2}\,\pi\,s^2$, and that for $s \geq \frac{D}{\sqrt{8}}$ the velocity is equal to v_0! For $s = -\frac{D}{\sqrt{8}}$, $v = v_0$.

2.17 Liquid flows out of a large container through a hose, lying horizontally on the ground, in steady motion into the open air. The end of the hose is suddenly lifted up to the height of the liquid level.

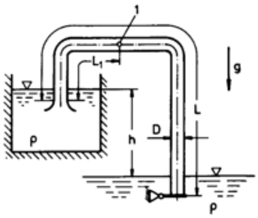


$L = 10\,\mathrm{m} \quad h = 5\,\mathrm{m} \quad D = 0.16\,\mathrm{m}$
$g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine

1. the velocity v_0 immediately after lifting up the hose,
2. the time, in which the velocity decreases to $\frac{v_0}{2}$,
3. the fluid volume that flowed through the hose during this time!

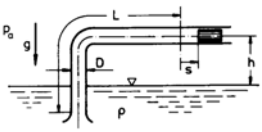
2.18 The discharge pipe of a large water container is led to a lake. The throttle valve at the end of the pipe is suddenly opened.



$L = 20\,\mathrm{m} \gg D \quad h = 5\,\mathrm{m} \quad L_1 = 5\,\mathrm{m}$
$p = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

1. After what time are 99% of the final velocity attained?
2. How much does the pressure at the position 1 differ from its final value?

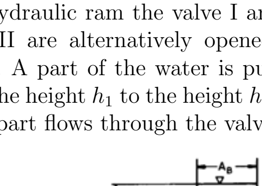
2.19 A piston is moving sinusoidally in a pipe $s = s_0\,\sin\omega\,t$.



$p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad L = 10\,\mathrm{m} \gg D \quad h = 2\,\mathrm{m}$
$s_0 = 0.1\,\mathrm{m} \quad \rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad p_v = 2500\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$
$g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

At what angular velocity ω is the pressure at the piston head equal to the vapor pressure p_v?

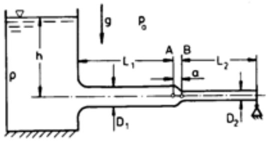
2.20 In a hydraulic ram the valve I and the valve II are alternatively opened and closed. A part of the water is pumped from the height h_1 to the height h_2. The other part flows through the valve I.



$h_1 = h_2 = 5\,\mathrm{m} \quad L = 10\,\mathrm{m} \gg D$
$A = 0.1\,\mathrm{m}^2 \ll A_B \quad T_1 = 1\,\mathrm{s}$
$g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

(a) First the valve I is opened for the time T. Determine the volume Q_I of the water discharged!
(b) After closing the valve I, the valve II is opened until the velocity in the pipe is decreased to zero. How large is the discharge volume Q_{II}?

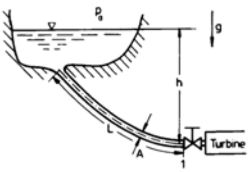
2.21 The flap at the end of a discharge pipe of a large container is suddenly opened.



$p_a = 10^5 \dfrac{\mathrm{N}}{\mathrm{m}^2}$ $L_1 = L_2 = 5\,\mathrm{m} \gg a$
$D_1 = 0.1\,\mathrm{m}$ $D_2 = 0.05\,\mathrm{m}$ $h = 2\,\mathrm{m}$
$\rho = 10^3 \dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $g = 10 \dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine
(a) the time T, in which the velocity attains 99% of its final value,
(b) the volume of the fluid discharged,
(c) the pressures p_A and p_B immediately after opening the flap and at time T!
(d) Sketch the pressure at the positions A and B as a function of time!

2.22 The pressurized pipe system of a storage power station is closed with a valve. During the closure (shut-down time T_s) the discharge volume decreases linearly from $\dot{Q}_0$ to zero.



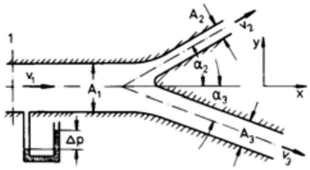
$h = 200\,\mathrm{m}$ $L = 300\,\mathrm{m}$ $A = 0.2\,\mathrm{m}^2$
$\dot{Q} = 3 \dfrac{\mathrm{m}^3}{\mathrm{s}}$ $\rho = 10^3 \dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $g = 10 \dfrac{\mathrm{m}}{\mathrm{s}^2}$
$\Delta p_{save} = (p_1 - p_a)_{save} = 2 \cdot 10^7 \dfrac{\mathrm{N}}{\mathrm{m}^2}$

Determine
(a) the excess pressure in front of the open valve for steady flow,
(b) the pressure variation $p_1(t)$ during closure of the valve (Sketch the result!),
(c) the closure time of the valve so that the excess prssure does not exceed the safe value Δp_{safe}!

3.1.3 Momentum and Moment of Momentum Theorem

In this chapter the friction forces are neglected in comparison to the volume, pressure, and inertia forces, but not the pressure losses, resulting from flow separation.

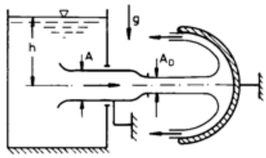
3.1 Water flows out of a bifurcated pipe into the open air. The pressure in the inflow stem is higher by the amount Δp than in the surrounding air.



$A_1 = 0.2\,\mathrm{m}^2$ $A_2 = 0.03\,\mathrm{m}^2$
$A_3 = 0.07\,\mathrm{m}^2$ $\alpha_2 = 30°$ $\alpha_3 = 20°$
$\Delta p = 10^4 \dfrac{\mathrm{N}}{\mathrm{m}^2}$ $\rho = 10^3 \dfrac{\mathrm{kg}}{\mathrm{m}^3}$

Determine
(a) the velocities v_1, v_2, v_3,
(b) the force F in the cross section 1,
(c) the angle α_3, for which F_{sy} vanishes!

3.2 Water flows out of a large container through a pipe under the influence of gravity in steady motion into the open air. Downstream from the nozzle the water jet is deflected by 180°. The flow is assumed to be two-dimensional.

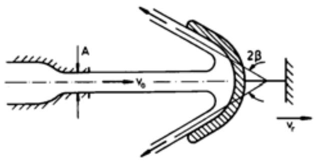


$A = 0.2\,\mathrm{m}^2$ $A_D = 0.1\,\mathrm{m}^2$ $h = 5\,\mathrm{m}$
$\rho = 10^3 \dfrac{\mathrm{kg}}{\mathrm{m}^3}$ $g = 10 \dfrac{\mathrm{m}}{\mathrm{s}}$

Determine the forces retaining the pipe and the guide vane
(a) for the sketched configuration,

(b) for the case that the inlet of the pipe and the nozzle are removed!

3.3 Water flows out of a two-dimensional nozzle in steady motion with the velocity v_0 against a guide vane, moving with the velocity v_r.

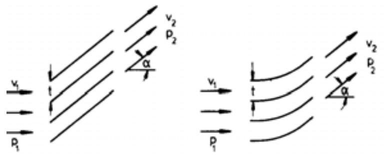


$A = 0.1\,\mathrm{m}^2 \quad v_0 = 60\,\dfrac{\mathrm{m}}{\mathrm{s}} \quad 2\,\beta = 45°$

$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

(a) At what velocity v_r does the performance of the vane attain its maximum value?
(b) How large is then the force acting on the vane?

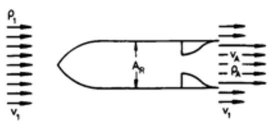
3.4 Two two-dimensional cascades (infinitely many blades) with width B and spacing t deflect a flow by the angle α.



cascade I cascade II

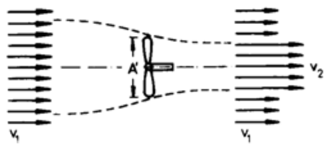
Given: ρ, v_1, α, B, t
Determine
(a) the velocity v_2,
(b) the pressure difference $p_1 - p_2$,
(c) the pressure loss $p_{01} - p_{02}$,
(d) the force exerted by the flow on a blade!

3.5 A rocket moves with constant velocity. The air flowing past the rocket is displaced in the radial direction. The velocity in the jet is v_A, around it v_1.



Given: v_1, v_A, ρ_1, ρ_A, A_R
Determine
(a) the mass of air displaced,
(b) the thrust and the net performance!

3.6 The constant free-stream velocity of a propeller is v_1. A certain distance downstream from the propeller the velocity in the slipstream is v_2, outside of it v_1.

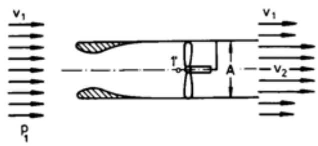


$A' = 7.06\,\mathrm{m} \quad v_1 = 5\,\dfrac{\mathrm{m}}{\mathrm{s}} \quad v_2 = 8\,\dfrac{\mathrm{m}}{\mathrm{s}}$

$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

Determine
(a) the velocity v' in the cross-sectional plane of the propeller,
(b) the efficiency!

3.7 A ducted propeller is positioned in a free stream with constant velocity. The inlet lip is well rounded.

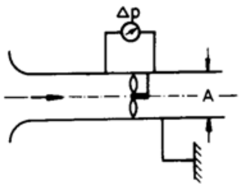


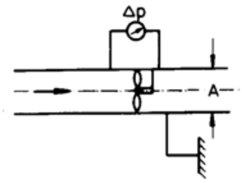
$A = 1\,\mathrm{m}^2 \quad v_1 = 10\,\dfrac{\mathrm{m}}{\mathrm{s}} \quad p_{1'} = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad p_1 = 1.345 \cdot 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

(a) Sketch the variation of the static pressure along the axis!
Determine
(b) the mass flow,
(c) the thrust,
(d) the power transferred by the propeller to the flow!

3.8 Two blowers, drawing air from the surroundings, differ in the shape of their inlets.





Given: ρ, A, Δp

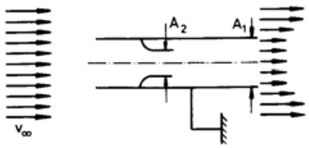
Determine

(a) the discharge volume,

(b) the power of the blowers,

(c) the retaining force!

3.9 A pipe with an inserted nozzle is positioned in a free stream with constant velocity.



$A_1 = 0.2\,\mathrm{m}^2 \quad A_2 = 0.1\,\mathrm{m}^2 \quad v_\infty = 40\,\dfrac{\mathrm{m}}{\mathrm{s}}$

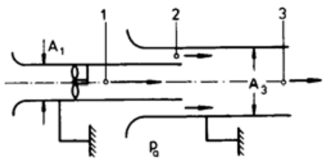
$\rho = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

Determine

(a) the velocity in the cross sections A_1 and A_2,

(b) the retaining force!

3.10 A jet apparatus, which is driven with a blower, sucks the volume rate of flow $\dot{Q}_2$ through a ring-shaped inlet.



$A_1 = 0.1\,\mathrm{m}^2 \quad A_3 = 0.2\,\mathrm{m}^2$

$p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

$\dot{Q}_2 = 4\,\dfrac{\mathrm{m}^3}{\mathrm{s}} \quad \rho = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

Determine

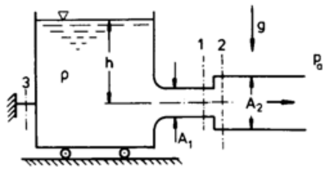
(a) the velocity v_2 and the pressure $p_{2'}$,

(b) the velocities v_1 and $v_{3'}$,

(c) the power of the blower,

(d) the retaining force of the blower casing (traction or compressive force?)!

3.11 Water flows out of a large frictionless supported container through a pipe, with a discontinuous increase of the cross section, into the open air.



$h = 5\,\mathrm{m} \quad A = 0.1\,\mathrm{m}^2 \quad p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

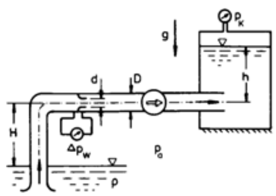
(a) For what cross-sectional area A_2 does the volume rate of flow attain its maximum value?

With A_2 determined under a) compute

(b) the pressure p_1,

(c) the cutting forces F_{s1}, F_{s2}, F_{s3} (Traction or compressive forces?)!

3.12 A pump is feeding water from a lake into a large pressurized container. The volume rate of flow is measured with a standard nozzle (discharge coefficient α).



$H = 5\,\mathrm{m} \quad h = 3\,\mathrm{m} \quad d = 0.07\,\mathrm{m}$

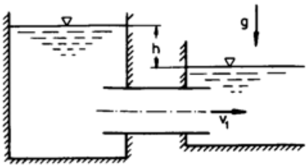
$p_K = 2 \cdot 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad \Delta p_w = 3160\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

$p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad \rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

$D = 0.1\,\mathrm{m} \quad \alpha = 1.08$

Determine
(a) the velocity in the pipe,
(b) the static pressure upstream and downstream from the pump,
(c) the net performance of the pump!

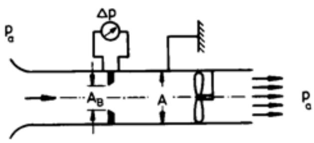
3.13 Water flows out of a large container through a pipe with a Borda mouthpiece into a lake.



$h = 1\,\mathrm{m} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine
(a) the contraction,
(b) the out-flow velocity v_1

3.14 The volume rate of flow of a ventilation blower is measured with an orifice (discharge coefficient α, contration coefficient Ψ).



$p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad \Delta p_w = 300\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

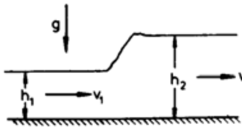
$\rho = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

$\alpha = 0.7 \quad \Psi = 0.66 \quad A = 10^{-2}\,\mathrm{m}^2$

$m = \dfrac{A_B}{A} = 0.5$

(a) Sketch the variation of the static and total pressure along the axis of the pipe!
Determine
(b) the volume rate of flow,
(c) the pressure upstream of the blower,
(d) the performance of the blower!

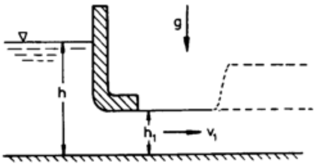
3.15 A hydraulic jump occurs in an open channel.



$h_1 = 0.1\,\mathrm{m} \quad h_2 = 0.2\,\mathrm{m} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine
(a) the velocities v_1 and v_2,
(b) the Froude numbers Fr_1 and Fr_2,
(c) the energy loss $H_1 - H_2$!

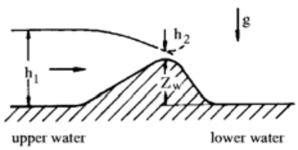
3.16 The volume of water flowing out of a storage pond is controlled with a wicket.



$h = 7.5\,\mathrm{m} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Determine
(a) the out-flow velocity v_1 as a function of the height of level h_1 (why is v_1 constant in the cross section?),
(b) the height of level, for which the volume rate of flow attains a maximum,
(c) the height of level, for which the hydraulic jump does not occur,
(d) the depth of water and the velocity downstream from the jump for $h_1 = 2.5\,\mathrm{m}$!

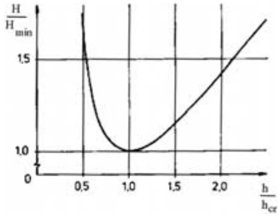
3.17 The depth of water h_1 of an open channel with constant volume rate of flow is controlled by changing the height Z_w of a weir. For $Z_w = 0$ the depth of water is h_0.



$\dot{Q} = 80 \, \frac{\mathrm{m}^3}{\mathrm{s}} \quad B = 20\,\mathrm{m} \quad h_0 = 2\,\mathrm{m}$

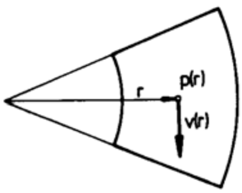
$\rho = 3 \, \frac{\mathrm{kg}}{\mathrm{m}^3}$

(a) Sketch the variation of the depths of water for $Z_w < Z_{crit.}$ and $Z_w > Z_{crit.}$!
Determine for $Z_w = 1\,m$
(b) the limiting height $Z_{crit.}$ of the dam,
(c) the depths of water h_1 and h_2,
(d) the difference of the energy heights between upper and lower water,
(e) the force acting on the weir!



Hint: If a hydraulic jump occurs, it will be at the downstream face of the weir; the depth of the lower water is h_0.

3.18 Assume that in a rotating flow pressure and velocity depend only on the radius.

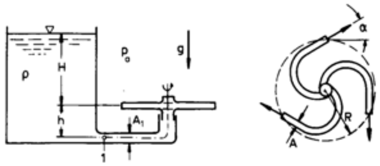


(a) Choose the segment of a circular ring as control surface and, by using the momentum theorem, derive the relation

$$\frac{dp}{dr} = \rho \, \frac{v^2}{r}!$$

(b) For what velocity distribution $v(r)$ does the Bernoulli constant have the same value for all streamlines?

3.19 A lawn sprinkler is fed from a large reservoir. The water jets are inclined to the circumferential direction by the angle α. The friction torque of the bearing is M_r.



$H = 10\,\mathrm{m} \quad h = 1\,\mathrm{m} \quad R = 0.15\,\mathrm{m}$
$A = 0.5 \cdot 10^4\,\mathrm{m}^2 \quad A_1 = 1.5 \cdot 10^4\,\mathrm{m}^2$

$|M_r| = 3.6\,\mathrm{Nm} \quad p_a = 10^5 \, \frac{\mathrm{N}}{\mathrm{m}^2}$

$\rho = 10^3 \, \frac{\mathrm{kg}}{\mathrm{m}^3} \quad g = 10 \, \frac{\mathrm{m}}{\mathrm{s}^2} \quad \alpha = 30°$

Determine
(a) the number of revolutions,
(b) the rate of volume flow,
(c) the pressure p_1,
(d) the maximum angular velocity, if the friction torque is assumed to be zero!

3.1.4 Laminar Flow of Viscous Fluids

4.1 Determine the following quantities for a fully developed laminar pipe flow of a Newtonian fluid
(a) the velocity distribution

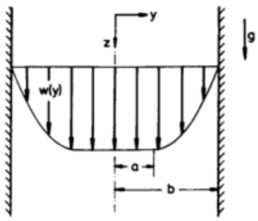
$$\frac{u(r)}{u_{max}} = f(\frac{r}{R}) \quad , \tag{3.1}$$

(b) the ratio

$$\frac{u_m}{u_{max}} \quad , \tag{3.2}$$

(c) the dependence of the pipe friction coefficient on the Reynolds number!

4.2 A Bingham fluid is driven by gravity between two parallel, infinitely wide plates.

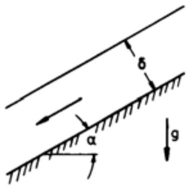


Given: b, ρ, μ, τ_0, g, $\frac{dp}{dz} = 0$

Assume that the flow is fully developed and determine

(a) the distance a,
(b) the velocity distribution!

4.3 An oil film of constant thickness and width flows down on an inclined plate.



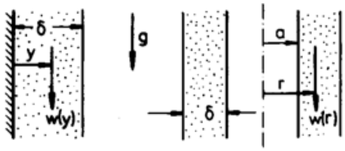
$\delta = 3 \cdot 10^3\,\mathrm{m} \quad B = 1\,\mathrm{m} \quad \alpha = 30°$

$\mu = 30 \cdot 10^{-3}\,\frac{\mathrm{Ns}}{\mathrm{m}^2} \quad \rho = 800\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

$g = 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$

Determine the volume rate of flow!

4.4 An oil film is driven by gravity.

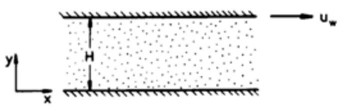


Given: δ, a, ρ, μ, g.

Determine the velocity distribution in the oil film

(a) on a plane vertical wall,
(a) on a wall of a vertically standing circular cylinder!

4.5 A Newtonian fluid flows in the gap between two horizontal plates. The upper plate is moving with the velocity u_w, the lower is at rest. The pressure is linearly decreasing in the x-direction.

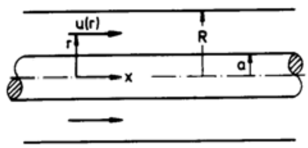


Given: H, u_w, ρ, μ, $\frac{dp}{dx}$

Assume fully developed laminar flow and determine

(a) the velocity distribution,
(b) the ratio of the shear stresses for $y = 0$ and $y = H$,
(c) the volume rate of flow for a width of the plates B,
(d) the maximum velocity for $u_w = 0$,
(e) the momentum flux for $u_w = 0$,
(f) the wall-shear stress in dimensionless form for u_w
(g) sketch the velocity and shear stress distribution for $u_w > 0$, $u_w = 0$, and $u_w < 0$!

4.6 A Newtonian fluid flows between two coaxial cylinders.

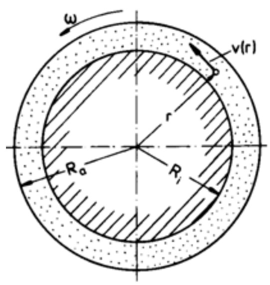


Given: R, a, μ, $\frac{dp}{dx}$

Assume fully developed laminar flow and determine

(a) the velocity distribution (sketch the result!),
(b) the ratio of the shear stresses for $r = a$ and $r = R$,
(c) the mean velocity!

4.7 A Couette viscosimeter consists out of two concentric cylinders of length L. The gap between them is filled with a Newtonian fluid. The outer cylinder rotates with the angular velocity ω, the inner is at rest. At the inner cylinder the torque M_z is measured.



$R_a = 0.11\,\mathrm{m}$ $\quad R_i = 0.1\,\mathrm{m}$ $\quad L = 0.1\,\mathrm{m}$

$\omega = 10\,\frac{1}{\mathrm{s}}$ $\quad M_z = 7.246\,10^{-3}\,\mathrm{Nm}$

Determine

(a) the velocity distribution,

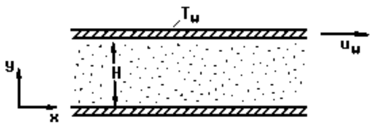
(b) the dynamic shear viscosity of the fluid!

Hint: The differential equations for the velocity and shear-stress distributions are:

$$\frac{d}{dr}\left[\frac{1}{r}\,\frac{d}{dr}\,(rv)\right] = 0 \quad ,$$

$$\tau = -\mu\, r\,\frac{d}{dr}\left(\frac{v}{r}\right)$$

4.8 A gas of thermal conductivity λ and specific heats c_p and c_v flows in the gap between two horizontal plates. The upper plate is moving with velocity u_w at temperature T_w, the lower is at rest and is thermally isolated.



Given: u_w, $\quad H$, $\quad T_w$, $\quad \frac{dp}{dx} = 0$,
ρ, $\quad \mu$, $\quad \lambda$, $\quad c_p$, $\quad c_v$

Assume fully developed laminar flow and determine for vanishing convective heat flux and constant material properties

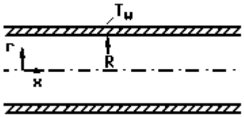
(a) the velocity and temperature distribution,

(b) the heat flux per unit area through the upper plate!

(c) Show that the stagnation enthalpy has the same value everywhere for $Pr = 1$!

(d) Determine the time-dependent temperature variation, if both plates are thermally isolated, and if at the time $t = 0$ the temperature in the flow field is T_0!

4.9 A Newtonian fluid with thermal conductivity λ flows through a pipe. The wall temperature is kept constant by cooling the wall.



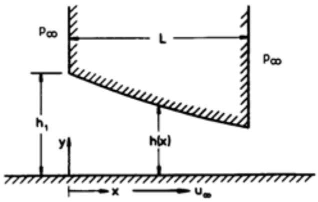
Given: R, $\quad T_w$, $\quad \lambda$, $\quad \mu$, $\quad \frac{dp}{dx}$

(a) Assume fully developed laminar flow and derive the differential equations for the velocity and temperature distribution for a ring-shaped volume element for vanishing convective heat flux and constant material properties! State the boundary conditions!

(b) Determine the temperature distribution

$$\frac{T - T_w}{T_{max} - T_w}$$

4.10 Under a gun slide a plane wall is moving with the velocity u_∞.



$h(x) = h_1\, e^{-\frac{x}{5L}}$ $\quad L = 5 \cdot 10^{-2}\,\mathrm{m}$

$h_1 = 10^{-4}\,\mathrm{m}$ $\quad u_\infty = 1\,\frac{\mathrm{m}}{\mathrm{s}}$ $\quad \mu = 10^{-1}\,\frac{\mathrm{Ns}}{\mathrm{m}^2}$

$\rho = 800\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

Determine

(a) the similarity parameter of the problem,

(b) the volume rate of flow per unit width,

(c) the pressure distribution in the gap,

(d) the pressure force per unit width bearing the gun slide,

(e) the performance loss per unit width t due to bearing friction!

4.11 From the momentum equation in integral form

$$\begin{aligned}\frac{d\boldsymbol{I}}{dt} &= \int_\tau \frac{\partial}{\partial t}\,(\rho\,\boldsymbol{v})\,d\tau + \int_A \rho\,\boldsymbol{v}\,(\boldsymbol{v}\,\boldsymbol{n})dA \\ &= \sum \boldsymbol{F}\end{aligned}$$

derive the differential form of the momentum equation in the x-direction for an infinitesimally small volume element

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}\right)$$

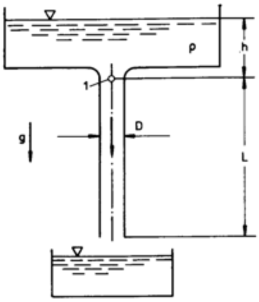
4.12 The x-component of the friction force acting on a volume element is given by

$$\begin{aligned} F_{fx} &= \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\nabla\cdot\boldsymbol{v}\right)\right]+ \\ &+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+ \\ &+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial x}\right)\quad .\right] \end{aligned}$$

Reduce this equation for an incompressible fluid with constant viscosity!

3.1.5 Pipe Flows

5.1 The viscosity of an oil is to be measured with a capillary viscosimeter.
This is done by measuring the time T, in which a small part of the oil (volume τ) flows through the capillary. Assume that the flow is loss-free upstream of the position 1!

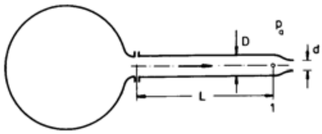


$\tau = 10\ \mathrm{cm}^3 \quad \rho = 900\ \dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad L = 0.1\ \mathrm{m}$

$h = 0.05\ \mathrm{m} \quad D = 1\ \mathrm{mm} \quad T = 254\ \mathrm{s}$

$g = 10\ \dfrac{\mathrm{m}}{\mathrm{s}^2}$

5.2 Water flows out of a large container through a hydraulically smooth pipe, with a nozzle fixed to its end. The pressure upstream of the nozzle is p_1. The friction losses in the intake of the pipe and in the nozzle can be neglected. Assume that the flow in the pipe is fully developed!



$L = 100\ \mathrm{m} \quad D = 10^{-2}\ \mathrm{m}$

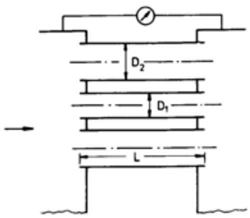
$d = 0.5\cdot 10^{-2}\ \mathrm{m} \quad p_a = 10^5\ \dfrac{\mathrm{N}}{\mathrm{m}^2}$

$p_1 = 1.075\cdot 10^5\ \dfrac{\mathrm{N}}{\mathrm{m}^2} \quad \rho = 10^3\ \dfrac{\mathrm{kg}}{\mathrm{m}^3}$

$\mu_1 = 10^{-3}\ \dfrac{\mathrm{Ns}}{\mathrm{m}^2}$

Determine
(a) the velocity in the pipe and at the exit of the nozzle,
(b) the pressure in the container,
(c) the velocity at the exit of the nozzle for $L = 0$ and the same pressure in the container!

5.3 Two containers are connected with each other by 25 pipes with diameter D_1 and 25 pipes with diameter D_2. A pressure difference of Δp is measured between the containers. The pressure-loss coefficient of the intake is ζ.



$D_1 = 0.025\ \mathrm{m} \quad D_2 = 0.064\ \mathrm{m}$

$L = 10\ \mathrm{m} \quad \rho = 10^3\ \dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \lambda = 0.025$

$\Delta p = 10^5\ \dfrac{\mathrm{N}}{\mathrm{m}^2} \quad \zeta = 1$

Determine the volume rate of flow!

5.4 A fluid flows through a hydraulically smooth pipe. The pressure drops by the amount Δp over the length L.

$L = 100\mathrm{m} \quad D = 0.1\mathrm{m} \quad \Delta p = 5\cdot 10^4\ \dfrac{\mathrm{N}}{\mathrm{m}^2}$

$$\text{Oil}: \quad \mu = 10^{-1}\,\frac{\text{Ns}}{\text{m}^2} \quad \rho = 800\,\frac{\text{kg}}{\text{m}^3}$$
$$\text{Water}: \quad \mu = 10^{-3}\,\frac{\text{Ns}}{\text{m}^2} \quad \rho = 10^3\,\frac{\text{kg}}{\text{m}^3}$$

Determine the velocities of the flow!
Hint: Use the Prandtl resistance law for super-critical Reynolds numbers!

5.5 Compressed air is pumped through a hydraulically smooth pipe. The pressure p_1, the density ρ_1 and the velocity $\bar{u}_{m1}$ are assumed to be known in the intake cross section.

$D = 10^{-2}\,\text{m} \quad p_1 = 8\cdot 10^5\,\dfrac{\text{N}}{\text{m}^2}$

$\bar{u}_{m1} = 10\,\dfrac{\text{m}}{\text{s}}$

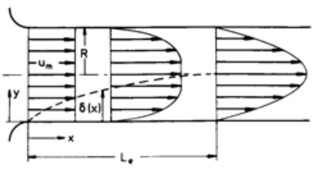
$\rho_1 = 10\,\dfrac{\text{kg}}{\text{m}^3} \quad \mu = 1.875\cdot 10^{-5}\,\dfrac{\text{Ns}}{\text{m}^2}$

(a) Derive the following relation with the aid of the momentum theorem:

$$\frac{dp}{dx} + \rho_1\,\bar{u}_{m1}\,\frac{d\bar{u}_m}{dx} + \frac{\lambda}{D}\,\frac{\rho}{2}\,\bar{u}_m^2 = 0$$

(b) Determine the length, over which the pressure drops by one half of its initial value for compressible flow with constant temperature and for incompressible flow!

5.6 The velocity distribution in the intake region of a laminar pipe flow is described by the following approximation:



$$\frac{u}{u_m} = \frac{f\left(\frac{y}{\delta}\right)}{1 - \frac{2}{3}\frac{\delta}{R} + \frac{1}{6}\left(\frac{\delta}{R}\right)^2}$$

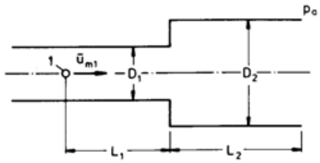
$$f\left(\frac{y}{\delta}\right) = \begin{cases} 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 & 0 \le y \le \delta(x) \\ 1 & \delta(x) \le y \le R \end{cases}$$

Given: u_m, R, ρ, μ

Determine the following quantities for the intake cross section, the end of the intake region, and for $\frac{\delta}{R} = 0.5$

(a) the momentum flux,
(b) the wall shear stress!

5.7 Water flows through a hydraulically smooth pipe with an discontinuous widening of the cross section into the open air.



$D = 0.02\,\text{m} \quad D_2 = 0.04\,\text{m} \quad L = 0.2\,\text{m}$

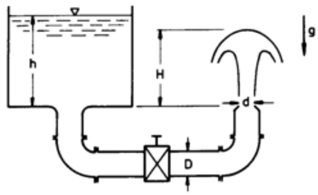
$\bar{u}_{m1} = 0.5\,\dfrac{\text{m}}{\text{s}} \quad \rho = 10^3\,\dfrac{\text{kg}}{\text{m}^3}$

$\mu = 10^{-3}\,\dfrac{\text{Ns}}{\text{m}^2}$

(a) At what length L_2 does the pressure difference $p_1 - p_a$ vanish?
(b) How large is the corresponding pressure loss?

Hint: Assume, that the wall shear stress in the widened part of the pipe can be determined with the equations for fully developed pipe flow!

5.8 The feed pipes of a fountain consist out of four straight pipes of total length L, two bends (loss coeficient ζ_K) and a valve (ζ_V).



$h = 10\,\text{m} \quad D = 0.05\,\text{m} \quad L = 4\,\text{m}$
$\zeta_K = 0.25 \quad \zeta_V = 4.5 \quad \lambda = 0.025$

Determine the volume rate of flow and the height H for dissipative and nondissipative flow with
(a) $d = \frac{D}{2}$,
(b) $d = D$!

Hint: Assume that the flow in the intake and in the nozzle is loss-free and fully developed in the straight pipes!

5.9 Water flows through a hydraulically smooth pipe.

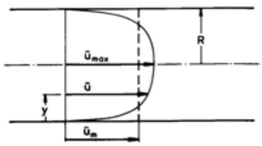
$D = 0.1\,\mathrm{m} \quad Re = 10^5 \quad \rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

$\mu = 10^{-3}\,\dfrac{\mathrm{Ns}}{\mathrm{m}^2}$

Determine
(a) the wall shear stress,
(b) the ratio of the velocities $\frac{\bar{u}_m}{\bar{u}_{max}}$,
(c) the velocity for $\frac{y\,u_*}{\nu} = 5$ and for $\frac{y\,u_*}{\nu} = 50$,
(d) the mixing length for $\frac{y\,u_*}{\nu} = 100$!

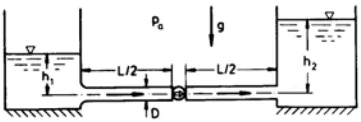
5.10 The velocity distribution of a turbulent pipe flow is approximately described by the ansatz

$$\frac{\bar{u}_m}{\bar{u}_{max}} = \left(\frac{y}{R}\right)^{\frac{1}{7}} \quad .$$



Determine
(a) the ratio of the velocities $\frac{\bar{u}_m}{\bar{u}_{max}}$,
(b) the ratio of the momentum fluxes $\frac{\dot{I}}{\rho\,\bar{u}_m^2\,\pi R^2}$!

5.11 A pump is feeding water through a rough pipe (equivalent sand roughness k_s) from the level h_1 to the level h_2.



$\dot{Q} = 0.63\,\dfrac{\mathrm{m}^3}{\mathrm{s}} \quad h_1 = 10\,\mathrm{m} \quad h_2 = 20\,\mathrm{m}$

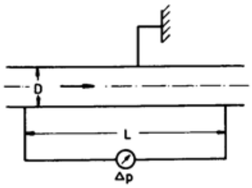
$L = 20\,\mathrm{km} \quad D = 1\,\mathrm{m} \quad k_s = 2\,\mathrm{mm}$

$\rho = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \nu = 10^{-6}\,\dfrac{\mathrm{m}^2}{\mathrm{s}}$

$p_a = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad g = 10\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

(a) Sketch the variation of the static pressure along the axis of the pipe!
Determine
(b) the pressure at the intake of the pump,
(c) the pressure at the exit of the pump,
(d) the net performance of the pump!

5.12 In a fully developed pipe flow with volume rate of flow $\dot{Q}$ a pressure drop Δp is measured over the distance L.



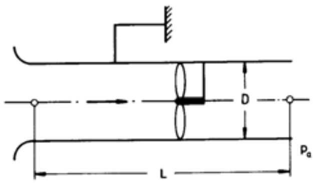
$\dot{Q} = 0.393\,\dfrac{\mathrm{m}^3}{\mathrm{s}} \quad L = 100\,\mathrm{m} \quad D = 0.5\,\mathrm{m}$

$\Delta p = 12820\,\dfrac{\mathrm{N}}{\mathrm{m}^2} \quad \rho = 900\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

$\mu = 5 \cdot 10^{-3}\,\dfrac{\mathrm{Ns}}{\mathrm{m}^2}$

Determine
(a) the pipe friction coefficient,
(b) the equivalent sand roughness of the pipe,
(c) the wall shear stress and the retaining force!
(d) How large would the pressure drop be in a hydraulically smooth pipe?

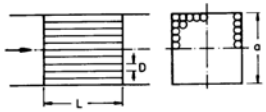
5.13 Air is to be conveyed through a rough pipe with a well rounded intake with the aid of a blower.



$L = 200\,\mathrm{m} \quad k_s = 1\,\mathrm{mm}$

Determine the ratio of the blower performance for the diameters $D = 0.1\,\mathrm{m}$ and $D = 0.2\,\mathrm{m}$ for the same volume rate of flow and very large Reynolds numbers!

5.14 Water is fed through a system of 100 pipes into a channel with quadratic cross section.

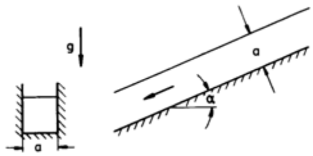


$$\dot{Q} = 0.01\,\frac{\mathrm{m}^3}{\mathrm{s}} \quad L = 0.5\,\mathrm{m} \quad a = 0.1\,\mathrm{m}$$

$$D = 0.01\mathrm{m} \quad \rho = 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3} \quad \mu = 10^{-3}\,\frac{\mathrm{Ns}}{\mathrm{m}^2}$$

For what length does the pressure loss of the channel become equal to that of the pipe system?

5.15 An open channel with quadratic cross section is inclined by the angle α.



$$\dot{Q} = 3 \cdot 10^{-4}\,\frac{\mathrm{m}^3}{\mathrm{s}} \quad a = 0.05\,\mathrm{m}$$

$$\rho = 10\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

$$\mu = 10^{-3}\,\frac{\mathrm{Ns}}{\mathrm{m}^2} \quad g = 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

Determine the angle of inclination and the pressure loss per unit length!

3.1.6 Similar Flows

6.1 Derive the dimensionless similarity parameters with the momentum equation for the x-direction

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad !$$

6.2 In hydraulically smooth pipes of different lengths and diameters the pressure loss is measured for different velocities, densities, and viscosities. The flow is fully developed, incompressible, and steady.
How can the results of the measurements be presented in a single curve?

6.3 A fluid flows slowly and steadily through a hydraulically smooth pipe. The flow is laminar and fully developed.
(a) Derive the Hagen-Poisseuille law with the aid of the dimensional analysis from the ansatz

$$\dot{Q} = \left(\frac{\Delta p}{L}\right)^{\alpha} \mu^{\beta}\, D^{\gamma} \quad !$$

(b) Show that the pipe friction coefficient is inversely proportional to the Reynolds number!

6.4 What is the drag of two spheres of different diameter but the same Reynolds number, if one moves in air and the other in water, and if the drag coefficient depends on the Reynolds number only?

$$\frac{\rho_a}{\rho_w} = 0.125 \cdot 10^{-2} \qquad \frac{\mu_a}{\mu_w} = 1.875 \cdot 10^{-2}$$

6.5 In an incompressible flow about a circular cylinder the frequency, with which vortices are shed, depends on the free-stream velocity, density, and viscosity. Determine the similarity parameters with the aid of the dimensional analysis!

6.6 The pipes in heat exchangers can oscillate due to excitation by the cross-flow they are exposed to. It is known, that in a flow about a circular cylinder, with its axis normal to the direction of the flow, the Strouhal number is constant for $200 \leq Re \leq 10^5$.

$$D = 0.1\,\mathrm{m} \quad v = 1\,\frac{\mathrm{m}}{\mathrm{s}} \quad v' = 3\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$\nu = 10^{-6}\,\frac{\mathrm{m}^2}{\mathrm{s}} \quad \nu' = 1.5 \cdot 10^{-5}\,\frac{\mathrm{m}^2}{\mathrm{s}}$$

Determine
(a) the minimum diameter of the model cylinder,
(b) the excitation frequency f, if for the smallest model $f' = 600\,\frac{1}{\mathrm{s}}$!

6.7 The power needed to overcome the aerodynamic drag of an automobile with quadratic cross-sectional area A is to be determined in a wind-tunnel experiment. The cross-sectional area of the model cannot exceed A_m in order to avoid blockage effects in the wind-tunnel.

$A = 4\,\mathrm{m}^2 \quad A_m = 0.6\,\mathrm{m}^2 \quad v = 30\,\dfrac{\mathrm{m}}{\mathrm{s}}$

(a) What speed has to be chosen for the measurement in the wind tunnel?
(b) Determine the power needed, if with a larger model the drag $F' = 810\,\mathrm{N}$ is measured!

6.8 Water flows through a model of a valve (volume rate of flow $\dot{Q}'$). Between intake and exit the pressure difference $\Delta p'$ is measured. The valve is supposed to be used in an air pipe.

$A = 0.18\,\mathrm{m}^2 \quad A' = 0.02\,\mathrm{m}^2$
$\rho = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \rho' = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$
$\mu = 1.875 \cdot 10^{-5}\,\dfrac{\mathrm{Ns}}{\mathrm{m}^2} \quad \mu' = 10^3\,\dfrac{\mathrm{Ns}}{\mathrm{m}^2}$
$\dot{Q}' = 0.2\,\dfrac{\mathrm{m}^3}{\mathrm{s}} \quad \Delta p' = 1.58 \cdot 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

(a) For what volume rate of flow are the flows through the model and the full-scale configuration similar?
(b) What is the pressure difference between intake and exit?

6.9 An axial blower (diameter D, number of revolutions n) is to be designed for air. In a model experiment with water (reduction scale 1:4) the increase of the total pressure $\Delta p_0'$ is measured.

$\dot{Q} = 30\,\dfrac{\mathrm{m}^3}{\mathrm{s}} \quad D = 1\,m \quad n = 12.5\,\dfrac{1}{\mathrm{s}}$
$\rho = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \mu = 1.875 \cdot 10^{-5}\,\dfrac{\mathrm{Ns}}{\mathrm{m}^2}$
$\rho' = 10^3\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \mu' = 10^{-3}\,\dfrac{\mathrm{Ns}}{\mathrm{m}^2}$
$\Delta p_0' = 0.3 \cdot 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

Determine
(a) the volume rate of flow and the number of revolutions during the experiment,
(b) the change of the total pressure of the blower,
(c) the power and the torque needed for driving the model and the main configuration!

6.10 The power of a propeller of an airplane is to be determined in a wind-tunnel experiment (Model scale 1:4) for the flight velocity v. In the test section of the wind tunnel the velocity can be varied between 0 and $300\,\dfrac{\mathrm{m}}{\mathrm{s}}$, the pressure between $0.5 \cdot 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$ and $5 \cdot 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$, and the temperature between 250 K and 300 K. The viscosity of air is described by the relation

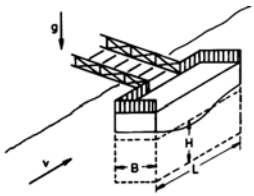
$$\frac{\mu}{\mu_{300K}} = \left(\frac{T}{300K}\right)^{0.75} .$$

$D = 1\,\mathrm{m} \quad n = 100\,\dfrac{1}{\mathrm{s}} \quad v = 200\,\dfrac{\mathrm{m}}{\mathrm{s}}$
$T = 300\,\mathrm{K} \quad R = 287\,\dfrac{\mathrm{Nm}}{\mathrm{kg\,K}} \quad \gamma = 1.4$
$p = 10^5\,\dfrac{\mathrm{N}}{\mathrm{m}^2}$

(a) Determine an operating point (v', p', T') such that the results of the measurements can be applied to the full-scale configuration!
(b) What number of revolutions must be used in the experiment?
(c) Determine the power, if in the experiment the power P' was measured!

6.11 A model experiment is to be carried out prior to the construction of a tanker (Model scale 1 : 100) in a towing basin.
(a) How large would the ratio of the kinematic viscosities $\frac{\nu'}{\nu}$ have to be?
(b) How large must the towing velocity in water be, if the aerodynamic drag can be neglected and if only the wave drag or only the frictional drag is taken into account?

6.12 A docking pontoon is fastened at the bank of a river. An experiment is to be carried out with a model scaled down to 1 : 16.



$L = 3.6\,\mathrm{m} \quad B = 1.2\,\mathrm{m} \quad H = 2.7\,\mathrm{m}$
$v = 3\,\frac{\mathrm{m}}{\mathrm{s}} \quad F'_D = 4\,\mathrm{N} \quad h' = 2.5\,\mathrm{cm}$
$\rho = 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

Determine

(a) the flow velocity in the model experiment,
(b) the force acting on the pontoon, if the force F' is measured in the experiment,
(c) the drag coefficient of the pontoon,
(d) the height of a wave h to be expected at the side of the pontoon, facing the oncoming flow, if the height h' was measured in the experiment!

6.13 In a refinery oil flows through a horizontal pipe line into a reservoir with pressure p_R, with the pressure at the intake being p_0. A safety valve is attached to the end of the pipe line, which in the case of emergency can close the pipe line within the time T. In a model experiment with water (diminution scale 1 : 10) the maximum pressure in front of the safety valve, measured during the shut-down procedure, is p'_{max}.

$p = 1.5 \cdot 10^5\,\frac{\mathrm{N}}{\mathrm{m}^2} \quad p_R = p'_R = 10^5\,\frac{\mathrm{N}}{\mathrm{m}^2}$
$\rho = 880\,\frac{\mathrm{kg}}{\mathrm{m}^3} \quad \rho' = 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3}$
$\mu = 10^{-1}\,\frac{\mathrm{Ns}}{\mathrm{m}^2} \quad \mu' = 10^{-3}\,\frac{\mathrm{Ns}}{\mathrm{m}^2}$
$T = 0.5\,\mathrm{s} \quad p'_{max} = 1.05 \cdot 10^5\,\frac{\mathrm{N}}{\mathrm{m}^2}$

Determine

(a) the pressure p'_0 and the shut-down time in the model experiment,
(b) the maximum pressure in the full-scale configuration!

6.14 In a petroleum pipe line (diameter D) the volume rate of flow is to be determined with a measuring throttle (diameter d). In a model experiment with water (diminution scale 1 : 10) a differential pressure $\Delta p'_D$ and a pressure loss $\Delta p'_l$ is measured at the measuring throttle.

$\dot{Q} = 1\,\frac{\mathrm{m}^3}{\mathrm{s}} \quad D = 1\,\mathrm{m} \quad d = 0.4\,\mathrm{m}$
$\rho = 800\,\frac{\mathrm{kg}}{\mathrm{m}^3} \quad \mu = 10^{-1}\,\frac{\mathrm{Ns}}{\mathrm{m}^2}$
$\rho' = 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3} \quad \mu' = 10^{-3}\,\frac{\mathrm{Ns}}{\mathrm{m}^2}$
$\Delta p'_D = 500\,\frac{\mathrm{N}}{\mathrm{m}^2} \quad \Delta p'_l = 400\,\frac{\mathrm{N}}{\mathrm{m}^2}$

Determine

(a) the flow velocity and the volume rate of flow in the model experiment,
(b) the discharge coefficient and the loss coefficient of the measuring throttle,
(c) the differential pressure and the pressure loss for the full-scale configuration!

3.1.7 Potential Flows of Incompressible Fluids

7.1 A cyclone is assumed to have the following velocity distribution:

$$v_\theta(r) = \begin{cases} \omega\, r & r \le r_0 \\ \frac{\omega\, r_0^2}{r} & r > r_0 \end{cases} \qquad v_r = 0$$

$r_0 = 10\,\mathrm{m} \quad \omega = 10\,\frac{1}{\mathrm{s}}$
$H = 100\,\mathrm{m} \quad \rho = 1.25\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

(a) Sketch $v_\theta(r)$!
(b) Determine the circulation for a circle around the axis of the cyclone for $r < r_0$, $r = r_0$ and $r > r_0$!
(c) Show that for $r > r_0$ the flow is irrotational!
(d) How large is the kinetic energy in a cylinder with radius $R = 2\,r_0$ and height H?

7.2 (a) State the definitions of potential and stream function for two-dimensional flow! What conditions have to be satisfied so that they can exist?
(b) How are $\nabla \cdot \boldsymbol{v}$ and $\nabla^2 \Phi$ and ∇ x $\boldsymbol{v}$ and $\nabla^2 \Psi$ related to each other?

7.3 Examine, whether potential and stream function exist for the following velocity fields!
(a) $u = x^2\,y \quad v = y^2\,x$
(b) $u = x \quad v = y$

(c) $u = y \quad v = -x$
(d) $u = y \quad v = x$
Determine potential and stream-function!

7.4 A two-dimensional flow is described by the stream function $\Psi = \left(\frac{U}{L}\right) xy$. In the point $x_{ref} = 0$, $y_{ref} = 1\,\mathrm{m}$ the pressure is $p_{ref} = 10^5 \,\frac{\mathrm{N}}{\mathrm{m}^2}$.

$$U = 2\,\frac{\mathrm{m}}{\mathrm{s}} \quad L = 1\,\mathrm{m} \quad \rho = 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

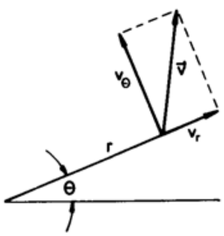
(a) Examine, whether the flow possesses a potential
Determine
(b) the stagnation points, the pressure coefficient, and the isotachs,
(c) velocity and pressure for $x_1 = 2\,m, \quad y_1 = 2\,m$,
(c) the coordinates of a particle, which at time $t = 0$ passes through the point x_1, y_1, for the time $t = 0.5\,s$,
(e) the pressure difference between these two points!
(f) Sketch the streamlines!

7.5 Given the potential

$$\Phi = yx^2 - \frac{y^3}{3} \quad .$$

(a) Determine the velocity components and examine, whether the stream function exists!
(b) Sketch the streamlines!

7.6 Determine the velocity fields
$v_r = \frac{c}{r} \quad v_\theta = 0$ and $v_r = 0 \quad v_\theta = \frac{c}{r}$
(a) ∇ x $\boldsymbol{v}$ and $\nabla \cdot \boldsymbol{v}$,
(b) potential and stream funktion,
(c) the circulation along a curve around the origin!



Hint: The following relations are valid for polar coordinates:

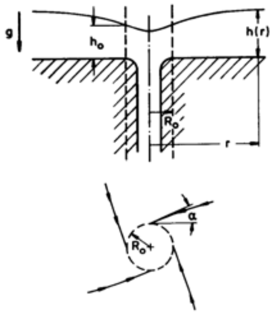
$$v_r = \frac{\partial \Phi}{\partial r} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad v_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -\frac{\partial \Psi}{\partial r}$$

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial (r\,v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \boldsymbol{v} = \left(\frac{1}{r}\frac{\partial (r\,v_\theta)}{\partial\, r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\boldsymbol{k}$$

7.7 Given the stream function $\Psi(r,0) = \frac{1}{n}\, r^n \,\sin(n\,\theta)$.
(a) Sketch the streamlines for $n = 0.5$; $n = 1$ und $n = 2$!
(b) Determine the pressure coefficient for the point $x = 0$, $y = 0$, if pressure and velocity are known for the point $x_{ref} = 1$, $y_{ref} = 1$!

7.8 Consider a large basin with an outlet. The flow outside of the outlet ($r > R_0$) can be described by superposition of a plane sink and a potential vortex. For $r = R_0$ the in-flow angle is α and the depth of water is h_0. The volume rate of flow of the discharging water is $\dot{Q}$.



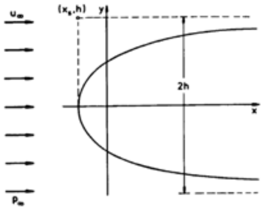
$$R_0 = 0.03\,\mathrm{m} \quad h_0 = 0.02\,\mathrm{m} \quad g = 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$\dot{Q} = 0.5 \cdot 10^{-3}\,\frac{\mathrm{m}^3}{\mathrm{s}} \qquad \alpha = 30^\circ$$

Determine
(a) the circulation Γ
(b) the shape of the water surface $h(r)$ for $r \geq R_0$,
(c) the depth of water at large distances from the outlet!

Hint: The discharge volume of the sink is to be determined for the radius $r = R_0$!

7.9 The free-stream velocity of a two-dimensional half-body with width $2h$ is u_∞.



Determine

(a) the stagnation point and the velocity in the point $x = x_s$, $y = h$,
(b) the contour of the half-body,
(c) the pressure distribution on the contour,
(d) the isobars,
(e) the curve along which the pressure is larger by the amount $\frac{\varrho}{4}\,u_\infty^2$ than the pressure p_∞ of the free-stream,
(f) the isotachs,
(g) the part of the flow field, in which the velocity component v is larger than $\frac{u_\infty}{2}$,
(h) the curve, which is inclined to the streamlines by 45°,
(i) the maximum deceleration a particle moving along the line of symmetry is experiencing between $x = -\infty$ and the stagnation point!

7.10 Given the stream function

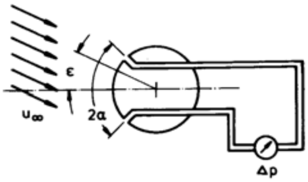
$$\Psi = u_\infty\, y \left(1 - \frac{R^2}{x^2 + y^2}\right) \quad .$$

(a) Sketch the streamlines for $x^2 + y^2 \geq R^2$!
Determine
(b) the pressure distribution on the contour $\Psi = 0$,
(c) the time it takes for a particle to move from the point $x = -3\,R$, $y = 0$ to the point $x = 2\,R$, $y = 0$!

7.11 Consider a parallel free-stream with velocity u_∞ flowing around a circular cylinder with radius R, with its axis normal to the direction of the oncoming flow being in the origin of coordinates. Determine

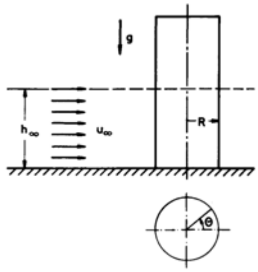
(a) the curve along which the pressure equals the free-stream pressure p_∞,
(b) the pressure on a circle around the origin of coordinates with radius $2R$!

7.12 The pressure difference Δp between two boreholes in a circular cylinder, with its axis normal to the direction of the oncoming flow, is a measure for the angle ϵ between the free-stream direction and the axis of symmetry.



(a) What is the relation between the pressure difference and the angle of attack?
(b) At what angle α does Δp attain its maximum value for every ϵ?

7.13 Consider a flow around a bridge pile with circular with cross section. The free-stream velocity is u_∞. The depth of water far upstream is h_∞.



$$u_\infty = 1\,\frac{\mathrm{m}}{\mathrm{s}} \quad h_\infty = 6\,\mathrm{m} \quad R = 2\,\mathrm{m}$$

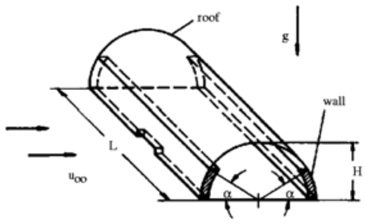
$$\rho = 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3} \quad g = 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

Determine

(a) the height of the water surface at the wall of the pile as function of θ,
(b) the height of the water surface at the stagnation points,
(c) the lowest depth of water, measured from the ground!

Hint: Assume two-dimensional flow!

7.14 The roof (weight G) of a hangar of length L and semi-circular cross section rests on the walls of the hangar, without being fixed. The hangar is completely closed except for a small opening on the leeward side.

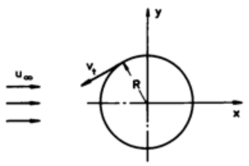


$L = 100\,\mathrm{m} \quad H = 10\,\mathrm{m} \quad G = 10^7\,\mathrm{N}$

$\alpha = 45° \quad u_\infty = 50\,\dfrac{\mathrm{m}}{\mathrm{s}} \quad \rho = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

Investigate, whether the roof has to be anchored to the walls!
Hint: Assume two-dimensional flow!

7.15 Consider a flow around a rotating circular cylinder of length L with its axis normal to the direction of the free-stream with velocity u_∞. The circumferential velocity of the flow on the surface of the cylinder caused by the rotation is v_t.



(a) Determine the circulation!
(b) Discuss the flow field for $v_t = u_\infty$!
(c) Determine the force acting on the cylinder!

3.1.8 Boundary Layers

8.1 Show that the drag coefficient of the flat plate at zero incidence is proportional to $\frac{1}{\sqrt{Re_L}}$ for a laminar boundary layer!

8.2 The surface of a flat plate is parallel to the direction of a free stream of air.
$u_\infty = 45\,\dfrac{\mathrm{m}}{\mathrm{s}} \quad \nu = 1.5 \cdot 10^{-5}\,\dfrac{\mathrm{m}^2}{\mathrm{s}}$
Determine
(a) the transition point for $Re_{crit.} = 5 \cdot 10^5$,
(b) the velocity at the point $x = 0.1$ m, $y = 2 \cdot 10^{-4}$ m with the aid of the Blasius solution! At what coordinate y does the velocity for $x = 0.15$ m attain the same value?
Sketch
(c) the variation of the boundary-layer thickness $\delta(x)$ and a velocity profile for $x < x_{crit.}$ and $x > x_{crit.}$,
(d) the wall-shear stress as a function of x for
$\dfrac{dp}{dx} < 0 \quad , \dfrac{dp}{dx} = 0$ and $\dfrac{dp}{dx} > 0$!

8.3 The surface of a flat plate is parallel to the dircetion of a free stream of water. Formulate the momentum thickness in terms of an integral over the wall-shear stress for a laminar boundary layer

$$-\int_0^x \frac{\tau(x; y = 0)}{\rho\, u_\infty^2}\, dx' \text{ dar!}$$

8.4 Air moves past a flat plate (length L, width B), with its surface parallel to the direction of the free stream.

$u_\infty = 10\,\dfrac{\mathrm{m}}{\mathrm{s}} \quad L = 0.5\,\mathrm{m} \quad B = 1\,\mathrm{m}$

$\rho = 1.25\,\dfrac{\mathrm{kg}}{\mathrm{m}^3} \quad \nu = 1.5 \cdot 10^{-5}\,\dfrac{\mathrm{m}^2}{\mathrm{s}}$

(a) Sketch the velocity profiles $u(y)$ for several values of x!
(b) State the boundary conditions for the boundary-layer equations!
(c) Sketch the distribution of the shear stress $\tau(y)$ for the position x!
(d) Compute the boundary-layer thickness at the trailing edge of the plate and its drag!

8.5 The velocity profile in the laminar boundary layer of a flat plate at zero incidence (length L) can be approximated by a polynomial of fourth degree

$$\begin{aligned}\frac{u}{u_\infty} &= a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + \\ &+ a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4 .\end{aligned}$$

(a) Determine the coefficients of the polynomial!
(b) Prove the validity of the following relations

$$\frac{\delta_1}{\delta} = \frac{3}{10},$$
$$\frac{\delta_2}{\delta} = \frac{37}{315},$$
$$\frac{\delta}{x} = \frac{5.84}{\sqrt{Re_x}},$$
$$c_D = \frac{1.371}{\sqrt{Re_L}}$$

8.6 The velocity profile in a laminar boundary layer on a flat plate at zero incidence of length L is approximated by
A) a polynom of fourth degree

$$\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

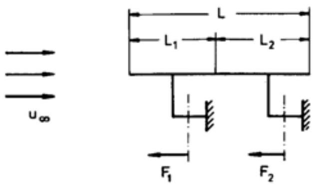
and
B) a sinosoidal ansatz

$$\frac{u}{u_\infty} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \quad .$$

(a) Determine δ_1, δ_2, δ, and c_w!
(b) Compute the boundary-layer thickness at the trailing edge of the plate and the drag for
$u_\infty = 1\,\frac{\text{m}}{\text{s}} \quad L = 0.5\text{m} \quad B = 1\text{m}$
$\rho = 10^3\,\frac{\text{kg}}{\text{m}^3} \quad \nu = 10^{-6}\,\frac{\text{m}^2}{\text{s}}$!

3.1.9 Drag

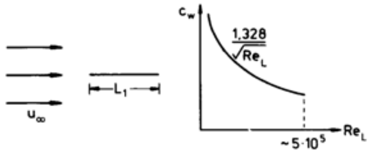
9.1 Two flat plates at zero incidence, one downstream from the other, are exposed to the free-stream velocity u_∞.

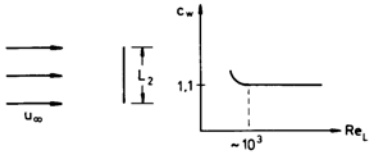


$u_\infty = 1\,\frac{\text{m}}{\text{s}} \quad L = L_1 + L_2 = 0.36\text{ m}$
$B = 1\text{ m} \quad \rho = 10^3\,\frac{\text{kg}}{\text{m}^3} \quad \nu = 10^{-6}\,\frac{\text{m}^2}{\text{s}}$

Determine
(a) the retaining forces F_1 and F_2 for $L_1 = L_2$,
(b) the lengths L_1 and L_1 for $F_1 = F_2$!

9.2 Two quadratic plates are exposed to a flow, one at zero incidence, the other with its surface normal to the direction of the free stream.





$u_\infty = 5\,\frac{\text{m}}{\text{s}} \quad L_1 = 1\text{ m} \quad \rho = 1.25\,\frac{\text{kg}}{\text{m}^3}$
$\nu = 15 \cdot 10^{-6}\,\frac{\text{m}^2}{\text{s}}$

(a) Explain the difference between frictional and pressure drag!
(b) How large must L_2 be, so that both plates generate the same drag?
(c) How does the drag depend on the free-stream velocity?

9.3 Two flat rectangular plates are exposed to a parallel flow. The plates have the same lateral lengths L_1 and L_2. The edge of plate 1, with lateral length L_1 is parallel to the direction of the free stream, and of plate 2 the edge with lateral length L_2. The free-stream velocity is u_∞.

$L_1 = 1\text{ m} \quad L_2 = 0.5\text{ m} \quad \nu = 10^{-6}\,\frac{\text{m}^2}{\text{s}}$

(a) Determine the ratio of the friction forces for $u_\infty = 0.4\,\frac{\text{m}}{\text{s}};\ 0.8\,\frac{\text{m}}{\text{s}};\ 1.6\,\frac{\text{m}}{\text{s}}$!
(b) How large would the free-stream velocity for the plate 2 have to be, if the free-stream velocity of plate 1 is $u_\infty = 0.196\,\frac{m}{s}$, and if both drag co-

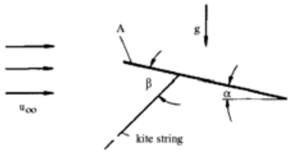
efficients are supposed to have the same value?

Hints:

$$c_D = \frac{0.074}{Re_L^{\frac{1}{5}}} - \frac{1700}{Re_L}$$

for $5 \cdot 10^5 < Re_L < 10^7$.

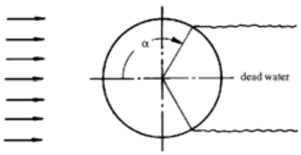
9.4 A kite (surface area A, weight W) generates the force F in the kite string at an angle of attack α.



$A = 0.5\,\mathrm{m}^2 \quad W = 10\,\mathrm{N} \quad u_\infty = 10\,\frac{\mathrm{m}}{\mathrm{s}}$
$\alpha = 10^\circ \quad \beta = 55^\circ \quad F = 42.5\,\mathrm{N}$
$\rho = 1.25\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

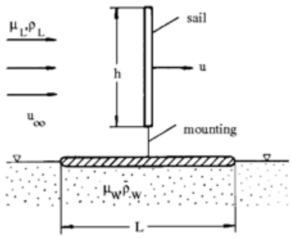
Determine the lift and drag coefficient of the kite!

9.5 Assume that the flow around a circular cylinder separates at $\alpha = 120^\circ$, that the pressure distribution up to the separation point can be determined with the potential flow theory, and that the pressure in the dead water region is constant!



Neglect the frictional drag and determine the drag coefficient of the cylinder!

9.6 A surfboard (width b) moves with the velocity u over the surface of quiescent water. The height of the triangular sail is h and its width b.



$L = 3.75\,\mathrm{m} \quad B = 0.5\,\mathrm{m} \quad u = 1.5\,\frac{\mathrm{m}}{\mathrm{s}}$
$\rho_w = 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3} \quad \mu_w = 10^{-3}\,\frac{\mathrm{Ns}}{\mathrm{m}^2}$
$\rho_a = 1.25\,\frac{\mathrm{kg}}{\mathrm{m}^3}$
$\mu_a = 1.875 \cdot 10^{-5}\,\frac{\mathrm{Ns}}{\mathrm{m}^2} \quad h = 4\mathrm{m} \quad b = 2\mathrm{m}$

Neglect the wave drag , the drag of the sail mounting, and the frictional drag of the upper side of the surfboard and determine

(a) the wind speed u_∞,
(b) the drag of the sail!
(c) How large would the frictional drag of the upper side of the board be?

Hints:

$$\begin{aligned} \text{Board} &: \; c_D = \frac{0.074}{Re_L^{\frac{1}{5}}} - \frac{1700}{Re_L} \\ (\text{for} &: \; 5 \cdot 10^5 < Re_L < 10^7) \\ \text{Sail} &: \; c_D = 1.2 \\ (\text{for} &: \; Re > 10^3) \end{aligned}$$

9.7 How large must the surface of the equivalent drag of a parachute at least be, in order to avoid the sinking speed in quiescent air to exceed v?

$v = 4\,\frac{\mathrm{m}}{\mathrm{s}} \quad W = 1000\,\mathrm{N} \quad c_D = 1.33$
$\rho = 1.25\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

9.8 A sphere and a circular cylinder of the same material fall with constant velocity in quiescent air. The axis of the cylinder is normal to the direction of the gravitational acceleration. For $0 < Re \leq 0.5$ the drag coefficient of a sphere is given by $c_D = \frac{24}{Re}$ and that of a circular cylinder with its axis normal to the oncoming flow by

$$c_D = \frac{8\,\pi}{Re(2 - \ln Re)} \quad .$$

$\rho = 800\,\dfrac{\text{kg}}{\text{m}^3} \quad \rho_a = 1.25\,\dfrac{\text{kg}}{\text{m}^3}$

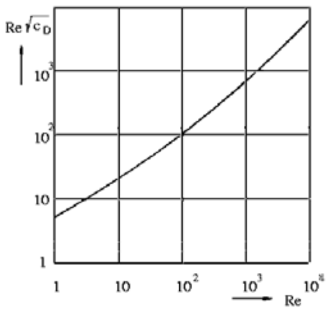
$\nu_a = 15 \cdot 10^{-6}\,\dfrac{\text{m}^2}{\text{s}} \quad g = 10\,\dfrac{\text{m}}{\text{s}^2}$

Determine

(a) the maximum diameters, for which these relations are valid,

(b) the corresponding sink velocities!

9.9 Determine with the aid of the diagram for the drag coefficient of a sphere



(a) the steady sink velocity of a spherical rain drop of diameter D in air,

(b) the steady ascending velocity of a spherical bubble of air of diameter D in water!

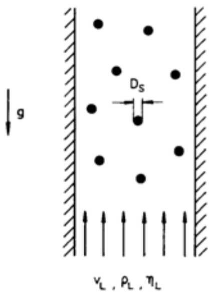
$D = 1\,\text{mm} \quad \rho_w = 10^3\,\dfrac{\text{kg}}{\text{m}^3}$

$\nu_w = 10^{-6}\,\dfrac{\text{m}^2}{\text{s}}$

$\rho_a = 1.25\,\dfrac{\text{kg}}{\text{m}^3} \quad \nu_a = 15 \cdot 10^{-6}\,\dfrac{\text{m}^2}{\text{s}}$

$g = 10\,\dfrac{\text{m}}{\text{s}}$

9.10 Spherically shaped dust particles (density ρ_s) are to be conveyed with a stream of air against the gravitational force.



$D_s = 5 \cdot 10^{-5}\,\text{m} \quad \rho_s = 2.5 \cdot 10^3\,\dfrac{\text{kg}}{\text{m}^3}$

$\rho_a = 1.25\,\dfrac{\text{kg}}{\text{m}^3} \quad \mu_a = 1.875 \cdot 10^{-5}\,\dfrac{\text{Ns}}{\text{m}^2}$

$g = 10\,\dfrac{\text{m}}{\text{s}^2}$

(a) At what velocity of the air v_1 are the dust particles suspended?

(b) How large is the velocity of the dust particles if the velocity of the air is $v_a = 3\,\dfrac{\text{m}}{\text{s}}$?

Hint:
Assume that the dust particles do not influence each other!

$$c_D = \frac{24}{Re}\,(1 + \frac{3}{16}\,Re) \qquad \text{for } 0 < Re < 1$$

9.11 A spherically shaped fog droplet (diameter D) is being suspended by an upward motion of air. At time $t = 0$ the air flow stops and the droplet begins to sink.

$D = 6 \cdot 10^{-5}\,\text{m} \quad \rho_w = 10^3\,\dfrac{\text{kg}}{\text{m}^3}$

$\rho_a = 1.25\,\dfrac{\text{kg}}{\text{m}^3} \quad \mu_a = 1.875 \cdot 10^{-5}\,\dfrac{\text{Ns}}{\text{m}^2}$

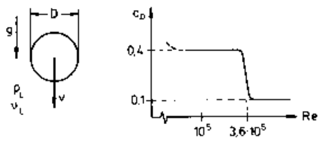
$g = 10\,\dfrac{\text{m}}{\text{s}^2}$

(a) How large is the velocity of the air flow prior to the lull?

(b) After what time does the droplet attain 99% of its steady sink velocity?

Hint: For $Re \leq 0.5$ the law $c_D = \frac{24}{Re}$ is valid for steady and unsteady flow.

9.12 A sphere is falling in steady motion in quiescent air with the velocity v_1. A downwash squall increases the velocity to v_2.



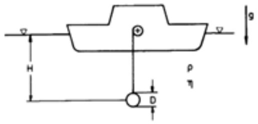
$D = 0.35\,\text{m} \quad G = 4.06\,\text{N} \quad v_1 = 13\,\dfrac{\text{m}}{\text{s}}$

$v_2 = 18\,\dfrac{\text{m}}{\text{s}} \quad \rho_a = 1.25\,\dfrac{\text{kg}}{\text{m}^3}$

$\nu_a = 15 \cdot 10^{-6}\,\dfrac{\text{m}^2}{\text{s}}$

(a) How large is the drag coefficient prior to the squall?
(b) What steady final velocity does the sphere attain after dying out of the squall?

9.13 A spherically shaped deep-sea probe is heaved with constant velocity from the depth H to the surface of the sea in the time T_1.



$D = 0.5\,\text{m} \quad H = 4000\,\text{m} \quad T_1 = 3\,\text{h}$
$\rho = 10^3\,\dfrac{\text{kg}}{\text{m}^3} \quad \mu = 10^{-3}\,\dfrac{\text{Ns}}{\text{m}^2} \quad g = 10\,\dfrac{\text{m}}{\text{s}^2}$

Assume constant density of the water, neglect the weight of the cable rope, and determine
(a) the power needed for heaving the probe, if the cable force is $F_1 = 2700$ N,
(b) the weight of the probe and the shortest heaving time, if the cable can take twice the value of the force,
(c) the power then needed!
Hint: Use the diagram of problem 9.12!

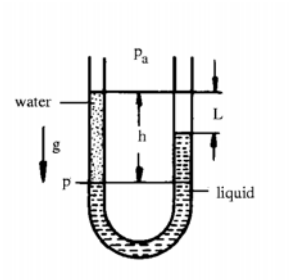
9.14 A sphere of diameter D and density ρ_s is vertically shot into quiescent air with initial velocity v_∞.
Assume a constant drag coefficient and determine
(a) the height of rise,
(b) the rise time,
(c) the velocity at impact on the ground,
(d) the falling time!
(e) What values do these quantities attain for a wooden sphere of density ρ_w and for a metal sphere of density ρ_m, if $c_D = 0.4$ and $c_w = 0$?

$D = 0.1\,\text{m} \quad v_0 = 30\,\dfrac{\text{m}}{\text{s}} \quad g = 10\,\dfrac{\text{m}}{\text{s}^2}$
$\rho_a = 1.25\,\dfrac{\text{kg}}{\text{m}^3} \quad \rho_w = 750\,\dfrac{\text{kg}}{\text{m}^3}$
$\rho_m = 7.5 \cdot 10^3\,\dfrac{\text{kg}}{\text{m}^3}$

3.2 Solutions

3.2.1 Hydrostatics

1.1



$$p = p_a + \rho_w\, g\, h = p_a + \rho_F\, g\, (h - L)$$
$$\rho_F = \rho_w \frac{h}{h-L} = 3 \cdot 10^3 \ \frac{\mathrm{kg}}{\mathrm{m}^3}$$

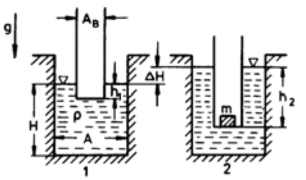
1.2

$$p_i = \frac{F_i}{A_i} + p_a$$
$$p_2 = p_1 + \rho\, g\, \Delta h_1 = p_3 - \rho\, g\, \Delta h_2$$
$$\Delta h_1 = \frac{p_2 - p_1}{\rho\, g} = 0.25 \ \mathrm{m}$$
$$\Delta h_2 = \frac{p_3 - p_2}{\rho\, g} = 0.33 \ \mathrm{m}$$

1.3

$$F_L = W$$
$$F_L = \rho_1\, h a^2\, g + \rho_2\, (a-h)\, a^2\, g$$
$$W = \rho_K\, a^3\, g$$
$$h = \frac{\rho_2 - \rho_K}{\rho_2 - \rho_1}\, a = 6.67 \cdot 10^{-2} \ \mathrm{m}$$

1.4



$$W_1 = F_{L1} = \rho\, A_B\, h_1\, g$$
$$W_2 = F_{L2} = \rho\, A_B\, h_2\, g$$
$$W_2 = W_1 + m\, g$$

The volume of the water remains constant.

$$A H - A_B\, h_1 = A\, (H + \Delta H) - A_B\, h_2$$
$$\Delta H = \frac{m}{\rho\, A}$$

1.5 (a)

$$\tau_0 = \frac{W_0}{\rho_m\, g} = 1.07 \cdot 10^5 \ \mathrm{m}^3$$

(b)

$$\Delta W = \rho_M\, A\, (h_0 - h_1)\, g$$
$$A = 1.95 \cdot 10^4 \ \mathrm{m}^2$$

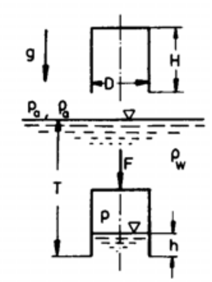
(c)

$$W_0 - \Delta W = \rho_M\, \tau_1\, g = \rho_F\, \tau_2\, g$$
$$\tau_2 - \tau_1 = \frac{W_0 - \Delta W}{\rho_M\, g}\left(\frac{\rho_M}{\rho_F} - 1\right)$$
$$= 2.44 \cdot 10^3 \ \mathrm{m}^3$$

(d)

$$\tau_2 - \tau_1 = (h_2 - h_1)\, A$$
$$h_2 = 10.625 \ \mathrm{m}$$

1.6



(a)

$$p_a \frac{\pi}{4} D^2\, H = p\, \frac{\pi}{4} D^2\, (H - h)$$
$$p = p_a + \rho_w\, g\, (T - h)$$

$$h = \frac{\rho_w\, g\, (T+h) + p_a}{2\, \rho_w\, g} - \sqrt{\left(\frac{\rho_w g (T+h) + p_a}{2 \rho_w g}\right)^2 - T H} = 2 \ \mathrm{m}$$

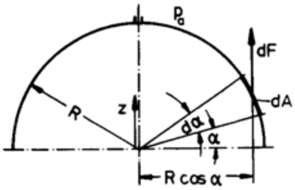
(b)

$$F = F_L - W_{air} - W$$
$$= \frac{\pi}{4} D^2\, g\, [(H-h)\ \rho_w - H\, \rho_a] - G$$
$$= -9.58 \cdot 10^3 \ \mathrm{N}$$

(c)

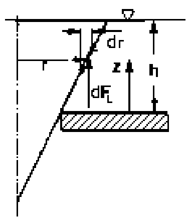
$$\begin{aligned} 0 &= F_L - W_{air} - W \\ h_0 &= H\left(1-\frac{\rho_a}{\rho_w}\right) - \frac{4\,W}{\pi\, D^2\,\rho_w\, g} \\ T_0 &= h_0\left(1+\frac{p_a}{\rho_w\, g\,(H-h_0)}\right) \\ &= 18.3\ \text{m} \end{aligned}$$

1.7



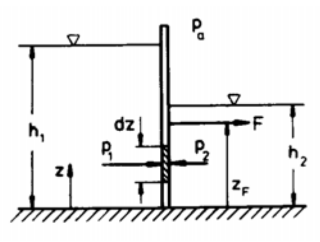
$$\begin{aligned} dF &= [p_i(z)-p_a]\, dA\, \sin\alpha \\ dA &= R\, d\alpha\, 2\,\pi\, R\, \cos\alpha \\ F &= \frac{\pi}{3}\, R^3\, \rho\, g = 1.05\cdot 10^4\ \text{N} \end{aligned}$$

1.8



$$\begin{aligned} dF_L &= \rho\, g\,(h-z)\, 2\pi\, r\, dr \\ \frac{r}{H-h+z} &= \frac{R}{H} \\ F_L &= \frac{\pi}{6}\, R^2\, H\, \rho\, g \\ F &= W - F_L \\ &= \frac{\pi}{3}\, R^2\, H\,(\rho_s - \frac{\rho}{2})\, g \\ &= 1.57\cdot 10^{-2}\ \text{N} \end{aligned}$$

1.9



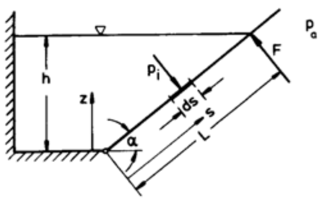
(a)

$$\begin{aligned} dF &= [p_1(z)-p_2(z)]\, B\, dz \\ z &\le h_1 : p_1(z) = p_a + \rho g(h_1 - z) \\ z &\le h_2 : p_2(z) = p_a + \rho g(h_2 - z) \\ h_2 &\le z \le h_1 :\ p_2 = p_a \\ F &= \frac{1}{2}\,\rho\, g\, B(h_1^2 - h_2^2) \\ &= 1.05\cdot 10^6\ \text{N} \end{aligned}$$

(b)

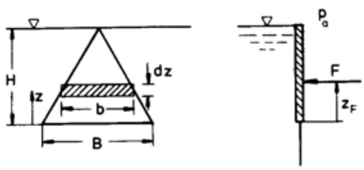
$$\begin{aligned} F\, z_F &= \int_{z=0}^{z=h_1} z\, dF \\ z_F &= \frac{1}{3}\,\frac{h_1^3-h_2^3}{h_1^2-h_2^2} = 1.86\ \text{m} \end{aligned}$$

1.10



$$\begin{aligned} F\, L &= \int_{s=0}^{s=L} s\, dF \\ dF &= [p_i(s)-p_a]\, B\, ds \\ p_i(s) &= p_a + \rho\, g\,[L\,\sin\alpha - z(s)] \\ F &= \frac{\rho\, g\, B\, h^2}{6\,\sin\alpha} = 3\cdot 10^4\ \text{N} \end{aligned}$$

1.11



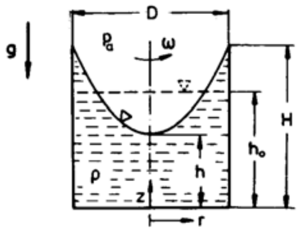
(a)

$$\begin{aligned} dF &= [p_i(z)-p_a]\, b(z)\, dz \\ p_i(z) &= p_a + \rho\, g\,(H-z) \\ F &= \frac{1}{4}\,\rho\, g\, B^3 = 2.5\cdot 10^3\ \text{N} \end{aligned}$$

(b)

$$\begin{aligned} F\, z_F &= \int_{z=0}^{z=H} z\, dF \\ z_F &= \frac{\sqrt{3}}{8}\, B = 0.217\ \text{m} \end{aligned}$$

1.12



Boundary condition:

$$
\begin{aligned}
r &= 0, \quad z = h: \quad p = p_a \\
p - p_a &= \rho\, g\, (h - z) + \frac{1}{2}\,\rho\,\omega^2\, r^2
\end{aligned}
$$

(a) Surface:

$$
\begin{aligned}
p &= p_a \\
z_0(r) &= h + \frac{\omega^2\, r^2}{2\, g}
\end{aligned}
$$

$$
\begin{aligned}
r = R: \quad z_0 &= H \\
\omega^2 &= 2\, g\, \frac{H - h}{R^2} \\
z_0(r) &= h + (H - h)\, \frac{r^2}{R^2}
\end{aligned}
$$

Volume of the water:

$$
\begin{aligned}
\pi R^2 h_0 &= \int_0^R z_0(r)\, 2\,\pi\, r\, dr \\
&= \pi R^2 h + \frac{1}{2}\pi R^2 (H - h)
\end{aligned}
$$

$$
\begin{aligned}
h &= 2\, h_0 - H = 0.4\ m \\
\omega &= \sqrt{\frac{4g}{R^2}\,(H - h_0)} = 13.9\ \frac{1}{\mathrm{s}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
r = R &: \quad p = p_a + \rho\, g\, (H - z) \\
z = 0 &: \quad p = p_a + \rho\, g\, h + \frac{\rho}{2}\,\omega^2\, r^2
\end{aligned}
$$

1.13

$$
\frac{dp}{dz} = -\rho(z)\, g
$$

(a)

$$
\rho(z) = \frac{p(z)}{R\, T_0} \qquad p = p_0\, e^{-\frac{gz}{RT_0}}
$$

(b)

$$
\begin{aligned}
\rho(z) &= \frac{p(z)}{R\,(T_0 - \alpha\, z)} \\
p &= p_0 \left(1 - \frac{\alpha\, z}{T_0}\right)^{\frac{g}{R\alpha}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\rho(z) &= \rho_0 \left(\frac{p}{p_0}\right)^{\frac{1}{\kappa}} \\
p &= p_0 \left(1 - \frac{\gamma - 1}{\gamma}\, \frac{gz}{R\, T_0}\right)^{\frac{\kappa}{\kappa - 1}}
\end{aligned}
$$

(d)

p/p_0	3000 m	6000 m	11000 m
a)	0.695	0.483	0.263
b)	0.686	0.457	0.215
c)	0.681	0.442	0.186

1.14 (a)

$$
\begin{aligned}
F &= F_L(z = 0) - W \\
&= (\rho_0\, \tau_0 - m)\, g \\
&= 10.6\ \mathrm{N}
\end{aligned}
$$

(b) for

$$
\begin{aligned}
z \le z_1: \quad p_1\, \tau_1 &= p_0\, \tau_0 \\
\frac{p_1}{p_0} &= e^{-\frac{g\, z_1}{R\, T_0}} \\
z_1 &= \frac{p_0}{\rho_0\, g}\, \ln \frac{\tau_1}{\tau_0} \\
&= 10.0\ \mathrm{km}
\end{aligned}
$$

(c)

$$
\begin{aligned}
F_L(z_2) &= W \\
z_2 &= \frac{p_0}{\rho_0\, g}\, \ln \frac{\rho_0\, \tau_1}{m} \\
&= 12.8\ \mathrm{km}
\end{aligned}
$$

1.15

$$
\begin{aligned}
F_L(z) &= W + W_{gas}(z) \\
F_L(0) &= W + W_{gas}(0) + F_s \\
F_L &= \rho\, \tau\, g \\
G_{gas} &= \rho_{gas}\, \tau\, g \\
\frac{\rho_{gas}(z)}{\rho_{gas}(0)} &= e^{-\frac{gz}{RT}} \\
z &= \frac{RT}{g}\, \ln\left(1 + \frac{F_s}{W}\right) \\
&= 7.84\ \mathrm{km}
\end{aligned}
$$

3.2.2 Hydrodynamics

2.1 (a)

$$\begin{aligned} \frac{dy}{dy} &= \frac{v}{u} = -\frac{v_0}{u_0}\ \tan(\omega\, t) \\ y &= \left[-\frac{v_0}{u_0}\ \tan(\omega\, t)\right]\ x + c \end{aligned}$$

Straight lines with slopes 0,−1,−∞

(b)

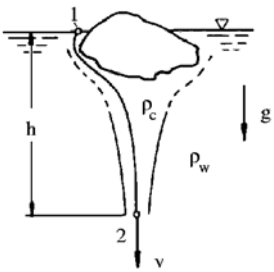
$$\begin{aligned} x(t) &= \int u\, dt + c_1 \\ &= \frac{u_0}{\omega}\ \sin(\omega\, t) + c_1 \\ y(t) &= \int v\, dt + c_2 \\ &= \frac{v_0}{\omega}\ \cos(\omega\, t) + c_2 \end{aligned}$$

$$\begin{aligned} \left(\tfrac{\omega}{u_0}\right)^2\ (x - c_1)^2 &\ \ + \\ \left(\tfrac{\omega}{v_0}\right)^2\ (y - c_2)^2 &\ = 1 \end{aligned}$$

Circles with radius 1 m.

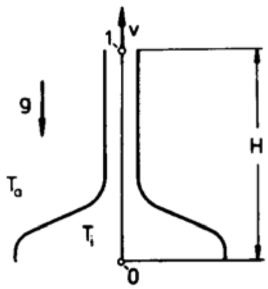
(c) Circle around the origin

2.2



$$\begin{aligned} p_1 + \rho_c\, g\, h &= p_2 + \rho_c\, \frac{v^2}{2} \\ p_2 &= p_1 + \rho_w\, g\, h \\ v &= \sqrt{2\, g\, h\, \frac{\rho_c - \rho_w}{\rho_c}} \\ &= 3.16\ \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

2.3



$$\begin{aligned} \frac{dp}{\rho} + \frac{1}{2}\, d\left(v^2\right) + g\, dz &= 0 \\ p &= \rho\, RT \end{aligned}$$

inner:

$$RT_i \ln\left(\frac{p_{1i}}{p_{0i}}\right) + \frac{v^2}{2} + g\, H = 0$$

outer:

$$\begin{aligned} RT_a \ln\left(\frac{p_{1a}}{p_{0a}}\right) + g\, H &= 0 \\ v = \sqrt{2\, g\, H\, \left(\frac{T_i}{T_a} - 1\right)} &= 31.6\ \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

2.4

$$\Delta p = \beta\, \frac{\rho}{2}\, v_\infty^2$$

(a) Assumption:

$$\frac{\rho\, v_\infty\, D}{\mu} > 250 \quad v_\infty = 0.5\ \frac{\mathrm{m}}{\mathrm{s}}$$
$$\left(\frac{\rho\, v_\infty\, D}{\mu} = 3000\right)$$

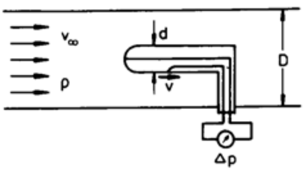
(b) Assumption:

$$\frac{\rho\, v_\infty\, D}{\mu} > 250 \quad v_\infty = 0.5\ \frac{\mathrm{m}}{\mathrm{s}}$$
$$\left(\frac{\rho\, v_\infty\, D}{\mu} = 300\right)$$

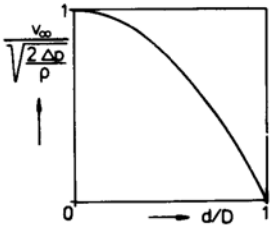
(c) Assumption:

$$2.5 < \frac{\rho\, v_\infty\, D}{\mu} < 250$$
$$v_\infty = 0.45\ \frac{\mathrm{m}}{\mathrm{s}} \qquad \left(\frac{\rho\, v_\infty\, D}{\mu} = 27\right)$$

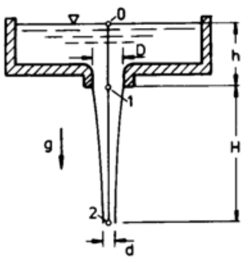
2.5



$$\Delta p = \frac{\rho}{2} v^2$$
$$v_\infty D^2 = v\,(D^2 - d^2)$$
$$\frac{v_\infty}{\sqrt{\frac{2\,\Delta p}{\rho}}} = 1 - \frac{d^2}{D^2}$$



2.6



$$\rho\, g\,(h + H) = \rho\, g\, H + \frac{\rho}{2} v_1^2$$
$$= \frac{\rho}{2} v_2^2$$
$$v_1 D^2 = v_2\, d^2$$
$$d = D \sqrt[4]{\frac{h}{h+H}} = 0.05 \text{ m}$$

2.7 (a)

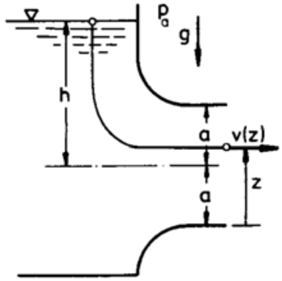
$$p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2$$
$$v_1 A_1 = v_2 A_2$$
$$= v_3 A_3$$

$$v_2 = \sqrt{\frac{2\,\Delta p}{\rho\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]}}$$
$$= 12 \frac{\text{m}}{\text{s}}$$
$$v_1 = 4 \frac{\text{m}}{\text{s}} \quad v_3$$
$$= 6 \frac{\text{m}}{\text{s}}$$

(b)

$$p_2 + \frac{\rho}{2} v_2^2 = p_3 + \frac{\rho}{2} v_3^2$$
$$p_3 = p_a = 10^5 \frac{\text{N}}{\text{m}^2}$$
$$p_2 = 0.46 \cdot 10^5 \frac{\text{N}}{\text{m}^2}$$
$$p_1 = 1.1 \cdot 10^5 \frac{\text{N}}{\text{m}^2}$$
$$p + \rho\, g\, h = p_a + \frac{\rho}{2} v_3^2$$
$$p = 1.08 \cdot 10^5 \frac{\text{N}}{\text{m}^2}$$

2.8



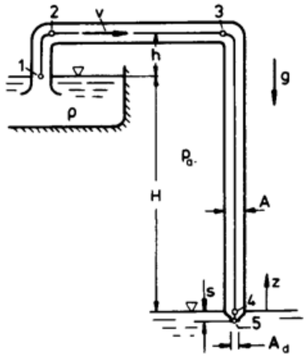
$$p_a + \rho\, g\,(a + h) = p_a + \rho\, g\, z + \tfrac{\rho}{2}\, v(z)^2$$

$$\dot{Q} = \int_0^{2a} v(z)\, B\, dz$$
$$= \frac{2}{3}\sqrt{2\,g}\, B \left[\sqrt{(h+a)^3} - \sqrt{(h-a)^3}\right]$$

$$\frac{\dot{Q}_0 - \dot{Q}}{\dot{Q}} = \frac{3\,\frac{a}{h}}{\sqrt{(1+\frac{a}{h})^3} - \sqrt{(1-\frac{a}{h})^3}} - 1$$

a/h	0.25	0.5
$\frac{\dot{Q}_0 - \dot{Q}}{\dot{Q}}$	0.264%	1.108%
a/h	0.75	1.0
$\frac{\dot{Q}_0 - \dot{Q}}{\dot{Q}}$	2.728%	6.066%

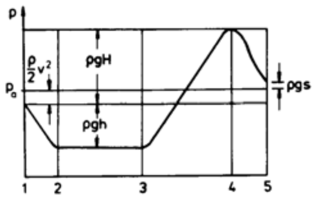
2.9



(a)

$$
\begin{aligned}
p_a + \rho\, g\, H &= p_5 - \rho\, g\, s + \frac{\rho}{2}\, v_5^2 \\
p_5 &= p_a + \rho\, g\, s \\
\dot{Q} &= A_d \sqrt{2\, g\, h} = 4\ \frac{\mathrm{m}^3}{\mathrm{s}}
\end{aligned}
$$

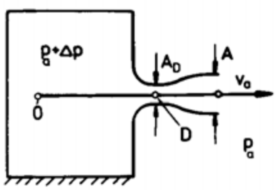
(b)



(c) Vapor bubbles are formed, if
$p_2 = p_3 = p_v$.

$$
\begin{aligned}
v_5^*\, A_d^* &= v^*\, A \\
p_a &= p_v + \rho\, g\, h + \frac{\rho}{2}\, v^{*2} \\
A_d^* &= A \sqrt{\frac{p_a - p_v}{\rho\, g\, H} - \frac{h}{H}} \\
&= 0.244\ \mathrm{m}^2
\end{aligned}
$$

2.10



(a)

$$
\begin{aligned}
v_D\, A_D &= v_a A \\
p_0 = p_a + \Delta p &= p_a + \frac{\rho}{2}\, v_a^2 \\
v_D = \sqrt{\frac{2\,\Delta p}{\rho}}\, \frac{A}{A_D} &= 4\, \frac{A}{A_D}\, \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

(b)

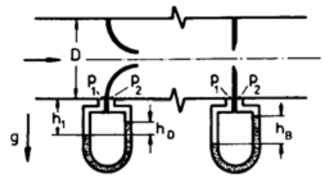
$$
\begin{aligned}
p_0 - p_a &= (p_0 - p_D) - (p_a - p_D) \\
&= \frac{\rho}{2}\, v_D^2 - \eta_D \left(\frac{\rho}{2}\, v_D^2 - \frac{\rho}{2}\, v_a^2\right)
\end{aligned}
$$

$$
\begin{aligned}
v_D &= \sqrt{\frac{2\Delta p}{\rho}}\, \frac{A}{A_D} \\
&\cdot \left[\left(\frac{A}{A_D}\right)^2 (1 - \eta_D) + \eta_D\right]^{-\frac{1}{2}} \\
v_D &= \frac{4\, \frac{A}{A_D}}{\sqrt{0.16\, \left(\frac{A}{A_D}\right)^2 + 0.84}}
\end{aligned}
$$

(c)

$$
\frac{A}{A_D} \to \infty : \quad v_D = 10\ \frac{\mathrm{m}}{\mathrm{s}}
$$

2.11



(a)

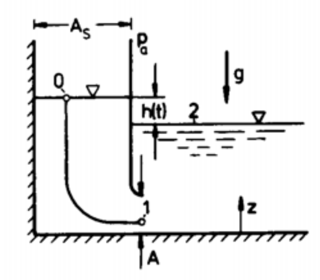
$$
\dot{Q} = m_D\, \frac{\pi\, D^2}{4}\, \alpha_D \sqrt{\frac{2\,(p_1 - p_2)_D}{\rho_w}}
$$

$$
\begin{aligned}
p_1 + \rho_w\, g\, h_1 &= p_2 + \rho_w\, g\, (h_1 - h_D) \\
&+ \rho_{Hg}\, g\, h_D \\
\dot{Q} &= 0.07\ \frac{\mathrm{m}^3}{\mathrm{s}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\dot{Q} &= m_B\, \frac{\pi\, D^2}{4}\, \alpha_B \sqrt{\frac{2\,(p_1 - p_2)_B}{\rho_w}} \\
\alpha_B &= \alpha_D\, \frac{m_D}{m_B} \sqrt{\frac{h_D}{h_B}} = 0.75
\end{aligned}
$$

2.12



$$
\begin{aligned}
p_a + \rho\, g\, z_0 + \frac{\rho}{2}\, v_0^2 &= p_1 + \rho\, g\, z_1 \\
&\quad + \frac{\rho}{2}\, v_1^2 \\
p_1 + \rho\, g\, z_1 &= p_a + \rho\, g\, z_2
\end{aligned}
$$

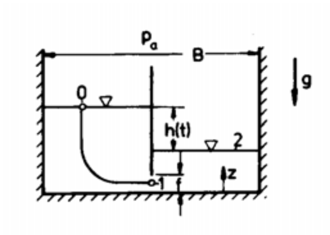
The assumption of quasi-steady flow requires that $\frac{A}{A_s} \ll 1$: $v_0^2 \ll v_1^2$.

$$v_1\, A = v_0\, A_s = -\frac{dh}{dt}\, A_s$$

$$T = -\frac{A_s}{A} \int_{h_0}^{0} \frac{dh}{\sqrt{2\, g\, h}} = \frac{A_s}{A} \sqrt{\frac{2\, h_0}{g}}$$

$$A = A_s\, \frac{\sqrt{\frac{2\, h_0}{g}}}{T} = 5\ \mathrm{m}^2$$

2.13

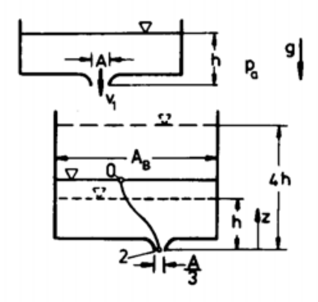


$$
\begin{aligned}
p_a + \rho\, g\, z_0 + \frac{\rho}{2}\, v_0^2 &= p_1 + \rho\, g\, z_1 \\
&\quad + \frac{\rho}{2}\, v_1^2 \\
p_1 + \rho\, g\, z_1 &= p_a + \rho\, g\, z_2
\end{aligned}
$$

$$
\begin{aligned}
v_0^2 &\ll v_1^2 \\
v_1\, f = v_0\, \frac{B}{2} &= -\frac{dz_0}{dt}\, \frac{B}{2} \\
\frac{dz_0}{dt} &= -\frac{dz_2}{dt} \\
\frac{dh}{dt} = \frac{d(z_0 - z_2)}{dt} &= 2\, \frac{dz_0}{dt}
\end{aligned}
$$

$$
\begin{aligned}
T &= -\frac{B}{4f} \int_{h_0}^{0} \frac{dh}{\sqrt{2\, g\, h}} \\
&= \frac{B}{2f} \sqrt{\frac{h_0}{2\, g}} = 100\ \mathrm{s}
\end{aligned}
$$

2.14



$$p_a + \rho\, g\, z_0 + \frac{\rho}{2}\, v_0^2 = p_a + \frac{\rho}{2}\, v_2^2$$

$$v_0^2 \ll v_2^2$$

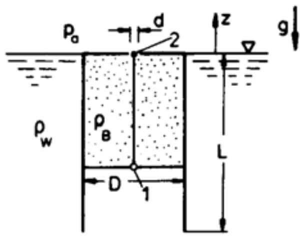
Volume rate of flow:

$$
\begin{aligned}
\frac{dz_0}{dt}\, A_B &= v_1\, A - v_2\, \frac{A}{3} \\
v_1 &= \sqrt{2\, g\, h}
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{3}{\sqrt{2\, g}}\, \frac{A_B}{A} \cdot \int_{h}^{4h} \frac{dz_0}{3\, \sqrt{h} - \sqrt{z_0}} \\
&= \frac{3}{\sqrt{2\, g}}\, \frac{A_B}{A} \cdot 2\, \Big[(3\, \sqrt{h} - \sqrt{z_0}) - \\
&\qquad -3\, \sqrt{h}\, \ln(3\, \sqrt{h} - \sqrt{z_0})\Big]_h^{4h}
\end{aligned}
$$

$$T = 6\, \frac{A_B}{A} \sqrt{\frac{h}{2\, g}}\, [3 \ln 2 - 1] = 108\ \mathrm{s}$$

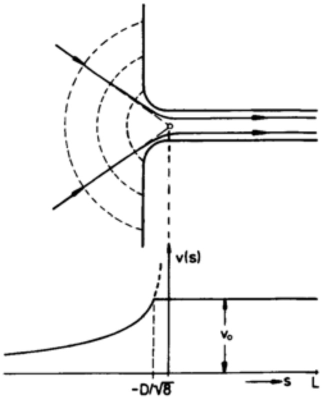
2.15



$$
\begin{aligned}
p_1 + \rho_B\, g\, z_1 + \frac{\rho_B}{2}\, v_1^2 &= p_a + \frac{\rho_B}{2}\, v_2^2 \\
p_a &= p_1 + \rho_w\, g\, z_1 \\
v_1^2 &\ll v_2^2 \\
v_2\, d^2 &= v_1\, D^2 \\
&= \frac{dz_1}{dt}\, D^2
\end{aligned}
$$

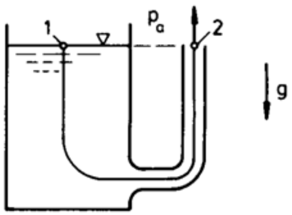
$$\begin{aligned}
T &= \left(\frac{D}{d}\right)^2 \sqrt{\frac{\rho_B}{2\,g\,(\rho_w-\rho_B)}} \\
&\quad \int_{-L}^{0} \frac{dz_1}{\sqrt{-z_1}} \\
&= \left(\frac{D}{d}\right)^2 \sqrt{\frac{\rho_B}{\rho_w-\rho_B}} \sqrt{\frac{2\,L}{g}} = 80\ \mathrm{s}
\end{aligned}$$

2.16



$$\begin{aligned}
\int_{-\infty}^{L} \frac{\partial v}{\partial t} ds &= \int_{-\infty}^{-\frac{D}{\sqrt{8}}} \frac{\partial}{\partial t} \left(\frac{\frac{v_0\,\pi\,D^2}{4}}{2\,\pi s^2}\right) ds + \\
&+ \int_{-\frac{D}{\sqrt{8}}}^{L} \frac{\partial v_0}{\partial t}\, ds = \\
&= \left(\frac{D}{\sqrt{2}} + L\right) \frac{dv_0}{dt}
\end{aligned}$$

2.17



(a)

$$v_0 = \sqrt{2\,g\,h} = 10\ \frac{\mathrm{m}}{\mathrm{s}}$$

(b)

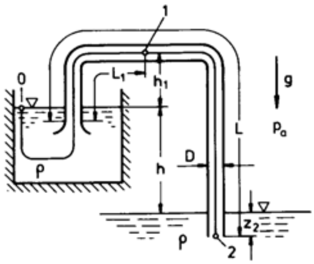
$$p_a + \frac{\rho}{2}\, v_1^2 = p_a + \frac{\rho}{2}\, v_2^2$$

$$\begin{aligned}
&\quad + \rho \int_{s_1}^{s_1} \frac{\partial v}{\partial t}\, ds \\
\int_{s_1}^{s_1} \frac{\partial v}{\partial t}\, ds &\approx L\, \frac{dv_2}{dt} \quad \left(\frac{D}{L} \ll 1\right) \\
T &= -2\,L \int_{v_0}^{\frac{v_0}{2}} \frac{dv_2}{v_2^2} \\
&= \frac{2\,L}{\sqrt{2\,g\,h}} = 2\,\mathrm{s}
\end{aligned}$$

(c)

$$\begin{aligned}
Q &= A \int_0^T v_2\, dt \\
&= -2\,A\,L \int_{v_0}^{\frac{v_0}{2}} \frac{dv_2}{v_2} \\
&= \frac{\pi}{2}\, L\, D^2\, \ln 2 = 0.279\ \mathrm{m}^3
\end{aligned}$$

2.18



(a)

$$\begin{aligned}
p_a + \rho_B\, g\,(h+z_2) &= p_a + \rho\, g\, z_2 \\
&+ \frac{\rho}{2}\, v_2^2 \\
&+ \rho \int_{s_0}^{s_2} \frac{\partial v}{\partial t}\, ds \\
\int_{s_0}^{s_2} \frac{\partial v}{\partial t}\, ds &\approx L\, \frac{dv_2}{dt}
\end{aligned}$$

$$\begin{aligned}
T &= 2\,L \int_0^{0.99\sqrt{2\,g\,h}} \frac{dv_2}{2\,g\,h - v_2^2} \\
&= \frac{L}{\sqrt{2\,g\,h}} \\
&\quad \ln \left[\frac{\sqrt{2\,g\,h} + v_2}{\sqrt{2\,g\,h} - v_2}\right]_0^{0.99\sqrt{2\,g\,h}} \\
&= 10.6\ \mathrm{s}
\end{aligned}$$

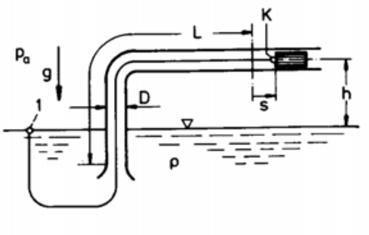
(b)

$$\begin{aligned}
p_a &= p_1 + \rho\, g\, h_1 + \frac{\rho}{2}\, v_2^2 \\
&\quad + \rho\, L_1\, \frac{dv_2}{dt} \\
p_a &= p_{1e} = \rho\, g\, h_1 + \frac{\rho}{2}\, v_{2e}^2
\end{aligned}$$

for:

$$\frac{dv_2}{dt} = \frac{1}{L}\left(g\,h - \frac{v_2}{2}\right)$$
$$v_2 = 0.99\sqrt{2\,g\,h}$$
$$p_1 - p_{1e} = \rho\,g\,h\left(1 - 0.99^2\right)\cdot$$
$$\cdot\left(1 - \frac{L_1}{L}\right)$$
$$= 746\,\frac{\mathrm{N}}{\mathrm{m}^2}$$

2.19

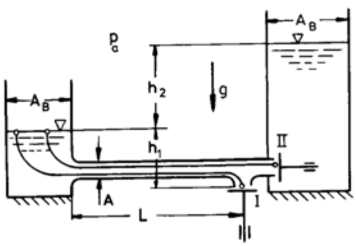


$$p_a = p_p + \rho\,g\,h + \frac{\rho}{2}\,v_p^2 + \rho\int_{s_1}^{s_p}\frac{\partial v}{\partial t}\,ds$$
$$\int_{s_1}^{s_p}\frac{\partial v}{\partial t} \approx L\,\frac{dv_p}{dt} \quad \left(\frac{s_0}{L} \ll 1\right)$$

$$p_p = p_a - \rho\,g\,h + \rho\,s_0\,\omega^2\times\left[L\,\sin\omega t - \frac{s_0}{2}\cos^2\omega t\right]$$
$$p_{pmin} = p_v$$
$$p_p = p_{pmin} \text{ with } \cos\omega t = 0$$

$$\text{(follows from}\frac{dp_p}{dt} = 0)$$
$$\omega = \sqrt{\frac{p_a - p_v - \rho\,g\,h}{\rho\,s_0\,L}} = 8.8\,\frac{1}{\mathrm{s}}$$

2.20



(a)

$$p_a{+}\rho\,g\,h_1 = p_a + \frac{\rho}{2}\,v^2 + \rho\,L\,\frac{dv}{dt}$$
$$Q_I = A\int_0^{T_I} v dt$$
$$= 2\,AL\int_0^{v_I}\frac{vdv}{2\,g\,h_1 - v^2}$$
$$= -AL\ln\left(1 - \frac{v_I^2}{2\,g\,h_1}\right)$$

Determination of v_I:

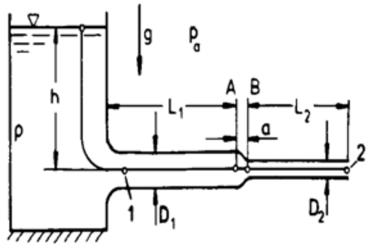
$$T_I = 2\,L\int_0^{v_I}\frac{dv}{2\,g\,h_1 - v^2}$$
$$= \frac{L}{\sqrt{2\,g\,h_1}}\,\ln\frac{\sqrt{2\,g\,h_1} + v_I}{\sqrt{2\,g\,h_1} - v_I}$$
$$v_I = \sqrt{2\,g\,h_1}\,\frac{e^{\frac{T_I\sqrt{2\,g\,h_1}}{L}} - 1}{e^{\frac{T_I\sqrt{2\,g\,h_1}}{L}} + 1}$$
$$Q_I = 0.240\ \mathrm{m}^3$$

(b)

$$p_a + \rho\,g\,h_1 = p_a + \rho\,g\,(h_1 + h_2) + \frac{\rho}{2}\,v^2 + \rho\,L\,\frac{dv}{dt}$$

$$Q_{II} = A\int_{T_I}^{T_{II}} v\,dt$$
$$= -2\,AL\int_{v_I}^{0}\frac{vdv}{2\,g\,h_2 + v^2}$$
$$= AL\ln\left(1 + \frac{v_I^2}{2\,g\,h_2}\right)$$
$$= 0.194\ \mathrm{m}^3$$

2.21



(a)

$$p_a + \rho\,g\,h = p_a + \frac{\rho}{2}\,v_2^2 + \rho\left(L_1\frac{dv_1}{dt} + L_2\frac{dv_2}{dt}\right)$$

$$v_1\, D_1^2 = v_2\, D_2^2$$

$$\begin{aligned} T &= 2\left[L_1\left(\frac{D_2}{D_1}\right)^2 + L_2\right]\cdot \\ &\cdot \int_0^{0.99\sqrt{2\,g\,h}} \frac{dv_2}{2\,g\,h - v_2^2} \\ &= \frac{L_1\left(\frac{D_2}{D_1}\right)^2 + L_2}{\sqrt{2\,g\,h}}\cdot \\ &\cdot \ln\left[\frac{\sqrt{2\,g\,h}+v_2}{\sqrt{2\,g\,h}-v_2}\right]_0^{0.99\sqrt{2\,g\,h}} \\ &= 5.231\,\mathrm{s} \end{aligned}$$

(b)

$$\begin{aligned} Q &= \left[L_1\left(\frac{D_2}{D_1}\right)^2 + L_2\right] A_2\cdot \\ &\cdot \int_0^{0.99\sqrt{2\,g\,h}} \frac{v_2\, dv_2}{2\,g\,h - v_2^2} \\ &= -\left[L_1\left(\frac{D_2}{D_1}\right)^2 + L_2\right] A_2\cdot \\ &\cdot \ln\left[2\,g\,h - v_2^2\right]_0^{0.99\cdot\sqrt{2\,g\,h}} \\ &= 0.048\ \mathrm{m}^3 \end{aligned}$$

(c)

$$p_A + \frac{\rho}{2}\, v_1^2 = p_a + \frac{\rho}{2}\, v_2^2 + \rho\, L_2 \frac{dv_2}{dt}$$
$$p_B + \frac{\rho}{2}\, v_2^2 = p_a + \frac{\rho}{2}\, v_2^2 + \rho\, L_2 \frac{dv_2}{dt}$$

as shown under (a):

$$\frac{dv_2}{dt} = \frac{2\,g\,h - v_2^2}{2\left[L_1\left(\frac{D_2}{D_1}\right)^2 + L_2\right]}$$

t = 0:

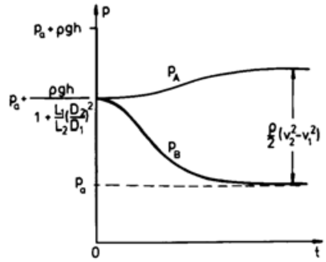
$$\begin{aligned} p_A &= p_B = p_A + \frac{\rho\,g\,h}{1+\frac{L_1}{L_2}\left(\frac{D_2}{D_1}\right)^2} \\ &= 1.16\cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

t = T:

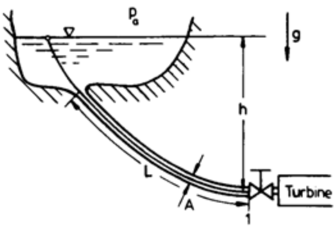
$$\begin{aligned} p_A &= p_a + \rho\,g\,h\left[0.99^2\left(1-\frac{D_2^4}{D_1^4}\right)\right. \\ &+ \left.\frac{1-0.99^2}{1+\frac{L_1}{L_2}\left(\frac{D_2}{D_1}\right)^2}\right] \\ &= 1.187\cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

$$\begin{aligned} p_B &= p_a + \frac{1-0.99^2}{1+\frac{L_1}{L_2}\left(\frac{D_2}{D_1}\right)^2}\rho\,g\,h \\ &= 1.003\cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

(d)



2.22

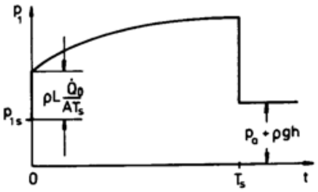


(a)

$$\begin{aligned} p_a + \rho\,g\,h &= p_{1s} + \frac{\rho}{2}\, v_{1s}^2 \\ v_{1s} &= \frac{\dot{Q}_0}{A} \\ p_{1s} - p_a &= \rho\left(g\,h - \frac{\dot{Q}_0^2}{2\,A^2}\right) \\ &= 18.875\cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

(b)

$$\begin{aligned} p_a + \rho\,g\,h &= p_1 + \frac{\rho}{2}\, v_1^2 + \\ &+ \rho\, L\, \frac{Dv_1}{dt} \\ \dot{Q}(t) &= \dot{Q}_0\left(1-\frac{t}{T_s}\right) \\ p_1(t) &= p_a + \rho\,g\,h + \rho\, L \frac{\dot{Q}_0}{A\,T_S} - \\ &- \frac{\rho}{2}\left(\frac{\dot{Q}_0}{A}\right)^2\left(1-\frac{t}{T_S}\right)^2 \end{aligned}$$

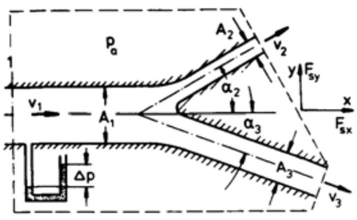


1.

$$
\begin{aligned}
\Delta p_{zul} &= p_{1max} - p_a \\
&= \rho\, g\, h + \rho\, L \frac{\dot{Q}_0}{A\, T_S} \\
T_S &= 0.25 \text{ s}
\end{aligned}
$$

3.2.3 Momentum and Moment of Momentum Theorem

3.1



(a)

$$
\begin{aligned}
p_1 + \frac{\rho}{2}\, v_1^2 &= p_a + \frac{\rho}{2}\, v_2^2 = p_a + \frac{\rho}{2}\, v_3^2 \\
\Delta p &= p_1 - p_a \\
v_1\, A_1 &= v_2\, A_2 + v_3\, A_3 \\
v_1 &= \sqrt{\frac{2\,\Delta p}{\rho}\; \frac{1}{\left(\frac{A_1}{A_2+A_3}\right)^2 - 1}} \\
&= 2.58\ \frac{\text{m}}{\text{s}} \\
v_2 &= v_3 = \frac{A_1}{A_2 + A_3}\, v_1 \\
&= 5.16\ \frac{\text{m}}{\text{s}}
\end{aligned}
$$

(b)

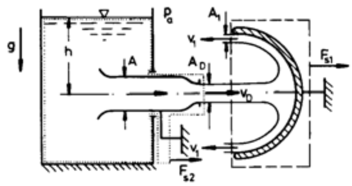
$$
\begin{aligned}
&\rho\, v_3^2\, A_3\, \cos\alpha_3 + \rho\, v_2^2\, A_2\, \cos\alpha_2 \\
&-\rho\, v_1^2\, A_1 = (p_1 - p_a)\, A_1 + F_{sx} \\
&F_{sx} = -866.4 \text{ N}
\end{aligned}
$$

$$
\begin{aligned}
&\rho\, v_2^2\, A_2\, \sin\alpha_2 - \rho\, v_3^2\, A_3\, \sin\alpha_3 = F_{sy} \\
&F_{sy} = -238.4 \text{ N}
\end{aligned}
$$

(c)

$$
\begin{aligned}
A_2\, \sin\alpha_2 - A_3\, \sin\alpha_3^* &= 0 \\
\alpha_3^* &= 12.37°
\end{aligned}
$$

3.2 (a)

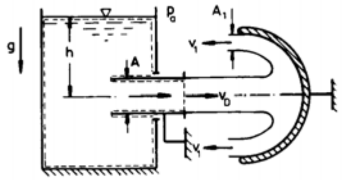


$$
-\rho\, v_D^2\, A_D - 2\,\rho\, v_1^2\, A_1 = F_{s1}
$$

$$
\begin{aligned}
v_D &= \sqrt{2\, g\, h} \\
v_D\, A_D &= 2\, v_1\, A_1 \\
p_a + \frac{\rho}{2}\, v_D^2 &= p_a + \frac{\rho}{2}\, v_1^2 \\
F_{s1} = -4\,\rho\, g\, h\, A_D &= -2 \cdot 10^4 \text{ N}
\end{aligned}
$$

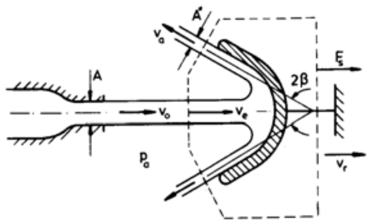
$$
\begin{aligned}
\rho\, v_D^2\, A_D &= (p_a + \rho\, g\, h - p_a)\, A + \\
&+ F_{s2} \\
F_{s2} &= \rho\, g\, h (2\, A_D - A) = 0
\end{aligned}
$$

(b)



$$
\begin{aligned}
\rho\, v_D^2\, A_D &= (p_a + \rho\, g\, h - p_a)\, A \\
F_{s1} &= -2\,\rho\, g\, h\, A \\
&= -2 \cdot 10^4 \text{ N} \\
F_{s2} &= 0
\end{aligned}
$$

3.3



(a)

$$P = -F_S\, v_r$$

Moving control surface:

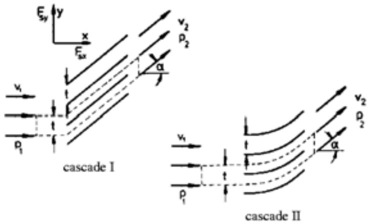
$$\begin{aligned} -\rho\, v_e^2\, A &- 2\,\rho\, v_a^2\, A^*\, \cos\beta = F_s \\ v_e &= v_0 - v_r \\ p_a + \frac{\rho}{2}\, v_e^2 &= p_a + \frac{\rho}{2}\, v_a^2 \\ v_e\, A &= 2\, v_a\, A^* \end{aligned}$$

$$\begin{aligned} P &= \rho\,(v_0 - v_r)^2\, v_r\, A\,(1 + \cos\beta) \\ \frac{dP}{dv_r} &= 0 \\ v_r &= \frac{v_0}{3} = 20\ \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

(b)

$$\begin{aligned} F_s &= -\frac{4}{9}\,\rho\, v_0^2\, A\,(1 + \cos\beta) \\ &= 3.08 \cdot 10^5\ \mathrm{N} \end{aligned}$$

3.4



(a)

$$\begin{aligned} v_1\, B\, t &= v_2\, B\, t\, \cos\alpha \\ v_2 &= \frac{v_1}{\cos\alpha} \end{aligned}$$

Cascade I

(b)

$$\begin{aligned} -\rho\, v_1^2\, B\, t + \rho\, v_1^2\, B\, t\, \cos^2\alpha \\ = (p_1 - p_2)\, B\, t + F_{sx} \\ \rho\, v_2^2\, B\, t\, \sin\alpha\, \cos\alpha = F_{sy} \end{aligned}$$

Force normal to the blade:

$$\tan\alpha = -\frac{F_{sx}}{F_{sy}}$$

$$p_1 - p_2 = -\rho\, v_1^2\, \tan^2\alpha$$

(c)

$$\begin{aligned} p_{01} - p_{02} &= (p_1 - p_2) + \frac{\rho}{2}\,(v_1^2 - v_2^2) \\ &= \frac{\rho}{2}\, v_1^2\, \tan^2\alpha \end{aligned}$$

(d)

$$\begin{aligned} F_x &= -F_{sx} = \rho\, v_1^2\, Bt\, \tan^2\alpha \\ F_y &= -F_{sy} = -\rho\, v_1^2\, Bt\, \tan\alpha \end{aligned}$$

Cascade II

(b)

$$\begin{aligned} p_1 + \frac{\rho}{2}\, v_1^2 &= p_2 + \frac{\rho}{2}\, v_2^2 \\ p_1 - p_2 &= \frac{\rho}{2}\, v_1^2\, \tan^2\alpha \end{aligned}$$

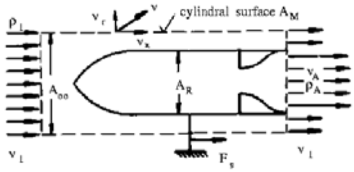
(c)

$$p_{01} - p_{02} = 0$$

(d) Momentum equ. as for cascade I

$$\begin{aligned} F_x &= \frac{\rho}{2}\, v_1^2\, Bt\, \tan^2\alpha \\ F_y &= -\rho\, v_1^2\, Bt\, \tan\alpha \end{aligned}$$

3.5



(a)

$$\begin{aligned} \rho_1\, v_1\, A_\infty &= \rho_1\, v_1\,(A_\infty - A_R) \\ &\quad + \Delta\dot{m} \\ \Delta\dot{m} &= \rho_1\, v_1\, A_R \end{aligned}$$

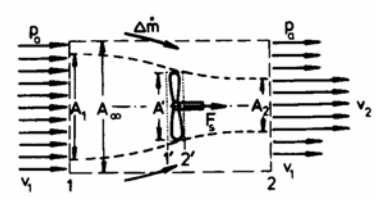
(b)

$$-\rho_1\, v_1^2\, A_\infty + \rho_1\, v_1^2\,(A_\infty - A_R) + \\ + \rho_A\, v_A^2\, A_R + \int_{A_M} \rho_1\, v_x\, v_r\, dA = F_s$$

$$\text{For} \quad \frac{A_\infty}{A_R} \gg 1 \quad : \quad v_x = v_1$$

$$\begin{aligned} \int_{A_M} \rho_1\, v_x\, v_r\, dA &= v_1 \int_{A_M} \rho_1\, v_r\, dA \\ &= v_1\, \Delta\, \dot{m} \\ F_s &= \rho_A\, v_A^2\, A_R \\ P &= F_s\, v_1 \\ &= \rho_A\, v_A^2\, v_1\, A_R \end{aligned}$$

3.6



(a)

$$\begin{aligned} p_a + \frac{\rho}{2}\, v_1^2 &= p_{1'} + \frac{\rho}{2}\, v'^2 \\ p_{2'} + \frac{\rho}{2}\, v'^2 &= p_a + \frac{\rho}{2}\, v_2^2 \\ v_1\, A_1 = v'\, A' &= v_2\, A_2 \end{aligned}$$

$$\begin{aligned} 0 &= (p_{1'} - p_{2'})\, A' + F_s \\ &- \rho\, v_1^2\, A_\infty + \rho\, v_2^2\, A_2 \\ &+ \rho\, v_1^2\, (A_\infty - A_2) \\ &- \Delta\, \dot{m}\, v_1 = F_s \end{aligned}$$

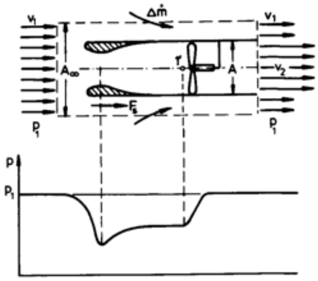
See problem 4.3

$$\begin{aligned} \rho\, v_1^2\, A_\infty + \Delta\, \dot{m} &= \rho\, v_2^2\, A_2 \\ &+ \rho\, v_1^2\, (A_\infty - A_2) \\ v' &= \frac{v_1 - v_2}{2} \\ &= 6.5\, \frac{\text{m}}{\text{s}} \end{aligned}$$

(b)

$$\eta = \frac{F_s\, v_1}{F_s\, v'} = \frac{v_1}{v'} = 0.769$$

3.7 (a)



(b)

$$\begin{aligned} p_1 + \frac{\rho}{2}\, v_1^2 &= p_{1'} + \frac{\rho}{2}\, v_{1'}^2 \\ v_{1'} = v_2 &= \sqrt{\frac{2}{\rho}\, (p_1 - p_{1'}) + v_1^2} \\ \dot{m} = \rho\, A\, v_{1'} &= 13 \cdot 10^3\, \frac{\text{kg}}{\text{s}} \end{aligned}$$

(c)

$$\begin{aligned} -\rho\, v_1^2\, A_\infty - \Delta\dot{m}\, v_1 + \rho\, v_2^2\, A \\ +\rho\, v_1^2\, (A_\infty - A) = F_s \end{aligned}$$

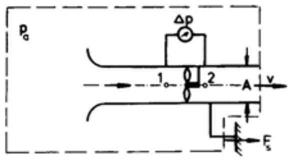
See problem 4.5.

$$\begin{aligned} \rho\, v_1\, A_\infty + \Delta\dot{m} &= \rho\, v_1\, (A_\infty - A) \\ &+\rho\, v_2\, A \\ F_s &= \rho\, v_2\, (v_2 - v_1)\, A \\ &= 0.39 \cdot 10^5\ \text{N} \end{aligned}$$

(d)

$$\begin{aligned} P &= \dot{Q}\, (p_{02} - p_{01'}) \\ &= \dot{Q}\, (p_1 - p_{1'}) = 448.5\ \text{kW} \end{aligned}$$

3.8



(a)

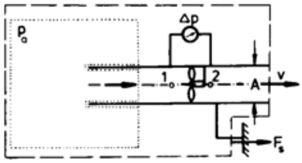
$$\begin{aligned} p_a &= p_1 + \frac{\rho}{2}\, v_1^2 \\ \Delta p &= p_2 - p_1 = p_a - p_1 \\ \dot{Q} &= v\, A = \sqrt{\frac{2\, \Delta\, p}{\rho}}\, A \end{aligned}$$

(b)

$$P = \dot{Q}\, (p_{02} - p_{01}) = \sqrt{\frac{2\, \Delta\, p}{\rho}}\, \Delta p\, A$$

(c)

$$\rho\, v^2\, A = F_s = 2\, \Delta p\, A$$



(d)

$$
\begin{aligned}
\rho\, v^2 A &= (p_a - p_1)\, A \\
\Delta p = p_2 - p_1 &= p_a - p_1 \\
\dot Q = v\, A &= \sqrt{\frac{\Delta\, p}{\rho}}\, A
\end{aligned}
$$

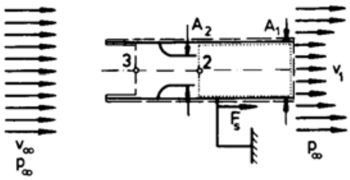
(e)

$$
P = \dot Q\, (p_{02} - p_{01}) = \sqrt{\frac{\Delta\, p}{\rho}}\, \Delta p\, A
$$

(f)

$$
\rho\, v^2\, A = F_s = \Delta p\, A
$$

3.9



(a)

$$
\begin{aligned}
p_\infty + \frac{\rho}{2}\, v_\infty^2 &= p_2 + \frac{\rho}{2}\, v_2^2 \\
-\rho\, v_2^2\, A_2 + \rho\, v_1^2\, A_1 &= (p_2 - p_\infty)\, A_1 \\
v_2\, A_2 &= v_1\, A_1
\end{aligned}
$$

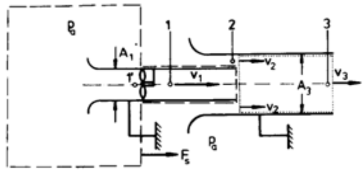
$$
\begin{aligned}
v_2 &= \frac{v_\infty}{\sqrt{1 - 2\,\frac{A_2}{A_1} + 2\,\left(\frac{A_2}{A_1}\right)^2}} \\
&= 56.6\ \frac{\mathrm{m}}{\mathrm{s}} \\
v_1 &= 28.3\ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

(b)

$$
-\rho v_3^2 A_1 + \rho v_1^2 A_1 = (p_3 - p_\infty) A_1 + F_s
$$

$$
\begin{aligned}
p_\infty + \frac{\rho}{2}\, v_\infty^2 &= p_3 + \frac{\rho}{2}\, v_3^2 \\
v_3 &= v_1 \\
F_s = \frac{\rho}{2}\, (v_1^2 - v_\infty^2)\, A_1 &= -100\ \mathrm{N}
\end{aligned}
$$

3.10



(a)

$$
\begin{aligned}
v_2 &= \frac{\dot Q_2}{A_3 - A_1} = 40\ \frac{\mathrm{m}}{\mathrm{s}} \\
p_a &= p_2 + \frac{\rho}{2}\, v_2^2 \\
p_2 &= 0.99 \cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
-\rho\, v_1^2\, A_1 - \rho\, v_2^2\, A_2 + \rho\, v_3^2\, A_3 & \\
= (p_2 - p_a)\, A_3 & \\
v_1\, A_1 + v_2\, A_2 = v_3\, A_3 &
\end{aligned}
$$

$$
\begin{aligned}
v_1 &= \left(1 + \sqrt{\frac{1}{2\,\frac{A_1}{A_3}\,\left(1 - \frac{A_1}{A_3}\right)}}\right)\, v_2 \\
&= 96.6\ \frac{\mathrm{m}}{\mathrm{s}} \\
v_3 &= v_1\, \frac{A_1}{A_3} + v_2 \left(1 - \frac{A_1}{A_3}\right) \\
&= 68.3\ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

(c)

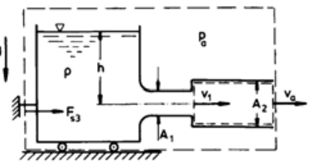
$$
\begin{aligned}
P &= \dot Q_1\, (p_{01} - p_{01'}) \\
&= \dot Q_1\, (p_1 - p_{1'}) \\
p_1 &= p_2 \\
p_a &= p_{1'} + \frac{\rho}{2}\, v_1^2 \\
p &= \frac{\rho}{2} \left(v_1^2 - v_2^2\right)\, v_1\, A_1 \\
&= 46.6\ \mathrm{kW}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\rho\, v_1^2\, A_1 &= (p_a - p_2)\, A_1 + F_s \\
F_s &= \rho\, \left(v_1^2 - \frac{v_2^2}{2}\right)\, A_1 \\
&= 1066\ \mathrm{N}
\end{aligned}
$$

(Traction force)

3.11



(a)

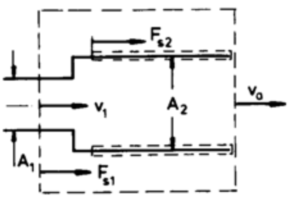
$$\begin{aligned} -\rho\, v_1^2\, A_1 + \rho\, v_a^2\, A_2 &= (p_1 - p_a)\, A_2 \\ p_a + \rho\, g\, h &= p_1 + \frac{\rho}{2}\, v_1^2 \\ v_1\, A_1 &= v_a\, A_2 \end{aligned}$$

$$\begin{aligned} \dot{Q} &= \frac{\sqrt{2\, g\, h}\, A}{\sqrt{2\, \left(\frac{A_1}{A_2}\right)^2 - 2\, \frac{A_1}{A_2} + 1}} \\ \frac{d\dot{Q}}{dA_2} &= 0 \\ A_2 &= 2\, A_1 = 0.2 \text{ m}^2 \end{aligned}$$

(b)

$$p_1 = p_a - \rho\, g\, h = 5\, 10^4\, \frac{\text{N}}{\text{m}^2}$$

(c)



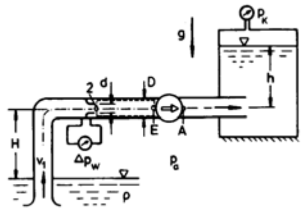
$$\begin{aligned} -\rho\, v_1^2\, A_1 + \rho\, v_a^2\, A_2 &= (p_1 - p_a)\, A_1 \\ &\quad + F_{s1} \end{aligned}$$

$$\begin{aligned} F_{s1} &= -\rho\, g\, h\, A_1 \\ &= -5\, 10^3 \text{ N (Traction force)} \end{aligned}$$

$$\begin{aligned} F_{s2} &= 0 \\ \rho\, v_a^2\, A_2 &= F_{s3} \end{aligned}$$

$$\begin{aligned} F_{s3} &= 2\, \rho\, g\, h\, A_1 \\ &= 10^4 \text{ N (Compressive force)} \end{aligned}$$

3.12



(a)

$$\begin{aligned} \dot{Q} &= v_1\, \frac{\pi\, D^2}{4} = \alpha\, \frac{\pi\, D^2}{4}\, \sqrt{\frac{2\, \Delta\, p_w}{\rho}} \\ v_1 &= \alpha\, \frac{d^2}{D^2}\, \sqrt{\frac{2\, \Delta\, p_w}{\rho}} = 1.33\, \frac{\text{m}}{\text{s}} \end{aligned}$$

(b)

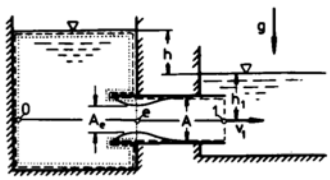
$$\begin{aligned} -\rho\, v_2^2\, A_2 &+ \rho\, v_1^2\, A_1 = (p_2 - p_E)\, A_1 \\ p_a &= p_2 + \rho\, g\, H + \frac{\rho}{2}\, v_2^2 \\ v_2\, A_2 &= v_1\, A_1 \end{aligned}$$

$$\begin{aligned} p_E &= p_a - \frac{\rho}{2}\, v_1^2 \left[\left(\frac{D^2}{d^2} - 1 \right)^2 + 1 \right] \\ &- \rho\, g\, H = 0.482 \cdot 10^5\, \frac{\text{N}}{\text{m}^2} \\ p_A &= p_K + \rho\, g\, h = 2.3 \cdot 10^5\, \frac{\text{N}}{\text{m}^2} \end{aligned}$$

(c)

$$\begin{aligned} P &= v_1\, \frac{\pi\, D^2}{4}\, (p_{oA} - p_{oE}) \\ &= 1.9 \cdot 10^3 \text{ kW} \end{aligned}$$

3.13



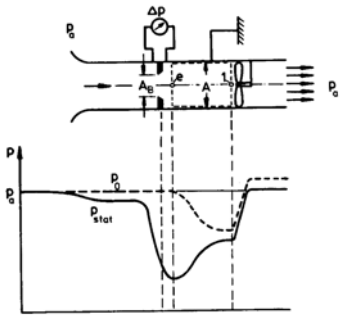
(a)

$$\begin{aligned} \rho\, v_e^2\, A_e &= (p_0 - p_e)\, A \\ p_0 &= p_e + \frac{\rho}{2}\, v_e^2 \\ \Psi &= \frac{A_e}{A} = 0.5 \end{aligned}$$

(b)

$$\begin{aligned}\rho\, v_1^2\, A &= (p_0 - p_1)\, A\\ p_0 &= p_a + \rho\, g\,(h + h_1)\\ p_1 &= p_a + \rho\, g\, h_1\\ v_1 &= \sqrt{g\, h} = 3.16\ \frac{\mathrm{m}}{\mathrm{s}}\end{aligned}$$

3.14 (a)



(b)

$$\begin{aligned}\dot{Q} &= \alpha\, m\, A \sqrt{\frac{2\,\Delta\, p_w}{\rho}}\\ &= 7.67 \cdot 10^{-2}\ \frac{\mathrm{m}^3}{\mathrm{s}}\end{aligned}$$

(c)

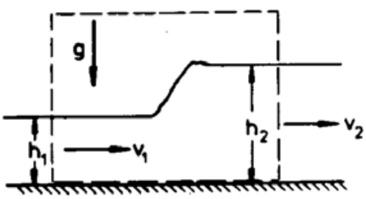
$$\begin{aligned}-\rho\, v_e^2\, A_e + \rho\, v_1^2\, A &= (p_e - p_1)\, A\\ p_a &= p_e + \frac{\rho}{2}\, v_e^2\end{aligned}$$

$$\begin{aligned}\dot{Q} &= v_1\, A_1 = v_e\, A_e = v_e\, \Psi\, m\, A\\ p_1 &= p_a - \alpha^2\, m^2 \cdot\\ &\cdot\ \Delta\, p_w \left[\left(1 - \frac{1}{\Psi\, m}\right)^2 + 1\right]\\ &= 0.998 \cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2}\end{aligned}$$

(d)

$$P = \dot{Q}\,(p_a - p_1) = 14.4\ \mathrm{W}$$

3.15



(a)

$$\begin{aligned}-\rho\, v_1^2\, B\, h_1 + \rho\, v_2^2\, B\, h_2 &= \rho\, g\, B\, \frac{h_1^2}{2} -\\ &- \rho\, g\, B\, \frac{h_2^2}{2}\end{aligned}$$

$$\begin{aligned}v_1\, B\, h_1 &= v_2\, B\, h_2\\ v_1 &= \sqrt{\frac{g}{2}\, \frac{h_2}{h_1}\,(h_1 + h_2)}\\ &= 1.73\ \frac{\mathrm{m}}{\mathrm{s}}\\ v_2 &= 0.87\ \frac{\mathrm{m}}{\mathrm{s}}\end{aligned}$$

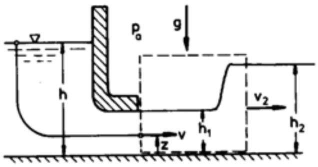
(b)

$$\begin{aligned}Fr_1 &= \frac{v_1}{\sqrt{g\, h}} = 1.73\\ Fr_2 &= 0.61\end{aligned}$$

(c)

$$\begin{aligned}H_1 - H_2 = h_1 - h_2 + \frac{1}{2\, g}\,(v_1^2 - v_2^2)\\ = \frac{(h_2 - h_1)^3}{4\, h_1\, h_2} = 0.0125\ \mathrm{m}\end{aligned}$$

3.16



(a) For each streamline it is,

$$\begin{aligned}p_a + \rho\, g\, h &= p + \rho\, g\, z + \frac{\rho}{2}\, v^2\\ p + \rho\, g\, z &= p_a + \rho\, g\, h_1\\ v &= \sqrt{2\, g\,(h - h_1)}\\ &= v_1 (\text{independent of z})\end{aligned}$$

(b)

$$\begin{aligned}\dot{Q} &= v_1\, B\, h_1\\ \frac{d\dot{Q}}{dh_1} &= 0\\ h_{1max} = \frac{2}{3}\, h &= 5\ \mathrm{m}\end{aligned}$$

(b)

$$\begin{aligned}Fr_1 &= \frac{v_1}{\sqrt{g\, h}} = 1\\ h_{1gr} &= \frac{2}{3}\, h = 5\ \mathrm{m}\end{aligned}$$

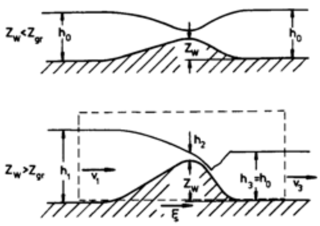
(c)

$$\rho\, g\, B \left(\frac{h_1^2}{2} - \frac{h_2^2}{2}\right) = -\rho\, v_1^2\, B\, h_1 + \rho\, v_2^2\, B\, h_2$$

$$v_1\, B\, h_1 = v_2\, B\, h_2$$

$$\begin{aligned} h_2 &= \frac{h_1}{2}\left(\sqrt{1+16\left(\frac{h}{h_1}-1\right)}-1\right) \\ &= 5.93 \text{ m} \\ v_2 &= 4.22\ \frac{\text{m}}{\text{s}} \end{aligned}$$

3.17 (a)



(b)

$$Z_{crit.} + H_{min} = H_0$$

$$\begin{aligned} Z_{crit.} &= h_0 + \frac{\dot{Q}^2}{2\, g\, B^2\, h_0^2} - \frac{3}{2}\sqrt[3]{\frac{\dot{Q}^2}{g\, B^2}} \\ &= 0.446 \text{ m} \end{aligned}$$

(c) $z_W > z_{crit.}$:

$$\begin{aligned} h_2 &= h_{crit.} = \sqrt[3]{\frac{\dot{Q}^2}{g\, B^2}} \\ &= 1.170 \text{ m} \\ H_1 &= Z_W + H_{min} \\ \frac{H_1}{H_{min}} &= \frac{Z_W}{H_{min}} + 1 = 1.570 \end{aligned}$$

from diagram:

$$\begin{aligned} \frac{h_1}{h_{crit.}} &= 2.26 \\ h_1 &= 2.64 \text{ m} \end{aligned}$$

(Elevation of the upper water)

(d)

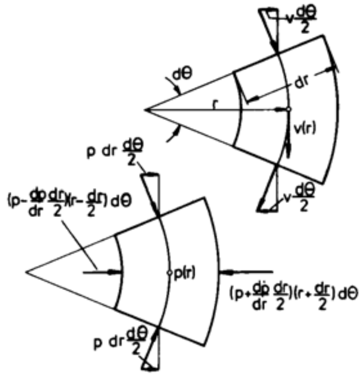
$$\begin{aligned} H_1 - H_3 &= H_1 - H_0 \\ &= Z_W - Z_{crit.} \\ &= 0.554 \text{ m} \end{aligned}$$

(e)

$$-\rho\, v_1^2\, B\, h_1 + \rho\, v_3^2\, B\, h_3 = \rho\, g\, B \left(\frac{h_1^2}{2} - \frac{h_3^2}{2}\right) + F_s$$

$$\begin{aligned} F = -\ F_s &= \frac{\rho g\, B}{2}\,(h_1 - h_0)\cdot \\ &\cdot \left(h_1 + h_0 - \frac{2\,\dot{Q}^2}{g\, B^2\, h_1\, h_0}\right) \\ &= 2.60 \cdot 10^5 \text{ N} \end{aligned}$$

3.18



(a)

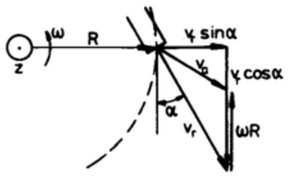
$$\begin{aligned} &- \rho\, v^2\, dr\, \frac{d\theta}{2} - \rho\, v^2\, dr\, \frac{d\theta}{2} = \\ &= \left(p - \frac{dp}{dr}\,\frac{dr}{2}\right)\left(r - \frac{dr}{2}\right) d\theta - \\ &- \left(p + \frac{dp}{dr}\,\frac{dr}{2}\right)\left(r + \frac{dr}{2}\right) d\theta + \\ &+ 2\, p\, dr\, \frac{d\theta}{2} \end{aligned}$$

$$\frac{dp}{dr} = \rho\, \frac{v^2}{r}$$

(b)

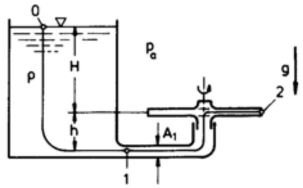
$$\begin{aligned} \frac{d}{dr}\left(p + \frac{\rho}{2}\, v^2\right) &= 0 \\ \frac{dp}{dr} = -\rho\, v\, \frac{dv}{dr} &= \rho\, \frac{v^2}{r} \\ v &= \frac{const.}{r} \end{aligned}$$

3.19 (a)



$$\begin{aligned} M_r &= 3\,\rho\, v_r A\,(\boldsymbol{r} \text{ x } \boldsymbol{v}_a)_z \\ &= 3\,\rho\, v_r A\, R(-v_r\, \cos\alpha + \omega\, R) \end{aligned}$$

Determination of v_r :



$$\begin{aligned} p_a &= p_a + \frac{\rho}{2}\, v_r^2 - \int_{s_0}^{s_2} \rho\,(\boldsymbol{b}\cdot d\boldsymbol{s}) \\ &= p_a + \frac{\rho}{2}\, v_r^2 - \rho\left(gH + \frac{\omega^2\, R^2}{2}\right) \\ v_r &= \sqrt{2\, g\, H + 2\,\omega^2\, R^2} \end{aligned}$$

with

$$\begin{aligned} \xi &= \frac{\omega\, R}{\sqrt{2\, g\, h}} \text{ and} \\ M_0 &= -3\,\rho\, A\, R\, 2\, g\, H\, \cos\alpha \quad : \\ \frac{M_r}{M_0} &= \sqrt{1+\xi^2}\left(\sqrt{1+\xi^2} - \frac{\xi}{\cos\alpha}\right) \end{aligned}$$

$$\xi^2 = \frac{2\,\frac{M_r}{M_0} + \tan^2\alpha - 1}{2\tan^2\alpha} \cdot$$

$$\cdot \left[\sqrt{1 + \left(\frac{\left(1 - \frac{M_r}{M_0}\right)\, 2\, \tan\alpha}{2\,\frac{M_r}{M_0} + \tan^2\alpha - 1}\right)^2} - 1\right]$$

$$\begin{aligned} \xi &= 0.07 \\ n &= 1.05\,\frac{1}{\text{s}} \end{aligned}$$

(b)

$$\dot{Q} = 3\, v_r\, A = 2.13 \cdot 10^{-3}\ \frac{\text{m}^3}{\text{s}}$$

(c)

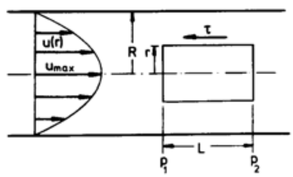
$$\begin{aligned} p_a + \rho\, g\,(h + H) &= p_1 + \frac{\rho}{2}\, v_1^2 \\ v_1 &= \frac{\dot{Q}}{A_1} \\ p_1 &= 1.095 \cdot 10^5\ \frac{\text{N}}{\text{m}^2} \end{aligned}$$

(d)

$$\begin{aligned} \xi_0 &= \frac{1}{\tan\alpha} = 1.73 \\ \omega_0 &= 163\ \frac{1}{\text{s}} \end{aligned}$$

3.2.4 Laminar Flow of Viscous Fluids

4.1



(a)

$$(p_1 - p_2)\,\pi\, r^2 - \tau\, 2\,\pi\, r\, L = 0$$

$$\begin{aligned} \tau &= -\mu\,\frac{du}{dr} \\ r = R: \quad u &= 0 \\ \frac{u(r)}{u_{max}} &= 1 - \left(\frac{r}{R}\right)^2 \end{aligned}$$

(b)

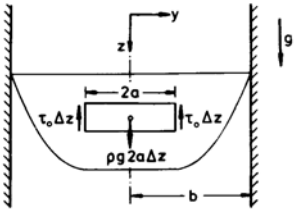
$$\begin{aligned} \frac{u(m)}{u_{max}} &= \frac{\dot{Q}}{u_{max}\,\pi\, R^2} \\ &= 2\int_0^1 \left[1 - \left(\frac{r}{R}\right)^2\right] \times \\ &\quad \times \frac{r}{R}\, d\left(\frac{r}{R}\right) \\ &= \frac{1}{2} \end{aligned}$$

(c)

$$\begin{aligned} \lambda &= \frac{8\,\tau_w}{\rho\, u_m^2} \\ \tau_w &= \frac{4\,\mu\, u_m}{R} \\ \lambda &= \frac{64}{Re} \end{aligned}$$

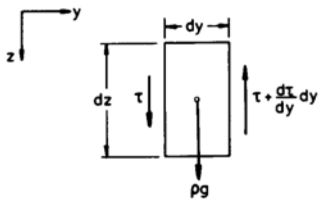
4.2 For $y \le a$ the fluid behaves as a rigid body.

(a)



$$\rho\, g\, a\, \Delta z = \tau_0 \Delta z$$
$$a = \frac{\tau_0}{\rho\, g}$$

(b)

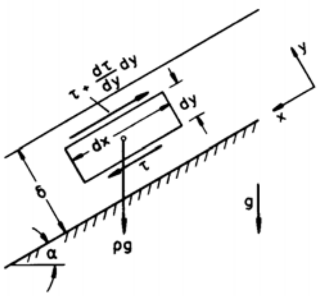


$$\begin{aligned} a \le y \le b: \quad \frac{d\tau}{dy} &= \rho\, g \\ \tau &= -\mu \frac{dw}{dy} + \tau_0 \\ y = a: \quad \tau &= \tau_0 \\ y = b: \quad w &= 0 \end{aligned}$$

$$w(y) = \frac{\rho\, g}{2\,\mu}[(b-a)^2 - (y-a)^2]$$

$$0 \le y \le a: \quad w(y) = \frac{\rho\, g}{2\,\mu}(b-a)^2$$

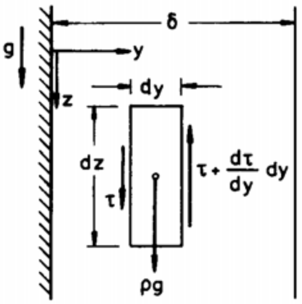
4.3



$$\begin{aligned} \dot{Q} &= B \int_0^\delta u(y)\, dy \\ \frac{d\tau}{dy} &= \rho\, g\, \sin\alpha \\ \tau &= -\mu \frac{du}{dy} \\ y = 0: \quad u &= 0 \\ y = \delta: \quad \tau &= 0 \end{aligned}$$

$$\begin{aligned} u(y) &= \frac{\rho\, g\, \sin\alpha}{\mu}\left[\delta y - \frac{y^2}{2}\right] \\ \dot{Q} &= \frac{\rho\, g\, B\, \sin\alpha}{3\,\mu}\,\delta^3 \\ &= 1.2 \cdot 10^{-3}\, \frac{\mathrm{m}^3}{\mathrm{s}} \end{aligned}$$

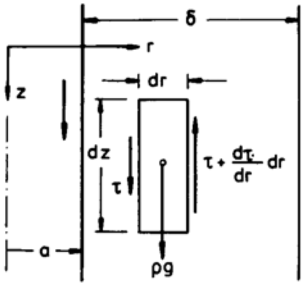
4.4 (a)



$$\begin{aligned} -\frac{d\tau}{dy} + \rho\, g &= 0 \\ \tau &= -\mu \frac{dw}{dy} \\ \frac{d^2 w}{dy^2} + \frac{\rho\, g}{\mu} &= 0 \\ y = 0: \quad w &= 0 \\ y = \delta: \quad \tau &= 0 \end{aligned}$$

$$w(y) = \frac{\rho\, g}{\mu}\,\delta^2 \left[\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]$$

(b)



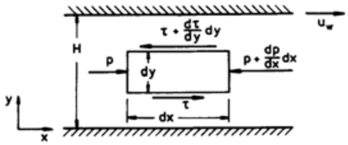
$$\tau\, 2\,\pi\, dz -$$
$$-\left(\tau + \frac{d\tau}{dr}\, dr\right)\, 2\,\pi\,(r+dr)\, dz +$$
$$+\rho\, g\, \pi\left[(r+dr)^2 - r^2\right]\, dz = 0$$

$$-r\,\frac{d\tau}{dr} - \tau + \rho\, g\, r \;=\; 0$$
$$\tau \;=\; -\mu\,\frac{dw}{dr}$$

$$\frac{d}{dr}\left(r\,\frac{dw}{dr}\right) \;+\; \frac{\rho\, g}{\mu}\, r = 0$$
$$r = a \;:\quad w = 0$$
$$r = a+\delta \;:\quad \tau = 0$$

$$w(r) \;=\; \frac{\rho\, g}{2\,\mu}\;\times$$
$$\times\left[(a+\delta)^2 \ln\frac{r}{a} + \frac{a^2 - r^2}{2}\right]$$

4.5 (a)



$$\frac{dp}{dx} + \frac{d\tau}{dy} \;=\; 0$$
$$\tau \;=\; -\mu\,\frac{du}{dy}$$
$$\frac{d^2u}{dy^2} \;=\; \frac{1}{\mu}\,\frac{dp}{dx}$$
$$y = 0: \quad u \;=\; 0$$
$$y = H: \quad u \;=\; u_w$$

$$u(y) \;=\; \frac{1}{2\ mu}\,\frac{dp}{dx}\, H^2\left[\left(\frac{y}{H}\right)^2 - \frac{y}{H}\right] +$$
$$+\;\; u_w\,\frac{y}{H}$$

(b)

$$\frac{\tau(y=H)}{\tau(y=0)} = \frac{u_w + \frac{1}{2\,\mu}\frac{dp}{dx}\, H^2}{u_w - \frac{1}{2\,\mu}\frac{dp}{dx}\, H^2}$$

(c)

$$\dot{Q} \;=\; u_m\, B\, H = B\int_0^H u(y)\, dy$$
$$\;=\; \left(\frac{u_w}{2} - \frac{dp}{dx}\,\frac{H^2}{12\,\mu}\right)\, BH$$

(d)

$$u_{max} = -\frac{dp}{dx}\,\frac{H^2}{8\,\mu}$$

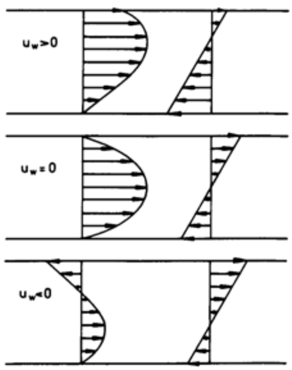
(e)

$$\frac{dI_x}{dt} = B\int_0^H \rho\, u(y)^2\, dy = \frac{6}{5}\,\rho\, u_m^2\, BH$$

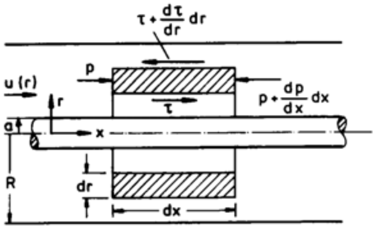
(f)

$$\frac{\tau_w}{\frac{\rho}{2}\, u_m^2} = \frac{12}{Re}$$

(g)



4.6



(a)

$$\frac{dp}{dx} + \frac{1}{r}\,\frac{d(\tau\, r)}{dr} = 0$$

Compare problem 5.4b

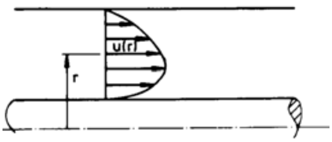
$$\tau = -\mu\,\frac{du}{dr}$$

$$\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{du}{dr}\right) - \frac{1}{\mu}\,\frac{dp}{dx} = 0$$

$$r = a: \quad u = 0$$

$$r = R: \quad u = 0$$

$$\begin{aligned} u(r) &= -\frac{1}{4\,\mu}\frac{dp}{dx}\,(R^2 - a^2)\times \\ &\times \left[\frac{R^2 - r^2}{R^2 - a^2} - \frac{\ln\left(\frac{r}{R}\right)}{\ln\left(\frac{a}{R}\right)}\right] \end{aligned}$$



(b)

$$\frac{\tau(r = a)}{\tau(r = R)} = \frac{R}{a}\;\frac{2\,a^2\;\ln\frac{a}{R} + R^2 - a^2}{2\,R^2\;\ln\frac{a}{R} + R^2 - a^2}$$

(c)

$$\begin{aligned} u_m &= \frac{\dot{Q}}{\pi\,(R^2 - a^2)} \\ &= \frac{1}{\pi\,(R^2 - a^2)}\int_a^R u(r)\,2\pi r\,dr \\ &= -\frac{1}{8\mu}\,\frac{dp}{dx}\,R^2\left[1 + \left(\frac{a}{R}\right)^2 + \right. \\ &\left. + \frac{1 - \left(\frac{a}{R}\right)^2}{\ln\left(\frac{a}{R}\right)}\right] \end{aligned}$$

4.7 (a)

$$\frac{d}{dr}\left[\frac{1}{r}\,\frac{d}{dr}\,(r\,v)\right] = 0$$

$$r = R_i \;:\quad v = 0$$

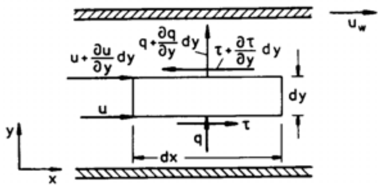
$$r = R_a \;:\quad v = \omega\,R_a$$

$$v(r) = \frac{\omega\,R_a^2}{r}\;\frac{r^2 - R_i^2}{R_a^2 - R_i^2}$$

(b)

$$\begin{aligned} \mu &= \left.\frac{\tau}{-r\,\frac{d}{dr}\,\left(\frac{v}{r}\right)}\right|_{r=R_i} \\ M_z &= -\tau(r = R_i)\,2\pi\,R_i^2\,L \\ \mu &= \frac{M_z}{4\,\pi\,\omega R_i^2\,L}\,\left[1 - \left(\frac{R_i}{R_a}\right)^2\right] \\ &= 10^{-2}\,\frac{\mathrm{Ns}}{\mathrm{m}^2} \end{aligned}$$

4.8



$$\rho\,\frac{de}{dt}\,dx\,dy =$$

$$\begin{aligned} &= \left[q - \left(q + \frac{\partial q}{\partial y}\,dy\right)\right]\,dx + \tau\,u\,dx - \\ &- \left(\tau + \frac{\partial \tau}{\partial y}\,dy\right)\,\left(u + \frac{\partial u}{\partial y}\,dy\right)\,dx \end{aligned}$$

$$\begin{aligned} \frac{d\tau}{dy} &= 0 \\ q &= -\lambda\,\frac{\partial T}{\partial y} \\ \tau &= -\mu\,\frac{\partial u}{\partial y} \\ \rho\,\frac{de}{dt} &= \lambda\,\frac{\partial^2 T}{\partial y^2} + \mu\,\left(\frac{\partial u}{\partial y}\right)^2 \end{aligned}$$

(a) It follows from problem 5.5a):

$$\begin{aligned} u(y) &= u_w\frac{y}{H} \\ \rho\,\frac{de}{dt} &= \rho\,c_v\left(\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\right) \\ &= 0 \\ \frac{d^2T}{dy^2} &= -\frac{\mu}{\lambda}\left(\frac{u_w}{H}\right)^2 \\ y &= H: \quad T = T_w \\ y &= 0: \quad q = 0 \\ T(y) &= T_w + \frac{\mu}{2\,\lambda}\left(\frac{u_w}{H}\right)^2 (H^2 - y^2) \end{aligned}$$

(b)

$$q_w = -\lambda \left.\frac{dT}{dy}\right|_{y=H} = \frac{\mu\, u_w^2}{H}$$

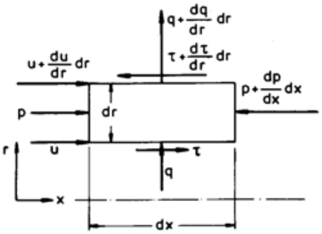
(c)

$$\begin{aligned}
h_0 &= c_p\, T + \frac{\boldsymbol{v}^2}{2} = c_p\, T + \frac{u^2}{2} \\
h_0 &= \text{const:} \quad \nabla\, h_0 = 0 \\
\frac{\partial h_0}{\partial x} &= c_p\, \frac{\partial T}{\partial x} + \frac{\partial(\frac{u^2}{2})}{\partial x} = 0 \\
\frac{\partial h_0}{\partial y} &= c_p\, \frac{\partial T}{\partial y} + \frac{\partial(\frac{u^2}{2})}{\partial y} \\
&= c_p\, \frac{dT}{dy} + u\, \frac{du}{dy} \\
&= \left(1 - \frac{\mu\, c_p}{\lambda}\right) \left(\frac{u_w}{H}\right)^2 y \\
Pr &= \frac{\mu\, c_p}{\lambda} = 1: \quad \frac{\partial h_0}{\partial y} = 0
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{\partial T}{\partial y} &= 0 \\
\frac{dT}{dt} &= \frac{\mu}{\rho\, c_v} \left(\frac{u_w}{H}\right)^2 \\
t &= 0: \quad T = T_0 \\
T(t) &= T_0 + \frac{\mu}{\rho\, c_v} \left(\frac{u_w}{H}\right)^2 t
\end{aligned}$$

4.9 (a)



Velocity distribution:

$$\begin{aligned}
\tau + r\, \frac{dp}{dx} + r\, \frac{d\tau}{dr} &= 0 \\
\tau &= -\mu\, \frac{du}{dr}
\end{aligned}$$

$$r\, \frac{dp}{dx} - \mu\, \frac{d}{dr} \left(r\, \frac{du}{dr}\right) = 0$$

$$\begin{aligned}
r = R &: \quad u = 0 \\
r = 0 &: \quad \frac{du}{dr} = 0
\end{aligned}$$

Temperature distribution:

$$\begin{aligned}
0 &= q\, 2\, \pi\, r\, dx + \tau\, u\, 2\pi\, r\, dx - \\
&- \left(q + \frac{dq}{dr}\, dr\right) 2\pi\, (r + dr)\, dx - \\
&- \left(\tau + \frac{d\tau}{dr}\, dr\right) \left(u + \frac{du}{dr}\, dr\right) \cdot \\
&\cdot\ 2\, \pi\, (r + dr)\, dx + \\
&+ \left[p - \left(p + \frac{dp}{dx}\, dx\right)\right] \cdot \\
&\cdot\ u\, \pi\, \left[(r + dr)^2 - r^2\right]
\end{aligned}$$

$$\begin{aligned}
q &= -\lambda \frac{dT}{dr} \\
\lambda\, \frac{d}{dr} \left(r\, \frac{dT}{dr}\right) &+ \mu\, r \left(\frac{du}{dr}\right)^2 = 0 \\
r = R &: \quad T = T_w \\
r = 0 &: \quad \frac{dT}{dr} = 0
\end{aligned}$$

(b) It follows from problem 5.1a

$$\begin{aligned}
\frac{u(r)}{u_{max}} &= 1 - \left(\frac{r}{R}\right)^2 \\
T - T_w &= \frac{\mu\, u_{max}^2}{4\, \lambda} \left[1 - \left(\frac{r}{R}\right)^4\right]
\end{aligned}$$

$r = 0:$

$$\begin{aligned}
T_{max} - T_w &= \frac{\mu\, u_{max}^2}{4\, \lambda} \\
\frac{T - T_w}{T_{max} - T_w} &= 1 - \left(\frac{r}{R}\right)^4
\end{aligned}$$

4.10 (a)

$$Re \left(\frac{h_1}{L}\right)^2 = 1.6 \cdot 10^{-3}$$

(b)

$$\begin{aligned}
\frac{\dot{Q}}{B} &= \int_0^{h(x)} u(x,y)\, dy \\
&= const. \\
\frac{dp}{dx} &= \mu\, \frac{\partial^2 u}{\partial y^2} \\
y = 0 &: \quad u = u_\infty \\
y = h(x) &: \quad u = 0
\end{aligned}$$

$$u(x,y) = u_\infty \left(1 - \frac{y}{h(x)}\right) - \\ -\frac{h^2(x)}{2\,\eta}\,\frac{y}{h(x)}\left(1 - \frac{y}{h(x)}\right)\frac{dp}{dx}$$

$$\frac{\dot{Q}}{B} = \frac{u_\infty\, h(x)}{2} - \frac{h^3(x)}{12\,\mu}\,\frac{dp}{dx}$$

$$x = 0: \quad p = p_\infty$$

$$\begin{aligned} p(x) &= p_\infty + 6\,\mu\,u_\infty \int_0^x \frac{dx'}{h^2(x')} - \\ &- 12\,\mu\left(\frac{\dot{Q}}{B}\right)\int_0^x \frac{dx'}{h^3(x')} \end{aligned}$$

With $p(x = L) = p_\infty$ yields:

$$\begin{aligned} \frac{\dot{Q}}{B} &= \frac{u_\infty}{2}\,\frac{\int_0^L \frac{dx}{h^2(x)}}{\int_0^L \frac{dx}{h^3(x)}} \\ \frac{\dot{Q}}{B} &= \frac{3}{4}\,u_\infty\,h_1 \frac{e^{\frac{2}{5}} - 1}{e^{\frac{3}{5}} - 1} \\ &= 4.49 \cdot 10^{-5}\,\frac{\mathrm{m}^2}{\mathrm{s}} \end{aligned}$$

(c) With

$$\begin{aligned} K &= \frac{e^{\frac{2}{5}} - 1}{e^{\frac{3}{5}} - 1}: \\ p(x) &= p_\infty + 15\frac{\mu\,u_\infty}{h_1}\frac{L}{h_1}\,\cdot \\ &\cdot \left[e^{\frac{2x}{5L}} - K\,e^{\frac{3x}{5L}} + K - 1\right] \end{aligned}$$

(d)

$$\frac{F_{py}}{B} = \int_0^L (p(x) - p_\infty)\,dx$$

$$= 15\mu\,u_\infty\left(\frac{L}{h_1}\right)^2\left[\frac{5}{6}\,(e^{\frac{2}{5}}-1)+K-1\right]$$

$$= 3035\,\frac{\mathrm{N}}{\mathrm{m}}$$

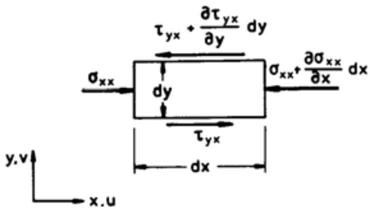
(e)

$$\begin{aligned} \frac{P}{B} &= u_\infty \int_0^L \tau_{xy}(x,y=0)\,dx \\ \tau_{xy} &= -\mu\,\frac{du}{dr} \end{aligned}$$

$$= \frac{\mu\,u_\infty}{h(x)}\left[1 + 3\left(1 - \frac{3K}{2}\,\frac{h_1}{h(x)}\right) \times \\ \times \left(1 - 2\,\frac{y}{h(x)}\right)\right]$$

$$\begin{aligned} \frac{P}{B} &= 20\,\mu\,u_\infty^2\,\frac{L}{h_1} \times \\ &\times \left[e^{\frac{1}{5}} - 1 - \frac{9K}{16}\,(e^{\frac{2}{5}} - 1)\right] \\ &= 55.9\,\frac{\mathrm{W}}{\mathrm{m}} \end{aligned}$$

4.11



$$\begin{aligned} \frac{dI_x}{dt} &= \int_\tau \frac{\partial(\rho\,u)}{\partial t}\,d\tau + \int_A \rho\,u\,(\boldsymbol{v}\cdot\boldsymbol{n})\,dA \\ &= \sum F_x \end{aligned}$$

$$\int_\tau \frac{\partial(\rho\,u)}{\partial t}\,d\tau = \left(\rho\,\frac{\partial u}{\partial t} + u\,\frac{\partial \rho}{\partial t}\right) dx\,dy\,dz$$

$$\int_A \rho\,u\,(\boldsymbol{v}\cdot\boldsymbol{n})\,dA =$$

$$\left[-\rho\,u^2 + \left(\rho\,u^2 + \frac{\partial(\rho\,u^2)}{\partial x}\,dx\right)\right]dy\,dz -$$

$$-\left[-\rho\,uv + \left(\rho\,uv + \frac{\partial(\rho\,uv)}{\partial y}\,dy\right)\right]dx\,dz$$

$$= \left[\rho\,u\,\frac{\partial u}{\partial x} + \rho\,v\,\frac{\partial u}{\partial y} + \\ + u\left(\frac{\partial(\rho\,u)}{\partial x} + \frac{\partial(\rho\,v)}{\partial y}\right)\right] \times \\ \times dx\,dy\,dz$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\,u)}{\partial x} + \frac{\partial(\rho\,v)}{\partial y} = 0$$

$$\sum F_x = -\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)\, dx\, dy\, dz$$

$$(\tau_{yx} = \tau_{xy})$$

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right) = -\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)$$

4.12

$$\begin{aligned}
\mu &= const.: \\
F_{rx} &= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \\
&\quad + \frac{\mu}{3}\,\frac{\partial}{\partial x}\,(\nabla\cdot\boldsymbol{v}) \\
\rho &= const.: \\
F_{rx} &= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)
\end{aligned}$$

3.2.5 Pipe Flows

5.1

$$\begin{aligned}
\mu &= \frac{\rho\,\bar{u}_m\,D}{Re} \\
\bar{u}_m &= \frac{4\,\tau}{\pi\,D^2\,T} \\
\rho\,g\,(h+L) &= (1+\lambda\,\frac{L}{D} + \zeta_e)\,\frac{\rho}{2}\,\bar{u}_m^2
\end{aligned}$$

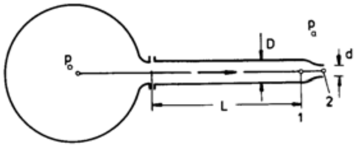
Assumption: Laminar flow

$$\begin{aligned}
\zeta_e &= 1.16 \\
\lambda &= 11.92 \\
Re &= \frac{64}{\lambda} = 5.37
\end{aligned}$$

Intake region:

$$\begin{aligned}
L_e &= 0.029\,Re\,D \\
&= 0.16\cdot 10^{-3}\ \mathrm{m} \ll L \\
\mu &= 8.40\cdot 10^{-3}\ \frac{\mathrm{Ns}}{\mathrm{m}^2}
\end{aligned}$$

5.2



(a)

$$p_1 + \frac{\rho}{2}\,\bar{u}_{m1}^2 = p_a + \frac{\rho}{2}\,\bar{u}_{m2}^2$$

$$\begin{aligned}
\bar{u}_{m1}^2\,\frac{\pi\,D^2}{4} &= \bar{u}_{m2}^2\,\frac{\pi\,d^2}{4} \\
\bar{u}_{m1}^2 &= \sqrt{\frac{2(p_1-p_a)}{\rho}\,\frac{1}{\left(\frac{D}{d}\right)^4-1}} \\
&= 1\ \frac{\mathrm{m}}{\mathrm{s}} \\
\bar{u}_{m2} &= 4\ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

(b)

$$\begin{aligned}
p_0 &= p_1 + (1+\lambda\,\frac{L}{D})\,\frac{\rho}{2}\,\bar{u}_{m1}^2 \\
Re_1 &= \frac{\rho\,\bar{u}_{m1}\,D}{\mu} = 10^4 \\
\lambda &= \frac{0.316}{\sqrt[4]{Re}} = 0.0316 \\
p_0 &= 2.66\cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2}
\end{aligned}$$

(c)

$$\bar{u}_{m2} = \sqrt{\frac{2\,(p_0-p_a)}{\rho}} = 18.22\ \frac{\mathrm{m}}{\mathrm{s}}$$

5.3

$$\begin{aligned}
\dot{Q} &= 25\frac{\pi}{4}\,(\bar{u}_{m1}\,D_1^2 + \bar{u}_{m2}\,D_2^2) \\
\Delta p &= (1+\lambda\,\frac{L}{D_1}+\zeta)\frac{\rho}{2}\,\bar{u}_{m1}^2 \\
&= (1+\lambda\,\frac{L}{D_2}+\zeta)\frac{\rho}{2}\,\bar{u}_{m2}^2 \\
\bar{u}_{m1} &= \sqrt{\frac{2\,\Delta p}{\rho\,(1+\lambda\,\frac{L}{D_1}+\zeta)}} \\
\bar{u}_{m2} &= \sqrt{\frac{2\,\Delta p}{\rho\,(1+\lambda\,\frac{L}{D_2}+\zeta)}} \\
\dot{Q} &= 0.518\ \frac{\mathrm{m}^3}{\mathrm{s}}
\end{aligned}$$

5.4

$$\begin{aligned}
\Delta p &= \lambda\,\frac{L}{D}\,\frac{\rho}{2}\,\bar{u}_m^2 \\
\bar{u}_m &= \frac{1}{\sqrt{\lambda}}\sqrt{\frac{2\,\Delta p}{\rho}\,\frac{D}{L}}
\end{aligned}$$

$$\bar{u}_m \frac{\rho D}{\mu} \sqrt{\lambda} = Re\sqrt{\lambda} = \sqrt{\frac{2\,\Delta p}{\rho}\frac{D}{L}}\frac{\rho D}{\mu}$$

$$(Re\sqrt{\lambda})_{crit} = 2300\sqrt{\frac{64}{2300}} = 384$$

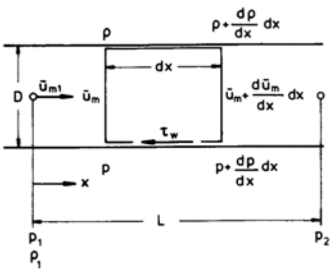
Oil:

$$Re\sqrt{\lambda} = 283$$
$$\frac{1}{\sqrt{\lambda}} = \frac{Re\sqrt{\lambda}}{64}$$
$$u_m = 1.56\ \frac{\mathrm{m}}{\mathrm{s}}$$

Water:

$$Re\sqrt{\lambda} = 3.16 \cdot 10^4$$
$$\frac{1}{\sqrt{\lambda}} = 2.0\ \log(Re\sqrt{\lambda}) - 0.8$$
$$\bar{u}_m = 2.59\ \frac{\mathrm{m}}{\mathrm{s}}$$

5.5 (a)



$$\rho\,\bar{u}_m = \left(\rho + \frac{d\rho}{dx}\,dx\right) \times \left(\bar{u}_m + \frac{d\bar{u}_m}{dx}dx\right) = \rho_1\,\bar{u}_{m1}$$

$$\left(\rho + \frac{d\rho}{dx}\,dx\right)\left(\bar{u}_m + \frac{d\bar{u}_m}{dx}dx\right)^2 \frac{\pi D^2}{4} - \rho\,\bar{u}_m^2 \frac{\pi D^2}{4} = -\tau_w\,\pi\,D\,dx + \left[p - \left(p + \frac{dp}{dx}dx\right)\right]\frac{\pi D^2}{4}$$

$$\lambda = \frac{8\,\tau_w}{\rho\,\bar{u}_m^2}$$
$$\frac{dp}{dx} + \rho_1\,\bar{u}_{m1}\frac{d\bar{u}_m}{dx} + \frac{\lambda}{D}\,\frac{\rho}{2}\,\bar{u}_m^2 = 0$$

(b) Compressible flow:

$$\rho\frac{d\bar{u}_m}{dx} + \bar{u}_m\frac{d\rho}{dx} = 0$$
(Continuity equation)

$$\rho = \frac{\rho_1}{p_1}\,p \quad (T = const.)$$
$$\frac{d\rho}{dx} = \frac{\rho_1}{p_1}\,\frac{dp}{dx}$$
$$\frac{dp}{dx} - \frac{\rho_1\,p_1\,\bar{u}_{m1}^2}{p^2}\,\frac{dp}{dx} + \frac{\lambda}{2\,D}\frac{\rho_1\,p_1\,\bar{u}_{m1}^2}{p} = 0$$

$$Re = Re_1 = 0.533 \cdot 10^5$$
$$\lambda = \frac{0.316}{\sqrt[4]{Re}} = 0.0208$$

$$\int_1^2 p\,dp - \rho_1\,p_1\,\bar{u}_{m1}^2 \int_1^2 \frac{dp}{p} + \frac{\lambda}{2\,D}\,\rho_1\,p_1\,\bar{u}_{m1}^2 \int_1^2 dx = 0$$

$$L = \frac{D}{\lambda}\,\frac{p_1}{\rho_1\,\bar{u}_{m1}^2}\left[1 - \left(\frac{p_2}{p_1}\right)^2\right] - \frac{D}{\lambda}\,\ln\left(\frac{p_1}{p_2}\right)^2 = 287.9\ \mathrm{m}$$

Incompressible flow:

$$\frac{d\bar{u}_m}{dx} = 0$$
$$\frac{dp}{dx} = -\frac{\lambda}{D}\,\frac{\rho_1}{2}\,\bar{u}_{m1}^2$$
$$L = 2\,\frac{D}{\lambda}\,\frac{p_1 - p_2}{\rho_1\,\bar{u}_{m1}^2} = 384.7\ \mathrm{m}$$

5.6 (a)

$$r = R - y$$

$$\dot{I} = \int_0^R \rho\,u^2\,2\pi\,r\,dr = 2\rho\,u_m^2\,\pi\,R\int_0^1 \left(\frac{u}{u_m}\right)^2 \frac{r}{R} d\left(\frac{r}{R}\right)$$

$$\delta = 0: \quad \frac{u}{u_m} = 1$$
$$\dot{I} = \rho\, u_m^2\, \pi\, R^2$$
$$\delta = R:$$
$$\frac{u}{u_m} = 2\left[1 - \left(\frac{r}{R}\right)^2\right]$$
$$\dot{I} = 1.33\,\rho\, u_m^2\, \pi\, R^2$$

$$\delta = \frac{R}{2}:$$
$$\frac{u}{u_m} = \begin{cases} \frac{96\,r}{17\,R}\left(1 - \frac{r}{R}\right) & \frac{R}{2} \le r \le R \\ \frac{24}{17} & 0 \le r \le \frac{R}{2} \end{cases}$$

$$\begin{aligned} \dot{I} &= 2\rho\, u_m^2 \pi\, R^2 \times \\ &\times \left[\int_0^{0.5} \left(\frac{24}{17}\right)^2 \frac{r}{R}\, d\left(\frac{r}{R}\right) + \right. \\ &+ \int_{0.5}^{1} \left[\frac{96}{17}\,\frac{r}{R}\left(1 - \frac{r}{R}\right)\right]^2 \times \\ &\left. \times \frac{r}{R}\, d\left(\frac{r}{R}\right)\right] \\ &= 1.196\,\rho\, u_m^2\, \pi\, R^2 \end{aligned}$$

(b)

$$\tau_w = \mu\, u_m \frac{\frac{2}{\delta}}{1 - \frac{2}{3}\,\frac{\delta}{R} + \frac{1}{6}\left(\frac{\delta}{R}\right)^2}$$

$$\delta \to 0 \quad : \quad \tau_w \to \infty$$
$$\delta \to R \quad : \quad \tau_w = 4\frac{\mu\, u_m}{R}$$
$$\delta \to \frac{R}{2} \quad : \quad \tau_w = 5.65\frac{\mu\, u_m}{R}$$

5.7 (a)

$$\begin{aligned} p_1 + \frac{\rho}{2}\,\bar{u}_{m1}^2 &= p_a + \left(1 + \lambda_2 \frac{L_2}{D_2}\right) \times \\ &\times \frac{\rho}{2}\,\bar{u}_{m2}^2 + \\ &+ \left(\zeta + \lambda_1 \frac{L_1}{D_1}\right) \frac{\rho}{2}\,\bar{u}_{m1}^2 \\ \zeta &= \left(1 - \frac{D_1^2}{D_2^2}\right)^2 \\ \bar{u}_{m1}\, D_1^2 &= \bar{u}_{m2}\, D_2^2 \end{aligned}$$

$$Re_1 = 10^4 \qquad \lambda_1 = 0.0316$$
$$Re_2 = 5 \cdot 10^3 \quad \lambda_1 = 0.0376$$

$$\begin{aligned} L_2 &= \frac{D_2}{\lambda_2}\left[\left(2\,\frac{D_1^2}{D_2^2} - \lambda_1\,\frac{L_1}{D_1}\right) \times \right. \\ &\times \left.\frac{D_2^4}{D_1^4} - 2\right] \\ &= 1.0\,\mathrm{m} \end{aligned}$$

(b)

$$\begin{aligned} \Delta p_v &= p_1 - p_a + \left(1 - \frac{D_1^4}{D_2^4}\right)\frac{\rho}{2}\,\bar{u}_{m1}^2 \\ &= 117.2\,\frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

5.8

$$\begin{aligned} \rho\, g\, h &= \left(\lambda \frac{L}{D} + 2\,\zeta_K + \zeta_v\right) \frac{\rho}{2}\,\bar{u}_m^2 + \\ &+ \frac{\rho}{2}\,u_d^2 \\ u_d\, d^2 &= \bar{u}_m\, D^2 \\ \dot{Q} &= u_d\,\frac{\pi\, d^2}{4} \end{aligned}$$

$$= \sqrt{\frac{2\, g\, h}{1 + \left(\frac{d}{D}\right)^4 \left(\lambda\,\frac{L}{D} + 2\,\zeta_K + \zeta_V\right)}}\;\frac{\pi\, d^2}{4}$$

$$H = \frac{u_d^2}{2\, g}$$

with losses:
(a)

$$\dot{Q} = 5.79 \cdot 10^{-3}\,\frac{\mathrm{m}^3}{\mathrm{s}} \qquad H = 6.96\ m$$

(b)

$$\dot{Q} = 9.82 \cdot 10^{-3}\,\frac{\mathrm{m}^3}{\mathrm{s}} \qquad H = 1.25\ m$$

loss-free:
(a)

$$\dot{Q} = 6.94 \cdot 10^{-3}\,\frac{\mathrm{m}^3}{\mathrm{s}} \qquad H = h$$

(b)

$$\dot{Q} = 27.77 \cdot 10^{-3}\,\frac{\mathrm{m}^3}{\mathrm{s}} \qquad H = h$$

5.9 (a)

$$\begin{aligned}\lambda &= \frac{0.316}{\sqrt[4]{Re}}\\ \bar{u}_m &= \frac{\mu}{\rho\, D}\, Re\\ \tau_w &= \frac{\lambda\, \rho\, \bar{u}_m^2}{8} = 2.22\ \frac{\mathrm{N}}{\mathrm{m}^2}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\bar{u}_m}{u_*} &= \frac{\bar{u}_{max}}{u_*} - 4.07\\ \lambda &= 8\left(\frac{u_*}{\bar{u}_m}\right)^2\\ \frac{\bar{u}_m}{\bar{u}_{max}} &= \frac{1}{1 + 4.07\sqrt{\frac{\lambda}{8}}} = 0.84\end{aligned}$$

(c)

$$\frac{y\, u_*}{\nu} = 5 = \frac{\bar{u}}{u_*} \quad \text{(Viscous sub-layer)}$$

$$\bar{u} = 5\sqrt{\frac{\lambda}{8}}\, \bar{u}_m = 0.236\ \frac{\mathrm{m}}{\mathrm{s}}$$

(for y = 0.11 mm)

$$\frac{y\, u_*}{\nu} = 50$$

(Logarithmic velocity distribution)

$$\begin{aligned}\frac{\bar{u}}{u_*} &= 2.5\ \ln\left(\frac{y\, u_*}{\nu}\right) + 5.5\\ \bar{u} &= 0.720\ \frac{\mathrm{m}}{\mathrm{s}}\end{aligned}$$

(for y = 1.1 mm)

(d)

$$\begin{aligned}l &= 0.4\, y = 0.4\ \frac{y\, u_*}{\nu}\ \frac{\nu}{\sqrt{\frac{\lambda}{8}}\, \bar{u}_m}\\ &= 0.85\ \mathrm{mm}\end{aligned}$$

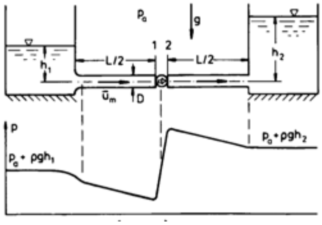
5.10 (a)

$$\begin{aligned}r &= R - y\\ \frac{\bar{u}_m}{\bar{u}_{max}} &= \frac{\dot{Q}}{\pi\, R^2\, \bar{u}_{max}}\\ &= 2\int_0^1 \left(1 - \frac{r}{R}\right)^{\frac{1}{7}} \frac{r}{R}\, d\left(\frac{r}{R}\right)\\ &= \frac{49}{60}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\dot{I}}{\rho\, \bar{u}_m\, \pi\, R^2} &= \frac{\int_0^R \rho\, \bar{u}^2\, 2\, \pi\, r\, dr}{\rho\, \bar{u}_m^2\, \pi\, R^2}\\ &= 2\left(\frac{\bar{u}_{max}}{\bar{u}}\right)^2 \cdot\\ &\cdot \int_0^1 \left(1 - \frac{r}{R}\right)^{\frac{2}{7}} \frac{r}{R}\, d\left(\frac{r}{R}\right)\\ &= \frac{50}{49}\end{aligned}$$

5.11 (a)



(b)

$$p_a + \rho\, g\, h_1 = p_1 + \left(1 + \lambda \frac{L}{2D}\right) \frac{\rho}{2}\, \bar{u}_m^2$$

$$\begin{aligned}\dot{Q} &= \bar{u}_m\, \frac{\pi\, D^2}{4}\\ Re &= \frac{\bar{u}_m\, D}{\nu} = 8 \cdot 10^5\\ \frac{R}{k_s} &= 250\end{aligned}$$

(From diagram, page 29)

$$\begin{aligned}\lambda &= 0.024\\ p_1 &= 1.22 \cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2}\end{aligned}$$

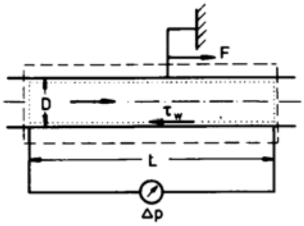
(c)

$$\begin{aligned}p_2 &= p_a + \rho\, g\, h_2 + \lambda\, \frac{L}{2D}\, \frac{\rho}{2}\, \bar{u}_m^2\\ &= 3.77 \cdot 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2}\end{aligned}$$

(d)

$$P = \dot{Q}(p_2 - p_1) = 160.5\ \mathrm{kW}$$

5.12



(a)

$$\begin{aligned}\Delta p &= \lambda \frac{L}{D} \frac{\rho}{2} \bar{u}_m^2 \\ \dot{Q} &= \bar{u}_m \frac{\pi D^2}{4} \\ \lambda &= \frac{\pi^2 \Delta p D^5}{8 \rho L \dot{Q}^2} = 0.0356\end{aligned}$$

(b)

$$Re = \frac{\rho \bar{u}_m D}{\eta} = 1.8 \cdot 10^5$$

(From diagram, page 29)

$$\begin{aligned}\frac{R}{k_s} &= 60 \\ k_s &= 4.2 \text{ mm}\end{aligned}$$

(c)

$$\begin{aligned}\Delta p \frac{\pi D^2}{4} - \tau_w \pi D L &= 0 \\ \tau_w = \Delta p \frac{D}{4L} &= 16 \frac{\text{N}}{\text{m}^2} \\ F = -\Delta p \frac{\pi D^2}{4} &= -2517 \text{ N}\end{aligned}$$

(d)

$$\lambda = 0.016$$

(From diagram page 29)

$$\Delta p = 5.8 \cdot 10^3 \frac{\text{N}}{\text{m}^2}$$

5.13

$$\begin{aligned}\frac{P_1}{P_2} &= \frac{\left(1 + \lambda_1 \frac{L}{D_1}\right) \frac{\rho}{2} \bar{u}_{m1}^2}{\left(1 + \lambda_2 \frac{L}{D_2}\right) \frac{\rho}{2} \bar{u}_{m2}^2} \\ \dot{Q} &= \bar{u}_m \frac{\pi D^2}{4} \\ \frac{1}{\sqrt{\lambda}} &= 2.0 \log\left(\frac{R}{k_s}\right) + 1.74 \\ \frac{P_1}{P_2} &= \frac{1 + \lambda_1 \frac{L}{D_1}}{1 + \lambda_2 \frac{L}{D_2}} \left(\frac{D_2}{D_1}\right)^4 = 39.2\end{aligned}$$

5.14

$$\begin{aligned}\lambda_{ps} \frac{L}{D} \frac{\rho}{2} \bar{u}_{mps}^2 &= \lambda_{ch} \frac{L_{ch}}{d_h} \frac{\rho}{2} \bar{u}_{mch}^2 \\ d_h &= a\end{aligned}$$

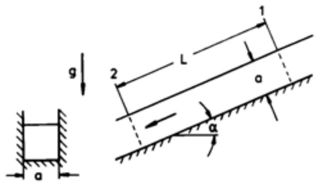
$$Re_{ch} = \frac{\rho a}{\mu} \frac{\dot{Q}}{a^2} = 10^5 \qquad \lambda_{ch} = 0.018$$

$$Re_{ps} = \frac{\rho D}{\mu} \frac{\dot{Q}}{100 \frac{\pi D^2}{4}} \qquad \lambda_{ch} = 0.030$$

$$= 1.27 \cdot 10^4$$

$$L_{ch} = 13.57 \text{ m}$$

5.15



$$\rho g L \sin\alpha + \frac{\rho}{2} \bar{u}_m^2 = \left(1 + \lambda \frac{L}{d_h}\right) \frac{\rho}{2} \bar{u}_m^2$$

$$d_h = \frac{4 a^2}{3 a}$$

$$\begin{aligned}Re_h &= \frac{\rho d_h}{\mu} \frac{\dot{Q}}{a^2} = 8 \cdot 10^3 \quad \lambda = 0.033 \\ \alpha &= 0.02^\circ \\ \frac{\Delta p_v}{L} &= \lambda \frac{1}{d_h} \frac{\rho}{2} \left(\frac{\dot{Q}}{a^2}\right)^2 = 3.61 \frac{\text{N}}{\text{m}^3}\end{aligned}$$

3.2.6 Similar Flows

6.1

$$\bar{u} = \frac{u}{v_1}, \quad \bar{v} = \frac{v}{v_1}, \quad \bar{p} = \frac{p}{\Delta p_1}, \quad \bar{\rho} = \frac{\rho}{\rho_1},$$
$$\bar{\mu} = \frac{\mu}{\mu_1}, \quad \bar{x} = \frac{x}{L_1}, \quad \bar{y} = \frac{y}{L_1}, \quad \bar{t} = \frac{t}{t_1}$$

$$\bar{\rho}\left(Sr \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}}\right)$$
$$= -Eu \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\bar{\mu}}{Re}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)$$

$$Sr = \frac{L_1}{v_1 t_1}, \quad Eu = \frac{\Delta p_1}{\rho_1 v_1^2},$$
$$Re = \frac{\rho_1 v_1 L_1}{\mu_1}$$

6.2

$$f_1(Re,Eu) = 0: \quad Eu = f_2(Re)$$
$$\lambda = \frac{D}{L}\frac{\Delta p}{\frac{\rho}{2}u_m^2} = 2\frac{D}{L}\,Eu = f_3(Re)$$

6.3 (a)

$$\begin{aligned} L^3\,T^{-1} &= (M\,L^{-2}\,T^{-2})^{\alpha} \\ &\cdot\ (M\,L^{-1}\,T^{-1})^{\beta}\,L^{\gamma} \\ \alpha &= 1 \quad \beta = -1 \quad \gamma = 4 \\ \dot{Q} &\sim \frac{\Delta p\,D^4}{L\,\mu} \end{aligned}$$

(b)

$$\begin{aligned} \lambda &= \frac{D}{L}\frac{\Delta p}{\frac{\rho}{2}u_m^2} \\ \frac{\Delta p\,D}{L} &\sim \frac{\dot{Q}\,\eta}{D^3} \sim \frac{u_m\,\mu}{D} \\ \lambda &\sim \frac{1}{Re} \end{aligned}$$

6.4

$$\begin{aligned} \frac{F_{Da}}{F_{Dw}} &= \frac{c_{Da}\,\frac{\pi}{4}\,D_a^2\,\frac{\rho_a}{2}\,u_{\infty L}^2}{c_{Dw}\,\frac{\pi}{4}\,D_w^2\,\frac{\rho_W}{2}\,u_{\infty w}^2} \\ &= \frac{c_{Da}\,\mu_a^2\,\frac{\rho_w}{2}\,Re_a^2}{c_{Dw}\,\mu_w^2\,\frac{\rho_a}{2}\,Re_w^2} \\ Re_a &= Re_w: \quad c_{Da} = c_{Dw} \\ \frac{F_{Da}}{F_{Dw}} &= 0.281 \end{aligned}$$

6.5

$$F(f,\mu,\rho,u_\infty,D) = 0$$
$$K_1 = f^{\alpha_1}\,\rho^{\beta_1}\,u_\infty^{\gamma_1}\,D^{\delta_1}$$

Take $\alpha_1 = 1$:

$$\begin{aligned} \beta_1 &= 0. \quad \gamma_1 = -1, \quad \delta_1 = 1 \\ K_1 &= \frac{f\,D}{u_\infty} = Sr \\ K_2 &= \mu^{\alpha_2}\,\rho^{\beta_2}\,u_\infty^{\gamma_2}\,D^{\delta_2} \end{aligned}$$

Take $\alpha_2 = -1$:

$$\begin{aligned} \beta_2 &= 1, \quad \gamma_1 = 1, \quad \delta_2 = 1 \\ K_2 &= \frac{\rho\,u_\infty\,D}{\mu} = Re \end{aligned}$$

6.6 (a)

$$\begin{aligned} Sr' &= Sr: \quad Re'_{min} = 200 \\ D'_{min} &= Re'_{min}\,\frac{\nu'}{v'} = 1\ mm \end{aligned}$$

(b)

$$Sr = Sr': \quad f = \frac{v\,D'_{min}}{v'\,D}\,f' = 2\,\frac{1}{s}$$

6.7 (a)

$$Re = Re': \quad v' = \sqrt{\frac{A}{A'}}\,v$$

Small, so that flow in wind-tunnel remains incompressible:

$$A' = A_m : v' = 77.46\ \frac{\mathrm{m}}{\mathrm{s}}$$

(b)

$$\begin{aligned} P &= \frac{F_D}{F'_D}\,F'_D\,v \\ &= \frac{c_D\frac{\rho}{2}\,v^2\,A}{c_{D'}\frac{\rho}{2}\,v'^2\,A_m}\,F'_D\,v \\ Re &= Re': \quad c_D = c'_D \\ P &= F'_D\,v = 24.3\ \mathrm{kW} \end{aligned}$$

6.8 (a)

$$\begin{aligned} Re = Re' &: \\ \dot{Q} &= \frac{vA}{v'A'}\,\dot{Q}' \\ &= \frac{\mu\,\rho'\,D'A}{\mu'\,\rho\,DA'}\,\dot{Q}' \\ \frac{A}{A'} &= \left(\frac{D}{D'}\right)^2 \\ \dot{Q} &= 9\ \frac{\mathrm{m}^3}{\mathrm{s}} \end{aligned}$$

(b)

$$\begin{aligned} Eu = Eu' : \ \Delta p &= \frac{\rho\,v^2}{\rho'\,v'^2}\,\Delta p' \\ &= \frac{\rho\,\dot{Q}^2\,A'^2}{\rho'\,\dot{Q}'^2\,A^2}\Delta p' \\ &= 4.94\cdot 10^3\ \frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

6.9 (a)

$$\begin{aligned} Re = Re' : \dot{Q}' &= \frac{v' D'^2}{v D^2} \dot{Q} \\ &= \frac{\eta' \rho D'}{\eta \rho' D} \dot{Q} \\ &= 0.5 \frac{\mathrm{m}^3}{\mathrm{s}} \\ Sr = Sr' : n' &= \frac{v' D}{v D'} n \\ &= 13.3 \frac{1}{\mathrm{s}} \end{aligned}$$

(b)

$$\begin{aligned} Eu = Eu' : \Delta p_0 &= \frac{\rho v^2}{\rho' v'^2} \Delta p_0 \\ &= 527 \frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

(c)

$$\begin{aligned} P &= \dot{Q} \Delta p_0 = 15.82 \text{ kW} \\ P' &= \dot{Q}' \Delta p_0' = 15 \text{ kW} \\ M &= \frac{P}{2 \pi n} = 201 \text{ Nm} \\ M' &= \frac{P'}{2 \pi n'} = 179 \text{ Nm} \end{aligned}$$

6.10 (a)

$$\begin{aligned} T' = T : \quad \mu' &= \mu_{300\ K} \\ Ma = Ma' : \quad v' &= v = 200 \frac{\mathrm{m}}{\mathrm{s}} \\ Re = Re' : \quad p' &= \frac{\rho'}{\rho} p = \frac{D}{D'} p \\ &= 4 \cdot 10^5 \frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

(b)

$$Sr = Sr' : \quad n' = \frac{v' D}{v D'} n = 400 \frac{1}{\mathrm{s}}$$

(c)

$$P = \frac{\frac{\rho}{2} v^3 D^2}{\frac{\rho'}{2} v'^3 D'^2} P' = 4 P'$$

6.11 (a)

$$\begin{aligned} Re &= Re' : \quad \frac{\nu'}{\nu} = \frac{v' L'}{v L} \\ Fr &= Fr' : \quad \frac{v'}{v} = \sqrt{\frac{L'}{L}} \\ \frac{\nu'}{\nu} &= 10^{-3} \quad (!) \end{aligned}$$

(b)

$$Fr = Fr' : \quad \frac{v'}{v} = \sqrt{\frac{L'}{L}} = 0.1$$

oder

$$Re = Re' : \quad \frac{v'}{v} = \frac{L}{L'} = 100$$

6.12 (a)

$$\begin{aligned} Fr_L &= Fr_L' : \quad v' = v \sqrt{\frac{L'}{L}} \\ &= 0.75 \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

(b)

$$\begin{aligned} c_D &= c_D' \\ F_D &= \frac{\frac{\rho}{2} v^2 B H}{\frac{\rho}{2} v'^2 B' H'} F_D' \\ &= 1.64 \cdot 10^4 \text{ N} \end{aligned}$$

(c)

$$c_D = 1.12$$

(d)

$$\frac{v}{\sqrt{g h}} = \frac{v'}{\sqrt{g h'}} : \ h = 16 h' = 0.4 \text{ m}$$

6.13 (a)

$$\begin{aligned} Eu &= Eu', \quad Re = Re' : \\ \frac{p_0' - p_R}{p_0 - p_R} &= \frac{\rho' v'^2}{\rho v^2} = \frac{\rho D^2 \mu'^2}{\rho' D'^2 \mu'^2} \\ p_0' - p_R &= 440 \frac{\mathrm{N}}{\mathrm{m}^2} \\ Sr &= Sr' : \\ T' &= \frac{v D'}{v' D} T = 0.57 \text{ s} \end{aligned}$$

(b) Analogously to a)

$$P_{max} = 6.68 \cdot 10^5 \frac{\mathrm{N}}{\mathrm{m}^2}$$

6.14 (a)

$$\begin{aligned} Re = Re' : \quad v' &= \frac{\rho \mu' D}{\rho' \mu D'} \frac{\dot{Q}}{\frac{\pi}{4} D^2} \\ &= 0.1 \frac{\mathrm{m}}{\mathrm{s}} \\ \dot{Q} &= 8 \cdot 10^{-4} \frac{\mathrm{m}^3}{\mathrm{s}} \end{aligned}$$

(c)

$$\alpha = \frac{\frac{4\dot{Q}}{\pi d^2}}{\sqrt{\frac{2\Delta p'_D}{\rho}}} = \frac{\left(\frac{D}{d}\right)^2}{\sqrt{2\,Eu_D}}$$

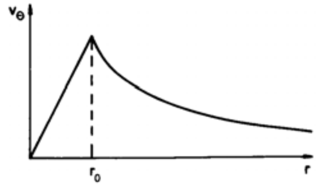
$$\begin{aligned} Eu_D &= Eu'_D: \quad \alpha = \alpha' = 0.6366 \\ Eu_l &= Eu'_l: \quad \zeta = \zeta' = 77.1 \end{aligned}$$

(d)

$$\begin{aligned} \Delta p_D &= \frac{\rho\, v^2}{\rho'\, v'^2}\,\Delta p'_D \\ &= 0.625 \cdot 10^5\,\frac{\mathrm{N}}{\mathrm{m}^2} \\ \Delta p_l &= \frac{\rho\, v^2}{\rho'\, v'^2}\,\Delta p'_l \\ &= 0.5 \cdot 10^5\,\frac{\mathrm{N}}{\mathrm{m}^2} \end{aligned}$$

3.2.7 Potential Flows of Incompressible Fluids

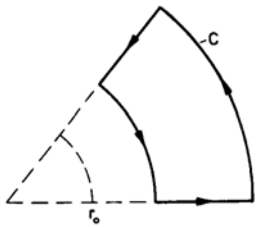
7.1 (a)



(b)

$$\begin{aligned} \Gamma &= \oint_c \boldsymbol{v}\, d\boldsymbol{s} \\ &= \int_0^{2\pi} v_\theta(r)\, r\, d\theta \\ &= \begin{cases} 2\,\pi\omega\, r^2 & r \le r_0 \\ 2\,\pi\omega\, r_0^2 & r > r_0 \end{cases} \end{aligned}$$

(c)



$$\begin{aligned} \Gamma &= \oint_c \boldsymbol{v}\, d\boldsymbol{s} = 0 \\ \boldsymbol{\omega} &= 0 \end{aligned}$$

(d)

$$\begin{aligned} E &= \int_0^{2\,r_0} \frac{\rho}{2}\, v_\theta^2\, H\, 2\,\pi\, r\, dr \\ &= \pi\,\rho\, H\,\omega^2\, r_0^2\,(0.25 + \ln 2) \\ &= 3.7 \cdot 10^8\ \mathrm{Nm} \end{aligned}$$

7.2 (a)

$$u = \frac{\partial \Phi}{\partial x} \qquad v = \frac{\partial \Phi}{\partial y}$$

Potential Φ exists, if

$$\nabla \times \boldsymbol{v} = 0\ .$$

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

Stream function exists, if,

$$\nabla \cdot \boldsymbol{v} = 0\ .$$

(b)

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= \nabla^2\,\Phi \\ \nabla \times \boldsymbol{v} &= -\nabla^2\,\Psi\,\boldsymbol{k} \end{aligned}$$

7.3

	$\nabla \cdot \boldsymbol{v}$	$\mid \nabla \times \boldsymbol{v} \mid$
a)	4xy	$y^2 - x^2$
b)	2	0
c)	0	-2
d)	0	0

Stream function exists, for c) and d), the potential for b) and d).

Determination of the stream function:

(a)

$$\begin{aligned} \Psi &= \int u\, dy + f(x) = \frac{y^2}{2} + f(x) \\ v &= -\frac{\partial \Psi}{\partial x} = -f'(x) = -x \\ \Psi &= \frac{1}{2}(x^2 + y^2) + c \end{aligned}$$

(b)

$$\Psi = \frac{1}{2}(y^2 - x^2) + c$$

Determination of the potential:

(c)

$$\Phi = \int u\,dx + f(y) = \frac{x^2}{2} + f(y)$$
$$v = \frac{\partial \Phi}{\partial y} = f'(y) = y$$
$$\Phi = \frac{1}{2}(x^2 + y^2) + c$$

(d)

$$\Phi = xy + c$$

7.4 (a)

$| \nabla \text{ x } \boldsymbol{v} |= 0 :$ Potential exists.

(b)

$$u = \frac{U}{L}\,x \qquad v = -\frac{U}{L}\,y$$

Stagnation points:

$$u = v = 0 : \quad x = y = 0$$

Pressure coefficient:

$$c_p = \frac{p - p_{ref}}{\frac{\rho}{2}\,\boldsymbol{v}^2_{ref}} = 1 - \frac{u^2 + v^2}{u^2_{ref} + v^2_{ref}} = 1 - \frac{x^2 + y^2}{x^2_{ref} + y^2_{ref}}$$

Isotachs:

$$\boldsymbol{v}^2 = u^2 + v^2 = \left(\frac{U}{L}\right)^2 (x^2 + y^2)$$
$$x^2 + y^2 = \left(\frac{\boldsymbol{v}\,L}{U}\right)^2$$

Circles around the origin of coordinates with radius

$$\frac{|\,\boldsymbol{v}\,|\;L}{U}$$

(c)

$$u_1 = 4\ \frac{\text{m}}{\text{s}} \quad v_1 = -4\ \frac{\text{m}}{\text{s}}$$
$$|\,\boldsymbol{v}_1\,|= 5.66\ \frac{\text{m}}{\text{s}}$$
$$p_1 = p_{ref} + c_{p1}\,\frac{\rho}{2}\,\boldsymbol{v}^2_{ref} = 0.86 \cdot 10^5\ \frac{\text{N}}{\text{m}^2}$$

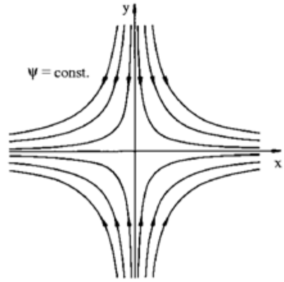
(d)

$$t = \int_{x_1}^{x_2} \frac{dx}{u} = \frac{L}{U}\ln\frac{x_2}{x_1}$$
$$x_2 = 5.44\ \text{m}$$
$$\Psi = const: \quad x_1\,y_1 = x_2\,y_2$$
$$y_2 = 0.74\ \text{m}$$

(e)

$$p_1 - p_2 = (c_{p1} - c_{p2})\,\frac{\rho}{2}\,\boldsymbol{v}^2_{ref} = 0.442 \cdot 10^5\ \frac{\text{N}}{\text{m}^2}$$

(f)



7.5 (a)

$$u = 2\,x\,y \qquad v = x^2 - y^2$$
$$\nabla\cdot\boldsymbol{v} = 0 : \text{Stream function exists.}$$

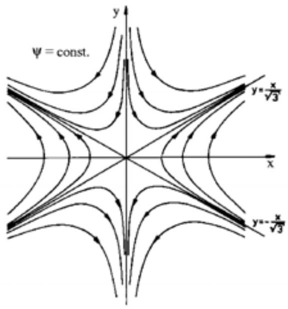
(b)

$$\Psi = x\,y^2 - \frac{x^3}{3} + c$$
$$\text{Streamlines: } \Psi = const.$$

$$y = \pm\sqrt{\frac{x^3}{3} + \frac{k}{x}} \quad (x \neq 0)$$

Asymptotes:

$$x \to \pm\infty : \quad y = \pm\frac{x}{\sqrt{3}}$$
$$x \to \pm 0 : \quad y \to \pm\infty$$



7.6

$$v_r = \frac{c}{r} \qquad v_\theta = 0$$

(a)

$$\nabla \text{ x } \boldsymbol{v} = 0$$
$$\nabla \cdot \boldsymbol{v} = 0$$

(b)

$$\begin{aligned} \Phi &= \int v_r \, dr + f_1(\theta) \\ &= c \ln r + f_1(\theta) \\ \frac{\partial \Phi}{\partial \theta} &= 0: \quad f_1(\theta) = k_1 \\ \Psi &= \int r \, v_r \, d\theta = c\theta + f_2(r) \\ \frac{\partial \Psi}{\partial r} &= 0: \quad f_2(r) = k_2 \end{aligned}$$

(c) Circle with radius r:

$$\begin{aligned} \Gamma &= \int_0^{2\pi} v_\theta \, r \, d\theta = 0 \\ v_r &= 0 \qquad v_\theta = \frac{c}{r} \end{aligned}$$

(a)

$$\begin{aligned} \nabla \text{ x } \boldsymbol{v} &= 0 \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}$$

(b)

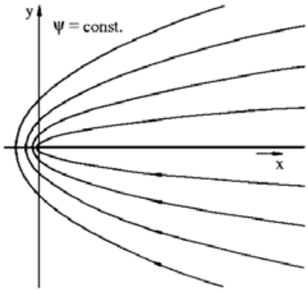
$$\begin{aligned} \Phi &= c\,\theta + k_3 \\ \Psi &= -c \, \ln x + k_4 \end{aligned}$$

(c)

$$\Gamma = 2\,\pi\,c$$

7.7 (a) n = 0.5:

$$\begin{aligned} \Psi &= 2\sqrt{r} \sin\left(\frac{\theta}{2}\right) \\ \Psi &= 0: \quad \theta = 0.2\,\pi \\ \Psi &= c: \quad r = \left(\frac{c}{2}\right)^2 \sin^{-2}\left(\frac{\theta}{2}\right) \end{aligned}$$



Parallel flow:

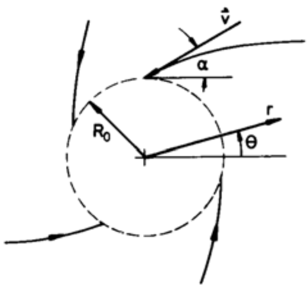
$$\begin{aligned} n &= 1: \Psi = r \, \sin\theta = y \\ n &= 2: \Psi = \frac{1}{2} r^2 \, \sin(2\,\theta) = xy \end{aligned}$$

See problem 8.4.

(b)

$$\begin{aligned} c_p &= \frac{p - p_{ref}}{\frac{\varrho}{2}\,\boldsymbol{v}_{ref}^2} = 1 - \frac{\boldsymbol{v}^2}{\boldsymbol{v}_{ref}^2} \\ v_\theta &= -r^{n-1} \, \sin(n\,\theta) \\ v_r &= r^{n-1} \, \cos(n\,\theta) \\ c_p &= 1 - \left(\frac{x^2 - y^2}{2}\right)^{n-1} \\ n &= 1: \quad c_p(0.0) = 0 \\ n &> 1: \quad c_p(0.0) = 1 \\ n &< 1: \quad c_p(0.0) = -\infty \end{aligned}$$

7.8 (a)



$$\begin{aligned} \Psi &= \frac{E}{2\,\pi}\,\theta - \frac{\Gamma}{2\,\pi}\, \ln\frac{r}{R} \\ v_r &= \frac{E}{2\,\pi\,r} \\ v_\theta &= \frac{\Gamma}{2\,\pi\,r} \end{aligned}$$

$$\tan\alpha = -\frac{v_r}{v_\theta}\bigg|_{r=R_0}$$

$$\begin{aligned} E &= -\frac{\dot{Q}}{h_0} \\ \Gamma &= \frac{\dot{Q}}{h_0\ \tan\alpha} = 4.33\cdot 10^{-2}\ \frac{\mathrm{m}^2}{\mathrm{s}} \end{aligned}$$

(b)

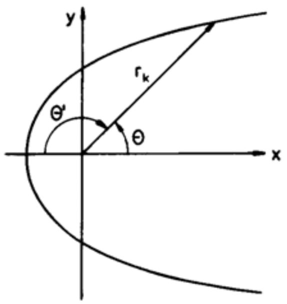
$$\begin{aligned} \rho\, g\, h + \frac{\rho}{2}\,\boldsymbol{v}^2 &= \rho\, g\, h_0 + \frac{\rho}{2}\,\boldsymbol{v}_0^2 \\ \boldsymbol{v}^2 &= v_r^2 + v_\theta^2 \\ \boldsymbol{v}_\theta^2 &= \boldsymbol{v}_{r=R_0}^2 \end{aligned}$$

$$h(r) = h_0 + \frac{1}{8g}\left(\frac{\dot{Q}}{\pi\, R_0\, h_0\ \sin\alpha}\right)^2 \cdot \cdot\left[1-\left(\frac{R_0}{r}\right)^2\right]$$

(c)

$$\begin{aligned} \lim_{r\to\infty} h(r) &= h_0 + \frac{1}{8g}\left(\frac{\dot{Q}}{\pi\, R_0\, h_0\ \sin\alpha}\right)^2 \\ &= 2.35\cdot 10^{-2}\ \mathrm{m} \end{aligned}$$

7.9



(a)

$$\begin{aligned} \Psi &= u_\infty\, y + \frac{E}{2\,\pi}\,\theta + c \\ &= u_\infty\left[y + \frac{h}{\pi}\arctan\left(\frac{y}{x}\right)\right] + c \\ u &= u_\infty\left(1+\frac{h}{\pi}\,\frac{x}{x^2+y^2}\right) \\ v &= u_\infty \frac{h}{\pi}\frac{y}{x^2+y^2} \end{aligned}$$

Stagnation point: $u = v = 0$:

$$\begin{aligned} x_s &= -\frac{h}{\pi},\quad y_s = 0 \\ u(x_s,h) &= u_\infty \frac{\pi^2}{1+\pi^2} \\ v(x_s,h) &= u_\infty \frac{\pi^2}{1+\pi^2} \end{aligned}$$

(b) Contour: Streamline through stagnation point

$$r_C = \frac{h}{\pi}\,\frac{\pi-\theta}{\sin\theta} = \frac{h}{\pi}\,\frac{\theta'}{\sin\theta}$$

with

$$\theta' = \pi - \theta$$

(c)

$$\begin{aligned} c_p &= 1 - \frac{u^2+v^2}{u_\infty^2} = -\frac{h}{\pi}\frac{2x+\frac{h}{\pi}}{x^2+y^2} \\ c_{pC} &= \frac{\sin(2\,\theta')}{\theta'} - \left(\frac{\sin\theta'}{\theta'}\right)^2 \end{aligned}$$

(d)

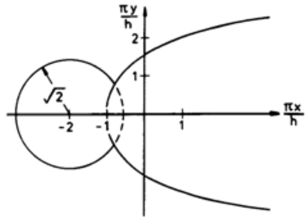
$$c_p = \text{const:}$$

$$\left(x+\frac{h}{\pi\, c_p}\right)^2 + y^2 = (1-c_p)\left(\frac{h}{\pi\, c_p}\right)^2$$

Circles around $\left(-\frac{h}{\pi\, c_p}.0\right)$ with radius $\frac{h\sqrt{1-c_p}}{\pi\, c_p}$

(e)

$$c_p = \frac{1}{2} : \left(x+\frac{2h}{\pi}\right)^2 + y^2 = 2\,\left(\frac{h}{\pi}\right)^2$$



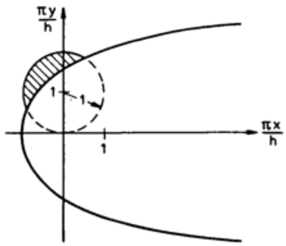
(f)

$$\frac{\sqrt{u^2+v^2}}{u_\infty} = k$$
$$\left(x - \frac{\frac{h}{\pi}}{k^2-1}\right)^2 + y^2 = \left(\frac{\frac{kh}{\pi}}{k^2-1}\right)^2$$

Circles around $\left(\frac{h}{\pi\,(k^2-1)}.0\right)$ with radius $\frac{kh}{\pi\,(k^2-1)}$

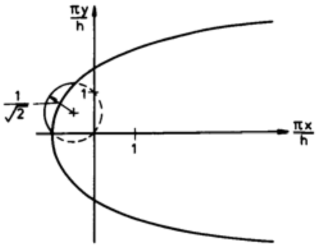
(g)

$$v = u_\infty \frac{h}{\pi}\,\frac{y}{x^2+y^2} > \frac{u_\infty}{2}$$
$$x^2 + \left(y - \frac{h}{\pi}\right)^2 < \left(\frac{h}{\pi}\right)^2$$



(h)

$$\tan\alpha = \frac{v}{u} = 1$$
$$\left(x + \frac{h}{2\,\pi}\right)^2 + \left(y - \frac{h}{2\pi}\right)^2 = \frac{1}{2}\left(\frac{h}{\pi}\right)^2$$



(i) Acceleration along the x-Axis:

$$b = \frac{du}{dt} = u\,\frac{\partial du}{\partial dx}$$
$$= -u_\infty \frac{h}{\pi}\left(\frac{1}{x^2} + \frac{h}{\pi}\,\frac{1}{x^3}\right)$$

$$\frac{db}{dx} = 0: \quad x_{max} = -\frac{3}{2}\,\frac{h}{\pi}$$
$$b_{max} = -\frac{4}{27}\,\frac{\pi}{h}\,u_\infty^2$$

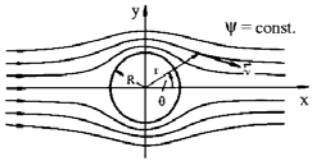
7.10 (a)

$$\Psi = 0: \quad y = 0$$
$$x^2 + y^2 = R^2$$

(Parallel flow)

$r = \sqrt{x^2+y^2} \to \infty: \quad \Psi \to u_\infty\, y$

Stream function describes the flow around a cylinder.



(b)

$$c_p = 1 - \frac{v_r^2 + v_\theta^2}{u_\infty^2}$$
$$v_r = u_\infty \left[1 - \left(\frac{R}{r}\right)^2\right] \cos\theta$$
$$v_\theta = -u_\infty \left[1 + \left(\frac{R}{r}\right)^2\right] \sin\theta$$

$$r = R: \quad c_p = 1 - 4\,\sin^2\theta$$

(c)

$$\Delta t = \int_{-3R}^{2R} \frac{dx}{u(x.0)}$$
$$u(x.0) = u_\infty \left(1 - \frac{R^2}{x^2}\right)$$

$$\Delta t = \frac{1}{u_\infty}\left[x + \frac{R}{2}\,\ln\,\frac{x-R}{x+R}\right]_{-3R}^{-2R}$$
$$= \frac{R}{u_\infty}\,(1 + 0.5\,\ln 1.5)$$

7.11 Determination of velocity components see problem 8.10.

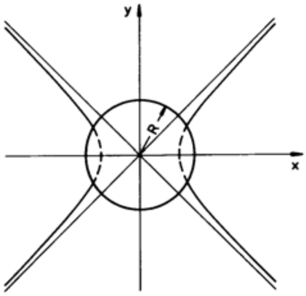
$c_p = \left(\frac{R}{r}\right)^2 \left[2\,\cos(2\,\theta) - \left(\frac{R}{r}\right)^2\right]$

(a)

$$c_p = 0$$
$$r = \frac{R}{\sqrt{2\,\cos(2\,\theta)}} \quad \text{or}$$

$$\left(\frac{\sqrt{2}\,x}{R}\right)^2 - \left(\frac{\sqrt{2}\,y}{R}\right)^2 = 1$$

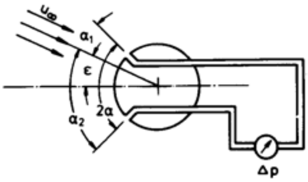
Hyperbola



(b)

$$c_p = \frac{7}{16} - \sin\theta$$

7.12 (a)



$$\begin{aligned}
\Delta p &= (c_{p1} - c_{p1})\frac{\rho}{2}u_\infty^2 \\
c_p &= 1 - 4\,\sin^2\alpha \\
\sin^2\alpha_2 - \sin^2\alpha_1 &= \sin(\alpha_1 + \alpha_2) \\
&\cdot\ \sin(\alpha_2 - \alpha_1)
\end{aligned}$$

$$\Delta p = 2\,\rho\,u_\infty^2\,\sin(2\,\alpha)\,\sin(2\,\epsilon)$$

(b)

$$\alpha = \frac{\pi}{2}$$

7.13 (a)

$$\begin{aligned}
\rho\,g\,h_\infty + \frac{\rho}{2}\,u_\infty^2 &= \rho\,g\,h(\theta) + \frac{\rho}{2}\,\boldsymbol{v}^2 \\
r = R : \boldsymbol{v}^2 &= v_\theta^2 = 4u_\infty^2\sin^2\theta \\
h(\theta) - h_\infty &= \frac{u_\infty^2}{2g}(1 - 4\sin^2\theta)
\end{aligned}$$

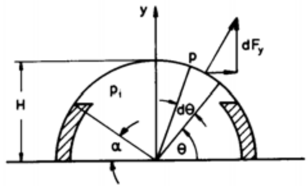
(b) Stagnation points:

$$\theta = 0 \text{ und } \theta = \pi$$
$$h = h_\infty + \frac{u_\infty^2}{2\,g} = 6.05\text{ m}$$

(c)

$$\theta_{min} = \frac{\pi}{2},\quad \frac{3\,\pi}{2}$$
$$h_{min} = h_\infty - \frac{3\,u_\infty^2}{2\,g} = 5.85\text{ m}$$

7.14



$$\begin{aligned}
dF_y &= (p_i - p)\,LH\,\sin\theta\,d\theta \\
p_i &= p_\infty + \frac{\rho}{2}\,u_\infty^2 \\
p &= p_\infty + c_p\,\frac{\rho}{2}\,u_\infty^2 \\
&= p_\infty + (1 - 4\,\sin^2\theta)\,\frac{\rho}{2}\,u_\infty^2 \\
F_y &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\,\rho\,u_\infty^2\,LH\,\sin^3\theta\,d\theta
\end{aligned}$$

$$\begin{aligned}
&= 2\,\rho\,u_\infty^2\,LH\left[-\frac{1}{3}\,\sin^2\theta\,\cos\theta -\right. \\
&\left.-\frac{2}{3}\,\cos\theta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
&= 7.37\cdot 10^6\text{ N} < G
\end{aligned}$$

Mooring not necessary.

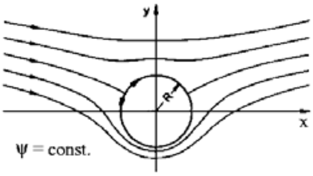
7.15 (a)

$$\begin{aligned}
\Psi &= u_\infty\,r\,\sin\theta\left[1 - \left(\frac{R}{r}\right)^2\right] - \\
&\quad -\frac{\Gamma}{2\,\pi}\,\ln\frac{r}{R} \\
v_r &= u_\infty\left[1 - \left(\frac{R}{r}\right)^2\right]\cos\theta \\
v_\theta &= -u_\infty\left[1 + \left(\frac{R}{r}\right)^2\right]\sin\theta\,* \\
&\quad + \frac{\Gamma}{2\,\pi\,r} \\
r &= R : \quad v_{\theta\ vortex} = v_t = \frac{\Gamma}{2\,\pi\,R} \\
\Gamma &= 2\,\pi\,R\,v_t
\end{aligned}$$

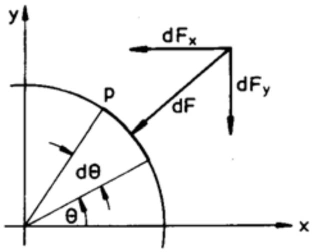
(b) Flow field for $v_t = u_\infty$:

$$\begin{aligned}
\Psi &= u_\infty\,r\,\sin\theta\left(1 - \left(\frac{R}{r}\right)^2\right) - \\
&- \ u_\infty\,R\ln\frac{r}{R} \\
v_r &= u_\infty\left[1 - \left(\frac{R}{r}\right)^2\right]\cos\theta \\
v_\theta &= u_\infty\left[\frac{R}{r} - \left[1 + \frac{R^2}{r^2}\right]\sin\theta\right] \\
\Psi &= 0 :
\end{aligned}$$

Contour: Circle around the origin of coordinates with radius R. 2 Stagnation points on the contour ($r = R$): $\theta_s = \frac{\pi}{6}, \frac{5\pi}{6}$; nor free Stagnation points



(c)



$$
\begin{aligned}
dF_x &= -pLR\,\cos\theta\,d\theta \\
dF_y &= -pLR\,\sin\theta\,d\theta \\
p &= p_\infty + c_p\,\frac{\rho}{2}\,u_\infty^2 \\
r &= R: \\
& c_p = 1 - \left(\frac{v_t}{u_\infty} - 2\sin\theta\right)^2
\end{aligned}
$$

$$
\begin{aligned}
F_x &= -LR\int_0^{2\pi}\frac{\rho}{2}\,u_\infty^2 \times \\
&\times \left[1 - \left(\frac{v_t}{u_\infty} - 2\,\sin\theta\right)^2\right] \times \\
&\times \cos\theta\,d\theta - \\
&- LR\int_0^{2\pi} p_\infty\,\cos\theta\,d\theta = 0 \\
F_y &= -LR\int_0^{2\pi}\frac{\rho}{2}\,u_\infty^2 \times \\
&\times \left[1 - \left(\frac{v_t}{u_\infty} - 2\,\sin\theta\right)^2\right] \times \\
&\times \sin\theta\,d\theta - \\
&- LR\int_0^{2\pi} p_\infty\,\sin\theta\,d\theta \\
&= -2\,\pi\,\rho LR\,v_t\,u_\infty \\
&= -\rho\,u_\infty\,\Gamma\,L
\end{aligned}
$$

3.2.8 Boundary Layers

8.1

$$
\begin{aligned}
c_D &\sim \frac{F_D}{\rho\,u_\infty^2\,BL} \sim \frac{\bar{\tau}_w}{\rho\,u_\infty^2} \\
&\sim \frac{\frac{\eta\,u_\infty}{\delta}}{\rho\,u_\infty^2} \sim \frac{\mu}{\rho\,u_\infty\,\delta}
\end{aligned}
$$

Inertia and frictional forces of equal order of magnitude:

$$
\begin{aligned}
\frac{\rho\,u_\infty^2}{L} &\sim \frac{\bar{\tau}_w}{\delta} \\
c_D &\sim \frac{1}{\sqrt{Re_L}}
\end{aligned}
$$

8.2 (a)

$$x_{crit.} = \frac{\nu\,Re_{crit}}{u_\infty} = 0.167\text{ m}$$

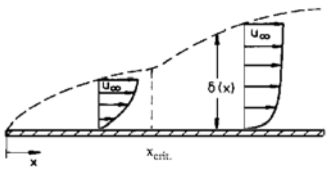
(b)

$$
\begin{aligned}
\eta = \frac{y}{x}\sqrt{Re_x} &= 1.095 \\
\frac{u}{u_\infty} = 0.36 : u &= 16.2\,\frac{\text{m}}{\text{s}}
\end{aligned}
$$

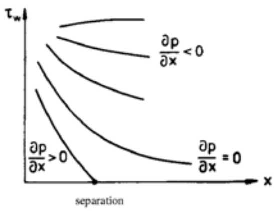
from diagram, page 60

$$
\begin{aligned}
x = 0.15\ m &: Re_x = 4.5\cdot 10^5 \\
\frac{y}{x}\sqrt{Re_x} = 1.095 &: y = 2.45\cdot 10^{-4}\text{m}
\end{aligned}
$$

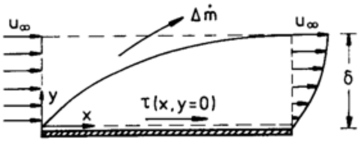
(c)



(d)



8.3



$$-\rho\, u_\infty^2\, \delta + \rho \int_0^\delta u^2\, dy + \varDelta\, \dot m\, u_\infty = \\ = \int_0^x \tau(x', y = 0)\, dx'$$

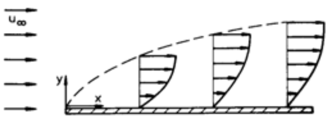
$$\varDelta\, \dot m = \rho \int_0^\delta (u_\infty - u)\, dy$$

$$\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) = \\ = \delta_2 = -\int_0^x \frac{\tau(x', y = 0)}{\rho\, u_\infty^2}\, dx'$$

8.4 (a)

$$Re_L = 3.33\, 10^5 :$$

Boundary layer laminar

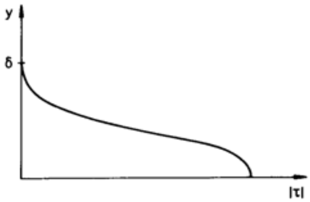


(b)

$$\begin{aligned} y = 0 \quad &: \quad u = v = 0 \\ y \to \infty \quad &: \quad u \to u_\infty \end{aligned}$$

(c)

From boundary-layer equation:
$\dfrac{\partial \tau}{\partial y} = 0$ for $y = 0$ und $y = \delta$



(d) From Blasius solution:

$$\begin{aligned} \delta(x = L) &= \frac{5L}{\sqrt{Re_L}} \quad c_f = \frac{0.664}{\sqrt{Re_x}} \\ \delta(x = L) &= 4.33 \text{ mm} \\ F_w &= 2 \int_0^L B\, \tau_w\, dx \\ &= \rho\, u_\infty^2\, B \int_0^L c_f\, dx \\ &= 0.144 \text{ N} \end{aligned}$$

8.5 (a) Boundary conditions:

$$\begin{aligned} \frac{y}{\delta} &= 0 : \quad \frac{u}{u_\infty} = 0, \quad \frac{v}{u_\infty} = 0 \\ \frac{y}{\delta} &= 1 : \quad \frac{u}{u_\infty} = 1 \end{aligned}$$

From boundary-layer equation:

$$\rho\left(u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y}\right) = \eta\, \frac{\partial^2 u}{\partial y^2} :$$

$$\begin{aligned} \frac{y}{\delta} = 0 : \quad u = v &= 0 : \\ \frac{\partial^2 \left(\frac{u}{u_\infty}\right)}{\partial \left(\frac{y}{\delta}\right)^2} &= 0 \\ \frac{y}{\delta} = 1 : \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} &= 0 : \\ \frac{\partial^2 \left(\frac{u}{u_\infty}\right)}{\partial \left(\frac{y}{\delta}\right)^2} &= 0 \end{aligned}$$

Inviscid external flow

$$\begin{aligned} \frac{y}{\delta} &= 1 : \quad \tau \sim \frac{\partial \left(\frac{u}{u_\infty}\right)}{\partial \left(\frac{y}{\delta}\right)} = 0 \\ \frac{u}{u_\infty} &= 2 \left(\frac{y}{\delta}\right) - 2 \left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \end{aligned}$$

(b)

$$\begin{aligned} \frac{\delta_1}{\delta} &= \int_0^1 \left(1 - \frac{u}{u_\infty}\right) d\left(\frac{y}{\delta}\right) \\ &= \frac{3}{10} \\ \frac{\delta_2}{\delta} &= \int_0^1 \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) d\left(\frac{y}{\delta}\right) \\ &= \frac{37}{315} \end{aligned}$$

Von Kármán integral relation

$$\frac{d\delta_2}{dx} + \frac{\tau(y=0)}{\rho\, u_\infty} = 0$$

$$\begin{aligned} \tau(y=0) &= -\frac{\mu\, u_\infty^2}{\delta} \left.\frac{d\left(\frac{u}{u_\infty}\right)}{d\left(\frac{y}{\delta}\right)^2}\right|_{\frac{y}{\delta}=0} \\ &= -2\,\frac{\mu\, u_\infty}{\delta} \end{aligned}$$

Integration:

$$\begin{aligned} \frac{\delta}{x} &= \frac{5.84}{\sqrt{Re_x}} \\ c_D &= \frac{2}{L}\int_0^L \frac{\tau_w}{\rho\, u_\infty^2}\, dx \\ &= -\frac{2}{L}\int_0^L \frac{\tau(y=0)}{\rho\, u_\infty^2}\, dx \\ &= \frac{1.371}{\sqrt{Re_L}} \end{aligned}$$

8.6 (a) Solution see problem 9.5

A)

$$\begin{aligned} \frac{\delta_1}{\delta} &= \frac{3}{8}, \\ \frac{\delta_2}{\delta} &= \frac{39}{280}, \\ \frac{\delta}{x} &= \frac{4.641}{\sqrt{Re_x}}, \\ c_D &= \frac{1.293}{\sqrt{Re_L}} \end{aligned}$$

B)

$$\begin{aligned} \frac{\delta_1}{\delta} &= 1 - \frac{2}{\pi} = 0.363 \\ \frac{\delta_2}{\delta} &= \frac{2}{\pi} - \frac{1}{2} = 0.137 \\ \frac{\delta}{x} &= \frac{\sqrt{\frac{2\,\pi^2}{4-\pi}}}{\sqrt{Re_x}} = \frac{4.795}{\sqrt{Re_x}} \\ c_D &= \frac{2\sqrt{2-\frac{\pi}{2}}}{\sqrt{Re_L}} = \frac{1.310}{\sqrt{Re_L}} \end{aligned}$$

(b) A)

$$\begin{aligned} \delta(x=L) &= 3.288 \text{ mm} \\ F_D &= c_D\, \rho\, u_\infty^2\, 2\, L\, B \\ &= 0.91\ N \end{aligned}$$

B)

$$\begin{aligned} \delta(x=L) &= 3.39 \text{ mm} \\ F_D &= 0.93 \text{ N} \end{aligned}$$

3.2.9 Drag

9.1 (a)

$$\begin{aligned} F_1 &= c_{D1}\,\frac{\rho}{2}\, u_\infty^2\, 2\, L_1\, B \\ Re_{L1} &= 1.8\cdot 10^5 < Re_{crit.} \\ c_{D1} &= \frac{1.328}{\sqrt{Re_{L1}}} = 3.13\cdot 10^{-3} \\ F_1 &= 0.564 \text{ N} \\ Re_L &= 3.6\cdot 10^5 < Re_{crit.} \\ F_{tot} &= F_1 + F_2 \\ &= \frac{1.328}{\sqrt{Re_L}}\,\frac{\rho}{2}\, u_\infty^2\, 2\, L\, B \\ F_2 &= 0.233 \text{ N} \end{aligned}$$

(b)

$$\begin{aligned} F_{tot} &= 2\, F_1 \\ L_1 &= \frac{L}{4} = 0.09 \text{ m} \\ L_2 &= 0.27 \text{ m} \end{aligned}$$

9.2 (a) The frictional drag results from the shear stresses acting on the body, the pressure drag results from change of the potential pressure distribution caused by the frictional force.

(b)

$$\begin{aligned} c_{D1}\,\frac{\rho}{2}\, u_\infty^2\, 2\, L_1^2 &= c_{D2}\,\frac{\rho}{2}\, u_\infty^2\, 2\, L_2^2 \\ Re_1 = \frac{u_\infty\, L_1}{\nu} &= 3.33\cdot 10^5 \\ c_{D1} = \frac{1.328}{\sqrt{Re_1}} &= 2.30\cdot 10^{-3} \end{aligned}$$

Assumption:

$$\begin{aligned} Re_2 &= \frac{u_\infty\, L_2}{\nu} > 10^3 : \quad c_{D2} = 1.1 \\ L_2 &= L_1\sqrt{\frac{2\, c_{D1}}{c_{D2}}} = 6.47\cdot 10^{-2} \text{ m} \\ Re_2 &= 2.16\cdot 10^4 > 10^3 \end{aligned}$$

(c)

$$\begin{aligned} F_{D1} &= \frac{1.328}{\sqrt{Re_1}}\,\frac{\rho}{2}\, u_\infty^2\, 2\, L_1^2 \sim u_\infty^{\frac{3}{2}} \\ F_{D2} &= 1.1\,\frac{\rho}{2}\, u_\infty^2\, L_2^2 \sim u_\infty^2 \end{aligned}$$

9.3 (a)

$$\frac{F_{D1}}{F_{D2}} = \frac{c_{D1}}{c_{D2}}$$

u_∞	Re_1	$10^3\, c_{D1}$
$0.4\ \frac{\text{m}}{\text{s}}$	$4 \cdot 10^5$	2.10
$0.8\ \frac{\text{m}}{\text{s}}$	$8 \cdot 10^5$	2.76
$1.6\ \frac{\text{m}}{\text{s}}$	$1.6 \cdot 10^5$	3.19
Re_1	$10^3\, c_{D2}$	$\frac{F_{D1}}{F_{D2}}$
$2 \cdot 10^5$	2.97	0.707
$4 \cdot 10^5$	2.10	1.313
$8 \cdot 10^5$	2.76	1.156

(b)

$$\begin{aligned} Re_1 &= 1.96 \cdot 10^5 \\ c_{D1} &= c_{D2} = 3.0 \cdot 10^{-3} \end{aligned}$$

1)

$$Re_2 = Re_1 : \quad u_{\infty 2} = 0.392\ \frac{\text{m}}{\text{s}}$$

2)

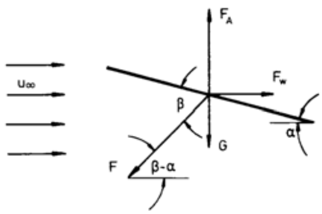
$$Re_2 \approx 1.3 \cdot 10^6 : u_{\infty 2} = 2.6\ \frac{\text{m}}{\text{s}}$$

3)

$$Re_3 \approx 9 \cdot 10^6 : \quad u_{\infty 2} = 18\ \frac{\text{m}}{\text{s}}$$

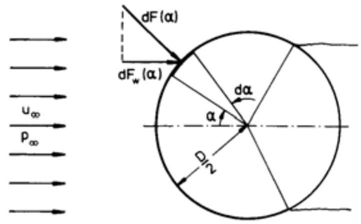
(2) and 3) from diagram page 64).

9.4



$$\begin{aligned} c_L &= \frac{F_L}{\frac{\rho}{2}\, u_\infty^2\, A} \\ F_L &- W - F\, \sin(\beta - \alpha) = 0 \\ c_L &= \frac{W + F\, \sin(\beta - \alpha)}{\frac{\rho}{2}\, u_\infty^2\, A} = 1.28 \\ c_D &= \frac{F_D}{\frac{\rho}{2}\, u_\infty^2\, A} \\ F_D &- F\, \cos(\beta - \alpha) = 0 \\ c_D &= \frac{F\, \cos(\beta - \alpha)}{\frac{\rho}{2}\, u_\infty^2\, A} = 0.96 \end{aligned}$$

9.5



$$\begin{aligned} c_D &= \frac{2\, \int_0^\pi dF_w(\alpha)}{\frac{\rho}{2}\, u_\infty^2\, DL} \\ dF_D(\alpha) &= dF(\alpha)\, \cos\alpha \\ &= p(\alpha)\, L\, \frac{D}{2}\, \cos\alpha\, d\alpha \\ p(\alpha) &= c_p(\alpha)\, \frac{\rho}{2}\, u_\infty^2 + p_\infty \end{aligned}$$

$$0 \le \alpha \le \frac{2}{3}\, \pi :$$

$$p(\alpha) = (1 - 4\, \sin^2\alpha)\, \frac{\rho}{2}\, u_\infty^2 + p_\infty$$

$$\frac{2}{3} \le \alpha \le \pi :$$

$$\begin{aligned} p(\alpha) &= \left[1 - 4\, \sin^2(\frac{2}{3}\, \pi)\right]\, \frac{\rho}{2}\, u_\infty^2 + p_\infty \\ &= -2\, \frac{\rho}{2}\, u_\infty^2 + p_\infty \\ c_D &= \sqrt{3} \end{aligned}$$

9.6 (a)

$$F_{DS} = F_{DB}$$

$$\begin{aligned} c_{DB}\frac{\rho_w}{2}u^2 LB &= c_{DS}\, \frac{\rho_a}{2}\, \times \\ &\quad \times (u_\infty - u)^2 \frac{hb}{2} \\ Re_L &= \frac{\rho_W\, u\, L}{\mu_W} \\ &= 5.625 \cdot 10^6 \\ c_{DB} &= \frac{0.074}{Re_L^{\frac{1}{5}}} - \frac{1700}{Re_L} \\ &= 3.0 \cdot 10^{-3} \end{aligned}$$

Assumption:

$$\begin{aligned} Re_b &= \frac{\rho_a\,(u_\infty - u)\,b}{\mu_a} > 10^3 \\ c_{DS} &= 1.2 \\ u_\infty &= u\left(1+\sqrt{\frac{c_{DB}}{c_{DS}}\,\frac{\rho_W}{\rho_a}\,\frac{2\,LB}{hb}}\right) \\ &= 2.95\ \frac{\mathrm{m}}{\mathrm{s}} \\ Re_b &= 1.94\cdot 10^5 > 10^3 \end{aligned}$$

(b)

$$F_{DS} = 6.33\ \mathrm{N}$$

(c)

$$\begin{aligned} F^*_{DB} &= c^*_{DB}\,\frac{\rho_a}{2}\,\left(u_\infty - u^2\right)\,LB \\ Re^*_L &= \frac{\rho_a\,(u_\infty - u)\,L}{\mu_a} = 3.63\cdot 10^5 \\ c^*_{DB} &= \frac{1.328}{\sqrt{Re^*_L}} = 2.20\cdot 10^{-3} \\ F^*_{DB} &= 5.45\cdot 10^{-3} \ll F_{DB} \end{aligned}$$

9.7

$$\begin{aligned} c_D\,\frac{\rho}{2}\,v^2\,A &= W \\ A = \frac{W}{c_D\,\frac{\rho}{2}\,v^2} &= 75.2\ \mathrm{m}^2 \end{aligned}$$

9.8 (a)

$F_D = W$ (Lift can be neglected)

Sphere:

$$c_D\,\frac{\rho_a}{2}\,v^2\,\frac{\pi\,D_S^2}{4} = \rho\,\frac{\pi\,D_S^3}{6}\,g$$

$$\begin{aligned} v &= Re\,\frac{\nu_a}{D_S} \\ D_S &= \sqrt[3]{18\,Re\,\frac{\rho_L}{\rho}\,\frac{\nu_a^2}{g}} \end{aligned}$$

$$Re = 0.5 :$$
$$D_{Smax} = 6.81\cdot 10^{-2}\ \mathrm{mm}$$

Cylinder (Length L):

$$c_D\,\frac{\rho_a}{2}\,v^2\,D_C\,L = \rho\,\frac{\pi\,D^2}{4}\,L\,g$$
$$D_C = \sqrt[3]{\frac{16\,Re}{2-\ln Re}\,\frac{\rho_a}{\rho}\,\frac{\nu_a^2}{g}}$$

$$Re = 0.5 :$$
$$D_{Cmax} = 4.71\cdot 10^{-2}\ \mathrm{mm}$$

(b)

$$v_S = 0.110\ \frac{\mathrm{m}}{\mathrm{s}}$$
$$v_C = 0.159\ \frac{\mathrm{m}}{\mathrm{s}}$$

9.9

$$v = Re\,\frac{\nu}{D}$$

(a) (Lift can be neglected:)

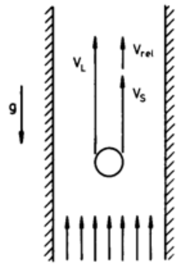
$$\begin{aligned} F_D &= W = \rho_W\,\frac{\pi\,D^3}{6}\,g \\ F_D &= c_D\,\frac{\rho_a}{2}\,v^2\,\frac{\pi\,D^2}{4} \\ &= c_D\,Re^2\,\frac{\pi}{8}\rho_a\,v_a^2 \\ Re\,\sqrt{c_D} &= \sqrt{\frac{8\,F_D}{\pi\,\rho_a}\,\frac{1}{\nu_a}} \\ &= \sqrt{\frac{4}{3}\,\frac{\rho_W}{\rho_a}\,D\,g}\,\frac{D}{\nu_a} \\ &= 217.7 \end{aligned}$$

$$\begin{aligned} \text{from diagram:}\quad Re &= 250 \\ v &= 3.75\ \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

(b) (Weight can be neglected)

$$\begin{aligned} F_D &= F_L = \rho_W\,\frac{\pi\,D^3}{6}\,g \\ F_D &= c_D\,\frac{\rho_W}{2}\,v^2\,\frac{\pi\,D^2}{4} \\ Re\,\sqrt{c_D} &= \sqrt{\frac{4}{3}\,D\,g}\,\frac{D}{\nu_W} = 115 \\ Re &= 113 \\ v &= 0.11\ \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

9.10



$$\begin{aligned} W &= F_D \quad \text{(Lift can be neglected)} \\ v_{rel} &= v_a - v_s \end{aligned}$$

(a)

$$\begin{aligned} v_s &= 0 \\ c_{DW} \frac{\rho_a}{2} v_1^2 \frac{\pi D_S^2}{4} &= \rho_S \frac{\pi D_S^3}{6} g \\ \text{Assumption: } Re &= \frac{\rho_a v_1 D_S}{\mu_a} < 1 \end{aligned}$$

$$\begin{aligned} v_1 &= -\frac{24}{9} \frac{\mu_a}{\rho_a D_S} + \\ &+ \sqrt{\left(\frac{24 \mu_a}{9 \rho_a D_S}\right)^2 + \frac{8}{27} \frac{\rho_S}{\rho_a} D_S g} \\ &= 0.168 \frac{\text{m}}{\text{s}} \\ Re &= 0.559 < 1 \end{aligned}$$

(b)

$$\begin{aligned} v_{rel} &= v_1 \\ v_s &= v_a - v_1 = 2.832 \frac{\text{m}}{\text{s}} \end{aligned}$$

9.11 (a)

$$F_D = W$$

(Lift can be neglected)

$$c_D \frac{\rho_a}{2} v_a^2 \frac{\pi D^2}{4} = \rho_W \frac{\pi D^3}{6} g$$

Assumption:

$$\begin{aligned} Re &= \frac{\rho_a v_a D}{\mu_a} < 0.5 \\ v_a &= \frac{\rho_W g D^2}{18 \mu_L} = 0.107 \frac{\text{m}}{\text{s}} \\ Re &= 0.427 < 0.5 \end{aligned}$$

(b)

$$\begin{aligned} \rho_W \frac{\pi D^3}{6} \frac{dv}{dt} = \rho_W \frac{\pi D^3}{6} g - \\ -\frac{24 \mu_a}{\rho_a v D} \frac{\rho_a}{2} v^2 \frac{\pi D^2}{4} \end{aligned}$$

Steady sinking velocity:
$v_s = v_a$

$$\begin{aligned} \frac{1}{g} \frac{dv}{dt} &= 1 - \frac{v}{v_a} \\ T &= -\frac{v_a}{g} \ln \left(1 - \frac{v}{v_a}\right)_0^{0.99 v_a} \\ &= 0.049 \,\text{s} \end{aligned}$$

9.12 (a)

$$W = F_{D1}$$

(Lift can be neglected)

$$\begin{aligned} F_{D1} &= c_{D1} \frac{\rho_a}{2} v_1^2 \frac{\pi D^2}{4} \\ c_{D1} &= 0.4 \qquad (Re_1 = 3.03 \cdot 10^5) \end{aligned}$$

(b)

$$\begin{aligned} Re_2 = \frac{v_2 D}{\nu_a} &= 4.2 \cdot 10^5 \\ \text{from diagram:} \quad c_{D2} &= 0.1 \end{aligned}$$

$$\begin{aligned} F_{D2} &= c_{D2} \frac{\rho_a}{2} v_2^2 \frac{\pi D^2}{4} \\ &= 1.95 \text{ N} < W \end{aligned}$$

Acceleration v_3, so that

$$\begin{aligned} G &= F_{D3} \\ G &= c_{D3} \frac{\rho_a}{2} v_3^2 \frac{\pi D^2}{4} \end{aligned}$$

$$\begin{aligned} \text{from diagram:} \quad c_{D3} &= 0.1 \\ v_3 &= 26.0 \frac{\text{m}}{\text{s}} \end{aligned}$$

9.13 (a)

$$P_1 = F_1 v_1 = F_1 \frac{H}{T_1} = 1000 \text{ W}$$

(b)

$$\begin{aligned} W &= F_1 + F_L - F_{D1} \\ &= F_1 + \rho \frac{\pi D^3}{6} g - \\ &- c_D \frac{\rho}{2} \left(\frac{H}{T_1}\right)^2 \frac{\pi D^2}{4} \\ Re &= \frac{\rho H D}{\mu T_a} = 1.85 \cdot 10^5 \end{aligned}$$

$$\begin{aligned} \text{from diagram:} \quad c_{D1} &= 0.4 \\ G &= 3349 \text{ N} \end{aligned}$$

(c)

$$\begin{aligned} F_{D2} &= c_{D2} \frac{\rho}{2} \left(\frac{H}{T_2}\right)^2 \frac{\pi D^2}{4} \\ &= 2 F_1 + F_L - W \end{aligned}$$

Assumption:

$$Re_2 = \frac{\rho\, H\, D}{\mu\, T_2} > 3.6 \cdot 10^5$$

from diagram: $c_{D2} \;=\; 0.1$

$$\begin{aligned} T_2 &= H D \sqrt{\frac{\pi\, \rho\, c_{D2}}{8\,(F_1 + F_{D1})}} \\ &= 241.0 \text{ s} \\ Re_2 &= 8.3 \cdot 10^6 \end{aligned}$$

(d)

$$P_2 = 2\, F_1\, \frac{H}{T_2} = 89.64 \text{ kW}$$

9.14 (a)

$$\begin{aligned} \rho_W\, \frac{\pi\, D^3}{6}\, \frac{dv}{dt} &= -\rho_S\, \frac{\pi\, D^3}{6}\, g - \\ &- c_{D2}\, \frac{\rho_a}{2}\, v^2\, \frac{\pi\, D^2}{4} \end{aligned}$$

Introduce steady sinking velocity:

$$\begin{aligned} v_s^2 &= \frac{4}{3}\, \frac{\rho_S}{\rho_a}\, \frac{D\, g}{c_D} - \\ -\frac{1}{g}\, \frac{dv}{1+\left(\frac{v}{v_s}\right)^2} &= dt = \frac{dz}{v} \end{aligned}$$

$$\begin{aligned} H &= -\frac{1}{g} \int_{v_0}^{0} \frac{v\, dv}{1+\left(\frac{v}{v_s}\right)^2} \\ &= \frac{v_s^2}{2\, g}\, \ln\left[1+\left(\frac{v}{v_s}\right)^2\right] \end{aligned}$$

(b)

$$\begin{aligned} T_H &= -\frac{1}{g} \int_{v_0}^{0} \frac{dv}{1+\left(\frac{v}{v_s}\right)^2} \\ &= \frac{v_s^2}{g}\, \arctan \frac{v_0}{v_s} \end{aligned}$$

(c)

$$\begin{aligned} \rho_S \frac{\pi\, D^3}{6}\, g \frac{dv}{dt} = &- c_D \frac{\rho_a}{2}\, v^2 \frac{\pi\, D^2}{4} + \\ &+ \rho_S\, \frac{\pi\, D^3}{6}\, g \\ \frac{1}{g}\, \frac{dv}{1-\left(\frac{v}{v_s}\right)^2} &= dt = \frac{dz}{v} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{g} \int_{0}^{v_B} \frac{v\, dv}{1-\left(\frac{v}{v_s}\right)^2} \\ &= -\frac{v_s^2}{2\, g}\, \ln\left[1-\left(\frac{v_B}{v_s}\right)^2\right] \\ v_B &= \frac{v_s}{\sqrt{1+\left(\frac{v}{v_s}\right)^2}} \end{aligned}$$

(d)

$$\begin{aligned} T_B &= -\frac{1}{g} \int_{0}^{v_B} \frac{dv}{1-\left(\frac{v}{v_s}\right)^2} \\ &= \frac{v_s^2}{g}\, \ln \frac{v_s + v_B}{v_s - v_B} \end{aligned}$$

(e)

	$c_D = 0.4$		$c_w = 0$
	wooden sphere	metal sphere	
H [m]	37.2	44.0	45
T_H [s]	2.64	2.96	3
v_B $\left[\frac{\text{m}}{\text{s}}\right]$	24.9	29.3	30
T_H [s]	2.81	2.98	3