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Application of Fractional Calculus to Fluid Mechanics

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In this note we present the application of fractional calculus, or the calculus of arbitrary (noninteger) differentiation, to the solution of time-dependent, viscous-diffusion fluid mechanics problems. Together with the Laplace transform method, the application of fractional calculus to the classical transient viscous-diffusion equation in a semi-infinite space is shown to yield explicit analytical (fractional) solutions for the shear-stress and fluid speed anywhere in the domain. Comparing the fractional results for boundary shear-stress and fluid speed to the existing analytical results for the first and second Stokes problems, the fractional methodology is validated and shown to be much simpler and more powerful than existing techniques.

1 Introduction

Fractional calculus is a mathematical concept of differentiation and integration to arbitrary (noninteger) order, such as $\partial^{-2/3}f/\partial x^{-2/3}$. Some useful definitions and properties of fractional derivatives are presented in the Appendix. Interest in fractional calculus became evident almost as soon as the ideas of classical calculus were known. In fact, Leibnitz [1] mentioned it in a letter to L'Hospital back in 1695.

Systematic studies of fractional calculus were undertaken during the first half of the 19th century [2–4]. Euler [5], Lagrange [6], and Fourier [7] mentioned the concept of derivatives of arbitrary order earlier in their studies without contemplating any specific application.

Notable contributions have been made to both the theory and application of fractional calculus during the 20th century when some rather special, but natural, properties of differintegrals (i.e., derivatives of arbitrary order) were examined with respect to arbitrary functions [8–11]. Applications include those to problems in rheology [12,13] to electrochemistry [14–16], and to chemical physics [17].

The lack of applications of fractional calculus to solving problems in engineering, and more particularly in fluid dynamics, is notorious. This note fulfills our objective to bring forth the concept of fractional calculus to the fluid mechanics community.

We consider the problem of time-dependent momentum diffusion with a semi-infinite Newtonian fluid exposed to a time-dependent excitation at the solid-fluid interface to show how fractional calculus, together with the Laplace transform method, can be utilized to reduce the order of the differential equation ruling the phenomenon. We also show how to obtain closed-form general analytical solutions of boundary shear-stress (when the boundary velocity is known) or boundary velocity (when the boundary shear-stress is known) to this problem. Finally, we validate the analysis by considering the classical 1st and 2nd Stokes problems, and comparing the solutions obtained with the fractional approach to the solutions obtained by different methods.

2 The Extraordinary Viscous-Diffusion Equation

To contemplate how fractional calculus can be useful in fluid dynamics, we consider a one-dimensional time-dependent viscous-diffusion problem of a semiinfinite fluid bounded by a flat plate. The momentum equation, assuming constant and uniform viscosity and neglecting convective inertia (advection) effects, is:

$$\frac{\partial F(y,t)}{\partial t}-v\,\frac{\partial^2 F(y,t)}{\partial y^2}=0 \tag{1}$$

where $F(y,t)$ is the fluid vorticity, or the fluid velocity in the case of negligible pressure effect, t is the time, v is the fluid kinematic viscosity, and y is the spatial coordinate normal to, and with origin at, the plate.

Assume the fluid to be (or, is) initially at equilibrium, so that $F(y,t<0)=F_0$, with F_0 being a constant value. Also, the condition far from the plate remains $F(\infty,t)=F_0$. The boundary of the fluid interfacing the plate is exposed to a time-dependent excitation $F(0,t>0)=F^+(t)$ caused by the plate movement. Changing the variables to $\xi=yv^{-1/2}$ and $G(\xi,t)=F(y,t)-F_0$, Eq. (1) becomes

$$\frac{\partial G(\xi,t)}{\partial t}-\frac{\partial^2 G(\xi,t)}{\partial \xi^2}=0. \tag{2}$$

The initial and boundary conditions are now written as $G(\xi,0)=0$, $G(\infty,t)=0$, and $G(0,t)=F^+(t)-F_0=G^+(t)$, respectively. The Laplace transform of Eq. (2) is

$$\frac{d^2 G^*(\xi,s)}{d\xi^2}-sG^*(\xi,s)=0 \tag{3}$$

where $G^*(\xi,s)$ is the Laplace transform of $G(\xi,t)$. Also, since $G(\infty,t)=0$ we then have $G^*(\infty,s)=0$. The solution of Eq. (3) is

$$G^*(\xi,s)=C_1(s)e^{[\xi(s^{1/2})]}+C_2(s)e^{[-\xi(s^{-1/2})]} \tag{4}$$

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where C_1 and C_2 are arbitrary functions of s. The boundary condition $G^*(\infty,s)=0$ requires that $C_1(s)=0$, and Eq. (5) simplifies to:

$$G^*(\xi,s)=C(s)e^{[-\xi(s^{1/2})]} \tag{5}$$

where $C(s)$ remains unspecified. Now, $C(s)$ can be eliminated by using Eq. (5) and the expression

$$\frac{\partial G^*(\xi,s)}{\partial \xi}=-s^{1/2}C(s)e^{[-\xi(s^{1/2})]} \tag{6}$$

which results upon differentiating Eq. (5). Combining Eqs. (5) and (6), the resulting equation in transform space is

$$\frac{\partial G^*(\xi,s)}{\partial \xi}=-s^{1/2}G^*(\xi,s). \tag{7}$$

Equation (7) can be inverted by recognizing that $L^{-1}[\partial G^*(\xi,s)/\partial \xi]=\partial\{L^{-1}[G^*(\xi,s)]\}/\partial \xi=\partial G(\xi,t)/\partial \xi$, and using the Laplace transform property $L[\partial^f G(\xi,t)/\partial t^f]=s^f L[G(\xi,t)]=s^f G^*(\xi,s)$, valid for a function $G(\xi,t)$ that satisfies $G(\xi,0)=0$, where $\partial^f()/\partial t^f$ is a differential operator of order f. Thus, Eq. (7) becomes:

$$\frac{\partial G(\xi,t)}{\partial \xi}=-\frac{\partial^{1/2}G(\xi,t)}{\partial t^{1/2}} \tag{8}$$

and, on restoring the original variables,

$$v^{1/2}\frac{\partial F(y,t)}{\partial y}=-\frac{\partial^{1/2}[F(y,t)-F_0]}{\partial t^{1/2}}. \tag{9}$$

Using the properties of fractional calculus ($A2$) and ($A7$), listed in the Appendix, Eq. (9) can be rewritten as

$$\frac{\partial F(y,t)}{\partial y}=-v^{-1/2}\frac{\partial^{1/2}F(y,t)}{\partial t^{1/2}}+(\pi v t)^{-1/2}F_0. \tag{10}$$

Thereby, the viscous-diffusion Eq. (1), which is an ordinary PDE of first order in time and second order in space, is transformed into an extraordinary PDE of half-th order in time and first order in space, Eq. (10). Observe that this transformation is valid anywhere in the domain, including at the fluid-plate interface.

Now, recalling the diffusion constitutive law for the flux $J(t)$ at the fluid-plate interface, namely,

$$J(t)=-\mu\frac{dF(0,t)}{dy} \tag{11}$$

where μ is the fluid dynamic viscosity, and using Eq. (10) to find an expression for $dF(0,t)dy$, the surface flux $J(t)$ can be directly computed from the surface excitation $F(0,t)$ using

$$J(t)=\frac{\mu}{v^{1/2}}\left[\frac{d^{1/2}F(0,t)}{dt^{1/2}}-\frac{1}{(\pi t)^{1/2}}F_0\right]. \tag{12}$$

Therefore, the flux at the fluid boundary can be obtained by simply semi-differentiating the intensive scalar quantity $F(0,t)$. Note that for a given flux excitation $J(t)$ at the boundary, the fluid response $F(0,t)$ can be obtained by taking $\partial^{-1/2}$ [Eq. (12)] $/\partial t^{-1/2}$ resulting in:

$$F(0,t)=\frac{v^{1/2}}{\mu}\frac{d^{-1/2}J(t)}{dt^{-1/2}}+F_0. \tag{13}$$

It is important to emphasize that the transformation of the diffusion Eq. (1) into the extraordinary PDE Eq. (10) is general and not restricted by any additional assumption on the physics of the process in question.

3 Validation

To validate the previous results, consider, for instance, the first Stokes problem, i.e., the case of a flat plate that is suddenly jerked in an infinite fluid domain and whose velocity for $t>0$ is constant and equal to U. The equation of motion for this simple case is

$$\frac{\partial u(y,t)}{\partial t}-v\frac{\partial^2 u(y,t)}{\partial y^2}=0 \tag{14}$$

where u is the local fluid velocity. The initial and boundary conditions for this problem are $u(y,0)=0$, $u(0,t)=U$, and $u(\infty,t)=0$.

Upon identification of u with F, Eq. (14) becomes identical to Eq. (1), with the same kind of initial and boundary conditions (with $F_0=0$). One can, therefore, use Eq. (12) with $J(t)$ replaced by the shear-stress at the surface $\tau_w(t)$, and write

$$\tau_w(t)=\mu v^{-1/2}\frac{d^{1/2}U}{dt^{1/2}} \tag{15}$$

or, using property ($A7$), obtain

$$\tau_w(t)=\mu(\pi v t)^{-1/2}U \tag{16}$$

which is exactly the same as the result obtained by solving Eq. (14) analytically for the velocity with the entire domain, using the similarity variable, and then obtaining an expression for the wall shear-stress via the constitutive relationship for a Newtonian fluid [18]. Observe that, with the fractional approach, the same result is obtained in one simple operation, i.e., finding the semi-derivative of the uniform fluid speed U (observe that the semi-derivative of a constant is nonzero—see Eq. ($A7$) in the Appendix).

From Eq. (16), one can observe that the displacement thickness, δ^*, for the case of a suddenly jerked flat plate, is proportional to $(\pi v t)^{1/2}$. By substituting t with x/U, one can find

$$\delta^*=(\pi v x/U)^{1/2} \tag{17}$$

or, after dividing Eq. (17) by x,

$$\frac{\delta^*}{x}=\pi^{1/2}\mathrm{Re}^{-1/2} \tag{18}$$

where $\mathrm{Re}=xU/v$. By noticing that $\pi^{1/2}\approx 1.77$, one can conclude that the result obtained by means of the fractional calculus approach is much closer to the precise value 1.721 than the value 1.83 which one can get after applying the more laborious integral method [19].

Consider now a more complicated problem (the so-called second Stokes problem) in which the plate velocity is time-dependent and varies as $U(t)=U\sin(\omega t)$. The wall shear-stress at the steady periodic regime can be obtained analytically in this case as well. The procedure to obtain this solution involves first modifying the viscous-diffusion equation to a complex velocity model, then solving the differential equation for the complex velocity, extracting the velocity solution from the complex velocity and finally using the constitutive relationship for a Newtonian fluid to obtain the corresponding shear-stress ([18], pp. 138–141). The result is

$$\tau_w(t)=U\mu\left(\frac{\omega}{2v}\right)^{1/2}[\sin(\omega t)+\cos(\omega t)]. \tag{19}$$

Using the fractional calculus approach, Eq. (12) gives for the shear-stress

$$\tau_w=\mu v^{-1/2}\frac{d^{1/2}[U\sin(\omega t)]}{dt^{1/2}}. \tag{20}$$

Now, using ($A3$), ($A4$), and ($A8$) with Eq. (20), one has

$$\tau_w(t)=\mu v^{-1/2}U\omega^{1/2}\left\{\sin\left(\omega t+\frac{\pi}{4}\right)-2^{1/2}\Lambda\left[\left(\frac{2\omega t}{\pi}\right)^{1/2}\right]\right\} \tag{21}$$

where Λ is the auxiliary Fresnel function (see Appendix). Observe that Eq. (21) is not identical to Eq. (19). This is because Eq. (21) is the general solution for the shear-stress, which includes the initial transient regime, see Fig. 1. On the other hand, Eq. (19) is the solution only of the steady-periodic regime, i.e., when time

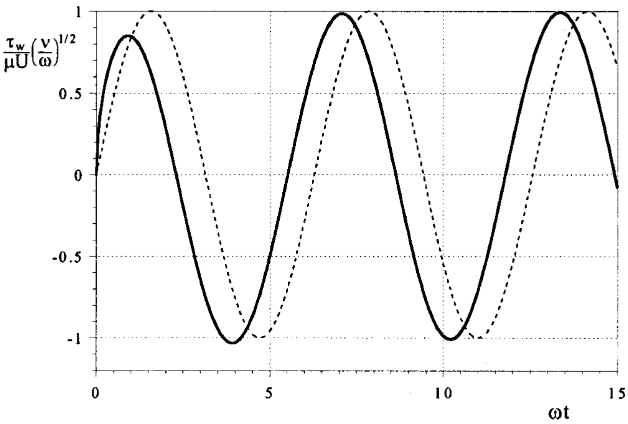

Fig. 1 Time evolution of surface shear stress, Eq. (21), and the imposed boundary condition (dashed line): $U(t)/U=\sin(\omega t)$.

is long enough for the flow process to become periodic. Notice that the term proportional to the auxiliary Fresnel function Λ of Eq. (21) governs the initial unsteady regime. When time is long enough, the contribution of the auxiliary Fresnel function to Eq. (21) becomes negligible because $\Lambda(z)$ approaches zero as z increases (for instance, at $t\sim 10\pi/\omega$, $|\max\{\Lambda[2\omega t/\pi)^{1/2}]\}|<0.001$). In this case, and upon expanding the sine function, one recovers from Eq. (21) exactly the solution for the steady-periodic regime presented in Eq. (19).

4 Summary and Conclusions

A brief historical overview of fractional calculus was presented. considering the viscous-diffusion problem of a semi-infinite fluid bounded by a moving solid surface, the simplified transport PDE, of first order in time and second order in space, was converted, via Laplace transform, into an extraordinary differential equation of half order in time and first order in space.

Closed-form analytical solutions for the flux and for the scalar response (fluid vorticity or velocity) at the fluid-solid interface were found. The results were validated considering the known solutions for the 1st and 2nd Stokes problems. The simplicity and accuracy involved in obtaining the local system response to a transient excitation within a semi-infinite viscous-diffusion system using the fractional approach was then established. Moreover, specifically regarding the 2nd Stokes problems, the fractional approach leads to an analytical solution for the entire regime, Eq. (21), including the nonperiodic initial regime. Observe that the nonperiodic initial regime is not covered by the analytical solution, Eq. (19).

The fractional approach presented here has the potential for becoming a powerful tool in solving other differential equations in fluid mechanics (such as time-diffusion vorticity equations with sources or sinks, linearized compressible aerodynamics equations, linearized viscous sublayer equations, and acoustic wave equations).

The application of fractional calculus need not be restricted to linear equations. Moreover, the factorization of time-diffusive operators (i.e., $\partial/\partial t-\nabla^2$), common in fluid dynamics equations, can be proposed as a more direct and general method (because it is not restricted to one-dimension) for obtaining the fractional equivalent to the original PDE's. Being of reduced order, the resulting fractional equation should require less computational effort to be solved. Finally, we point out that by using an extension to Taylor's series applied to fractional derivatives [20], the discretization of fractional equations does not require the use of transform definitions.

Appendix

In this Appendix, some useful definitions and properties of fractional derivatives are presented. From the several equivalent definitions of fractional derivatives, the most elegant is the Riemann-Liouville definition [21], namely:

$$\frac{d^f[g(t)]}{dt^f}=\frac{1}{\Gamma(-f)}\int_0^t\frac{g(\tau)}{(t-\tau)^{1+f}}d\tau \tag{A1}$$

where f is any negative number and Γ is the Gamma function. Some of the useful properties derived from Eq. (A1) are:

$$\frac{d^f[u(t)+v(t)]}{dt^f}=\frac{d^f[u(t)]}{dt^f}+\frac{d^f[v(t)]}{dt^f} \tag{A2}$$

$$\frac{d^f[Cg(t)]}{dt^f}=C\frac{d^fg(t)}{dt^f}\quad \frac{d^f[tg(t)]}{dt^f}=t\frac{d^fg(t)}{dt^f}+f\frac{d^{f-1}g(t)}{dt^{f-1}} \tag{A3}$$

$$\frac{d^h}{dt^h}\left(\frac{d^fg(t)}{dt^f}\right)=\frac{d^{h+f}g(t)}{dt^{h+f}}\quad \frac{d^f[g(Ct)]}{dt^f}=C^f\frac{d^fg(Ct)}{d(Ct)^f} \tag{A4}$$

$$\frac{d^f\delta(t-\tau)}{dt^f}=\frac{1}{\Gamma(-f)}(t-\tau)^{-f-1},\quad f<0 \tag{A5}$$

$$\frac{d^f[t^n]}{dt^f}=\frac{\Gamma(n+1)}{\Gamma(n+1-f)}t^{n-f}\quad \frac{d^f[C]}{dt^f}=\frac{Ct^{-f}}{\Gamma(1-f)} \tag{A6}$$

where $\delta(t-\tau)$ is the Dirac delta function, defined as $\delta(t-\tau)=\infty$, if $t=\tau$, otherwise, $\delta(t-\tau)=0$. In the previous formulas, C is a nonzero constant. Observe that the first expression in (A4) is not general (see [20] for limitations).

The semi-derivatives (case of f being $\pm 1/2$) of some common functions are:

$$\frac{\partial^{1/2}[C]}{\partial t^{1/2}}=C(\pi t)^{-1/2} \tag{A7}$$

$$\frac{d^{1/2}[\sin(t)]}{dt^{1/2}}=\sin\left(t+\frac{\pi}{4}\right)-2^{1/2}\Lambda\left[\left(\frac{2t}{\pi}\right)^{1/2}\right] \tag{A8}$$

$$\frac{d^{1/2}[\cos(t)]}{dt^{1/2}}=\frac{1}{(\pi t)^{1/2}}+\cos\left(t+\frac{\pi}{4}\right)-2^{1/2}\Omega\left[\left(\frac{2t}{\pi}\right)^{1/2}\right] \tag{A9}$$

$$\frac{d^{-1/2}[C]}{dt^{-1/2}}=2C\left(\frac{t}{\pi}\right)^{1/2} \tag{A10}$$

$$\frac{d^{-1/2}[\sin(t)]}{dt^{-1/2}}=\sin\left(t-\frac{\pi}{4}\right)+2^{1/2}\Omega\left[\left(\frac{2t}{\pi}\right)^{1/2}\right] \tag{A11}$$

$$\frac{d^{-1/2}[\cos(t)]}{dt^{-1/2}}=\cos\left(t-\frac{\pi}{4}\right)+2^{1/2}\Lambda\left[\left(\frac{2t}{\pi}\right)^{1/2}\right] \tag{A12}$$

In Eqs. (A8), (A9), (A11), and (A12), Ω and Λ are the auxiliary Fresnel integrals (function f and g, respectively, in [22], p. 300).

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Constant Pressure Laminar, Transitional and Turbulent Flows—An Approximate Unified Treatment

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A nondimensional number that is constant in two-dimensional, incompressible and constant pressure laminar and fully turbulent boundary layer flows has been proposed. An extension of this to constant pressure transitional flow is discussed.
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1 Introduction

For the two-dimensional (2-D), constant pressure and incompressible laminar flow and fully turbulent flow over a semi-infinite flat plate, a nondimensional number that is independent of the nature of these flows has been proposed here. This nondimensional number is based on the boundary layer momentum thickness, the shape factor, the skin-friction coefficient, and the streamwise distance.

Since the proposed nondimensional number is constant for the constant pressure laminar and turbulent flows, it is assumed to have the same value in constant pressure transitional boundary layer flows. This is utilized to propose an implicit solution of the momentum integral equation for the laminar, transition and turbulent regions.

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2 Analysis

We consider the 2-D constant pressure, and incompressible boundary layer flow over a semi-infinite plate. Let u and v denote the boundary layer velocity components in the x and y directions, respectively; x is the streamwise direction. The free-stream velocity is denoted by U. The constant pressure momentum integral equation considered here is [1],

$$d\theta/dx = C_f/2. \tag{1}$$

Here, θ and C_f denote the local momentum thickness, and the skin-friction coefficient, respectively:

$$\theta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy, \quad C_f = 2\nu U^{-2}(\partial u/\partial y)_{y=0},$$

where ν is the kinematic viscosity. Although the momentum integral equation (1) is valid for the laminar, turbulent and transitional flows, only the laminar and turbulent cases have been solved (see Ref. [1]), separately. Usually, one requires a correlation for C_f to obtain θ.

We propose the nondimensional quantity,

$$L = \theta/(xC_f H^{0.7}). \tag{2}$$

Here $H(=\delta^*/\theta)$ is the shape parameter, and δ^* is the local displacement thickness:

$$\delta^* = \int_0^\infty \left(1-\frac{u}{U}\right)dy.$$

For the laminar flow, the Blasius solution [1] ($C_f=\theta/x$, $H=2.6$) gives the value of this nondimensional quantity as $L \simeq 0.51$. Similarly, the 1/7th power-law for fully turbulent flow [1] ($\theta/x \simeq 0.036R_x^{-0.2}$, $C_f \simeq 0.0592R_x^{-0.2}$, $H=1.27$, where R_x $(=Ux/\nu)$ is the Reynolds number) gives $L \cong 0.51$. The 1/7th law being a popular one and in view of the availability of reliable experimental data, the 1/5th or 1/10th power-law has not been considered here. The experimental data for fully turbulent flows show that both H and C_f decrease slowly with x (see, for example, Proc. AFOSR-IFP-Stanford Confc. [2]). As shown in Fig. 1, the measured constant pressure data (number 1400 and 3000) of AFOSR-IFP-Stanford Confc. [2] also show that $L \simeq 0.5$ at high Reynolds number; in this figure R_θ denotes the Reynolds number based on the momentum thickness. (It may be noted that the AFOSR-IFP-Stanford Confc. Data are those carefully selected for the data bank value.) It can be seen in this figure that, except at $R_\theta \simeq 600$, the experimental data show an excellent collapse over a large Reynolds number range; the maximum deviation of 12% is attributed to the scatter usually associated with the experimental data. The behavior at $R_\theta \simeq 600$ is attributed to the low Reynolds number effect; for example, it is known [3] that the Cole's wake function agrees with the experimental data for fully turbulent flows at $R_\theta \simeq 1000$. The momentum equation (1) does not contain H, which is associated with the boundary layer velocity profile shapes. An inspection of the momentum integral equation (1) suggests that the quantity $\theta/(xC_f)$ can be of the order of unity. This

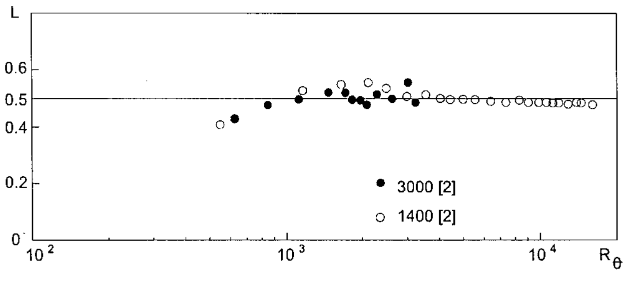

Fig. 1 Fully turbulent data showing a constant value of *L*

