On some control problems in fluid mechanics

Jean-Michel Coron

http://www.ann.jussieu.fr/~coron/

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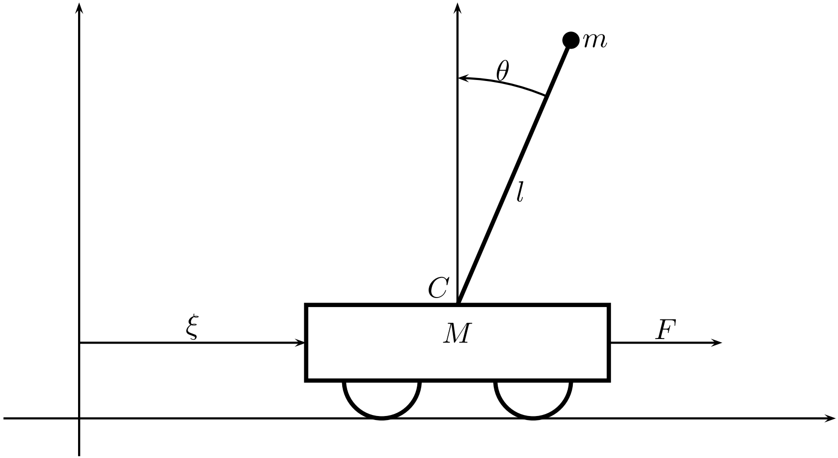
Control systems

A control system is usually a dynamical control system on which one can act by using suitable **controls**.
Mathematically it often takes the form

$$\dot{y} = f(y, u),$$

where y is called the **state** and u is the **control**. The state can be in finite dimension (then $\dot{y} = f(y, u)$ is an ordinary differential equation) or in infinite dimension (example: $\dot{y} = f(y, u)$ is a partial differential equation).

A first example: the cart-inverted pendulum



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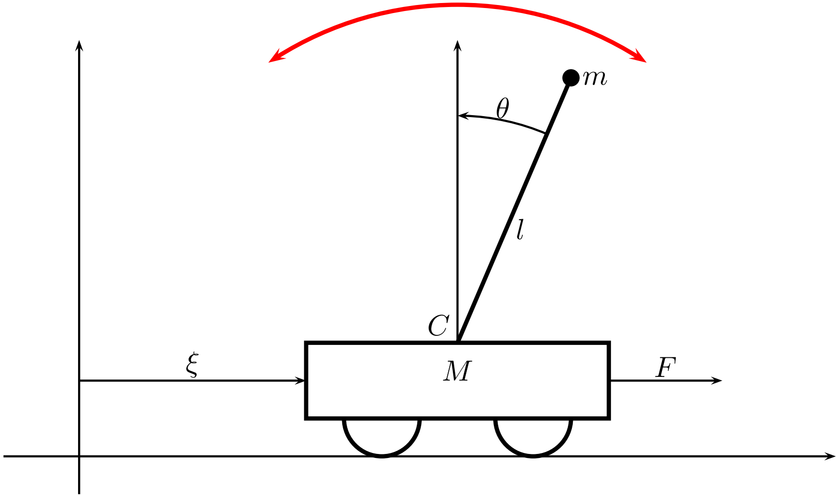


The Cart-inverted pendulum: The equations

Let

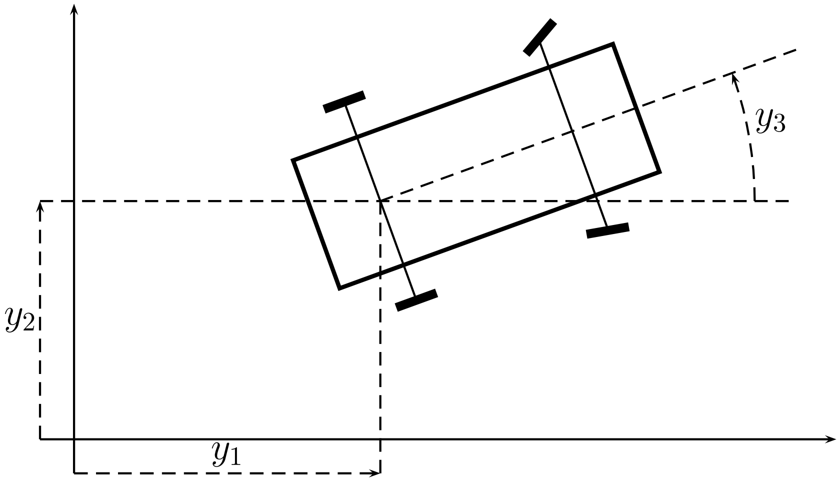
$$y_1 := \xi,\ y_2 := \theta,\ y_3 := \dot{\xi},\ y_4 := \dot{\theta},\ u := F,$$

The dynamics of the cart-inverted pendulum system is $\dot{y} = f(y,u)$, with $y = (y_1, y_2, y_3, y_4)^{\mathrm{tr}}$ and

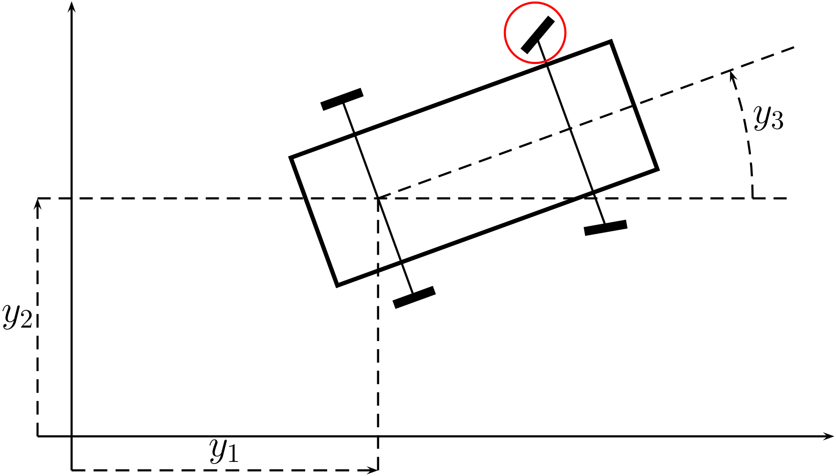
$$f := \begin{pmatrix} y_3 \\ y_4 \\ \dfrac{mly_4^2 \sin y_2 - mg \sin y_2 \cos y_2}{M + m \sin^2 y_2} + \dfrac{u}{M + m \sin^2 y_2} \\ \dfrac{-mly_4^2 \sin y_2 \cos y_2 + (M+m)g \sin y_2}{(M + m \sin^2 y_2)l} - \dfrac{u \cos y_2}{(M + m \sin^2 y_2)l} \end{pmatrix}.$$

The baby stroller

The baby stroller: The model



The baby stroller: The model



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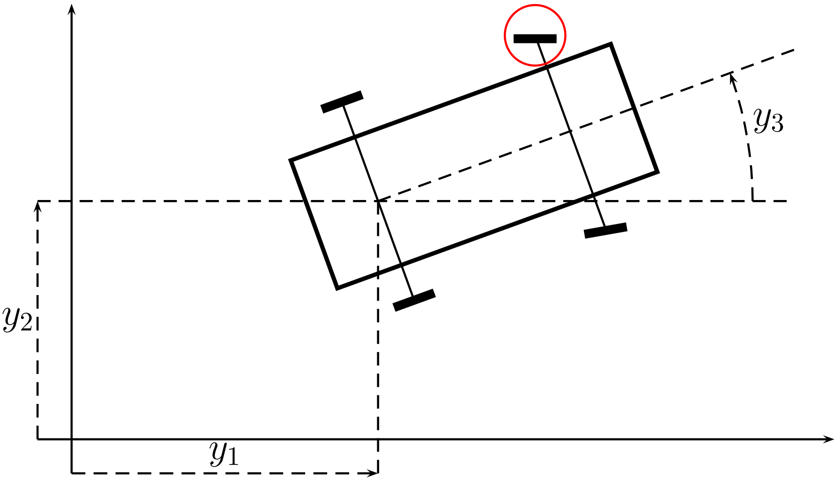

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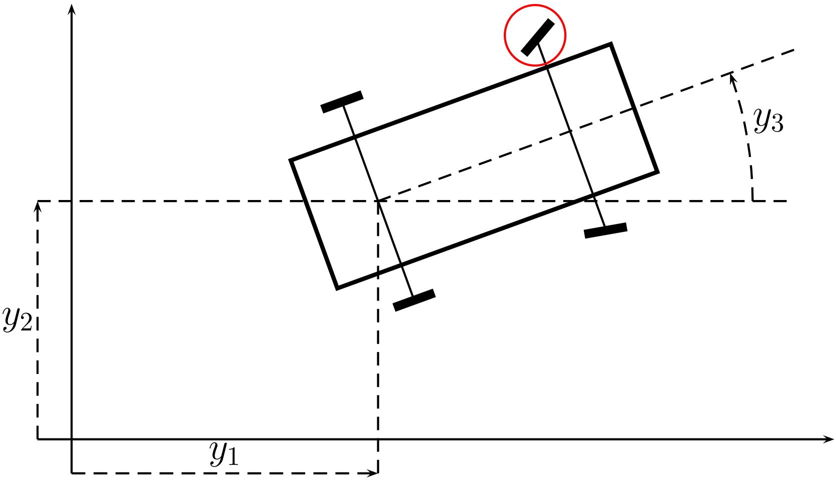
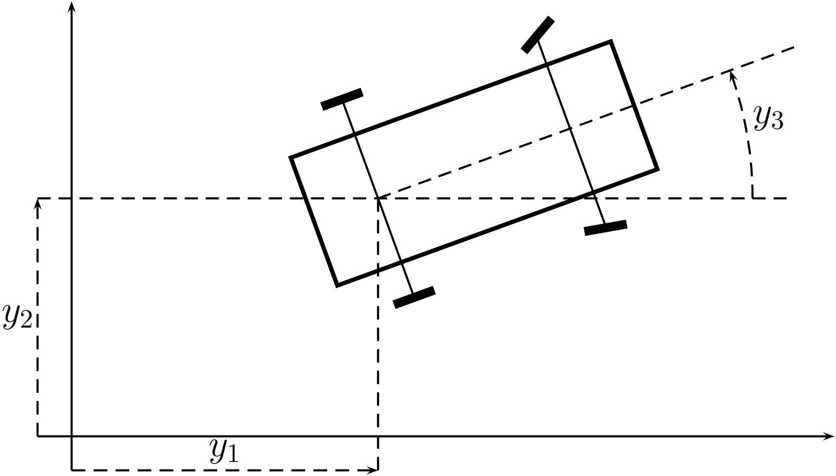


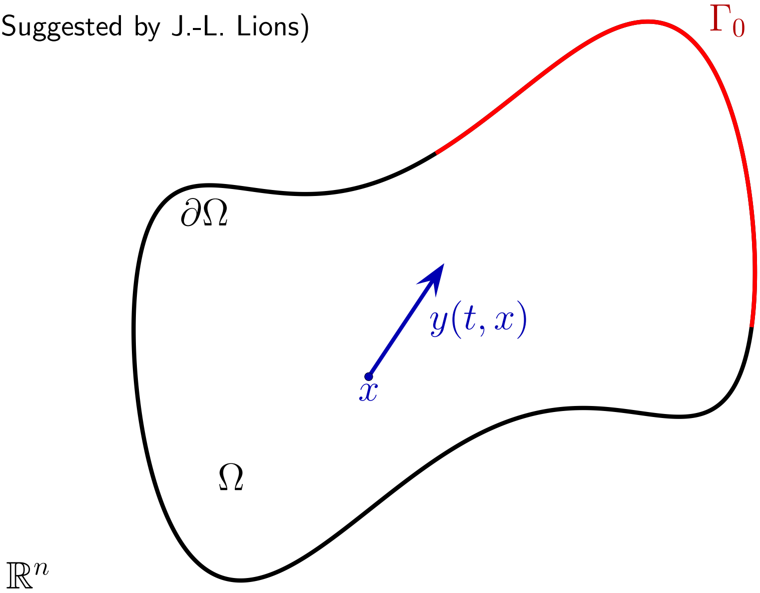
The baby stroller: The model



$$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3, \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

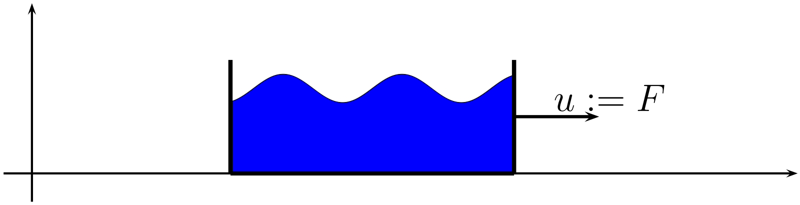
The Euler/Navier-Stokes control system

(Suggested by J.-L. Lions)



A water-tank control system

(Suggested by P. Rouchon)

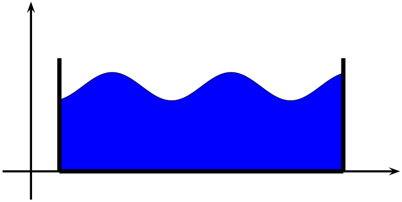


Controllability

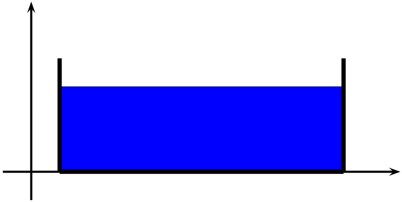
Given two states y^0 and y^1, does there exist a control $t \in [0,T] \mapsto u(t)$ which steers the control system from y^0 to y^1, i.e. such that

$$\big(\dot{y} = f(y,u(t)),\ y(0) = y^0\big) \Rightarrow \big(y(T) = y^1\big)?$$

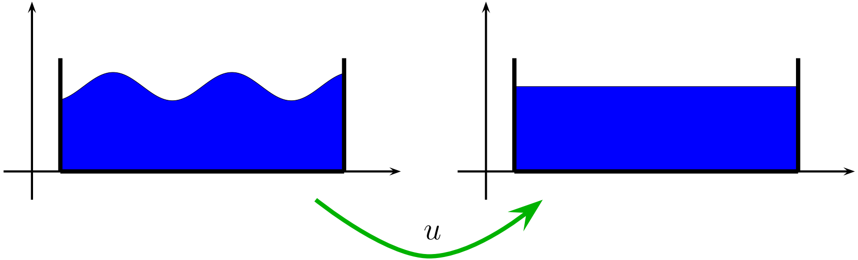
Destroy waves



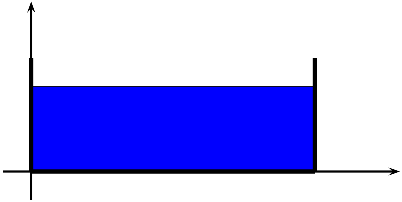
Destroy waves



Destroy waves



Steady-state controllability



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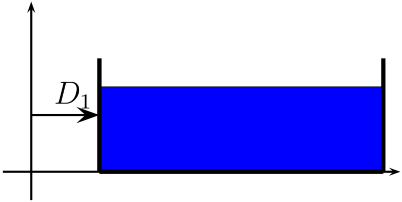
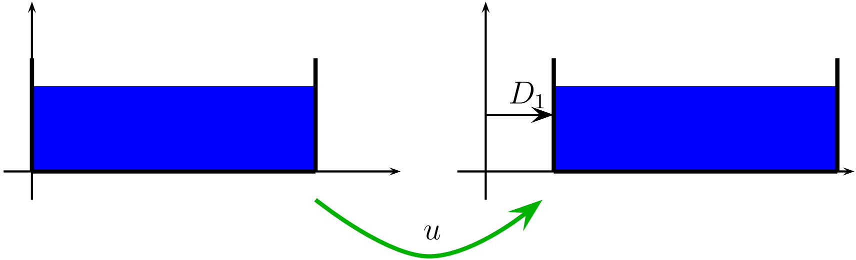


Steady-state controllability



Controllability of linear control systems

The control system is

$$\dot{y} = Ay + Bu,\ y \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

Theorem (Kalman's rank condition)

The linear control system $\dot{y} = Ay + Bu$ is controllable on $[0, T]$ if and only if

$$\textit{Span}\ \{A^i Bu; u \in \mathbb{R}^m,\ \ i \in \{0, 1, \dots, n-1\}\} = \mathbb{R}^n.$$

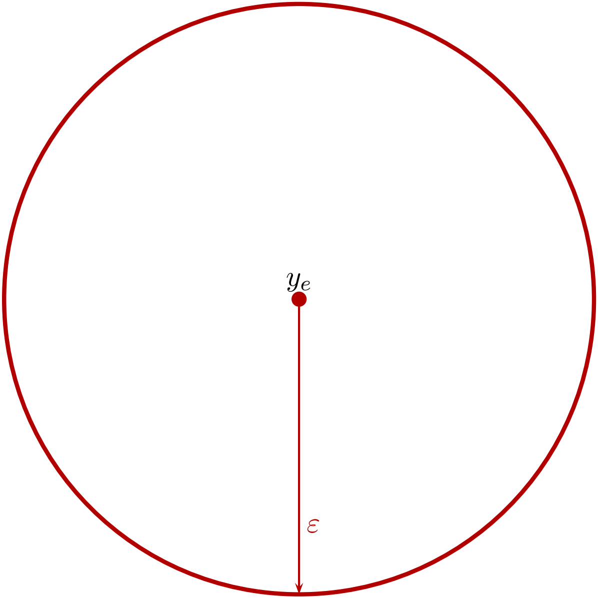
Remark

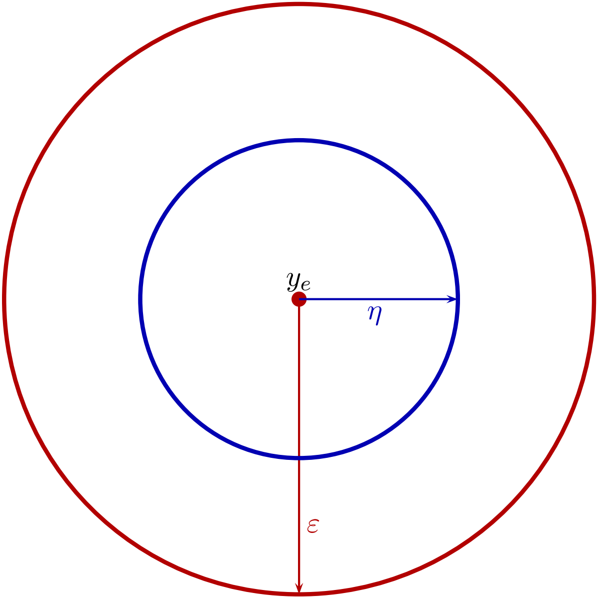
This condition does not depend on T. This is no longer true for nonlinear systems and for systems modeled by linear partial differential equations.

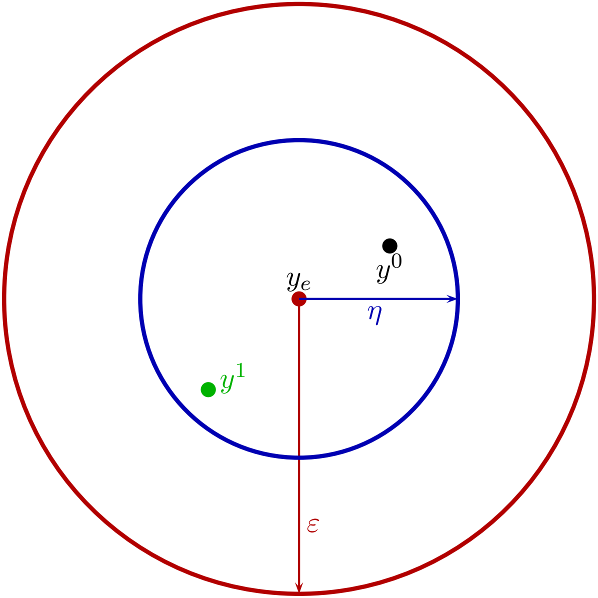
Small time local controllability

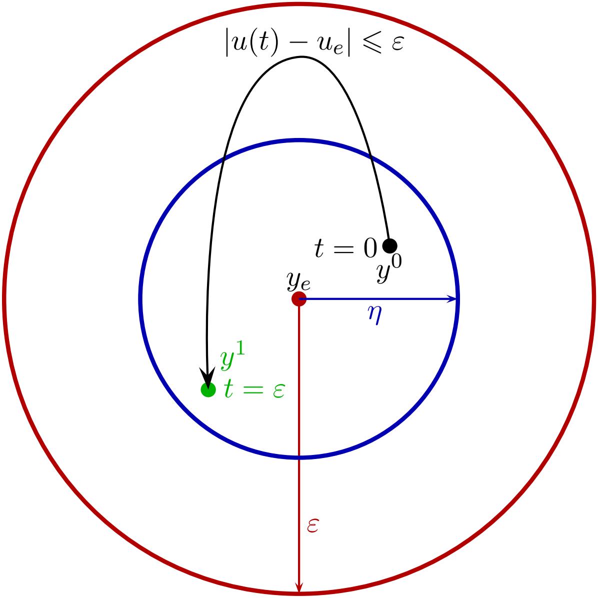
We assume that (y_e, u_e) is an equilibrium, i.e., that $f(y_e, u_e) = 0$. Many possible choices for natural definitions of local controllability. The most popular one is Small-Time Local Controllability (STLC): the state remains close to y_e, the control remains to u_e and the time is small.


y_e


y_e
ε


y_e
η
ε


y^0
y_e
η
y^1
ε


$|u(t) - u_e| \leqslant \varepsilon$
$t = 0$
y^0
y_e
η
y^1
$t = \varepsilon$
ε

The linear test

We consider the control system $\dot{y} = f(y,u)$ where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. Let us assume that $f(y_e, u_e) = 0$. The linearized control system at (y_e, u_e) is the linear control system $\dot{y} = Ay + Bu$ with

$$A := \frac{\partial f}{\partial y}(y_e, u_e),\ B := \frac{\partial f}{\partial u}(y_e, u_e).$$

If the linearized control system $\dot{y} = Ay + Bu$ is controllable, then $\dot{y} = f(y,u)$ is small-time locally controllable at (y_e, u_e).

Application to the cart-inverted pendulum

For the cart-inverted pendulum, the linearized control system around $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$ is $\dot{y} = Ay + Bu$ with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{mg}{M} & 0 & 0 \\ 0 & \dfrac{(M+m)g}{Ml} & 0 & 0 \end{pmatrix}, \; B = \frac{1}{Ml} \begin{pmatrix} 0 \\ 0 \\ l \\ -1 \end{pmatrix}.$$

One easily checks that this linearized control system satisfies the Kalman rank condition and therefore is controllable. Hence the cart-inverted pendulum is small-time locally controllable at $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$.

Application to the baby stroller

Let us recall that the baby stroller control system is

$$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3, \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

The linearized control system at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$ is

$$\dot{y}_1 = u_1,\ \dot{y}_2 = 0,\ \dot{y}_3 = u_2,$$

which is clearly not controllable. The linearized control system gives no information on the small-time local controllability at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$ of the baby stroller.

What to do if linearized control system is not controllable?

Question: What to do if

$$\dot{y} = \frac{\partial f}{\partial y}(y_e, u_e)y + \frac{\partial f}{\partial u}(y_e, u_e)u$$

is not controllable?

In finite dimension: one uses iterated Lie brackets.

Lie brackets and iterated Lie brackets

Definition (Lie brackets)

$$[X,Y](y) := Y'(y)X(y) - X'(y)Y(y).$$

Iterated Lie brackets: $[X,[X,Y]]$, $[[Y,X],[X,[X,Y]]]$ etc.
Why Lie brackets are natural objects for controllability issues? For simplicity, from now on we assume that

$$f(y,u) = f_0(y) + \sum_{i=1}^{m} u_i f_i(y).$$

Drift: f_0. Driftless control systems: $f_0 = 0$.

Lie bracket for $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$

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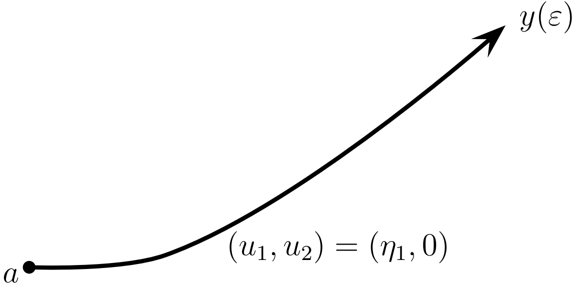
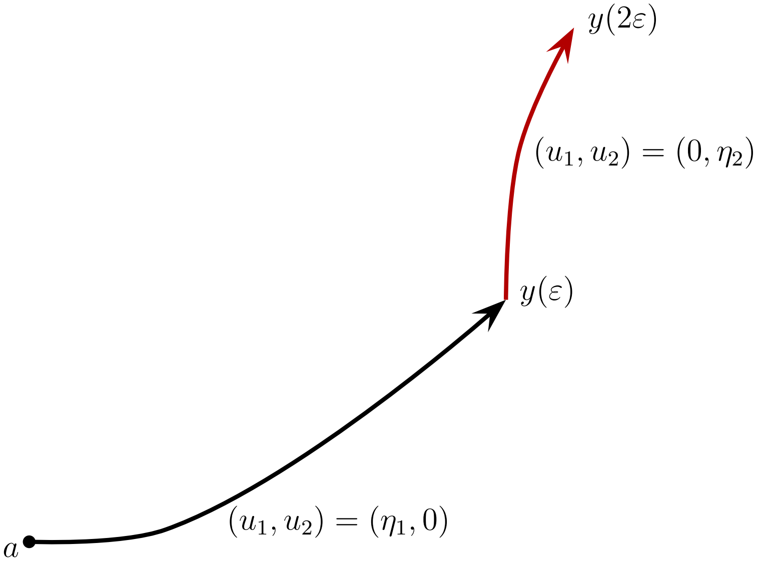


Lie bracket for $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$



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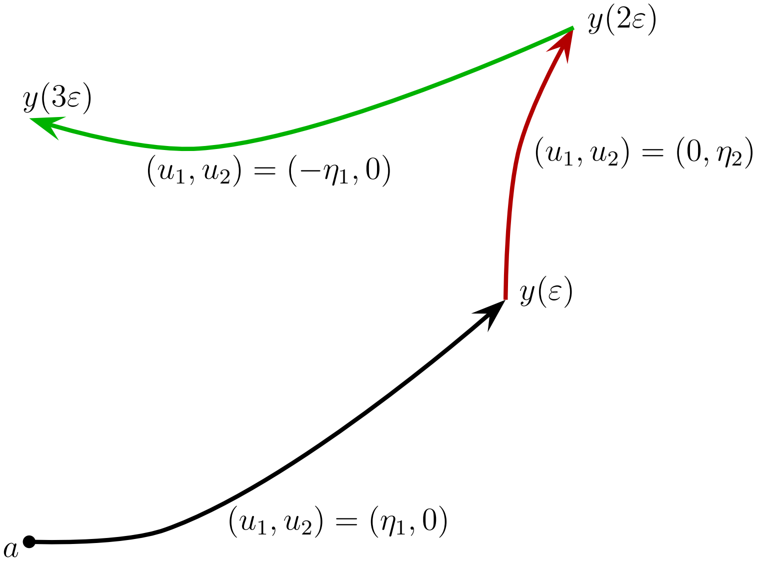
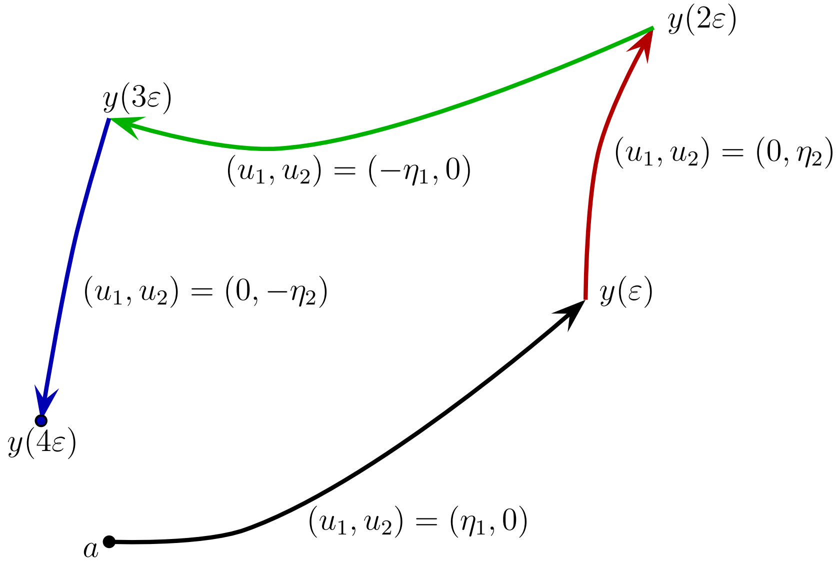
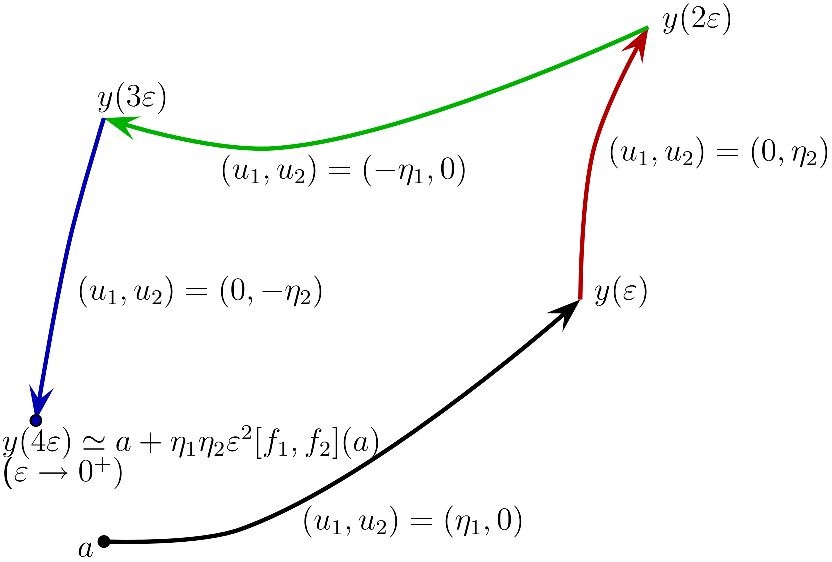


Lie bracket for $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$



Lie bracket for $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$



Controllability of driftless control systems: Local controllability

Theorem (P. Rashevski (1938), W.-L. Chow (1939))

Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}^n$ and let $y_e \in \mathcal{O}$. Let us assume that, for some $f_1, \ldots, f_m : \mathcal{O} \to \mathbb{R}^n$,

$$f(y,u) = \sum_{i=1}^{m} u_i f_i(y), \ \forall (y,u) \in \mathcal{O} \times \mathbb{R}^m.$$

Let us also assume that

$$\left\{ h(y_e);\ h \in \textit{Lie} \left\{ f_1, \ldots, f_m \right\} \right\} = \mathbb{R}^n.$$

Then the control system $\dot{y} = f(y,u)$ is small-time locally controllable at $(y_e, 0) \in \mathbb{R}^n \times \mathbb{R}^m$.

The baby stroller system: Controllability

$$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3, \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

This system can be written as $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$, with

$$f_1(y) = (\cos y_3, \sin y_3, 0)^{\text{tr}},\ f_2(y) = (0, 0, 1)^{\text{tr}}.$$

One has $[f_1, f_2](y) = (\sin y_3, -\cos y_3, 0)^{\text{tr}}$. Hence $f_1(0)$, $f_2(0)$ and $[f_1, f_2](0)$ span all of $\mathbb{R}^3$. This implies the small-time local controllability of the baby stroller at $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}^2$.

Controllability of driftless control systems: Global controllability

Theorem (P. Rashevski (1938), W.-L. Chow (1939))

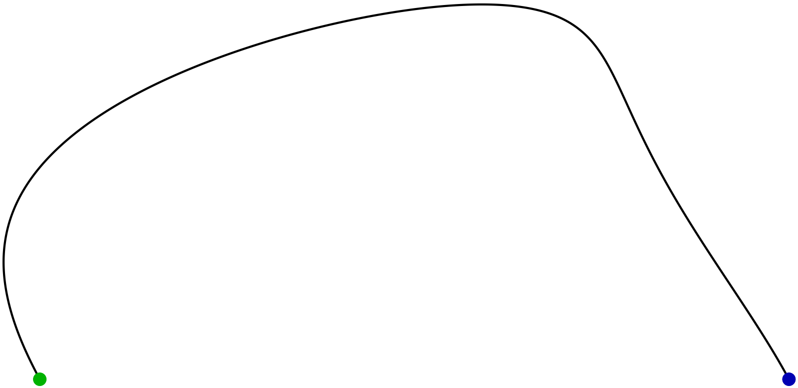
Let $\mathcal{O}$ be a connected nonempty open subset of $\mathbb{R}^n$. Let us assume that, for some $f_1, \ldots, f_m : \mathcal{O} \to \mathbb{R}^n$,

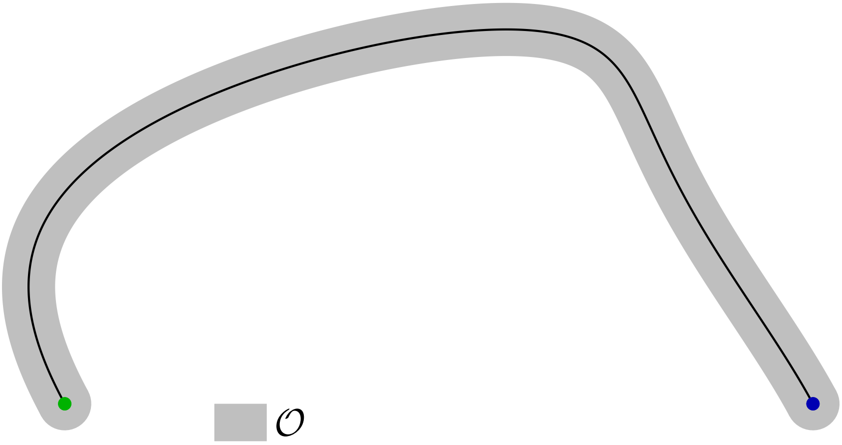
$$f(y,u) = \sum_{i=1}^{m} u_i f_i(y), \ \forall (y,u) \in \mathcal{O} \times \mathbb{R}^m.$$

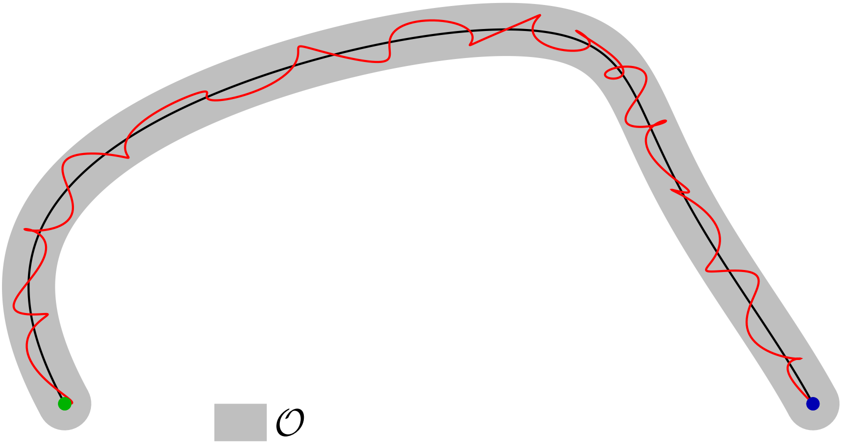
Let us also assume that

$$\{h(y);\ h \in \text{Lie}\ \{f_1, \ldots, f_m\}\} = \mathbb{R}^n, \ \forall y \in \mathcal{O}.$$

Then, for every $(y^0, y^1) \in \mathcal{O} \times \mathcal{O}$ and for every $T > 0$, there exists u belonging to $L^\infty((0,T);\mathbb{R}^m)$ such that the solution of the Cauchy problem $\dot{y} = f(y, u(t))$, $y(0) = y^0$, satisfies $y(t) \in \mathcal{O}$, $\forall t \in [0,T]$ and $y(T) = y^1$.




$\mathcal{O}$


$\mathcal{O}$

The Lie algebra rank condition

We consider the control affine system $\dot{y} = f_0(y) + \sum_{i=1}^m u_i f_i(y)$ with $f_0(0) = 0$. One says that this control system satisfies the Lie algebra rank condition at $0 \in \mathbb{R}^n$ if

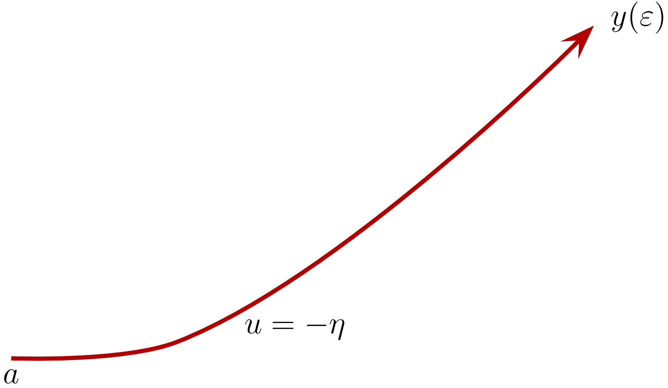
$$\Big\{h(0);\ h \in \text{Lie}\ \{f_0, f_1, \dots, f_m\}\Big\} = \mathbb{R}^n.$$

One has the following theorem

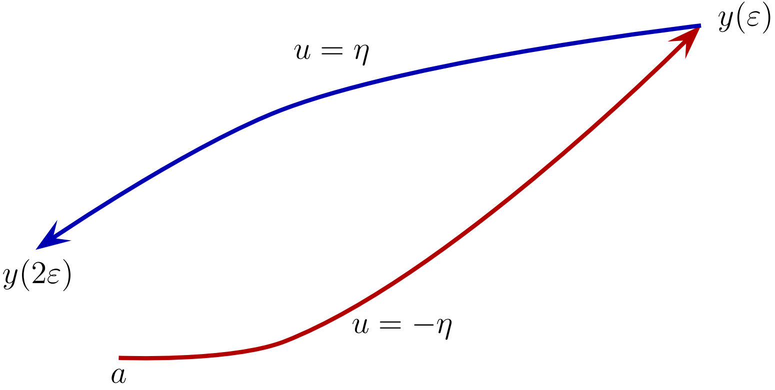
Theorem (R. Hermann (1963) and T. Nagano (1966))

If the f_i's are analytic in a neighborhood of $0 \in \mathbb{R}^n$ and if the control system $\dot{y} = f_0(y) + \sum_{i=1}^m f_i(y)$ is small-time locally controllable at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$, then this control system satisfies the Lie algebra rank condition at $0 \in \mathbb{R}^n$.

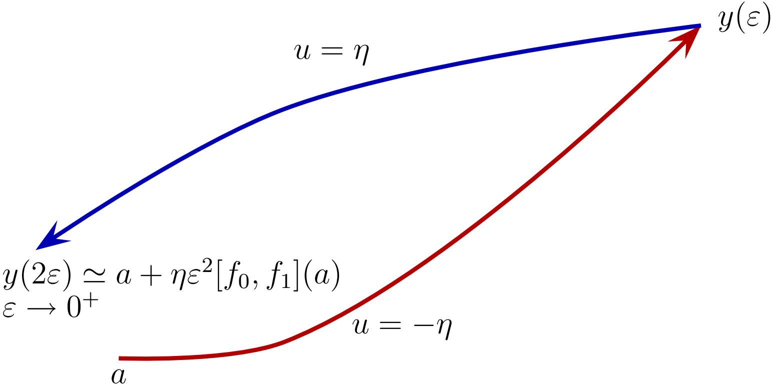
Lie bracket for $\dot{y} = f_0(y) + uf_1(y)$, with $f_0(a) = 0$



Lie bracket for $\dot{y} = f_0(y) + u f_1(y)$, with $f_0(a) = 0$



Lie bracket for $\dot{y} = f_0(y) + uf_1(y)$, with $f_0(a) = 0$



The Kalman rank condition and iterated Lie brackets

For $k \in \mathbb{N}$, $X : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^n$ and $Y : \mathcal{O} \to \mathbb{R}^n$, one defines $\text{ad}_X^k Y : \mathcal{O} \to \mathbb{R}^n$ by

$$\text{ad}_X^0 Y := Y,\ \text{ad}_X^1 Y := [X, Y],\ \text{ad}_X^2 Y = [X, [X, Y]],\ \text{etc.}$$

Let us write the linear control system $\dot{y} = Ay + Bu$ as $\dot{y} = f_0(y) + \sum_{i=1}^m u_i f_i(y)$ with

$$f_0(y) = Ay,\ f_i(y) = B_i,\ B_i \in \mathbb{R}^n,\ (B_1, \ldots, B_m) = B.$$

Then

$$\text{ad}_{f_0}^k f_i = (-1)^k A^k B_i.$$

Hence the Kalman rank condition can be written in the following way

$$\text{Span}\ \{\text{ad}_{f_0}^k f_i(0);\ k \in \{0, \ldots, n-1\},\ i \in \{1, \ldots, m\}\} = \mathbb{R}^n.$$

With a drift term: Not all the iterated Lie brackets are good

We take $n = 2$ and $m = 1$ and consider the control system

$$\Sigma : \quad \dot{y}_1 = y_2^2,\ \dot{y}_2 = u,$$

where the state is $y := (y_1, y_2)^{\text{tr}} \in \mathbb{R}^2$ and the control is $u \in \mathbb{R}$. This control system can be written as $\dot{y} = f_0(y) + u f_1(y)$ with

$$f_0(y) = (y_2^2, 0)^{\text{tr}}, f_1(y) = (0, 1)^{\text{tr}}.$$

One has $[f_1, [f_1, f_0]] = (2, 0)^{\text{tr}}$ and therefore $f_1(0)$ and $[f_1, [f_1, f_0]](0)$ span all of $\mathbb{R}^2$. However Σ is clearly not small-time locally controllable at $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}$.

References for sufficient or necessary conditions for small-time local controllability when there is a drift term

- A. Agrachev (1991),
- A. Agrachev and R. Gamkrelidze (1993),
- R. M. Bianchini and Stefani (1986),
- H. Frankowska (1987),
- M. Kawski (1990),
- H. Sussmann (1983, 1987),
- A. Tret'yak (1990).
- ...

Open problem

Let k be a positive integer. Let $\mathcal{P}_k$ be the set of vector fields in $\mathbb{R}^n$ whose components are polynomials of degree k. Let

$$S := \{(f_0, f_1) \in \mathcal{P}_k^2;\ f_0(0) = 0,\ \dot{y} = f_0(y) + u f_1(y) \text{ is STLC } \}.$$

Open problem

Is S a semi-algebraic subset of $\mathcal{P}_k^2$?

Theorem (J.-J. Risler, A. Gabrielov and F. Jean (1996 to 1999))

The set of $(f_0, f_1) \in \mathcal{P}_k^2$ satisfying the Lie algebra rank condition at 0 is a semi-algebraic subset of $\mathcal{P}_k^2$.

Controllability of control systems modeled by linear PDE

There are lot of powerful tools to study the controllability of linear control systems in infinite dimension. The most popular ones are based on the duality between observability and controllability (related to the J.-L. Lions Hilbert uniqueness method). This leads to try to prove observability inequalities. There are many methods to prove this observability inequalities. For example:

- Ingham's inequalities and harmonic analysis: D. Russell (1967),
- Multipliers method: Lop Fat Ho (1986), J.-L. Lions (1988),
- Microlocal analysis: C. Bardos-G. Lebeau-J. Rauch (1992),
- Carleman's inequalities: A. Fursikov, O. Imanuvilov, G. Lebeau, L. Robbiano (1993-1996).

However there are still plenty of open problems.

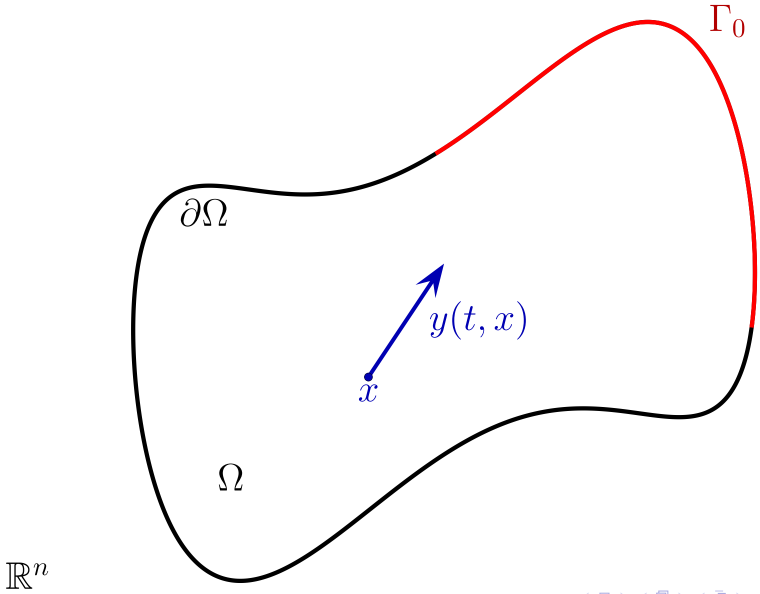
The linear test

Of course when one wants to study the local controllability around an equilibrium of a control system in infinite dimension, the first step is to again study the controllability of the linearized control system. If this linearized control system is controllable, one can usually deduce the local controllability of the nonlinear control system. However this might be sometimes difficult due to some loss of derivatives issues. One needs to use suitable iterative schemes.

Remark

If the nonlinearity is not too big, one can get a global controllability result (E. Zuazua (1988) for a semilinear wave equation).

The Euler control system



Controllability problem

We denote by $\nu : \partial\Omega \to \mathbb{R}^n$ the outward unit normal vector field to Ω. Let $T > 0$. Let y^0, $y^1 : \Omega \to \mathbb{R}^n$ be such that

$$\text{div } y^0 = \text{div } y^1 = 0,\ y^0 \cdot \nu = y^1 \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma_0.$$

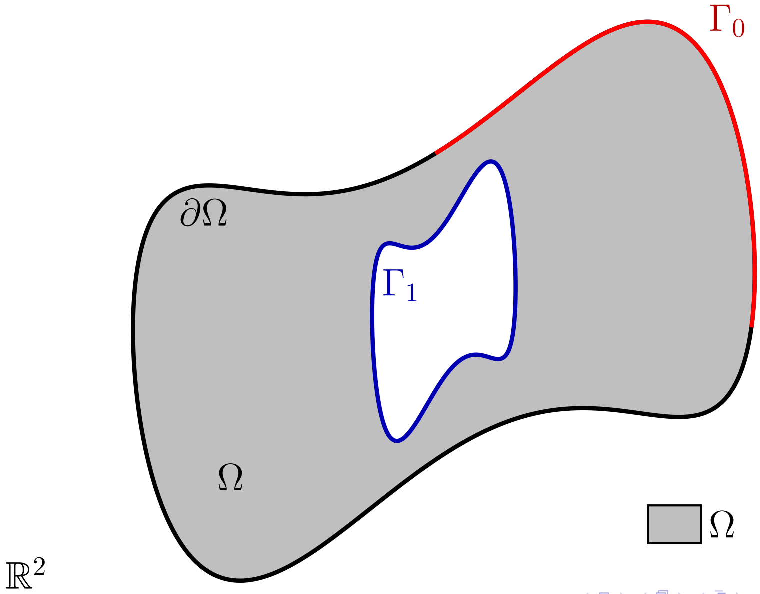
Does there exist $y : [0, T] \times \Omega \to \mathbb{R}^n$ and $p : [0, T] \times \Omega \to \mathbb{R}$ such that

$$y_t + (y \cdot \nabla)y + \nabla p = 0,\ \text{div } y = 0,$$
$$y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0)$$
$$y(0, \cdot) = y^0,\ y(T, \cdot) = y^1?$$

For the control, many choices are in fact possible. For example, for $n = 2$, one can take

1. $y \cdot \nu$ on Γ_0 with $\int_{\Gamma_0} y \cdot \nu = 0$,
2. $\mathrm{curl}\ y := \frac{\partial y^2}{\partial x_1} - \frac{\partial y^1}{\partial x_2}$ at the points of $[0, T] \times \Gamma_0$ where $y \cdot \nu < 0$.

A case without controllability



Proof of the noncontrollability

Let us give it only for $n = 2$. Let γ_0 be a Jordan curve in $\overline{\Omega}$. Let, for $t \in [0, T]$, $\gamma(t)$ be the Jordan curve obtained, at time $t \in [0, T]$, from the points of the fluids which, at time 0, were on γ_0. The Kelvin law tells us that, if $\gamma(t)$ does not intersect Γ_0,

$$\int_{\gamma(t)} y(t, \cdot) \cdot \overrightarrow{ds} = \int_{\gamma_0} y(0, \cdot) \cdot \overrightarrow{ds}, \forall t \in [0, T],$$

We take $\gamma_0 := \Gamma_1$. Then $\gamma(t) = \Gamma_1$ for every $t \in [0, T]$. Hence, if

$$\int_{\Gamma_1} y^1 \cdot \overrightarrow{ds} \neq \int_{\Gamma_1} y^0 \cdot \overrightarrow{ds},$$

one cannot steer the control system from y^0 to y^1.
More generally, for every $n \in \{2, 3\}$, if Γ_0 does not intersect every connected component of the boundary $\partial\Omega$ of Ω, the Euler control system is not controllable. This is the only obstruction to the controllability of the Euler control system.

Controllability of the Euler control system

Theorem (JMC for $n = 2$ (1996), O. Glass for $n = 3$ (2000))

Assume that Γ_0 intersects every connected component of $\partial\Omega$. Then the Euler control system is globally controllable in every time: For every $T > 0$, for every y^0, $y^1 : \Omega \to \mathbb{R}^n$ such that

$$\text{div } y^0 = \text{div } y^1 = 0,\ y^0 \cdot \nu = y^1 \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma_0,$$

there exist $y : [0,T] \times \Omega \to \mathbb{R}^n$ and $p : [0,T] \times \Omega \to \mathbb{R}$ such that

$$y_t + (y \cdot \nabla)y + \nabla p = 0,\ \text{div } y = 0,$$
$$y \cdot \nu = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma_0)$$
$$y(0,\cdot) = y^0,\ y(T,\cdot) = y^1.$$

Sketch of the proof of the controllability result

One first studies (as usual) the controllability of the linearized control system around 0. This linearized control system is the underdetermined system

$$y_t + \nabla p = 0, \text{div } y = 0,\ y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0).$$

For simplicity we assume that $n = 2$. Taking the curl of the first equation, on gets, with curl $y := \dfrac{\partial y^2}{\partial x_1} - \dfrac{\partial y^1}{\partial x_2}$,

$$(\text{curl } y)_t = 0.$$

Hence curl y remains constant along the trajectories of the Euler control system.

Iterated Lie brackets and PDE control systems

- Euler and Navier Stokes control systems: Andrei Agrachev and Andrei Sarychev (2005); Armen Shirikyan (2006, 2007),
- Schrödinger control system: Thomas Chambrion, Paolo Mason, Mario Sigalotti and Ugo Boscain (2009).

However it does not seem to work here.

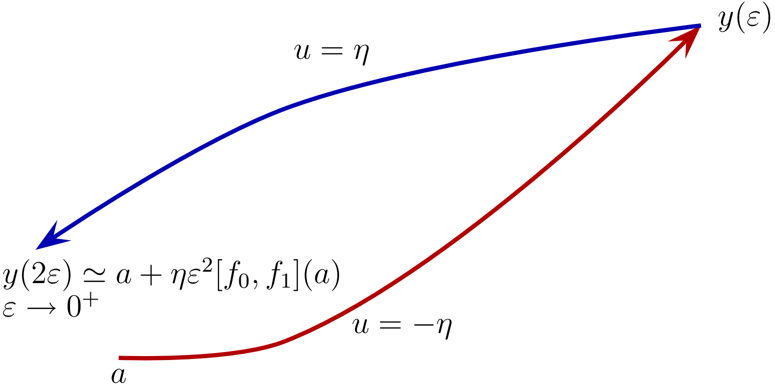
Problems of the Lie brackets for PDE control systems

Consider the simplest PDE control system

$$\Sigma: \; y_t + y_x = 0, \; x \in [0, L], y(t, 0) = u(t).$$

It is a control system where, at time t, the state is $y(t, \cdot) : (0, L) \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$.

Lie bracket for $\dot{y} = f_0(y) + u f_1(y)$, with $f_0(a) = 0$



Problems of the Lie brackets for PDE control systems (continued)

Let us consider, for $\varepsilon > 0$, the control defined on $[0, 2\varepsilon]$ by

$$u(t) := -\eta \text{ for } t \in (0, \varepsilon),\ u(t) := \eta \text{ for } t \in (\varepsilon, 2\varepsilon).$$

Let $y : (0, 2\varepsilon) \times (0, L) \to \mathbb{R}$ be the solution of the Cauchy problem

$$y_t + y_x = 0,\ t \in (0, 2\varepsilon),\ x \in (0, L),$$
$$y(t, 0) = u(t),\ t \in (0, 2\varepsilon),\ y(0, x) = 0,\ x \in (0, L).$$

Then one readily gets, if $2\varepsilon \leqslant L$,

$$y(2\varepsilon, x) = \eta,\ x \in (0, \varepsilon),\ y(2\varepsilon, x) = -\eta,\ x \in (\varepsilon, 2\varepsilon),$$
$$y(2\varepsilon, x) = 0,\ x \in (2\varepsilon, L).$$

Problems of the Lie brackets for PDE control systems (continued)

$$\left| \frac{y(2\varepsilon,\cdot)-y(0,\cdot)}{\varepsilon^2} \right|_{L^2(0,L)} \to +\infty \text{ as } \varepsilon \to 0^+.$$

For every $\phi \in H^2(0,L)$, one gets after suitable computations

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\int_0^L \phi(x)(y(2\varepsilon,x)-y(0,x))dx = -\eta\phi'(0).$$

So, in some sense, we could say that $[f_0, f_1] = \delta_0'$. Unfortunately it is not clear how to use this derivative of a Dirac mass at 0.

How to avoid the use of iterated Lie brackets?

The return method (JMC (1992))

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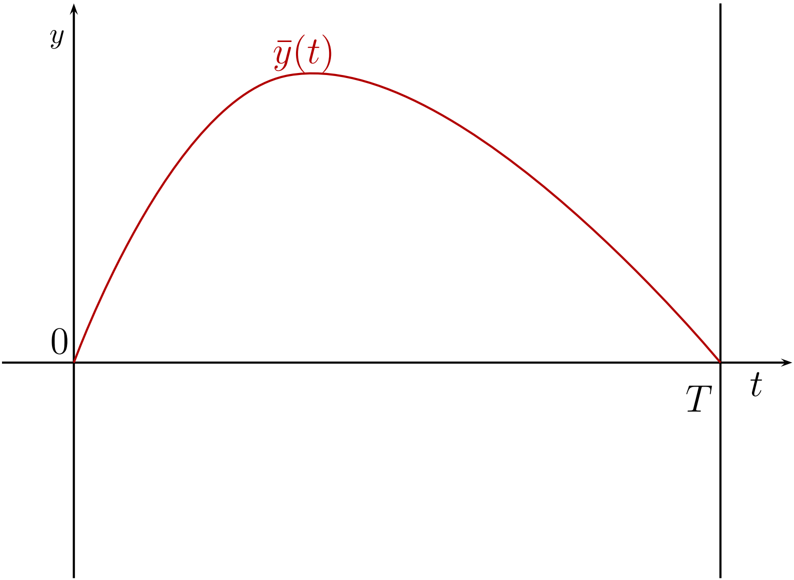
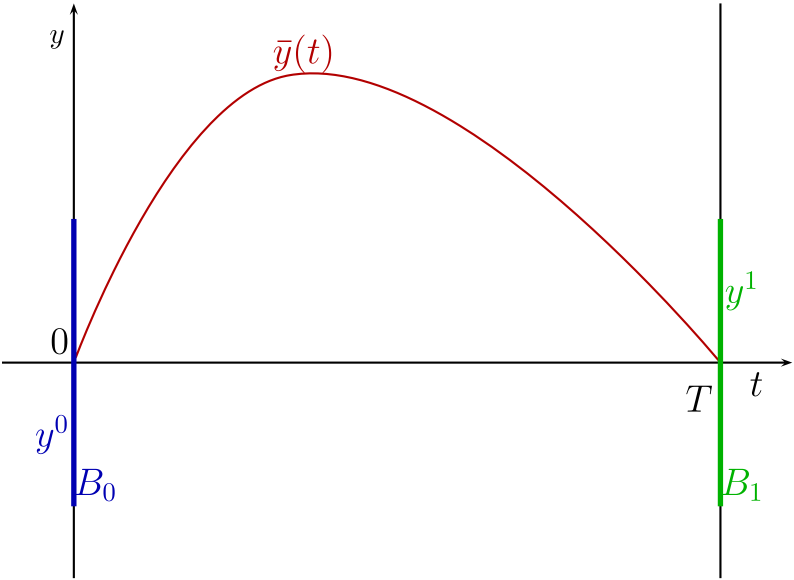


The return method (JMC (1992))



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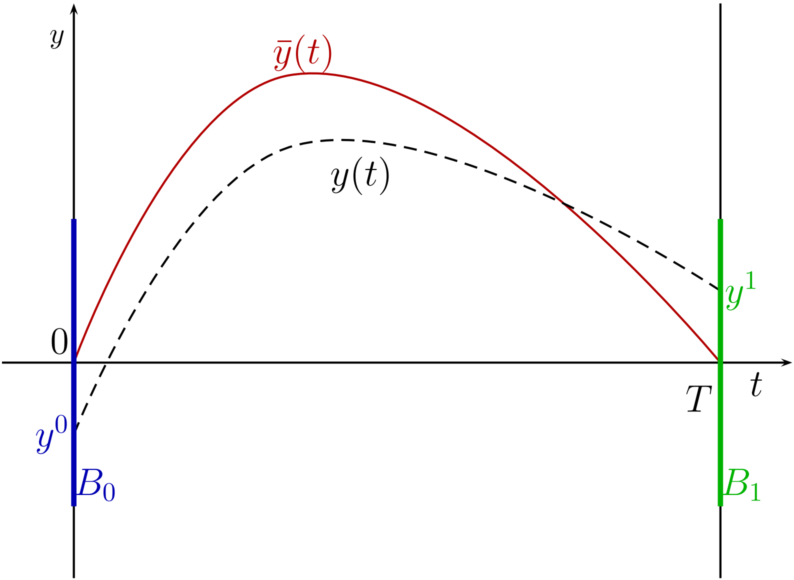

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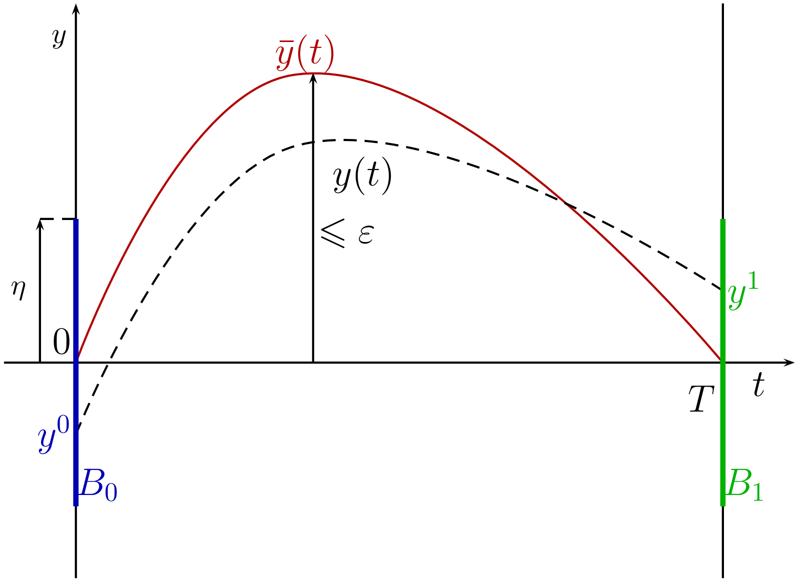


The return method: An example in finite dimension

We go back to the baby stroller control system

$$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3,\ \dot{y}_3 = u_2.$$

For every $\bar{u} : [0, T] \to \mathbb{R}^2$ such that, for every t in $[0, T]$, $\bar{u}(T - t) = -\bar{u}(t)$, every solution $\bar{y} : [0, T] \to \mathbb{R}^3$ of

$$\dot{\bar{y}}_1 = \bar{u}_1 \cos \bar{y}_3,\ \dot{\bar{y}}_2 = \bar{u}_1 \sin \bar{y}_3, \dot{\bar{y}}_3 = \bar{u}_2,$$

satisfies $\bar{y}(0) = \bar{y}(T)$. The linearized control system around $(\bar{y}, \bar{u})$ is

$$\dot{y}_1 = -\bar{u}_1 y_3 \sin \bar{y}_3 + u_1 \cos \bar{y}_3,\ \dot{y}_2 = \bar{u}_1 y_3 \cos \bar{y}_3 + u_1 \sin \bar{y}_3,\ \dot{y}_3 = u_2,$$

which is controllable if (and only if) $\bar{u} \not\equiv 0$. ...
We have got the controllability of the baby stroller system without using Lie brackets. We have only used controllability results for linear control systems.

No loss with the return method

We consider the control system

$$\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y), \qquad \Sigma$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. We assume that $f_0(0) = 0$ and that the f_i's, $i \in \{0, 1, \dots, m\}$ are of class C^∞ in a neighborhood of $0 \in \mathbb{R}^n$. One has the following proposition.

Proposition (E. Sontag (1988), JMC (1994))

Let us assume that Σ satisfies the Lie algebra rank condition at $0 \in \mathbb{R}^n$ and is STLC at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$. Then, for every $\varepsilon > 0$, there exists $\bar{u} \in L^\infty((0, \varepsilon); \mathbb{R}^m)$ satisfying $|u(t)| \leqslant \varepsilon$, $\forall t \in [0, T]$, such that, if $\bar{y} : [0, \varepsilon] \to \mathbb{R}^n$ is the solution of $\dot{\bar{y}} = f(\bar{y}, \bar{u}(t))$, $\bar{y}(0) = 0$, then

$$\bar{y}(T) = 0,$$

the linearized control system around $(\bar{y}, \bar{u})$ is controllable.

The return method and the controllability of the Euler equations

One looks for $(\bar{y}, \bar{p}) : [0, T] \times \Omega \to \mathbb{R}^n \times \mathbb{R}$ such that

$$\bar{y}_t + (\bar{y} \cdot \nabla \bar{y}) + \nabla \bar{p} = 0, \text{ div } \bar{y} = 0,$$
$$\bar{y} \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0),$$
$$\bar{y}(T, \cdot) = \bar{y}(0, \cdot) = 0,$$

the linearized control system around $(\bar{y}, \bar{p})$ is controllable.

Construction of $(\bar{y}, \bar{p})$

Take $\theta : \overline{\Omega} \to \mathbb{R}$ such that

$$\Delta\theta = 0 \text{ in } \Omega, \ \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \partial\Omega \setminus \Gamma_0.$$

Take $\alpha : [0, T] \to \mathbb{R}$ such that $\alpha(0) = \alpha(T) = 0$. Finally, define $(\bar{y}, \bar{p}) : [0, T] \times \Omega \to \mathbb{R}^2 \times \mathbb{R}$ by

$$\bar{y}(t, x) := \alpha(t)\nabla\theta(x), \ \bar{p}(t, x) := -\dot{\alpha}(t)\theta(x) - \frac{\alpha(t)^2}{2}|\nabla\theta(x)|^2.$$

Then $(\bar{y}, \bar{p})$ is a trajectory of the Euler control system which goes from 0 to 0.

Controllability of the linearized control system around $(\bar{y}, \bar{p})$ if $n = 2$

The linearized control system around $(\bar{y}, \bar{p})$ is

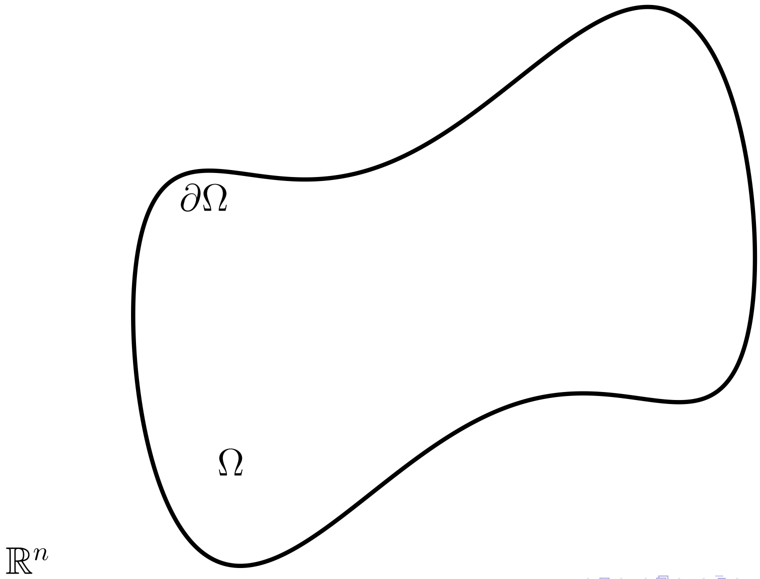
$$\left\{ \begin{array}{l} y_t + (\bar{y} \cdot \nabla) y + (y \cdot \nabla) \bar{y} + \nabla p = 0, \ \ \text{div } y = 0 \text{ in } [0, T] \times \Omega, \\ y \cdot \nu = 0 \text{ on } [0, T] \times (\partial \Omega \setminus \Gamma_0). \end{array} \right. \tag{1}$$

Again we assume that $n = 2$. Taking once more the curl of the first equation, one gets

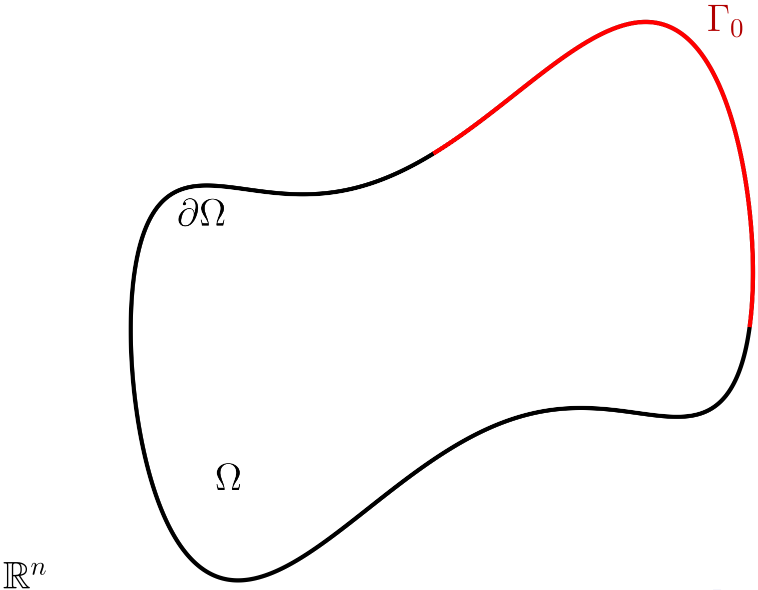
$$(\text{curl } y)_t + (\bar{y} \cdot \nabla)(\text{curl } y) = 0. \tag{2}$$

This is a simple transport equation on curl y. If there exists $a \in \overline{\Omega}$ such that $\nabla \theta(a) = 0$, then $\bar{y}(t, a) = 0$ and $(\text{curl } y)_t(t, a) = 0$ showing that (2) is not controllable. This is the only obstruction: If $\nabla \theta$ does not vanish in $\overline{\Omega}$, one can prove that (2) (and then (1)) is controllable if $\int_0^T \alpha(t) dt$ is large enough.

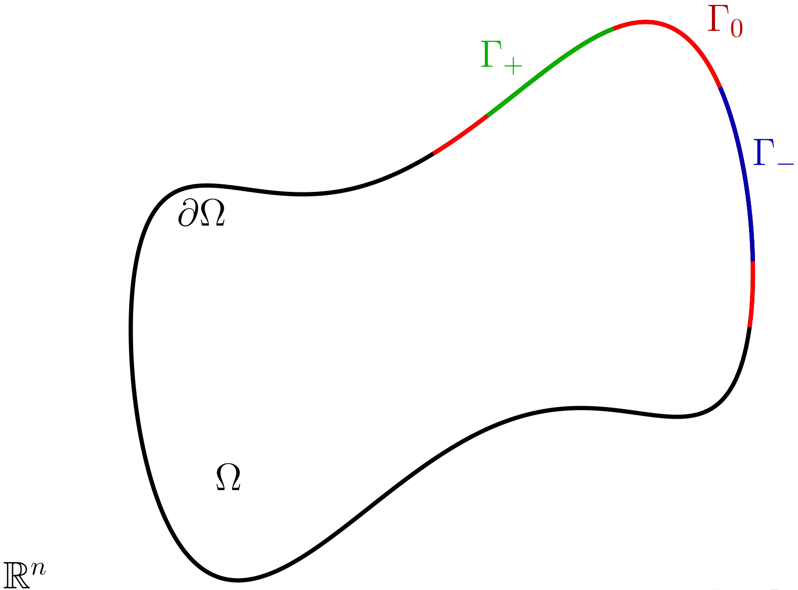
Construction of a good θ for $n = 2$



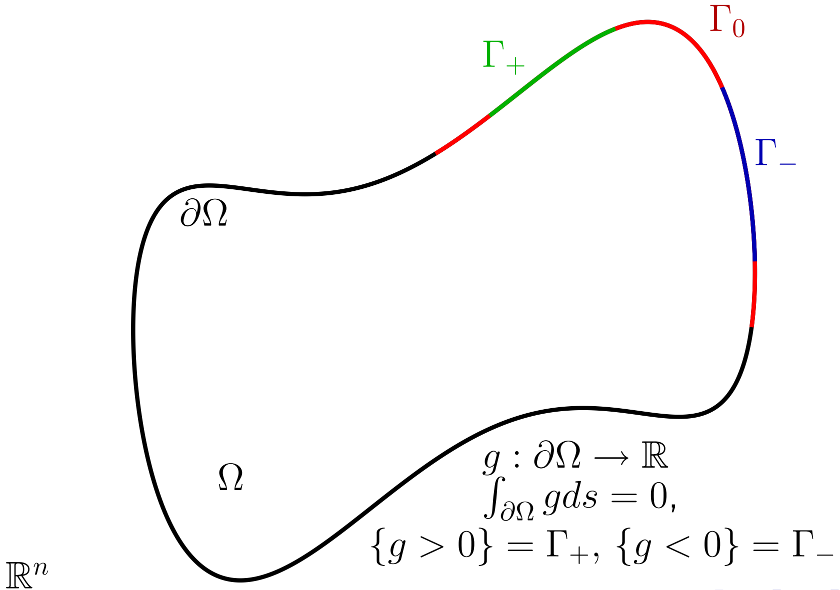
Construction of a good θ for $n = 2$



Construction of a good θ for $n = 2$



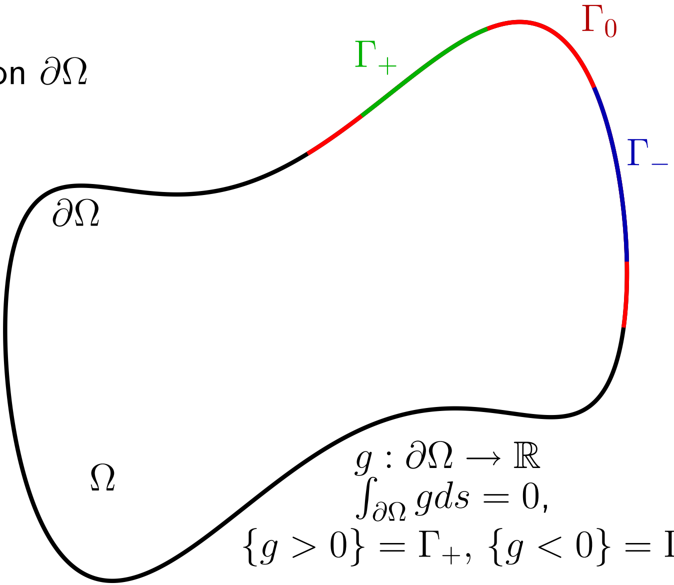
Construction of a good θ for $n = 2$



Construction of a good θ for $n = 2$

$\Delta\theta = 0$,

$\frac{\partial\theta}{\partial\nu} = g$ on $\partial\Omega$



$g : \partial\Omega \rightarrow \mathbb{R}$
$\int_{\partial\Omega} g ds = 0$,
$\{g > 0\} = \Gamma_+$, $\{g < 0\} = \Gamma_-$

$\mathbb{R}^n$

Construction of a good θ for $n = 2$

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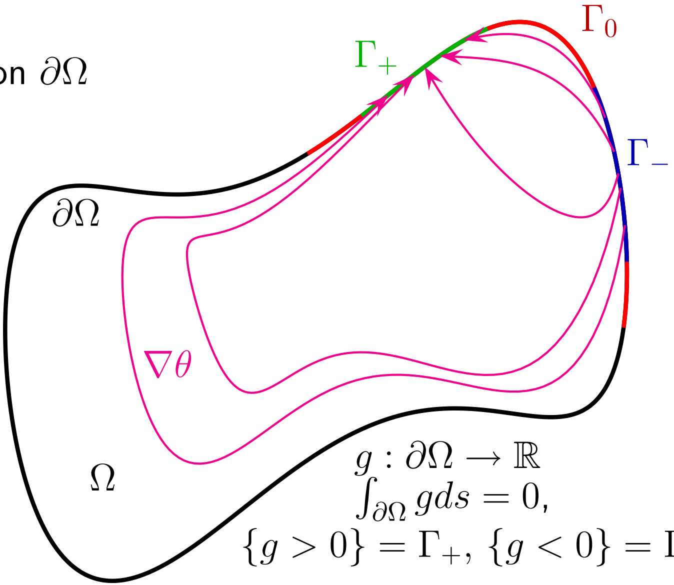


$g : \partial\Omega \to \mathbb{R}$

$\int_{\partial\Omega} g ds = 0,$

$\{g > 0\} = \Gamma_+, \{g < 0\} = \Gamma_-$

$\mathbb{R}^n$

From local controllability to global controllability

A simple scaling argument: if $(y,p):[0,1]\times\Omega :\longrightarrow \mathbb{R}^n\times\mathbb{R}$ is a solution of our control system, then, for every $\varepsilon>0$, $(y^\varepsilon,p^\varepsilon):[0,\varepsilon]\times\Omega\longrightarrow\mathbb{R}^n\times\mathbb{R}$ defined by

$$y^\varepsilon(t,x):=\frac{1}{\varepsilon}y\left(\frac{t}{\varepsilon},x\right),\ p^\varepsilon(t,x):=\frac{1}{\varepsilon^2}p\left(\frac{t}{\varepsilon},x\right)$$

is also a solution of our control system.

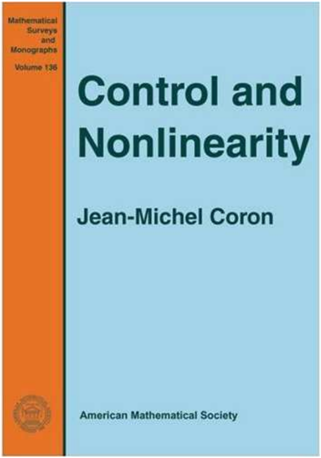
Return method: references

- Stabilization of driftless systems in finite dimension: JMC (1992).
- Euler equations of incompressible fluids: JMC (1993,1996), O. Glass (1997,2000).
- Control of driftless systems in finite dimension: E.D. Sontag (1995).
- Navier-Stokes equations: JMC (1996), JMC and A. Fursikov (1996), A. Fursikov and O. Imanuvilov (1999), S. Guerrero, O. Imanuvilov and J.-P. Puel (2006), JMC and S. Guerrero (2009), M. Chapouly (2009).
- Burgers equation: Th. Horsin (1998), M. Chapouly (2006), O. Imanuvilov and J.-P. Puel (2009).
- Saint-Venant equations: JMC (2002).
- Vlasov Poisson: O. Glass (2003).

Return method: references (continued)

- Isentropic Euler equations: O. Glass (2006).
- Schrödinger equation: K. Beauchard (2005), K. Beauchard and JMC (2006).
- Korteweg de Vries equation: M. Chapouly (2008).
- Hyperbolic equations: JMC, O. Glass and Z. Wang (2009).
- Ensemble controllability of Bloch equations: K. Beauchard, JMC and P. Rouchon (2009).
- Parabolic systems: JMC, S. Guerrero, L. Rosier (2010).

Return method: Commercial break



JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 pp.

The stabilizability problem

We consider the control system $\dot{y} = f(y, u)$ where the state is y in $\mathbb{R}^n$ and the control is u in $\mathbb{R}^m$. We assume that $f(0, 0) = 0$.

Problem

Does there exist $u : \mathbb{R}^n \to \mathbb{R}^m$ vanishing at 0 such that $0 \in \mathbb{R}^n$ is (locally) asymptotically stable for $\dot{y} = f(y, u(y))$? (If the answer is yes, one says that the control system is locally asymptotically stabilizable.)

Remark

The map $u : y \in \mathbb{R}^n \mapsto \mathbb{R}^m$ is called a feedback (or feedback law). The dynamical system $\dot{y} = f(y, u(y))$ is called the closed loop system.

Remark

The regularity of u is an important point. Here, we assume that the feedback laws are continuous.

Obstruction to the stabilizability

Theorem (R. Brockett (1983))

If the control system $\dot{y} = f(y, u)$ *can be locally asymptotically stabilized then*

(B) *the image by* f *of every neighborhood of* $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ *is a neighborhood of* $0 \in \mathbb{R}^n$*.*

Example: The baby stroller. The baby stroller control system

$$\dot{y}_1 = u_1 \cos y_3, \ \dot{y}_2 = u_1 \sin y_3, \dot{y}_3 = u_2$$

is small-time locally controllable at $(0, 0)$. However (B) does not hold for the baby stroller control system. Hence the baby stroller control system cannot be locally asymptotically stabilized.

A solution: Time-varying feedback laws

Instead of $u(y)$, use $u(t, y)$: E. Sontag and H. Sussmann (1980) for $n = 1$, C. Samson (1990) for the baby stroller. Note that asymptotic stability for time-varying feedback laws is also robust (there exists again a strict Lyapunov function).

Time-varying feedback laws for driftless control systems

Theorem (JMC (1992))

Assume that

$$\{g(y);\, g \in \mathit{Lie}\{f_1, \dots, f_m\}\} = \mathbb{R}^n,\ \forall y \in \mathbb{R}^n \setminus \{0\}.$$

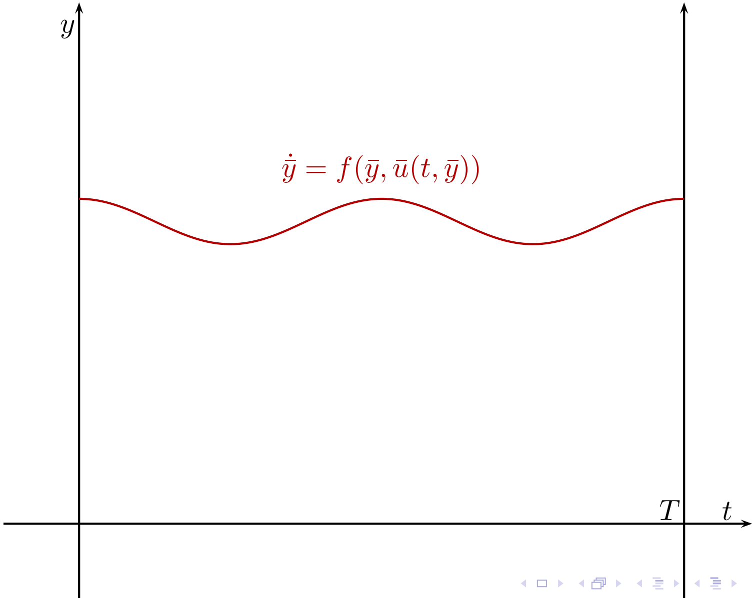
Then, for every $T > 0$, there exists u in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ such that

$$u(t, 0) = 0,\ \forall t \in \mathbb{R},$$

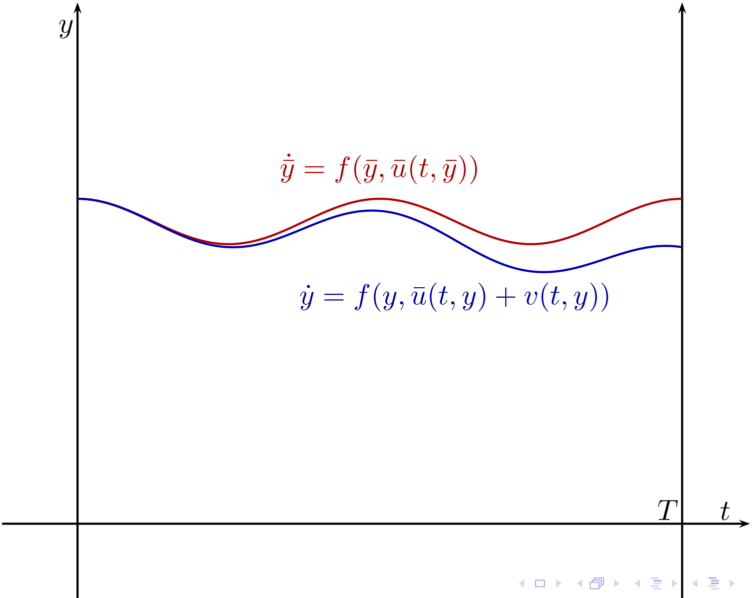
$$u(t + T, y) = u(t, y),\ \forall y \in \mathbb{R}^n,\ \forall t \in \mathbb{R},$$

$$0 \text{ is globally asymptotically stable for } \dot{y} = \sum_{i=1}^{m} u_i(t, y) f_i(y).$$

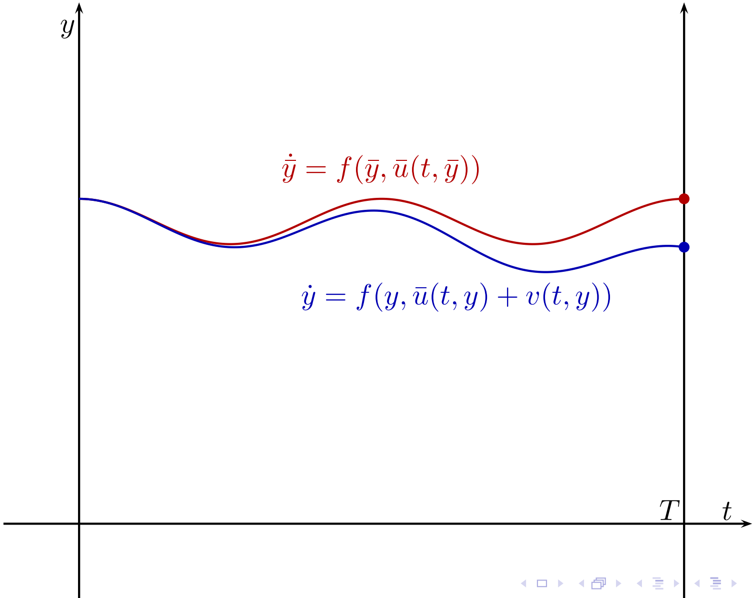
Sketch of proof



Sketch of proof



Sketch of proof



Construction of $\bar{u}$

In order to get periodic trajectories, one just imposes on $\bar{u}$ the condition

$$\bar{u}(t,y) = -\bar{u}(T-t,y),\ \forall (t,y) \in \mathbb{R} \times \mathbb{R}^n,$$

which implies that $y(t) = y(T-t)$, $\forall t \in [0,T]$, for every solution of $\dot{y} = f(y, \bar{u}(t,y))$, and therefore gives $y(0) = y(T)$.
Finally, one proves that for "generic" $\bar{u}$'s the linearized control systems around all the trajectories of $\dot{y} = f(y, u(t,y))$ except the one starting from 0 are controllable on $[0,T]$ (this is the difficult part of the proof).

The Navier-Stokes control system

The Navier-Stokes control system is deduced from the Euler equations by adding the linear term $-\Delta y$: the equation is now

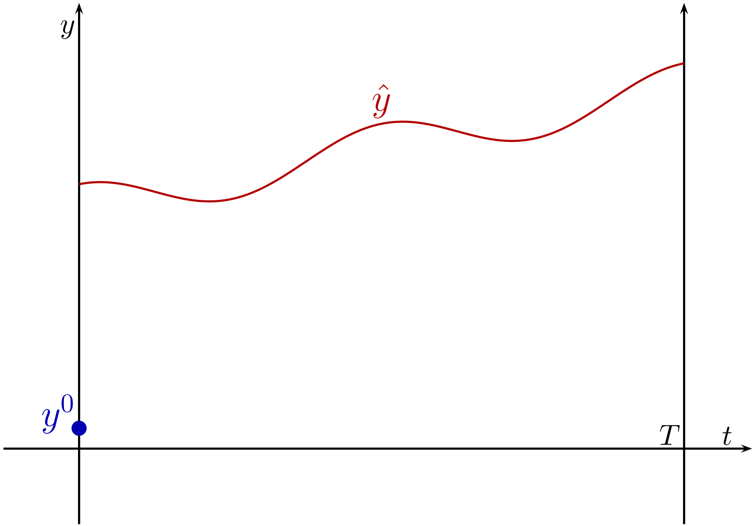
$$y_t - \Delta y + (y \cdot \nabla)y + \nabla p = 0, \text{ div } y = 0.$$

For the boundary condition, one requires now that

$$y = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0).$$

For the control, one can take, for example, y on $[0, T] \times \Gamma_0$.

Smoothing effects and a new notion of (global) controllability: A. Fursikov and O. Imanuvilov (1995)



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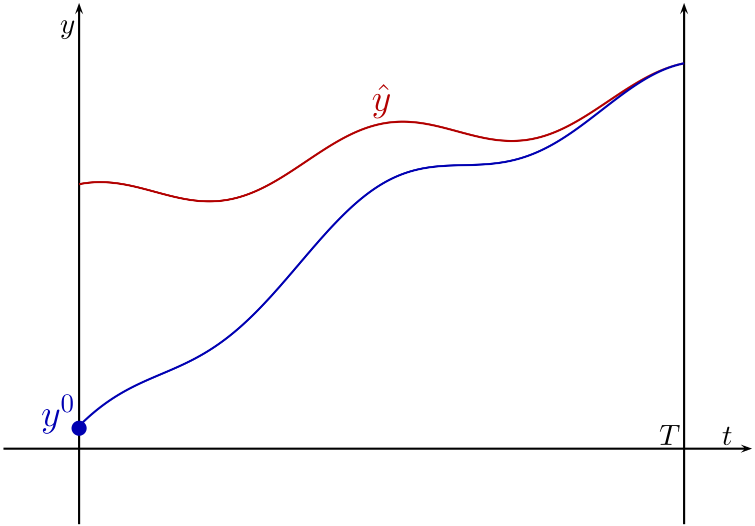
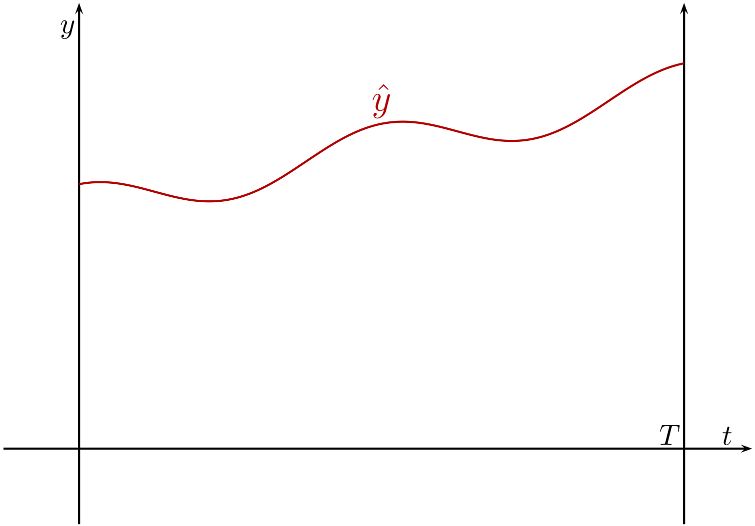


Smoothing effects and a new notion of local controllability A. Fursikov and O. Imanuvilov (1995)



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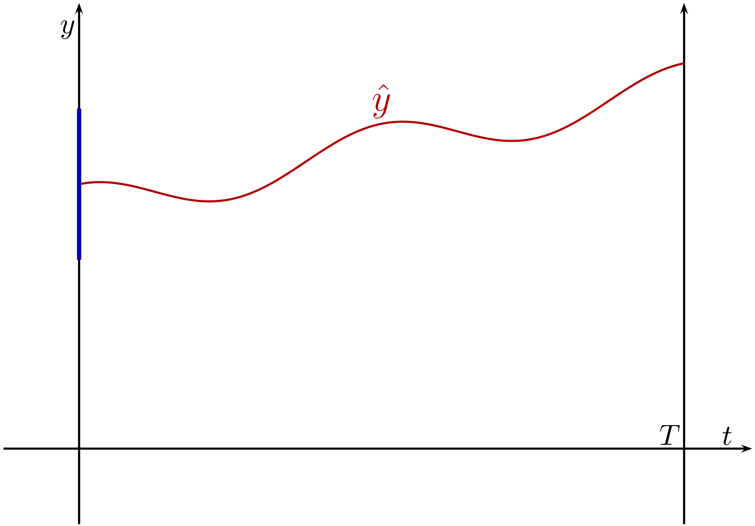

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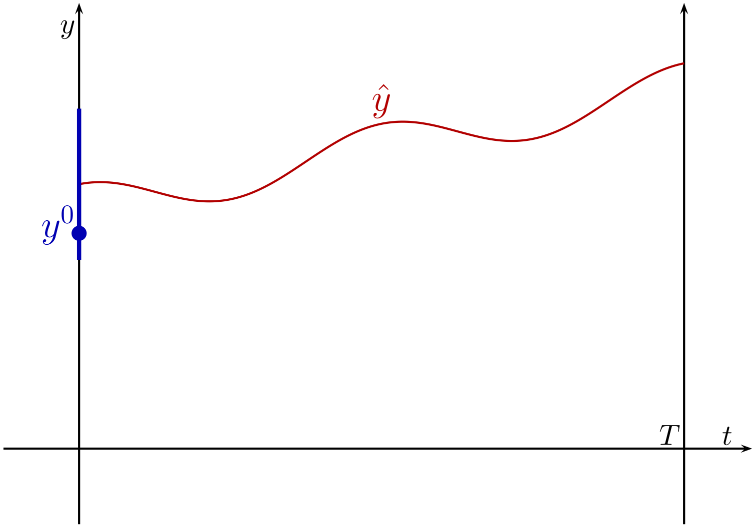

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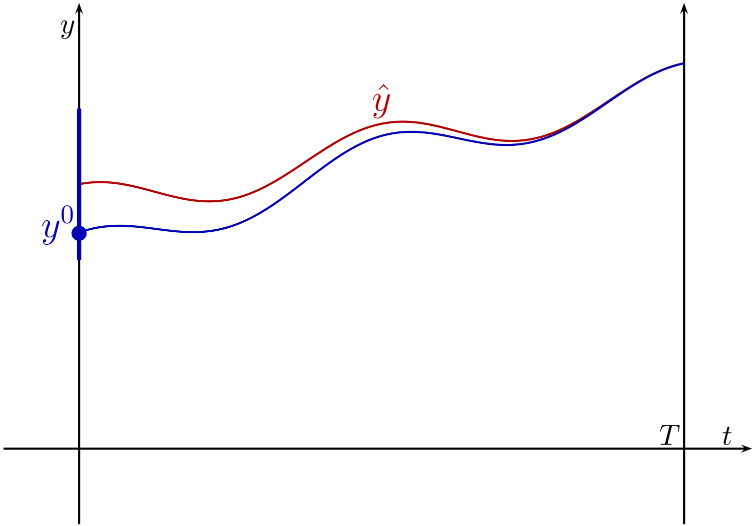


Local controllability

Theorem (A. Fursikov and O. Imanuvilov (1994), O. Imanuvilov (1998, 2001), E. Fernandez-Cara, S. Guerrero, O. Imanuvilov and J.-P. Puel (2004))

The Navier-Stokes control system is locally controllable.

The proof relies on the the controllability of the linearized control system around $\hat{y}$ (which is obtained by Carleman inequalities).

Global controllability

Theorem (JMC (1996), JMC and A. Fursikov (1996))

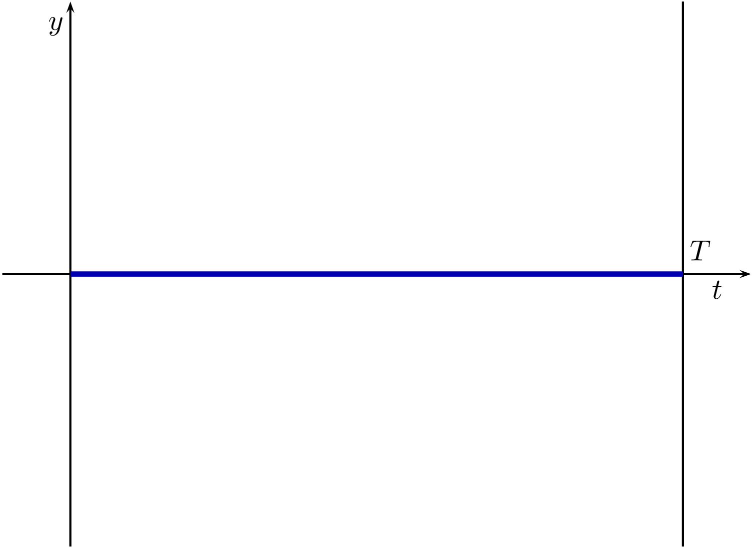
The Navier-Stokes control system is globally controllable if $\Gamma_0 = \partial\Omega$.

Open problem

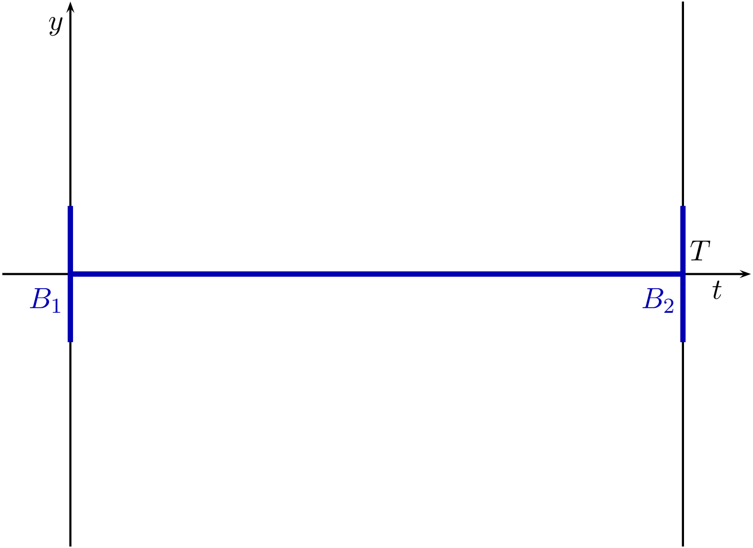
Does the above global controllability result hold even if $\Gamma_0 \neq \partial\Omega$?

Sketch of the proof of the global result

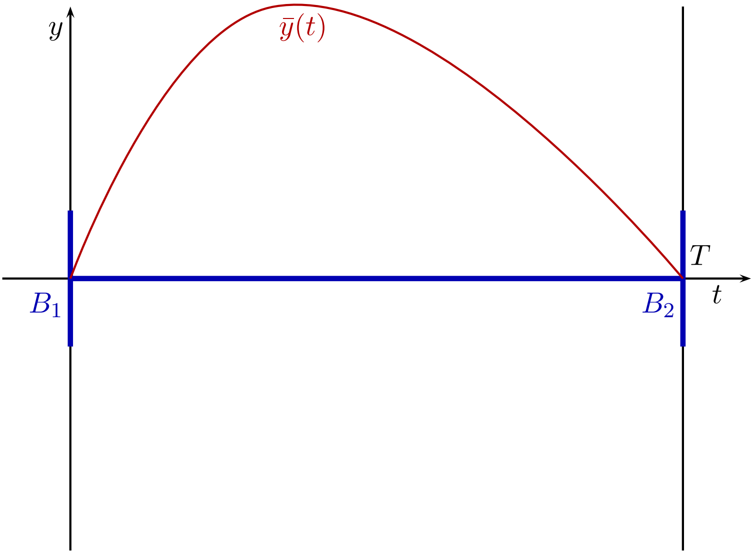
Sketch of the proof of the global result



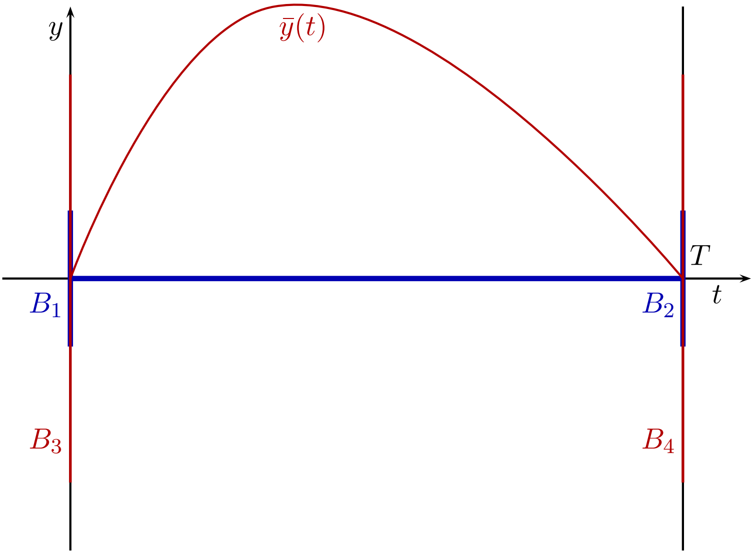
Sketch of the proof of the global result



Sketch of the proof of the global result



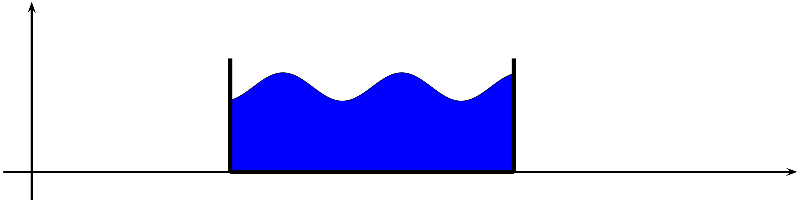
Sketch of the proof of the global result



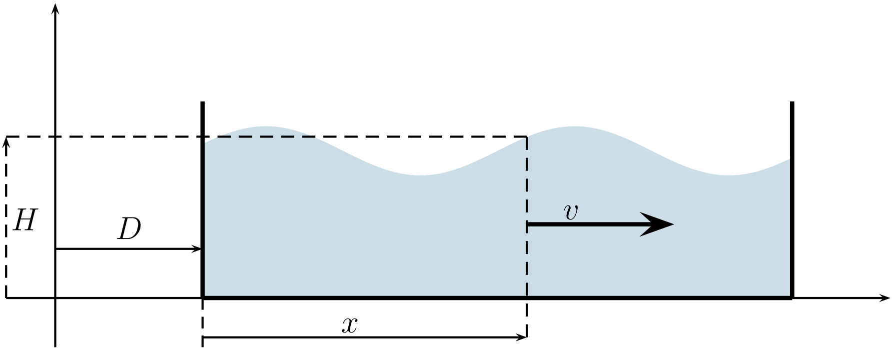
Main difficulty for the return method

It is often easy to leave the initial state and get a trajectory such that linearized control system around it is controllable. However it is then often difficult to return to the initial state.
To overcome this difficulty in some cases: Quasi-static deformations (JMC (2002)).

A water-tank control system



Saint-Venant equations: Notations



The model: Saint-Venant equations

$$
\begin{aligned}
H_t + (Hv)_x &= 0,\ t \in [0,T],\ x \in [0,L],\\
v_t + \left(gH + \frac{v^2}{2}\right)_x &= -u\,(t)\,,\ t \in [0,T],\ x \in [0,L],\\
v(t,0) = v(t,L) &= 0,\ t \in [0,T],\\
\dot{s}(t) &= u\,(t)\,,\ t \in [0,T],\\
\dot{D}(t) &= s\,(t)\,,\ t \in [0,T].
\end{aligned}
$$

- $u\,(t)$ is the horizontal acceleration of the tank in the absolute referential,
- g is the gravity constant,
- s is the horizontal velocity of the tank,
- D is the horizontal displacement of the tank.

State space

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H\left(t,x\right)\mathrm{d}x = 0,$$

$$H_x(t,0) = H_x(t,L) \quad (= -u(t)/g).$$

Definition

The state space (denoted $\mathcal{Y}$) is the set of
$Y = (H, v, s, D) \in C^1([0,L]) \times C^1([0,L]) \times \mathbb{R} \times \mathbb{R}$ satisfying

$$v(0) = v(L) = 0,\ H_x(0) = H_x(L),\ \int_0^L H(x)\mathrm{d}x = LH_e.$$

Main result

Theorem (JMC, (2002))

For $T > 0$ large enough the water-tank control system is locally controllable in time T around $(Y_e, u_e) := ((H_e, 0, 0, 0), 0)$.

Prior work: F. Dubois, N. Petit and P. Rouchon (1999): Steady state controllability of the linearized control system.

The linearized control system

Without loss of generality $L = H_e = g = 1$. The linearized control system around $(Y_e, u_e) := ((1,0,0,0),0)$ is

$$
\begin{aligned}
h_t + v_x = 0,\ t \in [0,T],\ x \in [0,L], \\
v_t + h_x = -u\,(t)\,,\ t \in [0,T],\ x \in [0,L], \\
v(t,0) = v(t,L) = 0,\ t \in [0,T], \\
\dot{S}\,(t) = u\,(t)\,,\ t \in [0,T], \\
\dot{D}\,(t) = S\,(t)\,,\ t \in [0,T].
\end{aligned}
$$

The linearized control system is not controllable

For a function $w:[0,1]\to\mathbb{R}$, we denote by w^{ev} "the even part" of w and by w^{od} the odd part of w:

$$w^{\text{ev}}(x):=\frac{1}{2}(w(x)+w(1-x)),\ w^{\text{od}}(x):=\frac{1}{2}(w(x)-w(1-x)).$$

$$\Sigma_1\begin{cases} h_t^{\text{od}}+v_x^{\text{ev}}=0,\\ v_t^{\text{ev}}+h_x^{\text{od}}=-u\left(t\right),\\ v^{\text{ev}}(t,0)=v^{\text{ev}}(t,1)=0,\ \dot{S}\left(t\right)=u\left(t\right),\ \dot{D}\left(t\right)=S\left(t\right),\end{cases}$$

$$\Sigma_2\begin{cases} h_t^{\text{ev}}+v_x^{\text{od}}=0,\\ v_t^{\text{od}}+h_x^{\text{ev}}=0,\\ v^{\text{od}}(t,0)=v^{\text{od}}(t,1)=0,\end{cases}$$

Water tank control system: Towards a toy model

If $h := H - 1$,

$$\Sigma_1 \begin{cases} h_t^{\text{od}} + v_x^{\text{ev}} = -(h^{\text{ev}} v^{\text{ev}} + h^{\text{od}} v^{\text{od}})_x, \\ v_t^{\text{ev}} + h_x^{\text{od}} = -u\,(t) - (v^{\text{ev}} v^{\text{od}})_x, \\ v^{\text{ev}}(t,0) = v^{\text{ev}}(t,1) = 0,\ \dot{s}\,(t) = u\,(t)\,,\ \dot{D}\,(t) = s\,(t)\,, \end{cases}$$

$$\Sigma_2 \begin{cases} h_t^{\text{ev}} + v_x^{\text{od}} = -(h^{\text{ev}} v^{\text{od}} + h^{\text{od}} v^{\text{ev}})_x, \\ v_t^{\text{od}}\,(t,x) + h_x^{\text{ev}} = -\frac{1}{2}((v^{\text{ev}})^2 + (v^{\text{od}})^2)_x, \\ v^{\text{od}}(t,0) = v^{\text{od}}(t,1) = 0. \end{cases}$$

Toy model (continued)

$$\mathcal{T}\left\{\begin{array}{l}\mathcal{T}_1\left\{\begin{array}{l}\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\\ \dot{s} = u,\ \dot{D} = s,\end{array}\right.\\ \mathcal{T}_2\left\{\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1y_2,\right.\end{array}\right.$$

where the state is $y = (y_1, y_2, y_3, y_4, s, D)^{\mathrm{tr}} \in \mathbb{R}^6$ and the control is $u \in \mathbb{R}$. The linearized control system of $\mathcal{T}$ around $(y_e, u_e) := (0, 0)$ is

$$\dot{y}_1 = \dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ \dot{s} = u,\ \dot{D} = s,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3,$$

which is not controllable.

Controllability of the toy model

If $y(0) = 0$,

$$y_3(T) = \int_0^T y_1^2(t)\cos(T-t)dt,$$
$$y_4(T) = y_1^2(T) - \int_0^T y_1^2(t)\sin(T-t)dt.$$

Hence $\mathcal{T}$ is not controllable in time $T \leqslant \pi$. Using explicit computations one can show that $\mathcal{T}$ is (locally) controllable in time $T > \pi$.

Remark

For linear systems in finite dimension, the controllability in large time implies the controllability in small time. This is no longer for linear PDE. This is also no longer true for nonlinear systems in finite dimension.

How to recover the large time local controllability of $\mathcal{T}$

We forget about S and D for simplicity and try to use the return method. The first point is at least to find a trajectory such that the linearized control system around it is controllable. We try the simplest possible trajectories, namely equilibrium points. Let $\gamma \in \mathbb{R}$ and define

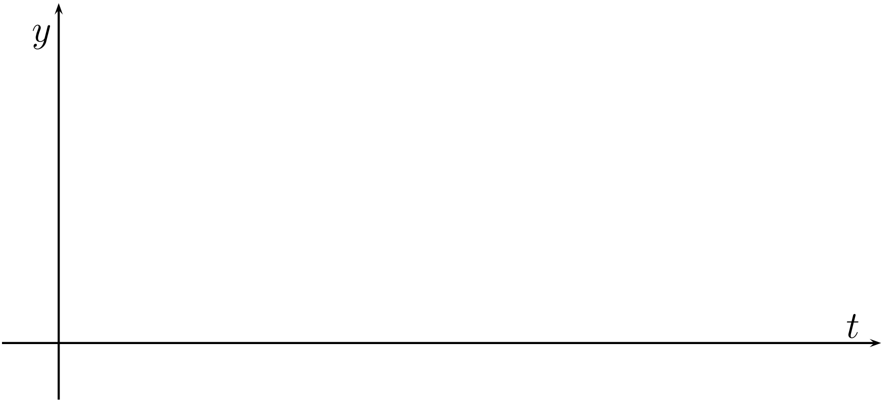
$$((y_1^\gamma, y_2^\gamma, y_3^\gamma, y_4^\gamma)^{\text{tr}}, u^\gamma) := ((\gamma, 0, 0, 0)^{\text{tr}}, \gamma).$$

Then $((y_1^\gamma, y_2^\gamma, y_3^\gamma, y_4^\gamma)^{\text{tr}}, u^\gamma)$ is an equilibrium of $\mathcal{T}$. The linearized control system of $\mathcal{T}$ at this equilibrium is

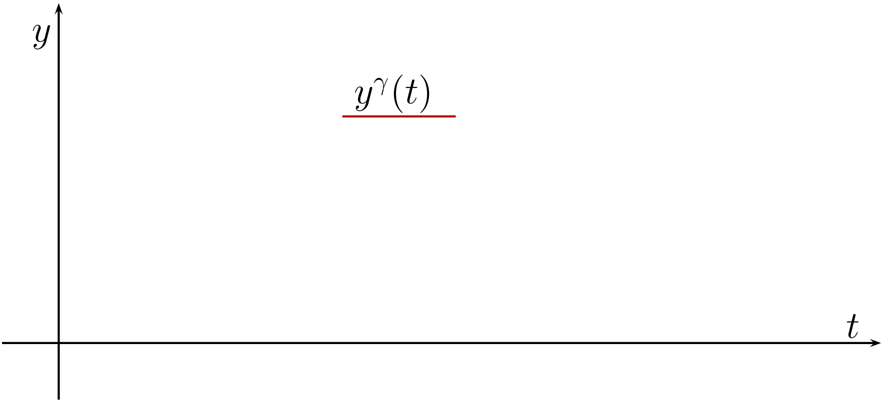
$$\dot{y}_1 = \dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2\gamma y_2,$$

which is controllable if (and only if) $\gamma \neq 0$.

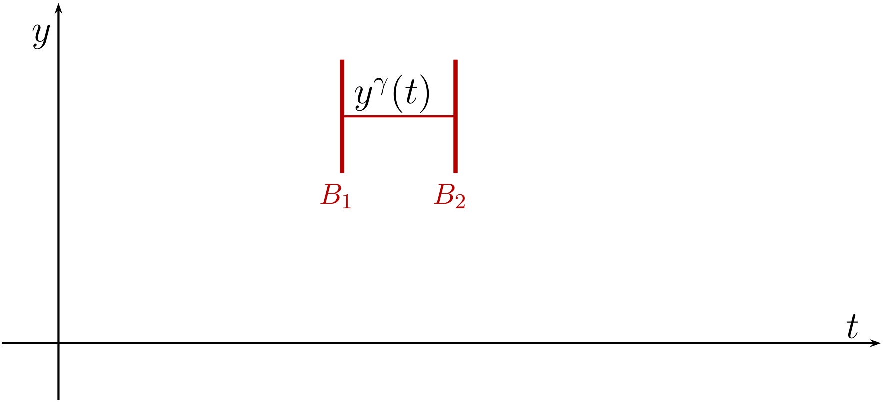
How to recover the large time local controllability of $\mathcal{T}$



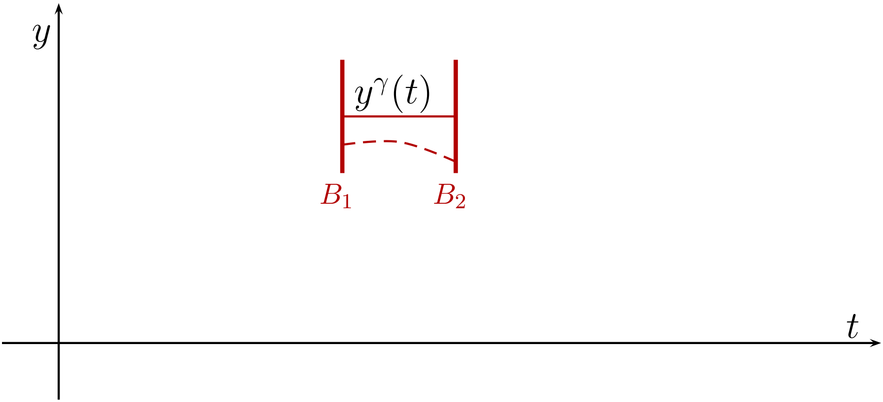
How to recover the large time local controllability of $\mathcal{T}$



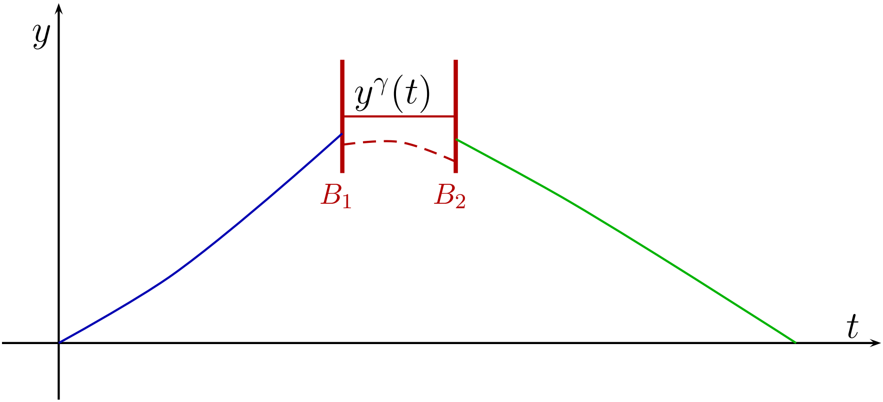
How to recover the large time local controllability of $\mathcal{T}$



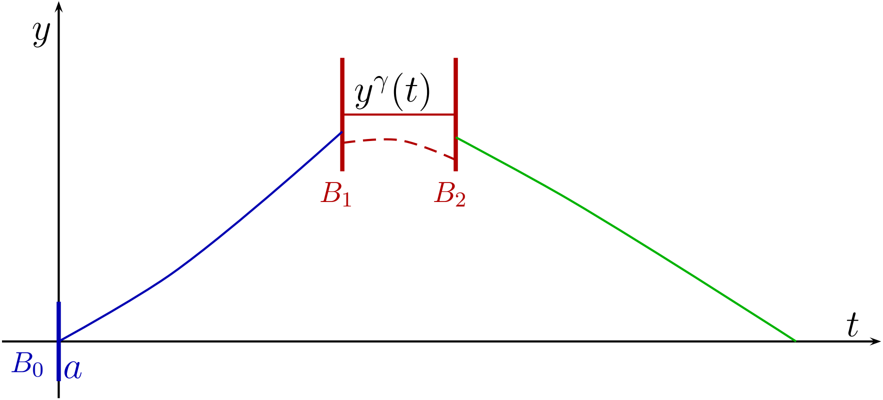
How to recover the large time local controllability of $\mathcal{T}$



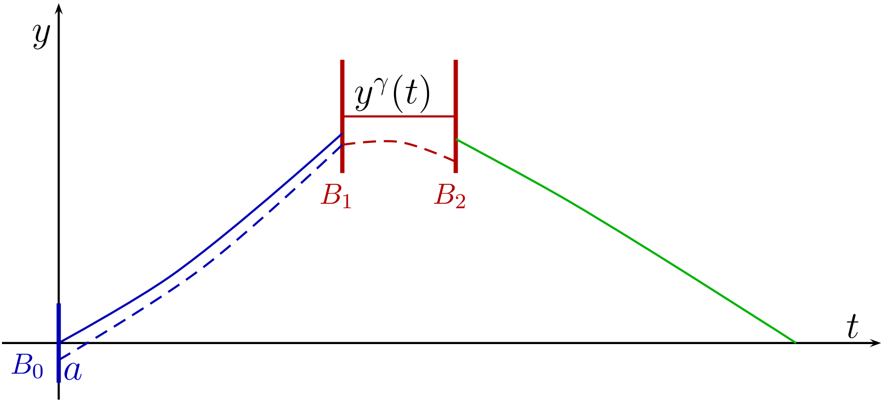
How to recover the large time local controllability of $\mathcal{T}$



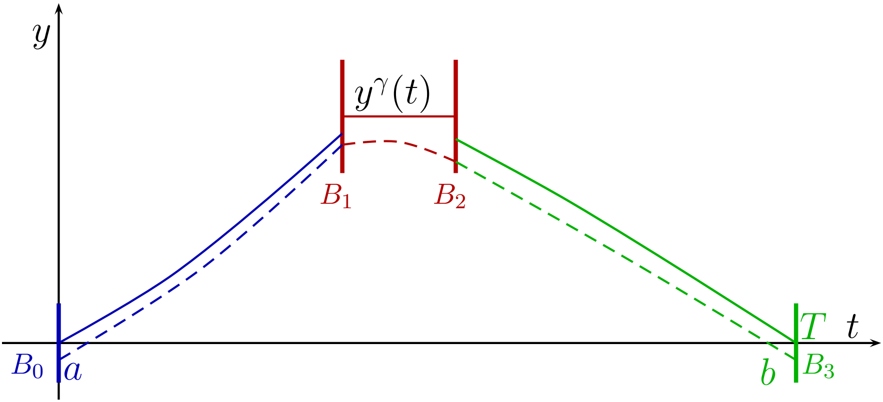
How to recover the large time local controllability of $\mathcal{T}$



How to recover the large time local controllability of $\mathcal{T}$



How to recover the large time local controllability of $\mathcal{T}$



Construction of the blue trajectory

One uses quasi-static deformations. Let $g \in C^2([0,1];\mathbb{R})$ be such that

$$g(0) = 0,\ g(1) = 1.$$

Let $\tilde{u} : [0, 1/\varepsilon] \to \mathbb{R}$ be defined by

$$\tilde{u}(t) := \gamma g(\varepsilon t),\ t \in [0, 1/\varepsilon].$$

Let $\tilde{y} := (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)^{\text{tr}} : [0, 1/\varepsilon] \to \mathbb{R}^4$ be defined by requiring

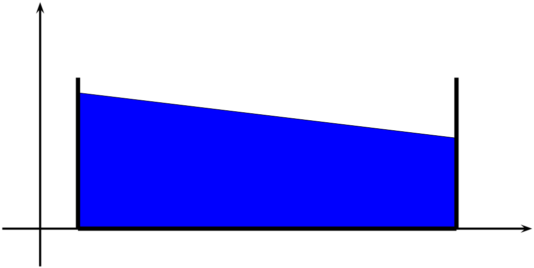
$$\dot{\tilde{y}}_1 = \tilde{y}_2,\ \dot{\tilde{y}}_2 = -\tilde{y}_1 + \tilde{u},\ \dot{\tilde{y}}_3 = \tilde{y}_4,\ \dot{\tilde{y}}_4 = -\tilde{y}_3 + 2\tilde{y}_1\tilde{y}_2,$$
$$\tilde{y}(0) = 0.$$

One easily checks that

$$\tilde{y}(1/\varepsilon) \to (\gamma, 0, 0, 0)^{\text{tr}} \text{ as } \varepsilon \to 0.$$

(y^γ, u^γ) for the water-tank

$$u(t) = \gamma,\ h = \gamma\left(\frac{1}{2} - y\right),\ v = 0.$$



Difficulties

Loss of derivatives. Solution: one uses the iterative scheme inspired by the usual one to prove the existence to $y_t + A(y)y_x = 0,\ y(0,x) = \varphi(x)$, namely

$$\Sigma_{n\text{linear}}: \quad y_t^{n+1} + A(y^n)y_x^{n+1} = 0,\ y^{n+1}(0,x) = \varphi(x).$$

However, I have only been able to prove that the control system corresponding to $\Sigma_{n\text{linear}}$ is controllable for (h^n, v^n) satisfying some resonance conditions. Hence one has also to insure that (h^{n+1}, v^{n+1}) satisfies these resonance conditions. This turns out to be possible. (For control system, resonance is good: when there is a resonance, with a small action we get a strong effect).

An open problem

What is the minimal time for the local controllability?

1. A simple observation on the speed of propagation shows that the time for local controllability is at least 1.
2. For the linearized control system around $h = \gamma\left((1/2) - x\right),\ v = 0$ the minimal time for controllability tends to 2 as $\gamma \to 0$.
3. For our toy model, there is no minimal time for the controllability around $((\gamma, 0, 0, 0)^{\text{tr}}, \gamma)$. However for the local the controllability of the nonlinear system the minimal time is $\pi > 0$.
4. For a related problem (a quantum particle in a moving box), there is again no minimal time for the controllability of the linearized control system around the analogue of $((\gamma, 0, 0, 0)^{\text{tr}}, \gamma)$ and there is a minimal time for the local controllability of the nonlinear system (JMC (2006)). Again the optimal time for the local controllability is not known.

Other references for quasi-static deformations

1. Semilinear heat equations: JMC and E. Trélat (2004),
2. Navier-Stokes equations for incompressible fluids: by M. Schmidt and E. Trélat (2006),
3. A quantum particle in a moving box: K. Beauchard (2005), K. Beauchard and JMC (2006),
4. Semilinear wave equations: JMC and E. Trélat (2006).

Power series expansions: The KdV control system

$$y_t + y_x + y_{xxx} + yy_x = 0,\ t \in [0,T],\ x \in [0,L],$$
$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T].$$

where, at time $t \in [0,T]$, the control is $u \in \mathbb{R}$ and the state is $y(t,\cdot) \in L^2(0,L)$.

Remark

Prior pioneer work on the controllability of the Korteweg-de Vries equation (with periodic boundary conditions and internal controls): D. Russell and B.-Y. Zhang (1996).

Controllability of the linearized control system

The linearized control system (around 0) is

$$y_t + y_x + y_{xxx} = 0,\ t \in [0,T],\ x \in [0,L],$$
$$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T].$$

where, at time $t \in [0,T]$, the control is $u \in \mathbb{R}$ and the state is $y(t,\cdot) \in L^2(0,L)$.

Theorem (L. Rosier (1997))

For every $T > 0$, the linearized control system is controllable in time T if and only

$$L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}},\ k \in \mathbb{N}^*,\ l \in \mathbb{N}^* \right\}.$$

Application to the nonlinear system

Theorem (L. Rosier (1997))

For every $T > 0$, the KdV control system is locally controllable (around 0) in time T if $L \notin \mathcal{N}$.

Question: Does one have controllability if $L \in \mathcal{N}$?

Controllability when $L \in \mathcal{N}$

Theorem (JMC and E. Crépeau (2004))

If $L = 2\pi$ (which is in $\mathcal{N}$: take $k = l = 1$), for every $T > 0$ the KdV control system is locally controllable (around 0) in time T.

Theorem (E. Cerpa (2007), E. Cerpa and E. Crépeau (2008))

For every $L \in \mathcal{N}$, there exists $T > 0$ such that the KdV control system is locally controllable (around 0) in time T.

Strategy of the proof: power series expansion.

Example with $L = 2\pi$. For every trajectory (y, u) of the linearized control system around 0

$$\frac{d}{dt}\int_0^{2\pi}(1-\cos(x))y(t,x)dx = 0.$$

This is is the only "obstacle" to the controllability of the linearized control system:

Proposition (L. Rosier (1997))

Let $H := \{y \in L^2(0,L);\ \int_0^L (1-\cos(x))y(x)dx = 0\}$. *For every* $(y^0, y^1) \in H \times H$, *there exists* $u \in L^2(0,T)$ *such that the solution to the Cauchy problem*

$$y_t + y_x + y_{xxx} = 0,\ y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T],$$
$$y(0,x) = y^0(x),\ x \in [0,L],$$

satisfies $y(T,x) = y^1(x)$, $x \in [0,L]$.

We explain the method on the control system of finite dimension

$$\dot{y} = f(y, u),$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. We assume that $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium of the control system $\dot{y} = f(y, u)$, i.e. that $f(0, 0) = 0$. Let

$$H := \text{Span}\,\{A^i Bu;\, u \in \mathbb{R}^m,\, i \in \{0, \ldots, n-1\}\}$$

with

$$A := \frac{\partial f}{\partial y}(0, 0),\; B := \frac{\partial f}{\partial u}(0, 0).$$

If $H = \mathbb{R}^n$, the linearized control system around $(0, 0)$ is controllable and therefore the nonlinear control system $\dot{y} = f(y, u)$ is small-time locally controllable at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$.

Let us look at the case where the dimension of H is $n-1$. Let us make a (formal) power series expansion of the control system $\dot{y}=f(y,u)$ in (y,u) around 0. We write

$$y=y^1+y^2+\dots,\ u=v^1+v^2+\dots.$$

The order 1 is given by (y^1,v^1); the order 2 is given by (y^2,v^2) and so on. The dynamics of these different orders are given by

$$\dot{y}^1=\frac{\partial f}{\partial y}(0,0)y^1+\frac{\partial f}{\partial u}(0,0)v^1,$$

$$\dot{y}^2=\frac{\partial f}{\partial y}(0,0)y^2+\frac{\partial f}{\partial u}(0,0)v^2+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)(y^1,y^1)$$
$$+\frac{\partial^2 f}{\partial y\partial u}(0,0)(y^1,v^1)+\frac{1}{2}\frac{\partial^2 f}{\partial u^2}(0,0)(v^1,v^1),$$

and so on.

Let $e_1 \in H^{\perp}$. Let $T > 0$. Let us assume that there are controls $v^1_{\pm}$ and $v^2_{\pm}$, both in $L^{\infty}((0,T);\mathbb{R}^m)$, such that, if $y^1_{\pm}$ and $y^2_{\pm}$ are solutions of

$$\dot{y}^1_{\pm} = \frac{\partial f}{\partial y}(0,0)y^1_{\pm} + \frac{\partial f}{\partial u}(0,0)v^1_{\pm},$$
$$y^1_{\pm}(0) = 0,$$

$$\dot{y}^2_{\pm} = \frac{\partial f}{\partial y}(0,0)y^2_{\pm} + \frac{\partial f}{\partial u}(0,0)v^2_{\pm} + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)(y^1_{\pm},y^1_{\pm})$$
$$+ \frac{\partial^2 f}{\partial y \partial u}(0,0)(y^1_{\pm},u^1_{\pm}) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(0,0)(u^1_{\pm},u^1_{\pm}),$$

$$y^2_{\pm}(0) = 0,$$

then

$$y^1_{\pm}(T) = 0,$$
$$y^2_{\pm}(T) = \pm e_1.$$

Let $(e_i)_{i\in\{2,...n\}}$ be a basis of H. By the definition of H, there are $(u_i)_{i=2,...,n}$, all in $L^\infty(0,T)^m$, such that, if $(y_i)_{i=2,...,n}$ are the solutions of

$$\dot{y}_i = \frac{\partial f}{\partial y}(0,0)y_i + \frac{\partial f}{\partial u}(0,0)u_i,$$
$$y_i(0) = 0,$$

then, for every $i \in \{2,\ldots,n\}$,

$$y_i(T) = e_i.$$

Now let

$$b = \sum_{i=1}^{n} b_i e_i$$

be a point in $\mathbb{R}^n$. Let v^1 and v^2, both in $L^\infty((0,T);\mathbb{R}^m)$, be defined by the following

- If $b_1 \geqslant 0$, then $v^1 := v^1_+$ and $v^2 := v^2_+$.
- If $b_1 < 0$, then $v^1 := v^1_-$ and $v^2 := v^2_-$.

Let $u:(0,T)\to\mathbb{R}^m$ be defined by

$$u(t) := |b_1|^{1/2}v^1(t) + |b_1|v^2(t) + \sum_{i=2}^{n} b_i u_i(t).$$

Let $y:[0,T]\to\mathbb{R}^n$ be the solution of

$$\dot{y} = f(y,u(t)),\ y(0)=0.$$

Then one has, as $b\to 0$,

$$y(T) = b + o(b).$$

Hence, using the Brouwer fixed-point theorem and standard estimates on ordinary differential equations, one gets the local controllability of $\dot{y}=f(y,u)$ (around $(0,0)\in\mathbb{R}^n\times\mathbb{R}^m$) in time T, that is, for every $\varepsilon>0$, there exists $\eta>0$ such that, for every $(a,b)\in\mathbb{R}^n\times\mathbb{R}^n$ with $|a|<\eta$ and $|b|<\eta$, there exists a trajectory $(y,u):[0,T]\to\mathbb{R}^n\times\mathbb{R}^m$ of the control system $\dot{y}=f(y,u)$ such that

$$y(0)=a,\ y(T)=b,$$
$$|u(t)|\leqslant\varepsilon,\ t\in(0,T).$$

Bad and good news for $L = 2\pi$

- Bad news: The order 2 is not sufficient. One needs to go to the order 3
- Good news: the fact that the order is odd allows to get the local controllability in arbitrary small time. The reason: If one can move in the direction $\xi \in H^{\perp}$ one can move in the direction $-\xi$. Hence it suffices to argue by contradiction (assume that it is impossible to enter in $H^{\perp}$ in small time etc.)

Open problems

Open problems

1. Is there a minimal time for local controllability for some values of L?
2. Do we have global controllability? This is open even with three boundary controls:

$$y_t + y_x + y_{xxx} + yy_x = 0,$$
$$y(t,0) = u_1(t),\ y(t,L) = u_2(t),\ y_x(t,L) = u_3(t).$$

Note that one has global controllability for

$$y_t + y_x + y_{xxx} + yy_x = u_4(t),$$
$$y(t,0) = u_1(t),\ y(t,L) = u_2(t),\ y_x(t,L) = u_3(t).$$

(M. Chapouly (2009)). The proof uses the return method as for the Navier-Stokes control system.