Fluid Mechanics

From lectures given by
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Contents

Editors' note

Welcome to fluids! Fluid mechanics is an exciting area of applied mathematics—after all, liquids and gases are pretty complicated—and there's a lot to find out. So, is it true that water goes down the plughole with the opposite spin in the Southern Hemisphere? How fast do waves move on the surface of water? How do planes fly? It is possible to give quantitative answers to such questions (as you would hopefully expect in the latter csae): this course deals with the simplest cases of fluid motion and is the foundation of more advanced study.

The good news if you struggled with applied maths in the first year is that there is very little in the way of Newtonian mechanics here. You won't be resolving in different directions all the time, and you won't have to memorise what $\mathbf{F} = m\mathbf{a}$ looks like in polar coordinates. Fluids is a newer subject and so we use newer maths: for example, we'll be seeing some complex numbers popping up. Have a little patience, if we can ask such a thing, while you go through the course. Sometimes you'll be wondering why we're going in a particular direction, but by Christmas the motivation should be more clear.

These notes were taken primarily from the MATH2301 course as taught by Professor Johnson in October–December 2009, but have had influence from different years' versions too. The notes are generally complete, and we've added in explanation in some important places, but there may be sections which you feel lack sufficiency. That's OK: that's why you go to the lectures! Also, at the end of the document, there are some extra practice sheets from bygone years, as well as their solutions.

If you notice any typos or mistakes (and it's very possible there are some, typesetting is a laborious task), please do let us know by emailing us at `adam@adamtownsend.com` and we'll fix it as soon as we can. You may find later versions of this document online, so check the cover page to see which version you're holding.

All the best,

GWG AKT
London, March 2012

Chapter 1

Specification and kinematics

The first question we will address is how we can use the mathematics that we know to describe the motion of a fluid. We make three assumptions in this course.

1.1 Assumptions

1.1.1 The continuum model

A *continuum* is, loosely speaking, a 'substance' that we take arbitrarily small volumes of, without changing its properties. That is to say, we can take a small volume δV and take the limit as it goes to zero.

We can now define properties of the fluid at a point $\mathbf{r} \in \delta V$ and time t, for example, its density ρ:

$$\rho(\mathbf{r}, t) = \lim_{\delta V \to 0} \left(\frac{\delta M}{\delta V} \right),$$

where δM is the mass enclosed by δV.

In reality, fluids are made up of atoms and molecules. The continuum model is therefore only a good approximation on scales greater than the mean free path—the mean distance travelled by molecules between collisions. In air at room temperature, the mean free path is about 68nm. In a liquid, it's about 0.3nm, so on the whole this is a good approximation! But we do have to be careful for rarefied (low density) gases.

1.1.2 The fluid is inviscid

A fluid is *inviscid* if it cannot support a *shear stress*—that is to say, it is not *viscous*. An inviscid fluid does not 'know' about the stress applied by the rough surface. In other words, the fluid flow along a boundary (wall) is not affected by the boundary.

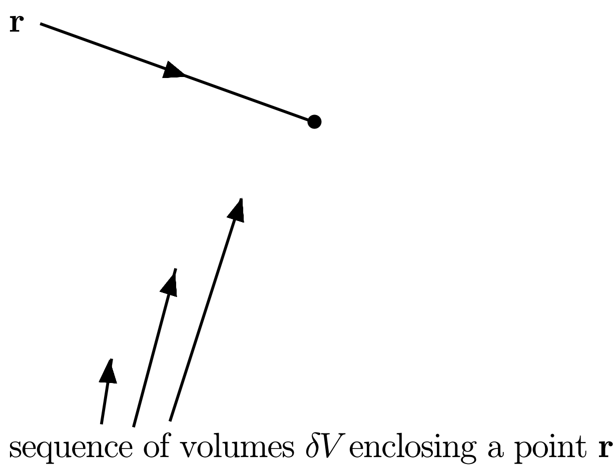

Figure 1.1: $\lim_{\delta V \to 0}$ exists

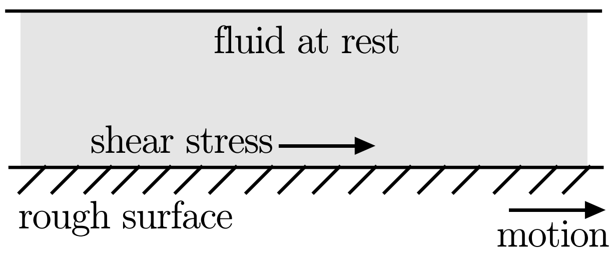

Figure 1.2: Inviscid fluids cannot support shear stress.

You should think of this as similar to an assumption that two objects move against each other without friction in solid mechanics.

In reality, all fluids are viscous. The inviscid approximation is good where the flow is fast, on large scales, and where the viscosity of the fluid is slow. Fluid dynamicists phrase this by saying

$$\mathrm{Re} := \frac{UL}{\nu} \gg 1,$$

where U is the flow speed, L is the flow scale and ν is the viscosity of the fluid. Re is the *Reynolds number*: this number becomes very important if you continue your study of fluids after this course (and we hope you do!).

1.1.3 The fluid is incompressible

A fluid is *incompressible* if the volume of a fluid element remains the same throughout the fluid motion. A fluid element conserves mass (as mass cannot be created or destroyed) and therefore conserves its density (density is not necessarily constant everywhere in the fluid, mind you, since different fluid elements can have different densities).

Figure 1.3: Lagrangian description of motion

In reality, all fluids are *compressible* (gases much more so than liquids). The incompressible model is good for flows which are slower than the speed of sound c, i.e.

$$U \ll c,$$

where U is the flow speed. We can write this in terms of the Mach number M,

$$M = \frac{U}{c} \ll 1.$$

The speed of sound in air is about $340\,\mathrm{m/s}$ and in water about $1500\,\mathrm{m/s}$, so again, this approximation is good for what we shall be looking at. Of course, sound wouldn't work with this model.

1.2 Describing fluid motion

Now we look at the two main ways of describing the motion of a fluid

1.2.1 Lagrangian description

In the Lagrangian description, all particles are labelled by their position $\mathbf{r}_0$ at some fixed time t_0. Subsequent positions are given by $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$.

Strengths: Conservation properties are easy, for example density $\rho = \rho(\mathbf{r}_0)$ at all times t as density is conserved by each particle.

Weaknesses: Simple motions can have complicated particle paths. It's hard to calculate spatial gradients of, for example, velocity fields at later times.

1.2.2 Eulerian description

In the Eulerian description, the motion of the fluid is described by a velocity vector field $\mathbf{u} = \mathbf{u}(x, y, z, t)$ with respect to a fixed frame.

Strengths: We can use vector calculus on the velocity $\mathbf{u}$.

Weaknesses: Conservation laws are more complicated. For example, the density $\rho = \rho(x, y, z, t)$ is not constant with time t for fixed x, y, z. This is because different fluid elements may flow through a point (x, y, z).

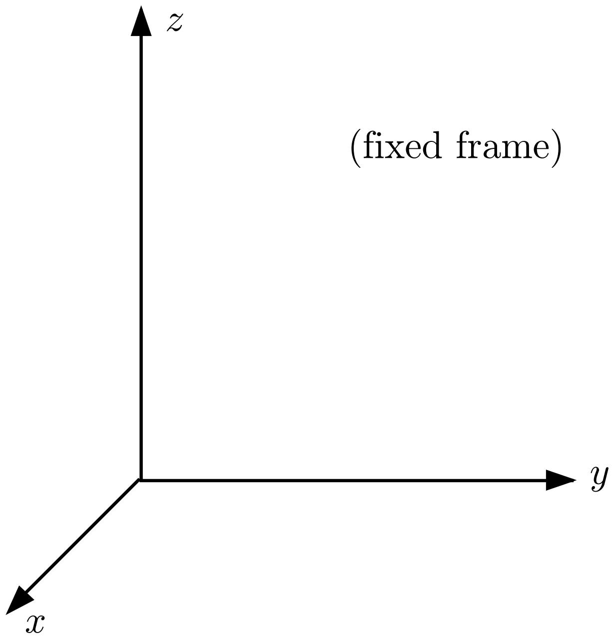

Figure 1.4: A fixed frame (x, y, z)

1.3 Visualising fluid flow

There are three methods of visualising fluid flow. You'd be well-advised to know these definitions for the exam.

1.3.1 Particle paths

Definition 1.1 The *particle path* is the path of a fluid particle over a given time interval.

To find the particle's path, we need to solve the ordinary differential equations given by

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t); \qquad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \qquad \mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

In 3D Cartesian components this is solving

$$\frac{dx}{dt} = u; \quad \frac{dy}{dt} = v; \quad \frac{dz}{dt} = w.$$

Example 1.2 Consider the flow

$$\mathbf{u} = \widehat{\mathbf{\imath}} - 2te^{-t^2}\widehat{\mathbf{\jmath}}.$$

This flow is two-dimensional and is spatially uniform (i.e. has no x, y-dependence).

Find the particle path for a particle released at $(1, 1)$ at $t = 0$

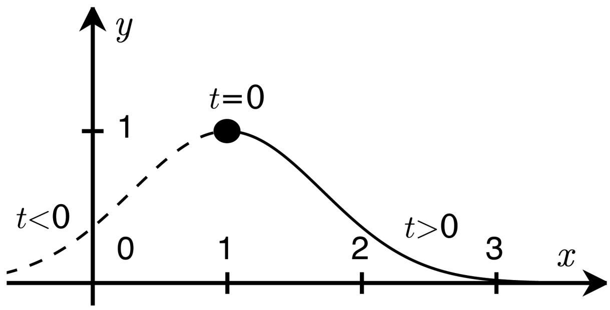

Figure 1.5: Particle path for this flow

Solution We need to solve for $\mathbf{r}$ given $\frac{d\mathbf{r}}{dt}$, i.e. solve

$$\frac{dx}{dt} = u(x, y, t) = 1; \qquad \frac{dy}{dt} = v(x, y, t) = -2te^{-t^2}$$

So we integrate to get

$$x(t) = t + x_0; \qquad y(t) = e^{-t^2} + y_0$$

and when we look at our initial condition, that $t = 0$ at $x = 1, y = 1$, we get that $x_0 = 1, y_0 = 0$ hence

$$x(t) = t + 1; \qquad y(t) = e^{-t^2}.$$

In general, particle paths are parameterised by time t, i.e. $\mathbf{r} = \mathbf{r}(t)$. Here however, we can rearrange and get, see figure 1.5

$$y = e^{-(x-1)^2}.$$

✓

1.3.2 Streaklines

Definition 1.3 A *streakline* is the locus of all particles passing a fixed point $\mathbf{r}_0$, during a given time interval.

To find the streakline at a time for particles released from (passing through) a point $\mathbf{r}_0$, we name the release time τ (where τ lies in some given interval) and solve

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t); \quad \mathbf{r}(\tau) = \mathbf{r}_0.$$

Example 1.4 Consider the same flow

$$\mathbf{u} = \hat{\imath} - 2te^{-t^2}\hat{\jmath}.$$

Find the streakline at $t = 0$ for particles released from $(1, 1)$ at some $t < 0$.

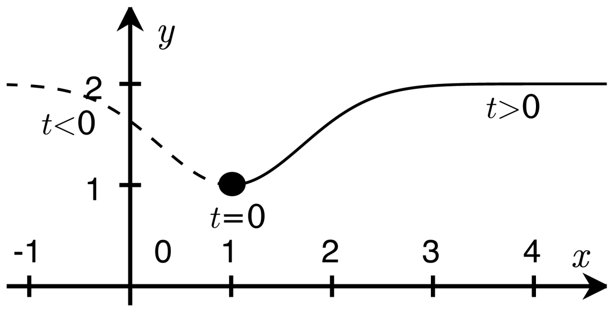

Figure 1.6: Plot of the Streakline for this flow

Solution Once again we solve

$$\frac{dx}{dt} = u(x, y, t) = 1; \qquad \frac{dy}{dt} = v(x, y, t) = -2te^{-t^2}$$

So we integrate to get

$$x(t) = t + x_0; \qquad y(t) = e^{-t^2} + y_0$$

and we use the initial conditions $x(\tau) = 1, y(\tau) = 1$ to get

$$x_0 = 1 - \tau; \qquad y_0 = 1 - e^{-\tau^2}$$

and so

$$x(t) = t + (1 - \tau); \qquad y(t) = e^{-t^2} + (1 - e^{-\tau^2}).$$

Set $t = 0$ to get the streakline at $t = 0$, and get

$$x(0) = 1 - \tau; \qquad y(0) = 2 - e^{-\tau^2}.$$

Streaklines are parameterised by release time, i.e. $\mathbf{r} = \mathbf{r}(\tau)$. Here we can rearrange

$$\tau = 1 - x \implies y(x) = 2 - e^{-(1-x)^2}$$

which is something we can then plot, see Figure 1.6. ✓

1.3.3 Streamlines

Definition 1.5 A *streamline* is a line parallel to the velocity vector $\mathbf{u}$ at a fixed time t_0.

To find the streamline, solve at a fixed time $t = t_0$,

$$\frac{d\mathbf{r}}{ds} = \mathbf{u}(\mathbf{r}, t_0)$$

where s is a parameter which varies along the streamline. We solve this to get $\mathbf{r} = \mathbf{r}(s)$.

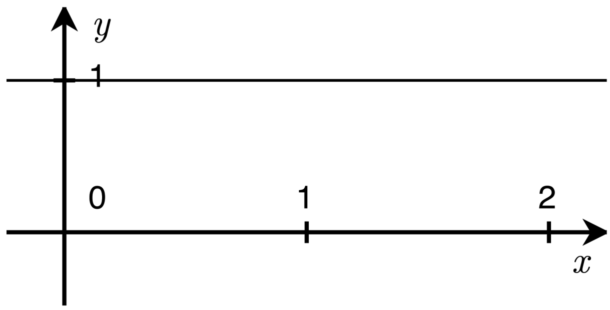

Figure 1.7: Plot of the Streamline for this flow

Example 1.6 Consider the same flow

$$\mathbf{u} = \hat{\imath} - 2te^{-t^2}\hat{\jmath}.$$

Find the streamline through $(1,1)$ at $t = 0$.

Solution At $t = 0$, the velocity $\mathbf{u} = \hat{\imath}$, i.e.

$$\frac{dx}{ds} = 1; \qquad \frac{dy}{ds} = 0.$$

Integrating ds we get

$$x = s + x_0; \qquad y = y_0.$$

We want the streamline through $(1,1)$. Say that at this point, our parameter $s = 0$. Therefore

$$x_0 = 1; \qquad y_0 = 1.$$

So we end up with

$$x = s + 1; \qquad y = 1$$

which tells us x goes from $-\infty$ to ∞ (not particularly interesting) and that y is constant at 1. We can plot this, see Figure 1.7. ✓

The particle path, streakline and streamline can all be different, or they could all be the same if the flow is *steady*—i.e. $\mathbf{u}$ has no time dependence, $\frac{d\mathbf{u}}{dt} = 0$.

Exercise 1.7 Show that the particle path, streakline and streamline of the same flow

$$\mathbf{u} = \hat{\imath} - 2te^{-t^2}\hat{\jmath}.$$

are all the same if the flow is steady.

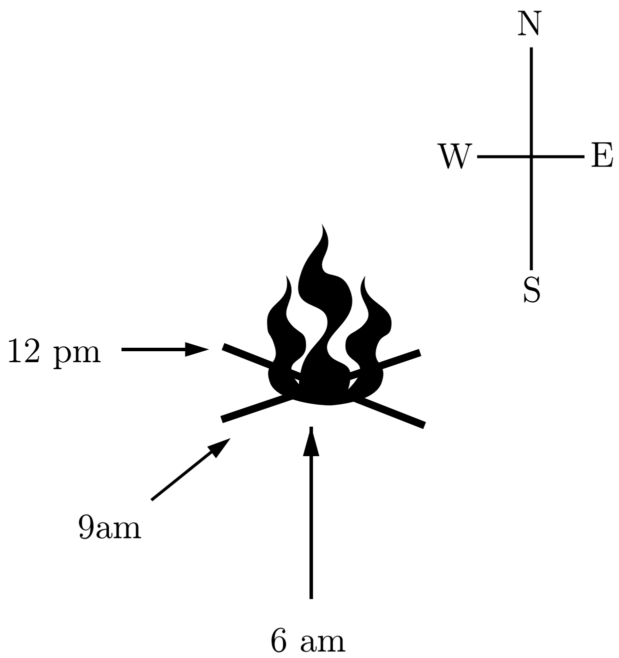

Figure 1.8: Image of the funeral

Example 1.8 A Viking funeral takes place early in the morning on a boat in the middle of a large lake, where the boat is anchored to be stationary and smoke emits from the funeral pyre. The wind blows from the south at 6am and veers round to be blowing from the west by 12 noon (see Figure 1.8). Sketch:

(i) The particle path of a smoke particle released at 6am,

(ii) The streamline at 9am,

(iii) The streakline at midday.

Solution See Figure 1.9. ✓

1.4 Incompressibility

The idea of incompressibility is obvious, we cannot compress the fluid: imagine water as opposed to air. More rigorously, this means that the volume of fluid elements cannot change or equivalently the flux of the fluid into any volume V must equal the flux out (we cannot fit any more fluid into a volume without taking some out).

Consider a fluid of constant density ρ flowing steadily though a pipe with upstream cross sectional area A_1 and velocity $u_1\,\widehat{\mathbf{\imath}}$ and downstream cross sectional area A_2 and velocity $u_2\,\widehat{\mathbf{\imath}}$.

In a small time δt upstream, the volume of fluid entering V is $(u_1\delta t)A_1$ and the volume of fluid leaving V downstream is $(u_2\delta t)A_1$.

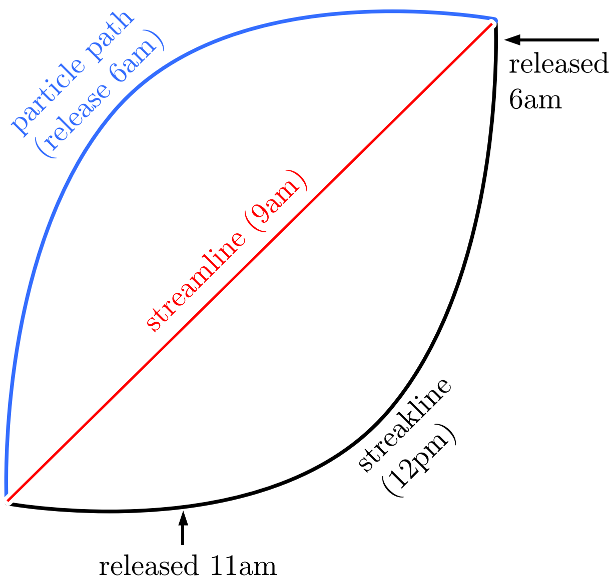

Figure 1.9: Solution to Exercise 1.8

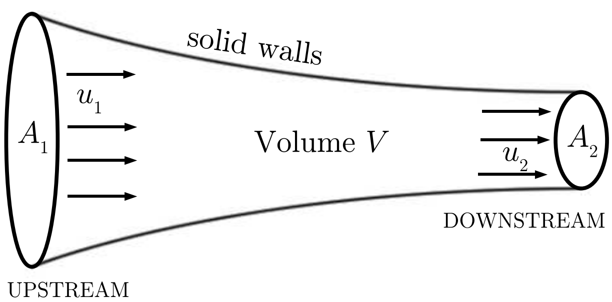

Figure 1.10: Fluid of constant density flows through this pipe

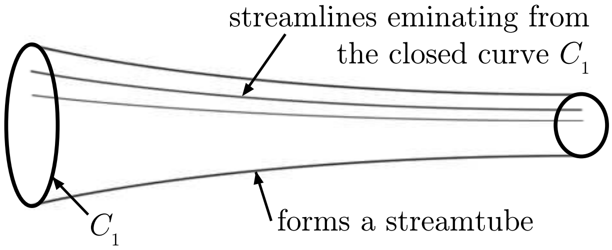

Figure 1.11: Streamtube

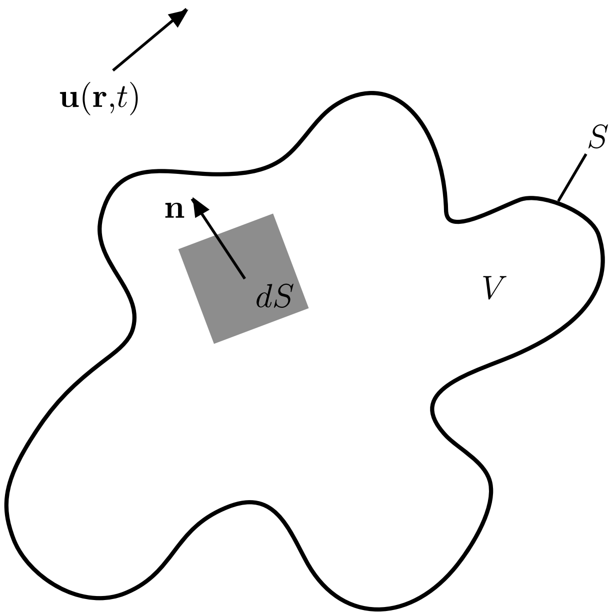

Figure 1.12: A piece of surface

Since the fluid is incompressible, these values must be equal, i.e.

$$u_1 A_1 = u_2 A_2$$
$$\frac{u_1}{u_2} = \left(\frac{A_1}{A_2}\right)^{-1}$$

So we see that velocity in a tube is inversely proportional to the cross-sectional area of the tube.

$$u \propto \frac{1}{A}$$

The same can be said of a steady flow with streamlines that emanate from a closed curve, a so called streamtube, see Figure 1.11.

1.4.1 Conservation of Mass for a Fixed Density Fluid

Consider a fluid of constant density flowing in a domain D. Let V be a volume in said domain with surface S, as in Figure 1.12.

We consider an infinitesimal area element dS. The volume of fluid flowing out of V through said element in a time δt is:

$$(\mathbf{u} \cdot \mathbf{n})\, \delta t\, dS$$

And so the total volume of fluid flowing out of V in a time δt is:

$$\delta t \iint_S \mathbf{u} \cdot \mathbf{n}\, dS$$

Since the fluid is incompressible and of constant density, the volume of fluid in V must remain unchanged, i.e.

$$\delta t \iint_S \mathbf{u} \cdot \mathbf{n}\, dS = 0$$

And by the divergence theorem (which you should recall from MATH1402, but can be found in Appendix A):

$$\iint_S \mathbf{u} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{u}\, dV = 0$$

This holds for any volume $V \subset D$, as our choice of V was arbitrary to begin with, so providing that $\mathbf{u}$ is differentiable with continuous derivatives (we justify this step in Lemma 1.9):

$$\nabla \cdot \mathbf{u} = 0 \quad \text{Everywhere in } D$$

Key Result: In an incompressible fluid with constant density, the divergence of the velocity vector field is zero everywhere.

Lemma 1.9 Let $f(x)$ be a continuous function on $[a, b]$. If, for any $(c, d) \subset [a, b]$,

$$\int_c^d f(x)\, dx = 0$$

then $f(x) \equiv 0$ in $[a, b]$.

Proof (by contradiction): Assume that $\exists \alpha \in [a, b]$ with $f(\alpha) \neq 0$, we assume $f(\alpha) > 0$ and leave the case $f(\alpha) < 0$ as a simple exercise in adapting the following proof.

$f(x)$ is continuous so:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. } \forall x : |x - \alpha| < \delta, |f(x) - f(\alpha)| < \epsilon$$

Take $\epsilon = \frac{f(\alpha)}{2} > 0$ then $\exists \delta$ such that:

$$|f(x) - f(\alpha)| < \frac{f(\alpha)}{2} \quad \forall x \in (\alpha - \delta, \alpha + \delta)$$

$$\implies -\frac{f(\alpha)}{2} < f(x) - f(\alpha) < \frac{f(\alpha)}{2} \quad \forall x \in (\alpha - \delta, \alpha + \delta)$$

Keeping only the important part of this inequality

$$f(x) > \frac{f(\alpha)}{2} \quad \forall x \in (\alpha - \delta, \alpha + \delta)$$

$$\implies \int_{\alpha-\delta}^{\alpha+\delta} f(x)\,dx \geq \int_{\alpha-\delta}^{\alpha+\delta} \frac{f(\alpha)}{2}\,dx = \delta f(\alpha) > 0$$

And this is a contradiction, so $f(x) \equiv 0$ everywhere in $[a, b]$. ☐

In the three dimensional case, we want to show that if $\nabla \cdot \mathbf{u}$ is differentiable with continuous derivatives then:

$$\iiint_V \nabla \cdot \mathbf{u}\,dV = 0 \quad \forall V \subset D \implies \nabla \cdot \mathbf{u} = 0 \quad \text{Everywhere in D}$$

Proof (sketch): Assume $\exists \mathbf{r_0} \in D$ where $(\nabla \cdot \mathbf{u})|_{\mathbf{r}=\mathbf{r_0}} = c > 0$. Continuity of $\nabla \cdot \mathbf{u}$ implies that $\exists \delta > 0$ such that $\nabla \cdot \mathbf{u} > \frac{c}{2}$ everywhere in the ball centred at $\mathbf{r_0}$ of radius δ (this is shown similarly to the one dimensional case). Then:

$$\iiint_{V(\mathbf{r_0},\delta)} \nabla \cdot \mathbf{u}\,dV > \iiint_{V(\mathbf{r_0},\delta)} \frac{c}{2}\,dV = \frac{4\pi\delta^3}{3}\frac{c}{2} > 0$$

Which again contradicts our initial assumption, so that $\nabla \cdot \mathbf{u} = 0$ everywhere in D. ☐

1.5 Stream Function

Consider a two dimensional incompressible flow

$$\mathbf{u} = u(x, y)\,\hat{\imath} + v(x, y)\,\hat{\jmath}$$

The continuity equation means that:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

If we consider the vector field $\mathbf{F} = -v(x, y)\,\hat{\imath} + u(x, y)\,\hat{\jmath}$ and apply Green's Theorem in the plane to a closed curve C enclosing a region R (again you should recall this from 1402, but it can be found in Appendix A):

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C -v\,dx + u\,dx \\
&= \iint_R \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\,dx\,dx \\
&= \iint_R \nabla \cdot \mathbf{u}\,dx\,dx \\
&= 0
\end{aligned}$$

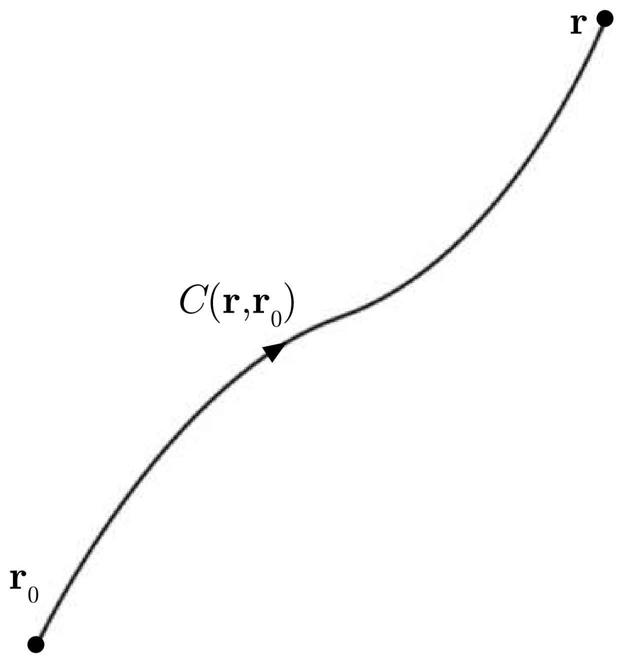

Figure 1.13: A curve C

Where $d\mathbf{r} = dx\,\widehat{\mathbf{\imath}} + dy\,\widehat{\mathbf{\jmath}}$. This means that $\mathbf{F}$ is a conservative vector field (as the integral over *any* closed curve C is zero). Recall that conservative vector fields can always be written as the gradient of a scalar field. Therefore there exists a $\psi = \psi(x, y)$ such that $\nabla\psi = \mathbf{F}$ everywhere, or in component form:

$$\frac{\partial\psi}{\partial x} = -v \quad \frac{\partial\psi}{\partial y} = u$$

ψ is known as the *Stream Function* and it would be in your best interests to memorise this definition now as we will use it repeatedly from now on.

We now begin an *Interlude*:

$\psi(\mathbf{r})$ can be defined (formally) as

$$\int_{C(\mathbf{r},\mathbf{r}_0)} \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{r}_0$ is any fixed point and $C(\mathbf{r}, \mathbf{r}_0)$ is any curve joining $\mathbf{r}$ and $\mathbf{r}_0$ (Figure 1.13).

This definition is sound, i.e. the choice of C does not affect the value of the integral since, for two curves C_1 and C_2, $C_1 - C_2$ is a closed path (Figure 1.14).

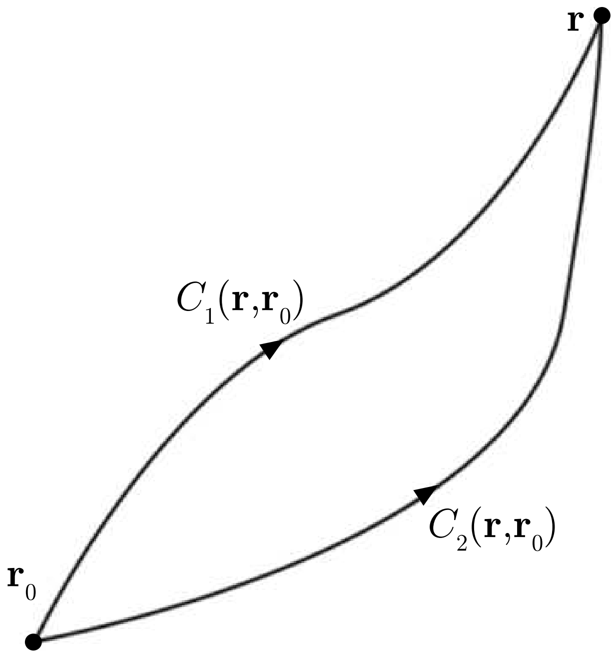

Figure 1.14: Two curves C_1 and C_2

So:

$$\oint_{C_1-C_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

$$\implies \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

The freedom of choice in our starting point for the curve C corresponds to the fact that ψ is defined only up to an arbitrary constant.

Thus concludes our *Interlude.*

Now we consider taking the vector product of $\nabla\psi = \mathbf{F} = -v\,\widehat{\imath} + u\,\widehat{\jmath}$ with $\widehat{\mathbf{k}}$:

$$\widehat{\mathbf{k}}\times\nabla\psi = \widehat{\mathbf{k}}\times(-v\,\widehat{\imath} + u\,\widehat{\jmath})$$

As $\widehat{\mathbf{k}}\times\widehat{\jmath} = -\widehat{\imath}$ and $\widehat{\mathbf{k}}\times\widehat{\imath} = \widehat{\jmath}$:

$$\begin{aligned} &= -u\,\widehat{\imath} - v\,\widehat{\jmath} \\ &= -\mathbf{u} \\ \implies \mathbf{u} &= -\widehat{\mathbf{k}}\times\nabla\psi \end{aligned}$$

So we see that:

$$\mathbf{u}\cdot\nabla\psi = (-\widehat{\mathbf{k}}\times\nabla\psi)\cdot\nabla\psi = 0$$

Note that $\nabla\psi$ is alway perpendicular to lines of constant ψ i.e. the level surfaces of ψ (this is a basic fact about the gradient operator). We have also just shown that $\nabla\psi$ is perpendicular to the velocity $\mathbf{u}$. So we see that the lines $\psi(x, y) = $ constant are parallel to $\mathbf{u}$, i.e they are *streamlines*.

Note that in fact ψ contains more information than just the streamlines as $-\widehat{\mathbf{k}}\times\nabla\psi = \mathbf{u}$.

In polar coordinates:

$$\begin{aligned}
\nabla\psi &= \frac{\partial\psi}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\widehat{\boldsymbol{\theta}} \\
\mathbf{u} &= u_r\widehat{\mathbf{r}} + u_\theta\widehat{\boldsymbol{\theta}} \\
\mathbf{u} &= -\widehat{\mathbf{k}}\times\nabla\psi \\
u_r\widehat{\mathbf{r}} + u_\theta\widehat{\boldsymbol{\theta}} &= \widehat{\mathbf{k}}\times\left[\frac{\partial\psi}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\widehat{\boldsymbol{\theta}}\right] \\
&= -\frac{\partial\psi}{\partial r}\widehat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\widehat{\mathbf{r}}
\end{aligned}$$

As $\widehat{\mathbf{k}}\times\widehat{\mathbf{r}} = \widehat{\boldsymbol{\theta}}$ and $\widehat{\mathbf{k}}\times\widehat{\boldsymbol{\theta}} = -\widehat{\mathbf{r}}$. And equating components we find that:

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = u_r, \qquad -\frac{\partial\psi}{\partial r} = u_\theta$$

Example 1.10 Show that $\mathbf{u} = x\widehat{\mathbf{\imath}} - y\widehat{\mathbf{\jmath}}$ is incompressible. Find ψ and sketch some of the streamlines. (Note that this flow is known as a *Uniform Strain Flow*).

Solution First we check incompressibility. $u = x$, $v = -y$, so $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1$ so that:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

And the fluid is therefore incompressible. Now we attempt to find ψ:

$$-\frac{\partial\psi}{\partial x} = v = -y \implies \frac{\partial\psi}{\partial x} = y \tag{1.1}$$

$$\frac{\partial\psi}{\partial y} = u = x \implies \frac{\partial\psi}{\partial y} = x \tag{1.2}$$

Integrating (1.1) we get that $\psi = xy + f(y)$ where f is an arbitrary function of y and integrating (1.2) we get $\psi = xy + g(x)$ where g is an arbitrary function of x.

Comparing these two, we find that $f(y) = g(x)$ and since the left hand side of this does not depend on y and the right hand side does not depend on x, we see that $f(y) = g(x) = $ const. And since the choice of this constant does not matter (ψ is only defined up to an a constant) we set it equal to 0 and we find that:

$$\psi(x, y) = xy$$

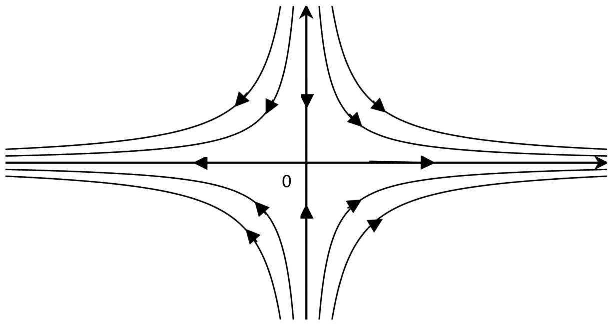

Figure 1.15: A plot of our streamlines for this flow

And the streamlines are lines where:

$$\psi = xy = c \text{ (const.)}$$

Note that the point $(0, 0)$ is a *Stagnation Point*, i.e. a point at which $\mathbf{u} = \mathbf{0}$, see Figure 1.15 for a plot of our streamlines. ✓

1.6 Inviscid Flow at a Solid Boundary

A solid boundary is a boundary that is impermeable to fluid flow, i.e. there is no flow through the boundary (see Figure 1.12).

If we consider the flow through each element dS on the above surface, we see that $\mathbf{u} \cdot \mathbf{n} = 0$ everywhere on S. This is known as the *No Normal Flow Condition*. In an inviscid fluid (i.e. for the rest of this course), the flow parallel to the boundary is unconstrained.

If you continue your study of fluid dynamics into the third year, in the Real Fluids course you will begin to study viscous fluids in which there is an different boundary condition at a solid boundary, that is the *No Flow Condition* under which $\mathbf{u} = 0$ on a solid boundary. (i.e. normal flow *and* tangential flow are 0).

In inviscid flows though, a solid boundary forms a streamline, since the flow is always parallel to the boundary (in three dimensions a solid surface is made up of streamlines). Any streamline in a two dimensional flow can be replaced by a streamline and the flow "would not know" (i.e. it would not be affected).

Example 1.11 For the flow $\mathbf{u} = 2y\hat{\mathbf{\imath}} - 2x\hat{\mathbf{\jmath}}$:

1. Show that the flow is incompressible
2. Find ψ
3. Sketch some streamlines
4. Find and sketch a solid body compatible with the flow

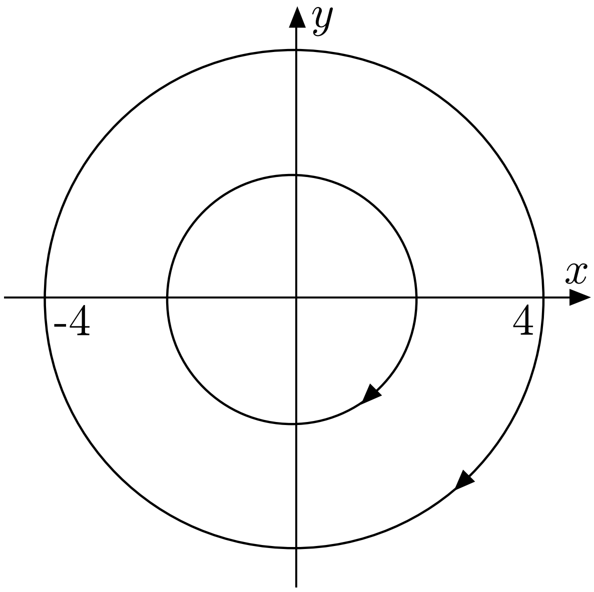

Figure 1.16: Plot of the streamlines of this flow

Solution First we show that the flow is incompressible. Note that $u = 2y$ and $v = -2x$, so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, so that:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

So the flow is incompressible. Now we attempt to find ψ:

$$\frac{\partial \psi}{\partial y} = 2y \tag{1.3}$$

$$-\frac{\partial \psi}{\partial x} = -2x \tag{1.4}$$

Integrating (1.3) we find that $\psi = y^2 + f(x)$ where f is an arbitrary function of x and integrating (1.4) we find that $\psi = x^2 + g(y)$ where g is an arbitrary function of y. Equating these, we find that $f(x) - x^2 = g(y) - y^2$ and as the left hand side does not depend on y and the right hand side does not depend on x:

$$f(x) - x^2 = g(y) - y^2 = c \text{ (const.)}$$

Since ψ is only defined up to a constant we set $c = 0$ and find that:

$$\psi = x^2 + y^2$$

And the streamlines are solutions to $\psi = x^2 + y^2 = \text{const}$ i.e. they are circles, this is known as *Solid Body Rotation Flow* (so called because it describes motion of a fluid that is stationary relative to a rotating bucket), see Figure 1.16 for a plot of the streamlines. ✓

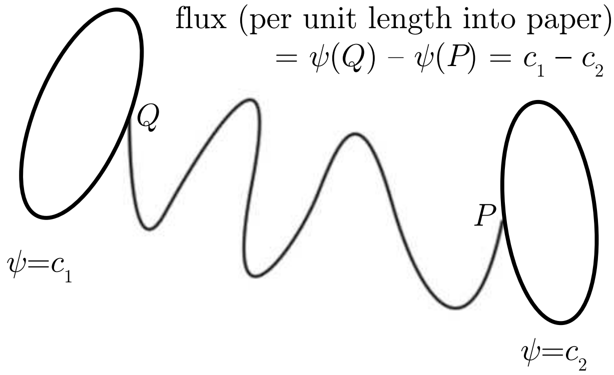

Figure 1.17: Flux between two points

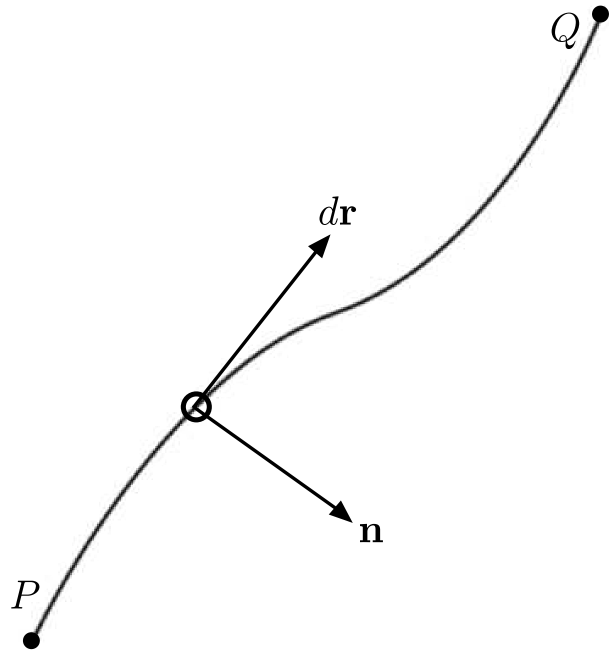

Figure 1.18: A directed line segment $d\mathbf{r}$ of the line joining P to Q

1.7 Physical Interpretation of the streamfunction

The volume flux in a clockwise direction across *any* line joining a point P to point Q in a flow field is given by $\psi(Q) - \psi(P)$ (see Figure 1.17).

Proof: Consider a directed line segment $d\mathbf{r}$ of the line joining P to Q (Figure 1.18), i.e.

$$d\mathbf{r} = dx\,\hat{\mathbf{\imath}} + dy\,\hat{\mathbf{\jmath}}.$$

Then it is clear that $\mathbf{n} = dy\,\hat{\mathbf{\imath}} - dx\,\hat{\mathbf{\jmath}}$ is normal to the curve $d\mathbf{r} \cdot \mathbf{n} = 0$, with length

$$ds = \sqrt{dx^2 + dy^2}.$$

So a unit normal is

$$\hat{\mathbf{n}} = \frac{dy}{ds}\hat{\mathbf{\imath}} - \frac{dx}{ds}\hat{\mathbf{\jmath}}$$

where positive $\hat{\mathbf{n}}$ crosses PQ in a clockwise direction.

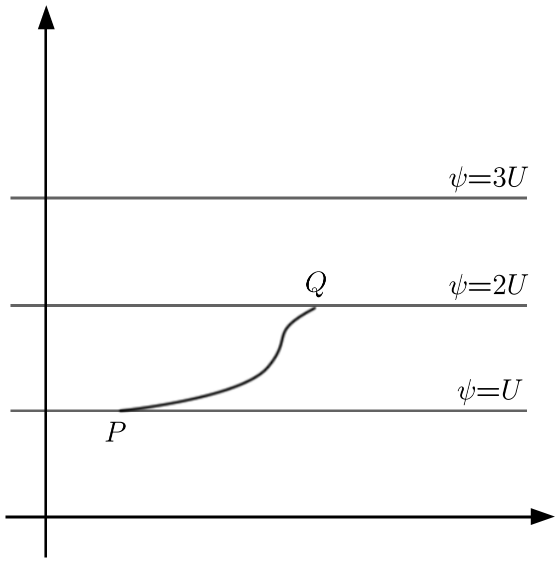

Figure 1.19: Flux over this line is U

The volume flux across an element ds of the line PQ is $(\mathbf{u}\cdot\widehat{\mathbf{n}})ds$ per unit distance into the paper, and has dimension [length^2 · time^{-1}].

We then calculate the total flux over PQ:

$$\begin{aligned}\int_P^Q (\mathbf{u}\cdot\widehat{\mathbf{n}})\,ds &= \int_P^Q \left(\frac{\partial\psi}{\partial y}\widehat{\imath} - \frac{\partial\psi}{\partial x}\widehat{\jmath}\right)\cdot\left(\frac{dy}{ds}\widehat{\imath} - \frac{dx}{ds}\widehat{\jmath}\right)ds \\ &= \int_P^Q \left(\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}\right)ds \\ &= \int_P^Q \frac{d\psi}{ds}\,ds \\ &= [\psi]_P^Q \\ &= \psi(Q) - \psi(P)\end{aligned}$$

i.e. it is path independent and we have conservation of mass. □

Example 1.12 Let us have a constant stream $\mathbf{u} = U\widehat{\imath}$. What is the flux between streamlines?

Solution We have $\mathbf{u} = U\widehat{\imath}$ so then

$$\psi_y = U; \qquad \psi_x = 0$$

which means $\psi = Uy$.

We have streamlines where ψ is constant, i.e. Uy is constant and hence $y = \text{const}$. If P is on the line $\psi = U$ and Q is on the line $\psi = 2U$ (as in Figure 1.19) then the flux over this line is simply $Q - P$, i.e. $2U - U = U$. You could have also worked this out as

$$\text{flux} = \text{speed} \times \text{width} = U \times 1 = U.$$

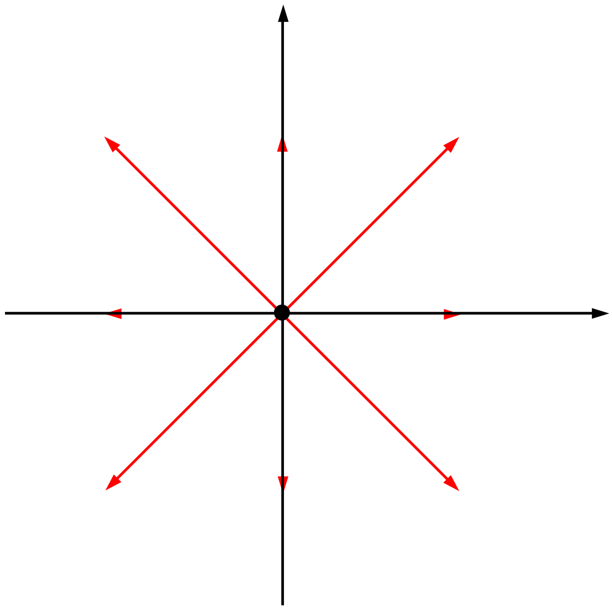

Figure 1.20: Isotropic source

✓

Example 1.13 Let us have an isotropic source, i.e. a source at the origin which produces radially outward velocity which is the same in each direction θ (see Figure 1.20). This has streamfunction $\psi = m\theta$ where m is a constant which affects us the strength of the source. Then, if $\mathbf{u} = u_r \,\widehat{\mathbf{r}} + \mathbf{u}_\theta \,\widehat{\boldsymbol{\theta}}$,

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

and so

$$u_r = \frac{m}{r}; \qquad u_\theta = 0.$$

The streamlines are where ψ is constant, i.e. where $\theta = \text{const}$. We need to choose a 2π-range for θ to keep θ single-valued, and so let's pick $-\pi < \theta \leq \pi$.

If P and Q touch so that P is at $\theta = \pi$ and Q is at of $\theta = -\pi + \varepsilon$ for really small ε, what is the flux across PQ?

Solution The flux crossing PQ is $\psi(Q) - \psi(P) = 2\pi m$. Since we chose our branch cut arbitrarily* (see Figure 1.21), the flux over any circuit enclsing the origin is $2\pi m$, no matter how small the circuit. We can say then that the fluid is being created at the origin at an area flux of $2\pi m$, and so $\psi = m\theta$ is an isotropic line source (where the line goes into the page) of strength $2\pi m$. ✓

*That is to say, we chose that $-\pi < \theta \leq \pi$. We could've chosen $0 < \theta \leq 2\pi$, in which case the branch cut would be along the 'positive x-axis'

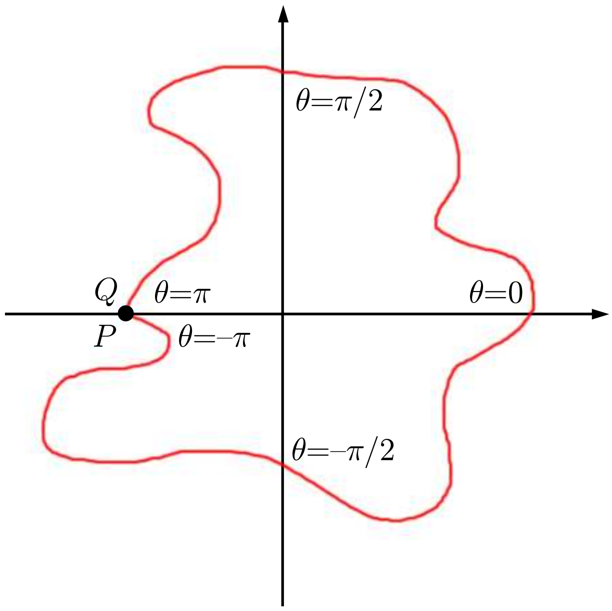

Figure 1.21: It doesn't matter which circuit we pick

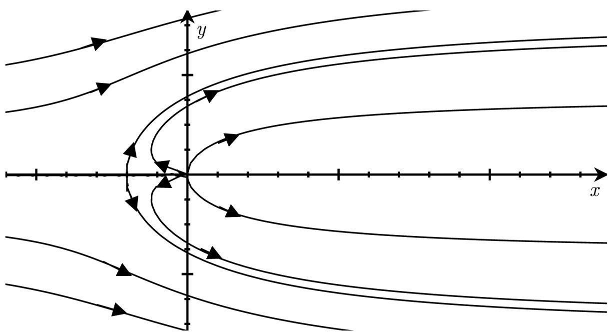

Figure 1.22: Plot of the streamlines for the flow in examples 1.14

Example 1.14 We now combine the last two examples. Consider an isotropic source in a uniform stream:

$$\psi_1 = Uy; \quad \psi_2 = m\theta; \quad \psi = \psi_1 + \psi_2.$$

What is the flux W over the line for downstream?

Solution

$$\text{The flux of shaded fluid over the line for downstream} = UW$$
$$\text{The flux of fluid from origin} = 2\pi m$$

And since these are equal, $W = 2\pi m/U$. See figure 1.22 for a plot of the streamlines for this flow. ✓

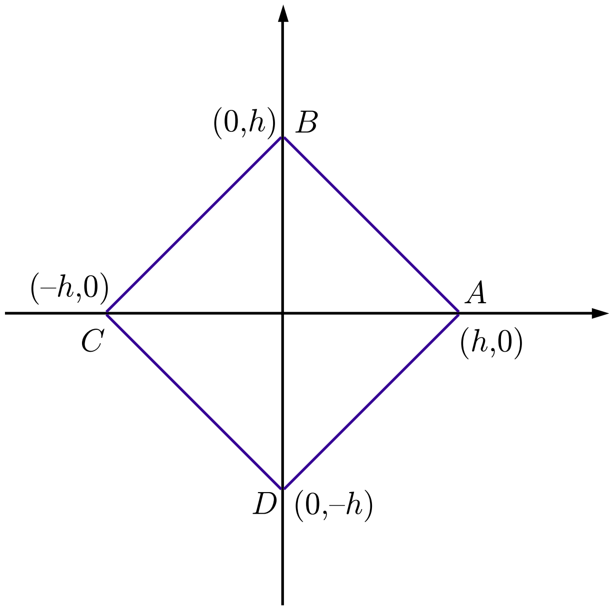

Figure 1.23: An arbitrarily small square fluid element

1.8 Local motion of a fluid element

What does an arbitrary 2D velocity field $\mathbf{u}(x, y, t)$ do to an arbitrarily small ($h \ll 1$) square fluid element (1.23)?

Consider the motion in a time interval δt where $\delta t \ll 1$. Hence, to order δt, we can take the flow to be steady. Now we Taylor expansion to expand the velocity field about the origin. Recall Taylor's theorem says in 1D:

$$f(a) = f(0) + f'(0)a + \frac{1}{2!}f''(0)a^2 + \cdots$$

and in 2D:

$$f(x, y) = f(0, 0) + \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}x^2 + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

So expanding $\mathbf{u}$ component-wise up to the first three terms,

$$u = U + \alpha x + \beta y + \cdots$$
$$v = V + \gamma x + \mu y + \cdots$$

where

$$\alpha = \frac{\partial u}{\partial x}(0, 0); \quad \beta = \frac{\partial u}{\partial y}(0, 0); \quad \gamma = \frac{\partial v}{\partial x}(0, 0); \quad \mu = \frac{\partial v}{\partial y}(0, 0).$$

Recall that in an incompressible fluid, $\nabla \cdot \mathbf{u} = 0$, i.e.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\alpha + \mu = 0$$
$$\mu = -\alpha$$

So we could write the equations for u and v in matrix form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now we use some cunning matrix algebra: for any matrix $\mathbf{A}$, we have

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right).$$

So if we write

$$\theta = \frac{1}{2}(\beta + \gamma); \qquad \phi = \frac{1}{2}(\beta - \gamma) = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right),$$

then

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} U \\ V \end{pmatrix}}_{\text{(I)}} + \left[\underbrace{\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{(II)}} + \underbrace{\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{(III)}} + \underbrace{\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{(IV)}} \right] \begin{pmatrix} x \\ y \end{pmatrix}.$$

Notice that terms (II) and (III) are symmetric, while term (IV) is antisymmetric.

Now we ask, what do each of these terms do to our diamond?

(I) In a time δt, you move a distance $\delta x = u\delta t$ and $\delta y = v\delta t$. The term

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} = \text{const.}$$

translates the element a distance

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} U\,\delta t \\ V\,\delta t \end{pmatrix}$$

i.e. a simple translation without change of shape or orientation.

(II) The term

$$\begin{pmatrix} u \\ v \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

we can write as

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \delta t$$

which is a function of position, so it does different things to different points. Let's take a look at what it does to the points A, B, C, D on our diamond.

$$A: \quad \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} \delta t = \begin{pmatrix} \alpha h\,\delta t \\ 0 \end{pmatrix}$$

$$C: \quad \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -h \\ 0 \end{pmatrix} \delta t = \begin{pmatrix} -\alpha h\,\delta t \\ 0 \end{pmatrix}$$

$$B: \quad \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \delta t = \begin{pmatrix} 0 \\ -\alpha h\,\delta t \end{pmatrix}$$

$$D: \quad \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -h \end{pmatrix} \delta t = \begin{pmatrix} 0 \\ \alpha h\,\delta t \end{pmatrix}$$

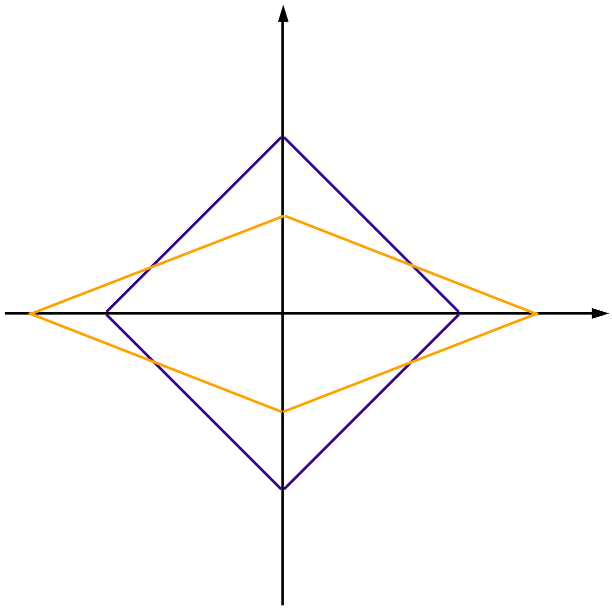

Figure 1.24: Term (II) represents a dilation

Thus term (II) represents a stretching in the x-direction and an equal and opposite contraction in the y-direction, i.e. a *dilation*. This can only happen in a fluid! See Figure 1.24.

(III) Let's do the same thing with term (III).

$$
\begin{aligned}
A: &\quad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}h\\ 0\end{pmatrix}\delta t = \begin{pmatrix}0\\ \theta h\,\delta t\end{pmatrix}\\
C: &\quad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-h\\ 0\end{pmatrix}\delta t = \begin{pmatrix}0\\ -\theta h\,\delta t\end{pmatrix}\\
B: &\quad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ h\end{pmatrix}\delta t = \begin{pmatrix}\theta h\,\delta t\\ 0\end{pmatrix}\\
D: &\quad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ -h\end{pmatrix}\delta t = \begin{pmatrix}-\theta h\,\delta t\\ 0\end{pmatrix}
\end{aligned}
$$

So what we have is another dilation which stretches along the line $y = x$ with an equal and opposite contraction along the line $y = -x$. Notice that dilations conserve volume! See Figure 1.25.

Also notice that the combination of two dilations is itself another dilation. If we add terms (II) and (III) we get

$$\begin{pmatrix}\alpha & \theta\\ \theta & \alpha\end{pmatrix}$$

which is real and symmetric, so it has real eigenvalues. The eigenvectors associated with the distinct eigenvalues are orthogonal, which we can show like this:

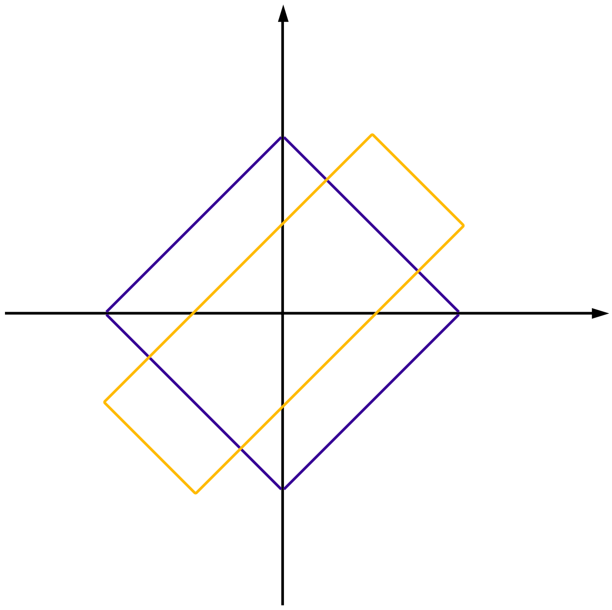

Figure 1.25: Term (III) also represents a dilation

First, find the eigenvalues:

$$\begin{aligned}\left|\begin{pmatrix}\alpha-\lambda & \theta \\ \theta & -\alpha-\lambda\end{pmatrix}\right| &= -(\alpha+\lambda)(\alpha-\lambda)-\theta^2 \\ &= \lambda^2-(\alpha^2+\theta^2) \\ \implies \lambda &= \pm\sqrt{\alpha^2+\theta^2}.\end{aligned}$$

So let our eigenvalues be

$$\lambda_1 = \sqrt{\alpha^2+\theta^2}; \qquad \lambda_2 = -\sqrt{\alpha^2+\theta^2}.$$

with corresponding eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.

Relative to the basis $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, the matrix is diagonal:

$$\begin{pmatrix}\lambda_1 & 0 \\ 0 & -\lambda_1\end{pmatrix}$$

and we have a stretching at rate λ_1 along $\boldsymbol{\xi}_1$ and an equal and opposite contraction at rate $-\lambda_1$ along $\boldsymbol{\xi}_2$, as in Figure 1.26.

(IV) Now let's look at what term four does in isolation:

$$\begin{pmatrix}u \\ v\end{pmatrix} = \phi\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$$

we can write as

$$\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = \phi\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}\delta t$$

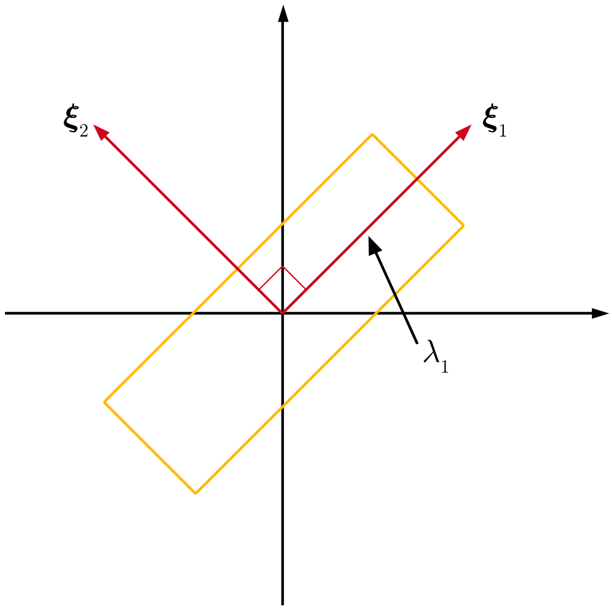

Figure 1.26: A stretch at rate λ_1 along $\boldsymbol{\xi}_1$ and an equal and opposite contraction at rate $-\lambda_1$ along $\boldsymbol{\xi}_2$

and again let's look at what happens to our points A, B, C, D.

$$
\begin{aligned}
A: &\qquad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \phi \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}h\\ 0\end{pmatrix}\delta t = \begin{pmatrix}0\\ \theta h\,\delta t\end{pmatrix}\\
C: &\qquad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \phi \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-h\\ 0\end{pmatrix}\delta t = \begin{pmatrix}0\\ -\theta h\,\delta t\end{pmatrix}\\
B: &\qquad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \phi \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ h\end{pmatrix}\delta t = \begin{pmatrix}-\theta h\,\delta t\\ 0\end{pmatrix}\\
D: &\qquad \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \phi \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ -h\end{pmatrix}\delta t = \begin{pmatrix}\theta h\,\delta t\\ 0\end{pmatrix}
\end{aligned}
$$

Thus the square has rotated through an angle $\phi\,\delta t$ in time δt, i.e. the element is rotating at a rate ϕ, see Figure 1.27.

So in conclusion, the local motion of an element consists of

1. Translation of the centre of mass
2. Dilation that conserves mass
3. Rotation about the centre of mass at a rate $\phi = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

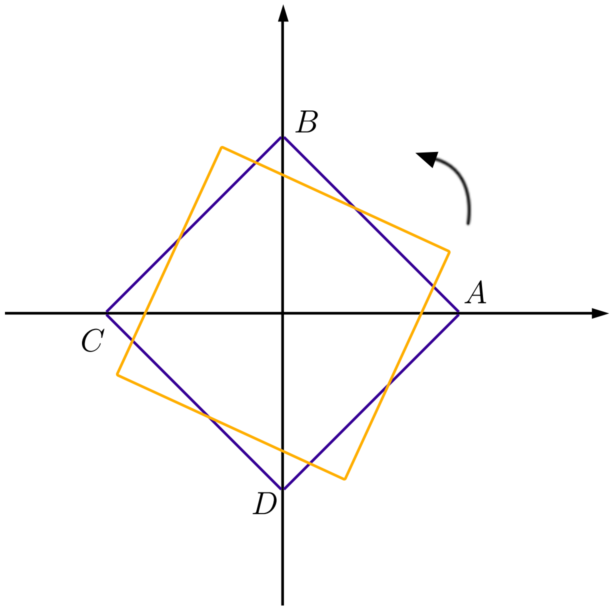

Figure 1.27: A rotation of rate ϕ

Chapter 2

Two-dimensional motion

2.1 Vorticity

Definition 2.1 The *vorticity* $\boldsymbol{\omega}$ at a point in a fluid with speed $\mathbf{u}$ is given by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

In 2D flow,

$$\mathbf{u} = u(x,y)\,\widehat{\mathbf{\imath}} + v(x,y)\,\widehat{\mathbf{\jmath}} + 0\,\widehat{\mathbf{k}}$$

So the vorticity is

$$\nabla \times \mathbf{u} = \begin{vmatrix} \widehat{\mathbf{\imath}} & \widehat{\mathbf{\jmath}} & \widehat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \widehat{\mathbf{k}}$$

Thus the vorticity, at a point, gives $2\times$ the rate of rotation of the fluid element at that point about its centre of mass.

We now present two arguments about vorticity in a 2D inviscid fluid:

1. Vorticity is conserved by fluid elements. This is because in an inviscid fluid, shear stress isn't supported, and the only way to change rate of rotation is to apply shear stress (see Figure 2.1). Obviously different elements can have different vorticities, though.

2. A flow that starts from rest ($\mathbf{u} = \mathbf{0}$ so $\nabla \times \mathbf{u} = \mathbf{0}$) has vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$ for all time. This follows from the argument above and is known as the *persistence of irrotationality.*

We will thus confine our attention to irrotational flows, i.e. where $\boldsymbol{\omega} = 0$. This is our governing equation.

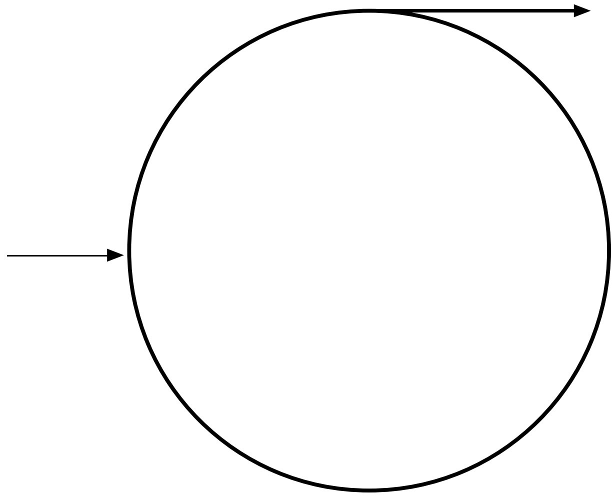

Figure 2.1: In an inviscid fluid, shear stress is not supported

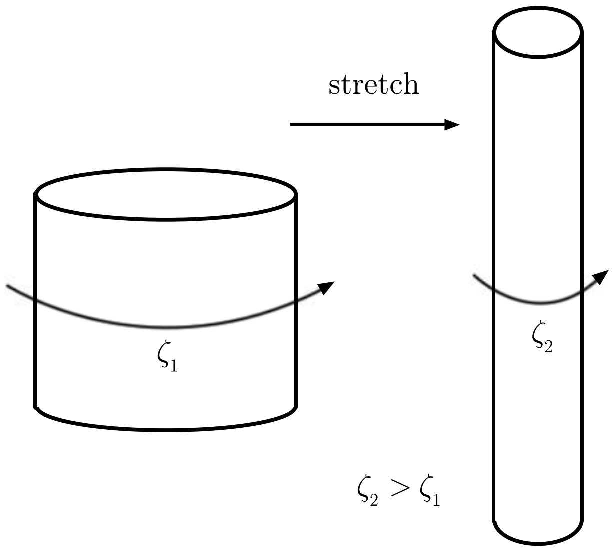

Figure 2.2: In 3D, argument 1 does not apply.

Note that argument 1 above does not apply in 3D, since if we take a fluid cylinder rotating with angular velocity ζ_1 and stretch it, we expect the cylinder to become thinner and spin with angular velocity ζ_2, where $\zeta_2 > \zeta_1$ (see Figure 2.2). (This is the spinning around with your arms out and then suddenly bringing your arms inward effect). Argument 2, however, does apply in 3D, since if there is no vorticity at the start, there is nothing to amplify.

So now we have, for an *incompressible*, *irrotational* flow, in both 2D and 3D, two governing equations (respectively):

$$\nabla \cdot \mathbf{u} = 0; \quad \nabla \times \mathbf{u} = 0.$$

So $\mathbf{u}$ is a *conservative* vector field, and hence it can be derived from a *potential* ϕ, with $\mathbf{u} = \nabla\phi$.

If we substitute $\mathbf{u} = \nabla\phi$ into $\nabla \cdot \mathbf{u} = 0$ we get

$$\nabla \cdot (\nabla\phi) = 0 \implies \nabla^2\phi = 0,$$

i.e. Laplace's equation.

Definition 2.2 The *velocity potential* is ϕ where $\mathbf{u} = \nabla\phi$.

Now we will stress two arguments which lead to something really useful. In 2D, for an incompressible flow we have $\nabla \cdot \mathbf{u} = 0$, i.e. there exists a streamfunction ψ such that

$$\mathbf{u} = -\widehat{\mathbf{k}}\times\nabla\psi.$$

In components, that is

$$u = \frac{\partial\psi}{\partial y}; \quad v = -\frac{\partial\psi}{\partial x}.$$

In 2D or 3D, for an irrotational flow, as we have just seen, we have $\nabla\times\mathbf{u} = \mathbf{0}$, hence there exists a velocity potential ϕ such that

$$\mathbf{u} = \nabla\phi.$$

In components, that is

$$u = \frac{\partial\phi}{\partial x}; \quad v = \frac{\partial\phi}{\partial y}.$$

Putting these together, we get

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$
$$v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

Do these equations look familiar? It's because ψ and ϕ satisfy the *Cauchy–Riemann equations*, which you've seen recently in MATH2101.

So we can say that ϕ and ψ are the real and imaginary parts (respectively) of a differentiable function of the complex variable $z = x+iy$. That is to say, we can write

$$w(z) = \phi(x, y) + i\psi(x, y).$$

Definition 2.3 The *complex velocity potential* is

$$w(z) = \phi(x, y) + i\psi(x, y).$$

Thus all 2D incompressible irrotational flows are expressible in terms of a complex velocity potential.

Note that dw/dz is uniquely defined:

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial w}{\partial x} \\ &= \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial x} \\ &= u - iv\end{aligned}$$

(the first line here you can verify by looking at your MATH2101 notes). This equivalence is very useful, and in fact the main reason we use w is that it has this very convenient property.

Example 2.4 A fluid moves with complex potential

$$w = \text{const}.$$

What is its velocity?

Solution Differentiating w,

$$\frac{dw}{dz} = 0$$

and so the fluid moves (or doesn't!) with velocity $\mathbf{u} = \mathbf{0}$. ✓

Example 2.5 A fluid moves with complex potential

$$w = Uz.$$

What is its velocity? Sketch the flow.

Solution Differentiating the complex potential,

$$\frac{dw}{dz} = U \implies u = U; \quad v = 0$$

so we have a uniform stream with velocity $U\hat{\mathbf{\imath}}$. Note that we can also compute the streamfunction ψ and potential ϕ:

$$\phi = \text{Re}(w) = Ux; \quad \psi = \text{Im}(w) = Uy.$$

See figure 2.3 for a sketch: The streamlines are where $\psi = \text{const.}$ and the equipotentials are the lines where $\phi = \text{const.}$

Note that $-\hat{\mathbf{k}}\times\nabla\psi = \nabla\phi = \mathbf{u}$: the level curves of the real and imaginary parts of a holomorphic complex function intersect at right angles. ✓

Example 2.6 A fluid moves with complex potential

$$w = z^2.$$

Sketch its velocity, streamfunctions and equipotentials.

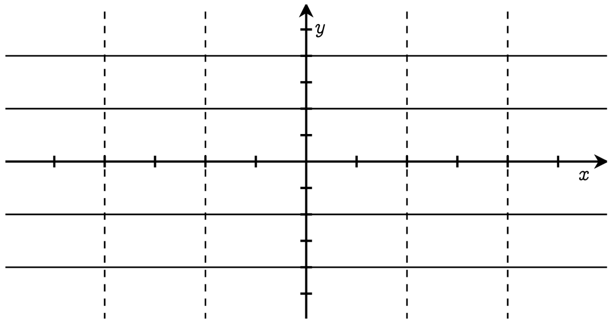

Figure 2.3: A sketch of the streamlines and equipotential lines for example 2.5

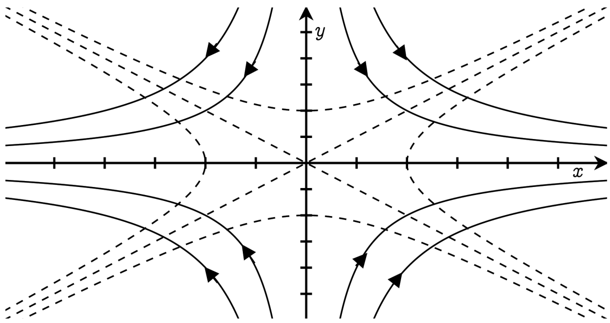

Figure 2.4: Plot of the streamlines and equipotential lines from example 2.6

Solution The velocity is

$$\frac{dw}{dz} = 2z = 2(x + iy),$$

which we can write as

$$u = 2x; \qquad v = -2y,$$

the streamfunction is

$$\psi = \mathrm{Im}(w) = \mathrm{Im}(x^2 + 2xyi - y^2) = 2xy,$$

and the velocity potential is

$$\phi = \mathrm{Re}(w) = \mathrm{Re}(x^2 + 2xyi - y^2) = x^2 - y^2.$$

See figure 2.4 for a plot: where again the streamlines are where ψ is constant, and the equipotentials are where ϕ is constant. Again we see that the equipotentials and strealines intersect orthogonally everywhere except the origin, where $dw/dz = 0$. At the origin, then, $\mathbf{u} = \mathbf{0}$ and this is a *stagnation point.* ✓

Example 2.7 A flow has complex potential

$$w = z^3.$$

Sketch the streamlines.

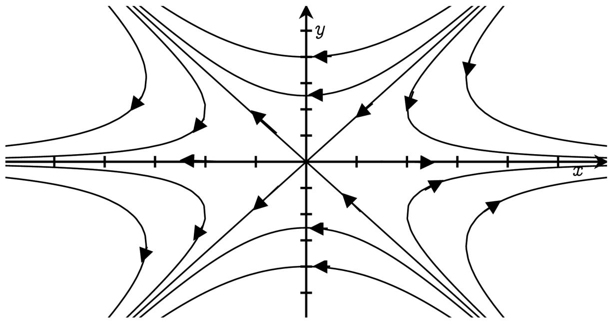

Figure 2.5: A plot of the streamlines of the flow from example 2.7

Solution We can rewrite w as

$$\begin{aligned} w &= z^3 \\ &= r^3 e^{3i\theta} \\ &= r^3(\cos 3\theta + i \sin 3\theta) \end{aligned}$$

so the streamfunction is

$$\psi = r^3 \sin 3\theta.$$

The streamlines are where ψ is constant, i.e. where $r^3 \sin 3\theta = \text{const.} = 0$, where we have picked our constant to be 0. For this to happen, either $r = 0$ (which is uninteresting) or $\sin 3\theta = 0$, i.e.

$$3\theta = \pi n \implies \theta = \frac{n\pi}{3} \qquad (n \in \mathbb{Z})$$

See figure2.5 for a plot of the streamlines of this flow.

The three streamlines intersect at $\theta = \pi/3$. ✓

Example 2.8 How does a fluid with complex potential Az^2, where $A = \alpha e^{i\varphi}$, compare to one with complex potential z^2?

Solution

$$\begin{aligned} w &= Az^2 \\ &= \alpha e^{i\varphi} r^2 e^{2i\theta} \\ &= \alpha r^2 e^{i(2\theta+\varphi)} \end{aligned}$$

so the effect of A is to rotate and amplify. ✓

This leads naturally into asking about what our equations in polar coordinates are. Our velocity is

$$\mathbf{u} = \nabla\phi = u_r\,\widehat{\mathbf{r}} + u_\theta\,\widehat{\boldsymbol{\theta}}.$$

First let's substitute in the potential ϕ. We know

$$\nabla\phi = \frac{\partial\phi}{\partial r}\,\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\widehat{\boldsymbol{\theta}}$$

(this is just the definition of the gradient function in polar coordinates.) So

$$u_r = \frac{\partial \phi}{\partial r}; \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}.$$

(Note: we can find the component of velocity in any direction $\widehat{\mathbf{b}}$ by simply dotting $\mathbf{u}$ with $\widehat{\mathbf{b}}$: $\mathbf{u} \cdot \widehat{\mathbf{b}} = \widehat{\mathbf{b}} \cdot \nabla\phi$.)

Now let's look at substituting in the streamfunction ψ. We have seen that

$$\begin{aligned}
\mathbf{u} &= -\widehat{\mathbf{k}} \times \nabla\psi \\
&= -\widehat{\mathbf{k}} \times \left(\frac{\partial \psi}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\widehat{\boldsymbol{\theta}} \right) \\
&= \frac{1}{r}\frac{\partial \psi}{\partial \theta}\widehat{\mathbf{r}} - \frac{\partial \psi}{\partial r}\widehat{\boldsymbol{\theta}}
\end{aligned}$$

so

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

and so putting that together with what we had above,

$$\left.\begin{aligned}
u_r &= \frac{\partial \phi}{\partial r} &&= \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\
u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} &&= -\frac{\partial \psi}{\partial r}
\end{aligned}\right\} \text{Cauchy–Riemann equations in polars}$$

i.e. $\phi(r, \theta)$ and $\psi(r, \theta)$ are the real and imaginary parts of a differentiable complex function $w(z) = \phi + i\psi$ where $z = re^{i\theta}$.

2.2 Finding the governing equations for ψ, ϕ, w

We now have three ways to solve any 2D irrotational incompressible problem:

1. Introduce the streamfunction ψ, or
2. Introduce the velocity potential ϕ, or
3. Introduce the complex velocity potential $w(z)$, where $z = x + iy$.

We will now find the governing equations for each of these.

1. **Streamfunction**: we have, for an incompressible flow, $\nabla \cdot \mathbf{u} = 0$, hence we were able to say that in 2D we have a streamfunction ψ such that

$$u = \frac{\partial \psi}{\partial y}; \qquad v = -\frac{\partial \psi}{\partial x}.$$

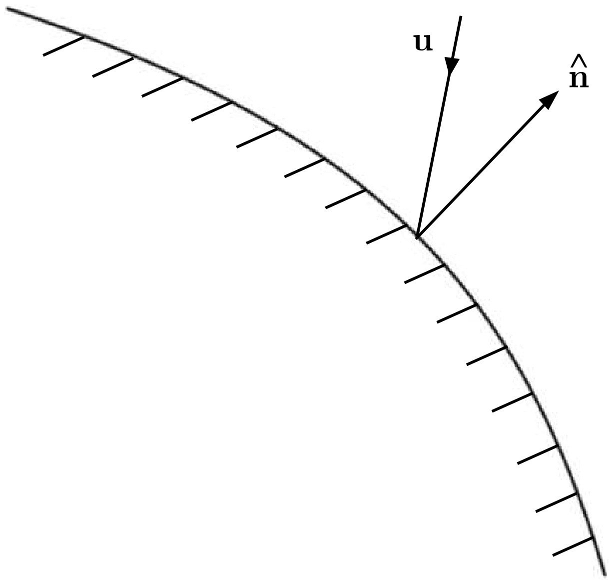

Figure 2.6: Boundary condition

For an irrotational flow, $\zeta = 0$, i.e.

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
$$\implies -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$$

i.e. $\nabla^2 \psi = 0$: Laplace's equation.

The boundary condition is that on a solid boundary, ψ is constant. If we only have one boundary, we can happily set $\psi = 0$ there.

2. **Velocity potential**: as we've done before, we have that $\mathbf{u} = \nabla\phi$, so

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla\phi) = \nabla^2\phi$$

so if $\nabla \cdot \mathbf{u} = 0$, then $\nabla^2\phi = 0$: Laplace's equation, again. And in fact this is true in 3D as well.

(Note: this shows the real and imaginary parts of a differentiable complex function satisfy Laplace's equation).

The boundary condition is that $\mathbf{u} \cdot \widehat{\mathbf{n}} = 0$, i.e. $\widehat{\mathbf{n}} \cdot \nabla\phi = 0$ (the rate of change of ϕ in the normal direction is zero). This is equivalent to saying

$$\frac{\partial \phi}{\partial n} = 0$$

on a solid boundary, i.e. the normal derivative vanishes (and this makes some intuitive sense— see Figure 2.6). Some examples:

Example 2.9 What are our boundary conditions for:

(i) a solid wall at $x = \text{const.}$,

(ii) a solid wall at $y = \text{const.}$,

(iii) a cylinder of radius a, centred at the origin.

Solution (i) Our $\widehat{\mathbf{n}}$ here is $\widehat{\mathbf{j}}$, so

$$\frac{\partial \phi}{\partial n} = \widehat{\mathbf{j}} \cdot \nabla \phi = 0$$

hence our boundary condition on ϕ is that $\partial \phi / \partial y = 0$.

(ii) Our $\widehat{\mathbf{n}}$ here is $\widehat{\mathbf{i}}$, so

$$\frac{\partial \phi}{\partial n} = \widehat{\mathbf{i}} \cdot \nabla \phi = 0$$

hence our boundary condition on ϕ is that $\partial \phi / \partial x = 0$.

(iii) Our $\widehat{\mathbf{n}}$ here is $\widehat{\mathbf{r}}$, so

$$\frac{\partial \phi}{\partial n} = \widehat{\mathbf{r}} \cdot \nabla \phi = 0$$

hence our boundary condition on ϕ is that $\partial \phi / \partial r = 0$.

✓

3. **Complex velocity potential**: Our governing equation is simply that w is differentiable.

   Our boundary condition can be either of those mentioned above.

The fastest way of finding a stagnation point is by looking at w and setting $dw/dz = 0$.

2.3 Laurent Series

A function that is analytic (holomorphic) within an annular (ring) region ($R_0 < |z| < R_1$) has a unique expansion of the form:

$$\begin{aligned} f(z) &= \cdots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 x + a_2 z^3 + \cdots \\ &= \sum_{n=-1}^{-\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^n \end{aligned}$$

We shall demand that the velocity field is analytic within an annular region, i.e. $u - iv$ is analytic, and so can be written as:

$$\frac{dw}{dz} = u - iv = \cdots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 x + a_2 z^3 + \cdots$$

We allow singularities inside a solid boundary as these are not part of the flow field. Then the most general form of the complex velocity potential is found by integrating the above to get:

$$w = \cdots + b_{-2}z^{-2} + b_{-1}z^{-1} + a_{-1}\log z + b_0 + b_1 z^1 + b_2 z^2 + \cdots$$

Now the real and imaginary parts of this series satisfy Laplace's equation:

$$\begin{aligned}
\mathrm{Re}(z^n) &= \mathrm{Re}(r^n e^{in\theta}) = r^n \cos n\theta \\
\mathrm{Im}(z^n) &= \mathrm{Im}(r^n e^{in\theta}) = r^n \sin n\theta \\
\mathrm{Re}(z^{-n}) &= \mathrm{Re}(r^{-n} e^{-in\theta}) = r^{-n} \cos n\theta \\
\mathrm{Im}(z^{-n}) &= \mathrm{Im}(r^{-n} e^{-in\theta}) = -r^{-n} \sin n\theta \\
\mathrm{Re}(\log z) &= \log r \\
\mathrm{Im}(\log z) &= \theta
\end{aligned}$$

(Recall that $\log z = \log(re^{i\theta}) = \log r + i\theta$)

Thus any solution of Laplace's equation in polar coordinates is a linear combination of functions drawn for the set $\{\log r, \theta, r^{\pm n}\cos n\theta, r^{\pm n}\sin n\theta\}$.

Example 2.10 Is the following a possible streamfunction:

$$7r^5 \sin 5\theta + 9r^{-3}\cos 3\theta + 15r^6 \sin 5\theta$$

Solution No, because of the $15r^6 \sin 5\theta$ term, since the power of the r and the coefficient of θ don't agree. ✓

Example 2.11 Consider a flow that, at infinity, is uniform with velocity U in the x-direction. Let there be an impermeable cylinder of radius a at the origin. Find the flow field.

Solution We will solve this problem using 3 different methods:

1. Finding the streamfunction
2. Finding the velocity potential
3. Finding the Complex Potential

1. **Streamfunction:**

   Our equation is $\nabla^2\psi = 0$ and the circular boundary implies that using polar coordinates is a good idea.

   (a) On the cylinder, $r = a$, $\psi =$ const. (since a solid boundary is a streamline). Since there is only one boundary, we take $\psi = 0$ on $r = a$

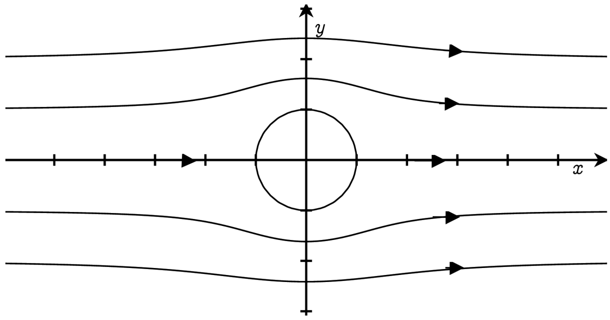

Figure 2.7: A plot of the streamlines of the flow in Example 2.11

(b) At a large distance $r \to \infty$ $u \to U$, $v \to 0$, i.e. $\frac{\partial \psi}{\partial y} \to U$, $\frac{\partial \psi}{\partial x} \to 0$
So $\psi \to Uy = Ur \sin\theta$ as $r \to \infty$

So we solve $\nabla^2 \psi = 0$ subject to $\psi = 0$ on $r = a$ and $\psi \to Ur\sin\theta$ as $r \to \infty$.

We approach this by looking for a solution of the form:

$$\psi = Ar\sin\theta + \frac{B}{r}\sin\theta$$

The first term is obviously chosen to agree with the far field boundary condition, while the second term is introduced in order to allow us to solve the homogeneous boundary condition on the cylinder whilst also maintaining the far field condition. Note that as we have taken these functions from the set $\{1, \theta, \log r, r^{\pm n}\cos n\theta, r^{\pm n}\sin n\theta\}$, $\nabla^2\psi = 0$ is satisfied automatically.

The far field condition gives:

$$\psi \to Ar\sin\theta = Ur\sin\theta \text{ as } r \to \infty$$

which implies that $A = U$. On $r = a$:

$$\psi = Ua\sin\theta + \frac{B}{a}\sin\theta = 0 \quad \forall\theta$$

Which gives us that:

$$\begin{aligned}\frac{B}{a} + Ua &= 0 \\ \implies B &= -Ua^2\end{aligned}$$

So we have that:

$$\psi = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right)$$

We can plot the resulting streamlines, as seen in figure 2.7. Note that we could have constructed these streamlines by plotting the streamlines ignoring the cylinder, and then distorting the streamlines the minimum amount. We find the following velocity profile:

$$\begin{aligned}u_r &= \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right) \\ u_\theta &= -\frac{\partial\psi}{\partial r} = -U\sin\theta - U\frac{a^2}{r^2}\sin\theta \\ &= -2U\sin\theta \quad \text{on } r = a\end{aligned}$$

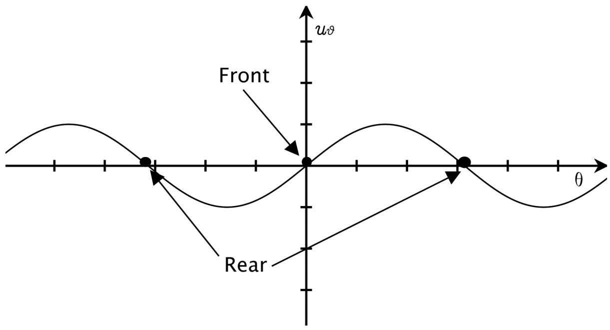

Figure 2.8: A plot of u_θ on the cylinder

We see there are stagnation points on the front and rear of the cylinder, as shown in figure 2.8.

2. **Velocity Potential**

Governing equation: $\nabla^2\phi = 0$, $\mathbf{u} = \nabla\phi$.

Far field boundary condition:

$$u \to U, \quad v \to 0 \text{ as } r \to \infty$$
$$\text{i.e. } \frac{\partial \phi}{\partial x} \to U, \quad \frac{\partial \phi}{\partial y} \to 0 \text{ as } r \to \infty$$
$$\text{i.e. } \phi \to Ux \text{ as } r \to \infty$$

Boundary condition on the cylinder:

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } r = a$$
$$\frac{\partial \phi}{\partial n} = 0 \text{ on } r = a$$
$$\frac{\partial \phi}{\partial r} = 0 \text{ on } r = a$$

In polars:

$$\phi \to Ur\cos\theta \text{ as } r \to \infty$$

So by the same logic as for the streamfunction, try:

$$\phi = Ar\cos\theta + \frac{B}{r}\cos\theta$$

The far field boundary condition gives $A = U$, while on $r = a$:

$$\frac{\partial \phi}{\partial r} = U\cos\theta - \frac{B}{a^2}\cos\theta = 0$$
$$\implies B = Ua^2$$

So:

$$\phi = Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right)$$

3. **Complex Potential**

$$\begin{aligned} w &= \phi + i\psi \\ &= Ur\cos\theta + \frac{Ua^2}{r}\cos\theta + i\left[Ur\sin\theta - \frac{Ua^2}{r}\sin\theta\right] \\ &= Ur\left(\cos\theta + i\sin\theta\right) + \frac{Ua^2}{r}\left[\cos\theta - i\sin\theta\right] \\ &= Ure^{i\theta} + \frac{Ua^2}{re^{i\theta}} \\ &= Uz + U\frac{a^2}{z} \\ &= U\left[z + \frac{a^2}{z}\right] \end{aligned}$$

✓

Example 2.12 A stationary cylinder with boundary $x^2 + y^2 = a^2$ is in a 2D irrotational flow with velocity components

$$u - ky \to 0; \quad v - kx \to 0, \qquad \text{as } x^2 + y^2 \to \infty.$$

Find the streamfunction, velocity potential, complex potential, and identify where the stagnation points are.

Solution We choose to solve this problem by first finding the streamfunction ψ. This will be the hard part of the question—the rest will fall out quite easily afterwards. The governing equation for ψ is

$$\nabla^2\psi = 0$$

with the far-field boundary equations

$$\frac{\partial\psi}{\partial y} \to ky; \quad \frac{\partial\psi}{\partial x} \to -kx, \qquad \text{as } r \to \infty.$$

We integrate these boundary equations to find the form of ψ that we can use. Integrating, we get

$$\psi \to \frac{1}{2}ky^2 + f(x); \quad \psi = -\frac{1}{2}kx^2 + g(y), \qquad \text{as } r \to \infty$$

and combining these, we get $f(x) = -\frac{1}{2}kx^2, g(y) = \frac{1}{2}ky^2$ and so

$$\psi = \frac{1}{2}k(y^2 - x^2), \qquad \text{as } r \to \infty.$$

So first let's plot the far field streamlines, as is seen in figure 2.9:

$$y^2 = x^2 = \text{const}.$$

We first take const. $= 0$ and so the streamlines are $y = \pm x$ We can ask ourselves,

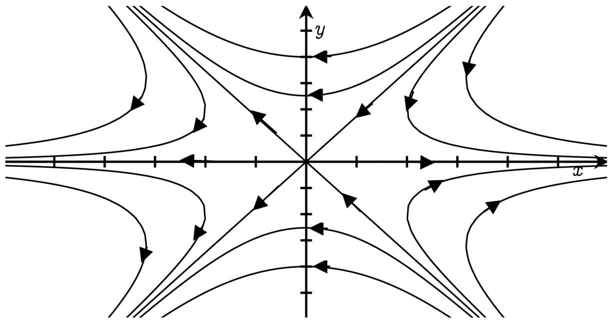

Figure 2.9: A plot of the streamlines of the flow in Example 2.12, ignoring the cylinder

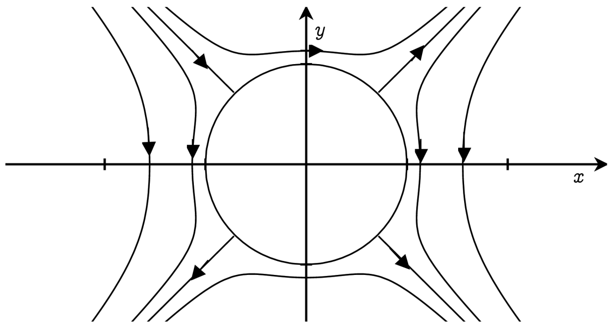

Figure 2.10: A plot of the streamlines of the flow in Example 2.12

which direction can we put on the streamlines? Well, for $y > 0, u > 0$ and $y < 0, u < 0$ so they point as shown in the diagram.

We now add the cylinder to the diagram, as shown in 2.10 and we say that the boundary of the cylinder is itself a streamline, where $\psi = \text{const}$. We are free to take $\psi = 0$ on $r = a$, and we probably should since this is a very sensible choice*!

We now try and find a form of ψ which will fit our governing equation ($\nabla^2\psi = 0$)—these are the ones that are solutions of Laplace's equation, as discussed earlier—and which will fit our far-field boundary equations. Polar coordinates seems sensible so let's look at our boundary equation again.

$$\begin{aligned}\psi &\to \frac{1}{2}k(y^2 - x^2)\\ &= \frac{1}{2}r^2(\sin^2\theta - \cos^2\theta)\end{aligned}$$

so we try

$$\psi = Ar^2\sin^2\theta + Br^2\cos^2\theta$$

but $\sin^2$ and $\cos^2$ are not solutions of Laplace's equation, so we have to do some algebraic manipulation. It turns out that we can use our double-angle identities† to

*In case you haven't seen why this is yet, we only have one 'physical' boundary condition here. Since the absolute value of ψ is unimportant at any point, merely the difference between them (much like gravitational potential energy), we can set $\psi = 0$ at the boundary, since it gives us a very easy boundary condition to play with in our equations.

†If, like one of the editors, you have difficulty remembering these identities, this is the place in the exam where you can slip up. I wouldn't expect to come across identities that are much harder

turn the above equation into

$$\psi \to \frac{1}{2}kr^2(-\cos 2\theta)$$

and we can try the form

$$\psi = Ar^2\cos 2\theta + \frac{B}{r^2}\cos 2\theta$$

which is composed of solutions to Laplace's equation.

If we let $r \to \infty$, then $A = -\frac{1}{2}k$ and

$$\psi = \underbrace{-\frac{1}{2}kr^2\cos 2\theta}_{\text{far field}} + \underbrace{\frac{B}{r^2}\cos 2\theta}_{\text{near field}}.$$

We now apply our cylinder boundary condition: that $\psi = 0$ on $r = a$:

$$-\frac{1}{2}ka^2\cos 2\theta + \frac{B}{a^2}\cos 2\theta = 0 \quad \forall\theta.$$

Since this condition applies for all values of θ, $B = \frac{1}{2}ka^4$ and we find our streamfunction to be

$$\psi = -\frac{1}{2}kr^2\cos 2\theta\left(1 - \frac{a^4}{r^4}\right).$$

And so now we can find ϕ and w quite easily. We know $w = \phi + i\psi$ so ψ is the imaginary part of w. So we write

$$\begin{aligned}\psi &= \operatorname{Im}(w)\\ &= -\frac{1}{2}k\left(r^2\cos 2\theta - \frac{a^4}{r^2}\cos 2\theta\right)\end{aligned}$$

Knowing that we have $\cos 2\theta$ in our equation (emphasis on the '2'), let's take a look at z^2 for complex z:

$$\begin{aligned}z^2 &= r^2e^{2i\theta}\\ &= r^2\cos 2\theta + ir^2\sin 2\theta\end{aligned}$$

And so plugging that into our equation for ψ,

$$\psi = -\frac{1}{2}k\left[\operatorname{Im}(iz^2) + ?\right]$$

but what is the second term? Let's look at z^{-2}:

$$\begin{aligned}z^{-2} &= r^{-2}e^{-2i\theta}\\ &= r^{-2}\cos 2\theta - ir^{-2}\sin 2\theta\end{aligned}$$

than these, but it's worth taking a short amount of time to flip back to MATH1401 notes and trying to learn a few.

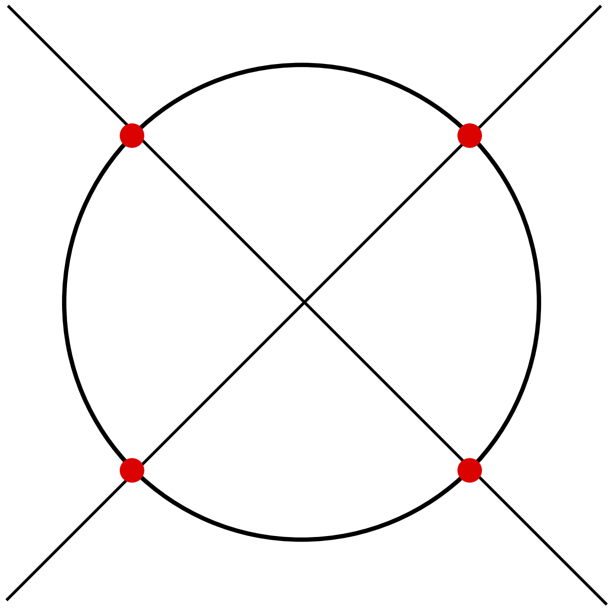

Figure 2.11: Stagnation points are marked by the red dots

and so

$$\psi = -\frac{1}{2}k\left[\mathrm{Im}(iz^2) + a^4\,\mathrm{Im}\left(\frac{i}{z^2}\right)\right].$$

Thus

$$w = -\frac{1}{2}ki\left[z^2 - \frac{a^4}{z^2}\right]$$

and

$$\phi = \mathrm{Re}(w) = \frac{1}{2}k(r^2\sin 2\theta + \frac{a^4}{r^2}\sin 2\theta).$$

Now where are the stagnation points? We could find the stagnation points by setting $dw/dz = 0$ and solving, but quite obviously by symmetry they are as shown in Figure 2.11. ✓

2.3.1 Streamlines at a stagnation point

Do streamlines always cross at right angles? The answer is **NO**!

Suppose we have a stagnation point in the flow. Without loss of generality, we can move it to the origin. Similarly we can take $w(0) = 0$ since w is arbitrary to within a constant. Near the stagnation point,

$$w = a_0 = a_1z + a_2z^2 + a_3z^3 + \cdots.$$

$w(0) = 0$ so $a_0 = 0$ and at a stagnation point, $dw/dz = 0$ so $a_1 = 0$. So let the first non-zero coefficient be a_n where $n \geq 2$.

For $|z|$ sufficiently small,

$$w \sim a_nz^n \quad (n \geq 2).$$

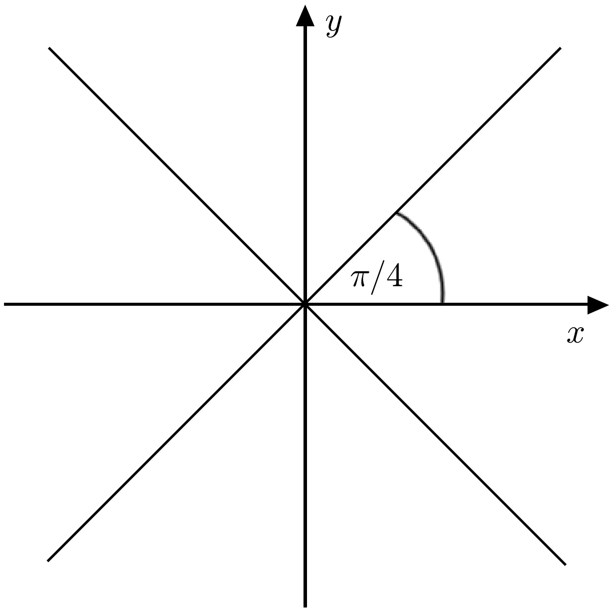

Figure 2.12: Here, streamlines are crossing at angle $\pi/4$

The factor a_n simply rotates and magnifies so let's just look at $w = z^n$.

$$\psi = \text{Im}(w) = r^n \sin n\theta$$

so $\psi = 0$ on $\theta = \frac{m\pi}{n}$, for $m = 0, \pm 1, \pm 2, \dots, \pm n$. That is to say, we've actually found something much more interesting: n streamlines cross at angle π/n.

2.3.2 Tennis balls, topspin and vortices

Professors Johnson and McDonald are playing tennis in the fifth floor common room after dark. In between discussing the latest cricket results, they find that if they give the ball topspin, the ball doesn't travel as far before hitting the ground as if they give the ball backspin. They are about to write a paper about why this is*, when they accidentally smash a window and leave quickly, hoping to blame it on a rowdy fresher (or Dr Bowles).

So far, we've been looking at cylinders in uniform streams where the flow is symmetric top and bottom, that is to say there is no upward or downward force. We first consider line sources again for some inspiration.

For a source of strength m, we know that

$$\begin{aligned} \psi &= \frac{m}{2\pi}\theta \\ &= \frac{m}{2\pi}\,\text{Im}(\log z) \\ \implies w &= \frac{m}{2\pi}\log z \end{aligned}$$

*which is really unnecessary since the phenomenon has been known for ages and we're about to explore it

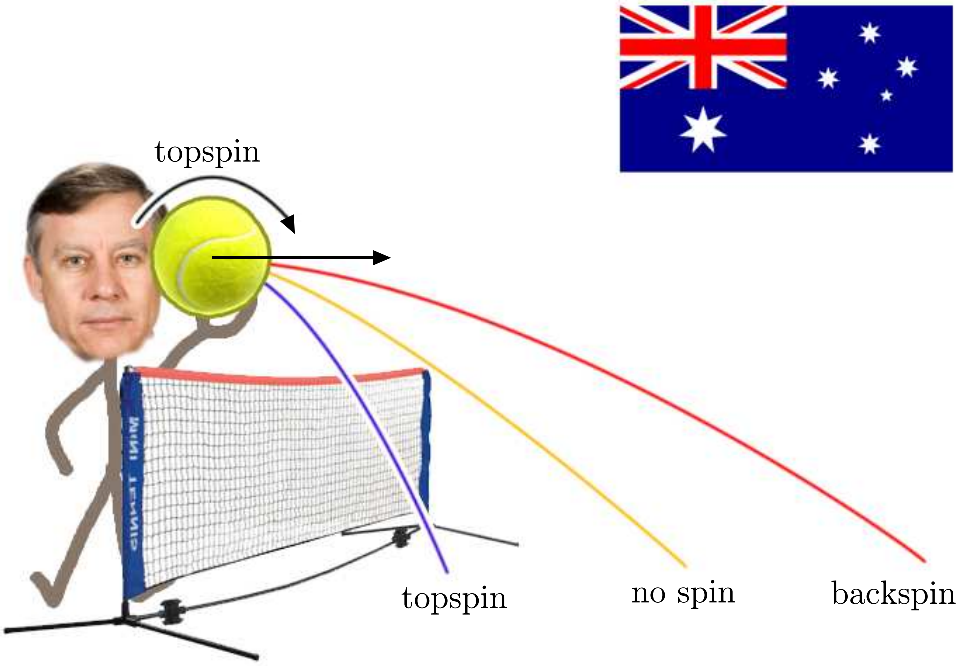

Figure 2.13: Professors playing tennis

$$\implies \phi = \frac{m}{2\pi} \log r.$$

Thus

$$\psi = \frac{m}{2\pi}\theta; \quad \phi = \frac{m}{2\pi} \log r$$

And we can find the flow velocity by looking at

$$\begin{aligned} \mathbf{u} &= \nabla\phi \\ &= \frac{m}{2\pi r}\widehat{\mathbf{r}} \\ \text{i.e. } u_r &= \frac{m}{2\pi r} \end{aligned}$$

And the flow through a circle of radius a (Figure 2.14)is

$$2\pi a \cdot u_r = 2\pi a \frac{m}{2\pi a} = m.$$

Now instead of the w above, consider

$$\begin{aligned} w &= \frac{-i\kappa}{2\pi} \log z \quad (\kappa \in \mathbb{R}) \\ &= \frac{-i\kappa}{2\pi}(\log r + i\theta) \\ &= \frac{-i\kappa}{2\pi} \log r + \frac{\kappa}{2\pi}\theta \end{aligned}$$

Thus

$$\phi = \frac{\kappa}{2\pi}\theta; \quad \psi = -\frac{\kappa}{2\pi} \log r.$$

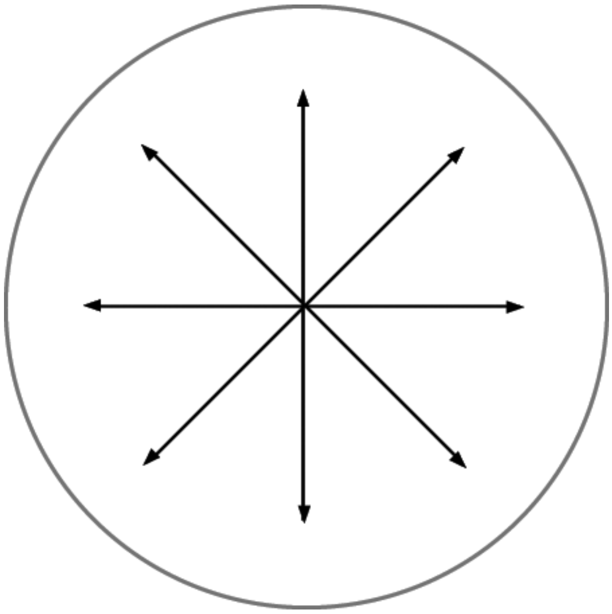

Figure 2.14: Flow through a circle of radius a

The streamlines are where $\psi =$ const., i.e. $r =$ const. so circles here. And our velocity is

$$\mathbf{u} = \nabla\phi = \frac{\kappa}{2\pi r}\,\widehat{\boldsymbol{\theta}}$$

which drops off as r^{-1} as $r \to \infty$. So our flow field looks like and we have a *vortex* or line vortex, as in Figure 2.15.

We measure the strength of a line vortex by its circulation Γ:

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s}$$

for any curve C containing the vortex. Now let's work out what this is. Around a circle of radius a, $|d\mathbf{s}| = a\,d\theta$ (Figure 2.16), and it is in direction $\widehat{\boldsymbol{\theta}}$ so

$$d\mathbf{s} = (a\,d\theta)\,\widehat{\boldsymbol{\theta}}$$

$$\mathbf{u} = \frac{\kappa}{2\pi a}\,\widehat{\boldsymbol{\theta}}$$

So plugging this into Γ,

$$\begin{aligned}\Gamma &= \int_0^{2\pi} \frac{\kappa}{2\pi a} \cdot a\,d\theta\ \underbrace{\widehat{\boldsymbol{\theta}} \cdot \widehat{\boldsymbol{\theta}}}_{1} \\ &= \frac{a\kappa}{2\pi a}\int_0^{2\pi} d\theta \\ &= \kappa\end{aligned}$$

i.e. the line vortex $w = -\frac{i\kappa}{2\pi}\log z$ has circulation or strength κ.

Note that this strength is totally independent of the choice of a, so we can say that all the circulation is generated at the origin. So long as we pick a so that it fits inside

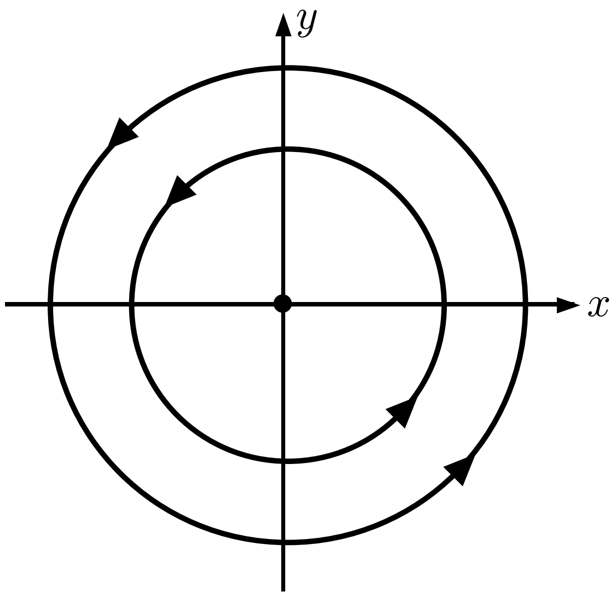

Figure 2.15: A vortex (or line vortex)

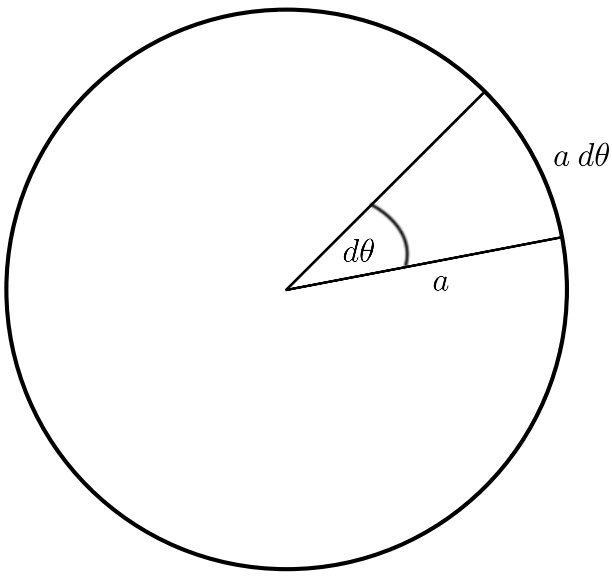

Figure 2.16: $|d\mathbf{s}| = a\,d\theta$ for a circle of radius a

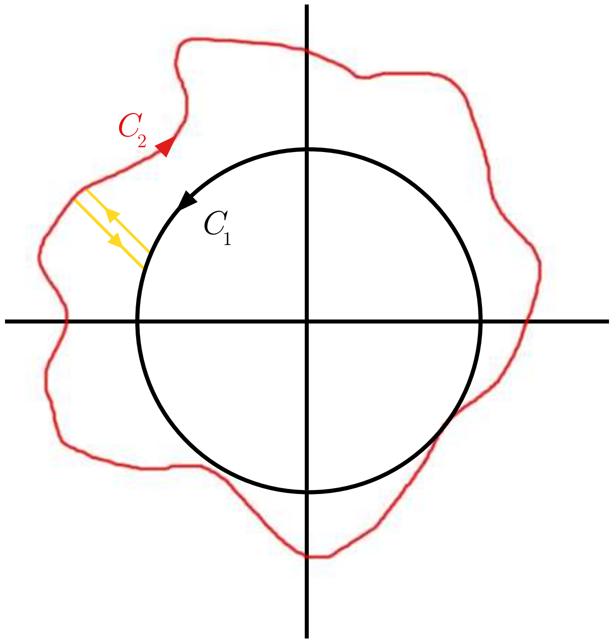

Figure 2.17: C_1 and C_2 as described

our bigger curve that we are looking at (see picture), we are fine. We can be more rigorous in saying this:

Consider the integral around the boundary of region A between the circuits, as in Figure 2.17.

$$\left[\oint_{C_2} - \oint_{C_1}\right] \mathbf{u}\cdot d\mathbf{s} = \iint (\nabla\times\mathbf{u})\cdot\widehat{\mathbf{n}}\,dA \quad \text{Stokes' theorem}$$
$$= 0$$

since $\nabla\times\mathbf{u} = 0$ because the flow is irrotational.

Thus our two solutions $\log r$ and θ represent line sources and line vortices. So,

Definition 2.13 A line source of strength m is

$$w = \frac{m}{2\pi}\log z$$

A line vortex of strength κ is

$$w = -\frac{i\kappa}{2\pi}\log z.$$

Now consider a cylinder with circulation κ in a flow that at infinity has velocity U in the x-direction, as in Figure 2.18. This is how we could model a ball hit from right to left with topspin (in a frame with the origin at the ball).

Note that Laplace's equation is linear, i.e. if $\nabla^2\phi_1 = 0$ and $\nabla^2\phi_2 = 0$, then

$$\nabla^2(\alpha_1\phi_1 + \alpha_2\phi_2) = 0$$

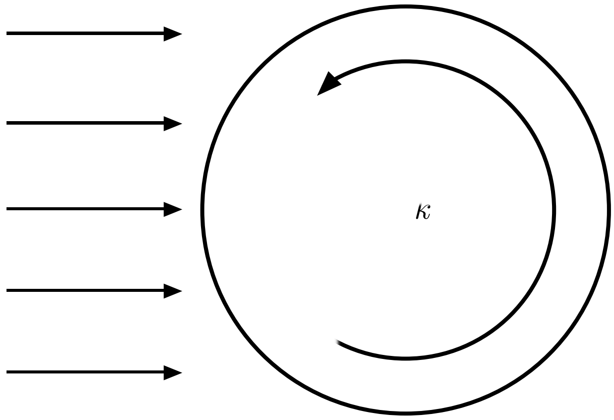

Figure 2.18: A cylinder with circulation κ in a flow that at infinity has velocity U in the x-direction

for constants α_1, α_2.

The flow past a cylinder has, (as proved already)

$$w_1(z) = Uz + \frac{Ua^2}{z}.$$

A line vortex with circulation κ has

$$w_2(z) = -\frac{i\kappa}{2\pi} \log z.$$

Theorem 2.14 We now claim that the flow past a cylinder with circulation κ has

$$\begin{aligned} w(z) &= w_1(z) + w_2(z) \\ &= Uz + \frac{Ua^2}{z} - \frac{i\kappa}{2\pi} \log z \end{aligned}$$

Proof: We require:

1. $u_r = 0$ on $r = a$ (the solid cylinder)
2. The circulation $\Gamma = \kappa$ where

$$\begin{aligned} \Gamma &= \oint \mathbf{u} \cdot d\mathbf{s} \\ &= \oint_{\theta=0}^{2\pi} u_\theta \, \widehat{\boldsymbol{\theta}} \cdot a \, d\theta \, \widehat{\boldsymbol{\theta}} \\ &= \int_0^{2\pi} u_\theta \, d\theta \end{aligned}$$

We want to use

$$\frac{dw}{dz} = u - iv,$$

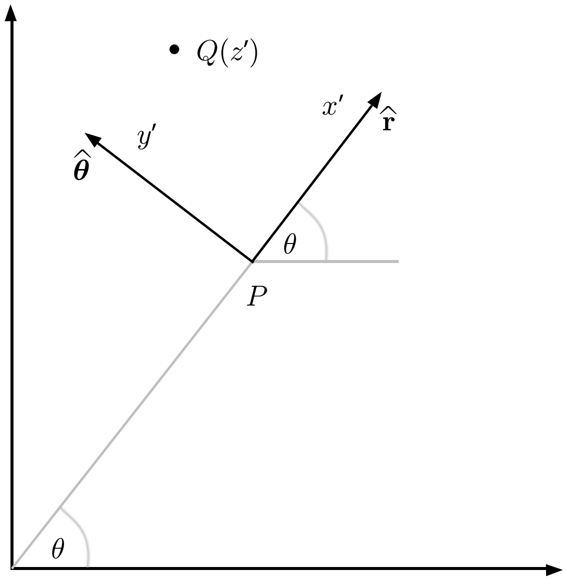

Figure 2.19: Let Q be z relative to Oxy and z' relative to $Ox'y'$.

but is there a polar version?

Consider the point Q. Let Q be z relative to Oxy and z' relative to $Ox'y'$ (see Figure 2.19). Then

$$|z| = |z'|; \quad \arg z = \arg z' + \theta$$

i.e.

$$\begin{aligned} z' &= |z'|e^{i\arg z'} \\ &= |z|e^{i(\arg z - \theta)} \\ &= ze^{-i\theta} \end{aligned}$$

Relative to x', y':

$$\begin{aligned} \frac{dw}{dz'} &= u' - iv' \\ &= u_r - iu_\theta \end{aligned}$$

Hence

$$\begin{aligned} u_r - iu_\theta &= \frac{dw}{dz} \\ &= \frac{dz}{dz'}\frac{dw}{dz} \\ &= e^{i\theta}\frac{dw}{dz} \end{aligned}$$

which gives us the important identity

$$u_r - iu_\theta = e^{i\theta}\frac{dw}{dz}.$$

Here we have

$$\begin{aligned} w &= Uz + \frac{Ua^2}{z} - \frac{i\kappa}{2\pi}\log z \\ \Longrightarrow \quad \frac{dw}{dz} &= U - \frac{Ua^2}{z^2} - \frac{i\kappa}{2\pi z} \end{aligned}$$

On the cylinder, $r = a$ or equivalently $z = ae^{i\theta}$, i.e.

$$\left.\frac{dw}{dz}\right|_{z=ae^{i\theta}} = U - Ue^{-2i\theta} - \frac{i\kappa}{2\pi a}e^{-i\theta}$$

Thus on the cylinder

$$\begin{aligned} u_r - iu_\theta &= e^{i\theta}\frac{dw}{dz} \\ &= Ue^{i\theta} - Ue^{-i\theta} - \frac{i\kappa}{2\pi a} \\ &= 2Ui\sin\theta - \frac{i\kappa}{2\pi a} \end{aligned}$$

So, taking real and imaginary parts, we get

$$u_r = 0; \qquad u_\theta = -2U\sin\theta + \frac{\kappa}{2\pi a}$$

And the circulation is

$$\begin{aligned} \Gamma &= a\int_0^{2\pi} u_\theta \, d\theta \\ &= a\frac{\kappa}{2\pi a}2\pi \\ &= \kappa \end{aligned}$$

□

The stagnation points occur on $r = a$ (so $u_r = 0$) where $u_\theta = 0$, i.e.

$$\sin\theta = \frac{\kappa}{4\pi aU} = s.$$

Clearly $-1 \leq s \leq 1$ by the properties of the sine function.

First sketch: $0 \leq s \leq 1$. The ball travelling right-to-left with top spin. See Figures 2.20 and 2.21.

If $s = 1$, we have a repeated root at $\theta = \pi/2$, as in Figure 2.22.

If $s > 1$, do we have no root for u_θ? No stagnation points on the surface, it lifts off. The best way to find it is to put $dw/dz = 0$. This gives $x = 0$ and $y = y_0 > a$, as in Figure 2.23.

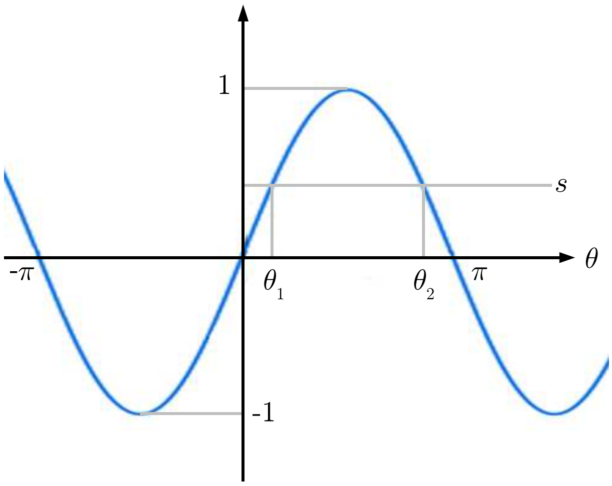

Figure 2.20: Definition of θ_1 in Figure 2.21.

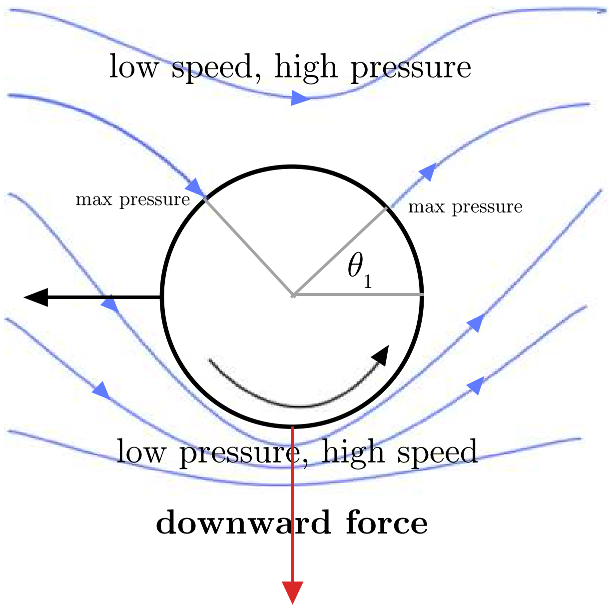

Figure 2.21: The ball is travelling right-to-left with topspin. $0 \leq s \leq 1$.

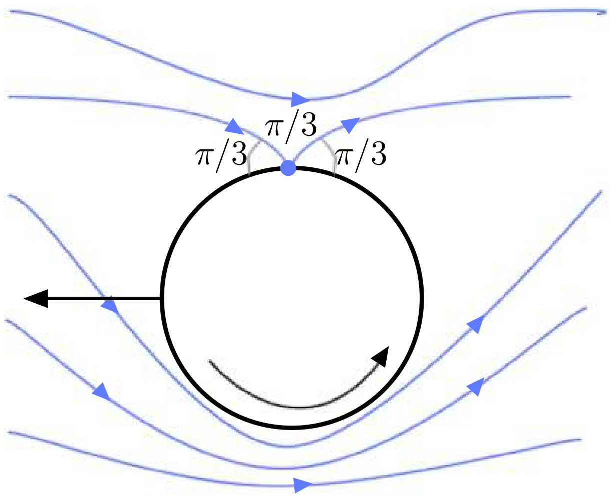

Figure 2.22: The ball is travelling right-to-left with topspin. $s = 1$.

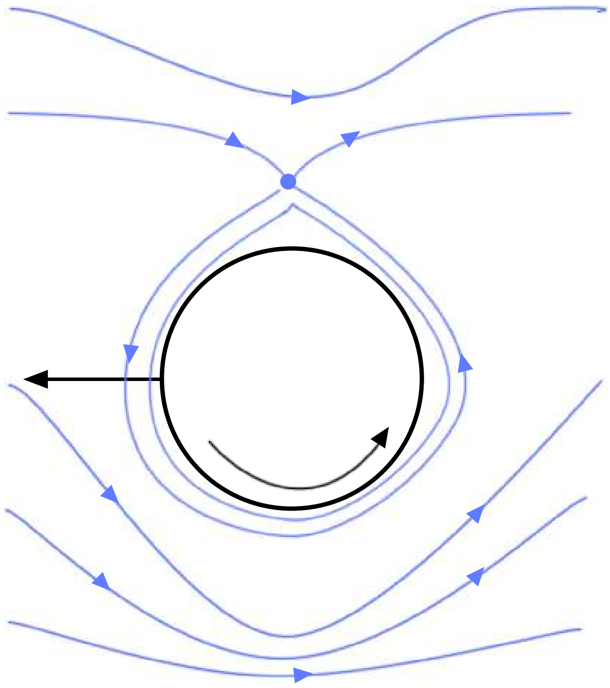

Figure 2.23: The ball is travelling right-to-left with topspin. $s > 1$.

2.4 The method of images

If the motion of a fluid in the plane (although this can work in 3D too) is due to a distribution of singularities (sources, sinks, vortices, all $\log z$ types, or higher singularities $(z - z_0)^n$) and there exists a curve C drawn in the plane *with no flow across* C, then the system of singularities on one side of C is the image of the systsem on the other side. (See Figure 2.24) The complex potential due to a source at $z = a$ of strength m (as in Figure 2.25) is

$$w_1(z) = \frac{m}{2\pi} \log(z - a)$$

where w_1 is in system A.

Example 2.15 Suppose we have a source at $z = a$ and suppose a solid wall lies along $x = 0$. Then there is no flow across $x = 0$, i.e. $x = 0$ is a curve C. What is system B? What is the image of system A? And what is the maximum speed along the wall?

Solution By symmetry, we expect system B to be a source of the same strength at $z = -a$, i.e.

$$w_2(z) = \frac{m}{2\pi} \log(z + a).$$

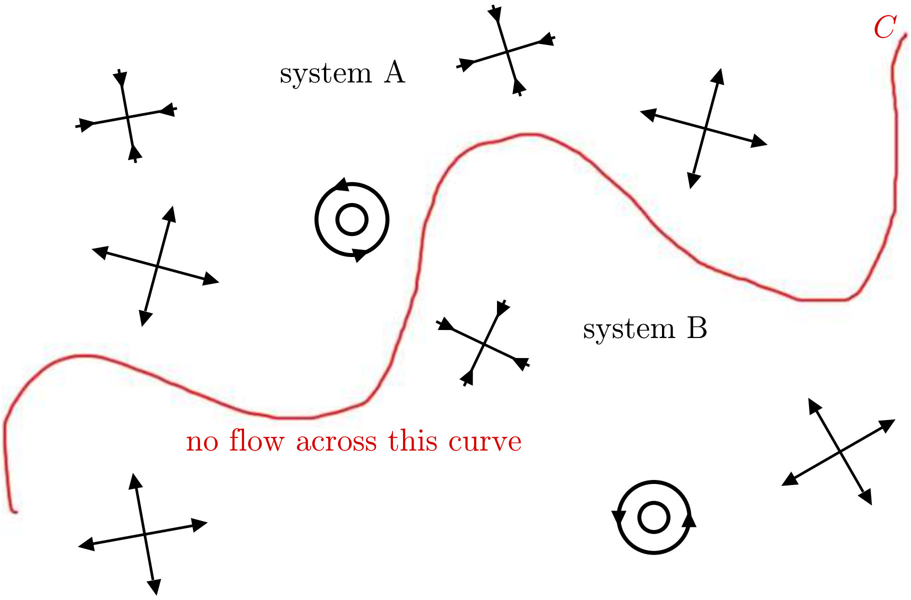

Figure 2.24: System A is the image of system B

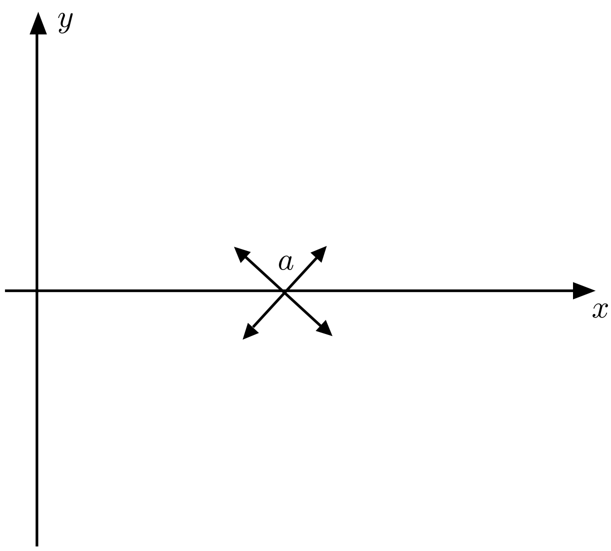

Figure 2.25: A source at $z = a$

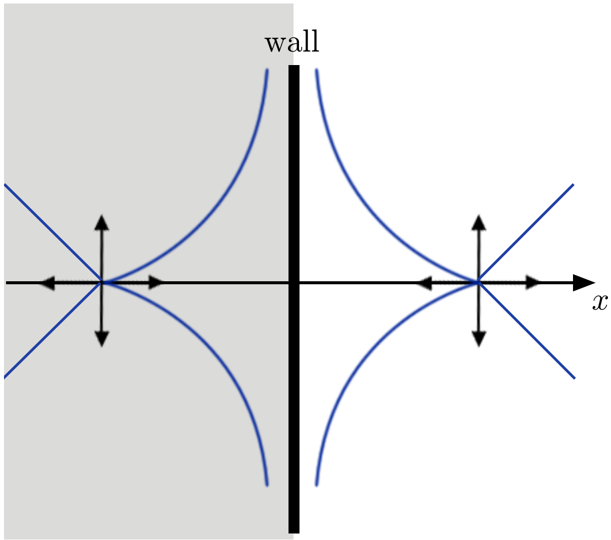

Figure 2.26: Note that we need the singularity at $z = -a$

Then the complex flow field is system A+ system B, i.e.

$$\begin{aligned} w(z) &= w_1(z) + w_2(z) \\ &= \frac{m}{2\pi}\log(z-a) + \frac{m}{2\pi}\log(z+a) \\ &= \frac{m}{2\pi}\log(z^2-a^2), \end{aligned}$$

as shown in Figure 2.26. Note that we need the singularity at $z = -a$, even though it is outside the flow field, since it lets us satisfy the boundary condition on the wall.

Now we find the maximum speed along the wall?

$$\begin{aligned} u - iv &= \frac{dw}{dz} \\ &= \frac{m \cdot 2z}{2\pi(z^2-a^2)} \end{aligned}$$

On the wall, $z = iy$, hence

$$u - iv = \frac{-iym}{\pi(y^2+a^2)}$$

Hence $u = 0$ as expected and

$$v = \frac{my}{\pi(a^2+y^2)}.$$

And so

$$v_{\max} = \frac{m}{2\pi a} \quad \text{at } y = a.$$

✓

Example 2.16 A source of strength m sits in the first quadrant at z_0 with walls $(x = 0, y > 0)$ and $(y = 0, x > 0)$, as in Figure 2.27.

What is the image system, and find the complex potential w at any point z.

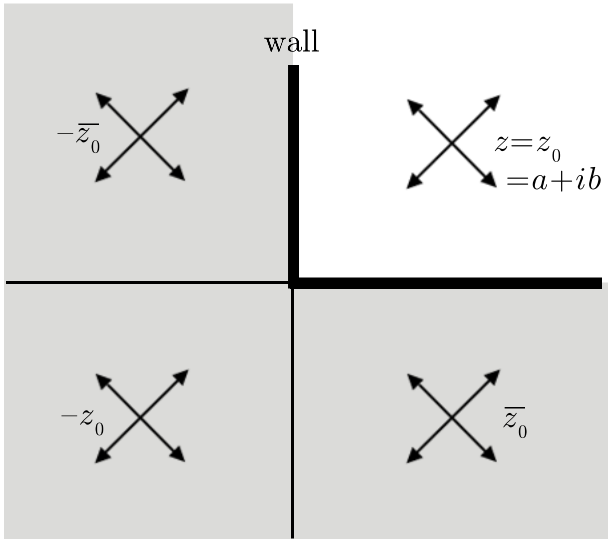

Figure 2.27: A source of strength m sits in the first quadrant at z_0 with walls $(x = 0, y > 0)$ and $(y = 0, x > 0)$

Solution The image system has 3 sources, each of strength m, at $-z_0, \pm\overline{z_0}$.

For our physical system, we have

$$w_1(z) = \frac{m}{2\pi}\log(z - z_0)$$

and our image system then has

$$w_2(z) = \frac{m}{2\pi}\log(z + z_0) + \frac{m}{2\pi}\log(z - \overline{z_0}) + \frac{m}{2\pi}log(z + \overline{z_0}).$$

So if we combine these, the full system has complex potential

$$\begin{aligned} w(z) &= w_1(z) + w_2(z) \\ &= \frac{m}{2\pi}\left[\log(z - z_0) + \log(z + z_0) + \log(z - \overline{z_0}) + \log(z + \overline{z_0})\right] \\ &= \frac{m}{2\pi}\left[\log(z^2 - z_0^2) + \log(z^2 - \overline{z_0}^2)\right] \end{aligned}$$

And we could now go on to find u, v on the walls by considering $dw/dz = u - iv$ on $x = 0$ and $y = 0$. The system is depicted in Figure 2.28 ✓

Example 2.17 A source sits between the lines $\theta = 0$ and $\theta = \pi/4$. What does the image source look like?

Solution See Figure 2.29. ✓

Example 2.18 A source sits between lines with arbitrary angle α. Can we use the same image method?

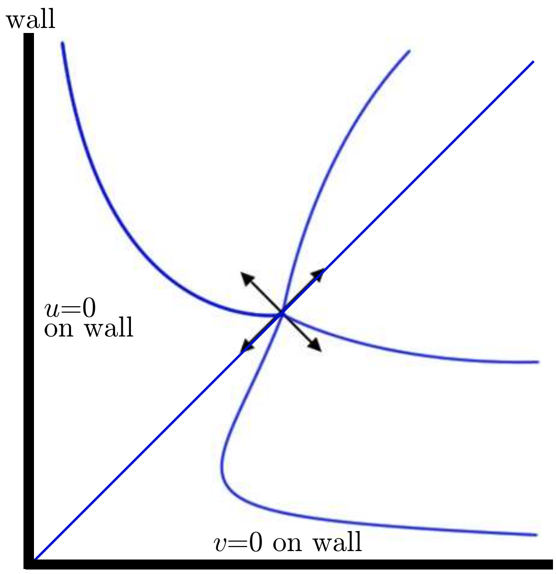

Figure 2.28: A source of strength m sits in the first quadrant

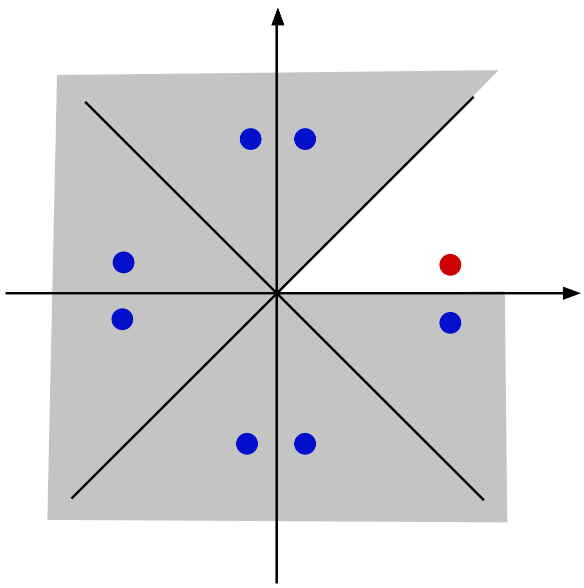

Figure 2.29: Solution to Example 2.17

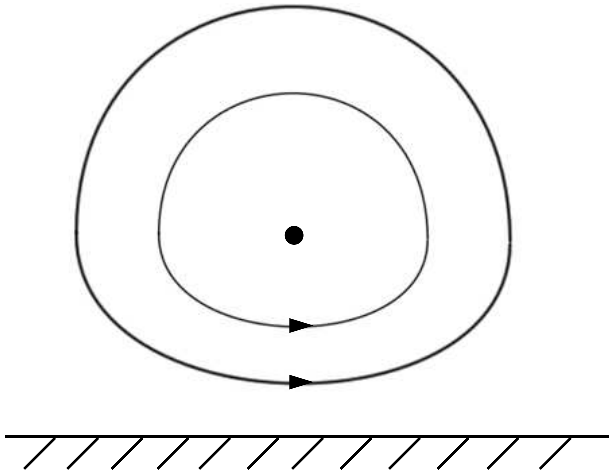

Figure 2.30: $\mathbf{u} \cdot \widehat{\mathbf{n}} = 0$, i.e. $v = 0$, on $y = 0$

Solution No—it only works with $\alpha = \pi/n$ for some integer n. ✓

Example 2.19 A vortex of strength κ sits at $z = z_0 = ia$ above the plane $y = 0$. What is the image? Find the complex potential for the whole system. How does the vortex propagate in time under the influence of its image?

Solution The image is a vortex of strength $-\kappa$ (because rotation is in the opposite direction) at $z = \overline{z_0} = -ia$. We know

$$\begin{aligned} w_1(z) &= -\frac{i\kappa}{2\pi}\log(z - z_0) \\ &= -\frac{i\kappa}{2\pi}\log(z - ia) \end{aligned}$$

and so

$$\begin{aligned} w_2(z) &= \frac{i\kappa}{2\pi}\log(z - (-ia)) \\ &= \frac{i\kappa}{2\pi}\log(z + ia) \\ &= \frac{i\kappa}{2\pi}\log(z - \overline{z_0}). \end{aligned}$$

Thus the complex potential for a vortex above a plate is

$$w(z) = w_1(z) + w_2(z) = \frac{i\kappa}{2\pi}\log\left(\frac{z + ia}{z - ia}\right),$$

as in Figure 2.30. We now find the velocity by differentiating:

$$\begin{aligned} u - iv &= \frac{dw}{dz} \\ &= -\frac{i\kappa}{2\pi(z - ia)} + \frac{i\kappa}{2\pi(z + ia)} \end{aligned}$$

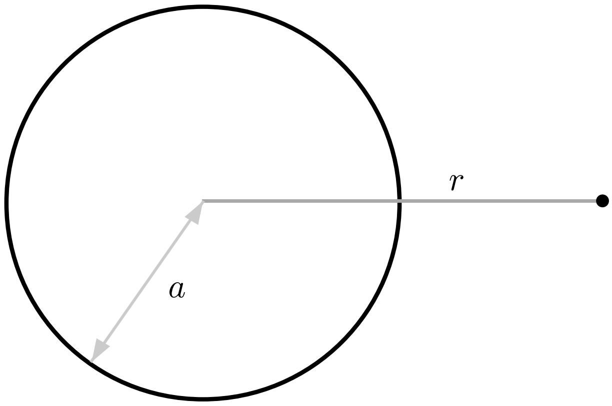

Figure 2.31: The optical image is at a^2/r along some line from the origin.

At the wall $y = 0$,

$$\begin{aligned} u - iv|_{y=0} &= -\frac{i\kappa}{2\pi(x - ia)} + \frac{i\kappa}{2\pi(x + ia)} \\ &= 2\,\mathrm{Re}\left(\frac{i\kappa}{2\pi(x + ia)}\right) \end{aligned}$$

and hence $v = 0$ as we would expect. Let's take a look at $u - iv$ again:

$$u - iv = \underbrace{-\frac{i\kappa}{2\pi(z - ia)}}_{\text{vortex}} + \underbrace{\frac{i\kappa}{2\pi(z + ia)}}_{\text{image}}$$

The vortex term is the velocity field if the plane were absent. This does not move the vortex. The image term is the velocity due to the image. At the vortex $z = ia$, it gives

$$\frac{i\kappa}{4i\pi a} = \frac{\kappa}{4\pi a}$$

i.e.

$$u = \frac{\kappa}{4\pi a}; \quad v = 0$$

and this moves the vortex.

Hence the vortex propagates to its right with speed $\kappa/4\pi a$ under the influence of its imagine in the plane. ✓

2.5 Circle theorem

Theorem 2.20 The image system for a circle $|z| = a$ of the complex potential $w_1(z) = f(z)$ where $f(z)$ has no singularities *inside* the circle $|z| < a$ is

$$w_2(z) = \overline{f}\left(\frac{a^2}{z}\right)$$

where for any function $g(z)$, $\overline{g}(z) = \overline{g(\overline{z})}$.

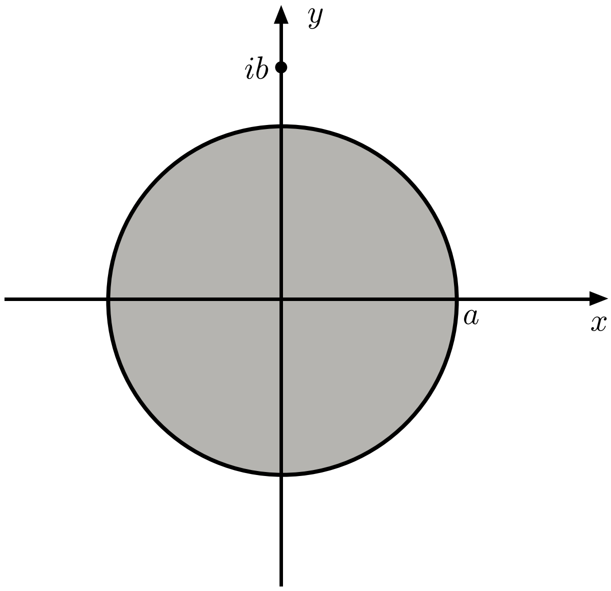

Figure 2.32: The image in Example 2.21

Proof: Notice $f(z)$ has no singularities in $|z| < a$, and so $f(a^2/z)$ has no singularities in $|z| > a$, so $w_2(z)$ has no singularities in $|z| > a$.

Thus w_1 and w_2 are candidates for image system in the curve C, $|z| = a$. We must show that there is no flow across C. Now,

$$\begin{aligned} w(z) &= w_1(z) + w_2(z) \\ &= f(z) + \overline{f}(a^2/z) \end{aligned}$$

which on $|z| = a$,

$$\begin{aligned} &= f(ae^{i\theta}) + \overline{f}(ae^{-i\theta}) \\ &= 2\,\mathrm{Re}\, f(ae^{i\theta}) \end{aligned}$$

which is purely real. But $w = \phi + i\psi$, hence $\psi = 0$ on $|z| = a$, i.e. $|z| = a$ is a streamline, i.e. there is no flow across it. □

Example 2.21 What is the flow field due to a source of strength m at $z = ib$ outside the cylinder $|z| = a$, where $a < b$?

Solution The image is shown in Figure 2.32.

The complex potential in absence of the cylinder is

$$w_1(z) = \frac{m}{2\pi} \log(z - ib)$$

i.e.

$$f(z) = \frac{m}{2\pi} \log(z - ib)$$

so

$$\begin{aligned}\overline{f}(z) &= \overline{\frac{m}{2\pi}\log\left(\overline{z}-ib\right)} \\ &= \frac{m}{2\pi}\log\left(\overline{\overline{z}-ib}\right) \\ &= \frac{m}{2\pi}\log(z+ib)\end{aligned}$$

And so

$$\begin{aligned}w_2(z) &= \overline{f}(a^2/z) \\ &= \frac{m}{2\pi}\log\left[\frac{a^2}{z}+ib\right] \\ &= \frac{m}{2\pi}\log\left[\frac{1}{z}ib\left(\frac{a^2}{ib}+z\right)\right] \\ &= \underbrace{-\frac{m}{2\pi}\log(z)}_{(1)} + \underbrace{\frac{m}{2\pi}\log(ib)}_{(2)} + \underbrace{\frac{m}{2\pi}\log\left(z-\frac{ia^2}{b}\right)}_{(3)}\end{aligned}$$

Thus the image consists of

(1) A sink of strength m at the origin: to absorb fluid emitted at the geometric image point

(2) A constant: doesn't do anything since $d(\text{const.})/dz \equiv 0$

(3) A source of strength m at ia^2/b: a geometric image point

So we can sketch the solution, shown in Figure 2.33. ✓

Example 2.22 Consider a vortex of strength κ outside a cylinder, $|z| = a$. What is the total complex potential?

Solution The solution can be seen in Figure 2.34.

$$\begin{aligned}w_1(z) &= \frac{i\kappa}{2\pi}\log(z-ib) \\ w_2(z) &= -\frac{i\kappa}{2\pi}\log(z) \\ &= \frac{i\kappa}{2\pi}\log(ib) \\ &= \frac{i\kappa}{2\pi}\log\left(z-\frac{ia^2}{b}\right)\end{aligned}$$

✓

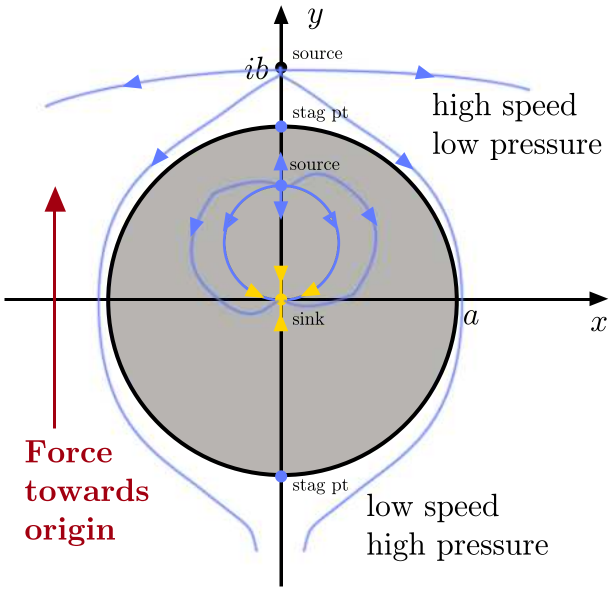

Figure 2.33: The solution to Example 2.21

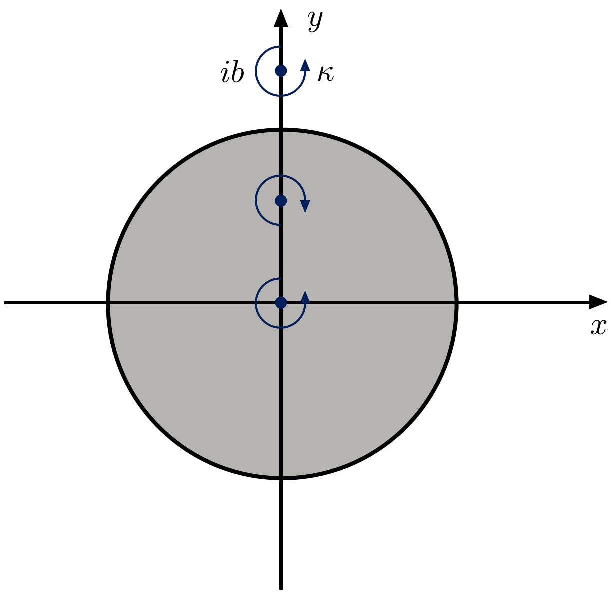

Figure 2.34: The solution to Example 2.22

Chapter 3

Dynamics

3.1 The material derivative

Consider this: we have a funnel (Figure 3.1) through which we pump water at a steady rate. The funnel goes inwards, so, as you expect, the water will speed up (it's like sticking your finger over the end of a hosepipe). If we observe a specific fluid particle (say we put a drop of pink dye in the water somewhere before it hits the funnel and follow its path), then as it goes through the funnel you will see that the particle speeds up—it accelerates. But we're pumping through the water at a steady rate. The problem, then, is this: how can the flow be both steady *and* accelerating. Surely they contradict each other?

This problem comes about because of the nature of liquids. In a solid, of course, you don't see this problem because solids don't change shape. So all the mechanics that you've done before this course haven't considered this problem. The answer, though, is a surprisingly nice one: if we're following a fluid particle (which we are), we have two different types of acceleration. The first type of acceleration is the one that you know and love: it's just $\partial\mathbf{u}/\partial t$, the totally standard time dependent acceleration. The second type, however, is the one that's new in fluids: it's acceleration *due to position*, and it's written $(\mathbf{u}\cdot\nabla)\mathbf{u}$. This notation is horrible, but don't worry about it. We're about to derive where it comes from, and then we'll see it over and over again in the next few weeks so you'll become more familiar with it. Just remember, it's acceleration due to position: this is the type of acceleration we see in our funnel.

When we talk about acceleration for a fluid particle, then, we consider *both* of these acceleration types together, and we have a notation for this,

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{due to time}} + \underbrace{(\mathbf{u}\cdot\nabla)\mathbf{u}}_{\text{due to position}}.$$

Now we'll derive where this comes from. For a particle,

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}).$$

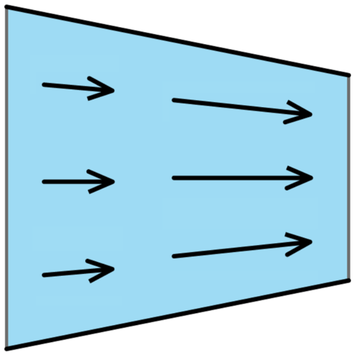

Figure 3.1: Consider fluid passing through a funnel.

We need to be able to discuss the rate of change with time following a fluid element. Suppose we have any quantity $\alpha(\mathbf{r}, t)$. Consider a fluid element that follows a path $\mathbf{r}(t)$. Then the value of α following this particle is $\alpha(\mathbf{r}(t), t)$. This is a function of time alone so we can take its deritvative with respect to time. Remember that

$$\frac{\partial \alpha}{\partial t} = \frac{d\alpha}{dt}(\mathbf{r}_0, t)$$

with $\mathbf{r}_0$ fixed.

We now introduce the operator

$$\frac{\mathrm{D}}{\mathrm{D}t}$$

for the rate of change *following a particle*. That is to say,

$$\begin{aligned}
\frac{\mathrm{D}\alpha}{\mathrm{D}t} &= \frac{d}{dt}\left[\alpha\big(\mathbf{r}(t), t\big)\right] \\
&= \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \alpha}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \alpha}{\partial z}\frac{\partial z}{\partial t} \\
&= \frac{\partial \alpha}{\partial t} + \nabla\alpha \cdot \frac{d\mathbf{r}}{dt} \\
&= \left[\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right]\alpha
\end{aligned}$$

i.e.

Definition 3.1 The *material derivative* is

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t} &= \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \\
\frac{\mathrm{D}}{\mathrm{D}t} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.
\end{aligned}$$

We can apply this to some fields to get a better idea of what this means intuitively.

Firstly, position, with displacement $\mathbf{r}$.

$$\begin{aligned}\frac{\mathrm{D}\mathbf{r}}{\mathrm{D}t} &= \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\left[x\,\widehat{\mathbf{\imath}} + y\,\widehat{\mathbf{\jmath}} + z\,\widehat{\mathbf{k}}\right] \\ &= u\,\widehat{\mathbf{\imath}} + v\,\widehat{\mathbf{\jmath}} + w\,\widehat{\mathbf{k}} \\ &= \mathbf{u}\end{aligned}$$

as expected!

Secondly, acceleration. Acceleration is the rate of change with time of $\mathbf{u}$ following a particle.

$$\begin{aligned}\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\left[\frac{dx}{dt}\widehat{\mathbf{\imath}} + \frac{dy}{dt}\widehat{\mathbf{\jmath}} + \frac{dz}{dt}\widehat{\mathbf{k}}\right] \\ &= \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\left[u\,\widehat{\mathbf{\imath}} + v\,\widehat{\mathbf{\jmath}} + w\,\widehat{\mathbf{k}}\right] \\ &= \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\widehat{\mathbf{\imath}} + \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\widehat{\mathbf{\jmath}} \\ &\quad + \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\widehat{\mathbf{k}} \\ &= \frac{\mathrm{D}u}{\mathrm{D}t}\widehat{\mathbf{\imath}} + \frac{\mathrm{D}v}{\mathrm{D}t}\widehat{\mathbf{\jmath}} + \frac{\mathrm{D}w}{\mathrm{D}t}\widehat{\mathbf{k}}\end{aligned}$$

which is half-way to Newton!

Remember that if

$$I(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$$

then

$$\begin{aligned}\frac{dI}{dt} &= \lim_{\delta t\to 0}\frac{I(t+\delta t) + I(t)}{\delta t} \\ &= \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(x,t)\,dx + f\big(\beta(t),t\big)\frac{d\beta}{dt} - f\big(\alpha(t),t\big)\frac{d\alpha}{dt}\end{aligned}$$

3.2 Reynolds transport theorem

Consider any quantity $\alpha(\mathbf{r},t)$ defined throughout a fluid domain D. Then the sub-volume V of D with surface S that always contains the same particles, i.e. that follows a material piece of the fluid.

Let the velocity field for the fluid motion be $\mathbf{u}(\mathbf{r},t)$. Let

$$I(t) = \int_{V(t)} \boldsymbol{\alpha}(\mathbf{r},t)\,dV.$$

What is $\frac{dI}{dt}$?

$$\frac{dI}{dt} = \lim_{\delta t \to 0} \frac{I(t+\delta t) - I(t)}{\delta t}$$

At time $t + \delta t$ let the particles occupy volume $V(t+\delta t) = V + \delta V$. Then

$$\frac{dI}{dt} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left(\int\limits_{V+\delta V} \alpha(\mathbf{r}, t+\delta t) dV - \int\limits_{V} \alpha(\mathbf{r}, t) dV \right)$$

Here

$$\begin{aligned} I(t+\delta t) &= \int\limits_{V(t+\delta t)} \alpha(\mathbf{r}, t+\delta t) dt \\ &= \int\limits_{V+\delta V} \left[\alpha(\mathbf{r}, t) + \delta t \frac{\partial \alpha}{\partial t}(\mathbf{r}, t) + \frac{1}{2}(\delta t)^2 \frac{\partial^2 \alpha}{\partial t^2}(\mathbf{r}, t) + \cdots \right] dV \end{aligned}$$

expanding as a Taylor series in δt about t.

Notice every quantity is evaluated at $\mathbf{r}$ and t so we take these as implicit, i.e.

$$\begin{aligned} I(t+\delta t) &= \int\limits_{V+\delta V} \left[\alpha + \delta t \frac{\partial \alpha}{\partial t} + \frac{1}{2}(\delta t)^2 \frac{\partial^2 \alpha}{\partial t^2} + \cdots \right] dV \\ &= \int\limits_{V} \left[\alpha + \delta t \frac{\partial \alpha}{\partial t} + \cdots \right] dV + \int\limits_{\delta V} \left[\alpha + \delta t \frac{\partial \alpha}{\partial t} + \cdots \right] dV \\ &= \int\limits_{V} \alpha \, dV + \delta t \int\limits_{V} \frac{\partial \alpha}{\partial t} dV + O[(\delta t)^2] + \int\limits_{\delta V} \alpha dV + \delta t \int\limits_{\delta V} \frac{\partial \alpha}{\partial t} dV + O[(\delta t)^2] \end{aligned}$$

and

$$I(t) = \int\limits_{V(t)} \alpha(\mathbf{r}, t) dV.$$

So

$$\begin{aligned} \frac{dI}{dt} &= \lim_{\delta t \to 0} \frac{1}{\delta t} \left[\delta t \int\limits_{V} \frac{\partial \alpha}{\partial t} dV + O[(\delta t)^2] + \int\limits_{\delta V} \alpha dV + \delta t \int\limits_{\delta V} \frac{\partial \alpha}{\partial t} dV + O[(\delta t)^2] \right] \\ &= \lim_{\delta t \to 0} \left[\int\limits_{V} \frac{\partial \alpha}{\partial t} dV + O(\delta t) + \frac{1}{\delta t} \int\limits_{\delta V} \alpha dV + \int\limits_{\delta V} \frac{\partial \alpha}{\partial t} dV + O(\delta t) \right] \\ &= \int\limits_{V} \frac{\partial \alpha}{\partial t} dV + \lim_{\delta t \to 0} \frac{1}{\delta t} \int\limits_{\delta V} \alpha \, dV + \lim_{\delta t \to 0} \int\limits_{\delta V} \frac{\partial \alpha}{\partial t} dV \end{aligned}$$

The last term is

$$\lim_{\delta t \to 0} \int\limits_{\delta V} \frac{\partial \alpha}{\partial t} dV.$$

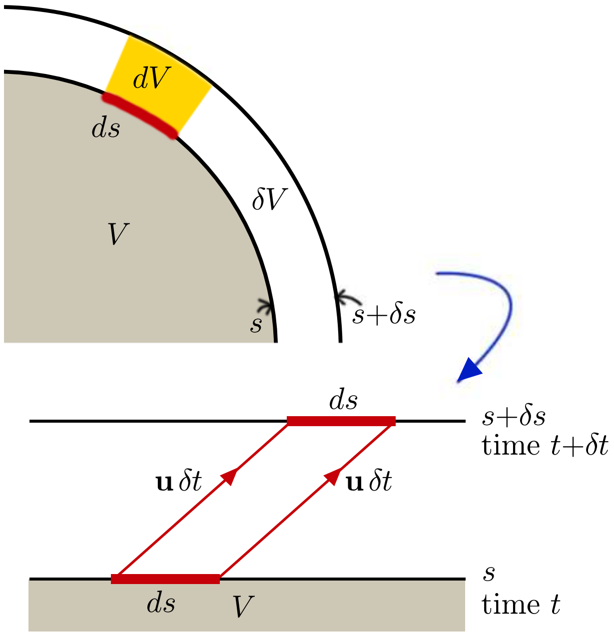

Figure 3.2: In time δt a particle moves a displacement $\mathbf{u}\,\delta t$. Thus a small element of area dS sweeps out a sliced cylinder with volume dV = area of base × height.

$$\left|\lim_{\delta t\to 0}\int_{\delta V}\frac{\partial\alpha}{\partial t}dV\right| = \lim_{\delta t\to 0}\left|\int_{\delta V}\frac{\partial\alpha}{\partial t}dV\right|$$

We expect the local rate of increase of α to be bounded, i.e.

$$\left|\frac{\partial\alpha}{\partial t}\right| < M$$

for some M. So

$$\begin{aligned}\lim_{\delta t\to 0}\left|\int_{\delta V}\frac{\partial\alpha}{\partial t}dV\right| &\le \lim_{\delta t\to 0}|\delta V|\cdot M\\ &= 0\end{aligned}$$

since $\delta V\to 0$ as $\delta t\to 0$. Thus

$$\frac{dI}{dt} = \int_V\frac{\partial\alpha}{\partial t}dV + \lim_{\delta t\to 0}\frac{1}{\delta t}\int_{\delta V}\alpha\,dV$$

as explained in Figure 3.2.

In time δt a particle moves a displacement $\mathbf{u}\,\delta t$. Thus a small element of area dS sweeps out a sliced cylinder with volume dV = area of base × height, or

$$dV = dS(\mathbf{u}\cdot\widehat{\mathbf{n}})\delta t.$$

The second term is

$$\lim_{\delta t\to 0}\frac{1}{\delta t}\int_S\alpha(\mathbf{u}\cdot\widehat{\mathbf{n}})dS\,\delta t = \int_S\alpha(\mathbf{u}\cdot\widehat{\mathbf{n}})dS$$

Hence

$$\frac{dI}{dt} = \int\limits_V \frac{\partial\alpha}{\partial t} dV + \int\limits_S \alpha(\mathbf{u} \cdot \widehat{n)} dS$$

i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t} \int\limits_V \alpha \, dV = \int\limits_V \frac{\partial\alpha}{\partial t} dV + \int\limits_S \alpha \mathbf{u} \cdot \widehat{\mathbf{n}} \, dS, \qquad \text{(RTT1)}$$

the sum of the local rate of change and the flux of α through the boundary of V.

We can use the divergence theorem:

$$\int\limits_V \nabla \cdot \mathbf{v} \, dV = \int\limits_S \mathbf{v} \cdot \widehat{\mathbf{n}} \, dS.$$

Thus

$$\frac{\mathrm{D}}{\mathrm{D}t} \int\limits_V \alpha \, dV = \int\limits_V \left[\frac{\partial\alpha}{\partial t} + \nabla \cdot (\alpha \mathbf{u}) \right] dV \qquad \text{(RTT2)}$$

But

$$\nabla \cdot (\alpha \mathbf{u}) = \alpha \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \alpha.$$

Thus

$$\begin{aligned} \frac{\partial\alpha}{\partial t} + \nabla \cdot (\alpha \mathbf{u}) &= \frac{\partial\alpha}{\partial t} + \alpha \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \alpha \\ &= \frac{\mathrm{D}\alpha}{\mathrm{D}t} + \alpha \nabla \cdot \mathbf{u} \end{aligned}$$

Hence

$$\frac{\mathrm{D}}{\mathrm{D}t} \int\limits_V \alpha \, dV = \int\limits_V \left[\frac{\mathrm{D}\alpha}{\mathrm{D}t} + \alpha \nabla \cdot \mathbf{u} \right] dV \qquad \text{(RTT3)}$$

3.2.1 RTT4

Now take $\alpha = \rho$, the fluid density.

Consider a fluid occupying a domain D. Take an *arbitrary* volume V of D (as in Figure 3.3). Then, because of the arbitrariness, if $\int_V f \, dV = 0$, we know $\int_V f \, dV$ for any $V \subset D$, and so if f is continuous, $f = 0$ in D. We're now going to use this argument.

Now by RTT2,

$$\frac{\mathrm{D}}{\mathrm{D}t} \int\limits_V \rho, dV = \int\limits_V \left[\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] dV.$$

By conservation of mass, the left hand side is always zero. Thus we have shown that for *all* subvolumes V of D,

$$\int\limits_V \left[\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] dV = 0.$$

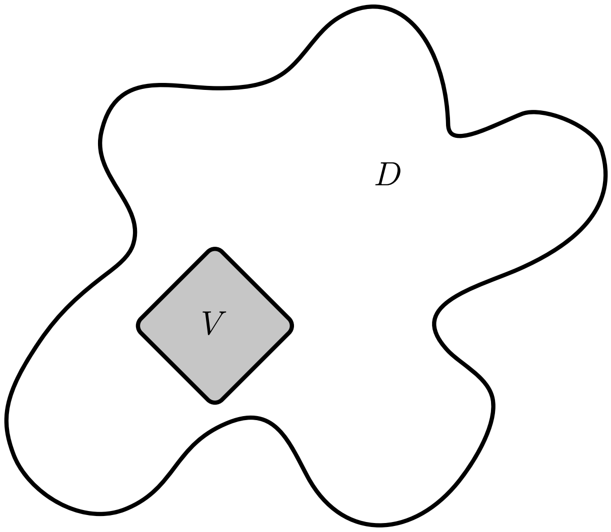

Figure 3.3: A domain D with arbitrary subdomain V.

Hence

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

everywhere in D.

This is conservation of mass for a fluid which may be compressible. Rearranging,

$$\frac{\partial \rho}{\partial t} + (u \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{u} = 0$$
$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u} = 0$$

and when we factor in incompressibility ($\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$), we get

$$\nabla \cdot \mathbf{u} = 0.$$

But notice this doesn't say that the density is constant.

So consider any quantity f, with density ρ.

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t} \int_V \rho f \, dV &= \int_V \left[\frac{\partial}{\partial t}(\rho f) + \nabla \cdot (\rho f \mathbf{u}) \right] dV \quad \text{(by RTT2)} \\
&= \int_V \left[\rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + \rho f \nabla \cdot \mathbf{u} + (\mathbf{u} \cdot \nabla)(f\rho) \right] dV \\
&= \int_V \left[f \left(\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \rho \right) + \rho \left(\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f \right) \right] dV
\end{aligned}$$

Now the first term,

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + u \cdot \nabla \rho = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

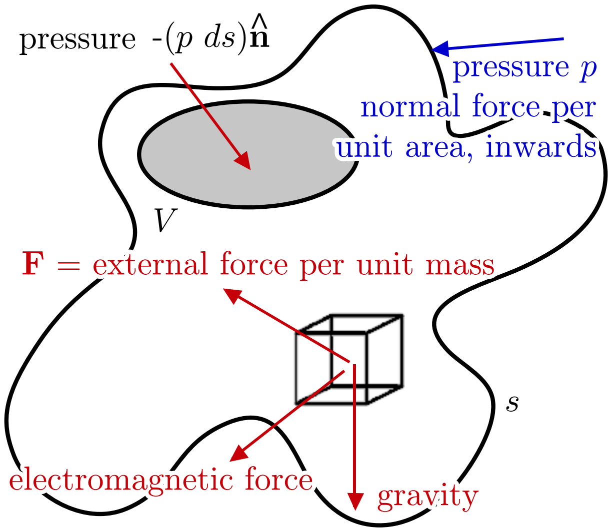

Figure 3.4: Forces acting on a blob.

by conservation of mass. Thus we get RTT4,

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V f\rho\, dV = \int_V \frac{\mathrm{D}f}{\mathrm{D}t}\rho\, dV \tag{RTT4}$$

Now what about momentum?

Take $\mathbf{f} = \mathbf{u}$ so $\rho\mathbf{f} = \rho\mathbf{u}$: momentum per unit volume.

Let our fluid domain by D. Take an arbitrary subvolume V of D. Consider

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{u}\, dV = \int_V \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\, dV$$

by RTT4. The left-hand side is the rate of change of momentum following the blob. By Newton, this is equal to the force acting on the blob (see Figure 3.4).

The total force acting on the blob is

$$\int_S -p\hat{\mathbf{n}}\, dS + \int_V \rho\mathbf{F}\, dV,$$

the sum of the pressure (left) and external (right) terms.*

*Thus begins the eternal battle between your handwriting's p and ρ. We'll be seeing a lot of these two throughout the rest of the course, so make sure that they are clearly distinguishable from each other—we recommend putting a tail on ρ, either inwards (ϱ) or outwards.

Thus

$$\int_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\, dV = \int_S -p\widehat{\mathbf{n}}\, dS + \int_V \rho \mathbf{F}\, dV$$
$$= -\int_V \nabla p\, dV + \int_V \rho \mathbf{F}\, dV$$

i.e.

$$\int_V \left(\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \nabla p - \rho \mathbf{F}\right) dV = 0$$

for *all* subvolumes V of D. Thus the integrand vanishes indentically everywhere in D, i.e.

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \nabla p - \rho \mathbf{F} = 0 \quad \text{in } D$$

which, rearranged gives us

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \mathbf{F}, \tag{Euler eqns}$$

the Euler equations, or Newton's law for an inviscid, possibly compressible fluid. So collectively we have

- Conservation of mass:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$
which gives us 2 scalar equations (in 2D),
- Euler equations:
$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \mathbf{F}$$
which gives us another 2 scalar equations (in 2D),
- Unknowns:
$$\rho, \quad p, \quad \mathbf{u}$$
which represent 5 unknowns,

In gas dynamics, we say that $p = f(\rho)$, which represents 5 unknowns. In geophysical fluid dynamics (GFD), we look at incompressible fluid, which gives us

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \quad \nabla \cdot u = 0$$

which represents 5 equations.

In homogeneous flow, $\rho =$ const. so $\nabla \cdot \mathbf{u} = 0$: incompressibility. So we have to solve $\nabla \cdot u = 0$ and the Euler equations.

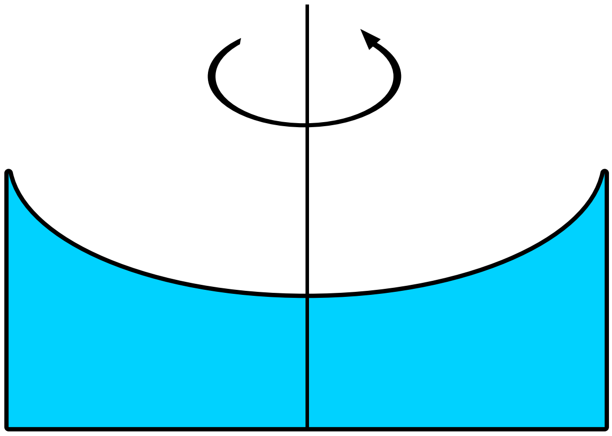

Figure 3.5: A fluid in solid body rotation.

Example 3.2 Consider a fluid in solid body rotation, with a free surface (see Figure 3.5). What is the shape of the surface if the only external force is gravity, i.e. $\mathbf{F} = -g\,\widehat{\mathbf{k}}$.

Solution Now in solid body rotation,

$$\begin{aligned}\mathbf{u} &= \boldsymbol{\Omega}\times\mathbf{r}\\ &= \Omega\,\widehat{\mathbf{k}}\times\mathbf{r}\\ &= \begin{vmatrix}\widehat{\imath} & \widehat{\jmath} & \widehat{\mathbf{k}}\\ 0 & 0 & \Omega\\ x & y & z\end{vmatrix}\end{aligned}$$

Note that this satisfies the incompressibility condition since

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0.$$

The flow is steady so $\frac{\partial}{\partial t} \equiv 0$. Then

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\\ &= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\\ &= -\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}\end{aligned}$$

First, in the x-direction, we get

$$\begin{aligned}-\frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{\mathrm{D}u}{\mathrm{D}t}\\ &= \left(-\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}\right)(-\Omega y)\\ &= -\Omega^2 x \qquad (3.1)\end{aligned}$$

and similarly in the y-direction we get

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = -\Omega^2 y \tag{3.2}$$

and in the z-direction we get

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

i.e.

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = g. \tag{3.3}$$

Now, taking equation 3.1 and integrating, we get

$$p = \frac{1}{2}\rho\Omega^2 x^2 + f(y,z)$$

Then

$$\frac{\partial p}{\partial y} = f_y = \rho\Omega^2 y$$

from equation 3.2. Thus

$$f(y,z) = \frac{1}{2}\rho\Omega^2 y^2 + f_2(z)$$

and

$$p = \frac{1}{2}\rho\Omega^2(x^2+y^2) + f_2(z).$$

This gives

$$\frac{\partial p}{\partial z} = f_2'(z) = -\rho g$$

from equation 3.3, i.e.

$$f_2 = -\rho g z + C$$

and

$$p = \frac{1}{2}\rho\Omega^2(x^2+y^2) - \rho g z + C.$$

The isobaric surfaces are the surfaces where $p = \text{const.}$, i.e.

$$\rho g z = \frac{1}{2}\rho\Omega^2(x^2+y^2) + \text{const.}$$

These are paraboloids of revolution, as in Figure 3.6.

$$z = \frac{\Omega^2}{2g}(x^2+y^2) + \text{const.}$$

At the surface, p is constant and is atmospheric prsesure, p_a (about 14psi). This surface is a paraboloid. ✓

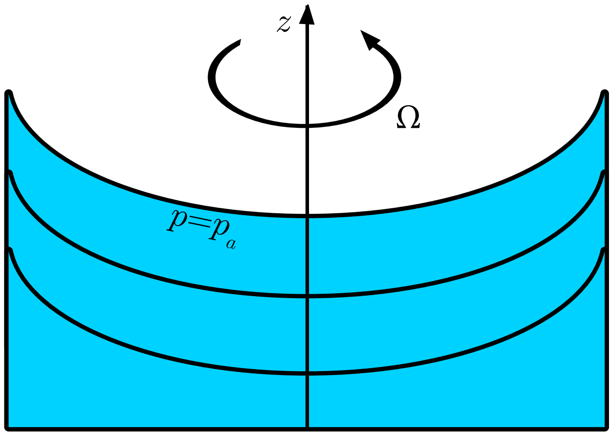

Figure 3.6: Paraboloids of revolution

3.3 Hydrostatic pressure

When the flow is at rest, $\mathbf{u} \equiv 0$. The Euler equations are

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - g\,\widehat{\mathbf{k}}$$

so

$$\nabla p = -\rho g\,\widehat{\mathbf{k}}$$

i.e.

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g$$

i.e.

$$\begin{aligned} p &= p_0 = \rho g z \\ &= p_h \quad \text{(hydrostatic pressure)} \end{aligned}$$

Example 3.3 Consider James Bond submerged in a fluid (say a shark tank) at rest (no sharks, yet)—see Figure 3.8.

Solution The fluid force on James's body is

$$\begin{aligned} \mathbf{B} &= \int_S (-p\widehat{\mathbf{n}})dS \\ &= -\int_V \nabla p\, dV \\ &= \int_V \rho g\,\widehat{\mathbf{k}}\, dV \\ &= \rho g\,\widehat{\mathbf{k}} \int_V dV \end{aligned}$$

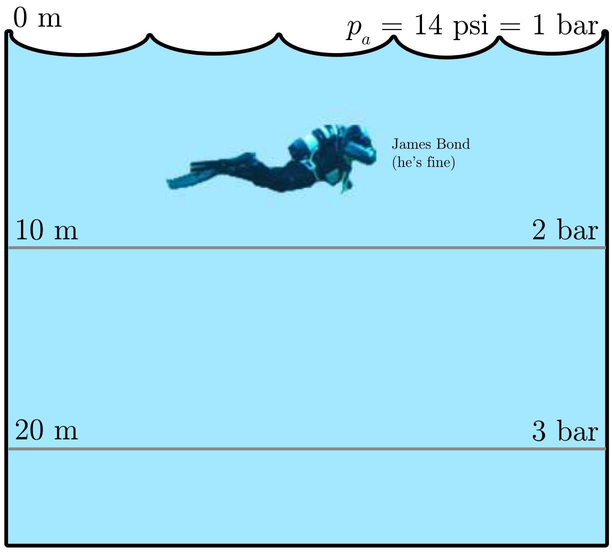

Figure 3.7: Pressure increases the further you swim down

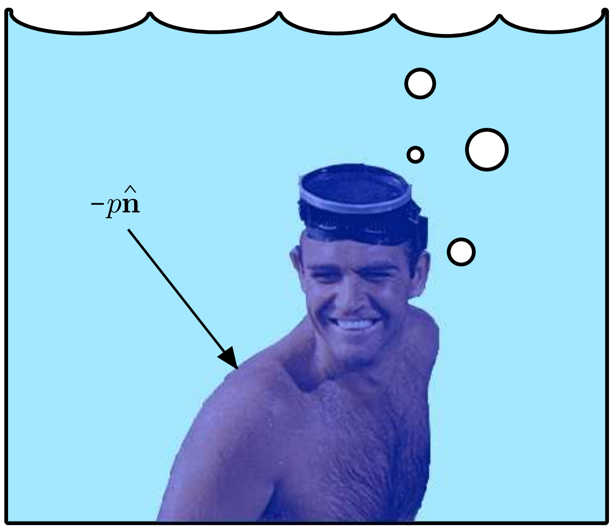

Figure 3.8: Bond is submerged in a shark tank

$$
\begin{aligned}
&= \rho g |V| \,\widehat{\mathbf{k}} \\
&= W \,\widehat{\mathbf{k}}
\end{aligned}
$$

where W is the weight of the water displaced. This is *Archimedes' principle!* ✓

The Euler equation, again, is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - g\,\widehat{\mathbf{k}}.$$

Introduce the dynamic pressure, p_d, which is the deviation from the hydrostatic pressure. Write

$$
\begin{aligned}
p &= p_h + p_d \\
\implies \quad \nabla p &= \nabla p_h + \nabla p_d \\
\nabla p &= -\rho g\,\widehat{\mathbf{k}} + \nabla p_d
\end{aligned}
$$

Then the Euler equation becomes

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p_d.$$

It is only p_d that accelerates the flow. The hydrostatic pressure simply balances the weight of the water. We can thus neglect gravity except where a free surface is involved, as there the boundary condition is $p = p_a$, i.e.

$$p_d + p_h = p_a.$$

3.4 Bernoulli equation

We have

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}^2\right) + \boldsymbol{\omega}\times\mathbf{u}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity. So

$$
\begin{aligned}
\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \\
&= \frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) + \boldsymbol{\omega}\times\mathbf{u}
\end{aligned}
$$

and the Euler equations become

$$\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F} - \nabla\left(\frac{1}{2}\mathbf{u}^2\right).$$

Let the external forces $\mathbf{F}$ be conservative and so derivable from a potential,

$$\mathbf{F} = -\nabla V_e$$

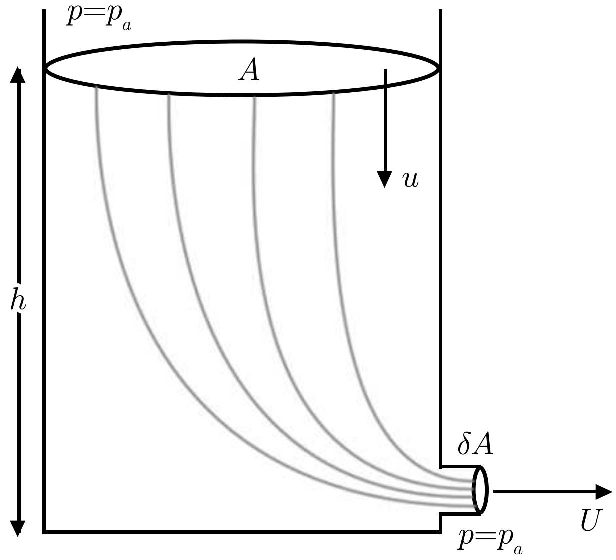

Figure 3.9: Our cup with a hole in the bottom, from Example 3.5. The streamlines are tangent to the velocity field and are marked in grey.

for some scalar function V_e (for gravity, $V_e = gz$). Then, where ρ is constant,

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u}\right] = -\nabla\left(p + \frac{1}{2}\rho\mathbf{u}^2 + \rho V_e\right) \tag{3.4}$$

$$= -\nabla H \tag{3.5}$$

for $H = p + \frac{1}{2}\rho\mathbf{u}^2 + \rho V_e$.

For steady flow, $\frac{\partial \mathbf{u}}{\partial t} = 0$ and if we dot equation 3.4 with $\mathbf{u}$ we get

$$\mathbf{u}\cdot\nabla H = 0$$

i.e.

$$\frac{\mathrm{D}H}{\mathrm{D}t} = \frac{\partial H}{\partial t} + (\mathbf{u}\cdot\nabla)H = 0$$

i.e. a particle retains its value of H; H is constant on particle paths; H is constant on streamlines (steady flow). This is Bernoulli's law.

Definition 3.4 *Bernoulli's law* says that, where ρ is constant,

$$H = p + \frac{1}{2}\rho\mathbf{u}^2 + \rho V_e$$

is constant on streamlines in *steady flow*.

Example 3.5 Consider a vessel open to the air with surface area A when the depth is h. Suppose a small hole of area δA is punctured at the bottom of the container, sufficiently small so that flow is approximately steady, with the surface falling at speed u, and the exit speed U. How fast does the fluid exit?

Solution Conservation of mass tells us that the volume flux at the surface and exit should be the same. The volume flux at the surface is uA and the volume flux at the exit is $U\,\delta A$. Equating the two gives us

$$u = \delta U$$

(i.e. $u \ll U$.)

We can now use Bernoulli's law but first we have to find streamlines. The streamlines join the surface to the outlet even though no particles have travelled from the surface to the outlet (for now!). But this is all we need for Bernoulli.

Thus $p + \frac{1}{2}\rho\mathbf{u}^2 + V_e$ is conserved along streamlines. Thus H is the same at the surface as it is at the exit.

$$\begin{aligned}
&\text{At surface} && p + \frac{1}{2}\rho\mathbf{u}^2 + V_e = p_a + \frac{1}{2}\rho u^2 + \rho g h \\
&\text{At exit} && p + \frac{1}{2}\rho\mathbf{u}^2 + V_e = p_a + \frac{1}{2}\rho U^2 + 0 \\
&\implies && \frac{1}{2}\rho(U^2 - u^2) = \rho g h \\
&\implies && U^2(1-\delta^2) = 2gh \\
&\implies \text{to } O(\delta^2) && U^2 \approx 2gh \\
&\implies && U \approx \sqrt{2gh} \\
&\implies && \text{Flux} = \sqrt{2gh}\,\delta A
\end{aligned}$$

The streamlines are marked in grey in Figure 3.9. Because pressure is the same at the surface and exit, the pressure energy cancels and so the problem is identical (for $\delta \ll 1$ to a free-falling particle. ✓

Example 3.6 Force on a spinning cylinder. Consider a cylinder of radius a, in a uniform stream of speed U in the $\widehat{\mathbf{\imath}}$ direction, and spinning such that the circulation about the cylinder is κ, as in Figure 3.10. What can we find out about it?

Solution The complex velocity potential is

$$w(z) = U\left[z + \frac{a^2}{z}\right] - \frac{i\kappa}{2\pi}\log z.$$

The total force per unit width into the page on the cylinder is

$$\mathbf{F} = \oint(-p\widehat{\mathbf{n}})d\ell.$$

Here, of course, $\widehat{\mathbf{n}} = \widehat{\mathbf{r}}$ (see Figure 3.11).

We call the component of $\mathbf{F}$ in the direction of the flow at infinity the drag, D, and the component of $\mathbf{F}$ in the perpendicular direction the lift, L:

$$D = \mathbf{F}\cdot\widehat{\mathbf{\imath}}, \qquad L = \mathbf{F}\cdot\widehat{\mathbf{\jmath}}.$$

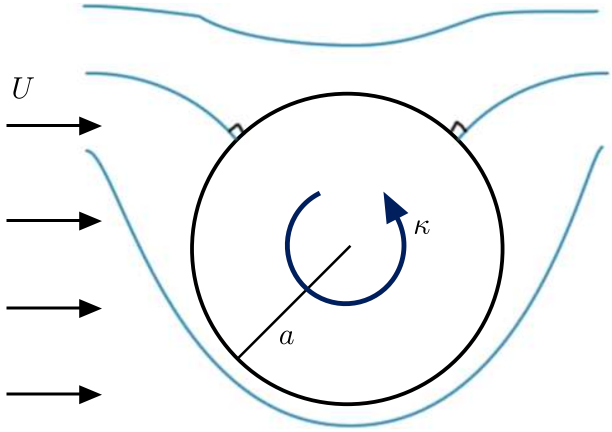

Figure 3.10: A spinning cylinder in a uniform stream.

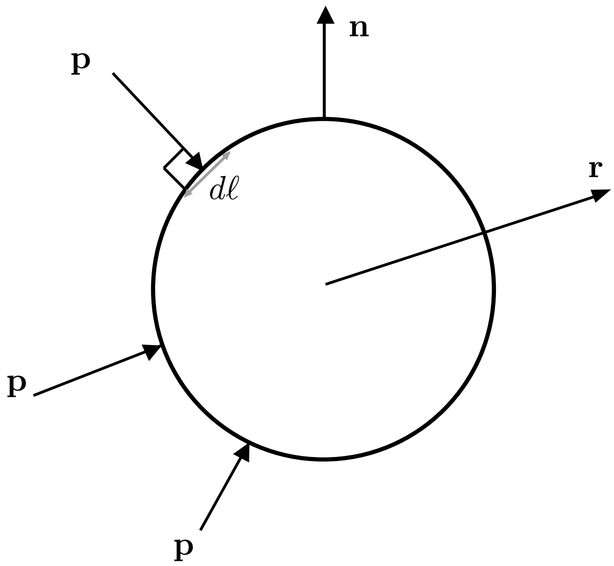

Figure 3.11: Pressure acts normal to the wall of the cylinder, in the $-\widehat{\mathbf{n}} = -\widehat{\mathbf{r}}$ direction.

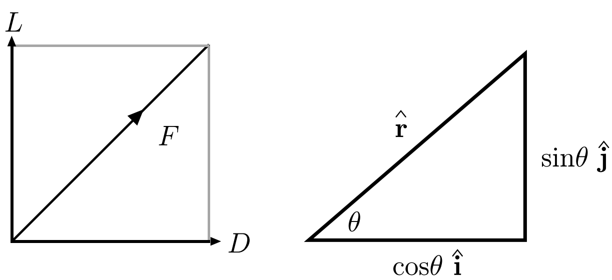

Figure 3.12: Geometrical considerations in our solution of Example 3.6.

(see Figure 3.12(a)). Now,

$$\mathbf{F} = -\oint p\,\widehat{\mathbf{r}}\,d\ell, \qquad \widehat{\mathbf{r}} = \widehat{\mathbf{r}}(\theta)$$

so

$$D = -\oint p\,\widehat{\mathbf{\imath}}\cdot\widehat{\mathbf{r}}\,d\ell, \qquad \widehat{\mathbf{\imath}} \neq \widehat{\mathbf{\imath}}(\theta)$$
$$L = -\oint p\,\widehat{\mathbf{\jmath}}\cdot\widehat{\mathbf{r}}\,d\ell$$

Now

$$\widehat{\mathbf{\imath}}\cdot\widehat{\mathbf{r}} = \cos\theta, \qquad \widehat{\mathbf{\jmath}}\cdot\widehat{\mathbf{r}} = \sin\theta, \qquad d\ell = a\,d\theta$$

(see Figure 3.12(b)). Thus

$$D = -a\int_{-\pi}^{\pi} p\cos\theta, d\theta$$
$$L = -a\int_{-\pi}^{\pi} p\sin\theta, d\theta$$

Bernoulli tells us that $p + \frac{1}{2}\rho\mathbf{u}^2$ is constant on streamlines. (We're ignoring gravity because the only effect of gravity is the buoyancy effect, and this is additive as Bernoulli is linear in V_e.)

All streamlines originate upstream where

$$\mathbf{u} = U\,\widehat{\mathbf{\imath}}, \qquad p = p_0 = \text{const.}$$

Thus

$$p + \frac{1}{2}\rho\mathbf{u}^2 = p_0 + \frac{1}{2}\rho U^2$$

and everywhere,

$$p = p_0 + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\mathbf{u}^2.$$

$\frac{1}{2}\rho\mathbf{u}^2$ here is non-negative, and is zero only at stagnation points. Thus the maximum pressure achieved is at a stagnation point, and we will call this p_s. Thus

$$p = p_s - \frac{1}{2}\rho\mathbf{u}^2$$

and we have the pressure on the cylinder provided that we have $\mathbf{u}^2$ on the cylinder.

On the cylinder, then,

$$\mathbf{u}^2 = u_r^2 + u_\theta^2$$
$$= u_\theta^2$$

as $u_r = 0$ on $r = a$. Also,

$$u_r - iu_\theta = e^{i\theta}\frac{\partial w}{\partial z}$$

where $z = ae^{i\theta}$.

On $r = a$, this gives

$$u_\theta = -2U\sin\theta + \frac{\kappa}{2\pi a}$$

(as we had worked out previously), and so

$$u_\theta^2 = 4U^2\sin^2\theta - \frac{4U\kappa}{2\pi a}\sin\theta + \frac{\kappa^2}{4\pi^2 a^2}$$

and

$$p = p_s - \frac{1}{2}\rho u_\theta^2 \qquad \text{on } r = a$$

Then the drag, D is

$$D = -a\int_{-\pi}^{\pi}\cos\theta p\, d\theta.$$

Now we know that, since $\{1, \cos n\theta, \sin n\theta\}$ is an orthogonal set over $\int_{-\pi}^{\pi}$,

$$\int_{-\pi}^{\pi}\cos\theta\, d\theta = 0$$
$$\int_{-\pi}^{\pi}\sin^2\theta\cos\theta\, d\theta = \int_{-\pi}^{\pi}\frac{1}{2}(1-\cos 2\theta)\cos\theta\, d\theta = 0$$
$$\int_{-\pi}^{\pi}\sin\theta\cos\theta\, d\theta = 0$$

(see Figure 3.13). Thus $D = 0$ as expected. The velocity field is symmetric fore and aft, so p is symmetric fore and aft. We have the same force on the front and back and so they cancel.

What about the lift L?

$$L = -a\int_{-\pi}^{\pi}\sin\theta p\, d\theta$$

and

$$p = p_s - \frac{1}{2}\rho u_\theta^2.$$

Now we know

$$\int_{-\pi}^{\pi}\sin\theta\, d\theta = 0$$
$$\int_{-\pi}^{\pi}\sin^3\theta\, d\theta = 0 \quad \text{(odd function)}$$
$$\int_{-\pi}^{\pi}\sin^2\theta\, d\theta = \frac{1}{2}2\pi = \pi$$

Thus

$$\begin{aligned} L &= -a\pi\cdot -\frac{1}{2}\rho\cdot -\frac{4\kappa U}{2\pi a} \\ &= -\rho U\kappa \end{aligned}$$

i.e. a downwards force (see Figure 3.13).

This is very important for aeroplanes... *and cricket!* ✓

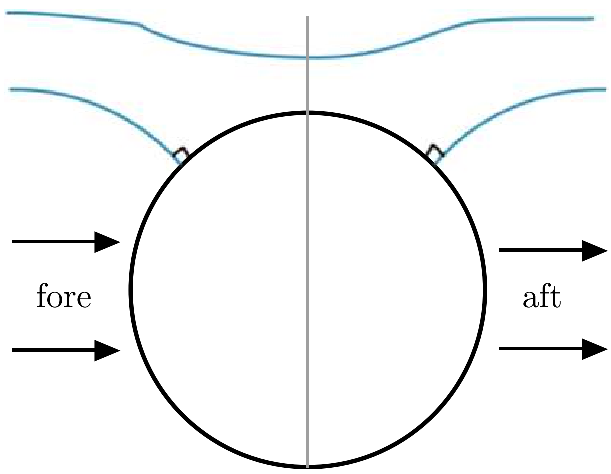

Figure 3.13: Fluid flowing past a cylinder

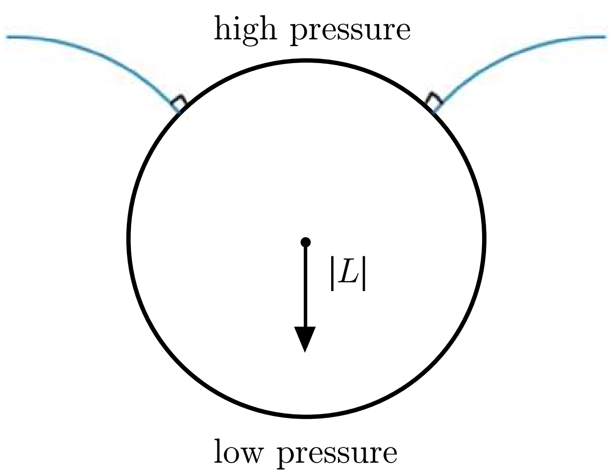

Figure 3.14: Pressure effects on the cylinder.

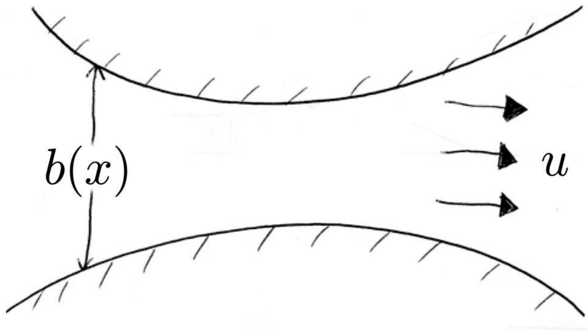

Figure 3.15: Plan view

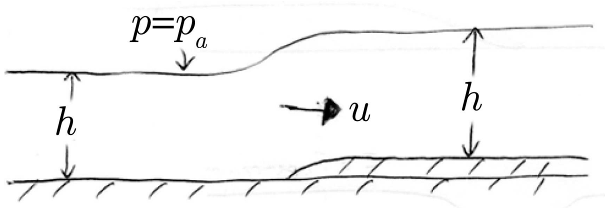

Figure 3.16: Elevation view

3.5 Open channel flow

This is the third, important example of Bernoulli.

We consider flow in a channel of possibly variable width b with a base which may also rise or fall, and a surface open to the atmosphere, as in Figures 3.15 and 3.16.

We consider channels where any changes in width or bottom height are so slow along the channel that the flow is independent of the coordinates across the channel and independent of depth.

$$u = u(x), \quad h = h(x)$$

where h is the local fluid depth.

Initially suppose the channel has constant width b, see figure 3.17

We first consider conservation of mass. The mass flux crossing the plane at station A is

$$\rho u_1 b h_1$$

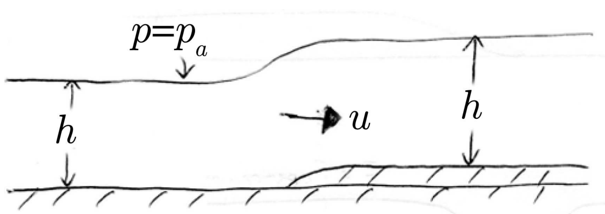

Figure 3.17: Elevation view of a channel of variable width

and the flux across the plane at B is

$$\rho u_2 b h_2.$$

These are the same by conservation of mass, and equating them gives

$$u_1 h_1 = u_2 h_2,$$

which we can take to mean

$$uh = Q = \text{const.}$$

where Q is the volume flux per unit width.

We now consider Bernoulli. We must consider steady flow. Then the surface is a streamline. Thus on the surface we have

$$p + \frac{1}{2}\rho \mathbf{u}^2 + V_e = \text{const.}$$

i.e.

$$p_a + \frac{1}{2}\rho u_1^2 + \rho g h_1 = p_a + \frac{1}{2}\rho u_2^2 + \rho g h_2$$

where we have taken the bottom to be horizontal. To repeat:

Important: this requires the bottom to be flat and we have taken V_e to be zero there.

This equation is equivalent, of course, to

$$u_1^2 + 2gh_1 = u_2^2 + 2gh_2$$

or

$$u^2 + 2gh = 2gH = \text{const.}$$

along the channel where H is a constant with dimensions height, i.e.

$$H = \frac{u^2}{2g} + h.$$

H provides an absolute maximum for the flow depth: it is the height of the surface, were the flow to be brought to rest. We call it the *pressure head.*

The motion, then, has two constants: Q and H.

We can eliminate u since $uh = Q$, hence $u = Q/h$. Then

$$H = \frac{Q^2}{2gh^2} + h$$

which we can plot as in figure 3.18

The graph has a minimum in $h > 0$ where $f(h) = 0$, i.e.

$$-\frac{Q^2}{gh_{\min}^3} + 1 = 0$$

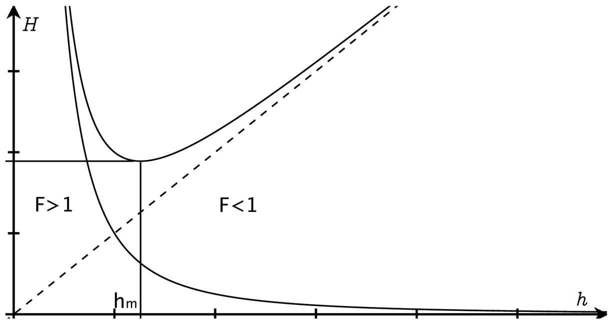

Figure 3.18: A plot of the pressure head

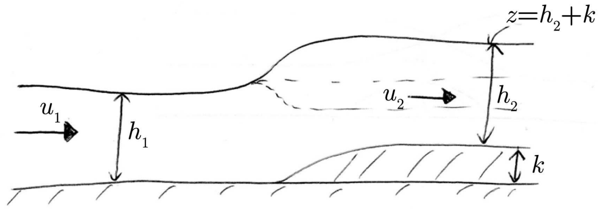

Figure 3.19: Will the surface rise, fall or stay level if we raise the floor of the channel?

i.e.

$$h_{\min}^3 = \frac{Q^2}{g}.$$

Here

$$\begin{aligned} H &= \frac{Q^2}{2gh_{\min}^3} h_{\min} + h_{\min} \\ &= \frac{3}{2} h_{\min}. \end{aligned}$$

Also

$$\begin{aligned} Q^2 &= gh_{\min}^3 \\ u_{\min}^2 h_{\min}^2 &= gh_{\min}^3 \\ \implies \frac{u_{\min}^2}{gh_{\min}} &= 1 \end{aligned}$$

At any point we write $F = u/\sqrt{gh}$, the Froude* number. Once again,

Definition 3.7 The *Froude* number, F, is

$$F = \frac{u}{\sqrt{gh}}.$$

Does the surface go up, does it go down, or does it stay flat in figure 3.18?

*Froude, to rhyme with food

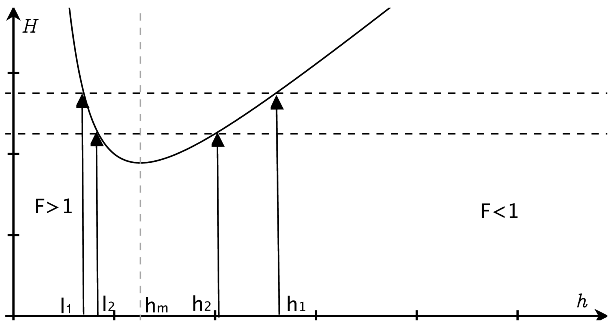

Figure 3.20: This is a graph of $y = f(h)$. The top horizontal line represents $y = f(h_1)$, the one below represents $y = f(h_2)$. The perpendicular distance between these lines is k.

Conservation of mass tells us

$$u_1 h_1 = u_2 h_2 = Q.$$

We want to use Bernoulli but for that we need a streamline. We use the surface as a streamline. It tells us, then, that

$$p_a + \frac{1}{2}\rho u_1^2 + \rho g h_1 = p_a + \frac{1}{2}\rho u_2^2 + \rho g(h_2 + k).$$

Eliminating u_1 and u_2 gives us

$$\frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + k$$

i.e.

$$f(h_1) = f(h_2) + k$$

where here $h_2 < h_1$. See Figure 3.20.

The rise in the surface is given by

$$\begin{aligned} r &= h_2 + k - h_1 \\ &= \frac{Q^2}{2g}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right) \\ &< 0 \end{aligned}$$

since $h_2 < h_1$.

Therefore it falls! See Figure 3.21. We start with a deep, slow flow with a high surface: low KE and high PE. We end with a shallow, fast flow with a lower surface, i.e. it converts some PE into KE to get over the bump.

This is what happens if the upstream flow is *subcritical*, which is what we call it when $F < 1$. More on that later.

Now, what if $h_2 > h_1$, so $r > 0$. Then the surface rises. But if you choose ℓ_1 and ℓ_2 instead of h_1 and h_2, we see Figure 3.22. We start with a shallow, fast flow with a low surface: low PE and high KE. We end with a deeper, slower flow and a high surface: high PE and low KE.

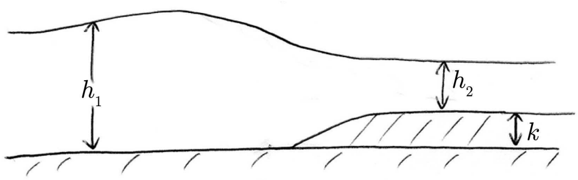

Figure 3.21: We start with a deep, slow flow with a high surface: low KE and high PE. We end with a shallow, fast flow with a lower surface, i.e. it converts some PE into KE to get over the bump.

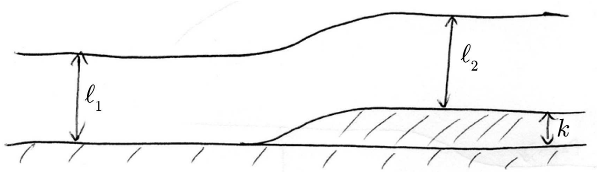

Figure 3.22: We start with a shallow, fast flow with a low surface: low PE and high KE. We end with a deeper, slower flow and a high surface: high PE and low KE.

This is what we get if the upstream flow is *supercritical*, which is what we say when $F > 1$. Again, more on that later.

Flow over a ridge then, can have two cases depending on F, shown in Figure 3.23.

However, there is a third option, as shown in Figure 3.24.

If $f(h_2) = f(h_m)$, then

$$k = f(h_1) - f(h_m)$$

But the flow which is observed is the flow where information about the obstacle height can propagate away from the obstacle, see Figure 3.25.

If $k > f(h_1) - f(h_m)$, the curve moves down, like in a weir, as in Figure 3.26. So if we know the depth h we know the speed U. So we know the flux $Q = Uh$. Flux is independent of the outer pressure.

3.5.1 Situation 2: A channel of variable width

Figures 3.27 and 3.28 show the plan and elevation views of a channel with variable width. For $F < 1$ we have shallow, and for $F > 1$ we have deeper. What is the upstream Froude number?

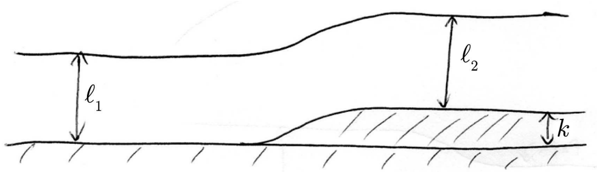

Figure 3.23: For flow over a ridge, there are two cases, depending on the value of F.

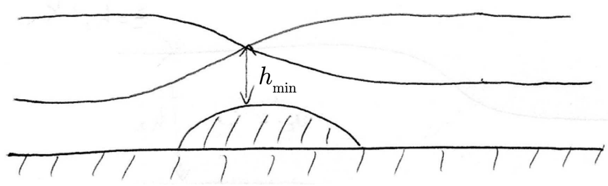

Figure 3.24: There is a third option.

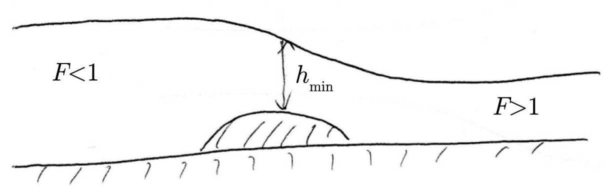

Figure 3.25: We observe a transition between $F < 1$ and $F > 1$. The critical point is directly above the centre of the lump.

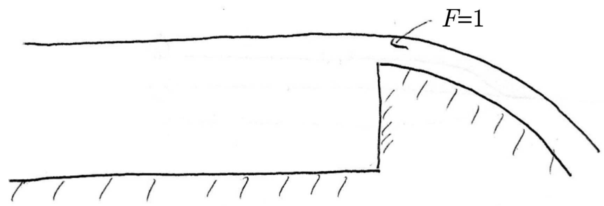

Figure 3.26: In a weir, the critical point, where $F = 1$, is directly above the edge of the top of the spout.

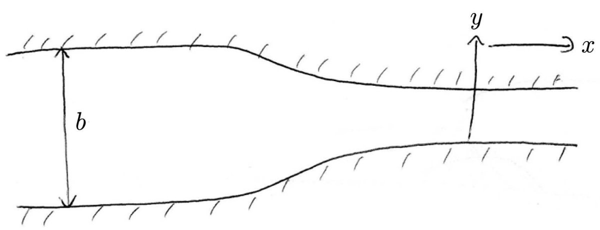

Figure 3.27: Plan view of a channel with variable width.

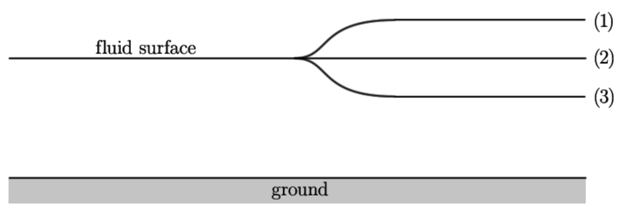

Figure 3.28: Elevation view of a channel with variable width.

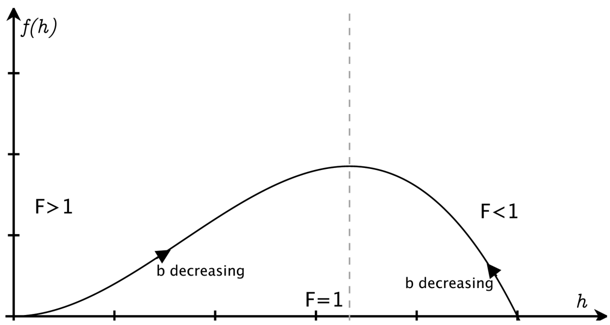

Figure 3.29: For a particle on the left-hand side, b decreasing leads to the flow deepening. For a particle on the right-hand side, b decreasing leads to the flow getting shallower.

Conservation of mass tells us

$$Q = ubh.$$

For Bernoulli, we need a streamline, so we pick the surface. We must be careful to use surface height and not depth: this is not always the same, although in this case it is. This tells us then that

$$p_a + \frac{1}{2}\rho u^2 + \rho g h = \text{const.}$$

i.e.

$$h + \frac{u^2}{2g} = H = \text{const.}$$

$$h + \frac{Q^2}{2gh^2b^2} = H$$

$$(H-h)h^2 = \frac{Q^2}{2gb^2} = f(h)$$

So our graph of $f(h)$ is given in Figure 3.29.

3.5.2 Situation 3

The next situation is shown in Figure 3.30. Once again,

$$f(h) = h^2(H-h) = \frac{Q^2}{2gb^2}$$

where H is constant. If b decreases, the RHS increases and our graph of $f(h)$ looks like Figure 3.31.

3.5.3 Situation 4

The next situation is shown in Figure 3.32. The graph, Figure 3.33, is the same graph as Figure 3.31, except the particles on the graph move in the opposite directions.

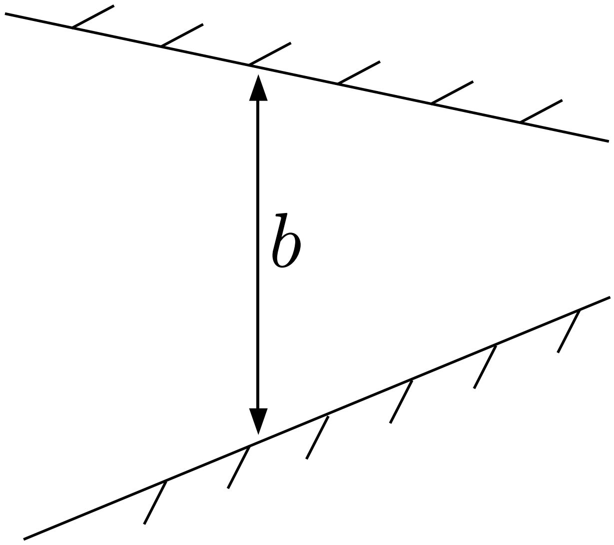

Figure 3.30: The banks are getting closer together.

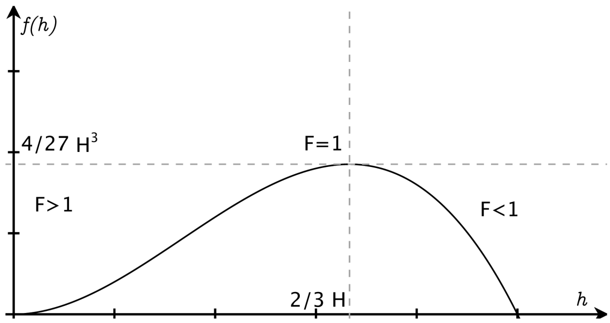

Figure 3.31: For a particle on the left-hand side, it travels upwards as b decreases. For a particle on the right-hand side, it also travels upwards (hence backwards) as b decreases.

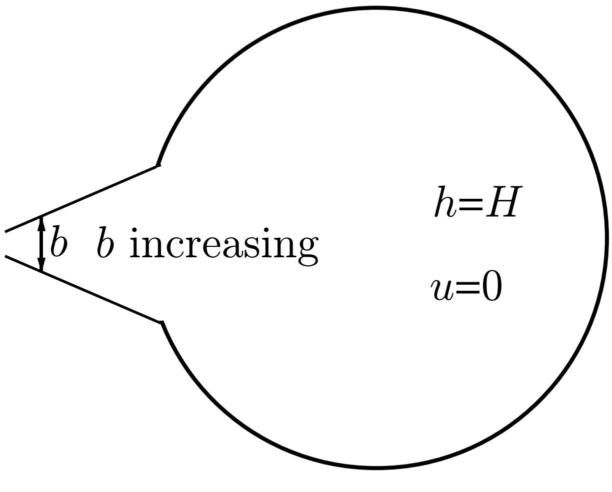

Figure 3.32: The banks are getting further away from each other.

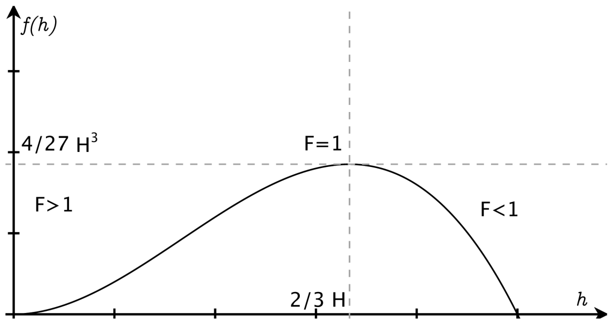

Figure 3.33: For a particle on the left-hand side, it travels downwards as b increases. For a particle on the right-hand side, it also travels downwards as b increases.

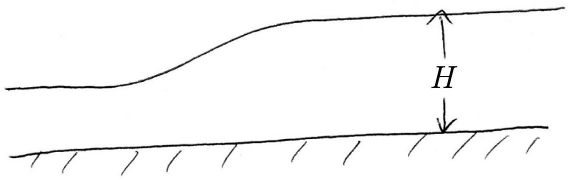

Figure 3.34: The depth smoothly increases to H as $b \to \infty$.

Subcritical upstream. The depth smoothly increases to H as $b \to \infty$, as in Figure 3.34.

Supercritical upstream. The depth smoothly decreases to 0 as $b \to \infty$, as in Figure 3.35. We get a jump, as in Figure 3.36.

Hydraulic jump: this is an exact analogy with a sonic boom, which you might be familiar with. The Mach number M is defined as

$$M = \frac{U}{c}$$

where c is the speed of *sound*.

If $M \sim 1$ in a compressible flow, we have transonic. If $F \sim 1$ in a free surface, we have transcritical.

In the wavy area in Figure 3.37 we have unsteady behaviour. Waves and sound are generated so energy is not conserved. Hence we cannot use Bernoulli.

Consider this sort-of conservation of momentum argument: The rate of change of x-momentum in the region between A and B = the force on the region in the x direction

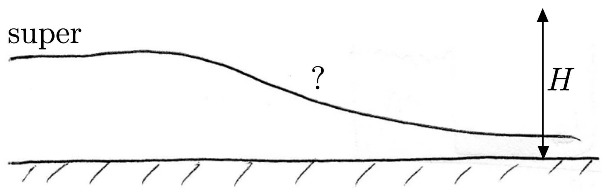

Figure 3.35: The depth smoothly increases to 0 as $b \to \infty$. Over on the left we have supercritical flow. No waves can travel upstream since $U > \sqrt{gh}$.

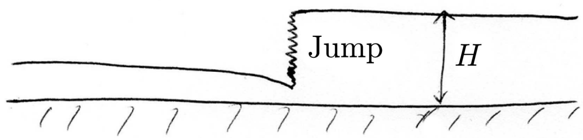

Figure 3.36: We end up with a jump, which is not Bernoulli!

$M = 1$	critical	sonic
$M > 1$	supercritical	supersonic
$M < 1$	subcritical	subsonic

$+$ momentum entering at A $-$ momentum leaving at B.

One side is sub, the other side is super.

Can we have downward jumps (Figure 3.38)? We can show that the change of energy ($E_{\text{in}} - E_{\text{out}}$) is proportional to $(h_2 - h_1)^3$ (this must be positive so $h_2 > h_1$, see Figure 3.39.

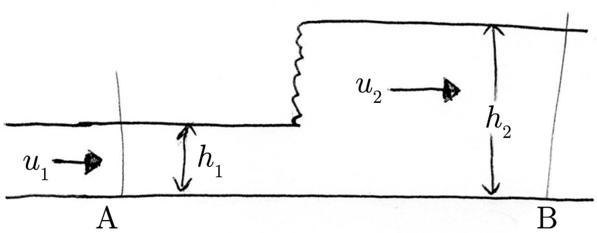

Figure 3.37: In the wavy region we have unsteady behaviour.

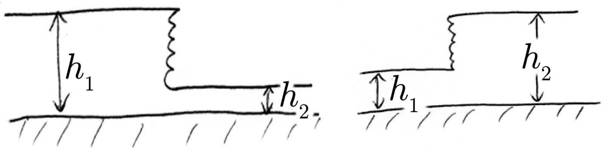

Figure 3.38: Can we have a downward jump?

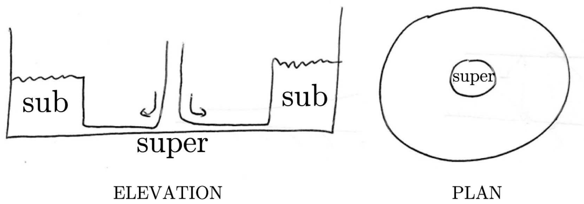

Figure 3.39: *Left:* If you turn the tap on in a sink (an actual sink), you see supercritical flow under the tap, and subcritical flow at the sides. *Right:* Top-down view of the sink.

Chapter 4

Free surface water waves

Figure 4.1 sets up our problem. We take our fluid to be 2D, incompressible, inviscid and thus irrotational. Recall that *irrotational* means that there exists a velocity potential ϕ such that $\mathbf{u} = \nabla\phi$, and that *incompressible* means that $\nabla \cdot \mathbf{u} = 0$. Putting these together, we get Laplace's equation, $\nabla^2\phi = 0$ on the interior $-h \leq y \leq \eta(x,t)$.

On the bottom we have the criterion that $\mathbf{u} \cdot \widehat{\mathbf{n}} = 0$, i.e. $\mathbf{u} \cdot \widehat{\mathbf{j}} = 0$. Since $\mathbf{u} = \nabla\phi$, we can write that last condition as

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = -h.$$

At the surface we have $y = \eta(x,t)$. We need, effectively one boundary condition to say where the surface is, and a second to say what is happening there. So there are *two* boundary conditions at $y = \eta(x,t)$:

1. $y = \eta(x,t)$ is the surface.

   A particle on the surface remains on the surface (see Figure 4.2). So following a

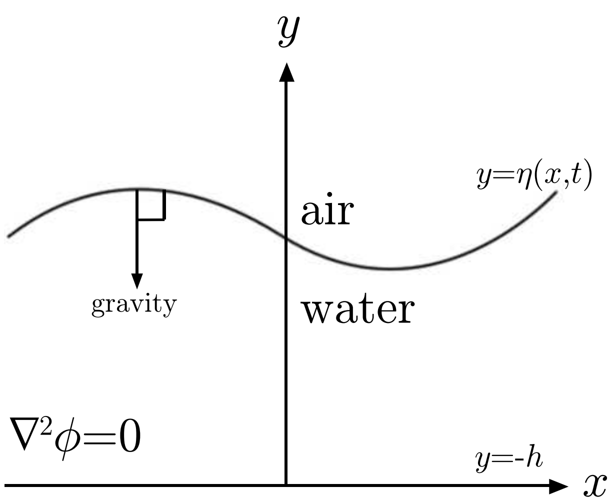

Figure 4.1: A free-surface water wave

Figure 4.2: The boundary condition on the surface

particle,

$$y - \eta(x,t) = 0 \quad \forall t$$

Thus

$$\frac{\mathrm{D}}{\mathrm{D}t}\Big(y - \eta(x,t)\Big) = 0,$$

i.e.

$$v - \frac{\mathrm{D}\eta}{\mathrm{D}t} = 0 \quad \text{on } y = \eta(x,t)$$

where v is the y-component of $\mathbf{u}$. Since $\mathbf{u} = \nabla\phi$, in components this is $u = \partial\phi/\partial x, v = \partial\phi/\partial y$, i.e.

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= \frac{\mathrm{D}\eta}{\mathrm{D}t} \\ &= \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} \\ &= \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} \quad \text{on } y = \eta(x,t)\end{aligned}$$

2. $p = p_a$ at $y = \eta(x,t)$.

   We need a new version of the Bernoulli equation, which we can't use in its existing form since the flow is not steady. Remember the Euler equation:

$$\begin{aligned}\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla p + \mathbf{F} \\ &= -\frac{1}{\rho}\nabla p + \nabla V_e\end{aligned}$$

   so

$$\frac{\partial\mathbf{u}}{\partial t} + \underbrace{(\boldsymbol{\omega}\times\mathbf{u})}_{0\text{ since irrotational}} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) = -\frac{1}{\rho}\nabla p + \mathbf{F}$$

   and $\mathbf{u} = \nabla\phi$, so $\partial u/\partial t = \nabla(\partial\phi/\partial t)$. Hence we have

$$\nabla\left(p + \frac{1}{2}\rho\mathbf{u}^2 + \rho g y + \rho\frac{\partial\phi}{\partial t}\right) = 0.$$

   Thus

$$p + \frac{1}{2}\rho\mathbf{u}^2 + \rho g y + \rho\frac{\partial\phi}{\partial t} = G(t).$$

   Without loss of generality, set $G \equiv 0$,* so on the surface $y = \eta(x,t)$,

$$\frac{\partial\phi}{\partial t} = \frac{1}{2}\left|\nabla\phi\right|^2 + g\eta = 0.$$

*You may not be happy with this. In that case, consider this. Say $G \not\equiv 0$. Then write $\widetilde{\phi} = \phi - \frac{1}{\rho}\int_0^t G(s)\,ds$, hence $\frac{\partial\widetilde{\phi}}{\partial t} = \frac{\partial\phi}{\partial t} - \frac{G}{\rho}$. Also $\nabla\widetilde{\phi} \equiv \nabla\phi$, so we can just use $\widetilde{\phi}$ as our velocity potential instead. This shows that, as stated, we lose no generality by setting $G \equiv 0$.

So in summary we have a system set up with four governing equations:

1. $\nabla^2\phi = 0$ on $-h \le y \le \eta(x,t)$.
2. $\frac{\partial \phi}{\partial y} = 0$ on $y = -h$.
3. $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = 0$ on $y = \eta(x,t)$ (kinematic boundary condition).
4. $\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0$ on $y = \eta(x,t)$ (Bernouilli unsteady dynamic boundary condition).

Note that equation 2 is linear, but equations 3 and 4, on the surface, are nonlinear.

Together, these four equations pose a very hard problem, and there is no known general solution. Finding a general solution is a challenge we leave to the analysts, and for now we simplify by only considering small waves, where $\eta \ll h$. We say that the amplitude of the small waves is ε, where $0 < \varepsilon \ll 1$. So $\eta, u, v, \phi \sim \varepsilon$, that is to say that they are all of the same 'size' as ε. Now what about our equations?

Our lower boundary condition, $\partial\phi/\partial y = 0$ on $y = -h$ is unchanged.

Our governing equation $\nabla^2\phi = 0$ on $-h \le y \le \eta$ is also unchanged.

However the next two equations will be different. Consider any function of y, $f(y)$, and Taylor expand it:

$$f(\eta) = \underbrace{f(0)}_{\sim 1} + \underbrace{\eta f'(0)}_{\sim\varepsilon} + \underbrace{\frac{\eta^2}{2!} f''(0)}_{\sim\varepsilon^2} + \cdots$$

For any function evaluated on the surface $y = \eta(x,t)$ (where $|\eta| < \varepsilon \ll 1$) we can replace this by the same function evaluated on $y = 0$, with error of order ε, which can be made arbitrarily small. So let's apply the surface boundary condition on $y = 0$ with error of order ε.

The kinematic boundary condition is

$$\underbrace{\frac{\partial \phi}{\partial y}}_{\sim\varepsilon} = \underbrace{\frac{\partial \eta}{\partial t}}_{\sim\varepsilon} + \underbrace{\underbrace{\frac{\partial \phi}{\partial x}}_{\sim\varepsilon}\underbrace{\frac{\partial \eta}{\partial x}}_{\sim\varepsilon}}_{\sim\varepsilon^2} = 0$$

So the terms are in ratio

$$1 : 1 : \varepsilon.$$

The third term, then, is arbitrarily small compared to the other two as $\varepsilon \to 0$, so we can *linearise* this boundary condition by saying

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{on } y = 0.$$

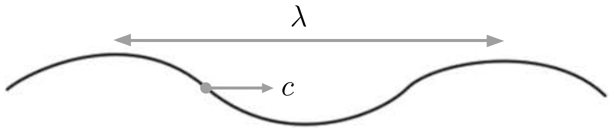

Figure 4.3: Wavelength and wave speed.

The dynamic boundary condition is

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\sim\varepsilon} + \underbrace{\frac{1}{2}|\nabla\phi|^2}_{\sim\varepsilon^2} + \underbrace{g\eta}_{\varepsilon} = 0$$

so the terms are in ratio

$$1 : \varepsilon : 1.$$

We neglect the middle term in the limit $\varepsilon \to 0$ and

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on } y = 0.$$

So in summary, for linear water waves in a fluid between $y = -h$ and $y = 0$, we have

1. $\nabla^2\phi = 0$ on $-h \leq y \leq 0$.
2. $\frac{\partial \phi}{\partial y} = 0$ on $y = -h$.
3. $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0$ (kinematic boundary condition).
4. $\frac{\partial \phi}{\partial t} + g\eta = 0$ on $y = 0$ (dynamic boundary condition).

It is sufficient to consider a sinusoidal wave as by Fourier superposition we can construct arbitrary initial conditions. Hence consider

$$\eta = \varepsilon \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

where ε is the amplitude, λ is the wavelength and c is the wave speed. The period $T = \lambda/c$ and the frequency $\omega = 2\pi c/\lambda$. We call $k = 2\pi/\lambda$ the *wavenumber* (thanks to physicists) and we rewrite the above equation as

$$\eta = \varepsilon \sin(kx - \omega t),$$

see Figure 4.3 for pictorial form.

The kinematic surface boundary condition is $\partial\phi/\partial y = \partial\eta/\partial t$ on $y = 0$, which becomes

$$\frac{\partial \phi}{\partial y} = -\varepsilon\omega\cos(kx - \omega t) \quad \text{on } y = 0.$$

So try

$$\phi(x, y, t) = -\varepsilon\omega\cos(kx - \omega t)Y(y).$$

Then we require $Y'(0) = 1$ to satisfy the kinematic boundary condition. Now

$$\nabla^2\phi = -\varepsilon\omega\left[-k^2\cos(kx-\omega t)Y + \cos(kx-\omega t)Y''\right]$$

(recalling that $\nabla^2\phi = \phi_{xx} + \phi_{yy}$) which must vanish for all x, t so

$$Y'' - k^2 Y = 0, \quad -h \le y \le 0.$$

The bottom boundary condition is that $\partial\phi/\partial y = 0$ on $y = -h$, which becomes

$$-\varepsilon\omega\cos(kx-\omega t)Y'(-h) = 0.$$

We need this to be true for all x, t, which is only true if

$$Y'(-h) = 0.$$

So in summary, we have

1. $Y'' - k^2 Y = 0$
2. $Y'(0) = 1$
3. $Y'(-h) = 0$.

and we know how to solve this! The complementary function is

$$Y(y) = A\sinh ky + B\cosh ky$$

but we could actually pick

$$Y(y) = A\sinh[k(y+h)] + B\cosh[k(y+h)]$$

because it behaves nicely at $y = -h$ where we have our homogeneous boundary condition. So then

$$Y'(y) = Ak\cosh[k(y+h)] + Bk\sinh[k(y+h)].$$

At $y = -h$, $Y' = Ak$ but $Y'(-h) = 0$, so $A = 0$.

At $y = 0$, $Y' = Bk\sinh kh$ but $Y'(0) = 1$, so $B = 1/(k\sinh kh)$.

Thus,

$$\begin{aligned}\phi(x,y,t) &= -\varepsilon\omega\cos(kx-\omega t)Y(y)\\ &= \varepsilon\omega\cos(kx-\omega t)\frac{\cosh[k(y+h)]}{k\sinh kh}\\ &= \varepsilon c\cos(kx-\omega t)\frac{\cosh[k(y+h)]}{\sinh kh}\end{aligned}$$

which, as we expected, has dimensions of speed c and amplitude ε.

We have yet to satisfy the condition

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on } y = 0,$$

i.e.

$$-\varepsilon c\omega \sin(kx - \omega t)\frac{\cosh kh}{\sinh kh} + g\varepsilon \sin(kx - ct) = 0 \quad \forall x, t.$$

Now because this is for all x, t, we can cancel out the sin terms, as well as the ε which obviously cancels normally, and we get

$$c\omega \coth(kh) = g.$$

Notice that $\omega = ck$, and then we get

$$c^2 k \coth(kh) = g,$$

we can rearrange this, and divide both sides by h to make it non-dimensional, to get

$$\begin{aligned}\frac{c^2}{gh} &= \frac{\tanh(kh)}{kh} \\ &= \frac{\lambda}{2\pi h}\tanh\left(\frac{2\pi h}{\lambda}\right)\end{aligned}$$

The relation says that the speed of a wave depends on its *wavelength*. These waves are known as *dispersive* waves. Different wave lengths travel at different speeds.

We can plot the equation above. Say $\theta = 2\pi h/\lambda$.

If $|\theta| \gg 1$, then $\tanh\theta \sim 1$, so

$$\frac{c^2}{gh} = \frac{\lambda}{2\pi h}$$

i.e. $c \sim \sqrt{\lambda}$. This corresponds to *deep water*.

If $|\theta| \ll 1$, then $\tanh\theta \sim \theta$, so

$$\frac{c^2}{gh} = \frac{\lambda}{2\pi h}\frac{2\pi h}{\lambda} = 1$$

i.e. $c \sim \sqrt{g\lambda}$. This corresponds to *shallow water*, see Figure 4.4.

This answers why waves come in parallel to the beach: see Figure 4.5.

And it also answers how a pulse spreads out: see Figure 4.6. Water does Fourier transforms for you!

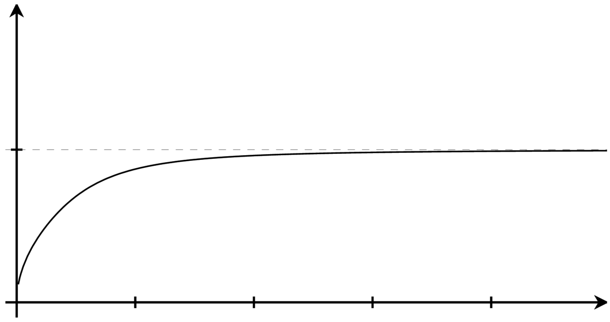

Figure 4.4: This corresponds to deep water

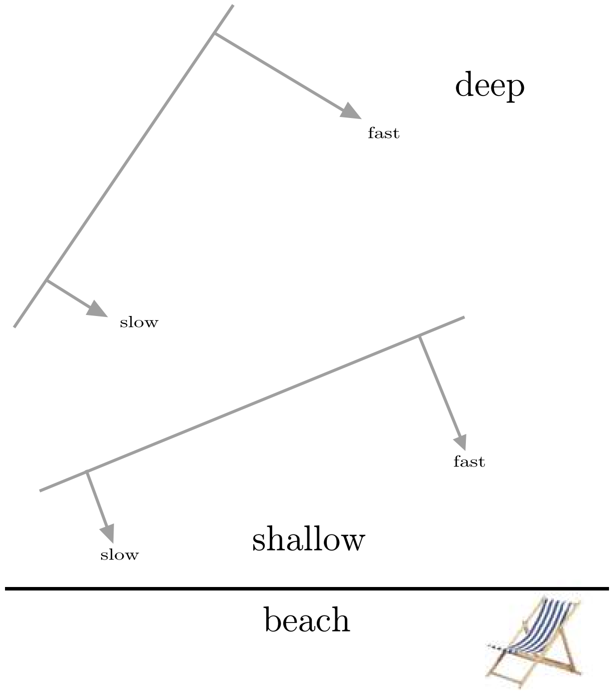

Figure 4.5: This is why waves come in parallel to the beach

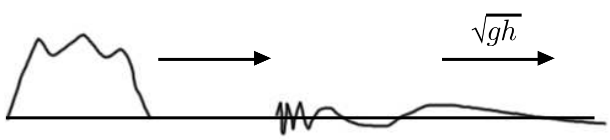

Figure 4.6: What about a pulse?

4.1 Particle paths

$$\frac{dx}{dt} = u(x,y,t)$$
$$\frac{dy}{dt} = v(x,y,t)$$

In infinitesimal waves, $\eta \sim \varepsilon$, i.e.

$$x = x_0 + \varepsilon X(x,y,t)$$
$$y = y_0 + \varepsilon Y(x,y,t)$$

Thus

$$\begin{aligned}\varepsilon\frac{dX}{dt} &= u(x,y,t)\\ &= u(x_0+\varepsilon x, y_0+\varepsilon y, t)\\ &= u(x_0,y_0,t)[1+O(\varepsilon)]\\ \varepsilon\frac{dY}{dt} &= v(x,y,t)\\ &= v(x_0+\varepsilon x, y_0+\varepsilon y, t)\\ &= v(x_0,y_0,t)[1+O(\varepsilon)]\end{aligned}$$

Remembering

$$\phi = \varepsilon c\cos(kx-\omega t)\frac{\cosh[k(y+h)]}{\sinh kh}$$

and

$$u = \frac{\partial\phi}{\partial x}; \quad v = \frac{\partial\phi}{\partial y},$$

we get

$$u = \varepsilon ck\sin(kx_0-\omega t)\frac{\cosh[k(y_0+h)]}{\sinh kh} + O(\varepsilon^2)$$
$$v = -\varepsilon ck\cos(kx_0-\omega t)\frac{\sinh[k(y_0+h)]}{\sinh kh} + O(\varepsilon^2)$$

and so to order ε,

$$\frac{dX}{dt} = A\sin(kx_0-\omega t)$$
$$\frac{dY}{dt} = B\cos(kx_0-\omega t)$$

where

$$A = ck\frac{\cosh[k(y_0+h)]}{\sinh kh}; \quad B = -ck\frac{\sinh[k(y_0+h)]}{\sinh kh}.$$

Thus, if we absorb the constant into the x_0,

$$X = \frac{A}{\omega}\cos(kx_0 - \omega t)$$
$$Y = -\frac{B}{\omega}\sin(kx_0 - \omega t)$$

and

$$\left(\frac{X\omega}{A}\right)^2 + \left(\frac{Y\omega}{B}\right)^2 = 1,$$

i.e. we have an ellipse with semi-minor axis

$$\frac{B}{\omega} = \beta = \frac{\sinh[k(y_0+h)]}{\sinh kh}$$

and semi-major axis

$$\frac{A}{\omega} = \alpha = \frac{\cosh[k(y_0+h)]}{\sinh kh}.$$

Let's check to see if this makes sense.

At the bottom, $y = -h$,

$$\left(\frac{X}{\alpha}\right)^2 + \left(\frac{Y}{\beta}\right)^2 = 1,$$

$\alpha > \beta$ since $\beta = 0$ and we are dragged along the bottom (the practical value of this calculation is that you shouldn't go diving in shallow water in a storm—you'll get dragged all over the place!)

At the surface, $y = 0$, $\alpha = \coth kh$, $\beta = 1$, i.e. Y has amplitude 1, i.e. y has amplitude ε and frequency ω, which is what we expect since $y = \eta$. In fact,

$$Y = \sin(kx_0 - \omega t)$$

which we can compare with

$$\eta = \varepsilon\sin(kx - \omega t).$$

So we have two limits: deep water and shallow water.

In deep water, $h \to \infty$.

$$\beta \to \frac{\frac{1}{2}e^{ky_0+kh}}{\frac{1}{2}e^{kh}}$$
$$= e^{ky_0}$$
$$\alpha \to \frac{\frac{1}{2}e^{ky_0+kh}}{\frac{1}{2}e^{kh}}$$
$$= e^{ky_0}$$
$$X^2 + Y^2 = e^{2ky_0}$$

which looks like Figure 4.7. We have circles of radius $e^{2ky_0} = e^{2\pi y_0/\lambda}$.

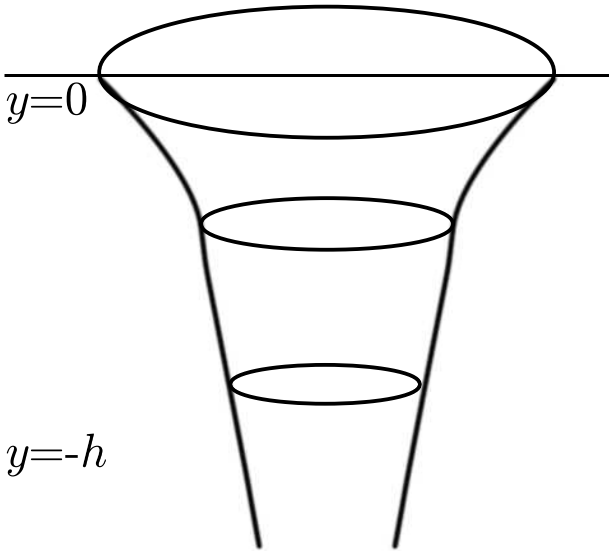

Figure 4.7: What we see if we look deep into the water

Now what if the problem were *reflection*? We have the same amplitude, same wavelength going on in opposite directions:

$$\eta_1 = \varepsilon \sin(kx - \omega t)$$
$$\eta_2 = \varepsilon \sin(kx + \omega t)$$

We solve for η_1 and get ϕ_1,

$$\phi_1 = -\varepsilon c \cos(kx - \omega t)\frac{\cosh[k(y_0 + h)]}{\sinh kh};$$

we solve for η_2 and get ϕ_2,

$$\phi_2 = \varepsilon c \cos(kx + \omega t)\frac{\cosh[k(y_0 + h)]}{\sinh kh}.$$

Hence for total reflection,

$$\eta = \eta_1 + \eta_2 \qquad \phi = \phi_1 + \phi_2$$

,

$$\begin{aligned}
\phi &= -\varepsilon c\frac{\cosh[k(y_0 + h)]}{\sinh kh}\left[\cos(kx - \omega t) - \cos(kx + \omega t)\right] \\
&= -\varepsilon c\frac{\cosh[\frac{2\pi}{\lambda}(y + h)]}{\sinh \frac{2\pi h}{\lambda}}\left[\cos\left(\frac{2\pi}{\lambda}(x - ct)\right) - \cos\left(\frac{2\pi}{\lambda}(x + ct)\right)\right] \\
&= -2\varepsilon c\frac{\cosh[\frac{2\pi}{\lambda}(y + h)]}{\sinh \frac{2\pi h}{\lambda}}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi ct}{\lambda}
\end{aligned}$$

Now, introduce κ, where

$$\kappa = \frac{\partial \phi}{\partial x} = -2\varepsilon c\frac{\cosh[\frac{2\pi}{\lambda}(y + h)]}{\sinh \frac{2\pi h}{\lambda}}\frac{2\pi}{\lambda}\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi ct}{\lambda}.$$

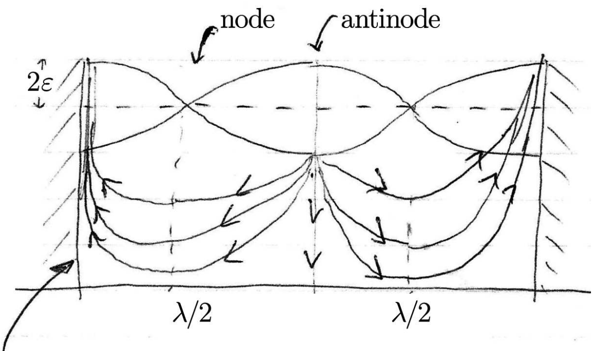

Figure 4.8: The walls can be solid. The maximum displacement is at the wall, where the slope is zero.

This vanishes wherever $\cos\frac{2\pi x}{\lambda} = 0$, i.e.

$$\frac{2\pi x}{\lambda} = \left(n + \frac{1}{2}\right)\pi \quad n \in \mathbb{Z}$$
$$\text{i.e. } x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}$$

i.e. every half-wavelength. And

$$\eta = \eta_1 + \eta_2 = 2\varepsilon \sin\frac{2\pi x}{\lambda}\cos\frac{2\pi ct}{\lambda}$$

is a standing wave. What does it look like?

$$\eta = \pm 2\varepsilon \cos\frac{2\pi ct}{\lambda}.$$

When $u = 0$, oscillates in place with period λ/c.

$$\frac{dx}{dt} = u = \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x} = -2\varepsilon c\frac{\cosh[\frac{2\pi}{\lambda}(y+h)]}{\sinh\frac{2\pi h}{\lambda}}\frac{2\pi}{\lambda}\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi ct}{\lambda}$$
$$\frac{dy}{dt} = v = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial y} = -2\varepsilon c\frac{\sinh[\frac{2\pi}{\lambda}(y+h)]}{\sinh\frac{2\pi h}{\lambda}}\frac{2\pi}{\lambda}\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi ct}{\lambda}$$

Hence

$$\frac{dy}{dx} = \frac{v}{u} = \tanh\left[\frac{2\pi}{\lambda}(y+h)\right]\tan\frac{2\pi x}{\lambda}$$

which gives the slope of the particle paths at any position: see Figure 4.8.

Chapter 5

Hydraulic jump/Energy sheet

5.1 Hydraulic jump

Consider the jump as shown in a channel of width d and horizontal bed. The suffixes 1 and 2 refer to conidtions before and after the jump, and we consider the fluid bounded by A_1B_1 and A_2B_2. In time δt, the fluid has moved to the region bounded by $A_1'B_1'$ and $A_2'B_2'$. Then at the two locations we have the following quantities:

Height	h_1	h_2
Mean velocity	u_1	u_2
Pressure	$\rho g(h_1 - z)$	$\rho g(h_1 - z)$
Thickness	$u_1\delta t$	$u_2\delta t$
Mass	$m_1 = \rho d h_1 u_1 \delta t$	$m = \rho d h_2 u_2 \delta t$
Flow rate	$dh_1u_1 = Q$	$dh_2u_2 = Q$
Momentum	m_1u_1	m_2u_2
Force in flow direction	$F_1 = \int_0^{h_1} \rho g d(h_1 - z)dz$ $= \frac{1}{2}\rho g d h_1^2$	$F_2 = -\int_0^{h_2} \rho g d(h_2 - z)dz$ $= -\frac{1}{2}\rho g d h_2^2$

Conservation of mass shows that $m_1 = m_2$ and the flow rate Q is the same at 1 and 2. Force equals rate of change of momentum gives

$$F_1 - F_2 = (m_2u_2 - m_1u_1)/\delta t$$

or

$$\frac{1}{2}\rho g d(h_1^2 - h_2^2) = \rho Q(u_2 - u_1) = \frac{\rho Q^2}{d}\left(\frac{1}{h_2} - \frac{1}{h_1}\right).$$

Hence either $h_2 - h_1 = 0$, in which case the flow is continuous and there is no jump, or

$$h_1h_2(h_1 + h_2) = \frac{2Q^2}{gd^2}.$$

For given upstream conditions this equation determines h_2 and hence $q_2 = Q/dh_2$. If

Kinetic energy	$\frac{1}{2}m_1u_1^2$	$\frac{1}{2}m_2u_2^2$
Work done by force	$F_1u_1\delta t$	$F_1u_1\delta t$

$D\delta t$ is the amount of kinetic energy lost in time δt, conservation of energy gives

$$(F_1u_1 - F_2u_2)\delta t = \frac{1}{2}m_2u_2^2 - \frac{1}{2}m_1u_1^2 + D\delta t,$$

from which it follows that

$$\begin{aligned} D &= \frac{1}{2}\rho gd(h_1^2u_1 - h_2^2u_2) + \frac{1}{2}\rho Q(u_1^2 - u_2^2) \\ &= \frac{1}{2}\rho gQ(h_1 - h_2) + \frac{\rho Q^3}{2d^2}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right) \end{aligned}$$

or

$$D = \frac{1}{2}\rho gQ\frac{(h_2 - h_1)^3}{2h_1h_2}.$$

Since $D \geq 0$, $h_2 \geq h_1$. In a hydraulic jump, the level of the water rises and the speed falls. For the flow upstream,

$$\begin{aligned} \frac{q_1^2}{gh_1} - 1 &= \frac{Q^2}{gd^2h_1^3} - 1 \\ &= \frac{h_1h_2(h_1 + h_2)}{2h_1^2} - 1 \\ &= \frac{(h_2 - h_1)(h_2 + 2h_1)}{2h_1^2} \end{aligned}$$

5.2 Energy

From mass and momentum conservation we have shown that

$$h_1h_2(h_1 + h_2) = \frac{2Q^2}{gd^2}. \tag{5.1}$$

Work done at A = force$\times$ distance, and distance = $u_1\delta t$ at all heights.

Force = pressure, integrated over area = $\rho g(h_1 - z)$ (taking $p_a = 0$). Hence force $= \frac{1}{2}\rho gh_1^2d$.

Thus work done at A is

$$\frac{1}{2}\rho gh_1^2u_1d\delta t = \frac{1}{2}\rho gQ\delta th_1$$

and similarly work done at B is

$$-\frac{1}{2}\rho gQ\delta th_2$$

since displacement is in the opposite direction to the force.

Thus the net work done on the fluid is

$$\frac{1}{2}\rho g Q\delta t(h_1 - h_2).$$

The kinetic energy in at A is $\frac{1}{2}\rho u_1^2$ per unit volume. Therefore the total kinetic energy in at A is

$$\frac{1}{2}\rho u_1^2 Q\delta t.$$

The potential energy in at A is $\rho g z$ per unit volume. Therefore the total potential energy in at A is

$$\frac{1}{2}\rho g h_1^2 d u_1 \delta t = \frac{1}{2}\rho g h_1 Q\delta t$$

(integrating from $0 to h_1$).

Hence the total energy in at A is

$$\frac{1}{2}\rho Q\delta t(u_1^2 + g h_1)$$

and the total energy in at B is

$$\frac{1}{2}\rho Q\delta t(u_2^2 + g h_2).$$

The energy lost is the work done + energy in − energy out, which is

$$(\rho Q\delta t)\left[g(h_1 - h_2) + \frac{1}{2}(u_1^2 - u_2^2)\right]$$

but

$$u_1^2 = \frac{Q^2}{h_1^2 d^2} = \frac{gh^2}{2h_1}(h_1 + h_2)$$

by equation 5.1, and

$$u_2^2 = \frac{gh_1}{2h_2}(h_1 + h_2)$$

similarly.

Thus the lost energy is

$$\frac{\rho g Q\delta t}{4h_1 h_2}\left[(h_1 + h_2)h_2^2 - (h_1 + h_2)h_1^2 + 4h_1 h_2(h_1 - h_2)\right]$$
$$= \frac{\rho g Q\delta t}{4h_1 h_2}(h_2 - h_1)^3$$

For energy to be lost, $h_2 > h_1$, i.e. an upward jump.

Chapter 6

Surface waves, from typewritten

6.1 Small amplitude gravity waves

The free surface of a body of water at rest is flat and horizontal. If the surface is disturbed, gravity will act to return the fluid towards equilibrium, but the motion so produced is oscillatory (compare the swinging of a pendulum). The oscillation can take the form of waves progressing along the water surface or, if the water is in a container, of standing waves. What is the frequency of these oscillations and with what speed does a water wave move along the surface?

Suppose we take the z-axis vertically upwards and the x-axis horizontal, with origin in the undisturbed surface. The top of the water is then in the plane z and let us suppose the water extends to infinity both horizontally and vertically. We shall deal with the cases of a finite depth of the water and of the presence of side walls later. Then, in equilibrium, the pressure is given by

$$p = p_0 - \rho g z,$$

where p_0 is the atmospheric pressure above the surface of the water. Now let us suppose that the water surface is displaced by a small amount, and that the resulting pressure deviation from equilibrium and the fluid velocities are all *small*. If they have a magnitude ε, say, we shall keep only terms of order ε and neglect terms in ε^2 and higher powers. This is what is meant by *small-amplitude*. The resulting equations will be linear, so that this procedure is also called *linearisation*.

We shall write the pressure as

$$p = p_0 - \rho g z + \rho\phi(t, x, z)$$

where ϕ is small. Then the Euler equations with second-order terms dropped become

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}, \quad \frac{\partial w}{\partial t} = -\frac{\partial \phi}{\partial z}, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

Note that the motion is confined to x, z planes with no variation in the y-direction.

From the first two equations we see that

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)=\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial z^2}$$

and from the third equation it follows that

$$\nabla^2\phi = 0$$

so we have to solve Laplace's equation. It is obvious that this equation has solutions of the form $X(x)Z(z)$, where

$$X = A\cos kx + B\sin kx, \quad Z = C\exp(kz)D\exp(-kz),$$

where A, B, C, D are constants (as far as x and z are concerned), and these are the solutions that are relevant to the water-wave problem.

We now need to specify the conditions that the solution must satisfy. The first is a *kinematic* condition. The free surface in the disturbed position is given by $z = \eta(t, x)$. Forming the convected derivative and dropping the quadratic terms, we find that

$$\frac{\partial \eta}{\partial t} - w = 0 \quad \text{on } z = O.$$

The other condition that must be satisfied on the free surface is a *dynamic* condition. It states that the pressure must be atmospheric there. Thus

$$-g\eta + \phi = 0 \quad \text{on } z = 0.$$

The final condition we need to close the problem is that, since the disturbance emanates from the free surface, the disturbance pressure should go to zero in the deep water, that is, $\phi \to 0$ as $z \to -\infty$.

We are looking for solutions that represent waves on the surface, that is we want η to have the form

$$\eta = a\cos(\sigma t - kx + \delta),$$

where σ is the frequency of the wave, k the wave number, a the amplitude and ϕ the phase. The period of the wave is $2\pi/\sigma$ and the wave length is $2\pi/k$. On $z = 0$ we must have

$$\phi = ag\cos(\sigma t - kx + \delta)$$

and the z-dependence that goes with this function of x is $\exp(kz)$. We cannot have $\exp(-kz)$ since we want ϕ to go to zero as $z \to -\infty$. Hence,

$$\phi = ag\exp(kz)\cos(\sigma t - kx + \delta).$$

The only condition we have not yet used is the kinematic condition on the free surface. From the equations of motion,

$$\frac{\partial w}{\partial t} = -\frac{\partial \phi}{\partial z} = -agk\exp(kz)\cos(\sigma t - kx + \delta),$$

so that, on $z = 0$,

$$w = -\frac{agk}{\sigma}\sin(\sigma t - kx + \delta).$$

But

$$\frac{\partial\eta}{\partial t} = -a\sigma\sin(\sigma t - kx + \delta),$$

so that the kinematic condition only holds if

$$\sigma^2 = gk$$

This is the equation relating the frequency of the wave to the wave number. The speed c of the wave is equal to σ/k, so that

$$c = \sqrt{\frac{g}{k}} \quad \text{or} \quad c = \sqrt{\frac{g\lambda}{2\pi}}$$

for a wave of length λ.

An important feature of this result is that water waves exhibit dispersion. This means that the speed of the waves varies with their length and is not a constant as it is for sound or light waves. Long waves travel faster than short ones.

Wavelength (m)	Speed (m/s)	Period (s)
100	12.5	8.0
1	1.25	0.8
0.01	0.125	0.08

Another feature of water waves is that the motion of the water is concentrated close to the surface. The velocity components decrease exponentially with depth, and at a depth equal to the length of the wave the amplitude is decreased by a factor $\exp(-k\lambda) = \exp(-2\pi) = -0.00187$. The sea is calm at a short depth below a rough surface. It also follows that the bottom of the sea does not have much effect on the waves unless the wave length is comparable with the depth of the sea.

Suppose we now consider waves on water of depth h. The formulation of the problem is the same as before, except that we must have zero vertical velocity on the bottom (since the fluid cannot go through it). Instead of requiring that the disturbance dies out with depth, we must now satisfy the condition $w = 0$ on $z = -h$, which is equivalent to the condition

$$\frac{\partial\phi}{\partial z} = 0 \text{ on } z = -h.$$

Hence, we can replace the previous form for ϕ by

$$\phi = ag\frac{\cosh k(z+h)}{\cosh kh}\cos(\sigma t - kx + \delta),$$

with η having the same form as before. When we apply the kinematic condition, we find that

$$\sigma^2 = gk\tanh kh$$

and the wave speed c is given by

$$c = \sqrt{\frac{g \tanh kh}{k}} \quad \text{or} \quad c = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}}$$

where λ is the wave length. The maximum wave speed is when λ is much larger than h, and for these long waves on shallow water the wave speed $c = \sqrt{gh}$. Note that in this limiting case the waves are not dispersive. For water of depth 1m the maximum wave speed is about 3.1m/s and if the depth is only 1cm the maximum speed is about 31 cm/s.

6.2 Particle paths and group velocity

It is important to distinguish between the velocity of the fluid in the presence of the wave and the velocity of the wave itself. The fluid velocity must be small since we are assuming small-amplitude waves. The wave velocity, or phase velocity, we have already determined and it does not depend on the amplitude of the wave, at least when the amplitude is small. We can define another velocity associated with the wave, known as the group velocity. Since the waves are dispersive, the phase and group velocities are different.

The horizontal and vertical components of the fluid velocity are given by

$$u = \frac{agk}{\sigma \cosh kh} \cosh k(z+h) \cos(\sigma t - kx + \delta),$$

$$w = -\frac{agk}{\sigma \cosh kh} \sinh k(z+h) \sin(\sigma t - kx + \delta).$$

Because these velocities are small and periodic, a particle of fluid does not move far from its mean position. Consider the particle moving about the point (x_0, z_0), so that its position is given by $x = x_0 + \xi, z = z_0 + \zeta$. Then the path of the particle is given by the solution of the equations

$$\frac{d\xi}{dt} = \sigma A \cos(\sigma t - kx_0 + \delta), \quad \frac{d\zeta}{dt} = -\sigma B \sin(\sigma t - kx_0 + \delta),$$

where

$$A = \frac{agk}{\sigma^2 \cosh kh} \cosh k(z_0 + h), \quad B = \frac{agk}{\sigma^2 \cosh kh} \sinh k(z_0 + h).$$

Integrating these equations we find that the particle paths are the ellipses

$$\frac{\xi^2}{A^2} + \frac{\zeta^2}{B^2} = 1.$$

The major axis is horizontal and the ratio of minor to major axis is B/A, which is equal to $\tanh k(z_0 + h)$. Thus the elliptical paths become increasingly compressed in the vertical direction until, at the bottom, the paths are straight lines along the bottom surface.

There are two special cases of this general result. For long waves in shallow water we take the limit $h \to 0$ and, since z_0 lies between 0 and $-h$, $B = 0$ and $A = agk/\sigma^2$. The particle paths are horizontal lines and the same at all depths. For fluid of infinite depth

$$A = \frac{agk}{\sigma} \exp(kz_0) = B,$$

so the paths are circles whose radius decreases exponentially with depth.

The phase velocity c for a wave of frequency σ and wave number k is defined by $c = \sigma/k$ and we have seen that for water waves $c = c(k)$. The term phase velocity is used because in a progressive wave any particular phase of the wave (a crest or trough or a point of zero displacement) advances with this speed. The group velocity is important when the motion consists of the superposition of a number of waves with differing wave lengths and not just of a single component. the definition of the group velocity c_g is

$$c_g = \frac{d\sigma}{dk}$$

Of course, if the waves are not dispersive, c is a constant and $\sigma = ck$ so that $c_g = c$. For water waves in deep water, $\sigma = \sqrt{gk}$ so that $c_g = \frac{1}{2}c$.

The significance of the group velocity is that it is the velocity with which energy is transported by the wave. It is also the velocity of a wave packet, or group of waves (hence the name). We can see this most easily by considering the sum of two waves of nearly equal wave numbers, say $k \pm \delta k$ with corresponding frequencies $\sigma \pm \delta\sigma$. Then the sum of the two waves with the same amplitude is proportional to

$$\cos[(\sigma + \delta\sigma)t - (k + \delta k)x] + \cos[(\sigma - \delta\sigma)t - (k - \delta k)x],$$

which can be simplified to give

$$2\cos(\sigma t - kx)\cos(\delta\sigma t - \delta k x),$$

This represents a wave with the mean wave number and frequency of the two components, but the amplitude is varying slowly in both space and time. The envelope of the wave is advancing with speed $\delta\sigma/\delta k$, which, in the limit as $\delta k \to 0$ is equal to c_g.

A full appreciation of the significance of the group velocity depends on being able to handle the superposition of a continuous spectrum of wave numbers. The technique required is called the Fourier transform and this a topic discussed in the MATH7402 ('Methods 4') course in the second term.

6.3 Standing waves

Consider the superposition of two waves with the same amplitude, frequency, wave number and phase, moving in opposite directions. The surface elevation is then given by

$$\eta = acos(\sigma t + kx + \delta) + acos(\sigma t - kx + \delta),$$

which can be simplified to the form

$$\eta = 2a\cos(\sigma t + \delta)\cos kx.$$

This is a different type of wave from the one we have been dealing with so far. It is not a progressive wave, but a *standing wave*. The surface elevation at each location oscillates with the same period $2\pi/\sigma$, but the amplitude of the oscillation varies with x. At certain locations, like $x = \pi/2k$, for example, the surface is always in its equilibrium position—this is called a node. At certain instants in time, $t = \left(\frac{1}{2}\pi - \delta\right)/\sigma$, for example, the whole of the water surface is flat.

The pressure disturbance for this combination is given by

$$\phi = 2ag\frac{\cosh k(z+h)}{\cosh kh}\cos(\sigma t + \delta) coskx,$$

and the horizontal velocity component is given by

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} = 2agk\frac{\cosh k(z+h)}{\cosh kh}\cos(\sigma t + \delta)\sin kx.$$

It follows that $u = 0$ at $x = 0$ for all t. This is just the condition that must be satisfied if there is a vertical barrier at $x = 0$ which the water cannot penetrate. hence, we have found the solution to the problem of a progressive wave moving from co towards a barrier at $x = 0$ and being reflected from it. The reflected wave is identical with the oncoming wave except in its direction of motion. The combination of the incident and reflected wave gives a standing wave. The elevation of the surface at the barrier has a maximum value equal to $2a$, which is twice the amplitude of the incident wave.

Suppose that now we introduce a second vertical barrier at $x = d$, say, and consider the fluid confined between these two walls. The conditions to be satisfied on the barriers is that $u = 0$ at $x = 0$ and at $x = d$. The solution we have just found satisfies the condition on $x = 0$ but not that on $x = d$ unless $\sin kd = 0$. This means that the wave number cannot now be chosen arbitrarily as it could when we were dealing with progressive waves, but it must take one of an infinite but discrete set of values. The possible wave numbers are

$$k = \pi/d, 2\pi/d, 3\pi/d, \ldots.$$

With each of these wave numbers there is a corresponding frequency, which is given by the dispersion relation we found earlier. It is a general principle that in a confined region the spectrum of wave numbers is discrete, and is only continuous when the region extends to infinity.

The general wave motion between the walls will consist of the superposition of waves with all the possible wave numbers. The relative amplitudes and phases would have to be computed from the manner in which the motion is initiated. Since each component has a different and non-commensurate frequency, the free surface will show a chaotic behaviour. However, the action of viscosity eventually damps out the oscillations and, since it is most effective in eliminating the high-frequency oscillations, after a time we expect to see only the slowest mode, $k = \pi/d$. This is known as the *sloshing mode*, since the fluid moves from one side of the channel to the other.

6.4 Three-dimensional waves

The waves considered so far depend on the horizontal coordinate x only. We can have a wave progressing in a general horizontal direction. Suppose $\mathbf{r}$ is the position vector in the plane $z = 0$, with $\mathbf{r} = x\hat{\imath} + y\hat{\jmath}$. Then the surface elevation for a progressive wave can be taken in the form

$$\eta = a\cos(\sigma t - k_1 x - k_2 y)$$

and ϕ will be as before. The previous set of equations has to be supplemented by $\frac{\partial v}{\partial t} = -\frac{\partial \phi}{\partial y}$ and the equation of continuity gives $\nabla^2\phi = 0$. The approriate solution of this equation have the form $X(x)Y(y)Z(z)$, with solutions

$$X = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} k_1 x, \quad Y = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} k_2 y, \quad Z = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} kz, \quad k^2 = k_1^2 + k_2^2.$$

If we define $\mathbf{k} = k_1\hat{\imath} + k_2\hat{\jmath}$, we call $\mathbf{k}$ the wavenumber vector, of magnitude k, and we can write η as $a\cos(\sigma t - \mathbf{k}\cdot\mathbf{r})$. The crests of this wave are moving in the direction of $\mathbf{k}$ with speed σ/k.

The two-dimensional standing-wave problem is that of finding the frequencies of waves on the surface of fluid inside a cylindrical container. Suppose the surface of the container $\mathcal{S}$ is defined in terms of Cartesian coordinates by $S(x, y) = 0$ for $0 > z > -h$ and by the plane bottom $z = -h$. The boundary condition that must be satisfied on $\mathcal{S}$ is $\mathbf{n}\cdot\mathbf{u} = 0$, where $\mathbf{n}$ is the normal to $\mathcal{S}$ and $\mathbf{u}$ is the fluid velocity on the surface. Because we are looking for standing waves, all physical quantities will have a similar time-dependence, so the surface elevation η and the disturbance pressure can be taken to have the forms

$$\phi = Ag\cos(\sigma t + \delta)F(x, y, z), \quad \eta = A\cos(\sigma t + \delta)F(x, y, 0),$$

where F is to be found and we have satisfied the dynamic condition on the free surface. The equations of motion in their linear form are $\frac{\partial \mathbf{u}}{\partial t} = -\nabla\phi$ and with the given form of ϕ the velocity components are given by

$$(u, v, w) = -\frac{Ag}{\sigma}\sin(\sigma t + \delta)\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right).$$

The equation of continuity is $\nabla\cdot\mathbf{u} = 0$, which gives

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0.$$

For the z-dependence we can see that the same form we used before will work, and we can write F in the form

$$F(x, y, z) = G(x, y)\frac{\cosh k(z + h)}{\cosh kh},$$

so that $F(x, y, 0) = G(x, y)$ and we have satisfied the condition on the bottom of the container which is that $w = 0$ there. The kinematic condition on the free surface is that $\frac{\partial \eta}{\partial t} = w$ on $z = 0$ which becomes

$$-A\sigma \sin(\sigma t + \delta)G(x, y) = \frac{Ag}{\sigma} k \tanh(kh)G(x, y).$$

We therefore have the same relation connecting σ and k that we had before, namely

$$\sigma^2 = gk \tanh kh.$$

The difference is that we have not yet found the possible values that k can take and until this has been done we cannot determine the spectrum of frequencies. Note that this relation is independent of the shape of the container, but the values of k have to be determined for each container shape. The problem of determining k is that of solving for $G(x, y)$ and satisfying the condition on the vertical wall of the container. The equations for G are

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + k^2 G = 0, \quad \mathbf{n} \cdot \nabla G = 0 \text{ on } S(x, y) = 0.$$

This is an eigenvalue problem, with a similar nature to eigenvalue problems for matrices. For general k, the unique solution is $G = 0$, the trivial solution. For certain values of k, known as the eigenvalues, a non-trivial solution exists with an arbitrary amplitude. These are known as the eigensolutions or eigenmodes for the problem. In the differential equation context, an eigenvalue problem consists of a homogeneous differential equation (ordinary or partial) and homogeneous boundary conditions. Usually, but not universally, the eigenvalues form an infinite discrete set.

To make any further progress we have to specify the shape of the container. Suppose first that the container has a rectangular cross-section, so that the vertical walls are at $x = \pm a, y = \pm b$. Then we must have

$$\frac{\partial G}{\partial x} = 0 \text{ on } x = \pm a, \quad \frac{\partial G}{\partial y} = 0 \text{ on } y = \pm b.$$

The form of the equation for G suggests that solutions in sines and cosines are possible. The boundary condition in x shows that we can have G proportional to $\cos(m\pi x/2a)$ with m an even integer or $\sin(m\pi x/2a)$ with m odd. Similarly for the y-dependence, $\cos(n\pi y/2b)$ and n even or $\sin(n\pi y/2b)$ and n odd. In addition, we could also have $m = 0$ and $n \neq 0$ or $m \neq 0$ and $n = 0$. When these solutions are put into the differential equation for G we see that

$$k^2 = \frac{\pi^2}{4} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

There are therefore a double infinity of possible modes for the standing waves on the surface of a rectangular container. The frequencies of these modes can be obtained from the expression for σ in terms of k. If $a > b$, the smallest possible value for k and the slowest mode is given by $m = 1, n = 0$. In this mode the fluid sloshes from side of the tank to the other, the surface elevation being proportional to $\sin \pi x/2a$.

Note that a possible solution is $k = 0$, with $m = n = 0$, and $G(x, y) = 1$. Then $\sigma = 0$ and $\eta = A\cos\delta, \phi = Ag\cos\delta$. This solution satisfies all the conditions we have imposed but it is not acceptable. The reason is that the volume of the fluid in the container remains constant, so that

$$\int_{-b}^{b}\int_{-a}^{a} \eta\, dx\, dy = 0$$

and $A\cos\delta = 0$. It is easy to see that, provided at least one of m and n is non-zero the integral vanishes, so this volume constraint does not impose any further restriction on the solutions.

As a second example, suppose the container has a circular cross-section of radius a. Then it is convenient to work in cylindrical polar coordinates and we have to solve the equations

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} + k^2 G = 0, \quad \frac{\partial G}{\partial r} = 0 \text{ on } r = a.$$

If the surface elevation is axisymmetric, G is a function of r only. The equation for G is then Bessel's equation of order zero and the solution that does not have a singularity at $r = 0$ is $J_0(kr)$. The boundary condition that determines the possible values of k is

$$J_0'(ka) = 0,$$

and there are an infinite number of roots of this equation. The non-axisyminetric solutions have the form $\cos(n\theta + \varepsilon)J_n(kr)$, where n is an integer since the solution must have period 2π in θ, and J_n is Bessel's function of order n. The equation to determine k is then $J_n'(ka) = 0$, which has an infinite number of roots for each n, so that there is a double infinity of possible values of k.

The smallest possible value for k comes from the equation with $n = 1$, and not from the axisymmetric solution. In this slowest mode, there is a nodal diameter, with the surface raised on one side and depressed on the other, that is, it is a sloshing mode. These modes for the gravity waves in a cylindrical container can easily be observed in a cup or glass. The reflection of an overhead light in the surface of black coffee shows them well.

Appendix A

Integral Theorems

You first saw these integral theorems in MATH1402 when you were (probably) told that you need to know all the theorems below very well. The sooner you start to learn them the better*. However, note that it is much easier to remember something when you really understand it.

Remember that there are two types of notation:

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} \qquad \operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} \qquad \operatorname{grad}\phi = \nabla\phi$$

In the main notes we tend to use the ∇ notation but here we've used the full names.

In all of the theorems below we assume that $\mathbf{A}$ is any smooth vector field.

A.1 Divergence theorem

Consider a volume V completely enclosed by a surface S with unit outward normal $\widehat{\mathbf{n}}$. Then

$$\int_V \operatorname{div}\mathbf{A}\, dV = \oint_S \mathbf{A}\cdot\widehat{\mathbf{n}}\, dS.$$

A.2 Stokes' theorem

Consider a surface S with unit normal $\widehat{\mathbf{n}}$ and boundary C, where the direction of C is consistent with the direction of $\widehat{\mathbf{n}}$. Then

$$\int_S (\operatorname{curl}\mathbf{A})\cdot\widehat{\mathbf{n}}\, dS = \oint_C \mathbf{A}\cdot d\mathbf{r}.$$

*Will you be one of the five members of the class who actually sits down and learns them?

A.3 Green's theorem in the plane

This theorem is in fact just a 2D version of Stokes' theorem, so it might be a useful test of your understanding to see whether you can prove it from Stokes' theorem; historically, however, the reverse was done.

Consider a region R in the xy-plane with boundary C taken in the positive direction, i.e. anti-clockwise. Then for any smooth functions $P(x, y)$ and $Q(x, y)$ we have

$$\oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

A.4 Related theorems

Using the notation of the above theorems, the following hold:

$$\int_V \operatorname{curl}\mathbf{A}\,dV = \oint_S \widehat{\mathbf{n}}\times\mathbf{A}\,dS, \qquad \oint_C \phi\,d\mathbf{r} = \int_S \widehat{\mathbf{n}}\times \operatorname{grad}\phi\,dS,$$

where ϕ is any smooth scalar function.

A.5 Applications

All of the above theorems have many applications in applied mathematics, but the applications that you are most likely to see in the near future are to fluid mechanics (obviously!), electromagnetic theory* and quantum mechanics.

*The editors strongly recommend the typeset notes for the relevant MATH3308 course

Appendix B

Practice sheets

These practice sheets are problem sheets from when Prof. Frank Smith gave this course many, many years ago. Not all of the questions written here were set to be handed in: as such, solutions are only given to those which were. These sheets are then for your own practice, which you might like to do as the course goes along, or more realistically as you prepare for the summer exam. Note that the order in which the sheets come might not necessarily correspond exactly with the order in which the material is taught in this course.

B.1 Practice sheet 1

1. A particle moves in 3D space so that at time t its position is given by

   $$x = 2t + 3, \quad y = t^2 + 3t, \quad z = t^3 + 2t^2.$$

   Find the components of its velocity and acceleration in the direction of the vector

   $$2\hat{\mathbf{\imath}} + 3\hat{\mathbf{\jmath}} + 4\hat{\mathbf{k}}$$

   when $t = 1$.

2. (a) If $\phi(x, y, z) = x^2yz^3 + xy^2z^2$, determine $\nabla\phi$ at the point P at $(1, 3, 2)$.
   (b) Find the unit normal vector to the surface $\phi = 2$ at the point $(1, 1, 1)$.

3. If $\mathbf{A} = x^2y\,\hat{\mathbf{\imath}} - xyz\,\hat{\mathbf{\jmath}} + yz^2\,\hat{\mathbf{k}}$ then write down
   (a) $\operatorname{div}\mathbf{A}$
   (b) $\operatorname{curl}\mathbf{A}$
   (c) $\operatorname{curl}\operatorname{curl}\mathbf{A}$
   (d) $\operatorname{div}\operatorname{curl}\mathbf{A}$

4. (a) Evaluate $\int_V \nabla\cdot\mathbf{F}\,dV$ where $\mathbf{F} = xy\,\hat{\mathbf{\imath}} + z\,\hat{\mathbf{\jmath}} - x^2\,\hat{\mathbf{k}}$ and V is the region bounded by the planes $x = 0, x = 2, y = 0, y = 3, z = 0, z = 4$.
   (b) By using the divergence theorem, or otherwise, evaluate $\int_S \mathbf{F}\cdot dS$ over the surface of this volume.

B.2 Practice sheet 2

1. (a) Sketch the pathline for the particle which starts at the origin in the flow

$$\mathbf{u} = (\alpha, kt, 0).$$

   (b) Sketch the pattern of streamlines for the above flow at any two different times.

   (c) Describe how the pathline of part (a) is related to the streamline of part (b).

2. In an unsteady 2D flow,

$$u = \frac{1}{1+t}, \quad v = 1, \quad w = 0.$$

   Sketch:

   (a) The pathline through $(1, 1, 0)$ at $t = 0$,

   (b) The streamline through $(1, 1, 0)$ at $t = 0$.

3. Calculate and describe (a) particle paths and (b) streamlines for the flow

$$\mathbf{u} = (ay, -ax, ct).$$

4. Sketch streamlines for

   (a) $\mathbf{u} = (a\cos\omega t, a\sin\omega t, 0)$

   (b) $\mathbf{u} = (x - Ut, y, 0)$

   (c) $\mathbf{u} = (r\cos\frac{1}{2}\theta, r\sin\frac{1}{2}\theta, 0)$, for $0 < \theta < 2\pi$

   Note that in (c) the velocity components are in the radial, azimuthal and vertical directions respectively.

5. Find streamlines and particle paths for the 2D flows

   (a) $\mathbf{u} = (xt, -yt, 0)$

   (b) $\mathbf{u} = (xt, -y, 0)$

B.3 Practice sheet 3

1. In a fluid montion, the velocity is given by

$$\mathbf{u} = \frac{-y\,\hat{\mathbf{\imath}} + x\,\hat{\mathbf{\jmath}}}{x^2 + y^2}.$$

   (a) Show $\operatorname{div}\mathbf{u} = \operatorname{curl}\mathbf{u} = 0$.

   (b) Determine the velocity potential and the equations of the streamlines.

2. A flow has velocity field $\mathbf{u} = y^2\,\widehat{\imath} + x^3\,\widehat{\jmath}$.

   (a) Show that this represents a montion that is incompressible and *rotational.*

   (b) Find a streamfunction for the flow or given a reason why none can exist.

   (c) Find a velocity potential for the flow or give a reason why none can exist.

3. Show that the following could *not* be the components of velocity for an incompressible fluid:

$$u = -\frac{2xyz}{(x^2+y^2)^2}, \quad v = \frac{(x^2-y^2)z}{x^2+y^2}, \quad w = \frac{y}{x^2+y^2}.$$

4. A fluid motion is irrotational with velocity potential ϕ. Show that the equipotential surfaces (i.e. the surfaces of constant ϕ) are orthogonal to the streamlines.

5. The velocity $\mathbf{u}$ in a narrow two-dimensional jet of incompressible fluid is given by $\mathbf{u} = u\,\widehat{\imath} + v\,\widehat{\jmath}$ where

$$u = U\beta x^{-1/3}\,\mathrm{sech}^2\left(\alpha y x^{-2/3}\right), \quad x \neq 0$$

   where α, β, U are constants. Find a streamfunction ψ for the flow such that $\psi = 0$ when $y = 0$ and calculate the velocity component v.

6. Show that the uniform flow of an incompressible fluid at a constant angle α to the x-axis is irrotational. Find the streamfunction and velocity potential.

7. The motion of an incompressible fluid is given by $\mathbf{u} = u\,\widehat{\imath} + v\,\widehat{\jmath}$ where

$$u(y) = U_0 + \frac{1}{2}(U_1 - U_0)\left(1 + \tanh\left(\frac{y}{a}\right)\right)$$

   and U_0, U_1, a are constants. Is the motion irrotational? If so, find a velocity potential. Find a streamfunction for the flow. Sketch the streamlines for $-2a < y < 2a$, indicating any asymptotes.

B.4 Practice sheet 4

1. The velocity potential for the flow around a cylinder with outward blowing at its surface, placed in a uniform stream, may be obtained by adding the velocity potentials for a uniform stream, a dipole and a source. Assume that the uniform stream has speed U, the normal (outward) velocity at the surface of the cylinder is V, the radius of the cylinder is a and the circulation around the cylinder is zero.

   (a) Write down the total velocity potential

   (b) Show that if V equals $4U$, then there is a stagnation point at a distance $(2+\sqrt{5})a$ upstream of the cylinder.

   (c) Sketch streamlines for the flow.

2. A source and a sink of equal strengths m are placed at a point A at $(s, 0)$ and the point B at $(-s, 0)$ respectively. A general point P is such that PA makes an angle θ_1 with the x-axis and PB makes an angle θ_2 with the x-axis, as shown below.

   (a) Write down the streamfunction ϕ at the point P in terms of the angle α.

   (b) Write down $\tan\theta_1$ and $\tan\theta_2$ in terms of x and y and hence write down expressions for α and ϕ.

   (c) By using the result of (b), show that the streamlines are circles whose centres lie along the y-axis.

   (d) The distance AB is now reduced, i.e. $2s$ tends to zero, and the strength m is increased in such a say that $2ms$ remains constant, and is denoted by μ. Give the limiting form for ϕ.

3. Show that the velocity potential for the flow past a circular cylinder, radius a, placed in a uniform flow of speed U, is given by

$$\phi = Ur\cos\theta + \frac{Ua^2\cos\theta}{r}.$$

4. A uniform stream at an angle α to the x-axis flows past a circular cylinder of radius a, which has circulation κ. Write down a streamfunction for the flow and give a condition for the existence of just one stagnation point on the cylinder.

B.5 Practice sheet 5

1. Write down the streamfunction for a line vortex of strength $2\pi m$.

   A line vortex of strength $2\pi m$ is placed at the point $(a, 0)$, equidistant from two walls which lie along the lines $y = x$ and $y = -x$ and which intersect at the origin.

   Write down a streamfunction at a general point P, for the flow due to the vortex itself together with its two image vortices in the walls.

   Show that the walls are not streamlines for this flow solution.

   Show that by including a third image vortex, being the image of the other two image vortices in the extensions of the walls into the region $x < 0$, a flow can be constructed which does have the walls as streamlines and hence which models a vortex in a corner.

2. A stationary circular cylinder with boundary $x^2 + y^2 = a^2$ is in a 2D irrotational flow field whose velocity has Cartesian components (u, v) such that

$$u + 2y \to 0, \quad v + 2x \to 0 \quad \text{as } x^2 + y^2 \to \infty.$$

   The circulation around the cylinder is κ.

(a) Show that a velocity potential for the motion in the farfield is given in polar coordinates by $\phi = -r^2 \sin 2\theta$ ($r \gg 1$).

(b) By solving Laplace's equation or by adding known solutions, obtain a velocity potential for the whole flow field.

(c) Show that the greatest value of the fluid speed on the cylinder is $4a + \frac{\kappa}{2\pi a}$.

(d) Show also that if $\kappa = 4\pi a^2$ there are 4 stagnation points on the cylinder and find their positions.

3. A stationary cylinder of radius a with circulation κ is placed in a 2D uniform flow, of speed U.

   (a) Write down the streamfunction and velocity potential for the flow.

   (b) Find the positions of the two stagnation points on the cylinder if $\frac{\kappa}{4\pi a U} = 0.5$.

4. The equation $\nabla^2\phi = 0$ is to be solved with the conditions

$$\frac{\partial \phi}{\partial x} = 0 \text{ on } x = \pm a, \quad \frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0$$

   with the solution being required in the region $-a < x < a$ and $0 < y < a$.

   Use separation of variables to derive the solution

$$\phi(x, y) = \sum_{n=0}^{\infty} c_n \cosh\left(\frac{n\pi y}{a}\right) \cos\left(\frac{n\pi x}{a}\right) + d_n \cosh\left[\frac{(2n+1)\pi y}{2a}\right] \sin\left[\frac{(2n+1)\pi x}{2a}\right].$$

B.6 Practice sheet 6

1. Consider a wave progressing on water of infinite depth. The velocity potential for the flow, $\phi(x, y, t)$, satisfies Laplace's equation. If the effects of surface tension are included, the pressure at the surface is reduced by an amount proportional to the curvature of the surface, i.e.

$$p(\text{surface}) = p_0 - \sigma \frac{\partial^2 \eta}{\partial x^2}$$

   where p_0 is the atmospheric pressure and σ is a constant. The boundary conditions for ϕ at $y = 0$ are therefore now

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{and} \quad \frac{\partial \phi}{\partial t} = -g\eta + \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2}.$$

   (a) What is the other boundary condition on $\phi(x, y, t)$?

   (b) If $\eta(x, t) = A \sin(kx - \omega t)$, show that the velocity potential is given by

$$\phi(x, y, t) = -\frac{\omega A}{k} e^{ky} \cos(kx - \omega t).$$

(c) Obtain the dispersion relation

$$\omega^2 = gk + \frac{\sigma k^3}{\rho}.$$

(d) Sketch the wave speed c $(= \omega/k)$ firstly for $\sigma = 0$ and secondly for a small value of σ. In the latter case, find the minimum wave speed. Which waves are most affected by the surface tension and why? Do affected waves travel faster or slower for non-zero σ than when $\sigma = 0$?

2. Fluid is oscillating in a container bounded by rigid walls at $x = 0$, $x = L$ and $y = -H$. The velocity potential $\phi(x, y, t)$ satisfies Laplace's equation and the following boundary conditions:

$$\frac{\partial \phi}{\partial y} = 0 \text{ at } y = -H, \quad \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \text{ at } y = 0,$$

$$\frac{\partial \phi}{\partial t} = -g\eta \text{ at } y = 0, \quad \frac{\partial \phi}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L,$$

the last condition ensuring no flow into the walls at $x = 0$ and $x = L$.

(a) Show that if $\eta(x, t) = A\cos(kx - \omega t)$ there is no solution for $\phi(x, y, t)$ which satisfies all the boundary conditions.

(b) Show that by adding two travelling waves of the form

$$\eta(x, t) = A\cos(kx - \omega t) \quad \text{and} \quad \eta(x, t) = A\cos(kx + \omega t)$$

we obtain a standing wave of the form

$$\eta(x, t) = 2A\cos(kx)\cos(\omega t).$$

(c) Use this form for η to obtain a velocity potential $\phi(x, y, t)$ which does satisfy all the boundary conditions. What restriction do you find on the wavenumber k?

3. An earthquake-generated ocean wave may be modelled by considering a fluid in water of depth H, with the lower boundary condition on the velocity potential $\phi(x, y, t)$ that

$$\frac{\partial \phi}{\partial y} = B\cos(kx - \omega t) \text{ at } y = -H,$$

where B is constant.

The other boundary conditions for ϕ are

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \text{ at } y = 0, \quad \frac{\partial \phi}{\partial t} = -g\eta \text{ at } y = 0.$$

By assuming a surface displacement $\eta(x, t) = A\sin(kx - \omega t)$, find the velocity potential, $\phi(x, y, t)$, and the dispersion relation.

B.7 Practice sheet 7

1. In this question you may use without proof the Euler equations*:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathbf{F},$$

and the identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times \omega = \nabla \left(\frac{1}{2} \mathbf{u}^2 \right).$$

Incompressible inviscid fluid flows steadily under the action of a conservative body force vF with potential G.

Show that

$$\nabla \left(\frac{p}{\rho} + \frac{1}{2} \mathbf{u}^2 + G \right) = \mathbf{u} \times \omega,$$

where $\omega = \nabla \times \mathbf{u}$ is the vorticity.

(a) Deduce that $\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2 + G$ is constant along any streamline.

(b) Water is being drained from a large tank of circular horizontal cross-section of radius R through a small circular hole of radius λR in the base, where $0 < \lambda \ll 1$. If the depth of the water in the tank is initially H and all free surfaces of the water are at atmospheric pressure, find the fluid velocity at the hole. (It may be assumed that the fluid motion is approximately steady.)

2. Liquid of density ρ is flowing along a horizontal pipe of variable cross-section. Pressure gauges are attached to the pipe at two points A and B. Show that if Δp is the pressure difference indicated by the gauges, the velocity at point B is equal to

$$\left[\frac{2a_1^2 \Delta p}{\rho(a_1^2 - a_2^2)} \right]^{1/2},$$

where a_1 and a_2 are the cross-sectional areas of the pipe at A and B respectively.

3. A water tap of diameter 1.5cm is 20m below the level of a reservoir which supplies water to a town. How many cubic metres of water can be delivered by the tap in one hour?

4. The equations governing the motion of an ideal fluid are

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathbf{F} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure and $\mathbf{F}$ is the body force. In two-dimensional motion, $\mathbf{u} = (u, v, 0)$ and u, v are functions of Cartesian coordinates x and y and time t. If the body force is conservative, eliminate the pressure from the governing equations and deduce that

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = 0,$$

*People tend to say 'equations' here even though this is obviously one equation, since it represents the two or three equations in component (that is, non-vector) form where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Interpret this result physically and deduce that the two-dimensional motion of an ideal fluid started from rest is irrotational.

B.8 Practice sheet 8

1. Consider a shallow stream flowing along a constant-width channel. Let the channel floor rise slowly to reach a small constant height k. Let the fluid depths upstream and downstream of the rise by h_1 and h_2 respectively.

   (a) Use Bernoulli's equation to show that

   $$\Phi(h_1) = \Phi(h_2) + k,$$

   where $\Phi(h) = h + Q^2/2gh^2$ and Q is the discharge per unit width.

   (b) Sketch the graph of $\Phi(h)$ for $h > 0$, obtaining the position h_m of the minimum of Φ. For which range of values of h is the flow supercritical and for which range is it subcritical?

   (c) Using the result of part (a) and the graph of part (b), describe the different behaviour of the flow depending on whether h_1 is greater than or less than h_m.

2. A stream of speed U and depth H flows along a channel of breadth B. It comes to a section where the breadth reduces slowly and smoothly and then increases slowly and smoothly to B. The flow is everywhere smooth and steady, and the upstream Froude number, $\frac{U}{\sqrt{gH}}$, is $1/\sqrt{3}$. Show that the downstream depth is either H or $\frac{1}{2}H$ and determine the two possible downstream Froude numbers.

3. The depth of water in a channel is equal to h at a location where the width is b.

   (a) If the volume rate of flow is equal to Q, show that

   $$\frac{Q^2}{2gb^2} = h^2(C - h),$$

   where C is a constant and g is gravity.

   (b) Show that, for a range of values of Q, there are two posible values of h, the larger of which lies between $\frac{2}{3}C$ and C.

   (c) If the width of the channel increaes by a small amount and if h lies in the range between $\frac{2}{3}$ and C, does the depth of the water increase or decrease?

4. When a ship travels through a narrow, shallow canal it is known that there is a danger that the ship will ground if it goes too fast. The ship has draught d and width b as shown below, while the canal has width B and the undisturbed water has depth H.

Show that the ship will group if its speed U is given by

$$U^2\left(\frac{B^2H^2}{d^2(B^2-b^2)}-1\right)=2g(H-d)$$

approximately. (Hint: consider the ship at rest and the water moving past it with speed U).

B.9 Practice sheet 9

1. Write down the complex potential for a line source of strength $2\pi m$ at the point $z=a$ in the z $(=x+iy)$ plane.

   A line source of strength $2\pi m$ is placed at the point $x=a$, symmetrically between the walls $z=re^{\pm i\pi/3}$ which intersect at the origin. All the fluid from this source passes into a line sink at a small hole in the walls at the origin.

   (a) Write down the complex potential for the combined flow.

   (b) Sketch the streamlines.

   (c) Calculate the fluid velocity at the point $z=ae^{i\pi/3}$.

   (d) If the line sink at the origin is replaced by a line source of the same strength, sketch the streamlines for this second flow and show that there is a stagnation point at the point $z=2^{-1/3}a$.

2. In this question you may use without proof the results:

$$\sinh(x+iy)=\sinh x\cos y+i\cosh x\sin y,$$

$$\cosh(x+iy)=\cosh x\cos y+i\sinh x\sin y.$$

   (a) Show that if $w(z)=m\log\sinh(z)$, then $w\sim m\log z$ as $|z|\to 0$.

   (b) Show that $\mathrm{Im}(w)$ is constant on the lines $y=\pm\frac{1}{2}\pi$.

   (c) Deduce that $w(z)$ is the complex potential for a source of strength $2\pi m$ at the origin in the presence of straight boundaries a distance π apart.

   (d) Find the speed on the boundaries and plot it as a function of x.

   (e) Derive the fluid velocity at a large distance from the source and give a physical argument to justify your answer.

3. Find the complex potential for the flow of a uniform stream of speed U at an angle α to the real axis, past a fixed cylinder of radius a, when there is no circulation round the cylinder.

4. A line source of strength $2\pi m$ is placed at the point $x=a$ symmetrically between walls at $\arg z=\pm\pi/4$ which intersect at 0. All the fluid passes into a link sink at 0.

   (a) Write down the complex potential for the combined flow.

(b) Sketch the streamlines.

(c) Calculate the fluid velocity at the point $z = ae^{i\pi/4}$.

(d) If the line sink at 0 is replaced by a line source of the same strength, sketch the streamlines for the second flow and show that there is a stagnation point at $z = 2^{-1/4}a$.

B.10 Practice sheet 10

1. Use the circle theorem to show that the complex potential for the flow of a uniform stream of speed U at angle α to the x-axis past a circular cylinder of radius a is
$$w(z) = U\left(ze^{ia} + \frac{a^2e^{i\alpha}}{z}\right) - i\kappa\log z,$$
where $2\pi\kappa$ is the circulation about the cylinder. Show that application of the transformation
$$\zeta = z + \frac{a^2}{z}$$
gives the complex potential for the corresponding flow past a flat plate which edges given by $\zeta = \pm 2a$. Determine the value of κ for which the velocity at the trailing edge, $\zeta = 2a$, is finite. Sketch the streamlines for the flow past the plate for this value of κ and also for the case $\kappa = 0$.

   What is the significance of this result in relation to the lift on an aerofoil?

2. (a) Use the circle theorem to show that the complex potential for a source of strength $2\pi m$ at $z = be^{i\alpha}$ outside the cylinder $|z| = a$ (where a, b, α are real, $a < b$ and $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$) can be written
$$w(z) = m\log(z - be^{i\alpha}) + m\log\left(\frac{a^2}{z} - be^{-i\alpha}\right) - i\kappa\log z.$$
   Here $2\pi\kappa$ is the circulation of the cylinder.

   (b) Show that the application of the transformation
$$\zeta = z + \frac{a^2}{z}$$
   gives the complex potential for the corresponding flow past a flat plate with edges given by $\zeta = \pm 2a$.

   (c) Determine the value for κ for wihch the velocity at the trailing edge $\zeta = -2a$ is finite and sketch the streamlines of the flow for this value of κ.

3. Use the circle theorem to show that hte complex potential for a source of strength $2\pi m$ at $z = b$ outside the cylinder $|z| = a$ (where a, b are real and $a < b$) can be written
$$w(z) = m\log(z - b) + m\log\left(\frac{a^2}{z} - b\right).$$
   Describe the image system within the cylinder. Sketch the streamlines for the flow outside the cylinder.

4. Find the complex potential for the flow of a uniform stream of speed U at an angle α to the real axis, past a fixed circular cylinder of radius c when there is no circulation around the cylinder.

   Show that the transformation
$$\zeta = z + \frac{k^2}{z}$$
   where $k < c$ maps the circle $|z| = c$ onto an ellipse in the ζ plane. Deduce the complex potential for the flow of a stream at an angle α to the major axis of an ellipse with semi-axes a and b, where $a > b$. Find the positions of the stagnation points on the ellipse and give a careful sketch of the streamline originating at the stagnation point.

Appendix C

Practice sheet solutions

Solutions are only given for the individual questions which were set as homework when the practice sheets were originally set. They have been typed up from handwritten sheets and so may have some errors in them.

C.1 Practice sheet 1

1. **Solution:**

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = (2, 2t+3, 3t^2+4t)$$

$$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2} = (0, 2, 6t+4).$$

When $t = 1$,

$$\mathbf{u} = (2, 5, 7); \quad \mathbf{a} = (0, 2, 10).$$

Let $\mathbf{v} = (2, 3, 4)$. Then

$$\text{Component of } \mathbf{u} \text{ in direction of } \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{47}{\sqrt{29}}$$

$$\text{Component of } \mathbf{a} \text{ in direction of } \mathbf{v} = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{46}{\sqrt{29}}.$$

2. **Solution:**

(a)

$$\begin{aligned}\nabla\phi &= (\phi_x, \phi_y, \phi_z)\\ &= (2xyz^3 + y^2z^2, x^2z^3 + 2xyz^2, 3x^2yz^2 + 2xy^2z)\end{aligned}$$

At $P(1,3,2)$,

$$\begin{aligned}\nabla\phi &= (48+36, 8+24, 36+36)\\ &= 84\,\widehat{\mathbf{\imath}} + 32\,\widehat{\mathbf{\jmath}} + 72\,\widehat{\mathbf{k}}\end{aligned}$$

(b)

$$\widehat{\mathbf{n}} = \left.\frac{\nabla\phi}{|\nabla\phi|}\right|_{(1,1,1)} = \frac{(3,3,5)}{\sqrt{43}}.$$

3. **Solution:**

(a)

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \frac{\partial x^2y}{\partial x} - \frac{\partial xyz}{\partial y} + \frac{\partial yz^2}{\partial z}\\ &= 2xy - xz + 2yz\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{curl}\mathbf{A} &= \begin{vmatrix}\widehat{\mathbf{\imath}} & \widehat{\mathbf{\jmath}} & \widehat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & -xyz & yz^2\end{vmatrix}\\ &= (z^2+xy, 0, -yz-x^2)\end{aligned}$$

(c)

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\mathbf{A} &= \begin{vmatrix}\widehat{\mathbf{\imath}} & \widehat{\mathbf{\jmath}} & \widehat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+xy & 0 & -yz-x^2\end{vmatrix}\\ &= (-z, 2z+2x, -x)\end{aligned}$$

(d) $\operatorname{div}\operatorname{curl}\mathbf{A} = 0$ (it always is)

4. **Solution:**

(a)

$$\begin{aligned}\int_V \nabla\cdot\mathbf{F}\,dV &= \int_0^3\int_0^4\int_0^2 y\,dx\,dz\,dy\\ &= \int_0^3\int_0^4 [xy]_0^2 dz\,dy\\ &= \int_0^3\int_0^4 2y\,dz,dy\\ &= \int_0^3 [2yz]_0^4 dy\\ &= \int_0^3 8y\,dy\\ &= [4y^2]_0^3\\ &= 36\end{aligned}$$

(b) The divergence theorem gives

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

(where $d\mathbf{S} = \widehat{\mathbf{n}}\, dS$). So from part (a),

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 36.$$

If you didn't notice this, you could have calculated $\int \mathbf{F} \cdot d\mathbf{S}$ over each of the six surfaces in turn.

C.2 Practice sheet 2

1. **Solution:**

(a) Pathlines (streaklines) are given by solving:

$$\frac{dx}{dt} = u,\ \frac{dy}{dt} = v,\ \frac{dw}{dt} = w. \tag{C.1}$$

In this case, we solve:

$$\frac{dx}{dt} = \alpha,\ \frac{dy}{dt} = kt,\ \frac{dw}{dt} = 0, \tag{C.2}$$

subject to $x = y = z = 0$ when $t = 0$.
We find that

$$x = \alpha t,\ y = \frac{1}{2}kt^2,\ z = 0, \tag{C.3}$$

having applied our initial conditions. This gives

$$y = \frac{kx^2}{2\alpha^2}. \tag{C.4}$$

Where $x \geq 0$ as $t \geq 0$. See Figure C.1.

(b) Streamlines are given by solving:

$$\frac{dx}{ds} = u,\ \frac{dy}{ds} = v,\ \frac{dz}{ds} = w. \tag{C.5}$$

In this case, we solve

$$\frac{dx}{ds} = \alpha,\ \frac{dy}{ds} = kt,\ \frac{dz}{ds} = 0. \tag{C.6}$$

We find that

$$x = \alpha s + x_0,\ y = kst + y_0,\ z = z_0 \tag{C.7}$$

At time $t = 1$, we see that

$$x = \alpha s + x_0,\ y = ks + y_0,\ z = z_0, \tag{C.8}$$

giving

$$y = \frac{kx}{\alpha} + C\ \text{(Const.)},\ z = z_0 \tag{C.9}$$

Similarly, at $t = 2$ we find that

$$y = \frac{2kx}{\alpha} + C\ \text{(Const.)},\ z = z_0 \tag{C.10}$$

See Figure C.2.

(c) The streaklines in 1a show the changing position of a particle over a time interval (in this case, the interval is $t \geq 0$). We see that as time increases, the particles turn towards the y-direction. the streamlines in 1b show the velocity field for all particles. Given the position of a particle at a specific time, the velocity of the particle can be found from the streamlines. In this case, the slope of the streamlines are given by kt/α.

2. **Solution:**

(a) Streakline is given by:

$$\frac{dx}{dt} = \frac{1}{1+t},\ \frac{dy}{dt} = 1, \tag{C.11}$$

Subject to $x = y = 1$ when $t = 0$. We find that

$$x = \ln(1+t) + 1,\ y = 1 + t, \tag{C.12}$$

or

$$y = e^{x-1}, \tag{C.13}$$

where $x \geq 1$ as $t \geq 0$. See Figure C.3.

(b) Streamlines are given by

$$\frac{dx}{ds} = \frac{1}{1+t},\ \frac{dy}{dt} = 1, \tag{C.14}$$

Subject to $x = y = 1$ at $t = 0$. We find that

$$x = \frac{s}{1+t} + x_0,\ y = s + y_0. \tag{C.15}$$

We measure s from 0 at $(1, 1)$, giving $x_0 = y_0 = 1$. Combining (C.15), after applying the initial condition gives

$$y = (1+t)(x-1) + 1. \tag{C.16}$$

At $t = 0$, this gives $y = x$. See Figure C.4.

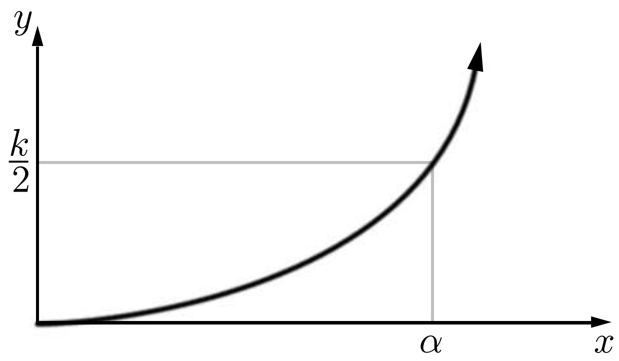

Figure C.1: Solution to Sheet 2, Q1a

3. **Solution:**

(a) Streaklines are given by

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (u, v, w) = (ay, -ax, ct). \tag{C.17}$$

Combining the above, we find that

$$\frac{dy}{dx} = -\frac{x}{y}, \ \frac{dz}{dt} = ct, \tag{C.18}$$

giving

$$y^2 + x^2 = C \text{ (const.)}, \ z = \frac{1}{2}ct^2 + z_0. \tag{C.19}$$

Hence the streaklines are spirals, with radius $\sqrt{C}$ in the xy-plane, where the z-spacing increases like t^2 as t increases. See Figure C.5.

(b) Streamlines are giving by:

$$\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = (u, v, w) = (ay, -ax, ct). \tag{C.20}$$

Again, we find that

$$y^2 + x^2 = D \text{ (const.)}, \tag{C.21}$$

however for z we find

$$z = cst + z_0. \tag{C.22}$$

So the streamlines again are spirals, but with the z-spacing increasing like t instead of t^2. See Figures C.6 and C.7.

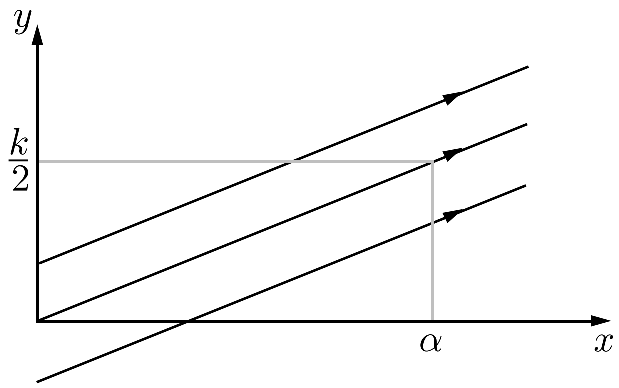

Figure C.2: Solution to Sheet 2, Q1b

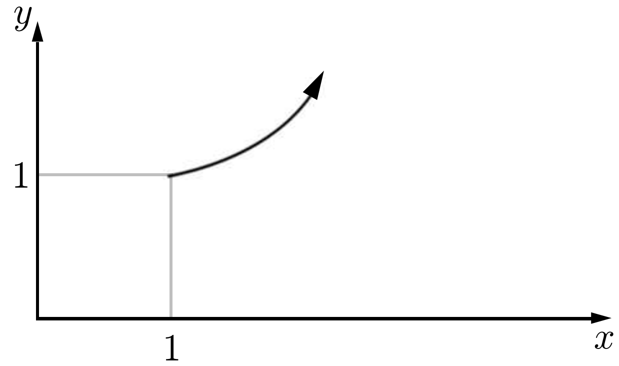

Figure C.3: Solution to Sheet 2, Q2a

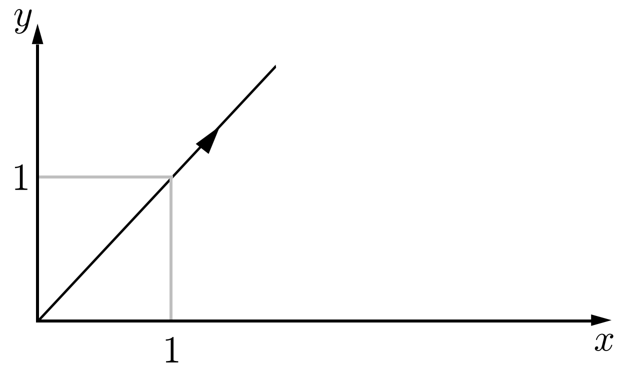

Figure C.4: Solution to Sheet 2, Q2b

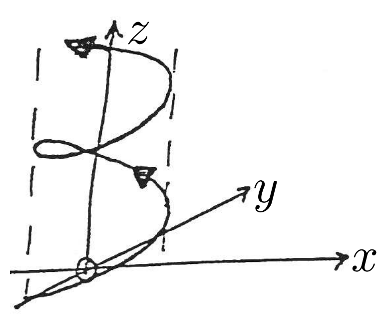

Figure C.5: Solution to Sheet 2, Q3a

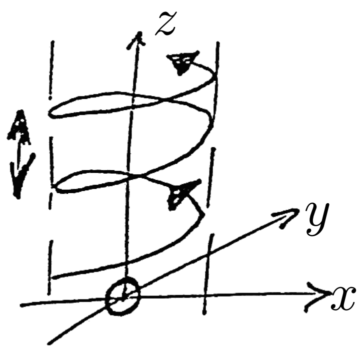

Figure C.6: Solution to Sheet 2, Q3b. At $t = 1$ (say), $z = cs + z_0$.

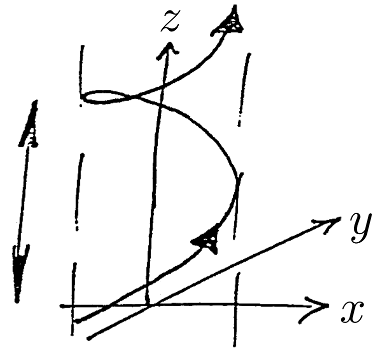

Figure C.7: Solution to Sheet 2, Q3b. At $t = 2$, $z = 2cs + z_0$.

C.3 Practice sheet 3

1. **Solution:**

(a)

$$\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \tag{C.23}$$

$$= \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0 \tag{C.24}$$

$$\nabla\times\mathbf{u} = \begin{vmatrix} \widehat{\mathbf{\imath}} & \widehat{\mathbf{\jmath}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} \tag{C.25}$$

$$= 0\,\widehat{\mathbf{\imath}} + 0\,\widehat{\mathbf{\jmath}} + \widehat{\mathbf{k}}\left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right) \tag{C.26}$$

$$= \widehat{\mathbf{k}}\left(-\frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \left(\frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}\right)\right) \tag{C.27}$$

$$= 0 \tag{C.28}$$

(b) To find the velocity potential, we solve

$$\mathbf{u} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right). \tag{C.29}$$

In this case

$$\frac{\partial \phi}{\partial x} = \frac{-y}{x^2+y^2}, \; \frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2}. \tag{C.30}$$

The first equation is easily solved to find

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) + f(y), \tag{C.31}$$

for some arbitrary $f(y)$. Substituting this into the second equation gives $f'(y) = 0$, so that f is constant. We may take this constant to be zero as the velocity potential is only defined up to a constant. So

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{C.32}$$

To find the streamlines, we first find the streamfunction. We solve

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right). \tag{C.33}$$

So

$$\frac{\partial \psi}{\partial y} = \frac{-y}{x^2+y^2}, \; \frac{\partial \psi}{\partial x} = \frac{-x}{x^2+y^2}. \tag{C.34}$$

The first equation gives us that

$$\psi = -\frac{1}{2}\ln(x^2+y^2) + f(x), \tag{C.35}$$

whilst the second gives

$$\psi = -\frac{1}{2}\ln(x^2+y^2) + g(y). \tag{C.36}$$

Equating these two gives that $f(x) = g(y) =$ constant, which we set to zero for the same reason as above. (Note that we could have found that f is constant via the same method as we did for the velocity potential, but it is worth seeing both techniques). We therefore see that

$$\psi = -\ln(r) \tag{C.37}$$

Which shows the the streamlines, which satisfy $\psi =$ constant, are $r =$ constant. This flow is known as a "line-vortex" flow. See Figure C.8.

2. **Solution:**

(a)

$$\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(x^3) = 0 \implies \text{incompressible.} \tag{C.38}$$

$$\nabla \times \mathbf{u} = 0\,\hat{\mathbf{\imath}} + 0\,\hat{\mathbf{\jmath}} + (3x^2+2y)\,\hat{\mathbf{k}} \neq 0 \implies \text{rotational.} \tag{C.39}$$

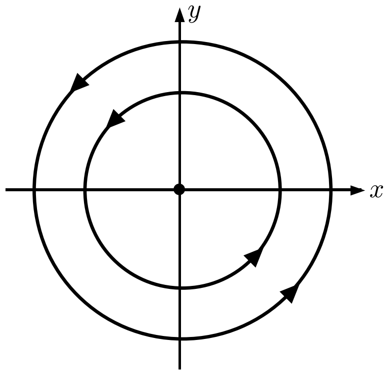

Figure C.8: Solution to Sheet 3, Q1b

(b) The flow is incompressible, so there exists a streamfunction, ψ. We solve

$$\psi_y = y^2,\ \psi_x = x^3. \tag{C.40}$$

This gives that

$$\psi = \frac{1}{3}y^3 + f(x) \tag{C.41}$$
$$\psi = \frac{1}{4}x^4 + g(y). \tag{C.42}$$

Equating these two equations and solving for f and g, we find that

$$\psi = \frac{1}{3}y^3 + \frac{1}{4}x^4, \tag{C.43}$$

where we have set the constant equal to 0.

(c) The flow is rotational, so there is no velocity potential.

3. The velocity field cannot be that of the flow of an incompressible fluid, as

$$\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x}\left(\frac{-2xyz}{(x^2+y^2)^2}\right) + \frac{\partial}{\partial y}\left(\frac{(x^2-y^2)z}{x^2+y^2}\right) + \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right) \tag{C.44}$$
$$= \frac{8x^2yz}{(x^2+y^2)^3} - \frac{2yz}{(x^2+y^2)^2} - \frac{2y\,(x^2-y^2)\,z}{(x^2+y^2)^2} - \frac{2yz}{x^2+y^2} \neq 0. \tag{C.45}$$

4. The surfaces of constant ϕ (equipotential surfaces) are normal to $\nabla\phi$, and $\mathbf{u} = \nabla\phi$. So the streamlines, which are parallel to $\mathbf{u}$, are parallel to $\nabla\phi$. Hence the streamlines are normal to the equipotential surfaces.

C.4 Practice sheet 4

1. **Solution:**

(a)
$$\phi = Ur\cos\theta + \frac{\mu\cos\theta}{r} + m\log r$$

We require the normal velocity at $r = a$ to be V.

$$U = \phi_r = U\cos\theta - \frac{\mu\cos\theta}{r^2} + \frac{m}{r}$$

Normal velocity is independent of θ therefore choose $\mu = Ua^2$ and $u = V$ at $r = a$. Therefore $m = aV$. Hence

$$\phi = Ur\cos\theta + \frac{Ua^2\cos\theta}{r} + aV\log r.$$

(b) If $V = 4U$,

$$\phi = Ur\cos\theta + \frac{Ua^2\cos\theta}{r} + 4aU\log r$$

So

$$u = \phi_r = U\cos\theta - \frac{Ua^2\cos\theta}{r^2} + \frac{4aU}{r}$$

$$v = \frac{1}{r}\phi_\theta = -U\sin\theta - \frac{Ua^2\sin\theta}{r^2}$$

$v = 0$ if

$$-U\sin\theta\left[1 + \frac{a^2}{r^2}\right] = 0$$

i.e. $\theta = 0, \pi$.

$u = 0$ if

$$U\left[\cos\theta - \frac{a^2}{r^2}\cos\theta + \frac{4a}{r}\right] = 0.$$

If $\theta = 0$ then

$$u = U\left[1 - \frac{a^2}{r^2} + \frac{4a}{r}\right]$$

Therefore $u = 0$ if

$$r^2 + 4ar - a^2 = 0$$

i.e.

$$r = -2a \pm \sqrt{5}a.$$

Take the positive root since $r > 0$. Therefore $r = a(-2 + \sqrt{5})$. **But** this is *inside* the cylinder (i.e. $r < a$). So we must use $\theta = \pi$ (i.e. upstream of cylinder):

$$u = U\left[-1 + \frac{a^2}{r^2} + \frac{4a}{r}\right]$$

and the stagnation point is where

$$r^2 - 4ar - a^2 = 0$$

and hence

$$r = a(2 + \sqrt{5}).$$

(c) See Figure C.9.

2. **Solution:**

(a) Adding the streamfunctions for a source and a sink gives

$$\psi = m(\theta_1 - \theta_2).$$

Angle α satisfies

$$\theta_2 + \pi - \theta_1 + \alpha = \pi$$

Hence $\alpha = \theta_1 - \theta_2$ and so $\psi = m\alpha$.

(b)

$$\tan\theta_1 = \frac{y}{x-s}, \qquad \tan\theta_2 = \frac{y}{x+s}$$

Hence

$$\begin{aligned} \tan\alpha &= \tan(\theta_1 - \theta_2) \\ &= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \\ &= \frac{2ys}{x^2 + y^2 - s^2} \end{aligned}$$

So

$$\alpha = \tan^{-1}\left[\frac{2ys}{x^2+y^2-s^2}\right], \qquad \psi = m\tan^{-1}\left[\frac{2ys}{x^2+y^2-s^2}\right]$$

(c) $\psi = \text{const.}$ implies

$$\frac{2ys}{x^2+y^2-s^2} = \text{const.} = A^{-1}$$

say. Then

$$x^2 + (y - sA)^2 = s^2(1 + A^2)$$

which are circles with centre $(0, sA)$.

(d)

$$\begin{aligned} \psi_{\text{lim}} &= \lim_{s\to 0, 2ms=\mu} m\tan^{-1}\left[\frac{2ys}{x^2+y^2-s^2}\right] \\ &= \frac{\mu r\sin\theta}{r^2} \\ &= \frac{\mu\sin\theta}{r} \end{aligned}$$

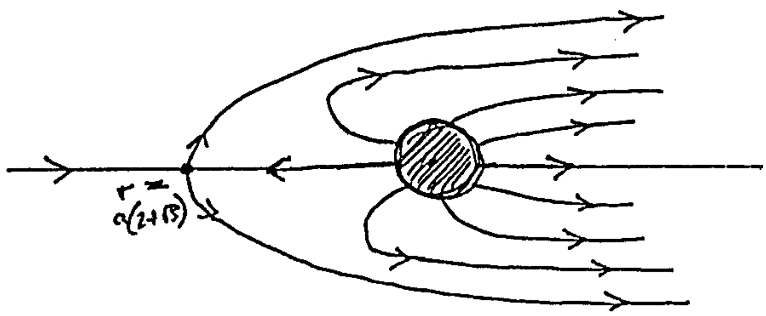

Figure C.9: Solution to Sheet 4, Q1c

C.5 Practice sheet 5

1. **Solution:** For a line vortex $\psi = -\beta \ln r$. The image vortices are located at $(0, \pm a)$ and have equal but opposite strength as shown in Figure C.10. Hence the streamfunction is given by

$$\psi = -\beta \ln r_1 + \beta \ln r_2 + \beta \ln r_3 \tag{C.46}$$

Where

$$r_1^2 = (x-a)^2 + y^2, \tag{C.47}$$
$$r_2^2 = x^2 + (y-a)^2, \tag{C.48}$$
$$r_3^2 = x^2 + (y+a)^2 \tag{C.49}$$

as shown in Figure C.11.

If we consider the line $y = x$, then $r_1 = r_2$ and so $\psi = \beta \ln r_3$ which is not constant. Similarly we find that on the line $y = -x$, ψ is not constant. Therefore the boundaries are not streamlines.

We now consider the extension of the boundaries into the region $x < 0$ and find the image of the image vortices at the point $(-a, 0)$, which has the same strength and orientation as the original vortex, as shown in Figure C.12. Therefore the full streamfunction can be written as

$$\psi = -\beta \ln r_1 + \beta \ln r_2 + \beta \ln r_3 - \beta \ln r_4 \tag{C.50}$$

Where

$$r_4^2 = (x+a)^2 + y^2 \tag{C.51}$$

Along the line $y = x$, $r_1 = r_2$ and $r_3 = r_4$, so that $\psi = 0$. Similarly for $y = -x$. We therefore see that this is the streamfunction for a flow near a corner.

2. **Solution:**

   (a) In the farfield, $u \to -2y$, $v \to -2x$, therefore $\phi_x = -2y$, $\phi_y = -2x$. We therefore find that (setting the constant of integration to 0)

$$\phi = -2xy \tag{C.52}$$
$$= -2r^2 \sin\theta \cos\theta \tag{C.53}$$
$$= -r^2 \sin 2\theta \tag{C.54}$$

(b) We solve $\nabla^2\phi = 0$ by separation, i.e. $\psi = R(r)S(\theta)$. Substitution leads to

$$r^2R'' + rR' - \lambda^2 R = 0 \tag{C.55}$$
$$S'' = -\lambda^2 S \tag{C.56}$$

Solving these, we find

$$R = Ar^{\lambda} + Br^{-\lambda} \tag{C.57}$$
$$S = C\cos\theta + D\sin\theta \tag{C.58}$$

Where we have the boundary conditions

$$\phi \to -r^2 \sin 2\theta \text{ as } r \to \infty, \tag{C.59}$$
$$\phi_r = 0 \text{ at } r = a, \tag{C.60}$$
$$\int_0^{2\pi} \phi \, d\theta = \kappa \text{ at } r = a. \tag{C.61}$$

Applying the first two boundary conditions, we find that

$$(C.59) \implies C = 0,\ \lambda = 2,\ AD = -1, \tag{C.62}$$
$$(C.60) \implies -2a - 2Ba^{-3} = 0 \implies B = -a^4. \tag{C.63}$$

We see that

$$\phi = -\left(r^2 + \frac{a^4}{r^2}\right)\sin 2\theta \tag{C.64}$$

However the circulation around the cylinder for this flow is 0. We must therefore look for another term in this solution. We consider the case $\lambda = 0$ and find that

$$R = A + B\ln r, \tag{C.65}$$
$$S = C + D\theta. \tag{C.66}$$

In order to satisfy boundary condition (C.59), we must have that $B = 0$ and ignoring the constant solution, we find the solution $\phi = \beta\theta$ (line-vortex flow). The circulation around the cylinder for this flow is $2\pi\beta$ and so we find, applying (C.61) that $\beta = \kappa/2\pi$. Our full solution then becomes

$$\phi = -\left(r^2 + \frac{a^4}{r^2}\right)\sin 2\theta + \frac{\kappa\theta}{2\pi}. \tag{C.67}$$

(c) On $r = a$,

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -4a\cos 2\theta + \frac{\kappa}{2\pi a}. \tag{C.68}$$

Therefore the maximum speed on the cylinder is

$$u_{\max} = 4a + \frac{\kappa}{2\pi a}. \tag{C.69}$$

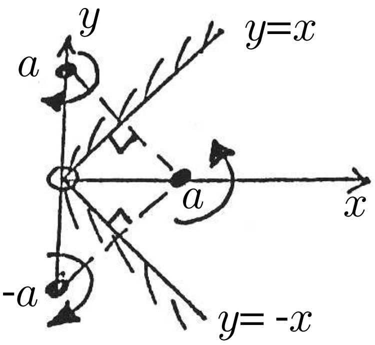

Figure C.10: Solution to Sheet 5, Q1. Diagram of image vortices.

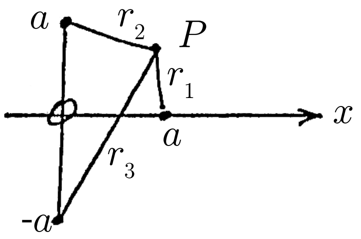

Figure C.11: Solution to Sheet 5, Q1. Definitions of r_1, r_2, r_3.

(d) Stagnation points exist if $u_\theta = 0$ on $r = a$. So (noticing that $\kappa = 4\pi a^2$)

$$\frac{\kappa}{2\pi a} = 4a\cos 2\theta \tag{C.70}$$

$$\implies \cos 2\theta = \frac{1}{2} \tag{C.71}$$

$$\implies \theta = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6}. \tag{C.72}$$

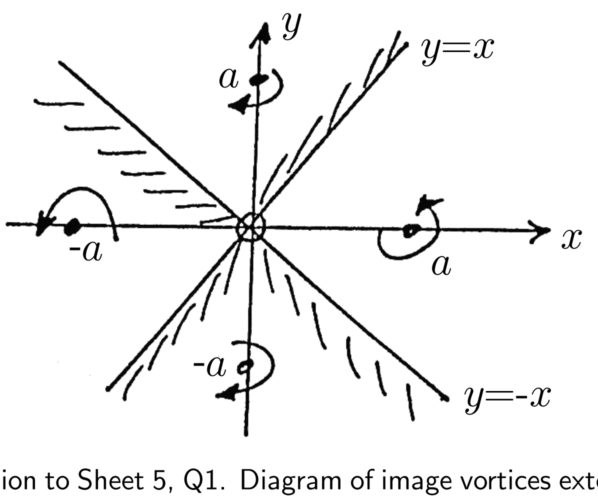

Figure C.12: Solution to Sheet 5, Q1. Diagram of image vortices extended into the $x < 0$ region.

C.6 Practice sheet 6

1. **Solution:**

   (a) The other boundary condition is that the motion must decay as $y \to -\infty$. In terms of the velocity potential, this means that $\nabla\phi \to 0$ as $y \to \infty$.

   (b) Taking inspiration from the first boundary condition on the surface, we try to set $\phi(x, y, t) = f(y)\cos(kx - \omega t)$. Then:

$$\nabla^2\phi = 0 \implies -k^2 f + f'' = 0 \tag{C.73}$$

$$\implies f(y) = Be^{ky} + Ce^{-ky}. \tag{C.74}$$

   The boundary condition from part 1a gives us that $C = 0$ and $\phi_y = \eta_t$ at $y = 0$ gives us that $kB = -\omega A$. Hence:

$$\phi = -\frac{\omega A}{k} e^{ky}\cos(kx - \omega t). \tag{C.75}$$

   (c) We apply the third boundary condition: $\phi_t = -g\eta + \sigma/\rho\eta_{xx}$ at $y = 0$, giving, after dropping a cos term:

$$-\frac{\omega^2 A}{k} = -gA - \frac{\sigma k^2}{\rho}A \tag{C.76}$$

   We rearrange to find:

$$\omega^2 = gk + \frac{\sigma}{\rho}k^3 \tag{C.77}$$

   (d)

$$c = \frac{\omega}{k} = \frac{1}{k}\left(gk + \frac{\sigma k^3}{\rho}\right)^{\frac{1}{2}} \tag{C.78}$$

$$= \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)^{\frac{1}{2}}. \tag{C.79}$$

   If $\sigma = 0$, then we get the graph in Figure C.13, since $c = (g/k)^{1/2}$.
   If $\sigma \neq 0$, the graph looks like Figure C.14. As $k \to \infty$, $c \sim (\sigma k/\rho)^{1/2}$. To find the minimum wave speed, we find the value of k such that:

$$\frac{dc}{dk} = 0 \tag{C.80}$$

$$\implies \frac{\frac{1}{2}\left(\frac{-g}{k^2} + \frac{\sigma}{\rho}\right)}{\left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)^{\frac{1}{2}}} = 0 \tag{C.81}$$

$$\implies -\frac{g}{k^2} + \frac{\sigma}{\rho} = 0 \tag{C.82}$$

$$\implies k = \left(\frac{g\rho}{\sigma}\right)^{\frac{1}{2}} \tag{C.83}$$

Therefore we find that the minimum wave speed is:

$$c_{\min} = 2^{\frac{1}{2}} \left(\frac{\sigma g}{\rho} \right)^{\frac{1}{4}} \tag{C.84}$$

2. **Solution:**

   (a) We try to set $\phi = f(y)\sin(kx - \omega t)$. Then:

$$\nabla^2 \phi = 0 \implies -k^2 f + f'' = 0 \tag{C.85}$$

$$\implies f(y) = Be^{ky} + Ce^{-ky}. \tag{C.86}$$

Then:

$$\phi_y = 0 \text{ at } y = -H \implies B = Ce^{2kH} \tag{C.87}$$

and

$$\phi_y = \eta_t \text{ at } y = 0 \implies C = \frac{\omega A}{2e^{kH} k \sinh(kH)}. \tag{C.88}$$

Hence

$$\phi = \frac{\omega A}{k \sinh(kH)} \cosh(k(H + y)) \sin(kx - \omega t). \tag{C.89}$$

Now

$$\phi_t = -g\eta \text{ at } y = 0 \implies \omega_2 = gk \tanh(kH). \tag{C.90}$$

But then the last boundary condition $\phi_x = 0$ at $x = 0, L$ requires:

$$\frac{\omega A}{k \sinh(kH)} \cosh(k(H + y)) \cos(\omega t) = 0 \ \forall t \tag{C.91}$$

and

$$\frac{\omega A}{k \sinh(kH)} \cosh(k(H + y)) \cos(kL - \omega t) = 0 \ \forall t. \tag{C.92}$$

This requires that $A = 0$, which would mean there is no wave.

   (b) Using the fact that $\cos(A) + \cos(B) = 2\cos((A + B)/2)\cos((A - B)/2)$:

$$A\cos(kx - \omega t) + A\cos(kx + \omega t) = 2A\cos(kx)\cos(\omega t). \tag{C.93}$$

This is a standing wave.

   (c) We try $\phi = f(y)\cos(kx)\sin(\omega t)$, and so $\nabla^2\phi = 0$ gives us that:

$$f(y) = Be^{ky} + Ce^{-ky}, \tag{C.94}$$

as above. Then $\phi_y = 0$ at $y = -H$ gives us that $B = Ce^{2kH}$, as above. $\phi_y = \eta_t$ at $y = 0$ gives us that:

$$C = -\frac{\omega A}{ke^H \sinh(kH)}. \tag{C.95}$$

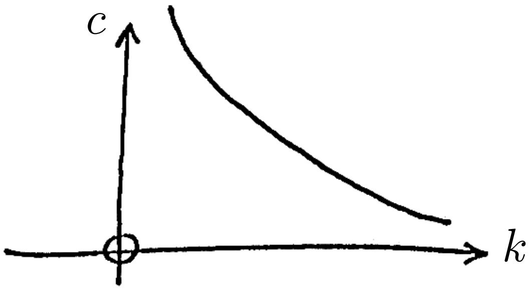

Figure C.13: Solution to Sheet 6, Q1d where $\sigma = 0$. Graph of $c = \left(\frac{g}{k}\right)^{1/2}$.

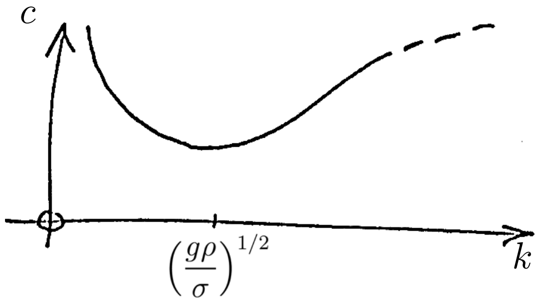

Figure C.14: Solution to Sheet 6, Q1d, where $\sigma \neq 0$. As $k \to \infty$, $c \sim \left(\frac{\sigma k}{\rho}\right)^{1/2}$.

Now, $\phi_y = \eta_t$ at $y = 0$ gives us that $\omega^2 = 2gk \tanh(kH)$. We therefore find that:

$$\phi = -\frac{2\omega A}{k \sinh(kH)} \cosh(k(H+y)) \cos(kx) \sin(\omega t). \tag{C.96}$$

Then $\phi_x = 0$ for $x = 0, L$ requires that:

$$\sin(0) = 0 \tag{C.97}$$

and

$$\sin(kL) = 0. \tag{C.98}$$

The first is obviously satisfied and the second gives us that $k = n\pi/L$.

C.7 Practice sheet 7

1. **Solution:**

   (a) From Euler's equations, since the flow is steady:

   $$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p - \rho \mathbf{F}. \tag{C.99}$$

   We use the given vector identity and write $\mathbf{F} = -\nabla G$, since G is the potential, to obtain:

   $$\nabla \left(\frac{1}{2}\mathbf{u}^2\right) - \mathbf{u}\times\mathbf{w} = -\frac{1}{\rho} - \nabla G. \tag{C.100}$$

Rearranging, we obtain:

$$\nabla\left(\frac{p}{\rho}+\frac{1}{2}\mathbf{u}^2+G\right)=\mathbf{u}\times\mathbf{w}. \quad (C.101)$$

(b) Taking the dot product with a vector $\mathbf{s}$ along a streamline, we see that since $\mathbf{u}$ is parallel to the streamline, $\mathbf{u}\times\mathbf{w}$ is perpendicular to it. Therefore the right hand side of the equation gives zero and we are left with:

$$\mathbf{s}\cdot\nabla\left(\frac{p}{\rho}+\frac{1}{2}\mathbf{u}^2+G\right)=0. \quad (C.102)$$

Which tells us that:

$$\frac{p}{\rho}+\frac{1}{2}\mathbf{u}^2+G \quad (C.103)$$

is conserved along steam lines. This is Bernoulli's law.

(c) Take a streamline from the surface at the upper surface to the hole. The initial depth is H, and we take the bottom of the cylinder to be $y=0$. The pressure at each point is p_0. Here, the external for is gravity, so the potential is $G=gy$. This is illustrated in Figure C.15. Then using Bernoulli's law, we find:

$$\frac{p_0}{\rho}+\frac{1}{2}U^2+gH=\frac{p_0}{\rho}+\frac{1}{2}V^2 \quad (C.104)$$

Where U and V are the velocities at the upper surface and hole respectively. Conservation of mass yields:

$$\pi R^2U=\pi\lambda^2R^2V. \quad (C.105)$$

Equations (C.104) and (C.105) yield:

$$V=\left(\frac{2gH}{1-\lambda^4}\right)^{\frac{1}{2}} \quad (C.106)$$

2. **Solution:** Consider a streamline connecting points A and B as in Figure C.16, and apply Bernoulli's law:

$$\frac{p_1}{\rho}+\frac{1}{2}U^2=\frac{p_2}{\rho}+\frac{1}{2}V^2. \quad (C.107)$$

Where p_1, p_2 are the pressures and U, V are the velocities at points A and B respectively. Conservation of mass gives us that $a_1U=a_2V$. We combine these to find:

$$\frac{p_1}{\rho}+\frac{1}{2}\frac{a_2^2}{a_1^2}V^2=\frac{p_2}{\rho}+\frac{1}{2}V^2. \quad (C.108)$$

Rearranging, we find:

$$V=\left(\frac{2a_1^2\Delta p}{\rho(a_1^2-a_2^2)}\right)^{\frac{1}{2}}. \quad (C.109)$$

Where $\Delta p=p_1-p_2$.

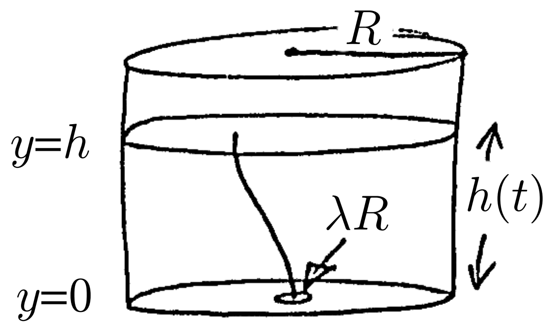

Figure C.15: Solution to Sheet 7, Q1c

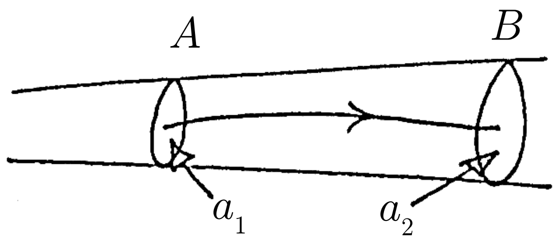

Figure C.16: Solution to Sheet 6, Q2

C.8 Practice sheet 8

1. **Solution:**

   (a) Figure C.17 shows us the situation we have. We apply Bernoulli's equation to the surface streamline:

   $$\frac{1}{2}u_1^2 + gh_1 = \frac{1}{2}u_2^2 + g(h_2 + k) \tag{C.110}$$

   Noting that $Q = uh$, so that $u_1h_1 = u_2h_2 = Q$. We substitute this into Bernoulli's equation and find:

   $$\frac{1}{2}\frac{Q^2}{h_1^2} + gh_1 = \frac{1}{2}\frac{Q^2}{h_2^2} + g(h_2 + k). \tag{C.111}$$

   We set:

   $$\Phi(h) = h + \frac{Q^2}{2gh} \tag{C.112}$$

   as shown in Figure C.18, and find that:

   $$\Phi(h_1) = \Phi(h_2) + k \tag{C.113}$$

(b) We look for h_m by setting:

$$\frac{d\Phi}{dh}(h_m) = 0 \tag{C.114}$$

$$\implies \frac{Q^2}{gh_m^3} = 1 \tag{C.115}$$

$$\implies h_m = \left(\frac{Q^2}{g}\right)^{\frac{1}{3}}. \tag{C.116}$$

If $h = h_m$, then, the Froude number is:

$$F^2 = \frac{U^2}{gh} = \frac{Q^2}{gh^3} = 1, \tag{C.117}$$

i.e. this is the critical point. If $h < h_m$, then $F > 1$, i.e. the flow is supercritical, and vice-versa.

(c) Note that part 1a shows us that $\Phi(h_2) < \Phi(h_1)$. So we see that if $h_1 > h_m$, we must travel from right to left on the graph and $h_2 < h_1$, however $h_2 > h_m$, so that the flow is everywhere subcritical. If $h_1 < h_m$, then we must move left to right on the graph, and so $h_2 > h_1$ and $h_2 < h_m$, i.e. the flow is everywhere supercritical.

2. **Solution:** Figure C.18 shows us what we are dealing with. The upstream Froude number is $1/\sqrt{3}$, so that:

$$\frac{U^2}{gH} = \frac{1}{3}. \tag{C.118}$$

Mass conservation implies that:

$$UBH = U_1BH_1 \implies UH = U_1H_1. \tag{C.119}$$

We apply Bernoulli to a surface streamline, and substitute the above expression in for U_1:

$$\frac{1}{2}U^2 + gH = \frac{1}{2}U_1^2 + gH_1 \tag{C.120}$$

$$\frac{1}{2}U^2 + gH = \frac{1}{2}\frac{U^2H^2}{H_1^2} + \frac{H_1}{H}. \tag{C.121}$$

Dividing through by gH:

$$\frac{U^2}{2gH} + 1 = \frac{U^2}{2gH}\frac{H^2}{H_1^2} + \frac{H_1}{H} \tag{C.122}$$

We set $H_1 = \alpha H$ and solve for α, using (C.118):

$$\frac{1}{6} + 1 = \frac{1}{6}\frac{1}{\alpha^2} + \alpha \tag{C.123}$$

$$\implies 6\alpha^3 - 7\alpha^2 + 1 = 0. \tag{C.124}$$

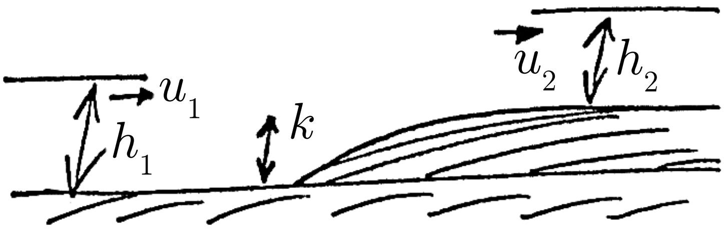

Figure C.17: Solution to Sheet 8, Q1a

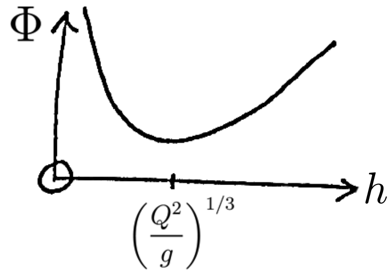

Figure C.18: Solution to Sheet 8, Q1b

We know that $\alpha = 1$ is a solution of this due to the upstream depth being H. Therefore we factor out $\alpha - 1$ and find:

$$(\alpha - 1)(6\alpha^2 - \alpha - 1) = 0. \tag{C.125}$$

The possible roots are therefore:

$$\alpha = 1 \text{ or } \alpha = \frac{1 \pm 5}{12}. \tag{C.126}$$

We need α to be positive, hence we have that $\alpha = 1, 1/2$. The downstream Froude number has:

$$F_1^2 == \frac{U_1^2}{gH_1^2} = \frac{U^2H^2}{gH_1^3} = \frac{F^2}{\alpha^3}. \tag{C.127}$$

Therefore if $\alpha = 1$ then $F_1 = F = 1/\sqrt{3}$ and if $\alpha = 1/2$ then $F_1 = F/\alpha^{3/2} = 2\sqrt{2}/\sqrt{3}$.

C.9 Practice sheet 9

1. **Solution:** $w(z) = m\ln(z - a)$.

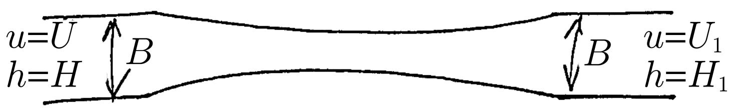

Figure C.19: Solution to Sheet 8, Q2

(a) See Figure C.20. Then

$$w(z) = m\ln(z-a) + m\ln\left[z - \left(-\frac{a}{2} + \frac{a\sqrt{3}}{2}i\right)\right] \tag{C.128}$$

$$+ m\ln\left[z - \left(-\frac{a}{2} - \frac{a\sqrt{3}}{2}i\right)\right] - 3m\ln z \tag{C.129}$$

$$= m\ln(z^3 - a^3) - 3m\ln z \tag{C.130}$$

(b) We show that the walls are streamlines as well as showing that $y = 0$ is a streamline. Recall that $\psi = \mathrm{Im}[w(z)]$. If $y = 0$, then $z = x$ so that $w(z) = m\ln(x^3 - a^3) - 3m\ln x = m\ln(1 - a^3/x^3)$. For $x > a$ this is real, and therefore $\psi = 0$ and for $x < a$, $\psi = m\pi$ (recall that for $x < 0$, $\ln x = \ln|x| + i\pi$). Therefore $y = 0$ is a streamline.
If $z = re^{\pm i\pi/3}$, then:

$$w(z) = m\ln(r^3e^{\pm i\pi} - a^3\_3m\ln(re^{\pm i\frac{\pi}{3}}) \tag{C.131}$$

$$= m\ln(r^3 - a^3) \pm mi\pi - 3m\ln r \mp mi\pi \tag{C.132}$$

$$= m\ln(r^3 - a^3) - 3m\ln r. \tag{C.133}$$

We apply the same argument as in the $y = 0$ case to see that the walls are streamlines. We then note that the flow near the source is radial and that all fluid must pass into the sink to sketch the rest of the streamlines. See Figure C.21.

(c) Recall that $u - iv = w'(z)$

$$w'(z) = \frac{3z^2m}{z^3 - a^3} - \frac{3m}{z}, \tag{C.134}$$

so that:

$$w'(ae^{i\frac{\pi}{3}}) = -\frac{3a^2e^{\frac{2\pi i}{3}}m}{2a^3} - \frac{3e^{-\frac{\pi}{3}}m}{a}. \tag{C.135}$$

Therefore:

$$u = \mathrm{Re}\,w' = -\frac{3m}{4a} \tag{C.136}$$

$$v = -\mathrm{Im}\,w' = -\frac{3\sqrt{3}m}{4a} \tag{C.137}$$

(d) Flow near sources is radial, so looking at the diagram, we expect there to be a stagnation point somewhere on the x-axis between the two source: see Figure C.22. In this case, $w(z) = m\ln(z^3 - a^3) + 3m\ln z$, so that:

$$w'(z) = \frac{3z^2m}{z^3 - a^3} + \frac{3m}{z} \tag{C.138}$$

$$= \frac{6z^3 - 3a^3}{z(z^3 - a^3)}. \tag{C.139}$$

Therefore we have stagnation point at $z = 1/2a^3$.

2. **Solution:**

(a)

$$w(z) = m \log \sinh(z) = m \log(\sinh x \cos y + i \cosh x \sin y). \tag{C.140}$$

As $x \to 0$ $\sinh x \to x$, $\cosh x \to 1$, and as $y \to 0$, $\sin y \to y$, $\cos y \to 1$. Therefore as $|z| \to 0$:

$$w(z) \to m \log(x + iy) = m \log z. \tag{C.141}$$

(b) At $y = \pm\pi/2$:

$$w(z) = m \log(\pm i \cosh x) \tag{C.142}$$

$$= m \log(re^{i\theta}), \tag{C.143}$$

where $r = \cosh x$ and $\theta = \pm\pi/2$. So:

$$\mathrm{Im}(w(z)) = m\theta = \pm\frac{m\pi}{2}. \tag{C.144}$$

(c) Set $w(z) = \phi + i\psi$ where ϕ is the potential and ψ is the streamfunction. As $\mathrm{Im}(z)$ is constant along the lines $y = \pm\pi/2$, the stream function is constant along these lines and hence they are streamlines. Therefore there is no flow across these lines and they are permissible boundaries. In order to show that the complex potential describes a point source at the origin of strength m, we require that as we approach the origin, the complex potential matches that of a point source, which was shown in part 2a. See Figure C.23.

(d) $u - iv = w'(z)$. On the boundaries, we need to find u.

$$w'(z) = \frac{m}{\sinh z} \cosh z = \frac{m(\cosh x \cos y + i \sinh x \sin y)}{\sinh x \cos y + i \cosh x \sin y}. \tag{C.145}$$

At $y = \pm\pi/2$, $\cos y = 0$ and $\sinh y = \pm 1$ so that:

$$u = \mathrm{Re}(w'(z)) = \mathrm{Re}\left(\frac{\pm im \sinh x}{\pm i \cosh x}\right) = m \tanh x \tag{C.146}$$

This is sketched in Figure C.24.

(e) Note that as $x \to \pm\infty$, $\tanh x \to \pm 1$.

$$w'(z) = \frac{m}{\sinh z} \cosh z = \frac{m(\cosh x \cos y + i \sinh x \sin y)}{\sinh x \cos y + i \cosh x \sin y} \tag{C.147}$$

$$= \frac{m(\cos y + i \tanh x \sin y)}{\tanh x \cos y + i \sin y} \tag{C.148}$$

So that as $x \to \pm\infty$, $w' \to \pm m$ i.e. $u \to \pm m$. The total flux far away from the source is $2\pi m$ which agrees with the flux out of the source.

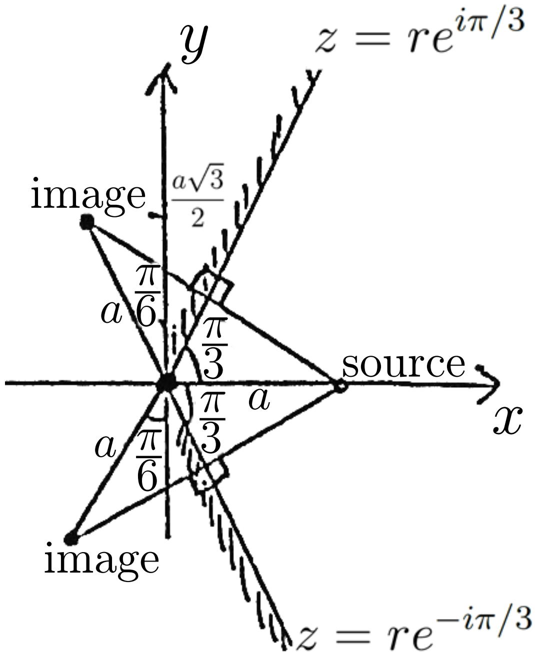

Figure C.20: Solution to Sheet 9, Q1a

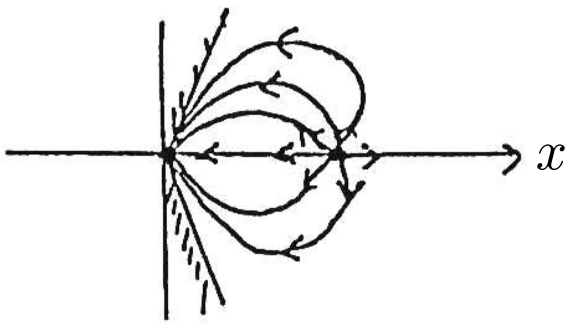

Figure C.21: Solution to Sheet 9, Q1b

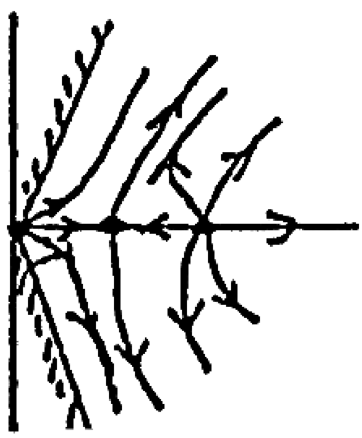

Figure C.22: Solution to Sheet 9, Q1d

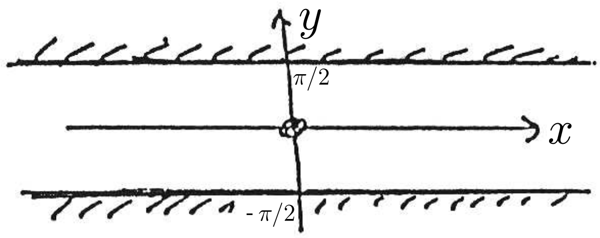

Figure C.23: Solution to Sheet 9, Q2c

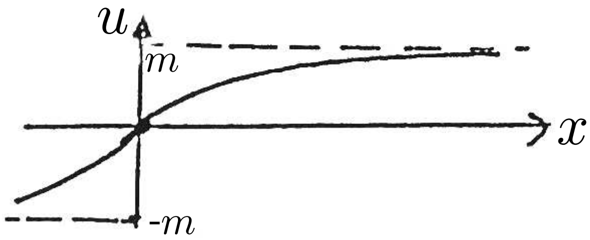

Figure C.24: Solution to Sheet 9, Q2d

C.10 Practice sheet 10

1. **Solution:** The complex potential for a uniform flow of speed U and at an angle α is $f(z) = Uze^{-i\alpha}$. The circle theorem implies that the complex potential for the flow past the cylinder is $w(z) = f(z) + \overline{f}(a^2/z)$. So, here:

$$w(z) = Uze^{-i\alpha} + U\frac{a^2}{z}e^{i\alpha}. \tag{C.149}$$

Adding circulation around the cylinder implies that we need to add a one vortex at the origin, so the complex potential for the whole flow becomes:

$$w = Uze^{-i\alpha} + U\frac{a^2}{z}e^{i\alpha} - i\kappa \ln z. \tag{C.150}$$

If $\zeta = z + a^2/z$, consider on the surface of the cylinder:

$$\zeta(ae^{i\theta}) = ae^{i\theta} + \frac{a^2}{a}e^{-i\theta} \tag{C.151}$$
$$= 2a\cos\theta \tag{C.152}$$

Hence ζ is real and varies between $\pm 2a$. To find the velocity, we us the fact that:

$$u - iv = \frac{dw}{d\zeta} \tag{C.153}$$
$$= \frac{dw}{dz}\left(\frac{d\zeta}{dz}\right)^{-1} \tag{C.154}$$
$$= \left(Ue^{-i\alpha} - \frac{Ua^2}{z^2}e^{i\alpha} - \frac{i\kappa}{z}\right)\left(1 - \frac{a^2}{z^2}\right)^{-1}. \tag{C.155}$$

At $\zeta = 2a$, $z = a$ so that we require the first bracket to be zero there in order to have a finite velocity. Therefore:

$$Ue^{-i\alpha} - Ue^{i\alpha} - \frac{i\kappa}{a} = 0. \tag{C.156}$$

Therefore $\kappa = 2aU \sin\alpha$. See Figures C.25 and C.26.

2. **Solution:**

(a) For a source of strength $2\pi m$, at $z = be^{i\alpha}$, $f(z) = m\ln(z - be^{i\alpha})$. The circle theorem gives us that:

$$\begin{aligned} w(z) &= f(z) + \overline{f}(a^2/z) &\text{(C.157)}\\ &= m\ln(z - be^{i\alpha} + m\ln\left(\frac{a^2}{z} - be^{-\alpha}\right). &\text{(C.158)}\end{aligned}$$

Adding the circulation from a line vortex, we find:

$$w(z) = m\ln(z - be^{i\alpha} + m\ln\left(\frac{a^2}{z} - be^{-\alpha}\right) - i\kappa\ln z. \tag{C.159}$$

(b) See question 1.

(c)

$$\begin{aligned} u - iv &= \frac{dw}{d\zeta} &\text{(C.160)}\\ &= \frac{dw}{dz}\left(\frac{d\zeta}{dz}\right)^{-1} &\text{(C.161)}\\ &= \left(\frac{m}{z - be^{i\alpha}} + \frac{m\left(-\frac{a^2}{z^2}\right)}{\frac{a^2}{z} - be^{-i\alpha}} - \frac{i\kappa}{z}\right)\left(\frac{z^2}{z^2 - a^2}\right) &\text{(C.162)}\end{aligned}$$

At $\zeta = -2a$, $z = -a$ so we require the first bracket to be zero at $z = -a$ in order to have a finite velocity.

$$-\frac{m}{a + be^{i\alpha}} + \frac{m}{a + be^{-i\alpha}} = -\frac{i\kappa}{a} \tag{C.163}$$

Let $g = m/(a + be^{i\alpha})$ so that:

$$\begin{aligned} -g + \overline{g} &= -\frac{i\kappa}{a} &\text{(C.164)}\\ \Longrightarrow \kappa &= 2a\,\mathrm{Im}\,g &\text{(C.165)}\\ &= \mathrm{Im}\left(\frac{m(a + be^{-i\alpha})}{(a + be^{i\alpha})(a + be^{-i\alpha})}\right) &\text{(C.166)}\\ &= \frac{-2amb\sin\alpha}{a^2 + b^2 + 2ab\cos\alpha}. &\text{(C.167)}\end{aligned}$$

The source is at $\xi = (b + a^2/b)\cos\alpha$, $\eta = (b - a^2/b)\sin\alpha$ and the streamlines are radial at the source. See Figure C.27.

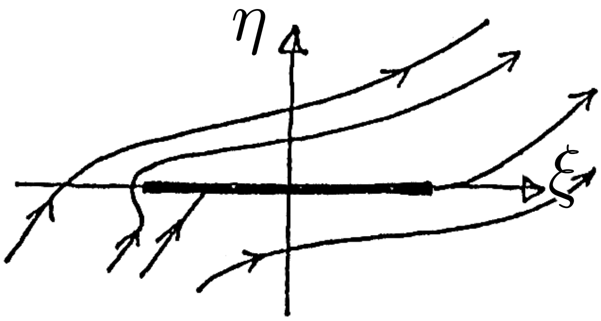

Figure C.25: Solution to Sheet 10, Q1. $\kappa = 2aU \sin\alpha$.

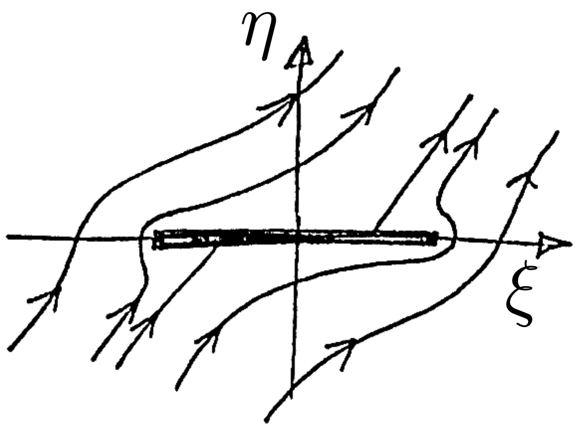

Figure C.26: Solution to Sheet 10, Q1. $\kappa = 0$.

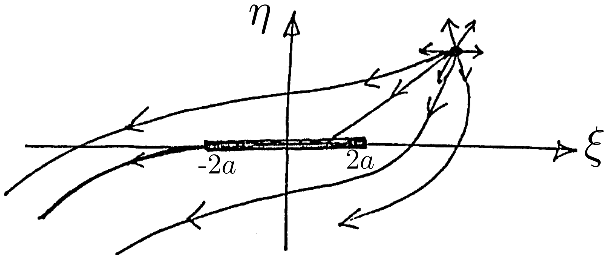

Figure C.27: Solution to Sheet 10, Q2. Streamlines are radial at source.