FOURIER ANALYSIS METHODS FOR MODELS IN FLUID MECHANICS

NONLINEAR EVOLUTION EQUATIONS AND APPLICATIONS
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Abstract. Fourier analysis methods and techniques based on Littlewood-Paley decomposition and paraproduct have known a growing interest in the last two decades for the study of nonlinear evolutionary equations. After a short presentation, we use these methods for proving a priori estimates for different types of linear PDEs. From them, in the case of small initial data, we deduce global well-posedness results in a *critical* functional framework for models of incompressible or compressible models viscous fluids. We end these notes with the study of the low Mach number asymptotics.

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Introduction

This course aims at presenting elementary Fourier analysis methods that proved to be particularly efficient and robust for investigating the Cauchy problem for nonlinear evolutionary PDEs. These techniques are relevant in any context where a good notion of Fourier transform is available. Here, for simplicity, we shall concentrate on the whole space case $\mathbb{R}^d$ (that is the space variable of the PDEs that are considered will be in $\mathbb{R}^d$). However, our approach may be easily adapted to periodic boundary conditions $x \in \mathbb{T}^d$ and more generally to $x \in \mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$ and so on.

We here chose to keep the course at an elementary level so as to give a general and as less technical as possible overview of how those techniques work. The reader may find more sophisticated results in e.g. [5], [17], [20] and in the references therein.

The first part of these notes is devoted to the presentation of the so-called *Littlewood-Paley decomposition* (see the first section) with many examples of applications to the proof of estimates for linear equations (second section). We chose to focus on the following types of linear equations:

- the heat equation,
- linear symmetric systems,
- the transport equation,
- the transport-diffusion equation,
- dispersive equations,

which are frequently encountered when linearizing systems coming from fluid mechanics (or, more generally, from mathematical physics).

Very often, solving a nonlinear PDE reduces to finding an appropriate functional framework in which one may combine a priori estimates for the linearized equation, product estimates for the nonlinear terms and a fixed point theorem (either the contracting mapping one or, if it is not possible, a Schauder-Tikhonoff type argument). In the second part of these notes, we give such examples. More precisely, in the third section, we focus on global well-posedness results for models of incompressible viscous fluids with small data. We first consider the case of homogeneous fluids – the celebrated *incompressible Navier-Stokes equations* – and next slightly nonhomogeneous incompressible fluids (that is the density may have small variations). In both case, we strive for a "critical functional framework".

In the fourth section, we tackle the study of the (barotropic, to simplify the presentation) compressible Navier-Stokes equations. We first establish a local-in-time result (just by noticing that the system is a coupling between a transport and a heat equation), and next a global result for small perturbations of a stable constant state. This latter result requires a fine analysis of the linearized system.

In the last section, we establish the convergence to the incompressible Navier-Stokes equations for general ill-prepared data in the low Mach number regime. There, the dispersive properties of the linearized system *in the whole space* play a fundamental role. In contrast with the results of the previous sections, those properties are specific to the whole space case.

1. The Fourier analysis toolbox

Here we introduce the Littlewood-Paley decomposition, define Besov spaces, state product estimates. More detailed proofs may be found in [5, 17].

1.1. **A primer on Littlewood-Paley theory.** The Littlewood-Paley decomposition is a localization procedure in the frequency space for tempered distributions. One of the main motivations for introducing such a localization when dealing with PDEs is that the derivatives act almost as homotheties on distributions with Fourier transform supported in a ball or an annulus.

In the L^2 framework, this noticeable property easily follows from Parseval's formula. The *Bernstein inequalities* below state that it is also true in the L^p framework:

Proposition 1.1 (Bernstein inequalities). *Let $0 < r < R$.*

- *Direct Bernstein inequality: a constant C exists so that, for any $k \in \mathbb{N}$, any couple (p,q) in $[1,\infty]^2$ with $q \geq p \geq 1$ and any function u of L^p with* $\operatorname{Supp}\widehat{u} \subset B(0,\lambda R)$ *for some $\lambda > 0$, we have*
$$\|D^k u\|_{L^q} \leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p};$$
- *Reverse Bernstein inequality: there exists a constant C so that for any $k \in \mathbb{N}$, $p \in [1,\infty]$ and any function u of L^p with* $\operatorname{Supp}\widehat{u} \subset \{\xi \in \mathbb{R}^d \,/\, r\lambda \leq |\xi| \leq R\lambda\}$ *for some $\lambda > 0$, we have*
$$\lambda^k \|u\|_{L^p} \leq C^{k+1}\|D^k u\|_{L^p}.$$

Proof. Changing variables reduces the proof to the case $\lambda = 1$. For proving the first inequality, we fix some smooth ϕ with compact support, and value 1 over $B(0,R)$. One may thus write
$$\widehat{u} = \phi\,\widehat{u} \quad \text{whence} \quad D^k u = (D^k\mathcal{F}^{-1}\phi) \star u.$$
Therefore using convolution inequalities, one may write
$$\|D^k u\|_{L^q} \leq \|D^k\mathcal{F}^{-1}\phi\|_{L^1}\|u\|_{L^p}$$
with $1 + 1/q = 1/p + 1/r$ (here we need $q \geq p$), and we are done.

For proving the second inequality, we now assume that ϕ is compactly supported away from the origin and has value 1 over the annulus $\mathcal{C}(0,r,R)$. We thus have
$$\widehat{u} = \left(-i\frac{\xi}{|\xi|^2}\phi(\xi)\right)\cdot\widehat{\nabla u(\xi)}.$$
Therefore, denoting by g the inverse Fourier transform of the first term in the r.h.s.,
$$\|u\|_{L^p} \leq \|g\|_{L^1}\|\nabla u\|_{L^p}.$$
This gives the result for $k = 1$. The general case follows by induction. □

As solutions to nonlinear PDE's need not be spectrally localized in annuli (even if we restrict to initial data with this property), it is suitable to have a device which allows for splitting any function into a sum of functions with this spectral localization. This is exactly what Littlewood-Paley's decomposition does.

To construct it, fix some smooth bump function χ with $\operatorname{Supp}\chi \subset B(0,\frac{4}{3})$ and $\chi \equiv 1$ on $B(0,\frac{3}{4})$, then set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ so that
$$\chi(\xi) + \sum_{j\in\mathbb{N}}\varphi(2^{-j}\xi) = 1 \qquad \text{and} \qquad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1 \ \text{ if } \ \xi \neq 0.$$
The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined by
$$\dot{\Delta}_j u := \varphi(2^{-j}D)u := \mathcal{F}^{-1}(\varphi(2^{-j}D)\mathcal{F}u) := 2^{jd}h(2^j\cdot)\star u \quad \text{with} \quad h := \mathcal{F}^{-1}\varphi.$$

We also define the low frequency cut-off operator $\dot S_j$ by

$$\dot S_j u := \chi(2^{-j}D)u := \mathcal{F}^{-1}(\chi(2^{-j}D)\mathcal{F}u) := 2^{jd}\check h(2^j\cdot)\star u \quad\text{with}\quad \check h := \mathcal{F}^{-1}\chi.$$

The nonhomogeneous dyadic blocks Δ_j are defined by

$$\Delta_j := \dot\Delta_j \ \text{ if } \ j\geq 0,\quad \Delta_{-1} := \dot S_0 = \chi(D) \quad\text{ and }\ \Delta_j = 0 \ \text{ if } \ j\leq -2,$$

and we set

$$S_j := \sum_{k\leq j-1}\Delta_k.$$

The homogeneous and nonhomogeneous Littlewood-Paley decomposition for u are

$$u=\sum_j \dot\Delta_j u \quad\text{and}\quad u=\sum_j\Delta_j u. \tag{1}$$

The second equality holds true in the set $\mathcal{S}'$ of tempered distributions.

This is not the case of the first one which holds true *modulo polynomials only* if no further assumptions. A way to overcome this is to restrict to the set $\mathcal{S}'_h$ of tempered distributions u such that

$$\lim_{j\to-\infty}\|\dot S_j u\|_{L^\infty}=0 \quad\text{with}\quad \dot S_j := \chi(2^{-j}D).$$

Note that loosely speaking, this condition on the low frequencies of u amounts to requiring u to tend to 0 at infinity (in the sense of distributions). Then the first equality (1) holds true whenever u is in $\mathcal{S}'_h$.

Owing to $\operatorname{Supp}\varphi\subset C(0,3/4,8/3)$ and $\operatorname{Supp}\chi\subset B(0,4/3)$ we have the following properties of quasi-orthogonality:

- $\dot\Delta_j\dot\Delta_k=0$ if $|j-k|>1$;
- $\dot\Delta_k(\dot S_{j-1}u\,\dot\Delta_j v)\equiv 0$ if $|k-j|>4$.

1.2. **Functional spaces.** Many classical norms may be written in terms of the Littlewood-Paley decomposition. This is e.g. the case of:

- the homogeneous Sobolev norm: $\displaystyle\|u\|_{\dot H^s}^2\approx\sum_j(2^{js}\|\dot\Delta_j u\|_{L^2})^2$;
- the homogeneous Hölder norm: $\displaystyle\|u\|_{\dot C^r}\approx\sup_j 2^{jr}\|\dot\Delta_j u\|_{L^\infty}$.

In effect, owing to $\operatorname{Supp}\varphi(2^{-j}\cdot)\cap\operatorname{Supp}(2^{-k}\cdot)=\emptyset$ if $|j-k|>1$, we have

$$\frac12\leq\sum_j\varphi^2(2^{-j}\xi)\leq 1\quad\text{for}\quad \xi\neq 0.$$

Hence, using the definition of Sobolev norm, of $\dot\Delta_j u$ and Parseval equality,

$$\|u\|_{H^s}^2=\int|\xi|^2|\widehat u(\xi)|^2\,d\xi\approx\sum_j\int|\xi|^2|\varphi(2^{-j})\widehat u(\xi)|^2\,d\xi\approx\sum_j 2^{2js}\|\dot\Delta_j u\|_{L^2}^2.$$

As for the Hölder norm, we notice that because h has average 0,

$$\dot\Delta_j u(x)=2^{jd}\int h(2^j(x-y))\,(u(y)-u(x))\,dy\quad\text{for all}\ \ j\in\mathbb{Z}.$$

Hence for all $x\in\mathbb{R}^d$ and $j\in\mathbb{Z}$,

$$|\dot\Delta_j u(x)|\leq 2^{-jr}\|u\|_{\dot C^{0,r}}\,2^{jd}\int|h(2^j(x-y))|(2^j|x-y|)^r\,dy\leq 2^{-jr}\|u\|_{\dot C^{0,r}}\||\cdot|^r h\|_{L^1}.$$

Conversely, if $C_r(u) := \sup_j 2^{jr}\|\dot\Delta_j u\|_{L^\infty} < \infty$ then we may write for any $N \in \mathbb{Z}$,

$$u(y) - u(x) = \sum_{j<N} (\dot\Delta_j u(y) - \dot\Delta_j u(x)) + \sum_{j\geq N} (\dot\Delta_j u(y) - \dot\Delta_j u(x)).$$

Hence

$$|u(y) - u(x)| \leq |y-x| \sum_{j<N} \|\nabla\dot\Delta_j u\|_{L^\infty} + 2\sum_{j\geq N} \|\dot\Delta_j u\|_{L^\infty}.$$

Therefore, taking advantage of Bernstein inequality for the terms in the first sum,

$$|u(y) - u(x)| \leq C_r(u)\biggl(|y-x| \sum_{j<N} 2^{j(1-r)N} + 2\sum_{j\geq N} 2^{-jr}\biggr).$$

Then taking the "best" N yields $\|u\|_{\dot C^{0,r}} \leq CC_r(u)$.

If looking at those two characterizations, we see that three parameters come into play: the "regularity" parameter s (or r), the Lebesgue exponent that is used for $\dot\Delta_j u$ and the type of summation that this done over $\mathbb{Z}$. This observation motivates the following definition:

Definition 1.1. *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we set*

$$\|u\|_{\dot B^s_{p,r}} := \biggl(\sum_j 2^{rjs}\|\dot\Delta_j u\|^r_{L^p}\biggr)^{\frac{1}{r}} \quad \text{if} \quad r<\infty \quad \text{and} \quad \|u\|_{\dot B^s_{p,\infty}} := \sup_j 2^{js}\|\dot\Delta_j u\|_{L^p}.$$

We then define the homogeneous Besov space $\dot B^s_{p,r}$ as the subset of distributions $u \in \mathcal{S}'_h$ such that $\|u\|_{\dot B^s_{p,r}} < \infty$.

Similarly we set

$$\|u\|_{B^s_{p,r}} := \biggl(\sum_j 2^{rjs}\|\Delta_j u\|^r_{L^p}\biggr)^{\frac{1}{r}} \quad \text{if} \quad r<\infty \quad \text{and} \quad \|u\|_{B^s_{p,\infty}} := \sup_j 2^{js}\|\Delta_j u\|_{L^p}.$$

and define the nonhomogeneous Besov space $B^s_{p,r}$ as the subset of distributions $u \in \mathcal{S}'$ such that $\|u\|_{B^s_{p,r}} < \infty$.

With this definition, we see that $\dot B^s_{2,2}$ coincides with the homogeneous Sobolev space $\dot H^s$ and it is true that $\dot B^r_{\infty,\infty}$ is the homogeneous Hölder space $\dot C^{0,r}$ if $r \in (0,1)$.

More generally, loosely speaking, having u in $\dot B^s_{p,r}$ means that u has s fractional derivatives in L^p.

Here are some important embedding properties:

- For any $p \in [1,\infty]$ we have the following chain of continuous embedding: $\dot B^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot B^0_{p,\infty}$;
- If $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, then $\dot B^s_{p_1,r_1} \hookrightarrow \dot B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$;
- the space $\dot B^{\frac{d}{p}}_{p,1}$ is continuously embedded in L^∞ (and even in the set of continuous functions decaying to 0 at infinity if $p<\infty$).

Proof. The left embedding of the first property follows from the triangle inequality for the L^p norm applied to
$$u=\sum_j \dot\Delta_j u$$
whereas the right inequality is a consequence of the convolution property $L^1\star L^p\to L^p$ which implies that
$$\|\dot\Delta_j u\|_{L^p}\leq \|2^{jd}h(2^j\cdot)\|_{L^1}\|u\|_{L^p}=\|h\|_{L^1}\|u\|_{L^p}.$$
As for the second property, we just have to use that, owing to Bernstein inequality,
$$\|\dot\Delta_j u\|_{L^{p_2}}\leq C2^{j(\frac{d}{p_1}-\frac{d}{p_2})}\|\dot\Delta_j u\|_{L^{p_1}}.$$
Finally, by combining the first two properties, we see that
$$\dot B^{\frac dp}_{p,1}\hookrightarrow B^0_{\infty,1}\hookrightarrow L^\infty.$$
Note in particular that this implies that if $u\in \dot B^{\frac dp}_{p,1}$ then the series $\sum_{j\in\mathbb{Z}}\dot\Delta_j u$ converges *uniformly* to u. In the case where $p<\infty$, each term $\dot\Delta_j u$ is continuous and goes to 0 at infinity, hence so does u. □

Here is a nonexhaustive list of classical (and important properties) of Besov spaces:

- $\dot B^s_{p,r}$ is a Banach space whenever $s<d/p$ or $s\leq d/p$ and $r=1$, and so does $B^s_{p,r}$ without any condition on (s,p,r);
- the following real interpolation property is satisfied for all $1\leq p,r_1,r_2,r\leq\infty$, $s_1\neq s_2$ and $\theta\in(0,1)$:
$$[\dot B^{s_1}_{p,r_1},\dot B^{s_2}_{p,r_2}]_{(\theta,r)}=\dot B^{\theta s_2+(1-\theta)s_1}_{p,r}\quad\text{and}\quad [B^{s_1}_{p,r_1},B^{s_2}_{p,r_2}]_{(\theta,r)}=B^{\theta s_2+(1-\theta)s_1}_{p,r};$$
- for any smooth homogeneous of degree m function F on $\mathbb{R}^d\setminus\{0\}$ the Fourier multiplier $F(D)$ maps $\dot B^s_{p,r}$ in $\dot B^{s-m}_{p,r}$. In particular, the gradient operator maps $\dot B^s_{p,r}$ in $\dot B^{s-1}_{p,r}$.

The following lemma ensures that the definition of Besov spaces is *independent* of the choice of $(\Delta_j)_{j\in\mathbb{Z}}$ or $(\dot\Delta_j)_{j\in\mathbb{Z}}$. It will be also very useful for proving nonlinear estimates (see the next paragraph).

Lemma 1.1. *Let $0<r<R$. Let $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$. Let $(u_j)_{j\geq-1}$ be such that* $\operatorname{Supp}\widehat{u_{-1}}\subset B(0,R)$ *and* $\operatorname{Supp}\widehat u_j\subset 2^j\mathcal{C}(0,r,R)$ *for all $j\in\mathbb{N}$. Then*
$$\Big\|2^{js}\|u_j\|_{L^p(\mathbb{R}^d)}\Big\|_{\ell^r(\mathbb{N}\cup\{-1\})}<\infty\quad\Longrightarrow\quad u:=\sum_{j\geq-1}u_j\ \textit{is in}\ B^s_{p,r}$$
and we have $\|u\|_{B^s_{p,r}}\approx\Big\|2^{js}\|u_j\|_{L^p(\mathbb{R}^d)}\Big\|_{\ell^r(\mathbb{N}\cup\{-1\})}.$

If $s>0$ then the result is still true under the weaker assumption that $\operatorname{Supp}\widehat u_j\subset B(0,2^jR)$.

A similar statement holds in the homogeneous setting.

Proof. In the first case, we notice that one may find some integer N (depending only on r and R) such that for all $k\geq -1$,

$$\Delta_k u=\sum_{|j-k|\leq N}\Delta_k u_j.$$

Therefore

$$\|\Delta_k u\|_{L^p}\leq \sum_{|j-k|\leq N}\|\Delta_k u_j\|_{L^p}\leq C\sum_{|j-k|\leq N}\|u_j\|_{L^p}$$

and we get the result.

If we only have $\operatorname{Supp}\widehat{u}_j\subset B(0,2^jR)$ then we just have for some integer N,

$$\Delta_k u=\sum_{j\geq k-N}\Delta_k u_j.$$

Therefore

$$2^{ks}\|\Delta_k u\|_{L^p}\leq \sum_{j\geq k-N}2^{(k-j)s}\,2^{js}\|u_j\|_{L^p}$$

and the convolution inequality $\ell^1\star\ell^r\to\ell^r$ gives the result if $s>0$. □

1.3. **Nonlinear estimates.** The basic question that we shall address in this subsection is: let u and v belong to two different Besov spaces:

- Does uv make sense ?
- If so, where does uv lie ?

Formally, any product of two distributions u and v may be decomposed into

$$uv=T_uv+R(u,v)+T_vu \tag{2}$$

with

$$T_uv:=\sum_j S_{j-1}u\Delta_jv\quad\text{and}\quad R(u,v):=\sum_j\sum_{|j'-j|\leq 1}\Delta_ju\,\Delta_{j'}v.$$

The above operator T is called "paraproduct" whereas R is called "remainder". The decomposition (2) has been first introduced by J.-M. Bony in [6].

Note that as T_uv involves product of functions with different spectral localizations, it is always defined (in Fourier variables, the sum is locally finite). At the same time, it cannot be smoother than what is given by high frequencies, namely v. As for the remainder, it may be not defined (think of the product of two Dirac masses at the same point). However, if it is defined then it is smoother than the paraproduct term. All this is detailed in the proposition below.

Proposition 1.2. *For any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ and $t<0$ we have*[1]

$$\|T_uv\|_{B^s_{p,r}}\lesssim\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}\quad\textit{and}\quad\|T_uv\|_{B^{s+t}_{p,r}}\lesssim\|u\|_{B^t_{\infty,\infty}}\|v\|_{\dot B^s_{p,r}}.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R}\times[1,\infty]^2$ we have

- *if $s_1+s_2>0$, $1/p:=1/p_1+1/p_2\leq 1$ and $1/r:=1/r_1+1/r_2\leq 1$ then*
$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}}\lesssim\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}};$$
- *if $s_1+s_2=0$, $1/p:=1/p_1+1/p_2\leq 1$ and $1/r_1+1/r_2\geq 1$ then*
$$\|R(u,v)\|_{B^0_{p,\infty}}\lesssim\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.$$

[1]The sign $\lesssim$ means that the l.h.s. is bounded by the r.h.s. up to a harmless multiplicative constant.

Similar results in homogeneous Besov spaces.

Proof. We just prove the first result of continuity for T and R. Both are consequences of Lemma 1.1. We first notice that the general term of $T_u v$ is supported in dyadic annuli whereas that of $R(u,v)$ is only supported in dyadic balls. Now, we see that

$$\|S_{j-1}u\Delta_j u\|_{L^p} \leq \|S_{j-1}u\|_{L^\infty}\|\Delta_j u\|_{L^p} \leq C\|u\|_{L^\infty}\|\Delta_j v\|_{L^p},$$

and thus

$$\|(2^{js}\|S_{j-1}u\Delta_j v\|_{L^p})\|_{\ell^r} \leq C\|u\|_{L^\infty}\|(2^{js}\|\Delta_j v\|_{L^p})\|_{\ell^r}$$

hence Lemma 1.1 gives the result.

For proving the first continuity result for R, we may write that

$$2^{j(s_1+s_2)}\|\Delta_j u\widetilde{\Delta_j}v\|_{L^p} \leq (2^{js_1}\|\Delta_j u\|_{L^{p_1}})\,(2^{js_2}\|\widetilde{\Delta_j}v\|_{L^{p_2}})$$

and use the last part of Lemma 1.1. □

Putting together decomposition (2) and the above results of continuity, one may deduce a number of continuity results for the product of two functions. For instance, one may get the following *tame estimate* which depends linearly on the highest norm of u and v:

Corollary 1.1. *Let u and v be in $L^\infty \cap B^s_{p,r}$ for some $s>0$ and $(p,r)\in[1,\infty]^2$. Then there exists a constant C depending only on d, p and s and such that*

$$\|uv\|_{B^s_{p,r}} \leq C\big(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}\big).$$

Proof. We proceed as follows:

1. Write Bony's decomposition $uv = T_u v + T_v u + R(u,v)$;
2. Use $T: L^\infty \times B^s_{p,r} \to B^s_{p,r}$;
3. Use $R: B^0_{\infty,\infty}\times B^s_{p,r} \to B^s_{p,r}$ if $s>0$;
4. Notice that $L^\infty \hookrightarrow B^0_{\infty,\infty}$.

□

Remark 1.1. *As a consequence of tame estimates and of the fact that $B^{n/p}_{p,1}$ and $\dot{B}^{n/p}_{p,1}$ are embedded in L^∞, we deduce that* both spaces $B^{d/p}_{p,1}$ and $\dot{B}^{d/p}_{p,1}$ are Banach algebra *if $p<\infty$.*

Finally, the following composition result will be needed for handling e.g. the pressure term when studying the compressible Navier-Stokes equations.

Proposition 1.3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a smooth function with $F(0)=0$. Then for all $(p,r)\in[1,\infty]^2$ and all $s>0$, there exists a constant C such that for all $u\in B^s_{p,r}\cap L^\infty$ we have $F(u)\in B^s_{p,r}$ and*

$$\|F(u)\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}$$

with C depending only on $\|u\|_{L^\infty}$, F, s, p and d.

Proof. We use Meyers's first linearization method:

$$F(u) = \sum_j F(S_{j+1}u) - F(S_j u) = \sum_j \underbrace{\Delta_j u \int_0^1 F'(S_j u + \tau\Delta_j u)\,d\tau}_{u_j}.$$

We notice that

$$\|u_j\|_{L^p} \leq C\|\Delta_j u\|_{L^p}.$$

Unfortunately, $\mathcal{F}u_j$ is not localized in a ball of size 2^j. However, after cumbersome computations, we find out that

$$\|D^k u_j\|_{L^p} \leq C2^{jk}\|\Delta_j u\|_{L^p}.$$

Hence everything happens as if the $\mathcal{F}u_j$ were well localized. This suffices to complete the proof. □

2. Linear estimates

2.1. A maximal regularity estimate for the heat equation. Consider the heat equation

$$\partial_t u - \Delta u = f, \qquad u_{|t=0} = u_0 \tag{3}$$

or, more generally[2]

$$\partial_t v + |D|^\sigma v = g, \qquad v_{|t=0} = v_0. \tag{4}$$

We want to establish estimates of the form

$$\|\partial_t u, D^2 u\|_{L^1(X)} \leq C\big(\|u_0\|_X + \|f\|_{L^1(X)}\big) \tag{5}$$

$$\|\partial_t v, |D|^\sigma v\|_{L^1(X)} \leq C\big(\|v_0\|_X + \|g\|_{L^1(X)}\big). \tag{6}$$

In the case of the heat equation, this gain of two derivatives compared to the source term when performing a L^1-in-time integration is the key to a number of well-posedness results in a critical functional framework for models arising in fluid mechanics.

Now, it is well known that if $r \in (1,\infty)$ and $X = L^q$ or $\dot W^{s,q}$ for some $s \in \mathbb{R}$ and $q \in (1,\infty)$ then, for the heat equation,

$$\|\partial_t u, D^2 u\|_{L^r(X)} \leq C\|f\|_{L^r(X)}.$$

This type of inequalities fails for the endpoint case $r = 1$ for those spaces (and more generally in any reflexive Banach space X). However, it has been noticed by J.-Y. Chemin in [9] that Inequality (5) is true for Besov spaces *with third index* 1. This is stated in the following theorem.

Theorem 2.1. *Estimates* (5) *hold true for any* $p \in [1,\infty]$, $\sigma \in \mathbb{R}$ *and* $s \in \mathbb{R}$ *if* $X = \dot B^s_{p,1}$.

Proving the theorem relies on the following:

Lemma 2.1. *There exist two positive constants* c *and* C *such that for any* $j \in \mathbb{Z}$, $p \in [1,\infty]$ *and* $\lambda \in \mathbb{R}^+$, *we have*

$$\|e^{-\lambda|D|^\sigma}\dot\Delta_j\|_{\mathcal{L}(L^p;L^p)} \leq Ce^{-c\lambda 2^{\sigma j}}.$$

Proof. If $p = 2$ this is a mere consequence of Parseval's formula.

In the general case, one may first reduce the proof to the case $j = 0$ (just perform a suitable change of variable) then consider a function ϕ in $\mathcal{D}(\mathbb{R}^d\setminus\{0\})$ with value 1 on a neighborhood of the support of φ so as to write

$$\begin{aligned} e^{\lambda\Delta}\dot\Delta_0 u &= \mathcal{F}^{-1}\left(\phi(\xi)e^{-\lambda|\xi|^\sigma}\widehat{\dot\Delta_0 u}(\xi)\right),\\ &= g_\lambda \star u \quad\text{with}\quad g_\lambda(x) := (2\pi)^{-d}\int e^{i(x|\xi)}\phi(\xi)e^{-\lambda|\xi|^2}d\xi. \end{aligned}$$

[2]With the convention that $\mathcal{F}(|D|^\sigma u)(\xi) := |\xi|^\sigma \mathcal{F}u(\xi)$.

If it is true that

$$\|g_\lambda\|_{L^1} \le Ce^{-c\lambda} \tag{7}$$

then the desired result follows just by using the convolution inequality $L^1 \star L^p \to L^p$.

Proving (7) follows from integration by parts: we have

$$g_\lambda(x) = (1+|x|^2)^{-d} \int_{\mathbb{R}^d} e^{i(x|\xi)} (\mathrm{Id}\, - \Delta_\xi)^d \Big(\phi(\xi) e^{-\lambda|\xi|^\sigma}\Big) d\xi.$$

So using Leibniz and Faá-di-Bruno's formulae, we conclude that

$$|g_\lambda(x))| \le C(1+|x|^2)^{-d} e^{-c\lambda}.$$

Therefore

$$\|e^{\lambda\Delta}\dot\Delta_0 u\|_{L^p} \le \|g_\lambda\|_{L^1}\|\dot\Delta_0 u\|_{L^p} \le Ce^{-c\lambda}\|\dot\Delta_0 u\|_{L^p},$$

and we are done. □

Proof of Theorem (2.1) To simplify the presentation, we focus on the case $\sigma = 2$ (heat equation). If u satisfies (3) then for any $j \in \mathbb{Z}$,

$$\partial_t \dot\Delta_j u - \Delta\dot\Delta_j u = \dot\Delta_j f.$$

Hence, according to Duhamel's formula

$$\dot\Delta_j u(t) = e^{t\Delta}\dot\Delta_j u_0 + \int_0^t e^{(t-\tau)\Delta}\dot\Delta_j f(\tau)\, d\tau.$$

According to the lemma, we thus have

$$\|\dot\Delta_j u(t)\|_{L^p} \lesssim e^{-c2^{2j}t}\|\dot\Delta_j u_0\|_{L^p} + \int_0^t e^{-c2^{2j}(t-\tau)}\|\dot\Delta_j f(\tau)\|_{L^p}\, d\tau. \tag{8}$$

Multiplying by 2^{js} and summing up over j yields

$$\sum_j 2^{js}\|\dot\Delta_j u(t)\|_{L^p} \lesssim \sum_j e^{-c2^{2j}t}2^{js}\|\dot\Delta_j u_0\|_{L^p} + \int_0^t e^{-c2^{2j}(t-\tau)}\sum_j \|\dot\Delta_j f(\tau)\|_{L^p}\, d\tau$$

whence

$$\|u\|_{L^\infty_t(\dot B^s_{p,1})} \le \|u\|_{\widetilde L^\infty_t(\dot B^s_{p,1})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|f\|_{L^1_t(\dot B^s_{p,1})}.$$

Note that combining (8) with convolution inequalities also yields

$$2^{2j}\|\dot\Delta_j u\|_{L^1_t(L^p)} \lesssim \Big(1-e^{-c2^{2j}t}\Big)\Big(\|\dot\Delta_j u_0\|_{L^p} + \|\dot\Delta_j f\|_{L^1_t(L^p)}\Big).$$

Now, multiplying by 2^{js} and summing over j yields

$$\|u\|_{L^1_t(\dot B^{s+2}_{p,1})} \lesssim \sum_j \Big(1-e^{-c2^{2j}t}\Big)2^{js}\Big(\|\dot\Delta_j u_0\|_{L^p} + \|\dot\Delta_j f\|_{L^1_t(L^p)}\Big), \tag{9}$$

which is slightly better than what we wanted to prove.[3] □

Remark 2.1. *Other maximal regularity estimates may be proved by the same token. For instance these ones:*

$$\|u\|_{\widetilde L^{\rho_1}_t(\dot B^{s+\frac{2}{\rho_1}}_{p,r})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|f\|_{\widetilde L^{\rho_2}_t(\dot B^{s-2+\frac{2}{\rho_2}}_{p,r})} \quad \text{for } \ 1\le\rho_2\le\rho_1\le\infty$$

with $\|v\|_{\widetilde L^a_t(\dot B^\sigma_{b,c})} := \Big\|2^{j\sigma}\|v\|_{L^a_t(L^b)}\Big\|_{\ell^c}.$

[3]As obviously $(1-e^{-c2^{2j}t})$ is bounded by 1 and tends to 0 when t goes to 0^+.

2.2. **Hyperbolic symmetric systems.** Hyperbolic symmetric systems arise naturally as baby models in a number of PDEs coming from physics or fluid mechanics. One may cite for instance the transport equation (see next paragraph) and the acoustic wave equation (which occurs when linearizing equations for a compressible fluid) or the wave equation. In this section, we focus on *linear hyperbolic symmetric systems* of the form:

$$\partial_t U + \sum_k \mathcal{A}^k \partial_k U + \mathcal{A}^0 U = F, \qquad U_{|t=0} = U_0 \tag{10}$$

with $F, U : [0,T] \to \mathbb{R}^N$ and $\mathcal{A}^k : [0,T] \times \mathbb{R}^d \to \mathcal{M}_N(\mathbb{R})$ for $0 \leq k \leq d$ reasonably smooth. The *symmetry assumption* means that matrices $\mathcal{A}^k$ for $k \in \{1, \cdots, d\}$ are symmetric.

Let us first derive the *basic energy inequality* associated to this system: taking the inner product in $L^2(\mathbb{R}^d; \mathbb{R}^N)$ of (10) with U yields

$$\frac{1}{2}\frac{d}{t}\|U\|_{L^2}^2 + \sum_{i,j,k} \int_{\mathbb{R}^d} U^i \mathcal{A}^k_{i,j} \partial_k U^j \, dx + (\mathcal{A}^0 \mid U) = (F \mid U).$$

Now, setting $(\operatorname{div} A)_{i,j} := \sum_k \partial_k \mathcal{A}^k_{i,j}$ and integrating by parts, we see that

$$\sum_{i,j,k} \int_{\mathbb{R}^d} U^i \mathcal{A}^k_{i,j} U^j \, dx = -\sum_{i,j,k} \int_{\mathbb{R}^d} \partial_k U^i \mathcal{A}^k_{i,j} U^j \, dx - \sum_{i,j} \int (\operatorname{div} \mathcal{A})_{i,j} U^i U^j \, dx$$

Owing to $\mathcal{A}^k_{i,j} = \mathcal{A}^k_{j,i}$ the first term of the r.h.s. (with sign $+$) is equal to the l.h.s. Therefore,

$$\frac{1}{2}\frac{d}{t}\|U\|_{L^2}^2 - \frac{1}{2}(\operatorname{div} \mathcal{A} U \mid U) + (\mathcal{A}^0 \mid U) = (F \mid U).$$

Integrating and using Gronwall's lemma thus implies that

$$\|U(t)\|_{L^2}^2 \leq \left(\|U_0\|_{L^2}^2 + \int_0^t e^{-\int_0^\tau a(\tau')\, d\tau'} \|F\|_{L^2}^2 \, d\tau \right) e^{\int_0^t a(\tau')\, d\tau'}$$

with $a(t) := \|\operatorname{div} \mathcal{A}\|_{L^\infty} + 2\|\mathcal{A}^0\|_{L^\infty}$.

This allows to prove the existence and uniqueness in $\mathcal{C}([0,T]; L^2)$ if the matrices $\mathcal{A}^k$ are smooth enough (see e.g. Chap. 4 of [5]).

We now want to prove *regularity estimates* for linear symmetric systems. The strategy is simple: we apply Δ_j to (10) then use energy estimates for bounding each term $\Delta_j U$ in L^2 then multiply by 2^{js} and perform a ℓ^2 summation to get a H^s norm. In passing, let us emphasize that performing instead a ℓ^r summation would yield a $B^s_{2,r}$ estimate.

Let us give more details in the case $\mathcal{A}^0 \equiv 0$ (to simplify the presentation). The main difficulty is that $\dot{\Delta}_j$ need not commute with the matrices $\mathcal{A}^k$ so we only get

$$\partial_t \Delta_j U + \sum_k \mathcal{A}^k \partial_k \Delta_j U = \Delta_j F + \sum_k \underbrace{[\mathcal{A}^k, \Delta_j] \partial_k U}_{R_j^k}.$$

The above energy inequality implies that

$$\|\Delta_j U(t)\|_{L^2} \leq \left(\|\Delta_j U_0\|_{L^2} + \int_0^t e^{-\frac{1}{2}\int_0^\tau a(\tau')\, d\tau'} \big(\|\Delta_j F\|_{L^2} + \|R_j\|_{L^2} \big) \, d\tau \right) e^{\frac{1}{2}\int_0^t a(\tau')}\, d\tau'.$$

Once R_j has been suitably bounded, in order to get H^s estimates, it suffices to multiply the inequality by 2^{js} and

We claim that there exists $(c_j)_{j\geq -1}$ in the unit sphere of ℓ^r such that

$$2^{js}\|R_j^k\|_{L^2}\leq Cc_j\|\nabla\mathcal{A}\|_{L^\infty\cap B^{\frac d2}_{2,\infty}}\|U\|_{B^s_{2,r}}\quad\text{if}\quad 0<s<d/2+1.$$

Proof. It is based on Bony's decomposition:

$$R_j^k=[T_{\mathcal{A}^k},\Delta_j]\partial_kU+T_{\partial_k\Delta_jU}\mathcal{A}^k-\Delta_jT_{\partial_kU}\mathcal{A}^k+R(\mathcal{A}^k,\partial_k\Delta_jU)-\Delta_jR(\mathcal{A}^k,\partial_kU).$$

Let us just explain how to bound the first term which is the only one where having a commutator improves the estimates (bounding the other terms stems mostly from Proposition 1.2). Using quasi-orthogonality and definition by convolution of dyadic blocks yields

$$[T_{\mathcal{A}^k},\Delta_j]\partial_kU(x)=\sum_{|j'-j|\leq 4}2^{jd}\int_{\mathbb{R}^d}h(2^j(x-y))\big(S_{j'-1}\mathcal{A}^k(x)-S_{j'-1}\mathcal{A}^k(y)\big)\partial_k\Delta_{j'}U(y)\,dy.$$

Hence, according to the mean value formula,

$$\begin{aligned}&[T_{A^k},\Delta_j]\partial_kU(x)\\&\quad=\sum_{|j'-j|\leq 4}2^{jd}\int_{\mathbb{R}^d}\int_0^1 h(2^j(x-y))\big((x-y)\cdot\nabla S_{j'-1}\mathcal{A}^k(y+\tau(x-y))\big)\partial_k\Delta_{j'}U(y)\,d\tau\,dy.\end{aligned}$$

So finally,

$$\|[T_{\mathcal{A}^k},\Delta_j]\partial_kU\|_{L^2}\lesssim 2^{-j}\,\|\nabla\mathcal{A}\|_{L^\infty}\sum_{|j'-j|\leq 4}\big\|\partial_k\Delta_{j'}U\big\|_{L^2}\lesssim\|\nabla\mathcal{A}\|_{L^\infty}\sum_{|j'-j|\leq 4}\big\|\Delta_{j'}U\big\|_{L^2}.$$

□

Finally inserting the commutator estimate in the inequality for Δ_jU, multiplying by 2^{js}, performing a ℓ^r summation and using Gronwall lemma eventually yields

$$\|U(t)\|_{B^s_{2,r}}\leq\left(\|U_0\|_{B^s_{2,r}}+\int_0^t e^{-\int_0^\tau a_{s,r}(\tau')\,d\tau'}\|F\|_{B^s_{2,r}}\,d\tau\right)e^{\int_0^t a_{s,r}(\tau')\,d\tau'}$$

for any $0<s<d/2+1$ (more positive s or negative s may be achieved as well if using stronger norms of $\mathcal{A}$).

2.3. **A priori estimates for transport equations.** In general, proving estimates for symmetric hyperbolic systems in spaces $B^s_{p,r}$ with $p\neq 2$, is hopeless. Indeed taking advantage of the antisymmetric character of the first order term requires some Hilbert structure.

A noticeable exception is the following type of transport equation which plays a fundamental role in fluid mechanics:

$$(T)\qquad\left\{\begin{array}{l}\partial_ta+v\cdot\nabla a=f\\ a_{|t=0}=a_0.\end{array}\right.$$

Roughly, if v is a Lipschitz time-dependent vector-field and if $a_0\in X$ and $f\in L^1(0,T;X)$, with X a Banach space then we expect (T) to have a unique solution $a\in\mathcal{C}([0,T);X)$ satisfying

$$\|a(t)\|_X\leq e^{CV(t)}\left(\|a_0\|_X+\int_0^t e^{-CV(\tau)}\|f(\tau)\|_X\,d\tau\right)$$

$$\text{with (say)}\quad V(t):=\int_0^t\|\nabla v(\tau)\|_{L^\infty}\,d\tau.\tag{11}$$

This is quite obvious if X is the Hölder space $C^{0,\varepsilon}$ (with $\varepsilon\in(0,1)$) as (in the case $f\equiv 0$ to simplify) the solution to (T) is given by

$$a(t,x)=a_0(\psi_t^{-1}(x))$$

where ψ_t stands for the flow of v at time t.

Therefore,

$$\begin{aligned}|a(t,x)-a(t,y)| &= |a_0(\psi_t^{-1}(x))-a_0(\psi_t^{-1}(y))|,\\ &\le \|a_0\|_{\dot C^{0,\varepsilon}}|\psi_t^{-1}(x)-\psi_t^{-1}(y)|^\varepsilon,\\ &\le \|a_0\|_{\dot C^{0,\varepsilon}}\|\nabla\psi_t^{-1}\|_{L^\infty}^\varepsilon|x-y|^\varepsilon.\end{aligned}$$

As $\|\nabla\psi_t^{-1}\|_{L^\infty}\le\exp(V(t))$, we get the result in this particular case.

Littlewood-Paley's decomposition will enable us to prove a similar result in a much more general framework.

Theorem 2.2. *The above inequality* (11) *holds true for* $X=\dot B^s_{p,r}$ *with*

$$V(t)=\int_0^t\|\nabla v(\tau)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\,d\tau$$

whenever

$$1\le p\le p_1\le\infty,\quad 1\le r\le\infty,\quad -\min\Bigl(\frac{d}{p_1},\frac{d}{p'}\Bigr)<s<1+\frac{d}{p_1}\cdot$$

If $r=1$ *(resp.* $r=\infty$*) then the case* $s=1+d/p_1$ *(resp.* $s=-\min\bigl(\frac{d}{p_1},\frac{d}{p'}\bigr)$*) also works.*

Proof. Applying $\dot\Delta_j$ to (T) gives

$$\partial_t\dot\Delta_j a+v\cdot\nabla\dot\Delta_j a=\dot\Delta_j f+\dot R_j\quad\text{with}\quad \dot R_j:=[v\cdot\nabla,\dot\Delta_j]a. \tag{12}$$

In the case $p\in(1,\infty)$, multiplying both sides by $|\dot\Delta_j a|^{p-2}\dot\Delta_j a$ and integrating over $\mathbb{R}^d$ yields

$$\frac1p\frac{d}{dt}\|\dot\Delta_j a\|_{L^p}^p+\frac1p\int v\cdot\nabla|\dot\Delta_j a|^p\,dx=\int\bigl(\dot\Delta_j f+R_j\bigr)|\dot\Delta_j a|^{p-2}\dot\Delta_j a\,dx. \tag{13}$$

Therefore

$$\|\dot\Delta_j a(t)\|_{L^p}\le\|\dot\Delta_j a_0\|_{L^p}+\int_0^t\Bigl(\|\dot\Delta_j f\|_{L^p}+\|\dot R_j\|_{L^p}+\frac{\|\operatorname{div}v\|_{L^\infty}}{p}\|\dot\Delta_j a\|_{L^p}\Bigr)\,d\tau. \tag{14}$$

Having p tend to 1 or ∞ implies that (14) also holds if $p=1$ or $p=\infty$.

Now, under the above conditions over s,p, the remainder term $\dot R_j$ satisfies

$$\|\dot R_j(t)\|_{L^p}\le Cc_j(t)2^{-js}\|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\|a(t)\|_{\dot B^s_{p,r}}\quad\text{with}\quad\|(c_j(t))\|_{\ell^r}=1. \tag{15}$$

The proof of this inequality is very similar to the corresponding one for general hyperbolic symmetric systems (this is only a matter of changing L^2 into L^p), and is thus omitted.

At the end, using (14) and (15), multiplying by 2^{js} then summing up over j yields

$$\|a\|_{L^\infty_t(\dot B^s_{p,r})}\le\|a_0\|_{\dot B^s_{p,r}}+\int_0^t\|f\|_{\dot B^s_{p,r}}\,d\tau+C\int_0^t V'\|a\|_{\dot B^s_{p,r}}\,d\tau.$$

Then applying Gronwall's lemma yields the desired inequality for a. □

One may wonder if, for certain choices of norms, one may avoid the exponential term (due to our using Gronwall lemma) in the estimates for the solution to the transport equation. The answer is yes if $X = L^p$ and, in addition $\operatorname{div} v = 0$. Note indeed that starting from the transport equation (T) *without any spectral localization* and arguing exactly as for proving (13) yields

$$\|a(t)\|_{L^p} \leq \|a_0\|_{L^p} + \int_0^t \|f\|_{L^p}\,d\tau.$$

This is in fact due to the fact that if $\operatorname{div} v = 0$ then the flow associated to v is *measure preserving.*

Does this still work in the spaces $\dot{B}^0_{p,r}$ which are very close to L^p ?

This question has been first answered by M. Vishik. Here we shall use the *dynamic interpolation method* of T. Hmidi and S. Keraani [31]. The key idea is that the linearity of the transport equation implies that $a = \sum_j a_j$ with

$$\partial_t a_j + v \cdot \nabla a_j = 0, \qquad a_{j|t=0} = \dot{\Delta}_j a_0.$$

Even though a_j is spectrally localized at time $t = 0$, there is no reason why this should be still true for $t \neq 0$. Therefore (we focus on estimates in $\dot{B}^0_{p,1}$ for simplicity, $\dot{B}^0_{p,r}$ works the same) one may just write that

$$\|a(t)\|_{\dot{B}^0_{p,1}} \leq \sum_j \|a_j(t)\|_{\dot{B}^0_{p,1}} \leq \sum_{j,k} \|\dot{\Delta}_k a_j(t)\|_{L^p}.$$

Fix some integer N. Then we may write

$$\|a\|_{\dot{B}^0_{p,1}} \leq \sum_{|j-k|\leq N} \|\dot{\Delta}_k a_j\|_{L^p} + \sum_{|j-k|>N} \|\dot{\Delta}_k a_j\|_{L^p}.$$

The first sum may be bounded by $CN\|a_j\|_{L^p}$ and we know that $\|a_j(t)\|_{L^p} = \|\dot{\Delta}_j a_0\|_{L^p}$. For the second sum, we use the fact that

$$\|a_j(t)\|_{\dot{B}^{\pm\frac12}_{p,1}} \leq \|\dot{\Delta}_j a_0\|_{\dot{B}^{\pm\frac12}_{p,1}} e^{C\int_0^t \|\nabla v\|_{L^\infty}\,d\tau}.$$

Coming back to the definition of $\dot{B}^0_{p,1}$, this yields

$$\|\dot{\Delta}_k a_j(t)\|_{L^p} \leq c_k 2^{\pm(\frac{j-k}{2})} \|\dot{\Delta}_j a_0\|_{L^p} e^{C\int_0^t \|\nabla v\|_{L^\infty}\,d\tau} \quad \text{with} \quad \sum c_k = 1.$$

Therefore we finally have

$$\|a(t)\|_{\dot{B}^0_{p,1}} \leq C\|a_0\|_{\dot{B}^0_{p,1}} \biggl(N + 2^{-N/2} e^{C\int_0^t \|\nabla v\|_{L^\infty}\,d\tau}\biggr).$$

Taking N so that $C2^{-N/2}e^{C\int_0^t \|\nabla v\|_{L^\infty}\,d\tau} \approx 1$ implies

$$\|a(t)\|_{\dot{B}^0_{p,1}} \leq \|a_0\|_{\dot{B}^0_{p,1}} \biggl(1 + C\int_0^t \|\nabla v\|_{L^\infty}\,d\tau\biggr). \tag{16}$$

Inequality (16) is of particular interest in the study of the lifespan of solutions to critical nonlinear PDEs. It has been first used by Vishik in [36] for proving the global well-posedness of the incompressible Euler equation in critical Besov spaces.

2.4. **A priori estimates for a convection-diffusion equation.** In fluid mechanics, it is frequent that both convection and diffusion have to be considered together. Typically, we have to consider the following type of convection-diffusion equations:

$$\partial_t u + v\cdot\nabla u - \nu\Delta u = f. \tag{TD_ν}$$

The following theorem (see [18, 30] asserts that one may get a family of a priori estimates in Besov spaces which are the optimal ones in the limit cases $v\equiv 0$ or $\nu=0$.

Theorem 2.3. *Let $1\leq p\leq p_1\leq\infty$ and $1\leq r\leq\infty$. Let $s\in\mathbb{R}$ satisfy*

$$-\min\Big(\frac{d}{p_1},\frac{d}{p'}\Big)<s<1+\frac{d}{p_1}\cdot$$

Then for any smooth solution of (TD_ν) with $\nu\geq 0$, we have

$$\|u\|_{\widetilde L^\infty_T(\dot B^s_{p,r})}+\nu\|u\|_{\widetilde L^1_T(\dot B^{s+2}_{p,r})}\leq Ce^{CV(T)}\bigg(\|u_0\|_{\dot B^s_{p,r}}+\|f\|_{\widetilde L^1_T(\dot B^s_{p,r})}\bigg)$$

with $V(T):=\displaystyle\int_0^T\|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\,dt.$

Proof. Let us first localize the equation about the frequency 2^j. We see that $u_j:=\dot\Delta_j u$ satisfies

$$\partial_t u_j + v_j\cdot\nabla u_j-\nu\Delta u_j = f_j+\dot R_j$$

with $v_j:=\dot S_{j-1}v,$ $f_j:=\dot\Delta_j f$ and $\dot R_j=(\dot S_{j-1}v-v)\cdot\nabla u_j+[v\cdot\nabla,\dot\Delta_j]u.$

A slight variation over the proof of (15) yields

$$\|\dot R_j\|_{L^p}\lesssim c_j 2^{-js}\|\nabla v\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\|u\|_{\dot B^s_{p,r}}\quad\text{with}\quad \|(c_j)\|_{\ell^r}=1.$$

Next, perform the following Lagrangian change of coordinates (with ψ_j being the flow of v_j):

$$\widetilde u_j:=u_j\circ\psi_j,\quad \widetilde f_j:=f_j\circ\psi_j,\quad \widetilde R_j:=\dot R_j\circ\psi_j.$$

We get

$$\partial_t\widetilde u_j-\nu\Delta\widetilde u_j=\widetilde f_j+\widetilde R_j+\nu T_j\quad\text{with}\quad T_j:=(\Delta u_j)\circ\psi_j-\Delta\widetilde u_j.$$

From the chain rule and Hölder inequality, we infer that

$$\|T_j\|_{L^p}\lesssim\big(1+\|\nabla\psi_j\|_{L^\infty}\big)\|\mathrm{Id}-\nabla\psi_j\|_{L^\infty}\|D^2u_j\circ\psi_j\|_{L^p}+\|\Delta\psi_j\|_{L^\infty}\|\nabla u_j\circ\psi_j\|_{L^p}.$$

The r.h.s. may be bounded according to the following classical flow estimates:

$$\begin{aligned}
\|\nabla\psi_j(t)\|_{L^\infty} &\leq \exp\Big(\int_0^t\|\nabla v_j\|_{L^\infty}\,d\tau\Big),\\
\|\mathrm{Id}-\nabla\psi_j(t)\|_{L^\infty} &\leq \exp\Big(\int_0^t\|\nabla v_j\|_{L^\infty}\,d\tau\Big)-1,\\
\|\nabla^2\psi_j(t)\|_{L^\infty} &\leq \exp\Big(2\int_0^t\|\nabla v_j\|_{L^\infty}\,d\tau\Big)\int_0^t\|\nabla^2 v_j\|_{L^\infty}\,d\tau.
\end{aligned}$$

Note that, according to Bernstein inequality,

$$\|\nabla^k v_j\|_{L^\infty}\lesssim 2^{j(k-1)}\|\nabla v\|_{L^\infty}\quad\text{for all}\quad k\geq 1.$$

Hence

$$\|T_j(t)\|_{L^p}\lesssim 2^{2j}\Big(e^{CV(t)-1}\Big)\|u_j(t)\|_{L^p}. \tag{17}$$

If $\widetilde{u}_j$ were spectrally localized in an annulus of size 2^j then the regularity estimates for the heat equation would enable us to gain the factor 2^{2j} and we would be done for t small as the term $(e^{CV(t)}-1)$ goes to 0 when t tends to 0.

As the Lagrangian change of variable destroys the spectral localization, the next idea is to localize again the equation for $\widetilde{u}_j$, namely

$$\partial_t\widetilde{u}_j-\nu\Delta\widetilde{u}_j=\widetilde{f}_j+\widetilde{R}_j+\nu T_j.$$

We may write

$$\partial_t\dot{\Delta}_{j'}\widetilde{u}_j-\nu\Delta\dot{\Delta}_{j'}\widetilde{u}_j=\dot{\Delta}_{j'}\widetilde{f}_j+\dot{\Delta}_{j'}\widetilde{R}_j+\nu\dot{\Delta}_{j'}T_j\quad\text{for } j'\in\mathbb{Z}$$

and use the smoothing properties of the heat equation for bounding each block, then sum over j' to bound $\widetilde{u}_j$.

If we simply use that

$$\|\dot{\Delta}_{j'}\widetilde{f}_j\|_{L^p}\lesssim\|\widetilde{f}_j\|_{L^p}$$

then, after summation, the contribution given by the terms $\dot{\Delta}_{j'}\widetilde{f}_j$ is infinite.

To overcome this, one may, in the light of Bernstein inequalities, write that

$$\begin{aligned}\|\dot{\Delta}_{j'}\widetilde{f}_j\|_{L^p}\lesssim 2^{-j'}\|\nabla\dot{\Delta}_{j'}\widetilde{f}_j\|_{L^p} &= 2^{-j'}\|\dot{\Delta}_{j'}((\nabla f_j\circ\psi_j)\cdot\nabla\psi_j)\|_{L^p},\\ &\lesssim 2^{-j'}\|\nabla f_j\circ\psi_j\|_{L^p}\|\nabla\psi_j\|_{L^\infty},\\ &\lesssim e^{CV}2^{j-j'}\|f_j\|_{L^p}.\end{aligned}$$

One may proceed in the same way for $\dot{\Delta}_{j'}\widetilde{R}_j$ and $\nu\dot{\Delta}_{j'}T_j$. Therefore, using the smoothing properties of the heat equation we get the following inequality for all $(j,j')\in\mathbb{Z}^2$:

$$\begin{aligned}\|\dot{\Delta}_{j'}\widetilde{u}_j\|_{L^\infty_t(L^p)}+\nu 2^{2j'}\|\dot{\Delta}_{j'}\widetilde{u}_j\|_{L^1_t(L^p)}\lesssim\|\dot{\Delta}_{j'}\dot{\Delta}_j u_0\|_{L^p}+2^{j-j'}e^{CV(t)}\|f_j\|_{L^1_t(L^p)}\\ +2^{2(j-j')}\nu 2^{2j'}\big(e^{CV(t)}-1\big)\|u_j\|_{L^1_t(L^p)}+2^{j-j'}\int_0^t c_j2^{-js}V'e^{CV}\|u\|_{\dot{B}^s_{p,r}}\,d\tau.\end{aligned}$$

This inequality is suitable if $j'\geq j-N_0$ (where N_0 fixed integer). To handle the low frequencies, one may merely bound $\dot{S}_{j-N_0}\widetilde{u}_j$ according to the following lemma (see the proof in [5, 36]):

Lemma 2.2. *For any $p\in[1,\infty]$, $N_0\in\mathbb{N}$ and $j\in\mathbb{Z}$, we have*

$$\|\dot{S}_{j-N_0}(\dot{\Delta}_j v\circ\phi)\|_{L^p}\lesssim\|J_{\phi^{-1}}\|_{L^\infty}\|\dot{\Delta}_j v\|_{L^p}\big(2^{-j}\|\nabla J_{\phi^{-1}}\|_{L^\infty}\|J_\phi\|_{L^\infty}+2^{-N_0}\|\nabla\phi\|_{L^\infty}\big).$$

Here we thus get

$$\|\dot{S}_{j-N_0}\widetilde{u}_j\|_{L^q_t(L^p)}\lesssim e^{CV(t)}\big(2^{-N_0}+e^{CV(t)}-1\big)\|u_j\|_{L^q_t(L^p)}\quad\text{for all }\ 1\leq q\leq\infty.$$

In order to bound u_j we split it into (N_0 is any fixed integer)

$$u_j=\dot{S}_{j-N_0}\widetilde{u}_j\circ\psi_j^{-1}+\sum_{j'\geq j-N_0}\dot{\Delta}_{j'}\widetilde{u}_j\circ\psi_j^{-1}.$$

Then putting together the previous computations yields

$$2^{js}\|u_j\|_{L_t^\infty(L^p)}+\nu 2^{2j}2^{js}\|u_j\|_{L_t^1(L^p)}\lesssim e^{CV(t)}\biggl(2^{js}\|\dot\Delta_j u_0\|_{L^p}+2^{3N_0}2^{js}\|f_j\|_{L_t^1(L^p)}$$
$$+\bigl(2^{-N_0}+2^{2N_0}\bigl(e^{CV(t)}-1\bigr)\bigr)\bigl(2^{js}\|u_j\|_{L_t^\infty(L^p)}+\nu 2^{2j}2^{js}\|u_j\|_{L_t^1(L^p)}\bigr)\biggr)$$
$$+2^{3N_0}\int_0^t c_jV'e^{CV}\|u\|_{\dot B^s_{p,r}}\,d\tau.$$

In order to conclude, it is only a matter of choosing N_0 large enough (say such that $16C2^{-N_0}\in[1,2)$) then t so small as the second line to be absorbed by the l.h.s. After performing a ℓ^r summation and using Gronwall lemma, we end up with

$$\|u\|_{\widetilde L_t^\infty(\dot B^s_{p,r})}+\nu\|u\|_{\widetilde L_t^1(\dot B^{s+2}_{p,r})}\le C_0\bigl(\|u_0\|_{\dot B^s_{p,r}}+\|f\|_{\widetilde L_t^1(\dot B^s_{p,r})}\bigr)$$

whenever $t\in[0,T_1]$ with T_1 s. t. $\displaystyle\int_0^{T_1}V'\,dt\approx\varepsilon$ with ε small enough.

Then one may split $[0,T]$ into

$$[0,T]=[0,T_1]\cup\cdots\cup[T_{k-1},T]\quad\text{with}\quad\int_{T_{j-1}}^{T_j}V'\,dt\approx\varepsilon$$

and repeat the argument on every subinterval. As $k\varepsilon\approx V$, this completes the proof. □

2.5. **Dispersive equations and Strichartz estimates.** Let $(U(t))_{t\in\mathbb{R}}$ be a group of unitary operators on $L^2(\mathbb{R}^d)$ satisfying the *dispersion inequality*:

$$\|U(t)f\|_{L^\infty}\le\frac{C}{|t|^\sigma}\|f\|_{L^1}\quad\text{for some}\ \ \sigma>0.$$

Interpolating between $L^2\mapsto L^2$ and $L^1\mapsto L^\infty$, we deduce that

$$\|U(t)f\|_{L^r}\le\left(\frac{C}{|t|^\sigma}\right)^{\frac{1}{r'}-\frac1r}\|f\|_{L^{r'}}\quad\text{for all}\ \ 2\le r\le\infty.$$

The basic examples are the groups generated by the Schrödinger equation

$$i\partial_t u+\Delta u=0\quad\text{in}\quad\mathbb{R}^d$$

for which $\sigma=d/2$, or the acoustic wave equations

$$\left\{\begin{array}{l}\partial_t a+\operatorname{div}u=0\\ \partial_t u+\nabla a=0\end{array}\right.\qquad\text{in}\ \ \mathbb{R}^d,$$

for which $\sigma=d/2-1$. The same holds for the classical wave equation $\partial^2_{tt}u-\Delta u=0$ if written as a system.

Definition 2.1. *A couple $(q,r)\in[2,\infty]^2$ is admissible if $1/q+\sigma/r=\sigma/2$ and $(q,r,\sigma)\neq(2,\infty,1)$.*

Theorem 2.4 (Strichartz estimates)**.** *Let $(U(t))_{t\in\mathbb{R}}$ satisfy the above hypotheses. Then*

(1) *For any admissible couple (q,r) we have $\|U(t)u_0\|_{L^q(L^r)}\le C\|u_0\|_{L^2}$;*
(2) *For any admissible couples (q_1,r_1) and (q_2,r_2) we have*

$$\Bigl\|\int_0^t U(t-\tau)f(\tau)\,d\tau\Bigr\|_{L^{q_1}(L^{r_1})}\lesssim\|f\|_{L^{q_2'}(L^{r_2'})}.$$

Remarks: 1. Compared to Sobolev embedding $H^{d(\frac12-\frac1r)}\hookrightarrow L^r$, Strichartz estimates provides a gain of $d(\frac12-\frac1r)=\frac{d}{q\sigma}$ derivative.

2. The relation $1/q+\sigma/r=\sigma/2$ may be guessed from a dimensional analysis.

Following the approach of Ginibre and Velo in [26], the proof of Strichartz estimates relies mainly on two ingredients:

(1) The $TT^\star$ argument (see below);
(2) The Hardy-Littlewood inequality.

Lemma 2.3 ($TT^\star$ argument)**.** *Let $T:\mathcal{H}\to B$ a bounded operator from the Hilbert space $\mathcal{H}$ to the Banach space B and $T^\star:B'\to\mathcal{H}$ the adjoint operator defined by*

$$\forall(x,y)\in B'\times\mathcal{H},\ (T^\star x\mid y)_{\mathcal{H}}=\langle x,\overline{Ty}\rangle_{B',B}.$$

Then we have

$$\|TT^\star\|_{\mathcal{L}(B';B)}=\|T\|^2_{\mathcal{L}(\mathcal{H};B)}=\|T^\star\|^2_{\mathcal{L}(B';\mathcal{H})}.$$

Proving the homogeneous Strichartz estimate: Let us introduce the operator $T:u_0\longmapsto U(t)u_0$. Hence, at least formally,

$$T^\star:\phi\longmapsto\int_{\mathbb{R}}U(-t')\phi(t')\,dt'\quad\text{and}\quad TT^\star:\phi\longmapsto\biggl[t\mapsto\int_{\mathbb{R}}U(t-t')\phi(t')\,dt'\biggr].$$

If we apply the $TT^\star$ argument with

$$\mathcal{H}=L^2(\mathbb{R}^d),\quad B=L^q(\mathbb{R};L^r(\mathbb{R}^d)),\quad B'=L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d),$$

we see that proving $\|Tu_0\|_{L^q(L^r)}\leq C\|u_0\|_{L^2}$ is equivalent to

$$\|TT^\star\phi\|_{L^q(L^r)}\leq C\|\phi\|_{L^{q'}(L^{r'})}. \tag{18}$$

Now, we have

$$\|TT^\star\phi(t)\|_{L^r}\leq\int_{\mathbb{R}}\|U(t-t')\phi(t')\|_{L^r}\,dt.$$

So taking advantage of the dispersion inequality $L^{r'}\to L^r$ and of the relation $\sigma(\frac{1}{r'}-\frac1r)=\frac2q$, we get

$$\|TT^\star\phi(t)\|_{L^r}\leq\int_{\mathbb{R}}\frac{1}{|t-t'|^{\frac2q}}\|\phi(t')\|_{L^{r'}}\,dt.$$

Applying the Hardy-Littlewood-Sobolev inequality gives (18) if $2<q<\infty$. □

Remarks:

(1) The endpoint $(q,r)=(\infty,2)$ is given by the fact that $(U(t))_{t\in\mathbb{R}}$ is unitary on L^2. The endpoint $(q,r)=(2,2\sigma/(\sigma-1))$ if $\sigma>1$ is more involved (Keel & Tao [32]).
(2) The nonhomogeneous Strichartz inequality follows from similar arguments.
(3) In the case of the linear wave or Schrödinger equation, using $(\dot\Delta_j)_{j\in\mathbb{Z}}$ allows to get Strichartz estimates involving Besov norms.

3. Incompressible models

As first examples of application of the results of the previous sections, we here aim at solving globally for small data two models for incompressible viscous fluids: first the classical *incompressible Navier-Stokes equations* and next the *density-dependent Navier-Stokes equations.*

3.1. **The incompressible Navier-Stokes equations.** The incompressible Navier-Stokes equations read:

$$(NS)\qquad \begin{cases} \partial_t u + \mathrm{div}\,(u\otimes u) - \mu\Delta u + \nabla P = 0, \\ \mathrm{div}\, u = 0. \end{cases}$$

Here $u : [0,T[\times\mathbb{R}^d \to \mathbb{R}^d$ stands for the velocity field, and $P : [0,T[\times\mathbb{R}^d \to \mathbb{R}$, for the pressure. The viscosity μ is a given positive number.

If we want to solve the Cauchy problem for (NS) then we have to prescribe some initial divergence-free velocity field u_0.

We shall see that, once a suitable functional framework has been found, System (NS) may be solved by means of the following abstract lemma.

Lemma 3.1. *Let X be a Banach space and $\mathcal{B} : X\times X \to X$ a continuous bilinear map. Then there exists a unique solution v in $B(0, 2\|v_0\|_X)$ to*

$$(E)\qquad v = v_0 + \mathcal{B}(v,v)$$

whenever

$$(19)\qquad 4\|\mathcal{B}\|\|v_0\|_X < 1.$$

Proof. This is a straightforward consequence of Picard's fixed point theorem in complete metric spaces. Indeed, denoting $F : v \to v_0 + \mathcal{B}(v,v)$, we see that

$$\|F(v) - v_0\|_X \le \|\mathcal{B}\|_X \|v\|_X^2.$$

Hence if (19) is satisfied and $\|v\|_X \le 2\|v_0\|_X$ then F maps the closed ball $\bar{B}(0, 2\|v_0\|_X)$ into itself.

Next, considering v_1 and v_2 in this closed ball, we see that

$$\|F(v_2) - F(v_1)\|_X \le \|\mathcal{B}\|(\|v_1\|_X + \|v_2\|_X)\|v_2 - v_1\|_X \le 4\|\mathcal{B}\|\|v_0\|_X\|v_2 - v_1\|_X.$$

Hence Condition (19) ensures that F is strictly contracting. □

Assume in addition that there exists a one-parameter family $(T_\lambda)_{\lambda>0}$ acting on X and which leaves (E) invariant that is:

$$v = v_0 + \mathcal{B}(v,v) \iff T_\lambda v = T_\lambda v_0 + \mathcal{B}(T_\lambda v, T_\lambda v) \quad \text{for all} \quad \lambda > 0.$$

Then the smallness condition (19) recasts in

$$4\|\mathcal{B}\|\|T_\lambda v_0\|_X < 1 \quad \text{for all} \quad \lambda > 0.$$

In other words, the norm in X has to be invariant (up to an irrelevant constant) by T_λ for all λ. If so then we shall call X *a scaling invariance space* for (E).

In the applications, a dimension analysis often allows to find such a family $(T_\lambda)_{\lambda>0}$. This is the case for instance if considering evolutionary equations such as the nonlinear Schrödinger, wave or heat equations.

Let us explain how to implement it on the incompressible Navier-Stokes equations. Introducing the Leray projector over divergence-free vector fields: $\mathcal{P} := \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}\,$, (NS) recasts in

$$\partial_t u + \mathcal{P}\mathrm{div}\,(u\otimes u) - \mu\Delta u = 0.$$

This equation enters in the class of *generalized Navier-Stokes equations*:

$$(GNS)\qquad \partial_t u + Q(u,u) - \mu\Delta u = 0$$

with $\mathcal{F}Q^j(u,v) := \sum \alpha^{j,m,n,p}_{k,\ell}\dfrac{\xi_n\,\xi_p\,\xi_m}{|\xi|^2}\,\mathcal{F}(u^k\,v^\ell)$. All the coefficients are supposed to be constant. Hence the entries of $Q(u,v)$ are first order homogeneous Fourier multipliers applied to bilinear expressions. From the point of view of homogeneous Besov spaces, the action of such multipliers is exactly the same as that of the gradient operator.

Let us now look for some scaling invariance for (GNS), if any. We notice that v is a solution if and only if $T_\lambda v$ is a solution (for all $\lambda>0$) with

$$T_\lambda v(t,x) := \lambda v(\lambda^2 t, \lambda x).$$

Hence one may tempt to solve (NS) or (GNS) in spaces X with norm invariant by the above transformation.

For (NS) this idea of combining the abstract lemma with dimensional analysis has been first implemented by H. Fujita and T. Kato in [23] (see also the work by J.-Y. Chemin in [8]). There

$$X = \bigl\{v\in\mathcal{C}(\mathbb{R}^+;\dot H^{\frac d2-1}) \text{ s. t. } t^{\frac14}v\in\mathcal{C}(\mathbb{R}^+;\dot H^{\frac d2-\frac12})\bigr\},$$

and the initial data is in the homogeneous Sobolev space $\dot H^{\frac d2-1}$

There are a number of critical functional spaces in which (NSI) may be globally solved for small data, for instance:

- $\mathcal{C}(\mathbb{R}^+;L^d)$ (see Giga [25], Kato [27], Furioli-Lemarié-Terraneo [24]);
- $\mathcal{C}(\mathbb{R}^+;\dot B^{\frac dp-1}_{p,1})\cap L^1(\mathbb{R}^+;\dot B^{\frac dp+1}_{p,1})$ and more general Besov spaces (see the works by Cannone-Meyer-Planchon in [7] and by H. Kozono and M. Yamazaki in [34]).

In these notes, we plan to prove the following statement.

Theorem 3.1. *Let $u_0\in\dot B^{\frac dp-1}_{p,r}$ with $\operatorname{div}u_0=0$. Assume that p is finite. There exists $c>0$ such that if*

$$\|u_0\|_{\dot B^{\frac dp-1}_{p,r}}\leq c\mu$$

then (GNS) has a unique global solution u in the space[4]

$$X := \widetilde L^\infty(\mathbb{R}^+;\dot B^{\frac dp-1}_{p,r})\cap\widetilde L^1(\mathbb{R}^+;\dot B^{\frac dp+1}_{p,r}).$$

Proof. We notice that solving (GNS) amounts to solving the equation (E) of the abstract lemma with

$$v_0 := e^{t\Delta}u_0\quad\text{and}\quad B(v,v)(t) = -\int_0^t e^{(t-\tau)\Delta}\mathcal{P}\mathrm{div}\,(v\otimes v)\,d\tau$$

where $(e^{t\Delta})_{t>0}$ stands for the heat semi-group.

We want to apply the abstract lemma with $X=\widetilde L^\infty(\mathbb{R}^+;\dot B^{\frac dp-1}_{p,r})\cap\widetilde L^1(\mathbb{R}^+;\dot B^{\frac dp+1}_{p,r})$,

$$v_0(t) := e^{\mu t\Delta}u_0\quad\text{and}\quad \mathcal{B}(u,v)(t) := -\int_0^t e^{\mu(t-\tau)\Delta}Q(u,v)\,d\tau.$$

Heat estimates imply that

$$\|v_0\|_X := \|v_0\|_{\widetilde L^\infty(\dot B^{\frac dp-1}_{p,r})}+\mu\|v_0\|_{\widetilde L^1(\dot B^{\frac dp+1}_{p,r})}\leq C\|u_0\|_{\dot B^{\frac dp-1}_{p,r}}.$$

[4] which corresponds to the norms defined in Remark 2.1.

That $\mathcal{B}: X\times X\to X$ is a consequence of continuity results for paraproduct and remainder. Indeed we have

$$\|Q(u,v)\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}-1}_{p,r})}\leq C\|u\otimes v\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}}_{p,r})}.$$

Now, using embedding and results of continuity for the paraproduct and remainder, we see that[5]

$$\begin{aligned}\|R(u,v)\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}}_{p,r})} &\lesssim \|u\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}-1}_{p,r})}\|v\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}+1}_{p,r})}\\ \|T_uv\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}}_{p,r})} &\lesssim \|u\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,\infty})}\|v\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}+1}_{p,r})}\end{aligned}$$

and a similar inequality for T_vu. Note that for the remainder we need that $(d/p-1)+(d/p+1)>0$ hence $p<\infty$. Because $\dot{B}^{\frac{d}{p}-1}_{p,r}\hookrightarrow\dot{B}^{-1}_{\infty,\infty}$, we eventually find that, for some $C=C(d,p,Q)$:

$$\|Q(u,v)\|_{\widetilde{L}^1(\dot{B}^{\frac{d}{p}-1}_{p,r})}\leq C\mu^{-1}\|u\|_X\|v\|_X.$$

Hence

$$\|\mathcal{B}(u,v)\|_X\leq C'\mu^{-1}\|u\|_X\|v\|_X.$$

Therefore $4\|\mathcal{B}\|\|v_0\|_X<1$ provided $\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}}\leq c\mu$ with c small enough. □

Remark 3.1. *The proof that we proposed is based on the abstract lemma and on product laws in Besov spaces that do not use the very structure of the nonlinearity. Hence it is very robust. This may be seen as an advantage as it applies indistinctly to any system (GNS) (even for those which have no physical meaning and that do not possess any conserved quantity), but also as an inconvenient because one cannot expect from this method much more than global existence for small data (or local existence for large data).*

Exercise: Adapt this approach to the following Keller-Segel model:

$$\partial_t\rho-\Delta\rho=-\chi\mathrm{div}\,(\rho\,\nabla(-\Delta)^{-1}\rho).$$

3.2. **The density-dependent incompressible Navier-Stokes equations.** Let us now look at a slightly more general model: the system for incompressible nonhomogeneous viscous fluids:

$$(INS)\qquad\left\{\begin{array}{l}\partial_t\rho+u\cdot\nabla\rho=0\\ \rho(\partial_tu+u\cdot\nabla u)-\mu\Delta u+\nabla P=0\\ \mathrm{div}\,u=0.\end{array}\right.$$

We restrict our attention to the case where the density ρ of the fluid goes to some positive constant $\bar\rho$ at infinity. After renormalization, one may take $\bar\rho=1$.

System (INS) is invariant by the rescaling

$$\rho(t,x)\to\rho(\lambda^2t,\lambda x),\qquad u(t,x)\to\lambda u(\lambda^2t,\lambda x).$$

From the dimensional analysis point of view, this means that the velocity has the same regularity as in the homogeneous case and that one more derivative is required for the density. In the Besov spaces scale, this induces to take data $(\rho_0=1+a_0,u_0)$ with

$$a_0\in\dot{B}^{\frac{d}{p_1}}_{p_1,r_1}\quad\text{and}\quad u_0\in\dot{B}^{\frac{d}{p_2}-1}_{p_2,r_2}.$$

[5] One may easily check that the product laws for $\widetilde{L}^\rho(\dot{B}^\sigma_{p,r})$ work the same as the usual ones, the time Lebesgue exponent just behaves according to Hölder inequality.

- To avoid vacuum (and loss of ellipticity), an L^∞ bound on a is needed. Note that $\dot B^{\frac{d}{p_1}}_{p_1,r_1}\hookrightarrow L^\infty$ if and only if $r_1=1$. Hence we take $r_1=1$.
- If $r_2=1$ then (neglecting the nonlinear terms), regularity properties of the heat equation give $u\in L^1_T(\dot B^{\frac{d}{p_2}+1}_{p_2,1})$. As $\dot B^{\frac{d}{p_2}+1}_{p_2,1}\hookrightarrow C^{0,1}$, this is exactly what we need to transport the Besov regularity of $a:=\rho-1$.
- Finally, owing to the coupling between the density and velocity equations, it is simpler (but not mandatory) to take $p_1=p_2=p$.

The rest of this section is devoted to proving the following global existence result for small data with critical regularity.

Theorem 3.2 (Global existence for small data)**.** *Let* $a_0\in\dot B^{\frac dp}_{p,1}$ *and* $u_0\in\dot B^{\frac dp-1}_{p,1}$ *with* $\operatorname{div}u_0=0$ *and* $1\leq p<2d$. *If in addition*

$$\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\mu^{-1}\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}\leq c$$

for a small enough $c>0$ *then* (INS) *has a unique global solution* (a,u) *with*

$$a\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac dp}_{p,1})\quad\textit{and}\quad u\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac dp-1}_{p,1})\cap L^1(\mathbb{R}^+;\dot B^{\frac dp+1}_{p,1}).$$

Proof. Before going further into the proof of existence, let us emphasize that one cannot expect to reduce System (INS) to the model problem presented at the beginning of this section. This is due to the hyperbolic nature of the density equation which entails a loss of one derivative in the Lipschitz-type stability estimates. Hence, existence will rather stem from bounds in high norm (that is in the space where we want to show the existence) for the solution and stability in low norm (typically the space of existence with one less of derivative). Proving existence may be alternately done by means of the Schauder-Tikhonoff fixed point theorem. However the main steps are more or less the same. Here we shall adopt the following scheme (that has to be slightly modified to provide a rigorous proof of existence) giving the result if $1\leq p<d$:

- 1) proving a priori estimates in *high norm* (that is in the space E of the statement) for a solution;
- 2) proving stability estimates in *low norm* (that is with one less derivative);
- 3) Use functional analysis (Fatou property) to justify that the constructed solution is in E;
- 4) Uniqueness : for $1\leq p\leq d$ this stems from stability estimates. For the full range $1\leq p<2d$, the system has to be considered in Lagrangian coordinates (recent joint work with P.B. Mucha [22]).

Step 1. A priori estimates in high norm. Estimates for the transport equation imply that $\|a(t)\|_{L^\infty}=\|a_0\|_{L^\infty}$ and

$$\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\leq e^{CU(t)}\|a_0\|_{\dot B^{\frac dp}_{p,1}}\quad\text{with}\quad U(t):=\|\nabla u\|_{L^1_t(\dot B^{\frac dp}_{p,1})}.$$

Hence if we have $CU(t)\leq\log 2$ then

$$\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\leq 2\|a_0\|_{\dot B^{\frac dp}_{p,1}}.\tag{20}$$

For the velocity, we have:

$$\partial_t u-\mu\Delta u=-\mathcal{P}(a\partial_t u)-\mathcal{P}((1+a)\,u\cdot\nabla u).$$

Hence regularity estimates for the heat equation imply that

$$X(t)\lesssim \|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\int_0^t\bigl(\|\mathcal{P}(a\partial_t u)\|_{\dot B^{\frac dp-1}_{p,1}}+\|\mathcal{P}((1+a)u\cdot\nabla u)\|_{\dot B^{\frac dp-1}_{p,1}}\bigr)\,dt$$

with $X(t):=\|u\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}+\mu\|u\|_{L^1_t(\dot B^{\frac dp+1}_{p,1})}+\|\partial_t u\|_{L^1_t(\dot B^{\frac dp-1}_{p,1})}.$

The l.h.s. may be bounded by using product laws (here we need $1\leq p<2d$ because of the remainder term $R(a,\partial_t u)$ coming from $a\,\partial_t u$). We get

$$X(t)\lesssim \|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+(1+\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})})\|u\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}\|u\|_{L^1_t(\dot B^{\frac dp+1}_{p,1})}+\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\|\partial_t u\|_{L^1_T(\dot B^{\frac dp-1}_{p,1})}.$$

According to (20), the last term may be absorbed by the left hand-side if $\|a_0\|_{\dot B^{\frac dp}_{p,1}}$ is small enough. Under this smallness condition, the second term of the r.h.s. may be bounded by $C\mu^{-1}X^2(t)$.

From an easy induction ("bootstrap") argument, we eventually get

$$\|a(t)\|_{\dot B^{\frac dp}_{p,1}}\leq 2\|a_0\|_{\dot B^{\frac dp}_{p,1}}\quad\text{and}\quad X^2(t)\leq 2CX(0)\quad\text{for all}\quad t\geq 0$$

provided $\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\mu^{-1}\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}\ll 1.$

Step 2. Stability estimates in small norm. Consider two solutions (a^1,u^1) and (a^2,u^2) of (NSI) bounded in E as in the first step. The difference $(\delta a,\delta u):=(a^2-a^1,u^2-u^1)$ satisfies

$$\begin{cases}\partial_t\delta a+u^2\cdot\nabla\delta a=-\delta u\cdot\nabla a^1,\\ \partial_t\delta u-\mu\Delta\delta u=-\mathcal{P}\bigl((1+a^1)(\delta u\cdot\nabla u^1+u^2\cdot\nabla\delta u)+a^1\partial_t\delta u+\delta a(\partial_t u^2+u^2\cdot\nabla u^2)\bigr).\end{cases}$$

Owing to the hyperbolic nature of the mass equation, one loses one derivative in the stability estimates: the r.h.s of the first equation has at most the regularity of ∇a^1. This induces a loss of one derivative for δu. Hence stability has to be proved in

$$F_T:=\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1})\times\bigl(\mathcal{C}([0,T];\dot B^{\frac dp-2}_{p,1})\cap L^1(0,T;\dot B^{\frac dp}_{p,1})\bigr)^d.$$

Therefore, as e.g. $\delta a\in\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1})$ and $\partial_t u^2\in L^1(0,T;\dot B^{\frac dp-1}_{p,1})$, the product laws in Besov spaces (see in particular the properties of continuity for the remainder in Theorem 2.1) enforce us to make *stronger* conditions on p and on d, namely

$$d>2\quad\text{and}\quad 1\leq p<d.$$

Then using the usual regularity estimates for the heat equation, one may conclude that

$$\|\delta a\|_{L^\infty(\dot B^{\frac dp-1}_{p,1})}+\mu\|\delta u\|_{L^\infty(\dot B^{\frac dp-2}_{p,1})}+\|\delta u\|_{L^1(\dot B^{\frac dp}_{p,1})}\lesssim\|\delta a(0)\|_{\dot B^{\frac dp-1}_{p,1}}+\|\delta u(0)\|_{\dot B^{\frac dp-2}_{p,1}}.$$

We skip Step 3 (showing that the constructed solution belongs to the desired functional space) which is a consequence of general properties of Besov spaces and Step 4 (uniqueness) which is very similar to Step 2. □

Let us end this section with a few remarks concerning the above global existence theorem for incompressible nonhomogeneous fluids:

- This global statement has been first proved by the author in [16] (in the case $p=2$ only) and by H. Abidi [1] (for general $p\in[1,2d[$ with uniqueness if $p\leq d$, and density dependent viscosity coefficients).

- A local-in-time statement is available for *large initial velocity* $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}$ with $1 \leq p < 2d$ [1]. If $p = 2$, also a_0 may be large provided $\inf_x(1 + a_0) > 0$ (no vacuum) [16].
- The Lebesgue exponents for a and u may be taken different (see [2]).
- Rewriting the system in Lagrangian coordinates allows to solve it by means of Picard's fixed point theorem. This improves the conditions for uniqueness and also allows to weaken the assumptions over the density: it may have jumps across an interface (recent joint work with P.B. Mucha [22]).
- There are many recent works devoted to the longtime asymptotics for (INS), see in particular the paper by Abidi-Gui-Zhang in [3].

4. The barotropic compressible Navier-Stokes equations

Proving in a similar result for *compressible* viscous fluids is the main goal of this section. To simplify the presentation, we focus on the *barotropic* Navier-Stokes equations :

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mu\Delta u - \mu'\nabla\mathrm{div}\, u + \nabla P = 0 \end{cases}$$

where

- $\rho = \rho(t, x) \in \mathbb{R}^+$ (with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$) is the density,
- $u = u(t, x) \in \mathbb{R}^d$ is the velocity field,
- the viscosity coefficients μ and μ' satisfy $\mu > 0$ and $\nu := \mu + \mu' > 0$.

In order to close the system, we assume the pressure P to be a given (suitably smooth) function of ρ. This is the so-called *barotropic assumption.* In the viscous case that we shall consider, this assumption is somewhat irrelevant from a physical viewpoint. However the above system already contains many features of the full model as far as mathematical results are concerned.

We supplement the system with the following boundary conditions:

- u decays to zero at infinity,
- ρ tends to some positive constant $\bar{\rho}$ at infinity. We shall take $\bar{\rho} = 1$ for simplicity.

Denoting $\rho = 1 + a$ and assuming that the density is positive everywhere the barotropic system rewrites

$$\begin{cases} \partial_t a + u \cdot \nabla a = -(1 + a)\mathrm{div}\, u, \\ \partial_t u - \mathcal{A}u = -u \cdot \nabla u - J(a)\mathcal{A}u - \nabla G(a) \end{cases} \tag{21}$$

where $\mathcal{A} := \mu\Delta - \mu'\nabla\mathrm{div}\,,$ $J(a) := a/(1 + a)$ and G is a smooth function with $G(0) = 0$.

This system has a number of similarities with (INS). The only difference is that $\mathrm{div}\, u$ is not prescribed any longer and therefore the pressure function is given whereas, in the incompressible case, ∇P was the Lagrange multiplier associated to the divergence free condition.

Because P is given, strictly speaking, System (21) does not own any scaling invariance property . However, up to a change of G into $\lambda^2 G$, it is invariant by the rescaling

$$a(t, x) \to a(\lambda^2 t, \lambda x), \qquad u(t, x) \to \lambda u(\lambda^2 t, \lambda x). \tag{22}$$

To some extent, the term G is lower order. Therefore, we expect the critical functional spaces for the velocity to be the same ones as in the incompressible case whereas one more derivative has to be taken for the density.

4.1. **The local existence theory.** At first sight, (NSC) looks very similar to (INS) hence we expect that it is globally well-posed if a_0 and u_0 are small in $\dot B^{\frac dp}_{p,1}$ and $\dot B^{\frac dp-1}_{p,1}$, respectively. The following computations show that whether such a statement may be true, is not so obvious.

1. A priori estimates in high norm for the density. Let $U(t):=\|\nabla u\|_{L^1_t(\dot B^{\frac dp}_{p,1})}$. Estimates for the transport equation imply that

$$\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\leq e^{CU(t)}\biggl(\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\int_0^t e^{-CU}\|(1+a)\mathrm{div}\,u\|_{\dot B^{\frac dp}_{p,1}}\,d\tau\biggr).$$

From product laws in Besov spaces, we have:

$$\|(1+a)\mathrm{div}\,u\|_{\dot B^{\frac dp}_{p,1}}\lesssim(1+\|a\|_{\dot B^{\frac dp}_{p,1}})\|\nabla u\|_{\dot B^{\frac dp}_{p,1}}.$$

Inserting this in the above inequality and applying Gronwall's lemma, we thus get

$$\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\leq e^{CU(T)}\|a_0\|_{\dot B^{\frac dp}_{p,1}}+e^{CU(T)}-1.$$

Hence, for any $\eta>0$, if $U(T)$ is small enough then

$$\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\leq 2\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\eta.$$

2. A priori estimates in high norm for the velocity. Let us first observe that even though $\mathcal A$ *is not* the Laplace operator, the associated regularity estimates are the same as if it were. Indeed, if z satisfies

$$\partial_t z-\mathcal A z=f$$

then we have, denoting $\nu:=\mu+\mu'$,

$$\partial_t\mathcal P z-\mu\Delta\mathcal P z=\mathcal P f\quad\text{and}\quad\partial_t\mathcal Q z-\nu\Delta\mathcal Q z=\mathcal Q f.$$

At this point it is of course fundamental that $\mu>0$ and $\nu>0$.

Coming back to the problem of bounding the velocity field in (NSC) we thus have

$$\|u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})\cap L^1_T(\dot B^{\frac dp+1}_{p,1})}\lesssim\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\int_0^T\|u\cdot\nabla u+J(a)\mathcal A u+\nabla(G(a))\|_{\dot B^{\frac dp-1}_{p,1}}\,dt.$$

Product and composition laws in Besov spaces yield if $d>1$ and $1\leq p<2d$,

$$\begin{aligned}&\|u\cdot\nabla u\|_{\dot B^{\frac dp-1}_{p,1}}\lesssim\|u\|_{\dot B^{\frac dp-1}_{p,1}}\|\nabla u\|_{\dot B^{\frac dp}_{p,1}},\\&\|J(a)\mathcal A u\|_{\dot B^{\frac dp-1}_{p,1}}\lesssim\|a\|_{\dot B^{\frac dp}_{p,1}}\|\nabla^2 u\|_{\dot B^{\frac dp-1}_{p,1}},\\&\|\nabla(G(a))\|_{\dot B^{\frac dp-1}_{p,1}}\lesssim\|a\|_{\dot B^{\frac dp}_{p,1}}.\end{aligned}$$

Hence

$$\begin{aligned}\|u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+\|u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}&\lesssim\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}\\&+\int_0^T\|u\|_{\dot B^{\frac dp-1}_{p,1}}\|u\|_{\dot B^{\frac dp+1}_{p,1}}\,dt+\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}\|u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}+\|a\|_{L^1_T(\dot B^{\frac dp}_{p,1})}.\end{aligned}$$

As in the incompressible case, the last but one term may be absorbed by the left hand-side if $\|a_0\|_{\dot B^{\frac dp}_{p,1}}$ is small enough, and the blue term is quadratic hence harmless if u is small. From this, we get

$$\|u\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+\|u\|_{L^1_T(\dot B^{\frac dp+1}_{p,1})}\lesssim \|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+T(\|a_0\|_{\dot B^{\frac dp}_{p,1}}+\eta).$$

At this point we see that the r.h.s. grows linearly in time hence we cannot expect to get any global-in-time control on u. In fact, the main problem is that estimates for the transport equation provide us with a bound for $\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}$ while $\|a\|_{L^1_T(\dot B^{\frac dp}_{p,1})}$ is needed. Here we only used that

$$\|a\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\lesssim T\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}.$$

It is not difficult to make the previous computations rigorous (at least if a_0 is small), and we end up with the following statement:

Theorem 4.1. *Assume that $a_0\in\dot B^{\frac dp}_{p,1}$ and that $u_0\in\dot B^{\frac dp-1}_{p,1}$ with $1\leq p<2d$. If in addition $1+a_0$ is bounded away from 0 then (NSC) has a local-in-time solution (a,u) with*

$$a\in\mathcal C([0,T];\dot B^{\frac dp}_{p,1})\quad\textit{and}\quad u\in\mathcal C([0,T];\dot B^{\frac dp-1}_{p,1})\cap L^1([0,T];\dot B^{\frac dp+1}_{p,1}).$$

Uniqueness holds true if $p\leq d$.

Remarks:

- This statement has been first established by the author in [15] (there also the full Navier-Stokes system is considered), *under some smallness condition over a_0*.
- The smallness condition over a_0 has been relaxed by the author in the case $p=2$ [19] and then for more general p by Chen-Miao-Zhang in [11] (in this latter work, the viscosity coefficients may depend on the density).
- Using Lagrangian coordinates allows to prove uniqueness whenever $1\leq p<2d$ and to consider nonconstant density coefficients as well (see the recent work [21]).
- The Lebesgue exponents for a and u may be taken different (see the recent work by B. Haspot [28]).

4.2. **The global existence theory.** The scaling invariance exhibited in (22) was imperfect inasmuch as it did not take the pressure term into consideration. This fact was quite obvious when proving local a priori estimates for (NSC) in the previous section because our estimates for the transport equation naturally provide bounds for $\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}$ whereas $\|a\|_{L^1_T(\dot B^{\frac dp}_{p,1})}$ is needed when bounding u by means of parabolic estimates. So far, we used that

$$\|\nabla(G(a))\|_{L^1_T(\dot B^{\frac dp-1}_{p,1})}\lesssim T\|a\|_{L^\infty_T(\dot B^{\frac dp}_{p,1})}.$$

Of course this is a good estimate if T is small, but this is not so clever when T goes to infinity. Hence it is very unlikely that one may get any global-in-time control on u by so rough a device. In other words, while the pressure term may be "neglected" in the linear analysis leading to local-in-time existence results, it has to be included in the linear analysis for the global existence theory. This is what we aim to do now.

4.2.1. *The linearized equations.* The linearized system about $(a,u)=(0,0)$ reads:

$$\begin{cases} \partial_t a + \operatorname{div} u = 0, \\ \partial_t u - \mathcal{A}u + \alpha\nabla a = 0 \end{cases} \qquad \text{with} \quad \alpha := P'(1). \tag{23}$$

Let $\nu := \mu + \mu'$. Applying operators $\mathcal{P}$ and $\mathcal{Q}$ to the second equation, the above system translates into

$$\begin{cases} \partial_t a + \operatorname{div} \mathcal{Q}u = 0, \\ \partial_t \mathcal{Q}u - \nu\Delta\mathcal{Q}u + \alpha\nabla a = 0, \\ \partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = 0. \end{cases} \tag{24}$$

In the homogeneous Besov spaces setting, it is equivalent to bound $\mathcal{Q}u$ or $v := |D|^{-1}\operatorname{div} u$, the advantage of the latter quantity being that it is real valued. So we are led to considering

$$\begin{cases} \partial_t a + |D|v = 0, \\ \partial_t v - \nu\Delta v - \alpha|D|a = 0, \\ \partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = 0. \end{cases} \tag{25}$$

Note that the last equation (that is the linearized equation for the vorticity part of the velocity field) is a mere heat equation with constant diffusion. So we have to concentrate on the linearized system for the density and the potential part of the velocity, namely

$$\begin{cases} \partial_t a + |D|v = 0, \\ \partial_t v - \nu\Delta v - \alpha|D|a = 0. \end{cases} \tag{26}$$

Taking the Fourier transform with respect to the space variable yields

$$\frac{d}{dt}\begin{pmatrix} \widehat{a} \\ \widehat{v} \end{pmatrix} = A(\xi)\begin{pmatrix} \widehat{a} \\ \widehat{v} \end{pmatrix} \quad \text{with} \quad A(\xi) := \begin{pmatrix} 0 & -|\xi| \\ \alpha|\xi| & -\nu|\xi|^2 \end{pmatrix}.$$

The characteristic polynomial of $A(\xi)$ is $X^2+\nu|\xi|^2X+\alpha|\xi|^2$, the discriminant of which is

$$\delta(\xi) := |\xi|^2(\nu^2|\xi|^2 - 4\alpha).$$

If $\alpha<0$ then there is one positive eigenvalue hence the linear system is unstable.

Therefore we assume from now that $\alpha>0$ (i.e. $P'(1)>0$), that is we focus on the case where the pressure law is increasing in some neighborhood of the reference density. Note also that a convenient change of variable reduces the study to the case $\alpha=1$, an assumption that we shall make from now on.

The low frequency regime $\nu|\xi|<2$. There are two distinct complex conjugated eigenvalues:

$$\lambda_\pm(\xi) = -\frac{\nu|\xi|^2}{2}(1\pm iS(\xi)) \quad \text{with} \quad S(\xi) := \sqrt{\frac{4}{\nu^2|\xi|^2}-1},$$

and we find that

$$\begin{aligned} \widehat{a}(t,\xi) = e^{t\lambda_-(\xi)}\biggl(\frac12\Bigl(1+\frac{i}{S(\xi)}\Bigr)\widehat{a}_0(\xi) - \frac{i}{\nu|\xi|S(\xi)}\widehat{v}_0(\xi)\biggr)& \\ +e^{t\lambda_+(\xi)}\biggl(\frac12\Bigl(1-\frac{i}{S(\xi)}\Bigr)\widehat{a}_0(\xi) + \frac{i}{\nu|\xi|S(\xi)}\widehat{v}_0(\xi)\biggr)&, \end{aligned}$$

$$\widehat{v}(t,\xi)=e^{t\lambda_-(\xi)}\left(\frac{i}{\nu|\xi|S(\xi)}\widehat{a}_0(\xi)+\frac12\Bigl(1-\frac{i}{S(\xi)}\Bigr)\widehat{v}_0(\xi)\right)\\ +e^{t\lambda_+(\xi)}\left(-\frac{i}{\nu|\xi|S(\xi)}\widehat{a}_0(\xi)+\frac12\Bigl(1+\frac{i}{S(\xi)}\Bigr)\widehat{v}_0(\xi)\right).$$

For $\xi\to0,$ we have

$$\widehat{a}(t,\xi)\sim\tfrac12e^{t\lambda_-(\xi)}\left(\widehat{a}_0(\xi)-i\widehat{v}_0(\xi)\right)+\tfrac12e^{t\lambda_+(\xi)}\left(\widehat{a}_0(\xi)+i\widehat{v}_0(\xi)\right),$$

$$\widehat{v}(t,\xi)\sim\tfrac12e^{t\lambda_-(\xi)}\left(i\widehat{a}_0(\xi)+\widehat{v}_0(\xi)\right)+\tfrac12e^{t\lambda_+(\xi)}\left(-i\widehat{a}_0(\xi)+\widehat{v}_0(\xi)\right).$$

Hence, the low frequencies of a and v have a similar behavior.

Note that $|e^{t\lambda_\pm(\xi)}|=e^{-\nu t|\xi|^2/2}$ and that

$$\operatorname{Re}\lambda_\pm(\xi)=-\frac{\nu|\xi|^2}{2},\qquad \operatorname{Im}\lambda_\pm(\xi)\sim\mp|\xi|\quad\text{for}\quad\xi\to0.$$

Hence we expect the system to have both parabolic and wave-like behavior.

For the time being, we just take advantage of the parabolic behavior. More precisely, we use the fact that, according to Parseval's formula,

$$\|(\dot\Delta_ja,\dot\Delta_jv)(t)\|_{L^2}\leq Ce^{-c\nu t2^{2j}}\|(\dot\Delta_ja_0,\dot\Delta_jv_0)\|_{L^2}\ \text{whenever}\ \ 2^j\nu\leq1. \tag{27}$$

The high frequency regime $\nu|\xi|>2$. There are two distinct real eigenvalues:

$$\lambda_\pm(\xi):=-\frac{\nu|\xi|^2}{2}(1\pm R(\xi))\quad\text{with}\quad R(\xi):=\sqrt{1-\frac{4}{\nu^2|\xi|^2}}$$

and after a lengthy computation, we find that

$$\widehat{a}(t,\xi)=e^{t\lambda_-(\xi)}\left(\frac12\Bigl(1+\frac{1}{R(\xi)}\Bigr)\widehat{a}_0(\xi)-\frac{1}{\nu|\xi|R(\xi)}\widehat{v}_0(\xi)\right)\\ +e^{t\lambda_+(\xi)}\left(\frac12\Bigl(1-\frac{1}{R(\xi)}\Bigr)\widehat{a}_0(\xi)+\frac{1}{\nu|\xi|R(\xi)}\widehat{v}_0(\xi)\right),$$

$$\widehat{v}(t,\xi)=e^{t\lambda_-(\xi)}\left(\frac{1}{\nu|\xi|R(\xi)}\widehat{a}_0(\xi)+\frac12\Bigl(1-\frac{1}{R(\xi)}\Bigr)\widehat{v}_0(\xi)\right)\\ +e^{t\lambda_+(\xi)}\left(-\frac{1}{\nu|\xi|R(\xi)}\widehat{a}_0(\xi)+\frac12\Bigl(1+\frac{1}{R(\xi)}\Bigr)\widehat{v}_0(\xi)\right).$$

For $|\xi|\to\infty,$ we have $R(\xi)\to1$ and $1-R(\xi)\sim2/(\nu\xi)^2$. Hence $\lambda_+(\xi)\sim-\nu|\xi|^2$ and $\lambda_-(\xi)\sim-\frac1\nu$. In other words, a parabolic and a damped mode coexist and the asymptotic behavior of (a,v) for $|\xi|\to\infty$ is given by

$$\widehat{a}(t,\xi)\sim e^{-\frac t\nu}\left(\widehat{a}_0(\xi)-(\nu|\xi|)^{-1}\widehat{v}_0(\xi)\right)+e^{-\nu t|\xi|^2}\left(-(\nu|\xi|)^{-2}\widehat{a}_0(\xi)+(\nu|\xi|)^{-1}\widehat{v}_0(\xi)\right),$$
$$\widehat{v}(t,\xi)\sim e^{-\frac t\nu}\left((\nu|\xi|)^{-1}\widehat{a}_0(\xi)-(\nu|\xi|)^{-2}\widehat{v}_0(\xi)\right)+e^{-\nu t|\xi|^2}\left(-(\nu|\xi|)^{-1}\widehat{a}_0(\xi)+\widehat{v}_0(\xi)\right).$$

At first, one would expect the damped mode to dominate as $e^{-\nu t|\xi|^2}$ is negligible compared to $e^{-\frac t\nu}$ for ξ going to infinity. This is true as far as a is concerned. This is not quite the case for v however owing to the negative powers of $\nu|\xi|$ in the formula. More precisely, by taking advantage of Parseval formula, we easily get

Lemma 4.1. *There exist two positive constants c and C such that for any $j \in \mathbb{Z}$ satisfying $2^j\nu \geq 3$ and $t \in \mathbb{R}^+$, we have*

$$\begin{aligned}
&\|\dot\Delta_j a(t)\|_{L^2} \leq Ce^{-\frac{t}{2\nu}}\left(\|\dot\Delta_j a_0\|_{L^2} + (2^j\nu)^{-1}\|\dot\Delta_j v_0\|_{L^2}\right),\\
&\|\dot\Delta_j v(t)\|_{L^2} \leq C\Big((2^j\nu)^{-1}e^{-\frac{t}{2\nu}}\|\dot\Delta_j a_0\|_{L^2} + \big(e^{-c\nu t2^{2j}} + (\nu 2^j)^{-2}e^{-\frac{t}{2\nu}}\big)\|\dot\Delta_j v_0\|_{L^2}\Big).
\end{aligned}$$

In fact, the same inequalities hold true for any $p \in [1,\infty]$. Indeed, following the proof of Lemma 2.1 yields

$$\dot\Delta_j a(t) = h_1^j(t) * \dot\Delta_j a_0 + h_2^j(t) * (\nu|D|)^{-1}\dot\Delta_j v_0 + h_3^j(t) * (|\nu|D|)^{-2}\dot\Delta_j a_0 + h_4^j(t) * (\nu|D|)^{-1}\dot\Delta_j v_0,$$

$$\dot\Delta_j v(t) = k_1^j(t) * (|\nu D|^{-1}\dot\Delta_j a_0) + k_2^j(t) * (\nu|D|)^{-2}\dot\Delta_j v_0 + k_3^j(t) * (|\nu D|^{-1}\dot\Delta_j a_0) + k_4^j(t) * \dot\Delta_j v_0$$

with

$$\begin{aligned}
\|h_1^j(t)\|_{L^1} + \|h_2^j(t)\|_{L^1} + \|k_1^j(t)\|_{L^1} + \|k_2^j(t)\|_{L^1} &\leq Ce^{-\frac{t}{2\nu}},\\
\|h_3^j(t)\|_{L^1} + \|h_4^j(t)\|_{L^1} + \|k_3^j(t)\|_{L^1} + \|k_4^j(t)\|_{L^1} &\leq Ce^{-c\nu t2^{2j}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\dot\Delta_j a\|_{L_t^\infty(L^p)} + \nu\|\dot\Delta_j a\|_{L_t^1(L^p)} &\lesssim \|\dot\Delta_j a_0\|_{L^p} + \|(\nu|D|)^{-1}\dot\Delta_j v_0\|_{L^p},\\
\|\dot\Delta_j u\|_{L_t^\infty(L^p)} + \nu 2^{2j}\|\dot\Delta_j u\|_{L_t^1(L^p)} &\lesssim \|\nu|D|\dot\Delta_j a_0\|_{L^p} + \|\dot\Delta_j v_0\|_{L^p}.
\end{aligned}$$

Hence, we recover that for high frequencies it is suitable to work at the same level of regularity for ∇a and v. At the same time, according to (27), one has to work at the same level of regularity for low frequencies, a fact which does not follow from our scaling considerations for (NSC).

Putting together all the estimates for the dyadic blocks and using Duhamel's formula, we conclude that whenever (a,u) satisfies

$$(LPH) \qquad \left\{\begin{aligned} &\partial_t a + \operatorname{div} u = F,\\ &\partial_t u - \mathcal{A}u + \nabla a = G,\end{aligned}\right.$$

we have for the low frequencies:

$$\|(a,u)\|^\ell_{\widetilde L_t^\infty(\dot B^{s'}_{2,r})\cap \widetilde L_t^1(\dot B^{s'+2}_{2,r})} \lesssim \|(a_0,u_0)\|^\ell_{\dot B^{s'}_{2,r}} + \|(F,G)\|^\ell_{\widetilde L_t^1(\dot B^{s'}_{2,r})}$$

and for the high frequencies,

$$\begin{aligned}
&\|a\|^h_{(\widetilde L_t^\infty\cap\widetilde L_t^1)(\dot B^{s+1}_{p,r})} + \|u\|^h_{\widetilde L_t^\infty(\dot B^s_{p,r})\cap\widetilde L_t^1(\dot B^{s+2}_{p,r})}\\
&\qquad\qquad \lesssim \|a_0\|^h_{\dot B^{s+1}_{p,r}} + \|u_0\|^h_{\dot B^s_{p,r}} + \|F\|^h_{\widetilde L_t^1(\dot B^{s+1}_{p,r})} + \|G\|^h_{\widetilde L_t^1(\dot B^s_{p,r})},
\end{aligned}$$

where the index ℓ (resp. h) means that only low (resp. high) frequencies have been taken into account when computing the norm[6].

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[6]We mean that $\|z\|^\ell_{\dot B^\sigma_{p,r}} = \Big(\sum_{2^j\nu\leq 1} 2^{jr\sigma}\|\dot\Delta_j z\|^r_{L^p}\Big)^{\frac1r}$ and that $\|z\|^h_{\dot B^\sigma_{p,r}} = \Big(\sum_{2^j\nu> 1} 2^{jr\sigma}\|\dot\Delta_j z\|^r_{L^p}\Big)^{\frac1r}$.

4.2.2. *The linearized system with convection terms.* In order to take advantage of the above a priori estimates for (LPH), let us rewrite the barotropic Navier-Stokes equations as follows:

$$\begin{cases} \partial_t a + \operatorname{div} u = -\operatorname{div}(au), \\ \partial_t u - \mathcal{A}u + \nabla a = -u\cdot\nabla u - J(a)\mathcal{A}u - \nabla(aK(a)) \quad \text{with} \quad K(0)=0. \end{cases}$$

As point out before, no gain of regularity is expected for the first equation. Hence the convection term may cause a loss of one derivative. This motivates our including it in our "linear analysis". So we consider:

$$(PL) \qquad \begin{cases} \partial_t a + v\cdot\nabla a + \operatorname{div} u = F, \\ \partial_t u + v\cdot\nabla u - \mathcal{A}u + \nabla a = G. \end{cases}$$

Then a new difficulty arises: the solution is not explicit any longer.

Now, keeping in mind that approach that we used for symmetric hyperbolic system, it is more or less clear that if the estimates for (LPH) may be proved by means of an energy method (which induces to consider only L^2 type Besov spaces) then the convection terms may be handled just by using suitable commutator estimates.

So we plan to prove estimates for

$$(PL) \qquad \begin{cases} \partial_t a + v\cdot\nabla a + \operatorname{div} u = F, \\ \partial_t u + v\cdot\nabla u - \mathcal{A}u + \nabla a = G, \end{cases}$$

by means of an energy method. At this point, we realize that the convection terms satisfy exactly the same commutator estimates as in our study of linear symmetric systems, so one may restrict our study to the case $v\equiv 0$. We also take $F\equiv 0$ and $G\equiv 0$ to simplify the presentation.

Then localizing the system by means of $\dot\Delta_j$ and setting $a_j := \dot\Delta_j a$ and $u_j := \dot\Delta_j u$, we get

$$(PL_j) \qquad \begin{cases} \partial_t a_j + \operatorname{div} u_j = 0, \\ \partial_t u_j - \mathcal{A}u_j + \nabla a_j = 0. \end{cases}$$

<u>Step 1.</u> The basic energy equality.

Taking the L^2 inner product of (PL_j) with (a_j, ju_j) yields

$$\frac{1}{2}\frac{d}{dt}\|(a_j,u_j)\|_{L^2}^2 + \mu\|\nabla\mathcal{P}u_j\|_{L^2}^2 + \nu\|\nabla\mathcal{Q}u_j\|_{L^2}^2 = 0.$$

This is good for u but does not give any decay for a.

<u>Step 2.</u> A second Lyapunov functional.

Set $Y_j^2 := \|a_j\|_{L^2}^2 + \|\mathcal{Q}u_j\|_{L^2}^2 + \alpha\nu(\nabla a_j|u_j)$ and

$$H_j^2 := \frac{\alpha\nu}{2}\|\nabla a_j\|_{L^2}^2 + \nu\left(1-\frac{\alpha}{2}\right)\|\nabla\mathcal{Q}u_j\|_{L^2}^2 - \frac{\alpha\nu^2}{2}(\Delta\mathcal{Q}u_j|\nabla a_j).$$

Then we have

$$\frac{1}{2}\frac{d}{dt}Y_j^2 + H_j^2 = 0.$$

This is good for low frequencies $2^j\nu\lesssim 1$ because if we take α small enough then

$$Y_j^2 \approx \|(a_j,\mathcal{Q}u_j)\|_{L^2}^2 \quad \text{and} \quad H_j^2 \approx \nu 2^{2j}\|(a_j,\mathcal{Q}u_j)\|_{L^2}^2.$$

Step 3. A third Lyapunov functional.

Set $Z_j^2 := \|\nu\nabla a_j\|_{L^2}^2 + 2\|\mathcal{Q}u_j\|_{L^2}^2 + 2\nu(\nabla a_j|u_j)$ and

$$K_j^2 := \nu\|\nabla a_j\|_{L^2}^2 + \nu\|\nabla\mathcal{Q}u_j\|_{L^2}^2 + 2(\nabla a_j|u_j).$$

Then we have

$$\frac{1}{2}\frac{d}{dt}Z_j^2 + K_j^2 = 0.$$

This is good for high frequencies $2^j\nu \gtrsim 1$ because

$$Z_j^2 \approx \|(\nu\nabla a_j, \mathcal{Q}u_j)\|_{L^2}^2 \quad \text{and} \quad K_j^2 \gtrsim \nu^{-1}\|(\nu\nabla a_j, \mathcal{Q}u_j)\|_{L^2}^2.$$

Putting all this together and integrating, we get

$$\|(a_j, \mathcal{Q}u_j)(t)\|_{L^2} + \nu 2^{2j}\int_0^t \|(a_j, \mathcal{Q}u_j)\|_{L^2}\,d\tau \lesssim \|(a_j, \mathcal{Q}u_j)(0)\|_{L^2} \qquad \text{for } 2^j\nu \lesssim 1,$$

$$\|(\nu\nabla a_j, \mathcal{Q}u_j)(t)\|_{L^2} + \nu^{-1}\int_0^t \|(\nu\nabla a_j, \mathcal{Q}u_j)\|_{L^2}\,d\tau \lesssim \|(\nu\nabla a_j, \mathcal{Q}u_j)(0)\|_{L^2} \quad \text{for } 2^j\nu \gtrsim 1.$$

In low frequency, this is what we want. In order to recover the expected parabolic smoothing for $\mathcal{Q}u$ in high frequency, we use the fact that

$$\partial_t\mathcal{Q}u_j - \mu\Delta\mathcal{Q}u_j = -\nabla a_j.$$

Hence, taking the L^2 inner product:

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}u_j\|_{L^2}^2 + \nu\|\nabla\mathcal{Q}u_j\|_{L^2}^2 \le |(\nabla a_j|\mathcal{Q}u_j)|,$$

whence, integrating and using the decay for ∇a_j that we have just proved,

$$\begin{aligned} \|\mathcal{Q}u_j(t)\|_{L^2} + \nu 2^{2j}\int_0^t \|\mathcal{Q}u_j\|_{L^2}\,d\tau &\lesssim \|\mathcal{Q}u_j(0)\|_{L^2} + \int_0^t \|\nabla a_j\|_{L^2}\,d\tau \\ &\lesssim \|(a_j, \mathcal{Q}u_j)(0)\|_{L^2}. \end{aligned}$$

The case with convection term may be handled by writing that

$$\begin{cases} \partial_t a_j + v\cdot\nabla a_j + \operatorname{div} u_j = F + R_j(a,v), \\ \partial_t u_j + v\cdot\nabla u_j - \mathcal{A}u_j + \nabla a = G + R_j(u,v). \end{cases}$$

The remainder terms $R_j(a,v)$ and $R_j(u,v)$ are exactly the same as for symmetric hyperbolic systems, so that we easily end up with:

Proposition 4.1. *Let $s \in]-\frac{d}{2}, \frac{d}{2}]$ and (a,u) be a solution of (PL). We have the following estimate with $V(t) := \int_0^t \|\nabla v(\tau)\|_{\dot B^{\frac{d}{2}}_{2,1}}\,d\tau$:*

$$\begin{aligned} &\|a\|^{\ell}_{L^\infty_t(\dot B^s_{2,1})\cap L^1_t(\dot B^{s+2}_{2,1})} + \|a\|^{h}_{L^\infty_t(\dot B^{s+1}_{2,1})\cap L^1_t(\dot B^{s+1}_{2,1})} + \|u\|_{L^\infty_t(\dot B^s_{2,1})\cap L^1_t(\dot B^{s+2}_{2,1})} \\ &\quad\lesssim e^{CV(t)}\bigg(\|a_0\|^{\ell}_{\dot B^s_{2,1}} + \|a_0\|^{h}_{\dot B^{s+1}_{2,1}} + \|u_0\|_{\dot B^s_{2,1}} + \int_0^t \big(\|F\|^{\ell}_{\dot B^s_{2,1}} + \|F\|^{h}_{\dot B^{s+1}_{2,1}} + \|G\|_{\dot B^s_{2,1}}\big)\,dt\bigg). \end{aligned}$$

Combining this statement with product estimates allows to prove global well-posedness in any dimension $d \ge 2$ for data (a_0, u_0) such that

$$\|a_0\|^{\ell}_{\dot B^{\frac{d}{2}-1}_{2,1}} + \|a_0\|^{h}_{\dot B^{\frac{d}{2}}_{2,1}} + \|u_0\|_{\dot B^{\frac{d}{2}-1}_{2,1}} \le c.$$

More precisely, we get the following statement (see [13]):

Theorem 4.2. *Assume that* $P'(1) > 0$, $a_0 \in \dot{B}^{\frac{d}{2}}_{2,1}$ *and* $u_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}$ *and that in addition* a_0^ℓ *is in* $\dot{B}^{\frac{d}{2}-1}_{2,1}$. *There exist two constants* c *and* M *depending only on* d, *and on the parameters of the system such that if*

$$\|a_0\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \leq c$$

then (NSC) *has a unique global-in-time solution* (a,u) *with*

$$a^\ell \in \mathcal{C}_b(\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\dot{B}^{\frac{d}{2}+1}_{2,1}), \quad a^h \in \mathcal{C}_b(\dot{B}^{\frac{d}{2}}_{2,1}) \cap L^1(\dot{B}^{\frac{d}{2}}_{2,1}),$$
$$u \in \mathcal{C}_b(\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\dot{B}^{\frac{d}{2}+1}_{2,1}).$$

4.2.3. *Global existence in a* L^p *framework.* We now aim at extending the above statement to a L^p type framework: we have in mind assumptions of the type a_0 and u_0 small in $\dot{B}^{\frac{d}{p}}_{p,1}$ and $\dot{B}^{\frac{d}{p}-1}_{p,1}$, respectively with an additional condition on low frequencies, if the case may be.

Preliminary remark: we have to keep in mind that the spectral analysis of (LPH): in low frequency, some eigenvalues have nonzero imaginary part. Hence it is hopeless to take a L^p framework with $p \neq 2$: we are stuck to the L^2 framework for the low frequencies of the solution.

We here follow the recent work by B. Haspot in [29]. In high frequency, the fundamental observations are that, at the linear level:

- $\mathcal{P}u$ satisfies a heat equation (hence parabolic smoothing in any Besov space);
- The "compressible" parabolic mode tends to be collinear to $\mathcal{Q}u + \nu^{-1}(-\Delta)^{-1}\nabla a$;
- The "damped" mode tends to be collinear to $\nu\nabla a + \mathcal{Q}u$.

Recall that

$$\begin{cases} \partial_t a + u\cdot\nabla a + (1+a)\operatorname{div}\mathcal{Q}u = 0, \\ \partial_t \mathcal{Q}u + \mathcal{Q}(u\cdot\nabla u) - \nu\Delta\mathcal{Q}u + \nabla(G(a)) = -\mathcal{Q}(J(a)\mathcal{A}u) \quad \text{with} \quad g'(0) = 1, \\ \partial_t\mathcal{P}u + \mathcal{P}(u\cdot\nabla u) - \mu\Delta\mathcal{P}u = -\mathcal{P}(J(a)\mathcal{A}u). \end{cases}$$

The last equation is a heat equation with quadratic terms. Hence one may expect that parabolic smoothing for $\mathcal{P}u$ holds in any Besov space.

As regards the second equation, we shall introduce a *modified* velocity field in order to express $-\nu\Delta\mathcal{Q}u + \nabla(G(a))$ as a Laplacian. This will enable us to exhibit the parabolic smoothing for the velocity.

Step 1. The effective velocity.

Let us introduce the *effective velocity* $w := u + \nu^{-1}(-\Delta)^{-1}\nabla(G(a))$. It satisfies the heat equation:

$$\begin{aligned}\partial_t\mathcal{Q}w - \nu\Delta\mathcal{Q}w = &-\mathcal{Q}(u\cdot\nabla u) - \mathcal{Q}(J(a)\mathcal{A}u)\\ &-\nu^{-1}(-\Delta)^{-1}\nabla(G'(a)\operatorname{div}(au) + \nabla(G'(a)-G'(0))\operatorname{div}u) - \nu^{-1}(-\Delta)^{-1}\nabla\operatorname{div}u.\end{aligned}$$

All the terms of the right-hand side (but the last one) are at least quadratic hence expected to be small if we start with small data. The last term is linear, but turns out to be lower order. Now, using regularity estimates for the heat equation yields for any p,

$$\|\mathcal{Q}w\|_{L^\infty(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \nu\|\mathcal{Q}w\|_{L^1(\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \|\mathcal{Q}w_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \nu^{-1}\|\mathcal{Q}u\|_{L^1(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \text{quadratic}.$$

The term involving $\mathcal{Q}u$ has not the right scaling. It has two more derivatives, hence it is good in high frequencies: if we put the threshold between low and high frequencies at j_0 s.t. $1 \ll 2^{j_0}\nu$ then

$$\nu^{-1}\|\mathcal{Q}u\|^h_{L^1(\dot B^{\frac{d}{p}-1}_{p,1})} \le \nu^{-1}2^{-2j_0}\|\mathcal{Q}u\|^h_{L^1(\dot B^{\frac{d}{p}+1}_{p,1})} \ll \nu\|\mathcal{Q}u\|^h_{L^1(\dot B^{\frac{d}{p}+1}_{p,1})}. \tag{28}$$

Hence, because $\mathcal{Q}u = \mathcal{Q}w - \nu^{-1}(-\Delta)^{-1}\nabla(G(a))$,

$$\|\mathcal{Q}w\|^h_{L^\infty(\dot B^{\frac{d}{p}-1}_{p,1})} + \nu\|\mathcal{Q}w\|^h_{L^1(\dot B^{\frac{d}{p}+1}_{p,1})} \lesssim \|\mathcal{Q}w_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} + \nu^{-2}\|G(a)\|^h_{L^1(\dot B^{\frac{d}{p}-2}_{p,1})} + \text{small},$$

and arguing as in (28), we see that the term involving $G(a)$ is very small compared to $\|G(a)\|^h_{L^1(\dot B^{\frac{d}{p}}_{p,1})}$.

Step 2. Parabolic estimates for $\mathcal{P}u$. Because

$$\partial_t \mathcal{P}u + \mathcal{P}(u\cdot\nabla u) - \mu\mathcal{P}u = -\mathcal{P}(J(a)\mathcal{A}u),$$

we readily have

$$\|\mathcal{P}u\|_{L^\infty(\dot B^{\frac{d}{p}-1}_{p,1})} + \nu\|\mathcal{P}u\|_{L^1(\dot B^{\frac{d}{p}+1}_{p,1})} \lesssim \|\mathcal{P}u_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}} + \text{quadratic}.$$

Step 3. Decay estimates for a.

We notice that

$$\partial_t a + u\cdot\nabla a + G'(0)a = -\operatorname{div}\mathcal{Q}w - (\operatorname{div} u + \nu^{-1}(G'(a) - G'(0)))a.$$

Given that $G'(0) = 1 > 0$, we deduce that

$$\|a\|^h_{L^\infty_t\cap L^1_t(\dot B^{\frac{d}{p}}_{p,1})} \lesssim \|a_0\|^h_{\dot B^{\frac{d}{p}}_{p,1}} + \|\operatorname{div}\mathcal{Q}w\|^h_{L^1(\dot B^{\frac{d}{p}}_{p,1})} + \text{quadratic}. \tag{29}$$

Recall that

$$\|\mathcal{Q}w\|^h_{L^\infty(\dot B^{\frac{d}{p}-1}_{p,1})} + \nu\|\mathcal{Q}w\|^h_{L^1(\dot B^{\frac{d}{p}+1}_{p,1})} \lesssim \|\mathcal{Q}w_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} + (\nu 2^{j_0})^{-2}\|a\|^h_{L^1(\dot B^{\frac{d}{p}}_{p,1})} + \text{small}. \tag{30}$$

Hence plugging (29) in (30) and taking j_0 large enough, we deduce that

$$\|\mathcal{Q}w\|^h_{L^\infty(\dot B^{\frac{d}{p}-1}_{p,1})} + \nu\|\mathcal{Q}w\|^h_{L^1(\dot B^{\frac{d}{p}+1}_{p,1})} + \|a\|^h_{L^\infty_t\cap L^1_t(\dot B^{\frac{d}{p}}_{p,1})} \lesssim \|\mathcal{Q}w_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} + \|a_0\|^h_{\dot B^{\frac{d}{p}}_{p,1}} + \text{quadratic},$$

Of course, as $\mathcal{Q}u = \mathcal{Q}w - \nu^{-1}(-\Delta)^{-1}\nabla g$, one may replace $\mathcal{Q}w$ by $\mathcal{Q}u$ in the above inequality.

Step 4. Low frequency estimates.

As explained before, we have to restrict to Besov spaces $\dot B^s_{2,1}$. By taking advantage of the previous energy method, we get:

$$\|(a,u)\|^\ell_{L^\infty_t(\dot B^{\frac{d}{2}-1}_{2,1})} + \|(a,u)\|^\ell_{L^1_t(\dot B^{\frac{d}{2}+1}_{2,1})} \lesssim \|(a_0,u_0)\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}} + \text{quadratic}.$$

Step 5. Putting everything together. Let

$$X(t) := \|(a,u)\|^\ell_{L^\infty_t(\dot B^{\frac{d}{2}-1}_{2,1})\cap L^1_t(\dot B^{\frac{d}{2}+1}_{2,1})} + \|a\|^h_{L^\infty_t\cap L^1_t(\dot B^{\frac{d}{p}}_{p,1})} + \|u\|^h_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})\cap L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}.$$

We use the previous steps and split all the nonlinear terms into low and high frequencies so has to bound them by $X^2(t)$. At this point, $p < 2d$ and $p \leq 4$ is needed. Then we eventually get

$$X \leq C\big(X(0) + X^2\big). \tag{31}$$

Now it is clear that as long as

$$2CX(t) \leq 1, \tag{32}$$

the Inequality (31) ensures that

$$X(t) \leq 2CX(0). \tag{33}$$

Using a bootstrap argument, one may conclude that if $X(0)$ is small enough then (32) is satisfied as long as the solution exists. Hence the (33) holds globally in time.

Let us state the result that we proved.

Theorem 4.3. *Let $p \in [2, 2d[\cap[2,4]$. Assume that $P'(1) > 0$, $a_0 \in \dot B^{\frac dp}_{p,1}$ and $u_0 \in \dot B^{\frac dp-1}_{p,1}$ and that in addition a_0^ℓ and u_0^ℓ are in $\dot B^{\frac d2-1}_{2,1}$. There exist two constants c and M depending only on d, and on the parameters of the system such that if*

$$\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}} + \|a_0\|^h_{\dot B^{\frac dp}_{p,1}} + \|u_0\|^h_{\dot B^{\frac dp-1}_{p,1}} \leq c$$

then (NSC) has a unique global-in-time solution (a,u) with

$$(a,u)^\ell \in \mathcal C_b(\dot B^{\frac d2-1}_{2,1}) \cap L^1(\dot B^{\frac d2+1}_{2,1}), \quad a^h \in \mathcal C_b(\dot B^{\frac dp}_{p,1}) \cap L^1(\dot B^{\frac dp}_{p,1}),$$
$$u^h \in \mathcal C_b(\dot B^{\frac dp-1}_{p,1}) \cap L^1(\dot B^{\frac dp+1}_{p,1}).$$

- The above statement has been first proved independently in a joint work with F. Charve [10] and by Q. Chen, C. Miao and Z. Zhang [12] in 2009.
- In these notes, we used Haspot's method [29].
- Uniqueness for the full range $p < 2d$ follows from Lagrangian approach [21].
- The smallness condition is satisfied for small densities and large highly oscillating velocities: take $u_0^\varepsilon : x \mapsto \phi(x)\sin(\varepsilon^{-1}x\cdot\omega)\,n$ with ω and n in $\mathbb S^{d-1}$ and $\phi \in \mathcal S(\mathbb R^d)$. Then
$$\|u_0^\varepsilon\|_{\dot B^{\frac dp-1}_{p,1}} \leq C\varepsilon^{1-\frac dp} \quad \text{if} \ \ p > d.$$
Hence such data with small enough ε generate global unique solutions.

5. On the incompressible limit

We now want to study the convergence of the barotropic Navier-Stokes equations when the Mach number ε tends to 0.

Given that the Mach number is the ratio of the typical velocity over the sound speed, in the small Mach number regime, we expect the relevant time scale to be $1/\varepsilon$. Therefore it is natural to set

$$(\rho, u)(t,x) = (\rho^\varepsilon, \varepsilon u^\varepsilon)(\varepsilon t, x).$$

With these new variables, the original system (NSC) recasts in

$$\begin{cases} \partial_t\rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu\Delta u^\varepsilon - (\lambda+\mu)\nabla\mathrm{div}\, u^\varepsilon + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} = 0. \end{cases}$$

In the case of *well-prepared* data:

$$\rho_0^\varepsilon = 1 + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad u_0^\varepsilon \;\; \text{with} \;\; \mathrm{div}\, u_0^\varepsilon = \mathcal{O}(\varepsilon),$$

the time derivatives may be bounded independently of ε for ε going to 0. Hence no acoustic waves have to be taken into account and one may prove that the solution tends to the incompressible Navier-Stokes equations when ε goes to 0 by a standard approach. Besides, asymptotic expansions may be derived if one has more information on the asymptotic expansions of the data.

Here, we shall rather consider *ill-prepared* data, namely

$$\rho_0^\varepsilon = 1 + \varepsilon b_0^\varepsilon \;\; \text{and} \;\; u_0^\varepsilon.$$

Initially, for such data, the time derivative of the solution is of order ε^{-1} and highly oscillating acoustic waves do have to be considered. Whether they may interact or not is the main problem. This is the question that we want to address now in the whole space framework.

To simplify, we take $(b_0^\varepsilon, u_0^\varepsilon) = (b_0, u_0)$ independent of ε. Note that it is not assumed that $\mathrm{div}\, u_0 = 0$. We still assume that $P'(1) = 1$.

Denoting $\rho^\varepsilon = 1 + \varepsilon b^\varepsilon$, it is found that $(b^\varepsilon, u^\varepsilon)$ satisfies

$$(NSC_\varepsilon) \qquad \begin{cases} \partial_t b^\varepsilon + \dfrac{\mathrm{div}\, u^\varepsilon}{\varepsilon} = -\mathrm{div}(b^\varepsilon u^\varepsilon), \\ \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \dfrac{\mathcal{A}u^\varepsilon}{1+\varepsilon b^\varepsilon} + (1+k(\varepsilon b^\varepsilon))\dfrac{\nabla b^\varepsilon}{\varepsilon} = 0, \\ (b^\varepsilon, u^\varepsilon)_{|t=0} = (b_0, u_0), \end{cases}$$

with $\mathcal{A} := \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$ and k a smooth function satisfying $k(0) = 0$.

According to the previous parts, System (NSC_ε) is locally well-posed for all small enough $\varepsilon > 0$. We want to study whether:

(1) we have $\liminf_{\varepsilon\to 0} T_\varepsilon \geq T$ where T_ε stands for the lifespan of $(b^\varepsilon, u^\varepsilon)$ and T stands for the lifespan of the solution v to the incompressible Navier-Stokes equation:

$$(NS) \qquad \begin{cases} \partial_t v + \mathcal{P}(v\cdot\nabla v) - \mu\Delta v = 0, \\ v_{|t=0} = \mathcal{P}u_0. \end{cases}$$

(2) $T_\varepsilon = +\infty$ for small ε if $T = +\infty$,

(3) u^ε tends to v and b^ε converges to 0.

To simplify the presentation, in these notes, we only consider the case where the data are so small that the solution to (NSC_ε) as well as that of (NS) are global. The reader may find results concerning large data in [14].

5.1. **Back to the linearized equations.** With the above scaling, the linearized compressible Navier-Stokes equations in terms of $(b^\varepsilon, u^\varepsilon)$ read

$$\begin{cases} \partial_t b^\varepsilon + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = -\mathrm{div}(b^\varepsilon u^\varepsilon), \\ \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \dfrac{\mathcal{A}u^\varepsilon}{1+\varepsilon b^\varepsilon} + (1+k(\varepsilon b^\varepsilon))\dfrac{\nabla b^\varepsilon}{\varepsilon} = 0 \end{cases}$$

and the linearized equations about $(0,0)$ are

$$\begin{cases} \partial_t b^\varepsilon + \dfrac{\mathrm{div}\,\mathcal{Q}u^\varepsilon}{\varepsilon} = 0, \\ \partial_t \mathcal{Q}u^\varepsilon - \nu\Delta\mathcal{Q}u^\varepsilon + \dfrac{\nabla b^\varepsilon}{\varepsilon} = 0, \\ \partial_t \mathcal{P}u^\varepsilon - \mu\Delta\mathcal{P}u^\varepsilon = 0. \end{cases}$$

As pointed out in the previous section, the last equation is the heat equation whereas denoting $v^\varepsilon := |D|^{-1}\mathrm{div}\,\mathcal{Q}u^\varepsilon$, the first two equations are equivalent to

$$(BM_\varepsilon) \qquad \begin{cases} \partial_t b^\varepsilon + \dfrac{|D|v^\varepsilon}{\varepsilon} = 0, \\ \partial_t v^\varepsilon - \nu\Delta v^\varepsilon - \dfrac{|D|v^\varepsilon}{\varepsilon} = 0. \end{cases}$$

This latter system may be solved explicitly by using the Fourier transform:

$$\frac{d}{dt}\begin{pmatrix} \widehat{b^\varepsilon}(\xi) \\ \widehat{v^\varepsilon}(\xi) \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon^{-1}|\xi| \\ \varepsilon^{-1}|\xi| & -\nu|\xi|^2 \end{pmatrix}\begin{pmatrix} \widehat{b^\varepsilon}(\xi) \\ \widehat{v^\varepsilon}(\xi) \end{pmatrix}.$$

As in the previous section, we discover that there are two regimes: in the high frequency regime $\nu\varepsilon|\xi| > 2$, the eigenvalues read

$$\lambda^\pm(\xi) = -\frac{\nu|\xi|^2}{2}\left(1 \pm \sqrt{1 - \frac{4}{\varepsilon^2\nu^2|\xi|^2}}\right)$$

whereas in the low frequency regime $\nu\varepsilon|\xi| < 2$, one has

$$\lambda^\pm(\xi) = -\frac{\nu|\xi|^2}{2}\left(1 \pm i\sqrt{\frac{4}{\varepsilon^2\nu^2|\xi|^2} - 1}\right).$$

Therefore

$$\lambda^+(\xi) \sim -\nu|\xi|^2 \quad \text{and} \quad \lambda^-(\xi) \sim -\frac{1}{\varepsilon^2\nu} \quad \text{for} \quad \xi \to \infty,$$

and

$$\lambda \pm (\xi) \sim -\nu\frac{|\xi|^2}{2} \mp i\frac{|\xi|}{\varepsilon} \quad \text{for} \quad \xi \to 0.$$

Hence, in high frequency we expect to have

- one parabolic mode with diffusion ν;
- one damped mode with coefficient $\frac{1}{\varepsilon^2\nu}$.

whereas, in low frequency (BM_ε) should behave like

$$\frac{d}{dt}z - \frac{\nu}{2}\Delta z \mp i\frac{|D|}{\varepsilon}z = 0.$$

The important fact is that the low frequency regime tends to invade the whole $\mathbb{R}^d$ when $\varepsilon \to 0$ as the threshold between the two regimes is at $|\xi| = 2(\nu\varepsilon)^{-1}$. Hence it has to be studied with more care than in the previous section. In $\mathbb{R}^d$, taking advantage of the large imaginary part of the eigenvalues for low frequencies turns out to be the key to proving convergence for ε tending to 0 as it supplies dispersion. Note that as our global existence theorem was based on L^2 type estimates as far as low frequencies were concerned, the imaginary part of the eigenvalues was not used so far.

5.2. **About dispersion.** In the whole space, the following Strichartz estimates are available for the acoustic wave system:

Proposition 5.1. *Let* $(b^\varepsilon, v^\varepsilon)$ *solve* $\begin{cases} \partial_t b^\varepsilon + \varepsilon^{-1}|D|v^\varepsilon = F, \\ \partial_t v^\varepsilon - \varepsilon^{-1}|D|b^\varepsilon = G. \end{cases}$

Then we have the inequality

$$\|(b^\varepsilon, v^\varepsilon)\|_{\widetilde{L}^r_T(\dot{B}^{s+d(\frac{1}{p}-\frac{1}{2})+\frac{1}{r}}_{p,1})} \lesssim \varepsilon^{\frac{1}{r}}\Big(\|(b_0,v_0)\|_{\dot{B}^s_{2,1}} + \|(F,G)\|_{L^1_T(\dot{B}^s_{2,1})}\Big)$$

whenever

$$p \geq 2, \quad \frac{2}{r} \leq \min\bigg(1, (d-1)\Big(\frac{1}{2} - \frac{1}{p}\Big)\bigg) \quad \textit{and} \quad (r,p,d) \neq (2,\infty,3).$$

Proof. Performing a suitable change of variables, one may assume that $\varepsilon = 1$ with no loss of generality. Now, as pointed out in Subsection 2.5, the acoustic wave equation generates a unitary group on $L^2(\mathbb{R}^d)$ which satisfies the dispersion inequality with $\sigma = (d-1)/2$. Therefore, localizing the system by means of $\dot{\Delta}_0$ and using Strichartz estimates, we see that in the case $\varepsilon = 1$,

$$\|(\dot{\Delta}_0 b, \dot{\Delta}_0 v)\|_{L^r_T(L^p)} \lesssim \|(\dot{\Delta}_0 b_0, \dot{\Delta}_0 v_0)\|_{L^2} + \|(\dot{\Delta}_0 F, \dot{\Delta}_0 G)\|_{L^1_T(L^2)}.$$

In fact, a simple scaling argument combined with this inequality gives also for all $j \in \mathbb{Z}$,

$$2^{j(d(\frac{1}{p}-\frac{1}{2})+\frac{1}{r})}\|(\dot{\Delta}_j b, \dot{\Delta}_j v)\|_{L^r_T(L^p)} \lesssim \|(\dot{\Delta}_j b, \dot{\Delta}_j v)\|_{L^2} + \|(\dot{\Delta}_j F, \dot{\Delta}_j G)\|_{L^1_T(L^2)},$$

from which one easily deduces the wanted inequality. □

The fundamental fact that we shall use for proving convergence is that the above statement implies that, compared to Sobolev embedding, dispersion gives a gain of $1/r$ derivative and the small factor $\varepsilon^{\frac{1}{r}}$. For instance, if the dimension is $d \geq 4$ then one may take $p = \infty$ and $r = 2$ so that, by virtue of functional embedding, one gets

$$\|(b^\varepsilon, v^\varepsilon)\|_{L^2_T(L^\infty)} \lesssim \varepsilon^{\frac{1}{2}}\Big(\|(b_0,v_0)\|_{\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1}} + \|(F,G)\|_{L^1_T(\dot{B}^{\frac{d}{2}-\frac{1}{2}}_{2,1})}\Big).$$

Similar gains may be obtained in dimension $d = 2, 3$. They are slightly more complicated to state, though.

5.3. **A global convergence statement for small critical data.** Let us now state our result of convergence for global small solutions.

Theorem 5.1. *There exist two positive constants* η *and* M *depending only on* d *and* G, *such that if*

$$C_0 := \|b_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1} \cap B^{\frac{d}{2}}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \leq \eta \tag{34}$$

then the following results hold:

(1) *System* (NSC_ε) *has a unique global solution* $(b^\varepsilon, u^\varepsilon)$ *with*

$$\|b^\varepsilon\|_{L^\infty(\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^2(\dot{B}^{\frac{d}{2}}_{2,1})} + \varepsilon\|b^\varepsilon\|_{L^\infty(\dot{B}^{\frac{d}{2}}_{2,1})} + \|u^\varepsilon\|_{L^\infty(\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\dot{B}^{\frac{d}{2}+1}_{2,1})} \leq MC_0$$

(2) *the incompressible Navier-Stokes equations* (NS) *with data* $\mathcal{P}u_0$ *have a unique solution* v *with*

$$\|v\|_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1})}\leq MC_0$$

(3) *for any* $\alpha\in]0,1/2]$ *if* $d\geq 4$, $\alpha\in]0,1/2[$ *if* $d=3$, $\alpha\in]0,1/6]$ *if* $d=2$, $\mathcal{P}u^\varepsilon$ *tends to* v *in* $\mathcal{C}(\mathbb{R}^+;\dot B^{-1-\alpha}_{\infty,1})$ *when* ε *goes to* 0.

(4) $(b^\varepsilon,\mathcal{Q}u^\varepsilon)$ *tends to* 0 *in some space* $L^r(\dot B^\sigma_{p,1})$ *(the value of* r *and* p *depending on the dimension) with an explicit rate of decay.*

Proof. Step 1. Uniform estimates.

Making the change of functions

$$c^\varepsilon(t,x):=\varepsilon b^\varepsilon(\varepsilon^2t,\varepsilon x),\ \ v^\varepsilon(t,x):=\varepsilon u^\varepsilon(\varepsilon^2t,\varepsilon x)$$

we notice that $(b^\varepsilon,u^\varepsilon)$ solves (NSC_ε) if and only if $(c^\varepsilon,v^\varepsilon)$ solves (NSC) with rescaled data $\varepsilon b_0^\varepsilon(\varepsilon\cdot)$, $\varepsilon u_0^\varepsilon(\varepsilon\cdot)$ and h^ε. Hence the global existence theorem for (NSC) (in the L^2 framework) ensures the first part of the theorem. We get a global solution $(b^\varepsilon,u^\varepsilon)$ such that

$$\|b^\varepsilon\|^\ell_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1})}+\varepsilon\|b^\varepsilon\|^h_{L^\infty(\dot B^{\frac d2}_{2,1})}+\varepsilon^{-1}\|b^\varepsilon\|^h_{L^1(\dot B^{\frac d2}_{2,1})}$$
$$+\|u^\varepsilon\|_{L^\infty(\dot B^{\frac d2-1}_{2,1})\cap L^1(\dot B^{\frac d2+1}_{2,1})}\leq M\big(\|b_0\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\varepsilon\|b_0\|^h_{\dot B^{\frac d2}_{2,1}}+\|u_0\|_{\dot B^{\frac d2-1}_{2,1}}\big).$$

Let us emphasize that in the above inequality the barrier between low and high frequencies is at ε^{-1}.

Step 2. Convergence to zero for the compressible modes $(b^\varepsilon,\mathcal{Q}u^\varepsilon)$.

We use the dispersive inequalities for the acoustic wave operator, given that

$$\left\{\begin{array}{l}\partial_tb^\varepsilon+\varepsilon^{-1}\mathrm{div}\,\mathcal{Q}u^\varepsilon=F^\varepsilon,\\ \partial_t\mathcal{Q}u^\varepsilon+\varepsilon^{-1}\nabla b^\varepsilon=G^\varepsilon,\end{array}\right.\tag{35}$$

with $F^\varepsilon:=-\mathrm{div}\,(b^\varepsilon u^\varepsilon)$ and

$$G^\cdot=-\mathcal{Q}\left(u^\varepsilon\cdot\nabla u^\varepsilon+\frac{1}{1+\varepsilon b^\varepsilon}\mathcal{A}u^\varepsilon+\frac{K(\varepsilon b^\varepsilon)\nabla b^\varepsilon}{\varepsilon}\right).$$

Taking $s=d/2-1$,

- $p\in[2(d-1)/(d-3),\infty]$ and $r=2$ if $d\geq 4$,
- $p\in[2,\infty[$ and $r=2p/(p-2)$ if $d=3$,
- $p\in[2,\infty]$ and $r=4p/(p-2)$ if $d=2$,

and using the fact that, according to product laws in Besov spaces and of the uniform estimates of the previous step we have

$$\|(F^\varepsilon,G^\varepsilon)\|_{L^1(\dot B^{\frac d2-1}_{2,1})}\leq CC_0,$$

one may conclude that

$$\|(b^\varepsilon,\mathcal{Q}u^\varepsilon)\|_{\widetilde L^r(\dot B^{\frac dp-\frac32+\frac1r}_{p,1})}\leq CC_0\varepsilon^{\frac1r}.$$

Step 3. Convergence of the incompressible part.

The vector-field $w^\varepsilon := \mathcal{P}u^\varepsilon - v$ satisfies

$$\partial_t w^\varepsilon - \mu\Delta w^\varepsilon = H^\varepsilon, \qquad w^\varepsilon_{|t=0} = 0, \tag{36}$$

with

$$H^\varepsilon := -\mathcal{P}(w^\varepsilon\cdot\nabla v) - \mathcal{P}(u^\varepsilon\cdot\nabla w^\varepsilon) - \mathcal{P}(\mathcal{Q}u^\varepsilon\cdot\nabla v) - \mathcal{P}(u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon) - \mathcal{P}\left(J(\varepsilon b^\varepsilon)\mathcal{A}u^\varepsilon\right).$$

There are three types of (quadratic) terms in H^ε:

- The first two terms are linear in w^ε, and their coefficient is small as u^ε and v are small. Hence one expect them to be negligible.
- Owing to $\mathcal{Q}u^\varepsilon$, the next two terms decay like some power of ε (previous step).
- The last term is small because $J(\varepsilon b^\varepsilon)$ is of order $\varepsilon b^\varepsilon$.

In order to make all this rigorous, one has to use appropriate norms. For instance, in the (nonphysical !) case $d \geq 4$, maximal regularity estimates for the heat equation ensure that

$$\|w^\varepsilon\|_{L^1(\dot B^{\frac dp+\frac12}_{p,1})} + \|w^\varepsilon\|_{L^\infty(\dot B^{\frac dp-\frac32}_{p,1})} \lesssim \|w_0^\varepsilon\|_{\dot B^{\frac dp-\frac32}_{p,1}} + \|H^\varepsilon\|_{L^1(\dot B^{\frac dp-\frac32}_{p,1})} \tag{37}$$

and the above heuristics combined with product laws in Besov spaces leads to

$$\|w^\varepsilon\|_{L^1(\dot B^{\frac dp+\frac12}_{p,1})} + \|w^\varepsilon\|_{L^\infty(\dot B^{\frac dp-\frac32}_{p,1})} \leq CC_0\varepsilon^{\frac12} \quad \text{for all} \ \ p\in[2(d-1)/(d-3),\infty].$$

□

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