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Analytical Study of Compact Mapping in Hilbert Space

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Abstract**-- This paper presents the study of the representation of compact mappings in Hilbert Space . Here we denote the Scalar Product of two elements (x,y) of a (real or complex) Hilbert Space by (x,y). Here it is proved in this paper that the study of compact mappings in Hilbert Space is a consequence of the spectral theory of compact Symmetric operators.**

Keywords**-- Hilbert Space, Compact mapping , Spectral theory of compact Symmetric operators, Orthogonal projection , Riesz Representation, Scalar Product.**

I. Introduction

Hall (1) and Kothe (2,3) are the pioneer worker of the present area . In fact , the present work is the extension of work done by Wong , Yau- Chuen (10) , Srivastava et al . (4), Srivastava et al. (5), Srivastava et al. (6) , Srivastava et al .(7), Kumar et al. (8) and Srivastava et al.(9). In this paper we have studied analytically about compact mapping in Hilbert Space.

Here , we use the following definitions, Notations and Fundamental ideas :

If M and N are subspaces of a Linear space X such that every $x \in X$ *can* be written uniquely as x = y + z where y ϵ M & z ϵN then the direct sum of M and N can also be written X= M ⊕ N where N is called complimentary subspace of M in X and if M ∩ N = {0 }, the decomposition x = y + z is unique.

A given subspace M has many complimentary subspaces and every complimentary subspace of M has the same dimension and the dimension of a complimentary subspace is called co-dimension of M in X , as if $X = R^3$ and M is a plane through the origin then any line through the origin that does not lie in M is a complimentary subspace.

If X = M ⊕ N then we define the projection P: X → X of X on to M along N by Px = y, where x = y+z with y ∈ M , Z ∈N which is Linear with ran P = M and ker P = N satisfying $P^2 = P$.

This property characterizes projections for which the following definitions and theorems follow : -

Definition 1: Any projection associated with a direct sum decomposition of a projection on a Linear space X is a linear map P:X → X such that $P^2 = P$

Definition 2: An orthogonal projection on a Hilbert space H is also a Linear mapping P:H → H satisfying $P^2 = P$, <Px,y> = <x , Py> for all x, y ∈ H.

"An orthogonal projection is necessarily bounded."

Theorem 1 : Let X be a linear space,

(i) If P:X → X is a projection then X = ran P ⊕ kerP

(ii) If X = M ⊕ N where M and N are Linear subspaces of X then there is a projection P:X → X with ran P = M and ker P = N.

Proof:

For (i) We show that x ∈ ran P if x = Px

If x = Px then clearly x ∈ ran P

If x∈ ran P then x = Py for some y ∈ x

And since $P^2 = P$ which follows that $Px = P^2y = Py = x$

If x ∈ ran P ∩ kerP then x = Px & Px = 0

So ran P ∩ kerP = {0}. If x ∈ X then

We have x = Px + (x- Px) ; where Px ∈ ran P and (x – Px) ∈ kerP .

Since $P(x- Px) = Px - P^2x = Px - Px = 0$

Thus X = ran P ⊕ kerP. ………………………(1.1)

Now for (ii)

We consider if X = M⊕ N then x ∈ N has unique decomposition x = y+z with y∈ M & Z ∈ N and Px = y defines the required Projection .

In particular, in orthogonal subspaces while using Hilbert Space, let us suppose that M is a closed subspace of Hilbert Space H then by well known property we have $H = M \oplus M^{\perp}$. We call the projection of H on to M along $M^{\perp}$ the orthogonal projection of H on to M.

If x= y+z and $x_1 = y_1 + z_1$ where y, $y_1 \in$ M and z, $z_1 \in M^{\perp}$ then by orthogonality of M and $M^{\perp} \Rightarrow$ $<Px, x_1> = <y, y_1 + z_1> = <y, y_1> = <y+z, y_1>$

$$= <x, Px_1> \ldots\ldots\ldots\ldots\ldots (1.2)$$

Which states that an orthogonal projection is self Adjoint. We show the properties (1.1) and (1.2) characterize orthogonal projections with Defn-2 .

Lemma :- If P is a non zero orthogonal projection then $\| P\| = 1$.

Proof : - If $x \in H$ and $Px \neq 0$ then by Cauchy Schwarz inequality ,

$$\| Px \| = \frac{< Px, Px>}{\| Px \|} = \frac{<x, P^2x >}{\| Px \|} = \frac{< x, Px >}{\| Px \|} \leq \| x\|$$

Therefore $\| P \| \leq 1$. If $P \neq 0$ then there is an $x \in H$ with $Px \neq 0$ and $\| P(Px) \| = \| Px \|$ so that $\| P\| \geq 1$.

Thus, the Orthogonal Projection P and closed subspace M of H such that ran P = M will must obey one –one correspondence, then the kernel of Orthogonal Projection is the Orthogonal Complement of M .

Example .1 – The space L^2 (R) is the Orthogonal direct sum of space M of even functions and the space N of odd functions .

The Orthogonal Projection P and Q of H onto M and N, respectively are given by

$$Pf (x) = \frac{f (x) + f(-x)}{2} , \quad Q f (x) = \frac{f (x) – f (-x)}{2}$$

Where I- P = Q .

Proposition: (a) A Linear functional on a Complex Hilbert space H is a Linear map from H to C. A Linear functional φ is bounded or continuous, if there exists a constant M such that $| \varphi (x) | \leq M \| x \|$ for all $x \in H$.

The norm of bounded linear functional φ is

$$\| \varphi \| = \sup_{\| x \| = 1} |\varphi (x) |$$

If $y \in H$ then $\varphi_y (x) = < y, x >$ is a bounded Linear functional on H, with

$\| \varphi_y \| = \| y\|$.

(b) If φ is a bounded Linear functional on a Hilbert space H, then there is a unique vector $y \in H$ such that

$\varphi (x) = < y, x>$ for all $x \in H$

Theorem.2 : (Riesz representation) If φ is a bounded linear functional on a Hilbert space H , then there is a unique vector $y \in H$ such that

$\varphi (x) = < y, x>$ for all $x \in H$. ………….(2.1)

Proof. If $\varphi = 0$, then $y = 0$, so we suppose that $\varphi \neq 0$. In that case , ker φ is a proper closed subspace of H. and , it implies that , there is a nonzero vector

$z \in H$ such that $z \perp \ker\varphi$. We define a linear map P : H →H by

$Px = \varphi (x) /\varphi (z) .z$

Then $P^2 = P$, so Theorem 1 implies that , H = ran P ⊕ kerP. Moreover,

ran P $=\{\alpha z|\alpha \in C\}$, kerP = kerφ

So that ran P ⊥ ker P. It follows that P is an orthogonal projection, and

$H =\{\alpha z|\alpha \in C\} \oplus \ker\varphi$ is an orthogonal direct sum. We can therefore write

$x \in H$ as $x= \alpha z +n$, $\alpha \in C$ and $n \in \ker\varphi$.

Taking the inner product of this decomposition with z, we get

$\alpha = < z, x >/\| z \|^2$, and evaluating φ on $x = \alpha z + n$, we find that

$\varphi (x) = \alpha \varphi (z)$.

The elimination α from these equations , and a rearrangement of the result

yields $\varphi (x) = < y, x >$, where $y = \overline{\varphi (z)}/\| z \|^2.z$

Thus, every bounded linear functional is given by the inner product with a fixed vector.We have already , seen that $\varphi_y (x) = < y, x >$ defines a bounded linear functional on H for every $y \in H$.

EXPLORING RESEARCH AND INNOVATIONS

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To prove that there is a unique y in H associated with a given linear functional, suppose that $\varphi_{y1} = \varphi_{y2}$. Then $\varphi_{y1}(y) = \varphi_{y2}(y)$. When $y= y_1 - y_2$, which implies that $\| y_1 - y_2 \|^2 = 0$, so $y_1 = y_2$.

The Map $J : H \to H^*$ given by $J_y = \varphi_y$, therefore identifies a Hilbert space H with its dual space H*. The norm of φ_y is equal to the norm of y, so j is an isometry . In this case of complex Hilbert spaces , J is antilinear , rather than linear, because $\varphi_{\lambda y} = \lambda \varphi_y$. Thus, Hilbert spaces are self – dual , meaning that H and H* are isomorphic as Banach spaces, and anti-isomorphic as Hilbert spaces. Thus Hilbert spaces are special in this respect. This completes the proof of the Theorem 2.

Proposition : (c) An important consequences of the Riesz representation theorem is the existence of the adjoint of a bounded linear operator on a Hilbert space. The defining property of the adjoint $A^* \in B(H)$ of an operator $A \in B(H)$ is that

$< x, Ay > = <A^*x ,y >$ for all $x, y \in H$ ....(2.2)

The Uniqueness of A* is obvious . The definition implies that

$(A^*)^* = A$, $(AB)^* = B^*A^*$.

To prove that A* exists , we have to show that for every $x \in H$, there is a vector $z \in H$, depending linearly on x such that

$< z, y > = < x, Ay >$ for all $y \in H$ .... (2.3)

For fixed x , the map φ_x defined by, $\varphi_x (y) = < x, Ay >$ is a bounded linear functional on H , with $\|\varphi_x\| \le \|A\| \|x\|$. By the Riesz representation Theorem, there is a unique $z \in H$ such that $\varphi_x (y) = < z, y >$. This z satisfies (2.3), So we get $A^*x = z$. The linearity of A* follows from the uniqueness in the Riesz representation theorem and the linearity of the inner product.

Thus, from above definitions , Theorems, Leema , example , Propositions (a), (b),& (C), which Shows the proof of the main result as "the representation of compact mappings of Hilbert Spaces is a Consequence of the Spectral theory of Compact symmetric operators.

1) Let H_1, H_2 be Hilbert spaces, $A \in £ (H_1,H_2)$ compact and not of finite rank . Then, there exists orthonormal systems , e_n , n = 1,2,.......... In H_1 and $\{f_n\}$, n= 1,2,........ in H_2 such that

2) $A x = \sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n$, $x \in H_1$ where $\lambda_n > 0$ and $\lambda_n \to 0$.

Proof :- Since A is Compact , A*A is Compact too and positive , where A* denotes the adjoint in the sense of the scalar product . It follows from Spectral theory that there exists an orthonormal sequence of eigen vectors e_n , n= 1,2,3,........ and eigen values $\lambda_n^2 > 0$, $\lambda_n^2 \to 0$ such that

$$A^* A_x = \sum_{n=1}^{\infty} \lambda_n^2 (x, e_n) e_n ,$$

A*A is zero on the orthonormal or complement H of the closed subspace spanned by all the e_n . But then A is zero too on H.

Take $y \in H$ and suppose $A_y \neq 0$.

Then $(A_y , A_y) = (y, A^*A_y) \neq 0$. But this would imply $A^*A_y \neq 0$, Therefore we have a representation

$$A_x = \sum_{n=1}^{\infty} \lambda_n (x, e_n) A e_n.$$

We now define

$f_n = (1/ \lambda_n)A e_n$. Then

$$A_x = \sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n$$

and other proposition will be proved if we Show that $\{ f_n \}$ is an orthonormal systems.

$$\begin{aligned} \text{But } (f_i ,f_k) &= (\lambda_i^{-1} Ae_i , \lambda_k^{-1} A e_k) \\ &= \lambda_i^{-1} \lambda_k^{-1}(A^*Ae_i,e_k) \\ &= \lambda_i^{-1} \lambda_k^{-1} (\lambda_i^2 e_i, e_k) \\ &= \delta_{ik} \end{aligned}$$

3) Conversely every mapping $A \in £(H_1,H_2)$ which has a representation (2) with

$\lambda_n > 0$, $\lambda_n \to 0$ is compact.

Let A_k be $\sum_{n=1}^{K} \lambda_n (x, e_n)f_n$, $\|(A - A_n)x \|^2$

$$\le \sum_{n=k+1}^{\infty} \lambda_n^2 | (x , e_n) |^2$$

$\le \epsilon^2 \| x \|^2$, it is $| \lambda_n | \le \epsilon$ for $n > k(\epsilon)$.

Thus A is compact as the limit of A_n in $£_b$ (H_1 , H_2) . From this proof and (1) follow immediately.

4) Let H_1, H_2 be Hilbert Spaces. Then every compact A ϵ $£_b$ (H_1, H_2) is the limit of a sequence of mappings of finite rank.

Then λ_n of (2) are called the singular values of A and the non- increasing sequence of all singular values of A is uniquely determined by A , the representation (2) can be written in a different way using linear forms instead of scalar product for the coefficients of the f_n .

The scalar product (x,y) in Hilbert space H is linear in x for y fixed , thus it defines a linear functional, $< \bar{y} , x > =$ (x,y), where $\bar{y}$ is uniqually determined . One calls $\bar{y}$ the Conjugate element to y. There exists an Orthonormal basis $\{e_\alpha\}$, α ϵ A , of H such that

For $x = \sum_\alpha \xi_\alpha e_\alpha$, $y = \sum_\alpha \eta_\alpha e_\alpha$

$$(x,y) = \sum_\alpha \xi_\alpha \bar{\eta}_\alpha = < \bar{y} , x >$$

Since this is true for all x ϵ H , if follows that $\bar{y} = \sum_\alpha \bar{\eta}_\alpha e_\alpha$:

The coefficients of $\bar{y}$ are the Conjugate of the coefficients of y.

Hence the Result.

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