Dottorato di Ricerca in Matematica
XI ciclo

Tesi per il conseguimento del titolo

SOME TOPICS IN FLUID MECHANICS

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Abstract

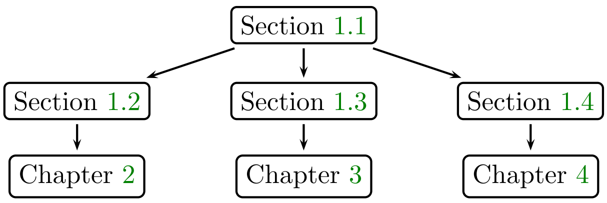
Fluid-mechanics is an “ancient science” that is incredibly alive today. The modern technologies require a deeper understanding of the behavior of real fluids; on the other hand new discoveries often pose new challenging mathematical problems.

In this framework a special role is played by *incompressible viscous* flows. The study of these flows has been attached with a wide range of mathematical techniques and, today, this is a stimulating part of both pure an applied mathematics.

The aim of this thesis is to furnish some results in very different areas, that are linked by the common scope of giving new insight in the field of fluid mechanics. Since the arguments treated are various, an extensive bibliography has been added. For the sake of completeness, there is an introductory chapter and each subsequent new topic is illustrated with the will of a self-contained exposition.

The reader’s background is a good understanding of the classical arguments of functional analysis and partial differential equations. In particular, it is needed a knowledge of the Sobolev spaces and of the variational formulation of linear elliptic and parabolic problems. The reader can find in the book by Dautray and Lions (the second of the series cited in the bibliography) almost all the required background material.

The first chapter is a reasonable introduction to few aspects of the mathematical theory of fluid mechanics. In the first Section 1.1 we introduce the Navier-Stokes equations, while in the other three sections of Chapter 1 we introduce the contents of the other three chapters, respectively. In each of the following Chapters 2,3, and 4 it is introduced a particular topic of fluid mechanics and some *original* results are given. The thesis can be read by following the natural order of the chapters, but also along the following paths:



We now describe the contents with more detail. In Chapter 1 the equations of motion of ideal and viscous fluids are derived. Then the weak formulation of the Navier-Stokes equations is introduced, together with some existence results. Some concepts regarding the long-time behavior are presented and finally, the basic concepts and results regarding the finite element method and the numerical approximation of the Navier-Stokes equations are given.

In Chapter 2 it is introduced the problem of the regularity and of the possible global existence of smooth solutions in the three dimensional case. Particular emphasis is given to the role of weak and strong solutions. The classical Prodi-Serrin condition is then introduced, because it is one of the best-known conditions which ensure the regularity of the solutions. Furthermore, the role of the pressure is discussed together with some regularity results for the Navier-Stokes equations. Finally, in Section 2.4.1 some *new* results regarding the possible regularizing effect of the pressure are given.

In Chapter 3 some results regarding the long-time behavior of solutions to the 2D Navier-Stokes equations are presented. Furthermore, the basic theory for stochastic partial differential equations is briefly recalled, together with the heuristic explanation of the role of random solutions in the theory of Navier-Stokes equations. Finally, in Section 3.3.6 and 3.3.7 some *new* results regarding the long-time behavior of the solution to the Stochastic Navier-Stokes equations are given.

In the last Chapter 4 a particular numerical strategy is proposed: the *domain decomposition method.* The techniques of domain decomposition are very interesting because they allow to split the computational effort into parallel steps and, consequently, they can use computational capabilities offered by *parallel computers.* A rather detailed introduction of the known results for the Poisson equation is given in the Section 4.2. Then, motivated by the study of non-symmetric problems (as the ones arising in the discretization of the Navier-Stokes equations), in Section 4.3.2 and 4.4 some *new* results regarding two classes of non-symmetric problems are presented. In particular, optimal convergence results for some iterative methods for the advection-diffusion equations are given. It is also proved a result concerning the time-harmonic Maxwell equations, which, though they have a different structure, can be studied with a similar approach. We emphasize that our interest for the advection-diffusion equations is due to the fact that they are a model-problem for transport equations and their solution gives also one of the main "computational kernels" of the *computational fluid dynamics.*

Acknowledgments

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Contents

Chapter 1

Navier-Stokes equations

The aim of this chapter is to present the Navier-Stokes equations, that are the equations governing the motion of viscous fluids. We briefly derive the Navier-Stokes equations and then we recall some classical results regarding different approaches to their study.

1.1 Derivation of the equations

In this section we follow essentially the book by Chorin and Marsden [CM93] and we explain the main features arising in the study of fluid-mechanics. We recall that the study of fluid-mechanics is one of the most challenging fields for mathematicians and also for physicists. In the preface of the classical book by Landau and Lifshitz [LL59] fluid-mechanics is intended to be a branch of theoretical physics, nevertheless difficult and still unsolved problems arise in the study of analytical, statistical and numerical properties of the solutions the equations of fluids.

1.1.1 Euler equations

We start with some basic facts of continuum mechanics. Let $D \subset \mathbb{R}^d$ be a region in the two ($d = 2$) or three ($d = 3$) dimensional space, filled with a fluid. Imagine a particle in the fluid and let $\mathbf{u}(\mathbf{x}, t) = (u_1, \dots, u_d)(\mathbf{x}, t)$ be a vector, depending on the space-time variable $(\mathbf{x}, t) = (x_1, \dots, x_d, t)$, denoting the velocity of a particle of fluid that is moving through $\mathbf{x}$ at time t. For each time t we assume that the fluid has a well-defined mass density $\rho(\mathbf{x}, t)$. Thus if W is any subregion of D, the mass fluid in W at time t is given by $m(W, t) := \int_W \rho(\mathbf{x}, t)\, d\mathbf{x}$, where $d\mathbf{x}$ is the volume element in the plane or in the space. The assumption that ρ exists is a *continuum assumption.* The derivation of the equations is based on three basic physical principles: 1) mass is never created or destroyed; 2) the rate of change of momentum of a portion of the fluid equals the force applied to it; 3) energy is neither created or destroyed.

Conservation of mass

As consequence of conservation of mass we have that

$$0 = \frac{d}{dt} m(W, t) = \int_W \frac{\partial \rho}{\partial t}\, d\mathbf{x}.$$

Let $\mathbf{n}$ denote the outward normal defined at points of ∂W and let dS denote the area (or $(d-1)$-surface) element on ∂W. Since the volume flow rate across ∂W per unit area is $\mathbf{u} \cdot \mathbf{n}$, the principle of conservation of mass can be stated as

$$\frac{d}{dt}\int_W \rho\, d\mathbf{x} = -\int_{\partial W} \rho\, \mathbf{u}\cdot\mathbf{n}\, dS.$$

By using the divergence theorem the last equation becomes

$$\int_W \left[\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\,\mathbf{u})\right] d\mathbf{x} = 0, \quad \text{where} \quad \mathrm{div}\,\mathbf{u} := \frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial u_d}{\partial x_d}.$$

Since the previous equality holds for all W it is equivalent to the following one

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\,\mathbf{u}) = 0,$$

called the "continuity equation."

Conservation of momentum

To define an ideal fluid we split the different forces acting on a piece of material into two classes: the stress forces (when a piece of material is acted on by forces across its surface, by the rest of the continuum) and the body forces (forces which exert a force per unit of volume as the gravity or an electro-magnetic field).

Definition 1.1.1. *We say that a continuum is an ideal fluid if for any motion of the fluid there is a function $p(\mathbf{x},t)$, called "pressure", such that if S is a surface in the fluid, with a chosen unit normal $\mathbf{n}$, the force of stress exerted across the surface S per unit area, at $\mathbf{x}\in S$ and at time t, is $p(\mathbf{x},t)\,\mathbf{n}$.*

Many papers have been written on the hypotheses underlying this definition. We do not enter into details, but we only remark that the absence of tangential forces implies that there is no way for rotation to start or to stop. In other words if $\mathbf{curl}\,\mathbf{u}$ vanishes at time $t = 0$, it must be identically zero for every time. We recall that in three dimensions the vorticity field $\mathbf{curl}\,\mathbf{u}$ is:

$$\mathbf{curl}\,\mathbf{u}(\mathbf{x},t) := \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)(\mathbf{x},t).$$

If W is a region in the fluid, at time t the total force exerted on the fluid inside W by the stresses is

$$S_{\partial W} = \big\{\text{force on } W\big\} = -\int_{\partial W} p\,\mathbf{n}\, dS. \tag{1.1}$$

We now impose the conservation of momentum: let us write $\boldsymbol{\phi}(\mathbf{x},t)$ (fluid flow map) for the trajectory followed by the particle that is at point $\mathbf{x}$ at time $t = 0$. We assume $\boldsymbol{\phi}$ to be smooth enough and we denote the map at fixed time t by $\boldsymbol{\phi}_t(\mathbf{x},t) : \mathbf{x} \to \boldsymbol{\phi}(\mathbf{x},t)$. We denote by W_t the *moving with the fluid region*, where $W_t := \boldsymbol{\phi}_t(W)$. If it is given a body force per unit mass $\mathbf{b}(\mathbf{x},t)$, the balance of momentum reads as

$$\frac{d}{dt}\int_{W_t} \rho\,\mathbf{u}\, d\mathbf{x} = S_{\partial W_t} + \int_{W_t} \rho\,\mathbf{b}\, d\mathbf{x}.$$

If $J(\mathbf{x},t)$ denotes the Jacobian of $\boldsymbol{\phi}_t$, with straightforward calculations we get the following formula

$$\frac{\partial}{\partial t}\boldsymbol{\phi}(\mathbf{x},t) = J(\mathbf{x},t)\,\mathrm{div}\,\mathbf{u}(\boldsymbol{\phi}(\mathbf{x},t),t). \tag{1.2}$$

This formula is interesting because it shows that a fluid is *incompressible* (*i.e.*, $\int_{W_t} d\mathbf{x}$ is constant in t) if and only if $J \equiv 1$ or if and only if $\operatorname{div} \mathbf{u} = 0$. By a change of variables and by using formula (1.2) we get (recall that $\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$)

$$\frac{d}{dt}\int_{W_t} \rho\, \mathbf{u}\, d\mathbf{x} = \int_W \left[\frac{D}{Dt}(\rho\, \mathbf{u}) + \operatorname{div} \mathbf{u}(\rho\, \mathbf{u})\right] \phi\, J\, d\mathbf{x} = \int_{W_t} \left[\frac{D}{Dt}(\rho\, \mathbf{u}) + (\rho \operatorname{div} \mathbf{u})\, \mathbf{u}\right] d\mathbf{x}.$$

and by using the conservation of mass we obtain

$$\frac{d}{dt}\int_{W_t} \rho\, \mathbf{u}\, d\mathbf{x} = \int_{W_t} \rho\, \frac{D\mathbf{u}}{Dt}\, d\mathbf{x}.$$

After reasoning on the integral formulation (since the last equation holds for each $W \subset D$) we establish the differential form of the conservation of momentum[1]

$$\rho\, \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\, \mathbf{b}.$$

Conservation of energy

We have developed $d+1$ equations with the $d+2$ unknowns ρ, p, and $\mathbf{u}$. Consequently we need another equation to avoid an over-determined problem. We suppose, as usual, that all the energy is the kinetic ($\mathrm{E_{kin}}$) one, and that the rate of change of the kinetic energy in a portion of fluid equals the rate at with the pressure and body forces work:

$$\frac{d}{dt}\mathrm{E_{kin}}(t) = \frac{d}{dt}\frac{1}{2}\int_{W_t} \rho(\mathbf{x},t)|\mathbf{u}(\mathbf{x},t)|^2\, d\mathbf{x} = -\int_{\partial W_t} p\, \mathbf{u}\cdot \mathbf{n}\, dS + \int_{W_t} \mathbf{u}\cdot \mathbf{b}\, d\mathbf{x}.$$

The application of the divergence theorem and of the formulas obtained before shows that necessarily $\operatorname{div} \mathbf{u} = 0$. The *Euler equations* for a fluid filling D, derived firstly by Euler [Eul1755], are finally

$$\begin{cases} \rho \dfrac{D\mathbf{u}}{Dt} = -\nabla p + \rho\, \mathbf{b}. \\[2ex] \dfrac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0. \\[2ex] \operatorname{div} \mathbf{u} = 0. \end{cases}$$

When the previous system is equipped with the tangential boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ on ∂D and initial conditions on ρ and $\mathbf{u}$, it is (in suitable spaces) a well-posed mathematical problem. For a collection of mathematical results regarding the Euler equations the reader can see the recent book by P.-L. Lions [PLL96].

1.1.2 Navier-Stokes equations

The Euler equations describe an ideal fluid, but if we want to consider a more general fluid we need different equations. The need of generalization comes from simple considerations about the kinetic

[1] The term $\frac{D}{Dt}$ is called the *material derivative*, because it takes into account the fact that the fluid is moving. If we denote by $\mathbf{a}(t)$ the acceleration of a particle we have, by the chain rule, that $\mathbf{a}(t) := \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + u_1 \frac{\partial \mathbf{u}}{\partial x_1} + \cdots + u_d \frac{\partial \mathbf{u}}{\partial x_d} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\, \mathbf{u} := \frac{D\mathbf{u}}{Dt}$.

theory of matter. We do not enter into details, but we simply change our previous assumption[2] (1.1) on the stress forces into the following one:

$$\{\text{force per unit of area}\} = -p(\mathbf{x},t)\,\mathbf{n} + \boldsymbol{\sigma}(\mathbf{x},t)\,\mathbf{n},$$

where $\boldsymbol{\sigma}$ is a matrix which depends only on the velocity gradient $\nabla\mathbf{u} = \partial u_i/\partial x_j$. The matrix $\boldsymbol{\sigma}$ must be symmetric and for physical reasons regarding the invariance (with respect to orthogonal transformations) of the equations we obtain

$$\boldsymbol{\sigma} = \lambda(\operatorname{div}\mathbf{u})\,\mathrm{I} + 2\,\mu\,\mathrm{D}^S,$$

where I is the identity matrix and D^S is the symmetric part of $\nabla\mathbf{u}$. The last equation is generally written by putting all the trace terms in one term

$$\boldsymbol{\sigma} = 2\,\mu\Big[\mathrm{D}^S - \frac{1}{3}(\operatorname{div}\mathbf{u})\Big] + \zeta(\operatorname{div}\mathbf{u})\,\mathrm{I},$$

where μ is the first coefficient of viscosity and $\zeta = \lambda + 2\mu/3$ is the second coefficient.

If we employ the transport theorem and the divergence theorem, as we did before, (we pass also from the integral formulation to the differential one) we get the following equation:

$$\rho\frac{D\mathbf{u}}{Dt} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla p + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}) + \mu\Delta\mathbf{u},$$

where $\Delta\mathbf{u} = \sum_{i=1}^d \partial^2\mathbf{u}/\partial x_i^2$ is the Laplacian of $\mathbf{u}$. We observe that the Laplacian raises the order of derivatives of $\mathbf{u}$ involved. This is accompanied by an increase in the number of boundary conditions: from the tangential condition we must pass to the *no-slip* condition $\mathbf{u} = \mathbf{0}$. The physical need for this boundary conditions comes from simple experiments involving flow past a solid wall. From the mathematical point of view other conditions are suitable, but we shall confine to the *no-slip* one.

Remark 1.1.2. *A crucial feature of the no-slip boundary condition is that it provides a mechanism by which a boundary can produce vorticity in the fluid.*

We are interested to incompressible problems and we mainly deal with homogeneous[3] (*i.e.*, $\rho = \rho_0 = const.$) viscous fluid. To study the scaling properties of the Navier-Stokes equations we must write the equations in a non-dimensional form. We set $\nu = \mu/\rho_0$ and $p^* = p/\rho_0$; for a given problem let L be a *characteristic length* and U a *characteristic velocity*. These numbers are chosen in a somewhat arbitrary way. If we measure $\mathbf{x}, \mathbf{u}$, and t as fraction of these scales we are changing the variables and introducing the following dimensionless quantities

$$\mathbf{u}^* := \frac{\mathbf{u}}{U}, \qquad \mathbf{x}^* := \frac{\mathbf{x}}{L}, \qquad t^* := \frac{t}{T}.$$

By straightforward computations and by suppressing the stars (with an abuse of notation) we obtain the following equations

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p - \frac{1}{R}\Delta\mathbf{u} = 0,$$

[2] The fact that the force acting on S should be a *linear* function of $\mathbf{n}$ is not an assumption, but it derives from balance of momentum. This result is known as Cauchy Theorem. Complete discussion with also historical remarks regarding the constitutive relation for $\boldsymbol{\sigma}$ can be found in Lamb [Lam93]. We recall that the Navier-Stokes equations were introduced by Navier [Nav1822] and, while Stokes studied in [Sto1849] mainly the linearized problem, the equation inherited the names of both.

[3] We recall that the incompressibility conditions implies that, if the density ρ is constant in space, it is also constant in time because $D\,\rho/Dt = 0$.

where the dimensionless quantity $R := LU/\nu$ is the Reynolds number.
With another abuse of notation (to have the equation as they are generally written in the mathematical literature) we write the equations with $\nu = 1/R$. The complete set of incompressible and homogeneous Navier-Stokes equations, driven by an external force $\mathbf{f}$ and with proper boundary and initial condition, is finally

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } D\times[0,T], \tag{1.3}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } D\times[0,T], \tag{1.4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial D\times[0,T], \tag{1.5}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } D\times\{0\}, \tag{1.6}$$

1.2 Main existence theorems

In this section we state some of the basic results regarding the mathematical approach to the Navier-Stokes equations. The main problem regarding the equations of incompressible fluid dynamics are: there exists a solution? Is it unique?

Many mathematicians have faced with this problem and the first satisfactory answer arrived from Leray [Ler33, Ler34a, Ler34b]. He proved the basic existence and uniqueness results by using the techniques of *hydrodynamic potentials*. These results were improved and the proof were simplified by Hopf [Hop51], by using a more functional approach. The role of the *weak solutions* became more and more important, especially after the appearance of the paper by Kiselev and Ladyženskaya [KL57] and the fundamental book by Ladyžhenskaya [Lad69]. We remark that in the same years appeared the extremely complete[4] paper by Berker [Berk63], in which big importance is given to explicit (classical) solutions of the problem, in particular geometric situations.

In this section we outline some basic facts regarding the functional approach to the Navier-Stokes equations. We only state some basic results; for their proof we refer to the book by Constantin and Foiaş [CF88], if no other explicit reference is given.

1.2.1 Function spaces and the Stokes operator

In the sequel we shall use extensively the customary Sobolev spaces $W^{m,p}(D)$ and $H^k(D)$, respectively with norm $\|\,.\,\|_{m,p,D}$ and $\|\,.\,\|_{k,D}$. For the reader not acquainted with these function spaces a classical reference is Adams [Ada75]. Since we deal with evolution equations it is classical to use the Banach spaces $L^p(0,T;B)$. They are the spaces of strongly (Lebesgue) measurable B-valued functions $v : [0,T] \to B$ such that

$$\int_0^T \|v(t)\|_B^p\,dt < \infty \quad \text{if} \quad 1 \le p < \infty \qquad \text{and} \qquad \operatorname*{ess\,sup}_{0<t<T} \|v(t)\|_B < \infty \quad \text{if} \quad p = +\infty.$$

These spaces are Banach spaces endowed with the norms:

$$\|v\|_{p,B} = \left[\int_0^T \|v(t)\|_B^p\,dt\right]^{1/p} \quad \text{if} \quad 1 \le p < \infty \qquad \text{and} \qquad \|v\|_{\infty,B} = \operatorname*{ess\,sup}_{0<t<T} \|v(t)\|_B \quad \text{if} \quad p = +\infty.$$

[4]The author would like to thank Prof. Cimatti for having pointed out this reference.

Remark 1.2.1. *We remark that for any $p \in [1, +\infty]$, $L^p(0, T; B)$ is the set of classes of functions induced by the equivalence relation*

$$u \sim v \quad \text{if and only if} \quad u = v \quad \text{a.e. in } (0, T),$$

and for simplicity we shall speak of functions instead of classes of functions.

In the treatment of the Navier-Stokes equations we shall use some appropriate Hilbert spaces. We define

$$\mathcal{V} := \left\{ \boldsymbol{\phi} \in (C_0^\infty(D))^d : \ \operatorname{div} \boldsymbol{\phi} = 0 \right\}.$$

Let us denote by H and V the closure of $\mathcal{V}$ in $(L^2(D))^d$ and $(H_0^1(D))^d$, respectively. We equip H with the $(L^2(D))^d$-norm denoted by $|\,.\,|$, induced by the usual scalar product $(\mathbf{u}, \mathbf{v}) := \int_D \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x}$. Since we deal essentially with problems in bounded domains, we equip V with the norm $\|\mathbf{u}\|^2 := \int_D |\nabla \mathbf{u}|^2\, d\mathbf{x}$. The norm in V is equivalent to that one in $(H_0^1(D))^d$ (by the Poincaré inequality) and it is induced by the scalar product $((\mathbf{u}, \mathbf{v})) = \int_D \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, d\mathbf{x}$.
We have the following proposition

Proposition 1.2.2 (Helmholtz decomposition)**.** *Let D be open, bounded, connected of class C^2. Then $(L^2(D))^d = H \oplus H_1 \oplus H_2$, where H_1, H_2 are the following mutually orthogonal spaces,*

$$H_1 := \left\{ \mathbf{u} \in (L^2(D))^d : \ \mathbf{u} = \nabla p, \ p \in H^1(D), \ \Delta p = 0 \right\},$$

$$H_2 := \left\{ \mathbf{u} \in (L^2(D))^d : \ \mathbf{u} = \nabla p, \ p \in H_0^1(D) \right\}.$$

The Stokes equations

The Stokes equations for $(\mathbf{u}, p)$ are

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial D. \end{cases}$$

If $(\mathbf{u}, p)$ are smooth then, after multiplying by $\mathbf{v} \in \mathcal{V}$ and by an integration by parts (recall that ∇p belongs to a space orthogonal to H) we obtain $((\mathbf{u}, \mathbf{v})) = (\mathbf{f}, \mathbf{v})$.

Definition 1.2.3. *We say that $\mathbf{u}$ is a weak solution of the Stokes equations if $\mathbf{u} \in V$ and*

$$((\mathbf{u}, \mathbf{v})) = (\mathbf{f}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in V.$$

We have the following proposition which states the role of the *weak solutions.*

Proposition 1.2.4. *Let D be open bounded and of class C^2. Then the following statements are equivalent*

i) $\mathbf{u}$ *is a weak solution of the Stokes equations;*

ii) $\mathbf{u} \in (H_0^1(D))^d$ *and satisfies: there exist $p \in L^2(D)$ such that*

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \textit{in the sense of distributions}, \\ \operatorname{div} \mathbf{u} = 0 & \textit{in the sense of distributions}, \\ \mathbf{u} = \mathbf{0} & \textit{in the trace sense}; \end{cases}$$

iii) $\mathbf{u} \in V$ *achieves the minimum of* $J(\mathbf{v}) := \nu \|\mathbf{v}\|^2 - 2(\mathbf{f}, \mathbf{v})$ *on* V.

By using the Lax-Milgram lemma in the separable Hilbert space V, we have the following theorem:

Theorem 1.2.5. *Let* D *be open and bounded in some direction. Then* $((\,.\,,\,.\,))$ *is a scalar product in* V *and for every* $\mathbf{f} \in (L^2(D))^d$ *there exists a unique weak solution of the Stokes equations.*

We have also the following regularity result, for which we refer[5] to Cattabriga [Cat61].

Theorem 1.2.6. *Let* $D \subset \mathbb{R}^d$ $d = 2, 3$ *be bounded and of class* C^r, $r = \max\{m+2, 2\}$, $m \geq -1$. *Let* $\mathbf{f}$ *belong to* $(W^{m,\alpha}(D))^d$. *Then there exists a unique* $\mathbf{u} \in (W^{m+2,\alpha}(D))^d)$ *and there exists a unique (up to an additive constant)* $p \in W^{m+1,\alpha}(D)$ *solutions of the Stokes equations. Moreover*

$$\|\mathbf{u}\|_{m+2,\alpha,D} + |||p|||_{m+1,\alpha,D} \leq C\|\mathbf{f}\|_{m,\alpha,D},$$

where $|||\,.\,|||_{m+1,\alpha,D}$ *is the norm in* $W^{m+1,\alpha}(D)/\mathbb{R}$.

The Stokes operator

We denote by P is the (Leray) orthogonal projection operator $P : (L^2(D))^d \to H$. Let us assume that D is bounded of class C^2.

Definition 1.2.7. *The Stokes operator* A *acting on* $\mathcal{D}(A) \subset H$ *into* H *is defined by*

$$A : \mathcal{D}(A) \to H, \qquad A := -P\Delta.$$

We have the following proposition.

Proposition 1.2.8. *The following results hold for the Stokes operator:*

1) *The Stokes operator is selfadjoint and* $\mathcal{D}(A) = (H^2(D))^d \cap V$;

2) *The inverse of the Stokes operator,* A^{-1}, *is a compact operator in* H;

3) *There exist* $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$, $\mathbf{w}_j \in \mathcal{D}(A)$ *and* $0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots$ *such that:*
   - *a)* $A\,\mathbf{w}_j = \lambda_j \mathbf{w}_j$,
   - *b)* $\lim_{j\to+\infty} \lambda_j = +\infty$[6],
   - *c)* $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$ *is an orthonormal basis of* H.

Remark 1.2.9. *More regularity of* ∂D *is inherited by* $\mathbf{w}_j$. *In particular we have that if* D *is of class* C^{l+2}, $l \geq 0$, *then* $\mathbf{w}_j$ *belongs also to* $(H^{l+2}(D))^d$.

[5] See also, in the case $\alpha = 2$, the simplified proof given in Beirão da Veiga [BdV97a], which avoids the methods of *potential theory*.

[6] We remark that the precise asymptotic behavior of the eigenvalues of the Stokes operator is known to be $\lim_{j\to+\infty} \left(\frac{|D|}{j}\right)^{2/d} \frac{\lambda_j}{(2\pi)^2} = ((n-1)\omega_d)^{-2/d}$, where $\omega_d = |B(0,1)|$ denotes the Lebesgue measure of the unitary ball and $|D|$ that one of D, see Kozhevnikov [Koz84].

Due to the previous result we can define, as usual, the fractional powers of A as follows:

Definition 1.2.10. *Let $\alpha > 0$ be real. For $\mathbf{u} \in \mathcal{D}(A^\alpha)$, where*

$$\mathcal{D}(A^\alpha) := \Big\{ \mathbf{u} \in H : \ \mathbf{u} = \sum_{j=1}^{+\infty} \mathbf{u}_j \mathbf{w}_j, \quad \sum_{j=1}^{+\infty} \lambda_j^{2\alpha} |\mathbf{u}_j|^2 < +\infty, \quad \mathbf{u}_j \in \mathbb{R}^d \Big\},$$

we define $A^\alpha \mathbf{u}$, by

$$A^\alpha \mathbf{u} := \sum_{j=1}^{+\infty} \lambda_j^\alpha \mathbf{u}_j \mathbf{w}_j \quad \textit{for} \quad \mathbf{u} := \sum_{j=1}^{+\infty} \mathbf{u}_j \mathbf{w}_j.$$

1.2.2 Inequalities for the nonlinear term

The presence of the nonlinear term is the most painful fact in the theory of Navier-Stokes equations. Its presence causes the lack of satisfactory existence and uniqueness theorems.

It is important to have good estimates on this term. In the Sobolev framework to treat the nonlinear term we introduce the following trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \sum_{i,j=1}^{d} \int_D u_j \frac{\partial v_i}{\partial x_j} w_i \, d\mathbf{x} = \int_D (\mathbf{u} \cdot \nabla) \, \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x}. \tag{1.7}$$

We recall the following definition.

Definition 1.2.11. *Let $\mathbf{u}, \mathbf{v} \in (C(\overline{D}))^d$. We define $B(\mathbf{u}, \mathbf{v})$ by*

$$B(\mathbf{u}, \mathbf{v}) := P((\mathbf{u} \cdot \nabla) \, \mathbf{u}),$$

where P is the Leray projector.

The trilinear term $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$ surely makes sense for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in (C^1(\overline{D}))^d$, and the following proposition states one important estimate.

Proposition 1.2.12. *Let D be bounded, open and of class C^l. Let s_1, s_2, s_3 be real numbers such that $0 \le s_1 \le l$, $0 \le s_2 \le l - 1$ and $0 \le s_3 \le l$. Let us assume that*

$$i) \qquad s_1 + s_2 + s_3 \le \frac{n}{2} \quad \textit{if } s_i \ne \frac{d}{2} \quad \textit{for } i = 1, 2, 3$$

or

$$ii) \qquad s_1 + s_2 + s_3 > \frac{n}{2} \quad \textit{if } s_i = \frac{d}{2} \quad \textit{for at least one } i.$$

Then $\forall \, \mathbf{u}, \mathbf{v}, \mathbf{w} \in (C^\infty(\overline{D}))^d$ there exists a constant c, depending on s_1, s_2, s_3, such that

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C \|\mathbf{u}\|_{[s_1],D}^{1+[s_1]-s_1} \|\mathbf{u}\|_{[s_1]+1,D}^{s_1-[s_1]} \|\mathbf{v}\|_{[s_2]+1,D}^{1+[s_2]-s_2} \|\mathbf{v}\|_{[s_2]+2,D}^{s_2-[s_2]} \|\mathbf{w}\|_{[s_3],D}^{1+[s_3]-s_3} \|\mathbf{w}\|_{[s_3]+1,D}^{s_3-[s_3]}.$$

The last proposition can be proven with a clever use of Hölder and interpolation inequalities. We recall also the orthogonality property

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \qquad \forall \, \mathbf{u} \in V, \quad \forall \, \mathbf{v} \in (H_0^1(D))^d,$$

which is of basic importance to get energy-type estimates.

1.2.3 Weak solutions

We now consider the Navier-Stokes equations in the particular cases of $d = 2, 3$, which are the most important from the physical point of view. The Navier-Stokes equations will be written in the abstract form as functional equations in the Hilbert space H as follows:

$$\frac{d\mathbf{u}}{dt} + \nu A\,\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f} \tag{1.8}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{1.9}$$

The solution will be a vector valued function $\mathbf{u}(t)$ such that $A\mathbf{u}(t)$ and $B(\mathbf{u}(t),\mathbf{u}(t))$ make sense.

We now define the notion of *weak solution* and then we outline the proof of an existence theorem.

Definition 1.2.13. *A weak solution of the Navier-Stokes equations (1.8) is a function $\mathbf{u}$ belonging to $L^2(0,T;V) \cap C_w(0,T;H)$, satisfying $d\mathbf{u}/dt \in L^1_{\text{loc}}(0,T;V')$ and*

$$< \frac{d\mathbf{u}}{dt}, \mathbf{v} > + \nu((\mathbf{u},\mathbf{v})) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \qquad \textit{a.e. in } t \quad \forall\, \mathbf{v} \in V, \tag{1.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{1.11}$$

where we denoted by V' the topological dual space of V, with pairing $< \,.\,,\,.\, >$. The space $C_w(0,T;H)$ is a subspace of $L^\infty(0,T;H)$ consisting of functions which are weakly continuous, i.e. $(\mathbf{u}(t),\mathbf{h})$ is a continuous function for all $\mathbf{h} \in H$. In particular the initial datum is taken in this sense.

The main result regarding weak solutions is the following, which is essentially due to Hopf [Hop51].

Theorem 1.2.14. *There exists at least a weak solution of (1.8), for every $\mathbf{u}_0 \in H$ and $\mathbf{f} \in L^2(0,T;V')$. Moreover, the energy inequality*

$$\frac{1}{2}|\mathbf{u}(t)|^2 + \nu \int_{t_0}^t \|\mathbf{u}(s)\|^2\, ds \le \frac{1}{2}|\mathbf{u}(t_0)|^2 + \int_{t_0}^t < \mathbf{f}(s), \mathbf{u}(s) > ds \tag{1.12}$$

holds for all $0 \le t_0 \le t \le T$, a.e. t_0 in $[0,T]$ and

$$\textit{if d=3 then} \qquad \frac{d\mathbf{u}}{dt} \in L^{4/3}(0,T;V')$$

$$\textit{if d=2 then} \qquad \frac{d\mathbf{u}}{dt} \in L^{2}(0,T;V').$$

We do not enter into details of the proof of Theorem 1.2.14, but we only outline that it is based on three steps:

i) a Faedo-Galerkin approximation with smooth functions $\mathbf{u}_n : (0,T) \to \mathbb{R}^k \subset V$, for some $k \in \mathbb{N}$;

ii) the energy-type estimate

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}_n|^2 + \nu\|\mathbf{u}_n\|^2 = < \mathbf{f}, \mathbf{u}_n > \tag{1.13}$$

to get that $\mathbf{u}_n$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$, uniformly in n;

iii) extraction of subsequences and additional compactness results ($d = 3$).

1.2.4 Strong solutions

In this section we introduce the notion of *strong solution* and we show some results which highlight the differences between problem in two and in three space dimensions.

Definition 1.2.15. *A strong solution of the Navier-Stokes equations is a function* $\mathbf{u}$ *satisfying* (1.10)-(1.11) *and belonging to* $L^\infty_{\text{loc}}(0,T;V) \cap L^2_{\text{loc}}(0,T;\mathcal{D}(A)) \cap L^2(0,T;V) \cap L^\infty(0,T;H)$.

The main tool to prove existence of *strong solutions* is an "high order" energy-type inequality. We consider again a Galerkin approximation, but this time we multiply the Navier-Stokes equations by $A\,\mathbf{u}_n$ and integrate over D. The "bad" term will obviously be $b(\mathbf{u}_n, \mathbf{u}_n, A\,\mathbf{u}_n)$.

The two dimensional case

If $d = 2$, by setting $s_1 = 1/2$, $s_2 = 1/2$, and $s_3 = 0$ in Proposition 1.2.12 we obtain

$$|b(\mathbf{u}_n, \mathbf{u}_n, A\,\mathbf{u}_n)| \leq C|\mathbf{u}_n|^{1/2}\|\mathbf{u}_n\||A\,\mathbf{u}_n|.$$

By using Young inequality we have

$$|(\mathbf{f}, A\,\mathbf{u}_n)| \leq \frac{\nu}{4}|A\,\mathbf{u}_n|^2 + \frac{\|\mathbf{f}\|^2_{\infty,H}}{\nu}.$$

These estimates lead to the inequality

$$\frac{d}{dt}\|\mathbf{u}_n\|^2 + \nu|A\,\mathbf{u}|^2 \leq \frac{2\|\mathbf{f}\|^2_{\infty,H}}{\nu} + \frac{c}{\nu^3}|\mathbf{u}_n|^2\|\mathbf{u}_n\|^4 \tag{1.14}$$

By using the estimates on $|\mathbf{u}_n|$ and $\|\mathbf{u}_n\|$, which are known for weak solutions, and by applying the Gronwall lemma, we get the estimates needed to prove the following result, see Kiselev and Ladyženskaya [KL57].

Theorem 1.2.16. *Let* $D \subset \mathbb{R}^2$ *be an open bounded set of class* C^2. *Let* $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^\infty(0,\infty;H)$. *Then* $\forall\, T > 0$ *there exists a solution* $\mathbf{u}$ *of the Navier-Stokes equations satisfying*

$$\mathbf{u} \in L^\infty_{\text{loc}}(0,T;V) \cap L^2_{\text{loc}}(0,T;\mathcal{D}(A)) \cap L^\infty(0,T;H) \cap L^2(0,T;V).$$

The three dimensional case

In the three dimensional case we estimate again the nonlinear term by using Proposition 1.2.12 with $s_1 = 1$, $s_2 = 1/2$, $s_3 = 0$, but we can obtain the following estimate:

$$|b(\mathbf{u}_n, \mathbf{u}_n, A\,\mathbf{u}_n)| \leq C\|\mathbf{u}_n\|^{3/2}|A\,\mathbf{u}_n|^{3/2}.$$

Reasoning as before on the energy-type estimate, derived by multiplying the Navier-Stokes equations by $A\,\mathbf{u}_n$, we get

$$\frac{d}{dt}\|\mathbf{u}_n\|^2 + \nu|A\,\mathbf{u}|^2 \leq \frac{2|\mathbf{f}|^2_{\infty,H}}{\nu} + \frac{c}{\nu^3}\|\mathbf{u}_n\|^6. \tag{1.15}$$

By using the last estimate (1.15) it is possible to prove the following theorem

Theorem 1.2.17. *Let $D \subset \mathbb{R}^3$ be an open bounded set of class C^2. There exists a positive constant C such that for $\mathbf{u}_0 \in V$ and $\mathbf{f} \in L^2(0,T;H)$ satisfying*

$$\frac{\|\mathbf{u}_0\|^2}{\nu^2\lambda_1^{1/2}} + \frac{2}{\nu^3\lambda_1^{1/2}}\int_0^T |\mathbf{f}(s)|^2\,ds \leq \frac{1}{4\sqrt{C}}, \tag{1.16}$$

there exists a solution $\mathbf{u}(t)$ of the Navier-Stokes equations belonging to $L^\infty(0,T;V) \cap L^2(0,T;\mathcal{D}(A))$.

The condition (1.16) can be interpreted in various way: small initial data and external force, but arbitrary T. The same inequality shows also that if $\|\mathbf{u}_0\|$ and $\|\mathbf{f}\|_{2,H}$ are not small with respect to suitable expression in ν^2 and λ_1, only local existence can be inferred. We understand the need to deal with weak solutions, which are defined for any time interval $[0,T]$ even for $d = 3$. As we shall see with more detail later (see Chapter 2)

> ... *even if $\mathbf{u}_0$ and $\mathbf{f}$ are very nice functions, in this case the existence of classical solutions of the Navier-Stokes equations is known, in general, only for short time intervals.*

Remark 1.2.18. *We remark that in the absence of boundaries the Leray projector P commutes with the Laplace operator Δ. By absence of boundaries we mean either the case $D = \mathbb{R}^d$ or the case $D = \mathbb{T}^d$, the d^{th} dimensional torus. In the latter case we can speak of periodic boundary conditions. In this case the Navier-Stokes equations are studied as equations on the whole space $\mathbb{R}^d$ with the following condition*

$$\mathbf{u}(x_1 + 2\pi, x_2, \dots, x_d) = \dots = \mathbf{u}(x_1, \dots, x_{d-1}, x_d + 2\pi) = \mathbf{u}(x_1, \dots, x_d) \qquad \forall\,(x_1, \dots, x_d) \in \mathbb{R}^d,$$

and there is no loss of generality to assume that $\int_D \mathbf{u}_0(\mathbf{x})\,d\mathbf{x} = 0$. We define H_{per} as the closure in $(L^2(D))^d$ of the set

$$\left\{\mathbf{u} \in (C^1_{per}(D))^d : \operatorname{div}\mathbf{u} = 0 \quad \textit{and} \quad \int_D \mathbf{u}(\mathbf{x})\,d\mathbf{x} = 0\right\},$$

where $C^1_{per}(D)$ is the space of differentiable periodic functions. We also define V_{per} as the divergence-free subspace of $(H^1_{per}(D))^d$, where $H^m_{per}(D)$ are the periodic functions in $H^m(D)$. By setting $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$, we define, for $m \geq 0$,

$$H^m_{per}(D) := \left\{u : u = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{i(k_1x_1+\dots+k_dx_d)},\ \sum_{k\in\mathbb{Z}^d} (1+|\mathbf{k}|)^{2m}|c_{\mathbf{k}}|^2 < \infty \quad \text{and } c_{\mathbf{0}} = 0\right\},$$

with the norm

$$\|u\|_m^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d} (1+|\mathbf{k}|)^{2m}|c_{\mathbf{k}}|^2 < \infty.$$

We recall that since $c_{\mathbf{k}} \in \mathbb{C}$ we have to impose $\bar{c}_{\mathbf{k}} = c_{-\mathbf{k}}$, to have real valued functions. We have again the Helmholtz decomposition $(L^2_{per}(D))^d = H_{per} \oplus G$, where G denotes a space of gradients, that is orthogonal to H_{per}.

With the periodic boundary conditions we have that if $d = 2$, then

$$b(\mathbf{u}_m, \mathbf{u}_m, A\,\mathbf{u}_m) \equiv 0.$$

The last equation shows one the main simplifications due to the use of periodic boundary conditions.

1.3 Long-time behavior

In this section we briefly explain the basic results regarding the long-time behavior of the Navier-Stokes equations. When dealing with long-time analysis we restrict to the two dimensional case. In this case the solution globally exist and it is unique. Further results will be presented when necessary. The main idea, underlying the results we shall show, is the following one: *since the Navier-Stokes equations are dissipative, "probably" the dynamical system generated by their solution, can be described (asymptotically) with a finite number of degrees of freedom*. The first results in this direction are due to Foiaş and Prodi [FP67] and Ladyženskaya [Lad72]. We shall present two of the main approaches: *attractors* and *determining projections*.

1.3.1 Attractors

In this section we describe the main features of the attractors in metric spaces, see Babin and Višhik [BV92]. We consider a dynamical system whose state is described by an element $\mathbf{u}(t)$ of a metric space H. The evolution of the system is described by the semigroup $S(t)$. We recall that a family of operators $\{S(t)\}_{t\geq 0}$ that maps H into itself for each t, is called a semigroup if $S(t+s) = S(t) \circ S(s)$, for $s, t \geq 0$ and $S(0)x = x$, $\forall\, x \in H$. We assume at least that $S(t)$ is a continuous nonlinear operator for $t \geq 0$. We give now the definition of ω-limit set.

Definition 1.3.1. *We say that the orbit starting at $\mathbf{u}_0$ is the set $\bigcup_{t\geq 0} S(t)\mathbf{u}_0$. For $\mathbf{u}_0 \in H$ or for $\mathcal{A} \subset H$ we define the ω-limit set of $\mathbf{u}_0$ and of $\mathcal{A}$ respectively as*

$$\omega(\mathbf{u}_0) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)\mathbf{u}_0} \qquad \textit{and} \qquad \omega(\mathcal{A}) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)\mathcal{A}},$$

where the closures are taken in H.

Remark 1.3.2. *We remark that $\phi \in \omega(\mathcal{A})$ if and only if there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and a real sequence $t_n \to +\infty$ such that $\lim_{t_n\to+\infty} S(t_n)\phi_n = \phi$.*

Another important definition is that one of *functional invariant* set.

Definition 1.3.3. *A set $X \subset H$ is a functional invariant set for the semigroup $S(t)$ if*

$$S(t)\,X = X \qquad \forall\, t \geq 0.$$

Trivial examples of a invariant set are

a) a singleton fixed point $\mathbf{u}_0$ or any union of fixed points;

b) any time-periodic orbit[7], when it exists.

The discussion of other examples, less trivial than the ones above, can be found in Temam [Tem97], §1. In the same reference one can also find the proof of all results of this section. We start with an abstract lemma.

Lemma 1.3.4. *Assume that for some nonempty subset $\mathcal{A} \subset H$ and for some $t_0 > 0$ the set $\bigcup_{t\geq t_0} S(t)\mathcal{A}$ is relatively compact in H. Then $\omega(\mathcal{A})$ is nonempty, compact and invariant.*

[7] An orbit is periodic if there exists $0 < T < +\infty$ such that $S(T)\mathbf{u}_0 = \mathbf{u}_0$.

Remark 1.3.5. *The lemma above provides us examples of invariant sets whenever we can show that $\cup_{t\geq t_0} S(t)\mathcal{A}$ is relatively compact. This set can consist of a single stationary solution $\mathbf{u}_*$, if all the orbits starting form $\mathcal{A}$ converge to $\mathbf{u}_*$ as $t \to +\infty$. It can also be a single periodic or a quasi-periodic*[8] *solution or even a more complex object.*

At this point it is clear that some compactness is needed. The problem reduces to show that the set $\bigcup_{t\geq t_0} S(t)\mathcal{A}$ is bounded if H is finite dimensional; the same set should be bounded in some subspace W, compactly embedded in H, if we deal with a problem in infinite dimensions.

Definition 1.3.6. *An attractor is a set $\mathcal{A} \subset H$ that enjoys the following properties*

i) $\mathcal{A}$ is a functional invariant set;

ii) $\mathcal{A}$ has an open neighborhood $\mathcal{U}$ such that, for every $\mathbf{u}_0$ in $\mathcal{U}$ $S(t)\mathbf{u}_0$ converges to $\mathcal{A}$ as t goes to $+\infty$, i.e.

$$d(S(t)\mathbf{u}_0, \mathcal{A}) \to 0 \qquad as \quad t \to +\infty,$$

where the distance is understood to be the distance of a point to a set

$$d(x, \mathcal{A}) = \inf_{y\in\mathcal{A}} d(x, y).$$

If $\mathcal{A}$ is an attractor, the largest open set $\mathcal{U}$ that satisfies ii) is called the *basin of attraction of* $\mathcal{A}$. We can express condition ii) by saying that $\mathcal{A}$ attracts the points of $\mathcal{U}$. We shall say that $\mathcal{A}$ uniformly attracts a set $\mathcal{B} \subset \mathcal{U}$ if

$$d(S(t)\mathcal{B}, \mathcal{A}) \to 0 \qquad \text{as} \quad t \to +\infty,$$

where $d(\mathcal{B}_0, \mathcal{B}_1)$ is the semidistance[9] of $\mathcal{B}_0, \mathcal{B}_1$, defined by $d(\mathcal{B}_0, \mathcal{B}_1) = \sup_{x\in\mathcal{B}_0} \inf_{y\in\mathcal{B}_1} d(x, y)$. We can now define the key concept of *global attractor.*

Definition 1.3.7. *We say that $\mathcal{A} \subset H$ is a global attractor for the semigroup $\{S(t)\}_{t\geq 0}$ if $\mathcal{A}$ is compact attractor that attracts the bounded sets of H. Its basin of attraction is then all of H.*

It is easy to prove that such a set is necessarily unique and that such a set is maximal for inclusion relation among the bounded attractors and among the bounded functional invariant sets. For this reason it is also called the *maximal attractor*

Existence of attractors

To prove the existence of attractors we introduce the notion of *absorbing sets* and that one of *uniformly compact semigroup.*

Definition 1.3.8. *Let $\mathcal{B}$ be a subset of H and $\mathcal{U}$ an open set containing $\mathcal{B}$. We say that $\mathcal{B}$ is absorbing in $\mathcal{U}$ if the orbit of any bounded set of $\mathcal{U}$ enters into $\mathcal{B}$ after a certain time, depending on the set:*

$$\forall\, \mathcal{B}_0 \subset \mathcal{U} \quad \mathcal{B}_0 \text{ bounded} \quad \exists\, t_*(\mathcal{B}_0) \quad \textit{such that} \quad S(t)\mathcal{B}_0 \subset \mathcal{B} \quad \forall\, t \geq t_*(\mathcal{B}_0).$$

[8] An orbit is quasi-periodic if the function $t \mapsto S(t)\mathbf{u}_0$ is of the form $g(\omega_1 t, \dots, \omega_d t)$ where g is a periodic with period 2π in each variable and the frequencies ω_j are rationally independent.

[9] We recall that the Hausdorff distance defined on the set of nonempty compacts subsets of a metric space is defined by $\delta(\mathcal{B}_0, \mathcal{B}_1) := \max(d(\mathcal{B}_0, \mathcal{B}_1), (\mathcal{B}_1, \mathcal{B}_0))$. We remark that d is not a distance as $d(\mathcal{B}_0, \mathcal{B}_1) = 0$ implies only $\mathcal{B}_0 \subset \mathcal{B}_1$.

Definition 1.3.9. *Let $\{S(t)\}_{t\geq 0}$ be a semigroup of operators from H into itself. We say that $\{S(t)\}_{t\geq 0}$ is a uniformly compact semigroup if for every bounded set $\mathcal{B}$ there exists t_0, which may depend on $\mathcal{B}$, such that:*

$$\bigcup_{t\geq t_0} S(t)\mathcal{B} \qquad \text{is relatively compact in } H.$$

The existence of a global attractor for a semigroup implies that of an absorbing set. Conversely the next theorem will show that a semigroup, which possesses an absorbing set and enjoys some other properties, has a global attractor.

Theorem 1.3.10. *Let us suppose that H is a metric space and that the operators $S(t)$ are continuous and satisfy the semigroup property. Let us suppose furthermore that the operators $S(t)$ are uniformly compact for t large.*

If we also assume that there exists an open set $\mathcal{U}$ and a bounded $\mathcal{B} \subset \mathcal{U}$ such that $\mathcal{B}$ is absorbing in $\mathcal{U}$, then the ω-limit set of $\mathcal{B}$ is the global attractor in $\mathcal{U}$. Furthermore if $\mathcal{U}$ is convex and connected, then $\mathcal{A} = \omega(\mathcal{B})$ is connected too.

With this abstract result it is immediate to prove that the *Lorenz system* has the global attractor. The equations of this system are

$$\begin{cases} x' &= -\sigma\, x + \sigma\, y, \\ y' &= r\, x - y - x\, z, \\ z' &= -b\, z + x\, y, \end{cases}$$

and this system is a three-mode Galerkin approximation (one in velocity and two in temperature of the Boussinesq equation, for a fluid heated from below). The numbers σ, r, b represent non-dimensional quantities. This model was proposed by Lorenz [Lor63], as an indication of the limits of predictability in weather prediction.

Attractors for the Navier-Stokes equations

We do not give the proof of the following result. The interested reader can find it in Temam [Tem97], §3–5. We only observe that it is based on application of the energy-type estimates (1.13)-(1.14).

Theorem 1.3.11. *The dynamical system associated to the two-dimensional Navier-Stokes equations possesses a global attractor. Furthermore the Hausdorff dimension of the global attractor $\mathcal{A}$ turns out to be finite. (See also the note at the end of page 16)*

In particular it is easy to check that the hypotheses of Theorem 1.3.10 are satisfied; more technical (not necessary in the sequel) tools are needs to show the finite dimensionality of the attractor.

1.3.2 Determining modes, nodes and volumes

The results of this section are mainly based on the results which followed the germinal paper by Foiaş and Prodi [FP67]. They proved that, at least asymptotically, the behavior of the solutions to the Navier-Stokes equations can be described by the behavior of a finite dimensional system or, in other words, by a system of ordinary differential equations.

General setting

We consider the Navier-Stokes equations with two external forces $\mathbf{f}, \mathbf{g}$ and we denote by $\mathbf{u}$ and $\mathbf{v}$ the relative solutions. We assume that D is a bounded smooth susbset of $\mathbb{R}^2$ and that

$$\lim_{t\to+\infty} |(\mathbf{f}-\mathbf{g})(t)| = 0.$$

We have various results on the behavior of $\mathbf{u}-\mathbf{v}$.

Determining modes

The basic result stated by Foiaş and Prodi [FP67] is the following one, which states that the behavior of the solutions is described by that one of the projection on a finite number of eigenfunctions of the Stokes operator.

Theorem 1.3.12. *Let $D, \mathbf{f}, \mathbf{g}$ satisfy the hypotheses described above. Then there exists N, which depends only on $\nu, D, \mathbf{f}, \mathbf{g}$, such that*

$$\lim_{t\to+\infty} \|P_N(\mathbf{u}-\mathbf{v})(t)\|_{R^{2N}} = 0 \qquad \text{implies} \qquad \lim_{t\to+\infty} |(\mathbf{u}-\mathbf{v})(t)| = 0,$$

where P_N denotes the projection operator on the subspace spanned by the first N eigenfunctions of the Stokes operator

$$P_N:\ V \to V_N := span < \mathbf{w}_1, \dots, \mathbf{w}_N > .$$

This result is very important, but of no practical use, because the eigenfunctions of the Stokes operator are no computable, unless we study problems with periodic boundary conditions.

Determining nodes

Another result in this direction was given by Foiaş and Temam [FT84]. They proved that if the large time behavior of the solutions is known on an appropriate discrete set (nodes), then the large time behavior of the solution itself is totally determined.

It is given a set of points $\mathcal{E}_N = \{\mathbf{x}^1, \dots, \mathbf{x}^N\} \subset D$, then the *density* of this set is measured in the following way. We associate to every point $\mathbf{x} \in D$ its distance to $\mathcal{E}_N$ by $d_N(\mathbf{x}) := \min_{1\le j\le N} |\mathbf{x}-\mathbf{x}^j|$ and we set

$$d_N := \max_{\mathbf{x}\in D} d_N(\mathbf{x}),$$

which will be the main parameter to measure density. We have the following theorem

Theorem 1.3.13. *Let the same hypotheses on $D, \mathbf{f}, \mathbf{g}$ of the previous Theorem 1.3.12 hold. If we assume that, as $t \to +\infty$,*

$$\mathbf{u}(\mathbf{x}^j, t) - \mathbf{v}(\mathbf{x}^j, t) \to 0 \qquad \text{for } j = 1, \dots, N,$$

then there exists a constant $\alpha = \alpha(\nu, D, \mathbf{f}, \mathbf{g})$ such that if $d_N \le \alpha$, then

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \to 0 \qquad \text{as} \quad t \to +\infty.$$

The interesting feature of this result is that the point $\mathbf{x}^j$ can be, for example, the nodal points for a finite element method or a collocation method. The result of Theorem 1.3.13 is then strictly linked with the numerical analysis of Navier-Stokes equations. We observe that regular solutions, say at least belonging to $(H^2(D))^d \subset (C(\overline{D}))^d$ a.e. in time, are needed to define the value at points in D.

Determining finite element volumes

A further generalization is based on the idea that the behavior of non-smooth solution cannot be characterized by nodal values, see Foiaş and Titi [FT91] and Jones and Titi [JT92]. We recall that $V \not\subset C(\overline{D})$, if $d \geq 2$ and that if $\mathbf{u}$ is a *weak solution*, then $\mathbf{u} \in V$ a.e. $t \in [0.T]$. A result, which does not use further regularity properties of the solutions, can be obtained by using the *spatial mean* of the solution.

We consider $D := (0, L)^2$ and we study the problem with periodic boundary conditions. We divide D into N equal squares of side $l = L/\sqrt{N}$, labelled by Q_j. We define the average of solutions on the square Q_j by

$$\langle \mathbf{u} \rangle_{Q_j} = \frac{N}{L^2} \int_{Q_j} \mathbf{u}(\mathbf{x})\, d\mathbf{x}, \qquad \text{for} \quad 1 \leq j \leq N.$$

We have the following theorem.

Theorem 1.3.14. *Let the same hypotheses on* $\mathbf{f}, \mathbf{g}$ *of Theorem 1.3.12 hold. There exists a natural number* $\overline{N} = \overline{N}(\nu, \mathbf{f}, \mathbf{g}, L)$ *such that if* $N \geq \overline{N}$ *then*

$$\lim_{t \to +\infty} \langle \mathbf{u} \rangle_{Q_j} - \langle \mathbf{v} \rangle_{Q_j} = 0 \qquad \textit{for} \quad 1 \leq j \leq N \quad \textit{implies} \quad \lim_{t \to +\infty} |\mathbf{u} - \mathbf{v}| = 0.$$

Precise estimates[10] on the number of degrees of freedom are known and the fundamental parameter is the *Grashof number* defined by

$$Gr := \frac{1}{\lambda_1\, \nu^2} \limsup_{t \to +\infty} |\mathbf{f}(t)|.$$

We do not enter into details, referring to the paper by Jones and Titi [JT93].

1.3.3 Determining projections

The results on nodes, modes and volumes can be generalized with a more abstract setting, which encompasses them. The definition of *determining projection* was given by Holst and Titi [HT97]

> ... *an operator which projects weak solutions onto a finite-dimensional space is determining if it annihilates the difference of two "nearby" weak solutions asymptotically, and if it satisfies a single approximation inequality.*

We now give the precise results.

Definition 1.3.15. *The projection operator* $\mathcal{R}_N : V \to V_N \subset (L^2(D))^d$, $N = \dim(V_N) < +\infty$ *is called a determining projection operator for weak solutions of the d-dimensional Navier-Stokes equations if*

$$\lim_{t \to +\infty} |\mathcal{R}_N(\mathbf{u}(t) - \mathbf{v}(t))| = 0,$$

implies that

$$\lim_{t \to +\infty} |(\mathbf{u}(t) - \mathbf{v}(t))| = 0.$$

[10] Sharp estimates (periodic boundary conditions) for the number of determining modes, nodes and volumes is Gr. This bound must be compared to the bound $Gr^{2/3}(1 + \log Gr)^{1/3}$, which holds for the global attractor. We recall that an estimate of order of Gr is in agreement with the heuristic estimates, which are based on physical arguments, that have been conjectured by Manley and Treve.

The result that can be proven is the following one, see Holst and Titi [HT97].

Theorem 1.3.16. *Let the same hypotheses on* $D, \mathbf{f}, \mathbf{g}$ *of the previous Theorem 1.3.12 hold. Let there exists a projection operator* $\mathcal{R}_N : V \to V_N \subset (L^2(D))^2$, $N = \dim(V_N) < +\infty$ *satisfying*

$$\lim_{t\to+\infty} |\mathcal{R}_N(\mathbf{u}(t) - \mathbf{v}(t))| = 0 \tag{1.17}$$

and satisfying the following approximation inequality

$$\exists\, \gamma > 0 \qquad \|\mathbf{u} - \mathcal{R}_N(\mathbf{u})\|_{(L^2(D))^2} \le C\frac{1}{N^\gamma}\|\mathbf{u}\|_{(H^1(D))^2} \qquad \forall\, \mathbf{u} \in (H^1(D))^2. \tag{1.18}$$

Then if $N > C(\lambda_1 Gr)^{1/\gamma}$, *where* C *is a constant independent of* $\nu, \mathbf{f}$ *and* $\mathbf{g}$, *the following estimate holds*

$$\lim_{t\to+\infty} |\mathbf{u}(t) - \mathbf{v}(t)| = 0.$$

We do not give the proof of this result here, because we shall analyze the problem in Chapter 3.

Remark 1.3.17. *We observe that:*

a) the projection operator acts on weak solutions (say only in V*);*

b) the definition of determining projection encompasses each of the notions of determining modes, nodes, volumes;

c) an operator satisfying (1.17)-(1.18) *can be explicitly constructed, as we shall show in the following Section 3.1.1.*

1.4 A review of numerical methods in Fluid Dynamics

In this section we review some basic techniques used in the numerical approximation of the Navier-Stokes equations. The basic framework will be that one of Finite Element Method. We only give the definition and the reader can find the details in the classical book by Ciarlet [Cia78].

Given a coercive bilinear form $a(\,.\,,\,.\,) : X \times X \to \mathbb{R}$ in the real Hilbert space X and given $f \in X$, the Faedo-Galerkin method reads as

$$\text{find } x_h \in X_h : \; a(x_h, y_h) = f(y_h) \qquad \forall\, y_h \in X_h,$$

where X_h is a suitable finite dimensional subspace of X. The major result is the following

Proposition 1.4.1. *Let* $a(\,.\,,\,.\,)$ *and* f *as before and let* X_h *be a family of finite dimensional subspaces of* X. *Assume that there exists a dense subset* $\mathcal{X} \subset X$ *such that*

$$\lim_{h\to+\infty} \inf_{x_h \in X_h} \|y - y_h\| = 0 \qquad \forall\, y \in \mathcal{X}.$$

Then, as $h \to 0$, x_h *converges in* X *to the solution* x *of the "continuous problem"*

$$\textit{find } x \in X : \; a(x, y) = f(y) \qquad \forall\, y \in X.$$

We recall that the *Finite Element Method* is a particular Faedo-Galerkin method in which, roughly speaking, the finite dimensional subspace, used to approximate the problem, is given and really computable.

The Finite Element Method

To introduce some concepts on finite element spaces we start from the following definition.

Definition 1.4.2. *The triple* $(\mathcal{K}, \mathcal{P}_\mathcal{K}, \mathcal{N}_\mathcal{K})$ *is called a finite element if*

i) $\mathcal{K} \subset \mathbb{R}^d$ *is a domain with piecewise smooth boundary (the element domain);*

ii) $\mathcal{P}_\mathcal{K}$ *is a finite dimensional space of functions on* $\mathcal{K}$ *(the shape functions);*

iii) $\mathcal{N}_\mathcal{K} = \{N_1, \dots, N_k\}$ *is a basis for* $\mathcal{P}'_\mathcal{K}$ *(the nodal variables).*

We define some other objects needed for the polynomial interpolation. We define P_s^d as the space of polynomials in d variables of degree less or equal than s. We have that P_s^d is a linear space, whose dimension is easily calculated to be $\binom{d+s}{d}$. As basic examples of finite element we recall the triangular Lagrange element, *i.e.*, $\mathcal{K}$ is a triangle, $\mathcal{P}_\mathcal{K} = P_1^2$ and $\mathcal{N}_\mathcal{K} = \{N_1, N_2, N_3\}$ with $N_i(v) = v(x_i)$, where x_i are the vertices of $\mathcal{K}$. Another example is the Crouzeix-Raviart element in which the vertices are replaced by the midpoints of the edges. This examples can be generalized to high order polynomials and to higher dimensional simplices.

We now define the notion of *local interpolant.*

Definition 1.4.3. *Given a finite element* $(\mathcal{K}, \mathcal{P}_\mathcal{K}, \mathcal{N}_\mathcal{K})$, *let the set* $\{\varphi_i\}_{i=1}^d \subseteq \mathcal{P}_\mathcal{K}$ *be the basis dual to* $\mathcal{N}_\mathcal{K}$ *(i.e.* $\langle N_j, \varphi_j, \rangle_{\mathcal{P}'_\mathcal{K}, \mathcal{P}_\mathcal{K}} = \delta_{ij}$*). If* v *is a function for which all* $N_i \in \mathcal{N}_\mathcal{K}$ *are defined, then we define the local interpolant by*

$$I_\mathcal{K} v := \sum_{i=1}^d N_i(v)\, \varphi_i.$$

Definition 1.4.4. *A subdivision* $\mathcal{T}$ *of a domain* D *is a finite collection of d-simplices* $\{\mathcal{K}_i\}$ *such that:*

i) $\mathcal{K}_i \cap \mathcal{K}_j = \emptyset \quad$ *if* $\quad i \neq j$;

ii) $\bigcup \overline{\mathcal{K}_i} = \overline{D}$;

iii) the faces (which are (d-1)-simplices) of each simplex $\mathcal{K}_i$ *lie on* ∂D *or are faces of another simplex* $\mathcal{K}_j$.

In this way it is is possible to create finite dimensional subspaces of some function spaces defined on D, by piecing together finite elements $(\mathcal{K}_i, \mathcal{P}_{\mathcal{K}_i}, \mathcal{N}_{\mathcal{K}_i})$, with $\mathcal{K}_i$ belonging to a given subdivision $\mathcal{T}$.

We are now in position to define the notion of *global interpolant.*

Definition 1.4.5. *Suppose* D *is a domain with a subdivision* $\mathcal{T}$. *Assume each element domain* $\mathcal{K}_i$ *in the subdivision is equipped with some type of shape functions* $\mathcal{P}_{\mathcal{K}_i}$ *and nodal variables* $\mathcal{N}_{\mathcal{K}_i}$ *such that* $(\mathcal{K}_i, \mathcal{P}_{\mathcal{K}_i}, \mathcal{N}_{\mathcal{K}_i})$ *form a finite element. Let* f *belong to a space on which the nodal variables are well defined. The global interpolant is defined by*

$$I_\mathcal{T} f|_{\mathcal{K}_i} = I_{\mathcal{K}_i} f, \qquad \textit{for all} \quad \mathcal{K}_i \in \mathcal{T}.$$

In our context we use as $\mathcal{P}_{\mathcal{K}_i}$ polynomials of a given degree (equal for every $\mathcal{K}_i$). It is easy to see that if v belongs just to $\mathcal{P}$, then its global interpolant is v itself.

Since without further assumptions on a subdivision no regularity properties can be asserted for the global interpolant we must give some conditions on the subdivision.
Let $D \subset \mathbb{R}^d$ be a connected, open bounded domain with Lipschitz polyhedral boundary.

Definition 1.4.6. *A simplicial subdivision $\mathcal{T}$ (i.e. a subdivision in which each $\mathcal{K}_i$ is a simplex) of D is regular if*

$$\max_{\mathcal{K}_i \in \mathcal{T}_h} \frac{h_{\mathcal{K}_i}}{\rho_{\mathcal{K}_i}} \leq \gamma_0,$$

with the constant $\gamma_0 \geq 1$ independent of h. With $\rho_{\mathcal{K}}$ we denote the radius of the largest closed ball contained in $\overline{\mathcal{K}}$ and with $h_{\mathcal{K}_i}$ the diameter of $\mathcal{K}_i$.
We define the mesh size h of a given subdivision $\mathcal{T}$ to be

$$h := \sup_{\mathcal{K}_i \in \mathcal{T}} h_{\mathcal{K}_i}$$

and we denote a subdivision $\mathcal{T}$ with mesh size h by $\mathcal{T}_h$.

The *regularity* condition means (roughly speaking) that the elements of $\mathcal{T}_h$ do not shrink too much. The main result is the following one, see Ciarlet [Cia78].

Theorem 1.4.7. *Let D be a polygonal domain of $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary and let $\mathcal{T}_h$ be a regular family of subdivisions of $\overline{D}$ such that each $\mathcal{K}_j$ is affine equivalent to the unit d-simplex. If the bilinear form $a(\,.\,,\,.\,)$ is continuous and coercive on $X = H_0^1(D)$ and*

$$X_h^r := \{x_h \in C^0(\overline{D}) : \; x_{h|\mathcal{K}} \in P_r^d \quad \forall \mathcal{K} \in \mathcal{T}_h\},$$

then the finite element method converges. Moreover if the exact solution belongs to $H^s(D)$ for some $s \geq 2$, then the following error estimate holds

$$\|x - x_h\|_{1,D} \leq C h^l \|x\|_{l+1}, \qquad \text{with} \quad l = \min(k, s-1).$$

1.4.1 Stokes equations

When dealing with the numerical analysis of the Stokes problem it is simple to apply an abstract Faedo-Galerkin method in V. On the other hand it is very difficult to find really computable finite dimensional subspaces of V, *i.e.*, to find divergence-free polynomial spaces. To overcome this problem it is generally used the so-called *mixed formulation* in which the approximate solution $\mathbf{u}_h$ belongs to $\widetilde{V}_h \not\subset V$ which is a (not *a-priori* divergence-free) finite dimensional space of $(H_0^1(D))^d$. The problem reads as: find $\mathbf{u}_h \in \widetilde{V}_h$ and $p_h \in Q_h \subset L_0^2(D)$ such that

$$(1.19) \qquad \nu((\mathbf{u}_h, \mathbf{v}_h)) + (\operatorname{div} \mathbf{u}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \widetilde{V}_h,$$

$$(1.20) \qquad (\operatorname{div} \mathbf{u}_h, q_h) = 0 \quad \forall\, q_h \in Q_h,$$

where $L_0^2(D) := Q = \{p \in L^2(D) : \; \int_D p\, d\mathbf{x} = 0\}$. We do not enter into details of the numerical approximation of the Stokes operator, because we shall not use it. We only recall the basic fact regarding the analysis of mixed problems, see Brezzi and Fortin [BF91].

Theorem 1.4.8. *Let us assume that the spaces $\widetilde{V}_h$ and Q_h satisfy the following compatibility condition (inf-sup or Ladyženskaya-Babuška-Brezzi condition): $\exists\, \beta_h > 0$ such that*

$$(1.21) \qquad \forall\, q_h \in Q_h \quad \exists\, 0 \neq \mathbf{v}_h \in \widetilde{V}_h : \qquad \int_D q_h \operatorname{div} \mathbf{v}_h\, dx \geq \beta_h \Big(\|\mathbf{v}_h\|_{0,D} + \|\operatorname{div} \mathbf{v}_h\|_{0,D} \Big) \|q_h\|_{0,D}.$$

Then the problem (1.19)-(1.20) has a unique solution $(\mathbf{u}_h, p_h) \in \widetilde{V}_h \times Q_h$.

It is easy to show that if the *inf-sup* condition is not satisfied, then the problem is ill-posed and great effort has been done to find appropriate couples of spaces $(\widetilde{V}_h, Q_h)$ satisfying (1.21). We do not give any detail regarding this topic, which is worth of a book itself, because we shall not use the mixed formulation. For an accurate analysis regarding the mixed formulation and the Stokes problem we refer, for example, to Brezzi and Fortin [BF91] and to Quarteroni and Valli [QV94], §7–9.

1.4.2 Navier-Stokes equations

In the study of time-dependent problems one possible approach is the *semi-discretization*, *i.e.*, the problem is discretized only with respect to the space variables. This approach leads to the study of systems of ordinary differential equations.

Semi-discrete approximation

In the numerical study of the Navier-Stokes equations we choose $\{V_h\}_{h\geq 0}$, a family of finite dimensional subspaces of the divergence free subspace $V \subset (H_0^1(D))^d$. With the semi-discrete approach we reduce to the following problem: for each $t \in (0, T)$ find $\mathbf{u}_h \in V_h$ such that

$$\begin{cases} \dfrac{d}{dt}(\mathbf{u}_h, \mathbf{v}_h) + \nu((\mathbf{u}_h, \mathbf{v}_h)) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}_h, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h, \\ \\ \mathbf{u}_h(0) = \mathbf{u}_{0h}, \end{cases}$$

where $\mathbf{u}_{0h}$ is any approximation of $\mathbf{u}_0$ in V_h. For the same reason explained for the Stokes problem the method described above is not suitable and a mixed formulation must be used. The *mixed formulation* reads as: for each $t \in (0, T)$ find $\mathbf{u}_h \in \widetilde{V}_h$ and $q_h \in Q_h$ such that

$$\begin{cases} \dfrac{d}{dt}(\mathbf{u}_h, \mathbf{v}_h) + \nu((\mathbf{u}_h, \mathbf{v}_h)) + \widetilde{b}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + (\operatorname{div} \mathbf{u}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \widetilde{V}_h, \\ \\ (\operatorname{div} \mathbf{u}_h, q_h) = 0 \quad \forall\, q_h \in Q_h, \\ \\ \mathbf{u}_h(0) = \mathbf{u}_{0h}, \end{cases}$$

We note that the trilinear form $b(\mathbf{u}, \mathbf{u}, \mathbf{v})$ has been replaced by the

$$\widetilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}\left[b(\mathbf{u}, \mathbf{v}, \mathbf{w}) - b(\mathbf{u}, \mathbf{w}, \mathbf{v})\right],$$

for stability purposes. The analysis of this method has been done by Heywood and Rannacher [HR82] and the basic result is that if $(\widetilde{V}_h, Q_h)$ satisfy the *inf-sup condition* (1.21), if

$$\inf_{\mathbf{v} \in \widetilde{V}_h} \|\mathbf{v} - \mathbf{v}_h\|_{1,D} + \inf_{q_h \in Q_h} \|q - q_h\|_{0,D} = O(h) \qquad \forall\, (\mathbf{v}, q) \in V \times Q,$$

if the initial datum is regular and if $\nabla \mathbf{u}$ belongs to $(L^\infty(0, T; L^2(D)))^{d \times d}$, then

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,D} \leq K \mathrm{e}^{K\,t} h^2, \qquad \|p - p_h\|_{0,D} \leq K \min\{t, 1\}^{-1/2} \mathrm{e}^{K\,t} h \qquad \forall\, t \in (0, T).$$

We remark that the results previously shown are suitable for moderately low Reynolds numbers. For high Reynolds number the convective term might induce numerical oscillations if not properly treated. Stabilization can be introduced by using implicit finite difference methods or stabilization terms. For the analysis of the numerical instability of advection-diffusion problems we refer to Section 4.4.2.

1.4.3 Operator splitting: the Chorin-Temam method

The *operator splitting* (also known as *fractional-step* or *splitting-up method*) is a method of approximation of evolution equations based on a decomposition of the operators.

We have to approximate a linear evolution equation

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + A\,\mathbf{u} = 0, & 0 < t < T \\ \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases}$$

where $\mathbf{u}$ belongs to a suitable Banach space X and A is a linear operator from X into itself. A first way is to introduce, with a standard Finite Differences Method[11], an implicit scheme and define a sequence of vectors $\mathbf{u}^m$, for $m = 0, \dots, N$, as follows:

$$\begin{cases} \mathbf{u}^0 = \mathbf{u}_0, \\ \dfrac{\mathbf{u}^{m+1} - \mathbf{u}^m}{k} + A\,\mathbf{u}^{m+1} = 0, & m = 0, \dots, N-1. \end{cases}$$

We recall that N is an integer, $T = k\,N$, and k is the *mesh-size.*

A second way is a *splitting-up* method, based on the existence of a decomposition of A as a sum $A = \sum_{j=1}^{q} A_j$. Starting again with $\mathbf{u}^0 = \mathbf{u}_0$ we recursively define a family of elements $\mathbf{u}^{m+j/q}$, for $M = 0, \dots, N-1$, and $j = 1, \dots, q$ as follows

$$\frac{\mathbf{u}^{m+j/q} - \mathbf{u}^{m+(j-1)/q}}{k} + A_j \mathbf{u}^{m+j/q} = 0, \qquad m = 0, \dots, N-1,\ j = 1, \dots, q.$$

When $\mathbf{u}^m$ is known $\mathbf{u}^{m+1}$ can be computed, in the case of an ordinary scheme, by the inversion of the operators $I + k\,A$. In the case of a fractional step method the computation of $\mathbf{u}^{m+1}$ requires the inversion of the q operators $I + k\,A_q$ and the algorithm is useful if all these operators are simpler to invert than $I + k\,A$.

The Chorin-Temam method

In the classical method introduced by Chorin [Cho67, Cho68] and Temam [Tem69], two[12] operators A_1 and A_2 are considered. The operator A_1 is defined by

$$A_1\mathbf{u} := -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u},$$

while the second one is an operator taking into account the term ∇p and the incompressibility condition $\operatorname{div}\mathbf{u} = 0$. This method is also called the *Projection Method.*

The interval $[0, T]$ is divided into $N \in \mathbb{N}$ intervals of length k and we set

$$\mathbf{f}^m := \frac{1}{k}\int_{(m-1)k}^{mk} \mathbf{f}(t)\,dt, \quad \text{for} \quad m = 1, \dots, N.$$

The projection method reads as follows: start with $\mathbf{u}^0 = \mathbf{u}_0$ and when $\mathbf{u}^m \in (L^2(D))^d$, $m \geq 0$, is known, define $\mathbf{u}^{m+1/2} \in (H_0^1(D))^d$ by

$$\frac{1}{k}(\mathbf{u}^{m+1/2} - \mathbf{u}^m, \mathbf{v}) + \nu((\mathbf{u}^{m+1/2}, \mathbf{v})) + \widetilde{b}(\mathbf{u}^{m+1/2}, \mathbf{u}^{m+1/2}, \mathbf{v}) = (\mathbf{f}^m, \mathbf{v}) \qquad \forall\,\mathbf{v} \in (H_0^1(D))^d,$$

[11] The derivative with respect to to time is approximated with an incremental ratio.
[12] In this case we speak of a two-steps method.

and then define $\mathbf{u}^{m+1} \in H$ by

$$(\mathbf{u}^{m+1}, \mathbf{v}) = (\mathbf{u}^{m+1/2}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in H.$$

The method is called a projection method because in the first step it is solved a non linear elliptic problem (without the incompressibility constraint); the second step amounts to project the solution onto H. If we introduce the "approximate functions" $\mathbf{u}_k^{(j)}$ from $[0,T]$ with values in $(L^2(D))^d$ such that $\mathbf{u}_k^{(j)} := \mathbf{u}^{m+j/2}$ for $mk \le t < (m+1)k$ we have the following convergence result, see Temam [Tem77] Ch. 3, §7.

Theorem 1.4.9. *Let* $\mathbf{f} \in L^2(0,T;H)$ *and* $\mathbf{u}_0 \in H$.
If the dimension of the space is $d = 2$*, then, as* $k \to 0$*, the following convergence results hold:*

$$\begin{aligned} &\mathbf{u}_k^{(j)} \to \mathbf{u} \quad \textit{strongly in } L^2(D \times (0,T)),\\ &\mathbf{u}_k^{(j)} \overset{*}{\rightharpoonup} \mathbf{u} \quad \textit{weakly-star in } L^\infty(0,T;(L^2(D))^2),\\ &\mathbf{u}_k^{(j)} \to \mathbf{u} \quad \textit{strongly in } L^\infty(0,T;(H_0^1(D))^2), \end{aligned}$$

where $\mathbf{u}$ *is the unique solution of the Navier-Stokes equations.*
If the dimension of the space is $d = 3$*, then there exists some sequence* $k' \to 0$ *such that:*

$$\begin{aligned} &\mathbf{u}_k^{(j)} \to \mathbf{u} \quad \textit{strongly in } L^2(D \times [0,T]),\\ &\mathbf{u}_k^{(j)} \overset{*}{\rightharpoonup} \mathbf{u} \quad \textit{weakly-star in } L^\infty([0,T];(L^2(D))^d),\\ &\mathbf{u}_k^{(j)} \rightharpoonup \mathbf{u} \quad \textit{weakly in } L^\infty([0,T];(H_0^1(D))^d), \end{aligned}$$

where $\mathbf{u}$ *is some solution of the Navier-Stokes equations.*

We remark that the condition which defines $\mathbf{u}^{m+1}$ can be written, with a strong formulation, as

$$\begin{cases} \mathbf{u}^{m+1} - \mathbf{u}^{m+1/2} + k\,\nabla p^{m+1} = 0 \quad \text{in } D,\\ \operatorname{div} \mathbf{u}^{m+1} = 0 \quad \text{in } D,\\ \mathbf{u}^{m+1} \cdot \mathbf{n} = 0 \quad \text{on } \partial D. \end{cases}$$

From this system it easily deduced that the approximate pressure p^{m+1} satisfies the *homogeneous* Neumann boundary value problem (A), which should be compared with the *non-homogeneous* Neumann boundary-value problem (E), satisfied by the exact pressure p.

$$\text{(A)} \begin{cases} -\Delta p^{m+1} = \dfrac{1}{k} \operatorname{div} \mathbf{u}^{m+1/2} \quad \text{in } D,\\ \dfrac{\partial p^{m+1}}{\partial \mathbf{n}} = \mathbf{u}^{m+1} \cdot \mathbf{n} = 0 \quad \text{on } \partial D, \end{cases} \qquad \text{(E)} \begin{cases} \Delta p = \operatorname{div} \mathbf{f} - \displaystyle\sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \quad \text{in } D,\\ \dfrac{\partial p}{\partial \mathbf{n}} = (\mathbf{f} + \nu \Delta \mathbf{u}) \cdot \mathbf{n} = 0 \quad \text{on } \partial D. \end{cases}$$

It is interesting to note that this discrepancy on the boundary conditions for the exact and the approximate problem implies that p^m converges only in a very weak sense to the exact pressure p; nevertheless, this does not affect the convergence of the scheme for the velocity field $\mathbf{u}$, as we have seen in the Theorem 1.4.9 above.

The Chorin-Temam (or projection) method is very useful because the first step involves a problem without the incompressibility constraint. In this way its discretization does not suffer of the problems arising in the numerical study of the Stokes problem. A further step can be introduced to *linearize* the equations. For simplicity we restrict to a two-dimensional problem and we refer to Temam [Tem77] Ch. III, §7 for more details. A three-steps method can be the following: start with $\mathbf{u}^0 = \mathbf{u}_0$ and when $\mathbf{u}^m \in ((L^2(D))^2$ is known, define $\mathbf{u}^{m+1/3} \in (H^1_0(D))^2$ by:

$$\frac{1}{k}(\mathbf{u}^{m+1/3} - \mathbf{u}^m, \mathbf{v}) + \nu((\mathbf{u}^{m+1/3}, \mathbf{v})) + \widetilde{b}_1(\mathbf{u}^m, \mathbf{u}^{m+1/3}, \mathbf{v}) = (\mathbf{f}^m, \mathbf{v}) \qquad \forall\, \mathbf{v} \in (H^1_0(D))^2.$$

Then find $\mathbf{u}^{m+2/3} \in (H^1_0(D))^2$ such that:

$$\frac{1}{k}(\mathbf{u}^{m+2/3} - \mathbf{u}^{m+1/3}, \mathbf{v}) + \nu((\mathbf{u}^{m+2/3}, \mathbf{v})) + \widetilde{b}_2(\mathbf{u}^{m+1/3}, \mathbf{u}^{m+2/3}, \mathbf{v}) = (\mathbf{f}^m, \mathbf{v}) \qquad \forall\, \mathbf{v} \in (H^1_0(D))^2,$$

and finally $\mathbf{u}^{m+1} \in V$ is the solution to the following problem:

$$(\mathbf{u}^{m+1}, \mathbf{w}) = (\mathbf{u}^{m+2/3}, \mathbf{w}) \qquad \forall\, \mathbf{w} \in V,$$

where we set

$$\widetilde{b}_i(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}\int_D \sum_{j=1}^{2} \left[u_i \frac{\partial v_j}{\partial x_i} w_j - u_i \frac{\partial w_j}{\partial x_i} v_j \right] d\mathbf{x} \qquad \text{for} \quad i = 1, 2.$$

Existence and uniqueness of the solutions of the first two steps follow in a standard manner from coercivity, by using the Lax-Milgram lemma. Furthermore, $\mathbf{u}^{m+1}$ is simply a $(L^2(D))^d$ orthogonal projection.

This method has a natural finite dimensional counterpart in which the space $(H^1_0(D))^2$ can be replaced by the polynomial Finite Element Spaces $(X^r_h)^2$, see Theorem 1.4.7. It is interesting to note that the first two steps involve the discretization of *standard* elliptic problems, *i.e.*, problems without conditions on the divergence of the solution. The third and last step of a discrete problem is again a projection on a divergence-free subspace.

The convergence of the method at a finite dimensional level (and for different discretization of the space variable), is discussed in Temam [Tem77] Ch. 3, §7.

In the concrete applications it is very important to have efficient numerical methods to solve the linear, non-symmetric and elliptic systems arising in the first two steps. Systems of this kind are known in literature as *advection-diffusion* systems. For these systems (but also the scalar non-symmetric problem presents the same pathologies) abstract results, as the Lax Milgram lemma, imply existence and uniqueness of the solution at both the infinite and finite dimensional level. On the other hand, their numerical approximation involves some difficult stability questions when the viscosity ν term is "small," see Section 4.4.2.

Having in mind the *Chorin Temam method* the importance of the numerical analysis of advection diffusion equations becomes clear. These equations are not only a linearized model for the fluid-dynamic equations, but they are also a basic tool in some numerical methods for the Navier-Stokes equations. In Chapter 4 we shall discuss some numerical methods for solving non-symmetric elliptic equations and the numerical difficulties arising in their study.

Chapter 2

Regularity results

In this chapter we recall some basic fact regarding uniqueness, and regularity for the solutions of the Navier-Stokes equations. We consider the problems regarding the possible global existence, in time, of smooth solutions in three dimensions. In particular we present the classical result regarding *strong solutions* and the uniqueness conditions due to Prodi and Serrin, which ensures also the regularity of weak solutions. Then we explain the special role played by the pressure in the system of Navier-Stokes equations and we show how to reconstruct the pressure from the velocity field. Some recent results concerning the regularity are presented. These results, due to Beirão da Veiga, use the truncation method and give sufficient conditions for the regularity. They are based on suitable combination of velocity and pressure. In the last section we reverse the standard approach and we obtain new results regarding the smoothness of the velocity, by starting from the pressure. In particular the smoothness of the velocity field is proved by starting only from summability conditions on the pressure.

2.1 Regular solutions

In this section we briefly explain how is it possible to prove more regularity for the solutions of the Navier-Stokes equations. We recall that if $\mathbf{f} = \mathbf{0}$, $\mathbf{u}_0 \in V$ and the boundary of $D \subset \mathbb{R}^3$ is smooth, then there exists a fully *classical solution* $(\mathbf{u}, p) \in (C^\infty(\overline{D} \times (0, T)))^4$, on a time interval $(0, T)$ with T bounded below in terms of the Dirichlet norm $\|\mathbf{u}_0\|$ of the initial datum. We present this result, that is due to Ladyženskaya [Lad66]; a simple proof and additional remarks regarding this result can be found in Heywood [Hey80].

Without entering into details of sharp results, we show how the *boot-strap* argument works. This is one of the most powerful tools to prove regularity results. We restrict ourselves to the steady state problem, since it is rather standard to pass to the non-stationary problem, see Temam [Tem77], Ch. III, §3. Some difficulties arise if we want to have full regularity also at time $t = 0$, provided the initial datum is smooth; for instance it is not sufficient that $\mathbf{u}_0 \in (C^\infty(\overline{D}))^3 \cap V$ and that ∂D is of class C^∞ to ensure that $\mathbf{u} \in (C^\infty(\overline{D} \times [0, T]))^3$, for some $T > 0$. In this case some *compatibility conditions* must be satisfied and these can be found in Temam [Tem82].

We remark that in the two dimensional case, the regularity results which hold for the stationary case can be extended to the time-dependent Navier-Stokes equations. In three spatial dimensions the regularity results hold only locally in time.

The main idea is to consider the first term $\mathbf{u}$ of the convective term $(\mathbf{u} \cdot \nabla)\, \mathbf{u}$ as a known term and then to study the linear problem

$$-\nu\Delta\mathbf{u} + \nabla p = -(\mathbf{v}\cdot\nabla)\,\mathbf{u}.$$

with $\mathbf{v} = \mathbf{u}$ as regular as weak solutions are. To show how this method work we give some of the calculations needed to prove the regularity. Some distinction, between the problem in two dimension and that one in three dimensions, is needed.

The two dimensional case

The term $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ equals $\sum_{i=1}^{d}\partial(u_i\mathbf{u})/\partial x_i$. If $d = 2$, we use the Sobolev embedding theorem to get that $u_i \in H_0^1(D) \subset L^\alpha(D)$, for any $1 \le \alpha < +\infty$. This implies that $\partial(u_iu_j)/\partial x_i$ belongs to $W^{-1,\alpha}(D)$. The regularity results for the Stokes operator show that u_i belongs to $W^{1,\alpha}(D)$ and p belongs to $L^\alpha(D)$. If now $\alpha > 2$, we have that $W^{1,\alpha}(D) \subset L^\infty(D)$, hence $u_i\,\partial u_j/\partial x_i$ belongs to $L^\alpha(D)$. This implies that $u_i \in W^{2,\alpha}(D)$ and $p \in W^{1,\alpha}(D)$. Repeating the same argument we find that $u_i\,\partial u_j/\partial x_i$ belongs to $W^{1,\alpha}(D)$ and consequently $u_i \in W^{3,\alpha}(D)$ and $p \in W^{2,\alpha}(D)$. The same argument can be used till the regularity of D and $\mathbf{f}$ allows to get regularity of the solutions of the Stokes problem, see Theorem 1.2.6.

The three dimensional case

In the three dimensional problem we have to use different estimates. In particular we can only infer that $u_i \in H_0^1(D) \subset L^6(D)$. This implies that $u_i\,\partial u_j/\partial x_i$ belongs to $L^{3/2}(D)$. By using again Theorem 1.2.6, we have that $u_i \in W^{2,3/2}(D)$ and this finally implies that $u_i \in L^\alpha(D)$ for any $1 \le \alpha < \infty$. Therefore, $\sum_{i=1}^{d}\partial(u_iu_j)/\partial x_i \in W^{-1,\alpha}(D)$ for any $1 \le \alpha < \infty$ and we can use the same proof given for $d = 2$.

2.1.1 On weak and strong solutions

In the previous Sections 1.2.3-1.2.4 we introduced the notion of weak and strong solution. The strong solutions are not *classical solutions*, but they are very important. We now give some additional results which can be useful to understand the role of weak and strong solutions, in the theory of Navier-Stokes equations. We recall the following result (a corollary of Proposition 1.2.12), which is useful to estimate the trilinear term and that is stated as Lemma 1 in Ladyženskaya book [Lad69].

Proposition 2.1.1. *For any open set $D \subset \mathbb{R}^2$, we have that:*

$$\|u\|_{L^4(D)} \le 2^{1/4}\|u\|_{L^2(D)}^{1/2}\|\nabla u\|_{L^2(D)}^{1/2} \qquad \forall\, u \in H_0^1(D).$$

Proof. It suffices to prove the last inequality for $v \in C_0^\infty(D)$. For such a v we write

$$|v(\mathbf{x})|^2 = 2\int_{-\infty}^{x_1} v(\zeta_1, x_2)\frac{\partial v(\zeta_1, x_2)}{\partial x_1}\,d\zeta_1$$

and therefore

$$|v(\mathbf{x})|^2 \le 2\,v_1(x_2),$$

where

$$v_1(x_2) = \int_{-\infty}^{+\infty} |v(\zeta_1, x_2)|\left|\frac{\partial v(\zeta_1, x_2)}{\partial x_1}\right| d\zeta_1.$$

By interchanging the role of x_1 and x_2 we have

$$|v(\mathbf{x})|^2 \leq 2\, v_2(x_1) = 2 \int_{-\infty}^{+\infty} |v(x_1,\zeta_2)| \left| \frac{\partial v(x_1,\zeta_2)}{\partial x_2} \right| d\zeta_2.$$

We finally obtain

$$\begin{aligned} \textstyle\int_{\mathbb{R}^2} |v(\mathbf{x})|^4 \leq \int_{\mathbb{R}^2} v_1(x_2) v_2(x_1)\, d\mathbf{x} \quad & \leq 4 \Big(\textstyle\int_{\mathbb{R}} v_1(x_2)\, dx_2 \Big) \Big(\int_{\mathbb{R}} v_2(x_1)\, dx_1 \Big) \\ & \leq 4 \|v\|^2_{L^2(\mathbb{R}^2)} \left\| \frac{\partial v}{\partial x_1} \right\|^2_{L^2(\mathbb{R}^2)} \left\| \frac{\partial v}{\partial x_2} \right\|^2_{L^2(\mathbb{R}^2)} \\ & \leq 2 \|v\|^2_{L^2(D)} \|\nabla v\|^2_{L^2(D)}. \end{aligned}$$

□

This inequality is very simple, but it plays a very big role in the theory of Navier-Stokes equations. With Proposition 2.1.1 we can easily prove the following uniqueness result.

Theorem 2.1.2. *Let $D \subset \mathbb{R}^2$ be open bounded and of class C^2. Let $\mathbf{f} \in L^2(0,T;V')$. Two weak solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ of the Navier-Stokes equations must coincide, or in other words, weak solutions are unique.*

Proof. The proof of this theorem is very simple and makes use of the classical methods for linear equations, joint with a simple estimate for the nonlinear term. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions, as usual, we define $\mathbf{w} := \mathbf{u}_1 - \mathbf{u}_2$. It is easily seen that $\mathbf{w}$ solves the following problem:

$$\begin{cases} \dfrac{d\mathbf{w}}{dt} + \nu A\, \mathbf{w} + B(\mathbf{u}_1, \mathbf{w}) + B(\mathbf{w}, \mathbf{u}_2) = 0, \\ \\ \mathbf{w}(0) = 0. \end{cases}$$

We take the scalar product with $\mathbf{w}$ and we obtain the following equality

$$\frac{1}{2}\frac{d}{dt} |\mathbf{w}|^2 + \nu \|\mathbf{w}\|^2 + b(\mathbf{w}, \mathbf{u}_2, \mathbf{w}) = 0.$$

We use the Hölder inequality and Proposition 2.1.1 to deduce that $|b(\mathbf{w}, \mathbf{u}_2, \mathbf{w})| \leq c|\mathbf{w}|\|\mathbf{w}\|\|\mathbf{u}_2\|$. By applying the Young inequality (with exponents $q = q' = 2$), we get

$$\frac{d}{dt} |\mathbf{w}|^2 \leq \frac{c}{\nu} \|\mathbf{u}_2\|^2 |\mathbf{w}|^2.$$

By using Gronwall lemma, we can infer the following inequality

$$|\mathbf{w}(t)|^2 \leq |\mathbf{w}(0)|^2 \, \mathrm{e}^{\dfrac{c}{\nu} \displaystyle\int_0^t \|\mathbf{u}_2(s)\|^2 \, ds}.$$

Since $\mathbf{w}(0) = 0$ and since $\mathbf{u}_2$ belongs to $L^2(0,T;H)$, we can conclude that $\mathbf{w}(t) \equiv 0$. □

We try to use (at least formally, because $d\mathbf{u}/dt$ is not regular) the same techniques for the three dimensional problems. If $d = 3$, by using the same argument of Proposition 2.1.1, we can prove the following estimate:

$$\|u\|_{L^4(D)} \leq \sqrt{2}\|u\|_{L^2(D)}^{1/4}\|\nabla u\|_{L^2(D)}^{3/4} \qquad \forall\, u \in H_0^1(D).$$

If we mimic the proof of Theorem 2.1.2, we get into troubles. In fact we obtain that

$$\frac{d}{dt}|\mathbf{w}(t)|^2 \leq \frac{c}{\nu^4}\|\mathbf{u}_2(t)\|^4|\mathbf{w}(t)|^2$$

and we do not know wether $\int_0^T \|\mathbf{u}_2(t)\|^4\, dt$ is finite[1] or not, and consequently we cannot use the Gronwall lemma to conclude.

This argument seems crude and one can think that sharper estimates can make the proof work. We recall that each of the sophisticated methods used in the last seventy years to try to prove the uniqueness if $d = 3$, failed in a similar way.

The problem of uniqueness of weak solution is still open. In three dimensions we can prove the following result, due to Kiselev and Ladyženskaya [KL57].

Theorem 2.1.3. *Let $D \subset \mathbb{R}^3$ be open bounded and of class C^2 and let $\mathbf{f}$ belong to $L^2(0, T; H)$. Two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ of the Navier-Stokes equations belonging to $L^2(0, T; \mathcal{D}(A)) \cap C_w(0, T; V)$ must coincide.*

Proof. We use the same techniques and the estimate $|b(\mathbf{w}, \mathbf{u}_2, \mathbf{w})| \leq c|\mathbf{w}|\|\mathbf{w}\|\|\mathbf{u}_2\|^{1/2}|A\,\mathbf{u}|^{1/2}$, to obtain

$$\frac{d}{dt}|\mathbf{w}|^2 \leq \frac{c}{\nu}\|\mathbf{u}_2\|\,|A\,\mathbf{u}_2|\,|\mathbf{w}|^2.$$

From the Gronwall lemma we have the following estimate

$$|\mathbf{w}(t)|^2 \leq |\mathbf{w}(0)|^2\ \mathrm{e}^{\displaystyle\int_0^t \frac{c}{\nu}\|\mathbf{u}_2(s)\|\,|A\,\mathbf{u}_2(s)|\,ds}$$

and, by using the hypothesis on $\mathbf{u}_2$, the integral is finite. Consequently we get that $\mathbf{w} \equiv 0$. □

Remark 2.1.4. *With a completely different technique it is possible to prove the same result by assuming that only one of the two solutions is strong: in other words, strong solutions are unique in the larger class of weak solutions.*

A continuation principle

If we are concerned to the regularity of the weak solutions of the Navier-Stokes equations, we may suppose $D, \mathbf{u}_0$ and $\mathbf{f}$ as smooth as we want (say again $\mathbf{f} = \mathbf{0}$, to avoid inessential technical arguments). We know that there exists a strong solution, at least in an time-interval $[0, T_0)$ and in particular $\sup_{0 \leq t \leq T_0} \|\mathbf{u}(t)\|$ is finite.

[1] By the way we found that $\mathbf{u}_2 \in L^4(0, T; V)$ is a sufficient condition for uniqueness, and strong solutions satisfy it. It is possible to make this formal argument rigorous and, to have uniqueness, it is sufficient the weaker assumption that only *one* of the two weak solution belongs to $L^8(0, T; (L^4(D))^3)$, see Serrin [Ser63].

Let now $T > T_0$; we know that there exists a weak solution $\widetilde{\mathbf{u}}$ on $(0, T)$ and, from the previous remark, we know that $\mathbf{u} \equiv \widetilde{\mathbf{u}}$ on $(0, T_0)$. We consider the maximal interval of existence $(0, T_*)$ (or *life-span*) of strong solutions, where

$$T_* := \max\big\{T > 0 : \text{ such that there exists a solution } \mathbf{u} \in L^\infty(0, T; V) \cap L^2(0, T; \mathcal{D}(A))\big\}.$$

We have the following result

Proposition 2.1.5. *Since strong solutions are unique, for a strong solution we have that necessarily*

$$\limsup_{t\to T_*} \|\mathbf{u}(t)\| = \infty \quad \textit{if} \quad T_* = \infty \qquad \textit{and} \qquad \limsup_{t\to \overline{T}} \|\mathbf{u}(t)\| < \infty \quad \textit{if} \quad \overline{T} < T_* < \infty.$$

Proof. The proof follows by a contradiction argument. Suppose that $[0, T_*)$ is the maximal existence interval of $\mathbf{u}$. If $\limsup_{t\to T_*} \|\mathbf{u}(t)\| < \infty$, then we could find $c \in \mathbb{R}$ and t as close as we want to T_* such that

$$\|\mathbf{u}(t)\| \leq c.$$

From the global existence theorem we have a *weak* solution $\mathbf{v}(s)$ with $\mathbf{v}(0) = \mathbf{u}(t_0) \in V$. The solution $\mathbf{v}(s)$ would be also strong in a time interval $(0, T_1)$, depending on $\|\mathbf{v}(0)\| = \|\mathbf{u}(t_0)\|$ through relation (1.16). Since $\|\mathbf{u}(t_0)\|$ is bounded from above as t tends to T_*, the corresponding T_1 is bounded below (in other words T_1 can be chosen uniformly for t_0 near T_*.) If $T_* - t_0 < T_1$, we obtain a strong solution $\widetilde{\mathbf{u}}(s) := \mathbf{v}(s + t_0)$ which coincides with $\mathbf{u}$ for $t < T_*$. Contradiction, because in this way we extended $\mathbf{u}(t)$ beyond T_*. The quantity $\|\mathbf{u}(\,\cdot\,)\|$ becoming infinite is then a necessary condition for loss of regularity. □

Remark 2.1.6. *We finally understand the role of strong solutions. A strong solution is not a classical solution at all, but it is unique in the class of weak solutions and furthermore a strong solution is smooth as the initial datum* $\mathbf{u}_0$*, provided the boundary* ∂D *and* $\mathbf{f}$ *are smooth enough. The last sentence says that if we are in the life-span of a strong solution, the solution belongs at least to* $L^\infty(0, T; V) \cap L^2(0, T; \mathcal{D}(A))$*, but it is also more regular. In particular it has the same regularity of the data, till the norm in* V *is bounded, recall the result of Ladyženskaya [Lad66] in Section 2.1.*

2.1.2 On the possible global existence of strong solutions

We start by recalling that in Section 1.2.4 we showed that in dimension two strong solutions are unique and exist globally in time. In three dimensions the situation changes drastically: strong solutions are unique, but they exist only for small initial data (or small times). In three dimensions we can only prove that weak solution are global in time, but we do not have a uniqueness result for them.

One of the main open problems is to show that a solution which is smooth at one instant cannot develop a singularity at a later time. Many authors tried to find appropriate *a-priori* estimates for smooth solutions, in order to use them in a continuation argument. All the efforts regarding this subject failed because many conditions on the velocity were found, but we are not able to check if they are satisfied by weak solutions.

By following the exposition given by Heywood [Hey90], we suppose $\mathbf{f} \equiv \mathbf{0}$ and the basic differential inequality we have to deal with is the following one:

$$\frac{d}{dt}\|\mathbf{u}(t)\|^2 \leq C \begin{cases} \|\mathbf{u}(t)\|^4 & \text{for } d = 2, \\[2ex] \|\mathbf{u}(t)\|^6 & \text{for } d = 3. \end{cases}$$

We essentially derived the last inequality in the previous chapter and in spite of its simplicity, it is the basic point to understand the problems arising in the study of three dimensional Navier-Stokes equations.

We consider a solution with an hypothetical singularity[2] at $t = t^*$, *i.e.*, a solution for which $\|\mathbf{u}(t)\| \to +\infty$ when $t \to t^*$. We consider the function $\phi(t)$ which satisfies the following ordinary differential equation

$$\frac{d}{dt}\phi(t) = C\,\phi^2(t) \qquad \text{for } d = 2,$$

$$\frac{d}{dt}\phi(t) = C\,\phi^3(t) \qquad \text{for } d = 3.$$

These equations correspond to the differential inequalities for $\|\mathbf{u}(t)\|^2$. Let $\phi_1(t) \in C([0,T))$ be the solution corresponding to the initial datum $\phi_1(0) = \|\mathbf{u}_0\|^2$. Furthermore, let $\phi_2(t)$ be a solution chosen to have a singularity at t^*. The graph of $\phi_2(t)$ lies below the graph of $\|\mathbf{u}(t)\|^2$ and

$$\int_{t^*-T}^{t^*} \phi_2(s)\,ds \le \int_{t^*-T}^{t^*} \|\mathbf{u}(s)\|^2\,ds.$$

Simple calculations show that

$$\int_{t^*-T}^{t^*} \phi_2(s)\,ds = \infty \qquad \text{if } d = 2,$$

$$\int_{t^*-T}^{t^*} \phi_2(s)\,ds = \frac{\sqrt{2}}{C}\sqrt{T} \qquad \text{if } d = 3.$$

By referring to the energy identity, which holds for smooth solutions,

$$\frac{1}{2}|\mathbf{u}(t)|^2 + \nu\int_0^t \|\mathbf{u}(s)\|^2\,ds = \frac{1}{2}|\mathbf{u}(0)|^2,$$

we see that a singularity is impossible if $d = 2$ and it is impossible in the three dimensional case if

$$\frac{1}{2}|\mathbf{u}(0)|^2 \le \frac{\sqrt{2}}{C}\sqrt{T}.$$

In the last case the development of a singularity requires more than the available energy. This simple argument shows the critical point in the theory of Navier-Stokes equations.

2.1.3 The Prodi-Serrin condition

In the previous section we pointed out the importance of strong solutions. We observe that the theory of Navier-Stokes equations is not satisfactory in the three dimensional case, because we have existence in a class (weak solutions) in which we do not have uniqueness and, on the other hand, we have uniqueness in a class (strong solutions) in which only local existence is known.

The problem was investigated in several directions and some sufficient conditions which ensure the uniqueness of weak solutions were found. The best known condition is that one known as *Prodi-Serrin condition.* The result, that we are going to show, was proved independently[3] by

[2] This is the condition for the blow-up of a strong solution.

[3] As it was pointed out to me by Prof. G.P. Galdi, the Prodi-Serrin condition (just for the pure Cauchy problem) is also given in a footnote of the 1934 seminal paper by Leray [Ler34b]. The Prodi-Serrin condition should be called, more properly, *Leray-Prodi-Serrin condition.*

Prodi [Pro59] and Serrin [Ser62]. The condition (2.1) seems very simple but, up to now, no result ensures that a weak solution satisfies it.

Theorem 2.1.7. *Let $d \geq 3$ and let $\mathbf{u}$ be a weak solution of the Navier-Stokes equations which satisfies the additional hypothesis*

$$u_j \in L^r(0,T;L^s(D))^d \qquad \text{for} \quad \frac{2}{r}+\frac{d}{s} \leq 1, \quad s \geq d. \tag{2.1}$$

Then such a solution is unique in the class

$$L^2(0,T;V) \cap L^\infty(0,T;H) \cap L^r(0,T;(L^s(D))^d)$$

Proof. We divide the proof into three steps.

Step 1) Estimate on the trilinear term.
We consider the interesting limit case of (2.1): $\frac{2}{r}+\frac{d}{s}=1$. By using the Hölder inequality we get

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leq C\|\mathbf{u}\|_{(L^s(D))^d}\|\mathbf{v}\|\|\mathbf{w}\|_{(L^\rho(D))^d}, \qquad \text{if} \quad \frac{1}{s}+\frac{1}{\rho}=\frac{1}{2}.$$

For each $\mathbf{u} \in (C^\infty(\overline{D}))^d$ we have, by the Hölder inequality,

$$\|\mathbf{u}\|_{(L^\rho(D))^d} \leq \|\mathbf{u}\|^{2/r}_{(L^2(D))^d}\|\mathbf{u}\|^{d/s}_{(L^{2d/(d-2)}(D))} \qquad \text{for} \quad \frac{1}{\rho}=\frac{2/r}{2}+\frac{d/s}{2d/(d-2)}.$$

By using the Sobolev embedding theorem, we have that $H^1_0(D) \subset L^{2d/(d-2)}(D)$ and consequently

$$\|\mathbf{u}\|_{(L^\rho(D))^d} \leq C\|\mathbf{u}\|^{2/r}_{(L^2(D))^d}\|\mathbf{u}\|^{d/s}.$$

The last estimate shows that

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leq C\|\mathbf{u}\|_{(L^s(D))^d}\|\mathbf{v}\||\mathbf{w}|^{2/r}\|\mathbf{w}\|^{d/s}. \tag{2.2}$$

Step 2) Regularity of the time-derivative.
With the preliminary estimate (2.2) we can prove additional regularity of $d\mathbf{u}/dt$. We recall that *a-priori* $d\mathbf{u}/dt \in L^{4/3}(0,T;V')$ while, if the Prodi-Serrin condition (2.1) is satisfied, we have that $d\mathbf{u}/dt \in L^2(0,T;V')$.

By using the estimate (2.2) and by recalling that $\mathbf{u} \in L^\infty(0,T;H)$ we have

$$\begin{aligned}|b(\mathbf{u},\mathbf{u},\mathbf{v})| = |-b(\mathbf{u},\mathbf{v},\mathbf{u})| &\leq C_1\|\mathbf{v}\|\|\mathbf{u}\|_{(L^s(D))^d}|\mathbf{u}|^{2/r}\|\mathbf{u}\|^{d/s},\\ &\leq C_2\|\mathbf{v}\|\|\mathbf{u}\|_{(L^s(D))^d}\|\mathbf{u}\|^{d/s}.\end{aligned}$$

By observing that $t \mapsto \|\mathbf{u}(t)\|_{(L^s(D))^d}$ belongs to $L^r(0,T)$ and that $t \mapsto \|\mathbf{u}(t)\|^{d/s}$ belongs to $L^{2s/d}(0,T)$, we get that

$$t \mapsto \|\mathbf{u}(t)\|_{(L^s(D))^d}\|\mathbf{u}(t)\|^{d/s} \quad \text{belongs to } L^2(0,T).$$

Consequently $(\mathbf{g},\mathbf{v}) := b(\mathbf{u},\mathbf{u},\mathbf{v})$ with $\mathbf{g}$ belonging to $L^2(0,T;V')$. This last result implies that $d\mathbf{u}/dt \in L^2(0,T;V')$.
Step 3) Proof of uniqueness.

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions. We define $\mathbf{w} := \mathbf{u}_1 - \mathbf{u}_2$. By using the regularity on the time derivative we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}(t)|^2 + \nu\|\mathbf{w}(t)\|^2 = -b(\mathbf{w}(t), \mathbf{w}(t), \mathbf{u}_2(t)).$$

An application of the method used in Step 1 (the following formula follows from (2.2) by interchanging the role of $\mathbf{u}$ and $\mathbf{w}$) gives

$$|b(\mathbf{w}, \mathbf{w}, \mathbf{u}_2)| \leq C\|\mathbf{u}_2\|_{(L^s(D))^d}|\mathbf{w}|^{2/r}\|\mathbf{w}\|^{1+d/s}.$$

We define $M(t) := \|\mathbf{u}_2\|^r_{(L^s(D))^d}$ and we use the Young inequality with exponents

$$p = r \qquad \text{and} \qquad p' \text{ defined by } \quad \frac{1}{p'} = 1 - \frac{1}{p} = 1 - \frac{1}{r} \leq \text{by (2.1)} \leq 1 - \frac{1}{2}\left(1 - \frac{d}{s}\right) = \frac{s+d}{2s}.$$

We obtain

$$\begin{aligned} |b(\mathbf{w}(t), \mathbf{w}(t), \mathbf{u}_2(t))| &\leq C\, M(t)^{1/r}|\mathbf{w}(t)|^{2/r}\|\mathbf{w}(t)\|^{1+d/s}, \\ &\leq \nu\|\mathbf{w}(t)\|^2 + C_3\, M(t)|\mathbf{w}(t)|^2, \end{aligned}$$

and finally

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}(t)|^2 \leq C_3\, M(t)|\mathbf{w}(t)|^2.$$

The last inequality and the fact that (by hypothesis) $M(t) := \|\mathbf{u}\|^r_{(L^s(D))^d}$ belongs to $L^1(0,T)$, proves that $\mathbf{w}(t) \equiv 0$. □

If $d = 3$ Theorem 2.1.7 can be compared with the results of uniqueness for strong solutions, that we know to exist locally. In fact if $\mathbf{u} \in L^\infty(0,T;V)$ then $\mathbf{u} \in L^\infty(0,T;(L^s(D))^d)$ with $1/s = 1/2 - 1/d$, and consequently

$$\mathbf{u} \in L^r(0,T;((L^s(D))^d), \qquad \frac{2}{r} + \frac{d}{s} = 1.$$

Remarks and references regarding the existence of solutions for $d \geq 4$ can be found in J.-L. Lions [JLL69], Ch. 1.

Further regularity by the Prodi-Serrin condition

The Prodi-Serrin condition was introduced to study the uniqueness of the solutions, but it was later discovered by Sohr [Soh83], that the condition

$$u_j \in L^r(0,T;L^s(D))^d \quad \text{for} \quad \frac{2}{r} + \frac{d}{s} = 1 \qquad s \in (d, \infty]$$

implies also regularity (provided the data are regular and the necessary compatibility conditions are satisfied). We do not give the proof of this sharp result, but we only say that its proof is based on fine estimates on the solution of linear Cauchy problems, see also Section 2.3.1.

2.2 A short digression on the role of the pressure

When dealing with the Navier-Stokes equations we pointed out in Chapter 1 that the problem is well-posed because we have $d+1$ equations and $d+1$ unknowns. The equations satisfied by $\mathbf{u}$ is a semilinear parabolic system, but when we try to apply the standard variational techniques we come into some troubles. The main problem is that the pressure p does not satisfy a parabolic equation, but it is a Lagrange multiplier associated to the constraint $\operatorname{div}\mathbf{u} = 0$. If we apply the divergence to the first (vectorial) equation we obtain the following:

$$-\Delta p = \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i}. \tag{2.3}$$

This last equation is very useful in the study of Euler equations ($\nu = 0$). In the case of inviscid fluids we have a well-posed elliptic problem if we couple (2.3) with the Neumann condition which arise by taking the scalar product by $\mathbf{n}$ of the equation for the conservation of momentum.

When dealing with the Navier-Stokes equations, the boundary condition is

$$\frac{\partial p}{\partial \mathbf{n}} = (\mathbf{f} + \nu\Delta\mathbf{u})\cdot\mathbf{n} \quad \text{on } \partial D,$$

which is of no practical[4] use (unless we restrict to $D = \mathbb{R}^d$ or to periodic boundary conditions), because the elliptic problem for the pressure is not well posed. When dealing with $D = \mathbb{R}^3$, for example, we can write the pressure as a function of $\mathbf{u}$ and $\mathbf{f}$,

$$p(\mathbf{x},t) = G(\mathbf{u},\mathbf{f}) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\operatorname{div}\Big[(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \mathbf{f}\Big](\mathbf{y},t)\,d\mathbf{y}$$

and we can study the equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla G(\mathbf{u},\mathbf{f}) = \mathbf{f} \quad \text{in } \mathbb{R}^3\times[0,T].$$

We observe that the operator $G(\mathbf{u},\mathbf{f})$ is *non-local, i.e.*, the pressure has a non-local effect on the velocity. In other words, the velocity field at *each point* $\mathbf{y}\in\mathbb{R}^3$ is responsible for the values of the pressure gradient at a *given point* $\mathbf{x}\in\mathbb{R}^3$; then the pressure gradients acts like a body force at the point $\mathbf{x}$.

Remark 2.2.1. *To see in more detail the role of the pressure, we consider a system of two coupled partial differential equations in two scalar variables u, v*

$$\begin{cases} \dfrac{\partial u}{\partial t} = F_1(D^\alpha u, D^\alpha v) & |\alpha| \le m, \\[2ex] \dfrac{\partial v}{\partial t} = F_2(D^\alpha u, D^\alpha v) & |\alpha| \le m, \end{cases}$$

where $\alpha := (\alpha_1,\dots,\alpha_d)$ is a multi-index, D^α is the differential operator $\partial^{|\alpha|}/\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}$ and m is a positive integer. When both u, v are known at time t, we can compute the right hand sides and we can find the rates of change (partial derivative with respect to time) of u and v at the same time. We can know the infinitesimal change of u and v at point $\mathbf{x}$. The effect of u and v, each

[4] The compatibility condition $\int_D \Delta p\,d\mathbf{x} = \int_{\partial D}\partial p/\partial\mathbf{n}\,dS$ is not *a-priori* satisfied.

on the other, is local. The influence of the pressure is different. It is not the effect of a variable coupled with the others, but it is a consequence of the incompressibility constraint $\operatorname{div}\mathbf{u} = 0$. *A way to impose local effect of the pressure is to come back to problems with variable density and to make the constitutive assumption*

$$p(\mathbf{x}, t) = \mathcal{F}(\rho(\mathbf{x}, t)).$$

In this way the variables $\mathbf{u}$ *and* p *are coupled in the ordinary (local) way. From the physical point of view the incompressibility constraint implies the infinite propagation-speed of perturbations. On the other hand, in compressible models with variable density, the perturbations travel with a finite speed. Unfortunately, the study of compressible models causes, up to now, mathematical difficulties bigger than the ones arising in the study of incompressible equations.*

2.2.1 Introduction of the pressure

The problems regarding the pressure were "solved" (or better hidden in the formulation) by using the Helmholtz decomposition, see Proposition 1.2.2. In this case the pressure disappears from the weak formulation, because gradients are orthogonal to divergence-free fields. The various theorems of existence, that we have shown, do not involve the pressure. We introduce now the pressure in the following way. Let us set:

$$\mathbf{U}(t) := \int_0^t \mathbf{u}(s)\,ds, \qquad \beta(t) := \int_0^t B(\mathbf{u}(s), \mathbf{u}(s))\,ds, \qquad \mathbf{F}(t) := \int_0^t \mathbf{f}(s)\,ds.$$

If $\mathbf{u}$ is a solution of the Navier-Stokes equations we have that $\mathbf{U}, \beta, \mathbf{F}$ belong to $C([0,T]; V')$. By integrating in time the abstract Navier-Stokes equations $d\mathbf{u}/dt + \nu A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f}$, we get

$$\nu((\mathbf{U}(t), \mathbf{v})) = < \mathbf{g}(t), \mathbf{v} > \qquad \forall\, \mathbf{v} \in V, \quad \forall\, t \in [0,T],$$

with

$$\mathbf{g}(t) := \mathbf{F}(t) - \beta(t) - \mathbf{u}(t) + \mathbf{u}_0, \qquad \mathbf{g} \in C([0,T]; V').$$

We recall the following proposition, see Duvaut and J.-L. Lions [DL76] Ch. 3.

Proposition 2.2.2. *Let* $D \subset \mathbb{R}^d$ *be an open set and let* $\mathbf{f} = (f_1, \dots, f_d)$ *such that* f_i *are distributions on* D, *for* $i = 1, \dots, d$. *Then we have the following result:*

$$\mathbf{f} = \nabla p, \ p \text{ being a distribution}, \quad \text{if and only if} \quad < \mathbf{f}, \mathbf{v} > = 0 \quad \forall\, \mathbf{v} \in \mathcal{V}.$$

Let furthermore $D \subset \mathbb{R}^d$ *be a bounded Lipschitz domain.*

a) If a distribution p *has all its first distributional derivatives in* $L^2(D)$, *then* p *itself belongs to* $L^2(D)$ *and*

$$\|p\|_{L^2(D)/\mathbb{R}} \leq c_1 \|\nabla p\|_{L^2(D)}, \qquad 0 < c_1 \in \mathbb{R}.$$

b) If a distribution p *has all its first distributional derivatives in* $H^{-1}(D)$, *then* p *belongs to* $L^2(D)$

$$\|p\|_{L^2(D)/\mathbb{R}} \leq c_2 \|\nabla p\|_{-1,D}, \qquad 0 < c_2 \in \mathbb{R}.$$

With the last proposition it is easy to conclude that for each $t \in [0,T]$ there exists $q(t) \in L^2(D)$, such that $\nabla q \in C([0,T]; (H^{-1}(D))^3)$ which satisfies

$$-\nu\Delta\mathbf{U}(t) + \nabla q(t) = \mathbf{g}(t).$$

This result imply that

$$q \in C([0,T]; L^2(D)).$$

By setting $p = \partial q/\partial t$ (the derivative is defined in the distributional sense) we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}.$$

In this way we can show a first regularity result for the pressure: it is the time derivative of a continuous function with value in $L^2(D)$. We observe that the same argument shows that more regularity on the velocity field implies more regularity on the pressure.

2.3 On the possible regularizing effect of the pressure

The problem of uniqueness of weak solutions or of their global regularity are basically open since appeared the papers by Leray cited in the bibliography. There were found many conditions of the following kind: if the velocity satisfies condition ... then it is regular. These results were found with the hope that, later, someone could be able to show that such a condition is satisfied by weak solutions.

In this section we show some results which are guided by the same idea. A new approach is, instead of looking for conditions for the velocity, to look for conditions involving the pressure or some combination of velocity and pressure.

We start by showing some recent results, which are obtained by using the classical truncation method. Then we show some results which are based on the use of L^α *energy-type* estimates.

2.3.1 Some results via the truncation method

The basic idea underlying the following results is to use (in a non standard way) some estimates for linear scalar parabolic equations. The idea used in the *boot-strap* argument is generalized and different methods are applied.

We consider the following parabolic equation for the scalar unknown u :

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{j=1}^{d} b_j \frac{\partial u}{\partial x_j} = \sum_{j=1}^{d} \frac{\partial f_j}{\partial x_j} - f \quad \text{in } D \times [0,T] := D_T, \tag{2.4}$$

and we treat in a classical way the linear initial-boundary value problem, which arise supplementing the equation above with bounded initial and boundary data. An application of Theorem 7.1. Ch. III §7 of the classical book by Ladyženskaja, Solonnikov and Ural'ceva [LSU67], gives the following result for (2.4).

Theorem 2.3.1. *Assume that*

$$b_i, f_i \in L^r(0,T; L^s(D)) \quad \textit{and} \quad f \in L^{r/2}(0,T; L^{s/2}(D)), \tag{2.5}$$

with

$$\frac{2}{r} + \frac{d}{s} < 1, \qquad r \in (2,\infty], \quad s \in (d,\infty]. \tag{2.6}$$

(Note that the pairs (r,s) can be different for different coefficients, i.e., each coefficient b_i, f_i, f may have its own couple (r,s), provided (2.5) and (2.6) are satisfied). Then the solution u of (2.4) belongs to $L^\infty(D_T)$.

Moreover, if we restrict to the coefficients b_i, we can replace (2.6) by the stronger condition

$$\frac{2}{r} + \frac{d}{s} = 1, \qquad s \in (d, \infty] \tag{2.7}$$

If we consider the system of the Navier-Stokes equations, the results above do not apply. One idea to understand what results should be expected for the Navier-Stokes equations is to apply "formally" the same results where $|\mathbf{u}|^2$ plays the role of the f_i, because $(\mathbf{u} \cdot \nabla)\,\mathbf{u} = \partial(u_i\mathbf{u})/\partial x_i$.

In this case we should get that $\mathbf{u}$ is bounded if $|\mathbf{u}|^2 \in L^r(0,T;L^s(D))$, with the condition r, s as in (2.6). For the Navier-Stokes equations the known result is that the solution is regular if $\mathbf{u}$ satisfy (2.7). This stronger result is also suggested by the use of the "linearized" equation

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} + \nabla p = 0 \quad \text{in } D \times [0,T],$$

where v_i plays the role of the b_i and $\mathbf{u} = \mathbf{v}$ plays the role of the unknown. In this case the techniques are similar to that one used in the *boot-strap* argument, see Sohr [Soh83].

Use of the pressure as a known term

Throughout this section $|\,.\,|$ will denote indifferently the absolute value of a scalar, the modulus of a vector or the Lebesgue measure of a set of $\mathbb{R}^d$.

A completely new device, in the study of regularity, is to use the pressure p as a "known smooth term." We recall that equation (2.3) suggests some correspondence between p and $|\mathbf{u}|^2$. The scalar equation (p corresponds now to the f_i's) suggests that the solution of the Navier-Stokes equations is regular if

$$p \in L^r(0,T;L^s(D)) \quad \text{for} \quad \frac{2}{r} + \frac{d}{s} < 2, \qquad r \in (2,\infty] \quad \text{and} \quad s \in (d,\infty].$$

We recall that if we consider $D = \mathbb{R}^d$ or a problem with periodic boundary conditions, from (2.3) (and if $\mathbf{f} = \mathbf{0}$) we can infer that $\|p\|_{L^2(D)} \leq C\||\mathbf{u}|^2\|_{L^2(D)}$. In these two cases the compatibility condition is satisfied (more precisely if $D = \mathbb{R}^d$ there are no conditions to be satisfied) and we can use the classical regularity result for elliptic equations, see also Section 2.2. On the other side, it is an open question if $\||\mathbf{u}|^2\|_{L^2(D)} \leq C\|p\|_{L^2(D)}$. We remark that this condition is true for inviscid irrotational stationary flows via the *Bernoulli law*, but there is no heuristic reason to believe that a similar relation holds for viscous fluids. The formal use of the results for linear parabolic equations suggests that $|p| \simeq |\mathbf{u}|^2$, at least as source of regularity. With this heuristic introduction we can show the first result, which is due to Beirão da Veiga [BdV97b].

Theorem 2.3.2. *Let $\mathbf{u}$ be a weak solution of the Navier-Stokes equations and let $\mathbf{u}_0$ be bounded. Assume that, for some positive real number k, the function ϕ_k, defined by*

$$\phi_k(\mathbf{x},t) = \begin{cases} \dfrac{|p(\mathbf{x},t)|}{1+|\mathbf{u}(\mathbf{x},t)|} & \text{if } |\mathbf{u}(\mathbf{x},t)| > k, \\[2ex] 0 & \text{otherwise,} \end{cases}$$

belongs to $L^r(0,T;L^s(D))$, where (r,s) satisfy (2.6). Then $\mathbf{u}$ is bounded in D_T.

As simple corollary, it is possible to prove the following result.

Corollary 2.3.3. *If the following condition*

$$\frac{|p|}{1+|\mathbf{u}|} \in L^r(0,T;L^s(D)) \tag{2.8}$$

holds, with (r,s) *satisfying* (2.6)*, then the solution* $\mathbf{u}$ *is bounded in* D_T.

This result gives a new insight into the possible regularity for the Navier-Stokes equations. Furthermore it involves (2.6) instead of condition (2.7), relative to the known results. The proof of Theorem 2.3.2 is rather technical and since in the sequel we do not use its methods we only give an idea of the proof. By using the *truncation method* introduced by De Giorgi [DG57] we set, for each $k > 0$

$$\mathbf{u}^{(k)} := \begin{cases} \left(1 - \dfrac{k}{|\mathbf{u}|}\right)\mathbf{u} & \text{if } |\mathbf{u}| \geq k, \\ \\ 0 & \text{otherwise.} \end{cases}$$

By using $\mathbf{u}^{(k)}$ as test function in the weak formulation of the Navier-Stokes equations, after some calculations it is possible to arrive to the following inequality

$$\frac{1}{2}\frac{d}{dt}\int_D |\mathbf{u}^{(k)}|^2\,d\mathbf{x} + \frac{\nu}{2}\int_D |\nabla \mathbf{u}^{(k)}|^2\,d\mathbf{x} \leq \frac{2}{\nu}k^2\int_{A_k}\frac{|p|^2}{|\mathbf{u}|^2}\,d\mathbf{x},$$

where $A_k(t) := \{\mathbf{x} \in D : \ |\mathbf{u}(\mathbf{x},t)| > k\}$. From this inequality it is standard to conclude by using the following theorem, that can be found in the book by Ladyženskaja, Solonnikov and Ural'ceva [LSU67], Ch. II §1.

Theorem 2.3.4. *Let* u *be a real function defined on* D_T *such that*

$$\int_{D_T} |u|^2 + |\nabla_x u|^2\,d\mathbf{x}\,dt < +\infty, \quad \textit{and} \quad \operatorname*{ess\,sup}_{0<t<T}\int_D |u|^2\,d\mathbf{x} + \int_{D_T}|\nabla_x u|^2\,d\mathbf{x}\,dt < \infty.$$

Suppose that $\operatorname{ess\,sup}_{\partial D\times(0,T)} u \leq \widehat{h}$, *for some* $\widehat{h} \geq 0$, *and that for certain positive constants* γ, κ *the following inequality holds:*

$$|u^{(k)}|_{D_T} \leq \gamma\,\kappa\,\mu^{\frac{1+\chi}{r}}(k), \quad k \geq \widetilde{k}.$$

Here $\mu(k) := \int_0^T |A_k(t)|^{r/q}$, *with* $1/r + d/2q = d/4$, *(with* $r > 2$ *if* $d > 2$*). Then* $\exists\,\beta > 0$ *such that:*

$$\operatorname*{ess\,sup}_{D_T}|u(\mathbf{x},t)| \leq 2\widehat{k}\left[1 + 2^{\frac{2}{\chi}+\frac{1}{\chi^2}}(\beta\,\gamma)^{1+\frac{1}{\chi}}\,T^{\frac{1+\chi}{r}}\,|D|^{\frac{1+\chi}{q}}\right].$$

2.3.2 A result in the framework of Marcinkiewicz spaces

In this section we show a regularity result, which holds in the Marcinkiewicz spaces. This result, which is strictly linked with that one of the previous section, is based on a different application of the truncation method. The result we present in this section, is due again to Beirão da Veiga [BdV98]. We start by defining the Marcinkiewicz spaces $L^p_*(E)$.

Definition 2.3.5. *Let* E *be a measurable bounded subset of* $\mathbb{R}^d$ *and let* $1 \leq p < +\infty$. *A measurable function* $f : E \to \mathbb{R}$ *belongs to* $L^p_*(E)$ *if there exists a constant* $[f]_p$ *such that*

$$|\{\mathbf{y} \in \mathbb{R}^d : \ |f(\mathbf{y})| > \sigma\}| \leq \left(\frac{[f]_p}{\sigma}\right)^{1/p} \qquad \forall\,\sigma > 0. \tag{2.9}$$

The smallest constant $[f]_p$ *for which* (2.9) *holds is called the "norm" of* f *in* $L^p_*(E)$.

The most important property of these spaces is that they are "very close" to the classical Lebesgue spaces $L^p(E)$. The following topological and algebraic inclusions hold:

$$L^p(E) \subset L^p_*(E) \subset L^{p-\varepsilon}(E), \qquad \forall\, \varepsilon > 0.$$

We now investigate on regularity properties of solutions belonging to $L^p(0,T;L^p(D)) := L^p(D_T)$ and $L^p_*(D_T)$ spaces. We recall that, up to the known results, a weak solution is regular if it satisfies

$$u_j \in L^{d+2}(D_T), \qquad j = 1, \dots, d,$$

because the condition

$$\frac{2}{r} + \frac{d}{r} = 1$$

implies $r = d + 2$. Furthermore a similar result it is not known for any exponent less than $d+2$. In the framework of Marcinkiewicz spaces the following theorem can be proved.

Theorem 2.3.6. *Let $(\mathbf{u}, p)$ be a weak solution of the Navier-Stokes equations. Assume that for some $\theta \in [0, 1[$ and some γ such that*

$$\frac{2(d+2)}{2\theta + (1-\theta)(d+2)} < \gamma < d + 2 \tag{2.10}$$

one has

$$\frac{p}{(1 + |\mathbf{u}|)^\theta} \in L^\gamma_*(D_T). \tag{2.11}$$

Then

$$u_i \in L^\mu(D_T), \qquad \mu = (1-\theta)\frac{(d+2)\gamma}{d+2-\gamma}. \tag{2.12}$$

Moreover if

$$\frac{p}{1 + |\mathbf{u}|} \in L^\gamma_*(D_T), \qquad \gamma > d + 2, \tag{2.13}$$

then $\mathbf{u}$ is bounded in D_T.

We omit the proof of this result and we remark that it is obtained by using the *truncation method* with the following test functions:

$$\mathbf{u}^{(k)}(\mathbf{x}, t) := \max\{|\mathbf{u}(\mathbf{x}, t)| - k, 0\}.$$

The proof follows with the application of a technique, which generalizes that one introduced by Stampacchia [Sta65]. We show the implications of Theorem 2.3.6, without other details about its proof.

Remark 2.3.7. *We start observing that if $\gamma > (d+2)/(2-\theta)$ then the solution is smooth, because in this case $\mu > d+2$. If we consider the result and we suppose the "formal" relation $|p| \simeq |\mathbf{u}|^2$, the assumption (2.11) corresponds to*

$$u_j \in L^{(2-\theta)\gamma}_*(D_T).$$

From Theorem 2.3.6 we can infer that:

the known regularity on $\mathbf{u}$ increases if	$\mu > (2-\theta)\gamma$	or equivalently	$(d+2)/(2-\theta) < \gamma$
the known regularity on $\mathbf{u}$ decreases if	$\mu < (2-\theta)\gamma$	or equivalently	$\gamma < (d+2)/(2-\theta)$
the known regularity on $\mathbf{u}$ does not change if	$\mu = (2-\theta)\gamma$	or equivalently	$\gamma = (d+2)/(2-\theta)$

and in the last case $\mu = (2-\theta)\gamma = d+2$ is the minimal exponent which gives regularity.

We start by observing that the case $\theta = 1$ and $\gamma > d+2$ falls within the range of applicability of Theorem 2.4.4 with $r = s = d+2$. It is interesting to consider the particular case with $\theta = 0$. In this case we have hypotheses which involve only p and we can infer the following corollary.

Corollary 2.3.8. *Let $(\mathbf{u}, p)$ be a weak solution of the Navier-Stokes equations. Assume moreover that*

$$p \in L^{\gamma}_{*}(D_T), \qquad \gamma \in]2, d+2[.$$

Then

$$u_j \in L^{\mu}_{*}(D_T), \qquad \mu = \frac{(d+2)\gamma}{d+2-\gamma}.$$

In particular if $p \in L^{d/2+1}_{*}(D_T)$, then $u_j \in L^{d+2}_{*}(D_T)$ and if $p \in L^{\gamma/2}_{*}(D_T)$, with $\gamma > d+2$, then $\mathbf{u}$ is "smooth".

Remark 2.3.9. *We remark that the last result gives the same regularity which can be obtained by solving the heat equation*

$$\frac{\partial u}{\partial t} - \nu \Delta u = \nabla p,$$

with a given p. This is interesting, because we obtain the same regularity which holds for the problem with the nonlinear term dropped out. However the result is different because in the Navier-Stokes equations the pressure p is an unknown and not a given datum.

2.4 Energy-type methods

In this section we present other results regarding the effect of the pressure. The methods used are now very similar to that one used in the construction of weak solutions. The *energy inequality* is now replaced by some $L^{\alpha}(D)$ estimates. This technique, which is the natural $L^{\alpha}(D)$, $\alpha \neq 2$, counterpart of the energy method, was introduced by Beirão da Veiga [BdV87] to obtain the estimates needed to prove the existence of suitable solutions in $\mathbb{R}^d$. This method consists in multiplying the Navier-Stokes equations by $|\mathbf{u}|^{\alpha-2}\mathbf{u}$, for suitable $\alpha > d$, and then integrating by parts. In this way the pressure[5] term does not disappear, but the problem is without boundaries and it is possible to use

[5] In this problem the pressure is determined by the condition $p \to 0$ as $|\mathbf{x}| \to +\infty$.

the *Poisson equation* (2.3) in the whole space $\mathbb{R}^d$. With the classical Calderón-Zygmund inequality, we have, for some $0 < c \in \mathbb{R}$, the following estimate

$$\|p\|_{L^{(\alpha+2)/2}(\mathbb{R}^d)} \leq c\|\mathbf{u}\|^2_{(L^{\alpha+2}(\mathbb{R}^d))^d},$$

which is strong enough to get (provided $\mathbf{f}$ and $\mathbf{u}_0$ are "smooth") existence and uniqueness of weak solutions such that

$$u_i \in L^2(0,T;H^1(\mathbb{R}^d)) \cap C([0,T];L^\alpha(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)).$$

2.4.1 Some regularity results

We start this section by recalling the first (and unique, up to 1997) regularity result based on the pressure. We have the following theorem, which was proved by Kaniel [Kan68]. As in the previous sections $D \subset \mathbb{R}^d$ will be "smooth" and bounded.

Theorem 2.4.1. *Let $\mathbf{u}$ be a weak solution of the three-dimensional Navier-Stokes equations, with smooth initial conditions. If the associate pressure p satisfies*

$$p \in L^\infty(0,T;L^q(D)) \quad \textit{for} \quad q > \frac{12}{5}, \tag{2.14}$$

then $\mathbf{u}$ is smooth.

This theorem is based on the following technique: the Navier-Stokes equations are multiplied by the vector $\mathbf{u}^3 := (u_1^3, \ldots, u_3^3)$; suitable calculations are performed to show that the V norm of the solution is bounded on $(0,T)$ and consequently to infer that the solution is strong. Then the passage to a smooth solution (if the data of the problem are smooth) is standard.

We now show some results which generalize that one of Theorem 2.4.1. The improvement is based on sharper energy-type estimates, joint with classical regularity results.

Remark 2.4.2. *To avoid non essential calculations we assume $\mathbf{f} = \mathbf{0}$, but it is easy to see that the same results holds if we add a forcing term belonging at least to $L^1(0,T;(L^\alpha(D))^d)$.*

To study the problem in the context of $L^\alpha(D)$ spaces, we need the space H_α, which is defined as the closure of $\mathcal{V}$ in $(L^\alpha(D))^d$. This space is the natural counterpart of the space H defined in Section 1.2.1. It is the space in which there were proved some existence and regularity results for problem in bounded domains. These results are shown by using the different method of semigroup in Banach spaces, see Giga and Miyakawa [GM85] and Miyakawa [Miy81].

We start by recalling a simple lemma, that follows from Sobolev embedding theorem and that will be crucial in the sequel.

Lemma 2.4.3. *Let $|\mathbf{u}|^{\alpha/2}$ belong to $H_0^1(D)$. Then*

$$\|\mathbf{u}\|^\alpha_{L^{\frac{\alpha d}{d-2}}(D)} \leq C_0 \int_D |\nabla|\mathbf{u}|^{\frac{\alpha}{2}}|^2 \, d\mathbf{x},$$

with C_0 a positive real constant.

Proof. We recall the following basic fact (see Adams [Ada75] for its proof):

$$H_0^1(D) \subset L^{p^*}(D) \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{2} - \frac{1}{d}.$$

This inclusion holds algebraically and topologically and we have

$$\|\phi\|^2_{L^{p^*}(D)} \le C_0 \|\nabla\phi\|^2_{(L^2(D))^d} \qquad \forall\, \phi \in H^1_0(D). \tag{2.15}$$

This last inequality (2.15) applied to $|\mathbf{u}|^{\alpha/2}$ gives the desired result. □

The exponent $(\alpha\, d)/(d-2)$ will play an important role in the sequel, similar to the exponent 6, which is the maximal one for the inclusion $V \subset (L^p(D))^3$, in three dimensions. Recall for instance the estimates for the nonlinear term we used in the previous sections.

We have the following results. The first one is due to Beirão da Veiga [BdV99] and it is in the same spirit of the results of the previous section. The second one is due to Berselli [Ber99] and it is the claimed generalization of Kaniel Theorem 2.4.1.

Theorem 2.4.4. *Let $\alpha > d$. Assume that $\mathbf{u}_0 \in H_\alpha$. Let $(\mathbf{u}, p)$ be a weak solution of the d-dimensional Navier-Stokes equations, such that*

$$\frac{p}{1+|\mathbf{u}|} \in L^r(0,T;L^s(D)), \qquad \frac{2}{r} + \frac{d}{s} < 1.$$

Then the solution is "smooth."

Theorem 2.4.5. *Let $(\mathbf{u}, p)$ be a weak solution of the d-dimensional Navier-Stokes equations with initial conditions $\mathbf{u}_0 \in H_\alpha$, for $\alpha > d$. If the pressure p satisfies*

$$p \in L^\alpha(0,T;L^{\frac{\alpha\, d}{\alpha+d-2}}(D)), \tag{2.16}$$

then the solution is "smooth."

Remark 2.4.6. *The condition $\mathbf{u}_0 \in H_\alpha$ is not restrictive and it is the natural one used to deal with problem in the Banach spaces $L^\alpha(D)$ with $\alpha \neq 2$, see Giga and Miyakawa [GM85].*

Theorem 2.4.4 should be compared with Theorem 2.3.2 and Theorem 2.3.6 of the previous section, since the results are very similar. From Theorem 2.4.5 we can immediately infer the following corollary.

Corollary 2.4.7. *Let $d = 3$, let p belong to $L^\alpha(0,T;L^p(D))$ with $9/4 < p < 3$ and let $\mathbf{u}_0$ belong to H_α, with $\alpha = p/(3-p)$. Then the weak solutions of the 3-dimensional Navier-Stokes equations are regular. In particular this holds if p belongs to $L^\infty(0,T;L^p(D))$.*

Proof. If we restrict the results of Theorem 2.4.5 to the problem with $d = 3$, we have that $L^{\frac{\alpha\, d}{\alpha+d-2}}(D) = L^{\frac{3\alpha}{1+\alpha}}(D)$ and

$$\inf_{\alpha > 3} \frac{3\alpha}{1+\alpha} = \frac{9}{4}.$$

The lowest upper bound is attained for $\alpha = 3$. We observe that

$$L^\alpha(0,T;L^{\frac{3\alpha}{1+\alpha}}(D)) \supset L^\infty(0,T;L^{\frac{3\alpha}{1+\alpha}}(D)).$$

Moreover, for every $\frac{9}{4} < p < 3$, if we set $\alpha = p/(p-3) > 3$, we get

$$L^\alpha(0,T;L^{\frac{3\alpha}{1+\alpha}}(D)) \supset \mathrm{L}^\infty(0,T;L^p(D)),$$

and this concludes the proof. □

Remark 2.4.8. *We remark that* $9/4$ *(the best exponent) is strictly lower than* $12/5$ *and in this special setting we improve the cited result* (2.14) *by Kaniel. He assumed that* $\mathbf{u}_0 \in H^1_0(D)$ *and this also implies that our summability condition on the initial datum* $\mathbf{u}_0$ *is satisfied, since* p *close to* $9/4$ *means that* α *is close to 3. Clearly, we also obtain a generalization of Kaniel result by taking* $\alpha = 4$*. In this case the regularity holds if* p *belongs to* $L^4(0, T; L^{12/5}(D))$.

Remark 2.4.9. *If we take our condition* (2.16) *and if we put* $r = \alpha$ *and* $s = \alpha\, d/(d + \alpha - 2)$*, we get as a sufficient condition for regularity that*

$$p \in L^r(0, T; L^s(D)) \quad \textit{with} \quad \frac{2}{r} + \frac{d}{s} = 1 + \frac{d}{\alpha}. \tag{2.17}$$

Note that the right hand side is a real number greater than one and strictly lower than two. It is worth noting that, at the light of the result shown before, (see Section 2.3.1) we expect that $2/r + d/s < 2$ *(or even equal to 2) should be sufficient here. Note that the exponent two corresponds exactly to the sufficient condition* (2.1)*, if* $p \simeq |\mathbf{u}|^2$*. Also note that we arrive at this exponent if we could take* $\alpha = d$*. In fact in that case we will get a pressure belonging to* $L^d(0, T; L^{d^2/2(d-1)}(D))$*, for which condition* (2.17) *gives exactly two, the number relative to the hypothesis* $p \simeq |\mathbf{u}|^2$.

We start by observing that if a weak solution satisfies

$$\mathbf{u} \in C(0, T; H_\alpha) \qquad \text{and} \qquad |\mathbf{u}|^{\alpha/2} \in L^2(0, T; H^1_0(D)), \tag{2.18}$$

if $\mathbf{u}_0$ and $\mathbf{f}$ are regular and if the necessary compatibility conditions are satisfied, then it is "smooth," see for example von Wahl [vW80]. Our aim will be to find some *a-priori* estimates to use the previous remark.

The following lemma is the core of the proof of Theorem 2.4.4 and Theorem 2.4.5.

Lemma 2.4.10. *Let* $(\mathbf{u}, p)$ *be a smooth solution of the Navier-Stokes equations in* $D\times]0, T]$*. Then the following estimate is satisfied*

$$\frac{1}{\alpha}\frac{d}{dt}\|\mathbf{u}\|^\alpha_{(L^\alpha(D))^d} + \frac{\nu}{2}N^\alpha_\alpha(\mathbf{u}) + 4\,\nu\frac{\alpha-2}{\alpha^2}M^\alpha_\alpha(\mathbf{u}) \le \frac{(\alpha-2)^2}{2\nu}\int_D p^2|\mathbf{u}|^{\alpha-2}\, d\mathbf{x}, \tag{2.19}$$

where

$$N_\alpha = \left[\int_D |\nabla \mathbf{u}|^2|\mathbf{u}|^{\alpha-2}\, d\mathbf{x}\right]^{1/\alpha} \qquad \textit{and} \qquad M_\alpha = \left[\int_D |\nabla|\mathbf{u}|^{\frac{\alpha}{2}}|^2\, d\mathbf{x}\right]^{1/\alpha}.$$

Proof. We apply the "energy" method previously described, even if we study the problem in a bounded domain $D \subset \mathbb{R}^d$. The tool needed to prove (2.19) is simply to multiply the Navier-Stokes equations by $|\mathbf{u}|^{\alpha-2}\mathbf{u}$ and to integrate over D. Then suitable integrations by parts are performed by taking into account that the fluid is incompressible (*i.e.*, $\operatorname{div}\mathbf{u} = 0$) and that the problem is equipped with no-slip (*i.e.*, $\mathbf{u}_{|\partial D} = \mathbf{0}$) boundary conditions.

We observe that, since $\operatorname{div}\mathbf{u} = 0$, with an integration by parts we get:

$$b(\mathbf{u}, \mathbf{u}, |\mathbf{u}|^{a-2}\mathbf{u}) = 0.$$

The term which requires more attention is

$$I := -\int_D \Delta\mathbf{u}\,\mathbf{u}|\mathbf{u}|^{\alpha-2}\, d\mathbf{x} = \int_D \sum_{i,k=1}^d \frac{\partial^2 u_i}{\partial x_k^2}\, u_i\Big(\sum_{j=1}^d u_j^2\Big)^{(\alpha-2)/2}\, d\mathbf{x}.$$

If we integrate by parts the boundary terms disappear and we obtain $I = I_1 + I_2$ with

$$I_1 := \int_D \sum_{i,k=1}^d \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}\Big(\sum_{j=1}^d u_j^2\Big)^{(\alpha-2)/2} d\mathbf{x},$$

and

$$I_2 := (\alpha-2)\int_D \sum_{i,k=1}^d u_i\frac{\partial u_i}{\partial x_k}u_j\frac{\partial u_j}{\partial x_k}\Big(\sum_{j=1}^d u_j^2\Big)^{(\alpha-2)/2-1} d\mathbf{x}.$$

The term I_1 is exactly $N_\alpha^\alpha(\mathbf{u})$. The term I_2 requires some cares. Expanding $|\nabla|\mathbf{u}|^{\alpha/2}|^2$ we get

$$\sum_{i=1}^d\Big(\frac{\partial}{\partial x_i}\big(\sum_{k=1}^d u_k^2\big)^{\alpha/4}\Big)^2 = \Big(\frac{\alpha}{4}\big(\sum_{k=1}^d u_k^2\big)^{\alpha/4-1}2u_k\frac{\partial u_k}{\partial x_i}\Big)^2 = \frac{\alpha^2}{4}\big(\sum_k u_k^2\big)^{(\alpha-4)/2}u_k^2\Big(\frac{\partial u_k}{\partial x_i}\Big)^2, \tag{2.20}$$

and finally we observe that

$$I_2 = 4\frac{\alpha-2}{\alpha^2}\int_D |\nabla|\mathbf{u}|^{\alpha/2}|^2\,d\mathbf{x} = 4\frac{\alpha-2}{\alpha^2}M_\alpha^\alpha(\mathbf{u}).$$

First we obtain that

$$\frac{1}{\alpha}\frac{d}{dt}\|\mathbf{u}\|^\alpha_{(L^\alpha(D))^d} + \nu N_\alpha^\alpha(\mathbf{u}) + 4\,\nu\frac{\alpha-2}{\alpha^2}M_\alpha^\alpha(\mathbf{u}) = -\int_D \nabla p\cdot\mathbf{u}|\mathbf{u}|^{\alpha-2}\,d\mathbf{x}. \tag{2.21}$$

We remark that the terms $N_\alpha(\mathbf{u})$ and $M_\alpha(\mathbf{u})$ are the same which appear if $D = \mathbb{R}^n$, since the boundary terms vanish. We also recall the following inequality (which derives from (2.20)), that will be used in the calculations

$$|\nabla|\mathbf{u}|^{\alpha/2}| \le \frac{\alpha}{2}|\mathbf{u}|^{\alpha/2-1}|\nabla\mathbf{u}| \qquad \text{a.e. in } D.$$

If we integrate by parts the right hand side of (2.21) we have

$$\begin{aligned}-\int_D \nabla p\cdot\mathbf{u}|\mathbf{u}|^{\alpha-2}\,d\mathbf{x} &= (\alpha-2)\sum_{i,j=1}^d\int_D p\frac{\partial\,u_j}{\partial\,x_i}u_i\,u_j|\mathbf{u}|^{\alpha-2}\,d\mathbf{x},\\ &= \frac{2(\alpha-2)}{\alpha}\int_D p|\mathbf{u}|^{\alpha/2-2}\left[\sum_{i=1}^d u_i\right]\left[\sum_{i=1}^d\frac{\partial}{\partial\,x_i}|\mathbf{u}|^{\alpha/2}\right]d\mathbf{x},\end{aligned}$$

and by observing that

$$\left|\sum_{i,j=1} u_i u_j\frac{\partial u_j}{\partial x_i}\right| \le |\mathbf{u}|^2|\nabla\mathbf{u}| \qquad \text{a.e. in } D,$$

we obtain

$$\left|\int_D \nabla p\cdot\mathbf{u}|\mathbf{u}|^{\alpha-2}\,d\mathbf{x}\right| \le (\alpha-2)\int_D |p||\nabla\mathbf{u}||\mathbf{u}|^{\alpha-2}\,d\mathbf{x}.$$

Finally, by using the Young inequality (q=q'=2) and Lemma 2.4.3, we arrive to the following inequality

$$\left|\int_D \nabla p\cdot\mathbf{u}|\mathbf{u}|^{\alpha-2}\,d\mathbf{x}\right| \le \frac{(\alpha-2)^2}{2\,\nu}\int_D p^2|\mathbf{u}|^{\alpha-2}\,d\mathbf{x} + \frac{\nu}{2}N_\alpha^\alpha(\mathbf{u}),$$

which proves the lemma. □

We now derive the two inequalities which prove the Theorems cited above. Given the a-priori estimates (2.22)-(2.24) both the proofs conclude by using the continuation argument and a regularity result.

Proof of Theorem 2.4.4: a-priori estimate. Let now (r, s) satisfy $2/r + d/s < 1$ and assume that $s < \infty$

$$\int_D p^2 |\mathbf{u}|^{\alpha-2}\, d\mathbf{x} \le \int_D \left(\frac{p}{1+|\mathbf{u}|}\right)^2 (1+|\mathbf{u}|)^{\alpha\frac{s-d}{d}} (1+|\mathbf{u}|)^{\frac{\alpha d}{s}}\, d\mathbf{x}.$$

Since $2/s + (s-d)/s + (d-2)/s = 1$, we can use the Hölder inequality to get

$$\int_D p^2 |\mathbf{u}|^{\alpha-2}\, d\mathbf{x} \le \|p/(1+|\mathbf{u}|)\|^2_{L^s(D)} \|(1+|\mathbf{u}|)\|^{\alpha(1-\frac{d}{s})}_{L^\alpha(D)} \|(1+|\mathbf{u}|)\|^{\frac{\alpha d}{s}}_{L^{\frac{\alpha d}{d-2}}(D)}.$$

By using the Young inequality with exponents $r/2$ and s/d we obtain, for $\varepsilon > 0$,

$$\int_D p^2 |\mathbf{u}|^{\alpha-2}\, d\mathbf{x} \le \varepsilon^{-\frac{s}{s-d}} \|p/(1+|\mathbf{u}|)\|^r_{L^s(D)} \|(1+|\mathbf{u}|)\|^\alpha_{L^\alpha(D)} + \varepsilon^{\frac{s}{d}} \|(1+|\mathbf{u}|)\|^\alpha_{L^{\frac{\alpha d}{d-2}}(D)},$$

and we have

$$\frac{1}{\alpha}\frac{d}{dt}\|\mathbf{u}\|^\alpha_{(L^\alpha(D))^d} + \frac{\nu}{2}N^\alpha_\alpha(\mathbf{u}) + 4\,\nu\frac{\alpha-2}{\alpha^2}M^\alpha_\alpha(\mathbf{u}) \le \varepsilon^{-\frac{s}{s-d}} \left\|\frac{p}{1+|\mathbf{u}|}\right\|^r_{L^s(D)} 2^{\alpha-1}\left(|D| + \|\mathbf{u}\|^\alpha_{L^\alpha(D)}\right) +$$
$$+\varepsilon^{\frac{s}{d}} 2^{\alpha-1}\frac{(\alpha-2)^2}{2\nu}\left(|D|^{\frac{d-2}{d}} + \|\mathbf{u}\|^\alpha_{L^{\frac{\alpha d}{d-2}}(D)}\right).$$

Finally, by fixing ε and by using Lemma 2.4.3, we get[6]

(2.22)
$$\frac{1}{\alpha}\frac{d}{dt}\|\mathbf{u}\|^\alpha_{(L^\alpha(D))^d} + \frac{\nu}{2}N^\alpha_\alpha(\mathbf{u}) + 4\,\nu\frac{\alpha-2}{\alpha^2}M^\alpha_\alpha(\mathbf{u}) \le c_1 + c_2 \left\|\frac{p}{1+|\mathbf{u}|}\right\|^r_{L^s(D)} \left(|D| + \|\mathbf{u}\|_{(L^\alpha(D))^d}\right).$$

□

Proof of Theorem 2.4.5: a-priori estimate. We use the Hölder inequality in the last term of (2.19) with exponents

$$p = \frac{\alpha\, d}{(\alpha-2)(d-2)} \quad \text{and} \quad q = \frac{p-1}{p} = \frac{\alpha\, d}{2(d+\alpha-2)}.$$

We observe that the condition $p \ge 1$ is satisfied since $\alpha > d \ge 3$. We obtain

$$\int_D p^2 |\mathbf{u}|^{\alpha-2}\, d\mathbf{x} \le \|\mathbf{u}\|^{\alpha-2}_{(L^{p(\alpha-2)}(D))^d} \|p\|^2_{L^{2q}(D)} \le \|\mathbf{u}\|^{\alpha-2}_{(L^{\frac{\alpha\, d}{d-2}}(D))^d} \|p\|^2_{L^{2q}(D)}. \tag{2.23}$$

Now we apply the Young inequality to (2.23) with exponents

$$p' = \frac{\alpha}{\alpha-2} \quad \text{and} \quad q' = \frac{\alpha}{2},$$

[6] With $\varepsilon = \left(\frac{\nu^2}{2^{\alpha-1}(\alpha-2)}\right)^{d/s}$, $c_1 = \frac{2\nu(\alpha-2)}{\alpha C_0}|D|^{\frac{d-2}{d}}$, and $c_2 = 2^{\alpha-2}\frac{(\alpha-2)^2}{\nu}\left[\frac{2^{\alpha-3}\alpha(\alpha-2)C_0}{\nu^2}\right]^{\frac{d}{s-d}}$.

to get

$$\frac{(\alpha-2)^2}{2\nu}\int_D p^2|\mathbf{u}|^{\alpha-2}\,d\mathbf{x} \leq 2\nu\frac{\alpha-2}{\alpha\,C_0}\|\mathbf{u}\|^{\alpha}_{(L^{\frac{\alpha\,d}{d-2}}(D))^d} + c_1\|p\|^{\alpha}_{L^{\frac{\alpha\,d}{\alpha+d-2}}(D)}$$

and the explicit value for c_1 is $c_1 = \frac{(\alpha-2)^2}{\nu\,\alpha}\left[\frac{(\alpha-2)^2C_0}{4\nu^2}\right]^{(\alpha-2)/2}$.

By collecting the previous results (Lemma 2.4.3 and inequality (2.19)) we finally obtain

$$\frac{1}{\alpha}\frac{d}{dt}\|\mathbf{u}\|^{\alpha}_{(L^{\alpha}(D))^d} + \frac{\nu}{2}N^{\alpha}_{\alpha}(\mathbf{u}) + 2\,\nu\frac{\alpha-2}{\alpha}M^{\alpha}_{\alpha}(\mathbf{u}) \leq c_1\|p\|^{\alpha}_{L^{\frac{\alpha\,d}{\alpha+d-2}}(D)}. \tag{2.24}$$

□

We can now conclude the proof of Theorems 2.4.4-2.4.5 by using the comparison theorems for ordinary differential equations. With inequality (2.22)-(2.24) and by using the hypotheses satisfied respectively by $p/(1+|\mathbf{u}|)$ and p in a Gronwall inequality, we have that $\mathbf{u}\in L^{\infty}(0,T;H_{\alpha})$ and that $|\nabla\mathbf{u}|^{\alpha/2}$ belongs to $L^2(0,T;H^1_0(D))$.

We denote by $\overline{t}$ the lowest upper bound of the values in $[0,T]$ for which the velocity satisfies the condition (2.18):

$$\mathbf{u}\in C(0,\overline{t};H_{\alpha}) \qquad \text{and} \qquad |\mathbf{u}|^{\alpha/2}\in L^2(0,\overline{t};H^1_0(D)).$$

We have that $\overline{t}>0$ and, as we claimed, the solution is regular in $]0,\overline{t}[$, see for example the classical references by Giga and Miyakawa [GM85], Miyakawa [Miy81], Sohr [Soh83] and von Wahl [vW80].

Since estimate (2.22)-(2.24) holds in $]0,\overline{t}[$, it follows that $\mathbf{u}\in L^{\infty}(0,\overline{t};H_{\alpha})$ and, as consequence of the hypothesis $\alpha>d$, we have $\mathbf{u}\in C(0,\overline{t};H_{\alpha})$. By taking now $\mathbf{u}(\overline{t})$ as initial datum we can construct a solution in a larger interval and it is easy to see that necessarily $\overline{t}=T$.

In the continuation argument we used the fact that if $\mathbf{u}$ is bounded with values in H_{α}, $\alpha>d$ then $\mathbf{u}$ is also continuous with values in H_{α}. The proof of this result of strong continuity, which is based on appropriate interpolation inequalities and a *boot-strap* argument, can be found in Beirão da Veiga [BdV99] and it is based on the argument developed for $D=\mathbb{R}^d$ in Beirão da Veiga [BdV87], Appendix B.

Chapter 3

On determining projections

In this chapter we analyze with more details the results regarding the long-time behavior of solutions. We start by presenting the results regarding the *determining projections* stated in Section 1.3.3. This result will be the starting point for some generalization. In particular we introduce several concepts regarding the analysis of stochastic partial differential equations and we study some aspects concerning the long-time behavior of their solutions. The new result we prove are the natural generalization to the stochastic framework of the result of Holst and Titi regarding the existence of *determining projections* for the two dimensional Navier-Stokes equations.

3.1 A result on determining projections

We start this chapter by proving Theorem 1.3.16, that we claimed in Section 1.3.3. We give the proof by Holst and Titi [HT97] regarding the existence of a determining projection operator. We start with a lemma, which is important in the analysis of the long time behavior of the Navier-Stokes equations. This lemma, whose proof can be found in Jones and Titi [JT93], is a generalization of the classical Gronwall lemma.

Lemma 3.1.1. *Let $T > 0$ be fixed, and let $\alpha(t)$ and $\beta(t)$ be locally integrable and real-valued on $(0, +\infty)$, satisfying*

$$\begin{gathered}\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)\,d\tau = m > 0, \qquad \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)\,d\tau = M < \infty, \\ \lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)\,d\tau = 0,\end{gathered} \tag{3.1}$$

where $\alpha^- := \max\{\alpha, 0\}$ and $\beta^+ := \max\{\beta, 0\}$. Then if $y(t)$ is an absolutely continuous, non-negative function on $(0,\infty)$ and if $y(t)$ satisfies

$$y'(t) + \alpha(t)\,y(t) \le \beta(t), \qquad a.e.\ on\ (0,\infty),$$

then $\lim_{t\to\infty} y(t) = 0$.

Proof. From the standard Gronwall lemma, we obtain that

$$0 \le y(t) \le y(t_0)\,\mathrm{e}^{-\int_{t_0}^t \alpha(\sigma)\,d\sigma} + \int_{t_0}^t \beta^+(\tau)\,\mathrm{e}^{-\int_\tau^t \alpha(\sigma)\,d\sigma}\,d\tau, \qquad \text{for } 0 < t \le t_0.$$

From the assumptions on α, we may choose t_0 large enough so that, for all $s \geq t_0$, we have

$$\int_s^{s+T} \alpha^-(\sigma)\, d\sigma \leq M+1 \quad \text{and} \quad \int_s^{s+T} \alpha(\sigma)\, d\sigma \geq \frac{m}{2}.$$

Hence, if $t_0 \leq \tau \leq t$ and if k is an integer chosen such that $\tau + k\,T \leq t \leq \tau + (k+1)\,T$, then

$$\mathrm{e}^{-\int_\tau^t \alpha(\sigma)\, d\sigma} = \mathrm{e}^{-\int_\tau^{\tau+k\,T} \alpha(\sigma)\, d\sigma}\, \mathrm{e}^{-\int_{\tau+k\,T}^t \alpha(\sigma)\, d\sigma} \leq \mathrm{e}^{-m\,k/2}\mathrm{e}^{M+1} \leq M'\mathrm{e}^{-m/2T\,(t-\tau)},$$

for some M'. By choosing an integer k_0 such that $t \leq t_0 + K_0 T \leq t + T$, we have that

$$\begin{aligned}\int_{t_0}^t \beta^+(\tau)\, M'\mathrm{e}^{-m\,(t-\tau)/2T}\, d\tau &\leq \sum_{k=1}^{k_0} \int_{t_0+(k-1)T}^{t_0+k\,T} \beta^+(\tau)\, M'\mathrm{e}^{-m\,(k_0T-kT-T)/2T}\, d\tau \\ &\leq \left(\sup_{t\geq t_0} \int_t^{t+T} \beta^+(\tau)\, d\tau\right) M'\frac{\mathrm{e}^{m/2}}{\mathrm{e}-1}.\end{aligned}$$

It follows that

$$\limsup_{t\to\infty} |y(t)| \leq \left(\sup_{t\geq t_0} \int_t^{t+T} \beta^+(\tau)\, d\tau\right) M'\frac{\mathrm{e}^{m/2}}{\mathrm{e}-1}$$

and by hypothesis (3.1) we can conclude that $|y(t)| \to 0$ as $t \to \infty$. □

We can now easily prove that the long-time behavior of the solutions of the Navier-Stokes equations is determined by their projection, if the projection operator satisfies:

$$\|\mathbf{u} - \mathcal{R}_N(\mathbf{u})\|_{(L^2(D))^2} \leq C\frac{1}{N^\gamma}\|\mathbf{u}\|_{(H^1(D))^2} \qquad \text{for } 0 < C \in \mathbb{R}. \tag{3.2}$$

Proof of Theorem 1.3.16. Let $\mathbf{u}$ and $\mathbf{v}$ be solutions, corresponding to $\mathbf{f}$ and $\mathbf{g}$ respectively. We set, as usual, $\mathbf{w} := \mathbf{u} - \mathbf{v}$. The application of the "energy method" and of the basic estimates for the nonlinear terms gives

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|^2 + \nu\|\mathbf{w}\|^2 \leq \|\mathbf{u}\||\mathbf{w}|\|\mathbf{w}\| + \|\mathbf{f}-\mathbf{g}\|_{V'}\|\mathbf{w}\|.$$

An application of the Young inequality leads to

$$\frac{d}{dt}|\mathbf{w}|^2 + \nu\|\mathbf{w}\|^2 - \frac{2}{\nu}\|\mathbf{u}\|^2|\mathbf{w}|^2 \leq \frac{2}{\nu}\|\mathbf{f}-\mathbf{g}\|^2_{V'} \tag{3.3}$$

To bound the term on the left we use (3.2) in the following form

$$|\mathbf{w}|^2 \leq \frac{2}{N^{2\gamma}}C\|\mathbf{w}\|^2 + 2\|\mathcal{R}_N\mathbf{w}\|^2_{(L^2(D))^2},$$

and we obtain

$$\frac{d}{dt}|\mathbf{w}|^2 + \left(\frac{\nu\,N^{2\gamma}}{2C} - \frac{2}{\nu}\|\mathbf{u}\|^2\right)|\mathbf{w}|^2 \leq \frac{2}{\nu}\|\mathbf{f}-\mathbf{g}\|^2_{V'} + \frac{\nu\,N^{2\gamma}}{C}\|\mathcal{R}_N\mathbf{w}\|^2_{(L^2(D))^2}.$$

The last inequality is of the form

$$\frac{d}{dt}|\mathbf{w}(t)|^2 + \alpha(t)|\mathbf{w}(t)|^2 \leq \beta(t).$$

To apply Lemma 3.1.1 to the last expression (and to prove that $\mathbf{w} \to 0$ as $t \to \infty$) we need to check that the hypothesis (3.1) on α and β is satisfied. The conditions on α^- and β^+ are trivially satisfied, because $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ belong to V. It remains to check that

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)\, d\tau = m > 0,$$

and the last reduces to verify that, for some fixed $T > 0$, the following hold:

$$N^\gamma > \frac{2C^2}{\nu} \limsup_{t\to\infty} \int_t^{t+T} \frac{2\|\mathbf{u}(\tau)\|^2}{\nu}\, d\tau = \frac{4C^2}{\nu^2} \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \|\mathbf{u}(\tau)\|^2\, d\tau.$$

Classical estimates on weak solutions yield

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \|\mathbf{u}(\tau)\|^2\, d\tau \leq \frac{2}{\nu^2} \limsup_{t\to\infty} \|\mathbf{f}(t)\|^2_{V'} \qquad \text{for } T = \frac{C(D)}{\nu},$$

where $C(D)$ is the best Poincaré constant. Therefore if

$$N^{2\gamma} \geq \frac{4C^2}{\nu^2}\left(\frac{2}{\nu^2}\right) \limsup_{t\to\infty} \|\mathbf{f}(t)\|^2_{V'},$$

then

$$\lim_{t\to\infty} |\mathbf{w}(t)| = 0.$$

□

Remark 3.1.2. *From the proof it will follow that N should be of order of Gr^2 (recall that Gr is the Grashof number, defined at page 10) The bound can be improved to the claimed $N \simeq Gr$ if the no-slip conditions are replaced with the periodic boundary conditions, mainly for the simplification due to $b(\mathbf{u}_m, \mathbf{u}_m, A\,\mathbf{u}_m) \equiv 0$, see Remark 1.2.18. This identity show the lack of a boundary vorticity shedding source, when the boundary is absent, recall also Remark 1.1.2.*

3.1.1 Scott and Zhang interpolant

The operator $\mathcal{R}_N$ was introduced in the previous sections as an abstract operator. We now explain why it is needed and how to construct it explicitly.

We recall that the basic finite element interpolant (Lagrange and Crouzeix-Raviart and some others) are defined for v at least in $C^0(\overline{\mathcal{K}})$ and the nodal interpolant (*i.e.*, the nodal variables are defined as evaluation at some points) are not well defined for functions in $H^1(D)$ with $D \subset \mathbb{R}^d$, $d \geq 2$. Since $H^1_0(D)$ (or more precisely V) is the natural space in which we want to study the Navier-Stokes equations (it is the space in which weak solution live) it is necessary to introduce some different interpolant. The Definition 1.4.2 in its generality encompasses also the setting needed to describe the *Scott and Zhang interpolant* we are going to introduce. This interpolant, which was introduced by Scott and Zhang [SZ90, SZ92], will turn out to be well defined on $H^1(D)$ and also to have optimal properties of approximation.

The interpolant we shall construct needs an additional hypothesis on the subdivision: the quasi uniformity.

Definition 3.1.3. *A simplicial subdivision $\mathcal{T}_h$ is said to be quasi-uniform if there exists $\eta > 0$ such that*

$$\min_{\mathcal{K}_i \in \mathcal{T}_h} (\rho_{\mathcal{K}_i}) \geq \eta\, h$$

we recall (see Section 1.4) that $\rho_{\mathcal{K}_i}$ is the radius of the largest ball contained in $\mathcal{K}_i$.

This condition means, roughly speaking, that all the simplices $\mathcal{K}_i$ are of comparable size and it is needed to make good estimates, since all the calculations can be done on a single reference element (the one for which $\mathcal{K}$ is the standard d-simplex). Observe also that if a family is *quasi-uniform* then it is *regular*, but not conversely. We point out that to construct the Scott and Zhang interpolant the mesh needs to be *regular.* Furthermore, to make the approximation property (3.2) hold, the mesh needs also to be *quasi-uniform.*

We consider a finite element space V_h consisting of continuous piecewise polynomials:

$$V_h := \left\{ v \in C(\overline{D}) : \, v_{|\mathcal{K}_i} \in \mathcal{P}_{\mathcal{K}_i} = P_s^d, \quad \forall\, \mathcal{K}_i \in \mathcal{T}_h \right\}, \tag{3.4}$$

and its subspace V_{0h}, consisting of functions vanishing on the boundary.

Now we present the Scott and Zhang interpolant, which defines an interpolation operator from $W^{k,p}(D)$ into V_h, with

$$k \geq 1 \quad \text{if} \quad p = 1 \qquad \text{and} \qquad k > 1/p \quad \text{otherwise.} \tag{3.5}$$

We construct the global interpolant I_h as follows: we shall use the same nodes $\{x_i\}_{i=1}^N$ of the Lagrange element and the nodal basis $\{\varphi_i\}_{i=1}^N$ of V_h to define the interpolation operator. We choose for every node x_i either a d-simplex (a triangle in our two dimensional problem) or a $(d-1)$-simplex (a side), according to the following rules

- if x_i is an interior point of some d-simplex $\mathcal{K}_i \in \mathcal{T}_h$ we let
$$\sigma_i = \mathcal{K}_i,$$
- if x_i is an interior point of some face (which is a $(d-1)$-simplex) $\mathcal{K}'$ of a d-simplex $\mathcal{K}_i \in \mathcal{T}_h$ we let
$$\sigma_i = \mathcal{K}',$$
- if x_i is on a $(d-2)$-simplex there are some degrees of freedom in the choice that produce different interpolant; we have in fact to pick one $(d-1)$-simplex $\mathcal{K}'$ in such a way that
$$\sigma_i \in \overline{\mathcal{K}'} \quad \text{with the restriction } \mathcal{K}' \subset \partial D \quad \text{if} \;\; x_i \in \partial D,$$
and we set
$$\sigma_i = \mathcal{K}'.$$

The condition $\mathcal{K}' \subset \partial D$ is made for the purpose to preserve homogeneous boundary conditions in a natural way. It is the main improvement with respect to the Clément interpolant [Clé75], since the Scott and Zhang interpolant allows us to treat non-homogeneous problems, as for example elliptic problems with non zero Dirichlet data or Navier-Stokes equations with periodic boundary conditions.

For each $(d-1)$-simplex $\mathcal{K}'$ of $\mathcal{K}$ there is a natural restriction of $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ that defines a finite element

$$(\mathcal{K}', \mathcal{P}_{\mathcal{K}'}, \mathcal{N}_{\mathcal{K}'}) = (\mathcal{K}, \mathcal{P}_{\mathcal{K}}, \mathcal{N}_{\mathcal{K}})|_{\mathcal{K}'}.$$

With $\mathcal{N}_{\mathcal{K}'}$ we denote the evaluation points of $\mathcal{N}_{\mathcal{K}}$ that lie on the simplex $\mathcal{K}'$ and $\mathcal{P}_{\mathcal{K}'} = P_s^{d-1}$. Let us denote by n_1 the dimension of P_s^d and by $n_0(\sigma_i)$ the dimension of $P_s^{dim\,\sigma_i}$ (that is $\binom{s+d}{d}$ or $\binom{s+d-1}{d-1}$). Let $x_{i,1} = x_i$ and let $\{x_{i,j}\}_{j=1}^{n_0}$ be the set of nodal points in σ_i, in which σ_i is the simplex associated to x_i.

We associate to the nodal basis $\{\varphi_{i,j}\}_{j=1}^{n_0}$, defined on σ_i, the $L^2(\sigma_i)$-dual basis $\{\psi_{i,j}\}_{j=1}^{n_0}$ (which exists by the Riesz representation theorem):

$$\int_{\sigma_i} \psi_{i,j}(x)\varphi_{i,k}(x)\,dx = \delta_{jk} \qquad j,k = 1,\dots,n_0. \tag{3.6}$$

We set $\psi_i = \psi_{i,1}$, for every $x_i \in \mathcal{N}_{\mathcal{K}_i}$, and we have the following relation

$$\int_{\sigma_i} \psi_i(x)\varphi_j(x)\,dx = \delta_{ij} \qquad i,j = 1,\dots,N.$$

We define the interpolation operator $\Pi_{\{\sigma_i\}} : \; W^{k,p}(D) \to V_h(D)$, as follows:

$$\Pi_{\{\sigma_i\}}\, v(x) = \sum_{i=1}^{N} \varphi_i(x) \int_{\sigma_i} \psi_i(\xi)v(\xi)\,d\xi.$$

Remark 3.1.4. *The operator $\Pi_{\{\sigma_i\}}$ depends on the choices of the simplices σ_i, but we do not write it explicitly. We refer to Π as the projection operator determined by a certain choice of the σ_i.*

Condition (3.5) also guarantees that the nodal values, $\{\Pi\, v(x_i)\}$, are well defined owing to the trace theorems for Sobolev spaces. Furthermore, the same condition guarantees the validity of the homogeneous boundary condition

$$\forall\, v \in W_0^{k,m}(D), \quad v_{|\partial D} = 0 \text{ in } L^1(\partial D).$$

The particular choice of σ_i, that we made, implies that Π preserves the homogeneous boundary condition:

$$\Pi : \; W_0^{k,p}(D) \to V_{0h}.$$

From the orthogonality condition (3.6) we can also conclude that Π is a projection operator on V_h or, in other words, $\Pi(v) = v$, for every $v \in V_h$. We now recall the following theorem, which states the approximation properties of Π, see Scott and Zhang [SZ90].

Theorem 3.1.5. *Let $v \in W^{k,p}(D)$ and let k, p satisfy (3.5) and let $\mathcal{T}_h$ be a regular simplicial subdivision of D. Then the Scott and Zhang interpolant Π made with polynomials of degree less or equal than s, satisfies*

$$\left(\sum_{\mathcal{K}_i \in \mathcal{T}_h} h_{\mathcal{K}_i}^{p(m-k)} \|v - \Pi\, v\|_{W^{m,p}(\mathcal{K})}^p \right)^{\frac{1}{p}} \le C\|v\|_{W^{k,p}(D)}^p \qquad 0 \le m \le k \le s+1,$$

where C denotes a constant that does depend nor on v nor on $\mathcal{T}_{\mathcal{K}}$.

Letting $m = k$, we obtain the following corollary

Corollary 3.1.6. *With the same hypotheses of Theorem 3.1.5 we have*

$$\left(\sum_{\mathcal{K}_i \in \mathcal{T}_h} \|\Pi\, v\|^p_{W^{k,p}(\mathcal{K}_i)} \right)^{\frac{1}{p}} \leq C \|v\|_{W^{k,p}(D)}.$$

Recalling that h is the mesh size, the statement of Theorem 3.1.5 can be written in the following form, more interesting for our applications:

$$\left(\sum_{\mathcal{K}_i \in \mathcal{T}_h} \|v - \Pi\, v\|^p_{W^{m,p}(\mathcal{K})} \right)^{\frac{1}{p}} \leq C\, h^{k-m} \|v\|_{W^{k,p}(D)}, \quad 0 \leq m \leq k \leq s+1.$$

Now, to fit this result into our problem, we must make some additional assumptions. First we consider the setting with $d = 2$ and we suppose that D is an exactly triangulated polygon (each $\mathcal{K}_i$ is a triangle) with a *regular* and *quasi-uniform* triangulation. We also restrict to linear polynomials, *i.e.*, we use as shape functions P_1^2. For quasi-uniform triangulation it holds that there exist two constants $C_0, C_1 > 0$, independent of both N and h, such that

$$C_0 \frac{|D|}{h^2} \leq N \leq C_1 \frac{|D|}{h^2}, \tag{3.7}$$

where N is the number of interpolation points (the vertices of the triangles $\mathcal{K}_i$). We must note that given some initial triangulation, by repeated bisection, we obtain a *quasi-uniform* family of triangulations, see Ciarlet [Cia78]. Furthermore, inequality (3.7) holds with the same constants for finer and finer meshes.

The results of Theorem 3.1.5 can be easily extended to vector valued functions and we can finally obtain an interpolant, $\boldsymbol{\mathcal{I}}_h : (H^1(D))^d \to V_h$, that satisfies

$$\|\mathbf{u} - \boldsymbol{\mathcal{I}}_h(\mathbf{u})\|_{(L^2(D))^d} \leq C\, h^{1+\alpha} |\mathbf{u}|_{(H^{1+\alpha}(D))^d}.$$

Now, by defining $\mathcal{R}_N = \boldsymbol{\mathcal{I}}_h$ with N that is of order of $1/h^2$, we get the desired inequality (3.2) with $\gamma = 1/2$.

As a final remark we observe that the operator $\mathcal{R}_N$ acts from $H^1(D)$ into a finite dimensional subspace of $(L^2(D))^d$. In particular this subspace does not need to be a subspace of divergence-free polynomials. In particular we point out that the shape functions need not to span a finite dimensional subspace of V : they can be external approximations. This is very interesting, because the problems arising with the divergence free constraint (recall the results for the Stokes operator in Section 1.4.1) are by-passed.

3.2 The Stochastic Navier-Stokes equations

In this section we introduce the Stochastic Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} + \frac{\partial \mathbf{g}}{\partial t} \quad \text{in } D \times [0, T], \tag{3.8}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D \times [0, T], \tag{3.9}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial D \times [0, T], \tag{3.10}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } D \times \{0\}. \tag{3.11}$$

The body forces are split into two terms: $\mathbf{f}$ is a classical term, and may represent slowly (differentiable) varying force, while $\partial\mathbf{g}/\partial t$ correspond to fast fluctuations of the force. It is possible to assume different assumptions to describe rapid fluctuations; we mainly assume that $\mathbf{g}$ is continuous, but not differentiable. Another possible choice is to take generalized stochastic processes, but we shall not enter into details, since our approach will be mainly the deterministic one, *i.e.*, we study the problem path-wise, for a fixed $\mathbf{g}$. Overview on stochastic partial differential equations can be found in Da Prato and Zabczyk [DPZ92, DPZ96].

The introduction of the stochastic Navier-Stokes equations is reasonable since the nonlinear nature of the equation leads naturally to the study of chaotic dynamical systems; a recent reference about chaos is Wiggins [Wig90]. The heuristic justification of the study of Navier-Stokes equations can be the following, see Chorin [Cho94]:

> *... we shall now consider random fields* $\mathbf{u}(\mathbf{x},\omega)$ *which, for each* ω *(i.e., for each experiment that produces them), satisfy the Navier-Stokes equations.* $\mathbf{u}$ *depends also on the time* t*; we shall usually not exhibit this dependence explicitly.*
>
> *There is an interesting question of principle that must be briefly discussed: why does it make sense to view solutions of the deterministic Navier-Stokes equations as being random? It is an experimental fact that the flow one obtains in the laboratory at a given time is a function of the experiment. The reason must be that the flow described by the Navier-Stokes equations for large* R *is chaotic; microscopic perturbations, even at a molecular scale, are amplified to macroscopic scales; no two experiments are truly identical and what one gets is a function of the experiment. The applicability of our constructions is plausible even if we do not know how to formalize the underlying probability space.*

Another justification is given by Barenblatt [Bar96] by considering the solution of the Navier-Stokes equations at high Reynolds number as a realization of a *turbulent*[1] flow:

> *...the flow properties for supercritical values of the Reynolds number undergo sharp and disorderly variations in space and in time, and the fields of flow properties,- pressure, velocity etc.-can to a good approximation be considered random. Such a regime of flow is called turbulent...*

We do not address to the result arising in the statistical study of Navier-Stokes equations, for which we refer to Višik and Fursikov [VF88]. The main result, which is not known in the deterministic case, is that in the presence of certain initial data and suitable "fast fluctuating terms" there is uniqueness of asymptotic behavior and an ergodic theorem holds. In particular there is a unique invariant measure associated to the Stochastic Navier-Stokes equations. We recall that if $S(t,\omega)$ is the transition semigroup associated to the Stochastic Navier-Stokes equations, then a measure μ on H is invariant if, for every time t, $S(t,\omega)$ satisfies the following equality:

$$\int S(t,\omega)\psi\, d\mu = \int \psi\, d\mu, \qquad \forall\psi \text{ belonging to the Borel bounded functions defined on } H,$$

see Cruzeiro [Cru89].

[1] Observe that, in his diaries, Leonardo Da Vinci used the word *turbulent* in the same sense we use it now.

3.2.1 Weak solutions of the Stochastic Navier-Stokes equations

We consider the Stochastic Navier-Stokes equations written as a functional differential equation

$$\frac{d\mathbf{u}}{dt} + \nu A\,\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f} + \frac{\partial \mathbf{g}}{\partial t}. \tag{3.12}$$

We assume that $\mathbf{u}_0 \in H$ and that $\mathbf{f} \in L^2(0,T;V')$. Furthermore, we assume that

$$\mathbf{g} \in C([0,T];V) \quad \text{and} \quad \mathbf{g}(0) = \mathbf{0}.$$

The equation above has now meaning only in an integral sense. To construct a weak solution, we project the Stochastic Navier-Stokes equations onto the space spanned by the first m eigenvectors of the Stokes operator and we consider the following integral system in $V_m := P_m(H)$:

$$\mathbf{u}_m(t) + \int_0^t A\,\mathbf{u}_m(s)\,ds + \int_0^t P_m(B(\mathbf{u}_m(s),\mathbf{u}_m(s))\,ds = P_m\,\mathbf{u}_0 + \int_0^t P_m\,\mathbf{f}(s)\,ds + P_m\,\mathbf{g}(t), \qquad t \geq 0,$$

which has a unique maximal solution $\mathbf{u}_m \in C(0,T;V_m)$. We define $\mathbf{v}_m := \mathbf{u}_m - P_m\,\mathbf{g} \in C(0,T;V_m)$, that satisfies

$$\begin{aligned}\mathbf{v}_m(t) + \int_0^t A\,\mathbf{v}_m(s)\,ds + &\int_0^t P_m(B(\mathbf{v}_m(s) + P_m\,\mathbf{g}(s), \mathbf{v}_m(s) + P_m\,\mathbf{g}(s))\,ds = \\ &= P_m(\mathbf{u}_0 - \mathbf{g}(0)) + \int_0^t P_m\,\mathbf{f}(s)\,ds + \int_0^t A\,P_m\,\mathbf{g}(s)\,ds, \qquad t \in [0,T].\end{aligned} \tag{3.13}$$

Therefore $\mathbf{v}_m$ is of class C^1 and it satisfies the equation

$$\frac{d\mathbf{v}_m}{dt} + \nu A\,\mathbf{v}_m + B(\mathbf{v}_m + P_m\,\mathbf{g}, \mathbf{v}_m + P_m\,\mathbf{g}) = \mathbf{f} + A\,\mathbf{g}, \qquad t \in [0,T].$$

We can use the "energy method" and we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{v}_m|^2 + \|\mathbf{v}_m\|^2 \leq -B(\mathbf{v}_m + P_m\,\mathbf{g}, P_m\,\mathbf{g}), \mathbf{v}_m > + \|\mathbf{v}_m\|\|\mathbf{f}\|_{V'} + \|\mathbf{v}_m\|\|A\,\mathbf{g}\|_{V'}.$$

We now observe that the term $B(\mathbf{v}_m + P_m\,\mathbf{g}, \mathbf{v}_m + P_m\,\mathbf{g})$ is no longer orthogonal to $\mathbf{v}_m$ and we need some special estimates. In particular, by using the Hölder[2] inequality, we get, for suitable positive constants

$$\begin{aligned}|< B(\mathbf{v}_m + P_m\,\mathbf{g}, P_m\,\mathbf{g}), \mathbf{v}_m >| \quad &= |< B(\mathbf{v}_m + P_m\,\mathbf{g}, \mathbf{v}_m), P_m\,\mathbf{g} >| \\ &\leq C_1\|\mathbf{v}_m\|\|\mathbf{v}_m\|_{L^4(D)}\|P_m\,\mathbf{g}\|_{L^4(D)} + C_1\|\mathbf{v}_m\|\|P_m\,\mathbf{g}\|^2_{L^4(D)} \\ &\leq C_2\|\mathbf{v}_m\|^{7/4}|\mathbf{v}_m|^{1/4}\|P_m\,\mathbf{g}\|_{L^4(D)} + C_1\|\mathbf{v}_m\|\|P_m\,\mathbf{g}\|^2_{L^4(D)}.\end{aligned}$$

By using the Young inequality, we have:

$$\frac{1}{2}\frac{d}{dt}|\mathbf{v}_m|^2 + \|\mathbf{v}_m\|^2 \leq C\left(|\mathbf{v}_m|^2\|P_m\,\mathbf{g}\|^8_{L^4(D)} + \|P_m\,\mathbf{g}\|^4_{L^4(D)} + \|A\,\mathbf{g}\|^2_{V'} + 2\|\mathbf{f}\|_{V'}\right). \tag{3.14}$$

[2] We restrict to $d = 3$ but, as in the deterministic problem, if $d = 2$ different (and more powerful) estimates hold.

From the last equation (3.14) if some estimate on $P_m\,\mathbf{g}$ is given, we can extract (as in the deterministic case) subsequences $\mathbf{v}_{m_k}$ that converge to some $\mathbf{v}$, which satisfies

$$< \mathbf{v}(t) - \mathbf{v}(t_0), \boldsymbol{\phi} > + \int_{t_0}^{t} < A^{1/2}\mathbf{v}(s), A^{1/2}\boldsymbol{\phi} > \, ds + \int_{t_0}^{t} < B(\mathbf{v}(s) + \mathbf{g}(s), \mathbf{v}(s) + \mathbf{g}(s)), \boldsymbol{\phi} > \, ds =$$

$$= \int_{t_0}^{t} < \mathbf{f}(s) + A\,\mathbf{g}(s), \boldsymbol{\phi} > \, ds, \qquad \forall\, t \geq t_0 \geq 0,\ \forall\, \boldsymbol{\phi} \in V.$$

Now by recalling that, for $m \in \mathbb{N}$, we defined $\mathbf{v}_m := \mathbf{u}_m - P_m\,\mathbf{g}$, we can give the definition of a weak solution by setting $\mathbf{u} := \mathbf{v} + \mathbf{g}$.

Definition 3.2.1. *Given $\mathbf{f} \in L^2(0,T;V')$ and $\mathbf{g} \in C([0,T];V)$, we say weak solution of the Stochastic Navier-Stokes equations (3.8) a function $\mathbf{u}$ belonging to $L^2(0,T;V) \cap L^\infty(0,T;H)$, which satisfies the following regularity property*

$$\text{if } d{=}3 \text{ then} \qquad \frac{d}{dt}(\mathbf{u} - \mathbf{g}) \in L^{4/3}(0,T;V')$$

$$\text{if } d{=}2 \text{ then} \qquad \frac{d}{dt}(\mathbf{u} - \mathbf{g}) \in L^{2}(0,T;V').$$

and such that:

a)

$$< \mathbf{u}(t) - \mathbf{u}(t_0), \boldsymbol{\phi} > + \int_{t_0}^{t} < A^{1/2}\mathbf{u}(s), A^{1/2}\boldsymbol{\phi} > \, ds + \int_{t_0}^{t} < B(\mathbf{u}(s), \mathbf{u}(s)), \boldsymbol{\phi} > \, ds =$$

$$=< \mathbf{g}(t) - \mathbf{g}(t_0), \boldsymbol{\phi} > + \int_{t_0}^{t} < \mathbf{f}(s), \boldsymbol{\phi} > \, ds \qquad \forall\, t \geq t_0 \geq 0,\ \forall\, \boldsymbol{\phi} \in V;$$

b) for almost all t and t_0, with $t \geq t_0 \geq 0$ it holds

$$|\mathbf{u}(t) - \mathbf{g}(t)|^2 \leq e^{\int_{t_0}^{t} (-\lambda_1 + C\|\mathbf{g}(s)\|_{L^4}^8)\, ds} |\mathbf{u}(t_0) - \mathbf{g}(t_0)|^2 +$$

$$+ \int_{t_0}^{t} e^{\int_{\sigma}^{t} (-\lambda_1 + C\|\mathbf{g}(s)\|_{L^4}^8)\, ds} C \left[\|\mathbf{g}(\sigma)\|_{L^4}^4 + \|A\,\mathbf{g}(\sigma)\|_{V'}^2 + \|\mathbf{f}(\sigma)\|_{V'}^2 \right] d\sigma;$$

c) for almost all t and t_0, with $t \geq t_0 \geq 0$ it holds

$$|\mathbf{u}(t) - \mathbf{g}(t)|^2 + \int_{t_0}^{t} \|\mathbf{u}(s) - \mathbf{g}(s)\|^2\, ds \leq |\mathbf{u}(t_0) - \mathbf{g}(t_0)|^2 +$$

$$+ C \int_{t_0}^{t} \left[|\mathbf{u}(\sigma) - \mathbf{g}(\sigma)|^2 \|\mathbf{g}(\sigma)\|_{L^4}^8 + 4\|\mathbf{g}(\sigma)\|_{L^4}^4 + 4\|A\,\mathbf{g}(\sigma)\|_{V'}^2 + 4\|A\,\mathbf{f}(\sigma)\|_{V'}^2 \right] d\sigma.$$

This is the first method introduced to study the Stochastic Navier-Stokes equations, see Bensoussan and Temam [BT73]. Another method, which is slightly different, is based on the introduction of an auxiliary (vector) Stokes equation, which is known as *Ornstein-Uhlenbeck* equation in the stochastic literature:

$$\frac{\partial \mathbf{z}}{\partial t} - \nu\Delta\mathbf{z} + \nabla q = \frac{\partial \mathbf{g}}{\partial t} \quad \text{in } D \times [0,T], \tag{3.15}$$

$$\operatorname{div}\mathbf{z} = 0 \quad \text{in } D \times [0,T], \tag{3.16}$$

$$\mathbf{z} = \mathbf{0} \quad \text{on } \partial D \times [0,T], \tag{3.17}$$

$$\mathbf{z}(\mathbf{x},0) = \mathbf{0} \quad \text{in } D \times \{0\}. \tag{3.18}$$

By setting $\mathbf{v} := \mathbf{u} - \mathbf{z}$ we have to study the modified Navier-Stokes equations:

$$\frac{d\mathbf{v}}{dt} + \nu A\,\mathbf{v} + B(\mathbf{v}+\mathbf{z},\mathbf{v}+\mathbf{z}) = \mathbf{f}. \tag{3.19}$$

The term $A\,\mathbf{g}$ is now absent and to obtain energy estimates we need some information on the regularity of $\mathbf{z}$. The application of the usual estimates on the nonlinear term gives the following condition

$$\mathbf{z} \in C([0,T];(L^4(D))^d),$$

which is sufficient to prove the existence of weak solutions.

Regularity results for the Ornstein-Uhlenbeck equations

We deal with the Ornstein-Uhlenbeck equations, in the abstract form:

$$\begin{cases} \dfrac{d\mathbf{z}}{dt} + A\,\mathbf{z} = \dfrac{d\mathbf{g}}{dt} & t \in [0,T] \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases}$$

and since $\mathbf{g} \in C([0,T];\mathcal{D}(A^{-1}))$ is not differentiable[3] with respect to time in the usual way, we must define the solution in an integral sense.

Definition 3.2.2. *A solution of the Ornstein-Uhlenbeck equations is a continuous function* $\mathbf{z}$ *on the interval* $[0,T]$, *with values in* $\mathcal{D}(A^\gamma)$ *for* $\gamma \geq 0$, *such that*

$$< \mathbf{z}(t) - \mathbf{z}(0), \boldsymbol{\phi} > + \int_0^t < \mathbf{z}(s), A\,\boldsymbol{\phi} > \, ds = < \mathbf{g}(t) - \mathbf{g}(0), \boldsymbol{\phi} > \qquad \forall\, t \in [0,T],\ \forall\, \boldsymbol{\phi} \in \mathcal{D}(A).$$

There are different methods to study the Ornstein-Uhlenbeck equations, for example by using the semigroup method. We write the solution by using the *variation of constants formula* below

$$\mathbf{z}(t) := \mathrm{e}^{-tA}\mathbf{z}_0 + \int_0^t \mathrm{e}^{-(t-s)A}\,\frac{d\mathbf{g}(s)}{dt}\,ds \tag{3.20}$$

and by an integration by parts we give meaning to the integral. The following result is well-known, see for instance Flandoli [Fla96].

Theorem 3.2.3. *Let* $\gamma \geq 0$ *be given. Assume that* $\mathbf{z}_0 \in \mathcal{D}(A^\gamma)$ *and*

i) assume that $\mathbf{g} \in C([0,T];\mathcal{D}(A^{\gamma+\varepsilon}))$. *Then the Ornstein-Uhlenbeck equations have a unique solution, which is given by* (3.20). *If* $\varepsilon \in (0,1/2]$, *we also have:*

$$\mathbf{z} \in L^2(0,T;\mathcal{D}(A^{\gamma+\varepsilon}));$$

ii) let $\varepsilon \in (0,1)$ *and let* $\beta \in (\gamma+\varepsilon-1, \gamma+\varepsilon)$ *be given. Assume that* $\mathbf{g} \in C^{\gamma+\varepsilon-\beta}([0,T];\mathcal{D}(A^\beta))$. *Then the Ornstein-Uhlenbeck equations has a unique solution given by*

$$\mathbf{z}(t) = e^{-tA}(\mathbf{z}_0 + \mathbf{g}(t) - \mathbf{g}(0)) + \int_0^t A\,e^{-(t-s)A}(\mathbf{g}(t) - \mathbf{g}(s))\,ds.$$

[3] This is the regularity assumption we generally made on the fast fluctuating force $\mathbf{g}$.

3.3 Determining projections for stochastic equations

In this section we consider the problem of determining projections for some dissipative stochastic equations. This problem was studied for the first time in the abstract context of random dynamical systems by Flandoli and Langa [FL99]. The results were simplified and improved by Berselli and Flandoli [BF99]. We basically refer to this paper in the following sections. We give now the definition of determining projection for a stochastic equation.

Definition 3.3.1. *The projection operator*

$$\mathcal{R}_N : V \to V_N \subset (L^2(D))^m, \qquad N = \dim(V_N) < \infty, \quad \textit{and} \quad D \subset \mathbb{R}^d,$$

is called a **determining projection** *for weak solutions of the d-dimensional stochastic dissipative equations if the following property holds true. Assume that the two initial conditions* $\mathbf{u}_0, \mathbf{v}_0 \in \mathrm{H}$ *are such that for* P*-a.e.* $\omega \in \mathbf{\Omega}$

$$\lim_{t\to+\infty} \|\mathcal{R}_N(\mathbf{u}(t,\omega) - \mathbf{v}(t,\omega))\|_{(L^2(D))^m} = 0. \tag{3.21}$$

Then for P*-a.e.* $\omega \in \mathbf{\Omega}$

$$\lim_{t\to+\infty} \|\mathbf{u}(t,\omega) - \mathbf{v}(t,\omega)\|_{(L^2(D))^m} = 0.$$

Here P is the Wiener probability measure on the space $\mathbf{\Omega} := C_0([0,T])$ of continuous functions vanishing at zero. We define a *cylindrical subset* of $\mathbf{\Omega}$ as

$$I(t_1,\dots,t_n;B) := \{\omega \in \mathbf{\Omega} : \ (w(t_1),\dots,w(t_n)) \in B, \ B \text{ Borel subset of } \mathbb{R}^n\}.$$

The Wiener measure is defined on cylindrical subsets as

$$P(I(t_1,\dots,t_n;B)) := \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \int_B \mathrm{e}^{-\frac{1}{2}[\frac{\xi_1^2}{t_1}+\cdots+\frac{(\xi_n-\xi_{n-1})^2}{t_n-t_{n-1}}]}\, d\xi_1\dots\xi_n,$$

and then extended by a standard argument to $\mathcal{F}$, the σ-algebra of Borel subsets of $\mathbf{\Omega}$.

We start by considering a very simple equation, for which there are satisfactory results. Then we shall study the Stochastic Navier-Stokes equations, in which the nonlinear term causes some additional difficulties.

3.3.1 The model problem: a reaction-diffusion equation

In this section we study a scalar reaction-diffusion equation. Let D denote a smooth bounded open set of $\mathbb{R}^d$ and let g be a polynomial of odd degree with a positive leading coefficient

$$g(s) := \sum_{j=0}^{2p-1} b_j s^j \quad \text{with} \quad b_{2p-1} > 0.$$

We consider the following boundary-initial value problem:

$$\frac{\partial u}{\partial t} - \nu\Delta u + g(u) = f + \sum_{i=1}^{n} \phi_i \frac{d\mathcal{W}_i}{dt} \quad \text{in } D\times(0,T), \tag{3.22}$$

$$u = 0 \quad \text{on } \partial D\times(0,T), \tag{3.23}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \text{in } D\times\{0\}. \tag{3.24}$$

The scalars $\mathcal{W}_i(t)$ are independent Brownian motions on a probability space $(\mathbf{\Omega}, \mathcal{F}, P)$, and the functions ϕ_i are smooth and depend only on the space variables. We take only a finite number of Brownian motions to avoid inessential problems. The reader, not acquainted with this topic, can view this additional forcing term again as the derivative of a certain not differentiable function.

Definition 3.3.2. *Let $(H, \|\,.\,\|)$ be a separable Banach space and let $(\mathbf{\Omega}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a probability space. A family of random variables $\{X(t)\}_{t\in I\subset\mathbb{R}}$ on this probability space, with values in $(H, \mathcal{B}(H))$ ($\mathcal{B}(H)$ are Borel sets) is a stochastic process. A Brownian motion is a real valued stochastic process such that*

a) $\mathcal{W}(0) = 0$;

b) for any $0 \leq s \leq t$ the random variable $\mathcal{W}(t) - \mathcal{W}(s)$ is independent of $\mathcal{F}_s$;

c) for any $0 \leq s \leq t$ the distribution of $\mathcal{W}(t) - \mathcal{W}(s)$ is $\mathcal{N}(0, t-s)$, i.e. a normal distribution with zero mean and variance $t - s$.

We refer to Billingsley for other details [Bil97]. We recall the following property satisfied by $g(s)$, that is easily obtained with repeated application of the Young inequality.

Proposition 3.3.3. *There exists $c > 0$ such that*

$$\frac{2p-1}{2} b_{2p-1} s^{2p-2} - c \leq g'(s) \leq \frac{3}{2}(2p-1) b_{2p-1} s^{2p-2} + c.$$

We consider the reaction-diffusion equation, because the nonlinear term is Lipschitz continuous and can be treated in a simple way. For the mathematical setting of this section we use the customary Sobolev space $(H, |\,.\,|) := (L^2(D), |\,.\,|)$ and $(V, \|\,.\,\|) := (H^1_0(D), \|\,.\,\|)$, where

$$|u| = \left(\int_D u^2\, d\mathbf{x}\right)^{1/2} \quad \text{and} \quad \|u\| = \left(\int_D \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right|^2 d\mathbf{x}\right)^{1/2}.$$

We consider the following abstract evolution equation in H

$$\frac{du}{dt} + \nu\, A\, u + B\,(u) = f + \sum_{i=1}^n \phi_i \frac{d\mathcal{W}_i}{dt}, \tag{3.25}$$

$$u(0) = u_0, \tag{3.26}$$

that is obtained from (3.22) by taking $B(u) = g(u)$, with u a scalar unknown and

$$A = D(A) \subset H \to H,$$

where $D(A) = H^2(D) \cap H^1_0(D)$ and $A\,u$ is defined by $A\,u = -\Delta u$. The main existence and regularity result we are going to use is the following one, see Crauel and Flandoli [CF94].

Theorem 3.3.4. *Let $u_0 \in H$, let $D \subset \mathbb{R}^d$ be a smooth bounded domain and let $f \in L^2(0, T; H)$. Then P-a.e. $\omega \in \mathbf{\Omega}$ there exists a unique solution to (3.25)-(3.26) such that*

$$u(t, \omega) \in C([0, T]; H) \cap L^2(0, T; V) \cap L^{2p}(0, T; L^{2p}(D)),$$

and

$$\frac{d}{dt}\left(u(t, \omega) - \sum_{i=1}^n \phi_i \mathcal{W}_i\right) \in L^2(0, T; V'),$$

where V' denotes the topological dual of V.

The proof of Theorem 3.3.4, that is a natural extension of the result that holds for the deterministic problem, is made with a standard Galerkin approximation on a modified equation in which the unknowns are translated. By using the techniques by Bensoussan and Temam [BT73], described in the previous Section 3.2.1, the new unknown

$$U = u - \sum_{i=1}^{n} \phi_i \mathcal{W}_i,$$

is introduced, for which the existence can be obtained by using the standard tools (with minor changes) that are needed in the deterministic setting, see J.-L. Lions [JLL69].

We can now prove the main result of this section, which is due to Berselli and Flandoli [BF99].

Theorem 3.3.5. *An operator satisfying the properties of the Scott and Zhang interpolant is determining for the stochastic reaction-diffusion equation* (3.25)-(3.26)*.*

Proof. We start by considering two solutions u and v corresponding to the initial data u_0 and v_0 and by defining $w := u - v$. We obtain the following equation

$$\frac{\partial w}{\partial t} - \nu \Delta w + g(u) - g(v) = 0.$$

We now multiply the last expression by w and we integrate over D to obtain

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu \|w\|^2 + \int_D g'(\xi_{u,v})\, w^2\, dx = 0, \tag{3.27}$$

where $\xi_{u,v}$ denotes a point depending on u and v. By using the result of Proposition 3.3.3 in (3.27) we obtain

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu \|w\|^2 + \int_D \xi_{u,v}^{2p-2}\, w^2\, dx \leq c|w|^2.$$

Now by using the following inequality (that holds for an interpolation operator like the Scott and Zhang one):

$$|w|^2 \leq |\mathcal{R}_N(w)|^2 + \frac{C}{N}\|w\|^2,$$

we get

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \left(\frac{\nu}{2}\frac{\|w\|^2}{|w|^2} - \frac{C_2}{N}\frac{\|w\|^2}{|w|^2}\right)|w|^2 \leq C_\nu |\mathcal{R}_N(w)|^2.$$

If we take N big enough we have that

$$\frac{\nu}{2} - \frac{C_2}{N} = \sigma > 0,$$

and we can write

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \lambda_1\left(\frac{\nu}{2} - \frac{C_2}{N}\right)|w|^2 \leq C_\nu |\mathcal{R}_N(w)|^2, \tag{3.28}$$

where $\lambda_1 > 0$ is the smallest eigenvalue of A. Finally, we can the use Gronwall lemma in (3.28) to get

$$|w(t)|^2 \leq |w(0)|e^{-\sigma\, t} + 2C_\nu \int_0^t e^{-\sigma(t-s)}|\mathcal{R}_N(w(s))|^2\, ds. \tag{3.29}$$

We easily obtain that $|w(t)|$ converges to zero when t goes to plus infinity. In fact, the first term decays exponentially and the integral appearing in (3.29) is seen to decay to zero, provided $\mathcal{R}_N(w(s)) \to 0$. If we split it as

$$\int_0^\tau e^{-\sigma(t-s)}|\mathcal{R}_N(w(s))|^2\, ds + \int_\tau^t e^{-\sigma(t-s)}|\mathcal{R}_N(w(s))|^2\, ds,$$

we can see that the first integral decays to zero since the integration is performed on a finite interval and a bounded term is multiplied by an exponential. The second integral is seen to be, in the limit of large t, smaller than every positive constant ϵ. In fact there exists $\epsilon_1 > 0$ such that

$$\lim_{t\to+\infty} \int_\tau^t e^{-\sigma(t-s)}\epsilon_1^2\, ds \leq \epsilon,$$

and we can choose $\tau = \tau(\epsilon_1)$ in such a way (by the hypothesis on $\mathcal{R}_N(w)$)

$$\forall\, s \geq \tau \qquad \mathcal{R}_N(w(s)) \leq \epsilon_1.$$

□

3.3.2 On determining projections for Stochastic Navier-Stokes equations

In this section we consider the problem of determining projections for the Stochastic Navier-Stokes equations. We start by recalling the following result, that extends in a natural way the result, we have seen previously for the deterministic problem, see Bensoussan and Temam [BT73].

Theorem 3.3.6. *Let $\mathbf{u}_0 \in H$, let $D \subset \mathbb{R}^2$ be a smooth bounded domain and let $\mathbf{f} \in L^2(0,T;\mathrm{H})$. Then P-a.e. $\omega \in \mathbf{\Omega}$ there exists a unique solution to (3.12) such that*

$$\mathbf{u}(t,\omega) \in C([0,T];\mathrm{H}) \cap L^2(0,T;\mathrm{V}),$$

and

$$\frac{d}{dt}\left(\mathbf{u}(t,\omega) - \sum_{i=1}^{n} \boldsymbol{\phi}_i \mathcal{W}_i\right) \in L^2(0,T;\mathrm{V}'),$$

where V' denotes the topological dual of V.

The proof of Theorem 3.3.6 (and in particular the uniqueness condition) follows essentially by the deterministic techniques and by the change of variables with a regular $\mathbf{z}$, as described in Section 3.2.1.

3.3.3 Stochastic framework

In this section we consider the Stochastic Navier-Stokes equations in the domain $D =]0,2\pi[\times]0,2\pi[$

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \mathbf{f} + \sum_{i=1}^{n} \boldsymbol{\phi}_i \frac{d\mathcal{W}_i}{dt} \qquad \text{in } D\times(0,T), \tag{3.30}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } D\times(0,T), \tag{3.31}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0 \qquad \text{in } D\times\{0\}, \tag{3.32}$$

and the vector $\mathbf{u} = (u_1,u_2)$ is equipped with periodic boundary conditions, see Remark 1.2.18 for the definition of the functional spaces needed. We use these simplifying boundary conditions since they allow us to get the estimates of Lemma 3.3.8.

Since we need to use some ergodic properties of the random attractor associated to the Stochastic Navier-Stokes equations, we introduce a suitable stochastic framework. Let $\mathbf{\Omega}$ be the space of all continuous functions $\omega : \mathbb{R} \to \mathbb{R}^n$ vanishing at the origin, endowed with the topology of uniform convergence on compact sets and let $\mathcal{F}$ be the Borel σ-algebra on $\mathbf{\Omega}$. Moreover let P be the Wiener measure on $(\mathbf{\Omega},\mathcal{F})$, namely the measure such that the $\mathbb{R}^n$-valued stochastic processes $\mathcal{W}^+(t,\omega)$ and $\mathcal{W}^-(t,\omega)$, defined, for $t \geq 0$ on $(\mathbf{\Omega},\mathcal{F},P)$, as

$$\mathcal{W}^+(t,\omega) := \omega(t), \quad \mathcal{W}^-(t,\omega) := \omega(-t),$$

are independent n-dimensional Brownian motions. The process

$$\mathcal{W}(t,\omega) := \omega(t) \qquad t \in \mathbb{R},$$

is now called a 2-sided n-dimensional Brownian motion. We remark that, by its definition, it has continuous trajectories. We use this 2-sided Brownian motion in the Stochastic Navier-Stokes equations. Furthermore we denote by $\mathbb{E}$ the *expectation* in $(\mathbf{\Omega},\mathcal{F},P)$.

3.3.4 The random attractor

In the previous Section 1.3.1 we described the attractors for deterministic dynamical systems. We now introduce the concept of *random attractor*, *i.e.*, the attractor for a random dynamical system, as that one generated by the Stochastic Navier-Stokes equations.

As we have stated above, for every given $\omega \in \mathbf{\Omega}$ one can solve uniquely the Navier-Stokes equations, with arbitrary initial time and initial condition, and construct a dynamical system, in the following sense. The introduction of these concepts is due essentially to Crauel, Debussche and Flandoli. For the results used in this section we refer to Crauel and Flandoli [CF94] or Crauel, Debussche and Flandoli [CDF97].

Random dynamical systems

There is a family $S(t,s,\omega)$ of continuous operators in the Hilbert space H, defined for all real $t \geq s$ and for all $\omega \in \mathbf{\Omega}$, such that for every $\mathbf{u}_0 \in$ H, the stochastic process $(t,\omega) \longmapsto S(t,s,\omega)\mathbf{u}_0$ is the solution of the Stochastic Navier-Stokes equations over the time interval $[s,\infty)$, with initial condition $\mathbf{u}_0$ at time s. This family of operators satisfies the usual evolution properties for all $\omega \in \mathbf{\Omega}$:

$$S(t,s,\omega)S(s,r,\omega) = S(t,r,\omega) \quad \text{for all } t \geq s \geq r,$$

$$S(t,t,\omega) = \text{identity in } H.$$

It also satisfies certain continuity and measurability properties in its variables, that we do not need to specify here.

It is convenient to express this random dynamical system with a different language which exploits the ergodicity of the noise. Over the path space $\mathbf{\Omega}$, a group of transformations ϑ_t is defined, for $t \in \mathbb{R}$, by

$$(\vartheta_t\,\omega)\,(s) = \omega(t+s) - \omega(t).$$

The system $(\mathbf{\Omega}, \mathcal{F}, P, \vartheta_t)$ is an ergodic dynamical system. For every $t \geq s$ and $\mathbf{u}_0 \in H$, we have

$$S(t,s,\omega)\mathbf{u}_0 = S(t-s,0,\vartheta_t\,\omega)\mathbf{u}_0 \quad P-a.e.$$

We recall the definition of random attractor.

Definition 3.3.7. *A compact-set valued (in H) stochastic process $A(t,\omega)$ (i.e. a mapping from $\mathbb{R} \times \mathbf{\Omega}$ to the family of compact subsets of H, such that for every $x \in H$ the real valued function $(t,\omega) \longmapsto d(x, A(t,\omega))$ is measurable) is called a random attractor if it is invariant:*

$$S(t,t_0,\omega)A(t_0,\omega) = A(t,\omega),$$

and attracts (at least) the bounded sets $B \subset H$:

$$\lim_{t_0 \to -\infty} d(S(t,t_0,\omega)B, A(t,\omega)) = 0,$$

where $d(.,.)$ is the semi-distance between sets in H *and the previous properties have to hold P-a.e.*

It is known that the Stochastic Navier-Stokes equations considered here has a random attractor. Many results on random attractors are known, including a finite Hausdorff dimension and an estimate on moments of its radius. The estimate on the radius are valid at least in the case of two dimensional Navier-Stokes equations on a torus with periodic boundary conditions. We recall such result and some of its consequences, see Berselli and Flandoli [BF99].

Lemma 3.3.8. *We denote by $R_A(t,\omega)$ the radius in L^4 of the random attractor $A(t,\omega)$ of the Stochastic Navier-Stokes equations. The stochastic process $R_A(t,\omega)$ is ergodic and satisfies*

$$\mathbb{E}\left(\sup_{t\in[0,1]} R_A(t)^p\right) \le C_p, \tag{3.33}$$

for all $p \ge 1$. In particular, there exist a constant C_A and a random variable $C_{A,1}(\omega)$, a.e. finite, such that P-a.e.

$$\int_t^0 R_A(s,\omega)^8\, ds \le (C_{A,1}(\omega) + C_A\,|t|) \quad \text{for all } t \le 0 \tag{3.34}$$

and

$$\int_0^t R_A(s,\omega)^8\, ds \le (C_{A,1}(\omega) + C_A t) \quad \text{for all } t \ge 0. \tag{3.35}$$

Moreover, there exists a random variable $C_{A,2}(\omega)$, a.e. finite, such that P-a.e.

$$R_A(t,\omega)^8 \le (C_{A,2}(\omega) + |t|)\,, \tag{3.36}$$

for all $t \in \mathbb{R}$ (any other power of $R_A(t,\omega)$ has a similar bound).

Proof. We have $R_A(s,\omega) = R_A(0,\vartheta_s\,\omega)$ (these two processes are modifications one of the other) since $A(s,\omega) = A(0,\vartheta_s\,\omega)$. Since ϑ_s is ergodic and $R_A(0,.)$ is measurable, $R_A(s,\omega)$ is an ergodic process.

From these facts it is standard to deduce the bounds of the lemma. Indeed, by the ergodic theorem, we have

$$\lim_{t\to-\infty} \frac{1}{|t|}\int_{-t}^0 R_A(s,\omega)^8\, ds = \mathbb{E}(R_A(0,.)^8),$$

which implies, by setting $C_A = \mathbb{E}(R_A(0,.)^8) + 1$, that the first bound is true for sufficiently large $|t|$; for smaller t we bound, path-wise, the continuous function

$$\int_{-t}^0 R_A(s,\omega)^8\, ds,$$

by a constant $C_{A,1}(\omega)$, and complete the proof of the bound (3.34). The proof of (3.35) is similar.

Finally, the proof of (3.36) is based on Borel-Cantelli lemma. We have, for $n \ge 0$, by using the stationarity and (3.33)

$$\begin{aligned} P(\sup_{t\in[n,n+1]} R_A(t,\omega)^8 > n) &= P(\sup_{t\in[0,1]} R_A(t,\omega)^8 > n) \\ &\le \frac{\mathbb{E}\left(\sup_{t\in[0,1]} R_A(t,\omega)^{16}\right)}{n^2} \le \frac{C_{16}}{n^2}. \end{aligned}$$

Since the last term is the term of a convergent series, the condition

$$\sup_{t\in[n,n+1]} R_A(t,\omega)^8 > n$$

holds true only for a finite number of n, giving $R_A(t,\omega)^8 \le [t] \le t$ for sufficiently large t. For smaller values of t, since $R_A(t,\omega)^8$ is locally bounded (again from (3.33)), we bound it by a constant $C_{A,2}(\omega)$. This completes the proof of (3.36) for positive values of t, while for negative ones the argument is similar. □

In the next section we shall use the random variable $C(\omega)$, a.e. finite, that is made precise in the definition below.

Definition 3.3.9. *The random variable $C(\omega)$ is defined by means of*

$$C(\omega) := e^{C_{A,1}(\omega)} \left(\int_{-\infty}^{0} \left(C_{A,2}(\omega) + |s|\right) e^{-\frac{\nu\lambda_1}{8}|s|} ds \right)^{\frac{1}{2}}. \tag{3.37}$$

We observe that $C(\omega)$ is defined only in terms of random constants related to the attractor.

3.3.5 Energy-type estimate

We now obtain a classical inequality that is the counterpart of the much more powerful (3.27) used for the reaction-diffusion equation. We consider $\mathbf{u}$ and $\mathbf{v}$ two solutions of (3.12) and we set $\mathbf{w} = \mathbf{u} - \mathbf{v}$. We can see that $\mathbf{w}$ satisfies the following equation

$$\frac{d\mathbf{w}}{dt} + \nu \mathrm{A}\,\mathbf{w} + B(\mathbf{u},\mathbf{w}) + B(\mathbf{w},\mathbf{v}) = 0. \tag{3.38}$$

As usual we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|^2 + \nu\|\mathbf{w}\|^2 \le |< B(\mathbf{w},\mathbf{v}),\mathbf{w} >|\,. \tag{3.39}$$

By recalling the definition of B and with the Hölder inequality with exponents $p = 4$ and $q = 4/3$, we get

$$|< B(\mathbf{w},\mathbf{v}),\mathbf{w} >| \le C_\nu |\mathbf{w}|^2 \|\mathbf{v}\|_{L^4}^4 + \frac{\nu}{2}\|\mathbf{w}\|^2,$$

with $C_\nu = 3^3/2^5\nu^3$.

By collecting the previous results we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|^2 + \frac{\nu}{2}\|\mathbf{w}\|^2 \le C_\nu |\mathbf{w}|^2 \|\mathbf{v}\|_{L^4}^4.$$

Now by using the following inequality, that holds for an interpolation operator like the Scott and Zhang one:

$$|\mathbf{w}| \le |\mathcal{R}_N(\mathbf{w})| + \frac{C}{\sqrt{N}}\|\mathbf{w}\|,$$

we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|^2 + \left(\frac{\nu}{2}\frac{\|\mathbf{w}\|^2}{|\mathbf{w}|^2} - \frac{C_2}{\sqrt{N}}\frac{\|\mathbf{w}\|}{|\mathbf{w}|}\|\mathbf{v}\|_{L^4}^4\right)|\mathbf{w}|^2 \le C_\nu |\mathbf{w}||\mathcal{R}_N(\mathbf{w})|\|\mathbf{v}\|_{L^4}^4,$$

Finally we can use the Gronwall lemma in

$$\frac{d}{dt}|\mathbf{w}| + \left(\frac{\nu\lambda_1}{4} - \frac{C_2}{N}\|\mathbf{v}\|_{L^4}^8\right)|\mathbf{w}| \le |\mathcal{R}_N(\mathbf{w})|\,\|\mathbf{v}\|_{L^4}^4,$$

where λ_1 is the smallest eigenvalue of the Stokes operator to obtain the following inequality

$$\begin{aligned} |\mathbf{u}(t,\omega) - \mathbf{v}(t,\omega)| &\le |\mathbf{u}(t_0,\omega) - \mathbf{v}(t_0,\omega)|\, e^{-\int_{t_0}^t \left(\frac{\nu\lambda_1}{4} - \frac{C_2}{N}\|\mathbf{v}(r,\omega)\|_{L^4}^8\right) dr} + \\ &+ \int_{t_0}^t |R_N\left(\mathbf{u}(s,\omega) - \mathbf{v}(s,\omega)\right)|\, \|\mathbf{v}(s,\omega)\|_{L^4}^4\, e^{\int_s^t \left(\frac{\nu\lambda_1}{4} - \frac{C_2}{N}\|\mathbf{v}(r,\omega)\|_{L^4}^8\right) dr} ds. \end{aligned} \tag{3.40}$$

We observe that the last expression (with a slight variation in the derivation) can be obtained with the term $\|\mathbf{u}\|_{L^4}^8$ instead of $\|\mathbf{v}\|_{L^4}^8$. In this way we can see that the subsequent properties, we shall obtain from (3.40), are based on the regularity or ergodicity of only one of the two solutions.

3.3.6 Determining projections forward in time

The property of determining projections proved in the following theorem is slightly weaker than the property introduced in Definition 1.1. We require indeed a small exponential decay of the projections, and we assume that one solution is on the random attractor. Restrictions of this kind are imposed also in Flandoli and Langa [FL99] (in fact stronger, since both solutions have to belong to the random attractor and the exponential decay of the projections cannot be arbitrarily small). To remove these restrictions is an open problem. We have the following result, see Berselli and Flandoli [BF99].

Theorem 3.3.10. *Let N be a given natural number such that*

$$\lambda := \frac{\nu\lambda_1}{4} - \frac{C_2}{N}\mathbb{E}\left(R_A(0,\omega)^8\right) > 0.$$

Let $\mathbf{u}(t,\omega)$ and $\mathbf{u}^\infty(t,\omega)$ be two solutions on $[0,\infty)$, such that

$$\mathbf{u}^\infty(t,\omega) \in A(t,\omega) \quad \textit{for all } t \ge 0,\ P\textit{-a.e.}$$

Assume that for some $\delta > 0$ and some random constant $C_\delta(\omega)$ we have

$$|\mathcal{R}_N\left(\mathbf{u}(t,\omega) - \mathbf{u}^\infty(t,\omega)\right)| \le C_\delta(\omega)e^{-\delta t} \quad \textit{for all } t \ge 0,\ P\textit{-a.e.}$$

Then, for every $\gamma < \min\{\delta,\lambda\}$ and for some random constant $C_\gamma(\omega)$, we have

$$|\mathbf{u}(t,\omega) - \mathbf{u}^\infty(t,\omega)| \le C_\gamma(\omega)e^{-\gamma t}$$

for all $t \ge 0$, P-a.e.

Proof. Since $\mathbf{u}^\infty(t,\omega) \in A(t,\omega)$, we have

$$\|\mathbf{u}^\infty(t,\omega)\|_{L^4} \le R_A(t,\omega).$$

Let

$$\mathbf{w}(t,\omega) := \mathbf{u}(t,\omega) - \mathbf{u}^\infty(t,\omega), \quad \text{and} \quad \rho(t,\omega) := \frac{\nu\lambda_1}{4} - \frac{C_2}{N}R_A(t,\omega)^8.$$

We have

$$-\left(\frac{\nu\lambda_1}{4}-\frac{C_2}{N}\left\|\mathbf{u}^\infty(s,\omega)\right\|_{L^4}^8\right)\leq-\rho(t,\omega).$$

Thus, from the inequality (3.40), we have

$$|\mathbf{w}(t,\omega)|\leq e^{-\int_0^t\rho(\sigma,\omega)d\sigma}\left|\mathbf{w}(0,\omega)\right|+\int_0^t e^{-\int_s^t\rho(\sigma,\omega)d\sigma}\left|\mathcal{R}_N\left(\mathbf{w}(s,\omega)\right)\right|\left\|\mathbf{u}^\infty(s,\omega)\right\|_{L^4}^4 ds,$$

for all $t\geq 0$, P-a.e., that we rewrite in the form

$$|\mathbf{w}(t,\omega)|\leq e^{-\int_0^t\rho(\sigma,\omega)d\sigma}\left(\left|\mathbf{w}(0,\omega)\right|+\int_0^t e^{\int_0^s\rho(\sigma,\omega)d\sigma}\left|\mathcal{R}_N\left(\mathbf{w}(s,\omega)\right)\right|\left\|\mathbf{u}^\infty(s,\omega)\right\|_{L^4}^4 ds\right).$$

From the assumption on the decay of $|\mathcal{R}_N(\mathbf{w}(s,\omega))|$ and from the last bound in Lemma 3.3.8, given any $\delta'<\delta$ there exists a random constant $C_{\delta'}(\omega)$ such that

$$\left|\mathcal{R}_N\left(\mathbf{w}(s,\omega)\right)\right|\left\|\mathbf{u}^\infty(s,\omega)\right\|_{L^4}^4\leq C_{\delta'}(\omega)e^{-\delta' s}\quad\text{for all } t\geq 0,\ P\text{-a.e.}$$

Therefore, we have the inequality

$$|\mathbf{w}(t,\omega)|\leq e^{-\int_0^t\rho(\sigma,\omega)d\sigma}\left(\left|\mathbf{w}(0,\omega)\right|+C_{\delta'}(\omega)\int_0^t e^{\int_0^s\rho(\sigma,\omega)d\sigma}e^{-\delta' s}ds\right)\quad\text{for all } t\geq 0,\ P\text{-a.e.}$$

By the ergodic theorem

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t\rho(\sigma,\omega)d\sigma=\mathbb{E}\left(\rho(0,\omega)\right)=\frac{\nu\lambda_1}{4}-\frac{C_2}{N}\mathbb{E}\left(R_A(0,\omega)^8\right)=\lambda>0.$$

Choose $\varepsilon>0$ such that $\varepsilon<\lambda$ and $2\varepsilon<\delta'$. There exists $t_0(\omega)$ such that

$$t\left(\lambda-\varepsilon\right)\leq\int_0^t\rho(\sigma,\omega)d\sigma\leq t\left(\lambda+\varepsilon\right)\quad\text{for all } t\geq t_0(\omega)\text{ and } P\text{-a.e.}$$

Therefore, for all $t\geq t_0(\omega)$, P-a.e.,

$$|\mathbf{w}(t,\omega)|\leq e^{-t(\lambda-\varepsilon)}\left(\left|\mathbf{w}(0,\omega)\right|+C_{t_0}(\omega)+C_{\delta'}(\omega)\int_{t_0(\omega)}^t e^{s(\lambda+\varepsilon)}e^{-\delta' s}ds\right),$$

where

$$C_{t_0}(\omega)=C_{\delta'}(\omega)\int_0^{t_0(\omega)}e^{\int_0^s\rho(\sigma,\omega)d\sigma}e^{-\delta' s}ds.$$

By explicit computation (we can always choose a smaller $\varepsilon>0$ such that $\lambda+\varepsilon-\delta'\neq 0$)

$$|\mathbf{w}(t,\omega)|\leq e^{-t(\lambda-\varepsilon)}\left(\left|\mathbf{w}(0,\omega)\right|+C_{t_0}(\omega)+C_{\delta'}(\omega)\frac{e^{t(\lambda+\varepsilon-\delta')}-e^{t_0(\omega)(\lambda+\varepsilon-\delta')}}{\lambda+\varepsilon-\delta'}\right)$$

$$=e^{-t(\lambda-\varepsilon)}\left(\left|\mathbf{w}(0,\omega)\right|+C_{t_0}(\omega)-C_{\delta'}(\omega)\frac{e^{t_0(\omega)(\lambda+\varepsilon-\delta')}}{\lambda+\varepsilon-\delta'}\right)+e^{-t(\delta'-2\varepsilon)}\frac{C_{\delta'}(\omega)}{\lambda+\varepsilon-\delta'}.$$

The arbitrariness of $\varepsilon>0$ and $\delta'<\delta$ proves the theorem. □

3.3.7 Determining projections backward in time

In the stochastic framework it makes sense to consider solutions defined for all $t \leq 0$. Roughly speaking, a property of determining projections over $(-\infty, 0]$ could say that two solutions are close for $t = 0$ if we know that their projections are close for $t < 0$. In the following theorem and its two corollaries we state some results in this direction, see again Berselli and Flandoli [BF99].

Theorem 3.3.11. *Let N be sufficiently large, depending on the constants of the problem (it is sufficient, for instance, that $N \geq C_2$ and $\frac{\nu\lambda_1}{4} - \frac{C_2}{N}C_A > 0$). Let $\mathbf{u}^\infty(t,\omega)$ be a solution of the Stochastic Navier-Stokes equations defined for $t \in \mathbb{R}$, such that P-a.e.*

$$\mathbf{u}^\infty(t,\omega) \in A(t,\omega) \quad \textit{for all } t \in \mathbb{R}.$$

Let $\mathbf{u}(t,\omega)$ be any other solution of the Stochastic Navier-Stokes equations, also defined for $t \in R$. Assume that

$$\lim_{t\to-\infty} \frac{\log^+ |\mathbf{u}(t,\omega)|}{|t|} = 0, \qquad P - a.e.\, \omega \in \mathbf{\Omega}.$$

Then P-a.e. (recall (3.37))

$$|\mathbf{u}^\infty(0,\omega) - \mathbf{u}(0,\omega)| \leq C(\omega) \left(\int_{-\infty}^{0} |\mathcal{R}_N (\mathbf{u}^\infty(s,\omega) - \mathbf{u}(s,\omega))|^2 e^{-\frac{\nu\lambda_1}{8}|s|} ds \right)^{\frac{1}{2}}. \tag{3.41}$$

This theorem shows that the difference of the solutions can be bounded by the difference of their projections. In particular, if the projections coincide, the solutions also coincide.

Corollary 3.3.12. *Under the same assumptions, if*

$$\mathcal{R}_N (\mathbf{u}^\infty(t,\omega) - \mathbf{u}(t,\omega)) = 0,$$

for all $t \leq 0$ (or at least for $t \leq t_0$ for some $t_0 \in R$), then

$$\mathbf{u}^\infty(t,\omega) - \mathbf{u}(t,\omega) = 0 \quad \textit{for all } t \in \mathbb{R}.$$

The previous corollary holds true either P-a.e., or even for a single ω satisfying (3.41). The next consequence of the first theorem of this section expresses the fact that better and better approximations of the *projections* of a solution $\mathbf{u}^\infty(t,\omega)$ give us better and better approximations of $\mathbf{u}^\infty(t,\omega)$ itself.

Corollary 3.3.13. *Let $\mathbf{u}_\varepsilon(t,\omega)$, $\varepsilon \in (0,1)$ be a family of solutions satisfying the assumptions of the previous theorem and in addition such that*

$$\lim_{\varepsilon\to 0} \mathcal{R}_N (\mathbf{u}^\infty(t,\omega) - \mathbf{u}_\varepsilon(t,\omega)) = 0 \quad \textit{for all } t \leq 0$$

and

$$|\mathcal{R}_N (\mathbf{u}^\infty(t,\omega) - \mathbf{u}_\varepsilon(t,\omega))| \leq C_1(\omega) e^{\delta|t|}$$

uniformly in ε, for some random constant $C_1(\omega)$ and a sufficiently small $\delta > 0$ (for instance $\delta < \frac{\nu\lambda_1}{8}$). Then

$$\lim_{\varepsilon\to 0} |\mathbf{u}^\infty(0,\omega) - \mathbf{u}_\varepsilon(0,\omega)| = 0.$$

We now prove Theorem 3.3.11 and we do not give the very easy proofs of the two corollaries.

Proof of Theorem 3.3.11. We estimate the difference between $\mathbf{u}^\infty(0,\omega)$ and $\mathbf{u}(0,\omega)$ with

$$|\mathbf{u}^\infty(0,\omega)-\mathbf{u}(0,\omega)| \le |\mathbf{u}^\infty(t_0,\omega)-\mathbf{u}(t_0,\omega)|\, e^{-\int_{t_0}^0\left(\frac{\nu\lambda_1}{4}-\frac{C_2}{N}\|\mathbf{u}^\infty(r,\omega)\|_{L^4}^8\right)dr}$$
$$+\int_{t_0}^0 |\mathcal{R}_N(\mathbf{u}^\infty(s,\omega)-\mathbf{u}(s,\omega))|\, \|\mathbf{u}^\infty(s,\omega)\|_{L^4}^4\, e^{\int_s^0\left(\frac{\nu\lambda_1}{4}-\frac{C_2}{N}\|\mathbf{u}^\infty(r,\omega)\|_{L^4}^8\right)dr}\, ds =$$
$$= I_1(t_0,\omega)+I_2(t_0,\omega)\le I_1(t_0,\omega)+I_2(\omega),$$

where

$$I_2(\omega)=\int_{-\infty}^0 |\mathcal{R}_N(\mathbf{u}^\infty(s,\omega)-\mathbf{u}(s,\omega))|\, \|\mathbf{u}^\infty(s,\omega)\|_{L^4}^4\, e^{\int_s^0\left(\frac{\nu\lambda_1}{4}-\frac{C_2}{N}\|\mathbf{u}^\infty(r,\omega)\|_{L^4}^8\right)dr}\, ds.$$

We prove that

$$\lim_{t_0\to-\infty} |\mathbf{u}^\infty(t_0,\omega)-\mathbf{u}(t_0,\omega)|\, e^{-\int_{t_0}^0\left(\frac{\nu\lambda_1}{4}-\frac{C_2}{N}\|\mathbf{u}^\infty(r,\omega)\|_{L^4}^8\right)dr}=0 \tag{3.42}$$

and that

$$I_2(\omega)\le C(\omega)\left(\int_{-\infty}^0 |\mathcal{R}_N(\mathbf{u}^\infty(s,\omega)-\mathbf{u}(s,\omega))|^2\, e^{-\frac{\nu\lambda_1}{8}|s|}\, ds\right)^{\frac{1}{2}}. \tag{3.43}$$

This implies the inequality of the theorem.

By the estimates on the radius in L^4 of the random attractor, we have for all $t\le 0$

$$\begin{aligned}
e^{-\int_t^0\left(\frac{\nu\lambda_1}{4}-\frac{C_2}{N}\|\mathbf{u}^\infty(r,\omega)\|_{L^4}^8\right)dr} &\le e^{-\frac{\nu\lambda_1}{4}|t|+\frac{C_2}{N}\int_t^0\|\mathbf{u}^\infty(r,\omega)\|_{L^4}^8 dr},\\
&\le e^{-\frac{\nu\lambda_1}{4}|t|+\frac{C_2}{N}\left(C_{A,1}(\omega)+C_A|t|\right)},\\
&\le e^{\frac{C_2}{N}C_{A,1}(\omega)}e^{-\left(\frac{\nu\lambda_1}{4}-\frac{C_2}{N}C_A\right)|t|},\\
&\le e^{C_{A,1}(\omega)}e^{-\frac{\nu\lambda_1}{8}|t|},
\end{aligned}$$

for N sufficiently large, depending only on the constants of the equation (C_2, ν, λ_1, C_A).

The limit (3.42) is now obvious, since $|\mathbf{u}^\infty(t_0,\omega)-\mathbf{u}(t_0,\omega)|$ grows less than $e^{\varepsilon|t_0|}$ as $t_0\to-\infty$, for every $\varepsilon>0$. To prove the inequality (3.43), it is sufficient to use the previous bound and the Hölder inequality. We get

$$\begin{aligned}
I_2(\omega) &\le e^{C_{A,1}(\omega)}\int_{-\infty}^0 |\mathcal{R}_N(\mathbf{u}^\infty(s,\omega)-\mathbf{u}(s,\omega))|\, \|\mathbf{u}^\infty(s,\omega)\|_{L^4}^4\, e^{-\frac{\nu\lambda_1}{8}|s|}\, ds,\\
&\le C^\infty(\omega)\left(\int_{-\infty}^0 |\mathcal{R}_N(\mathbf{u}^\infty(s,\omega)-\mathbf{u}(s,\omega))|^2\, e^{-\frac{\nu\lambda_1}{8}|s|}\, ds\right)^{\frac{1}{2}},
\end{aligned}$$

where

$$C^\infty(\omega)=e^{C_{A,1}(\omega)}\left(\int_{-\infty}^0 \|\mathbf{u}^\infty(s,\omega)\|_{L^4}^8\, e^{-\frac{\nu\lambda_1}{8}|s|}\, ds\right)^{\frac{1}{2}}.$$

Finally, to see that $C^\infty(\omega)\le C(\omega)$, we recall the definition (3.37). □

Chapter 4

Domain decomposition methods

In this chapter we present some numerical methods to solve the non-symmetric elliptic systems which arise in the first step of the Chorin-Temam method, see Section 1.4.3.

We start by recalling some results on linear systems and we explain the basic features of the *domain decomposition methods* on the model problem of the Poisson equation. We recall that *domain decomposition methods* are powerful iterative methods for solving large linear systems that arise from the discretization of partial differential equations. At each step of an iteration we solve smaller systems, which correspond to the restriction of the original problem to subregions. Some small "interface" problems are then considered. These *divide and conquer* methods are very interesting for parallel computation since many operations can be done in parallel. Another interesting feature is that domain decomposition methods lead to the construction of optimal (independent of h, the mesh size) preconditioners.

In the following sections, we shall present some possible extension to non-symmetric problems. Finally, we give some new results regarding the Maxwell equations and advection-diffusion equations and systems.

4.1 Linear systems

After a suitable discretization, a problem described by partial differential equations, can be reduced to the solution of a linear system. We have seen an example in Section 1.4 in which the approximating finite dimensional problem is obtained by using the Finite Element Method. In this section we recall some of the problems arising when dealing with a linear system with a $n \times n$ real matrix A

$$A\mathbf{x} = \mathbf{f}, \tag{4.1}$$

where $\mathbf{x}$ and $\mathbf{f}$, belonging to $\mathbb{R}^n$, are the unknown and the load vector respectively.

Error analysis

We recall that when we are faced with the problem of inverting a square matrix A (or equivalently to solve a linear system) one of the most important parameters is the *condition number*

$$\chi(A) := \|A\|\|A^{-1}\|,$$

where $\|\,.\,\|$ is any norm in the matrix space. We recall that a norm in the matrix space is a function $\mathcal{F} : \mathbb{R}^{n\times n} \to \mathbb{R}$, such that, for all $A, B \in \mathbb{R}^{n\times n}$:

a) $\mathcal{F}(A) \geq 0$ and $\mathcal{F}(A) = 0$ if and only if $A = 0$; b) $\mathcal{F}(A + B) \leq \mathcal{F}(A) + \mathcal{F}(B)$;

c) $\mathcal{F}(\alpha\, A) = |\alpha|\, \mathcal{F}(A), \quad \alpha \in \mathbb{R}$; d) $\mathcal{F}(AB) \leq \mathcal{F}(A)\mathcal{F}(B)$.

The *spectral radius* of A is defined as

$$\rho(A) := \max\{|\lambda_i(A)| : \quad \text{for } i = 1, \ldots, n\},$$

where $\lambda_i(A)$ are the eigenvalues of A. It is well-known that the spectral radius of A is the upper lower bound of $\|A\|$, taken over all possible *natural norms*[1]. We have the following result, see Isaacson and Keller [IK66].

Proposition 4.1.1. *Let A be non singular and suppose that $\|E\| < 1/\|A^{-1}\|$, for the natural norm induced by the vector norm $|\,.\,|$. Then suppose that in the linear system (4.1) the data A and $\mathbf{f}$ are perturbed by the matrix E and by the vector $\mathbf{e}$ respectively. If $\overline{\mathbf{x}}$ satisfies $(A+E)\overline{\mathbf{x}} = \mathbf{f}+\mathbf{e}$, then the following error estimate holds*

$$\frac{|\mathbf{x} - \overline{\mathbf{x}}|}{|\mathbf{x}|} \leq \frac{\chi(A)}{1 - \|A^{-1}\|\|E\|} \left(\frac{|\mathbf{e}|}{|\mathbf{x}|} + \frac{\|E\|}{\|A\|} \right).$$

We have to take into account this result, because it shows how the condition number affects the solution of a linear system. Recall also that, in general, the matrices arising in the finite element approximation of an elliptic problem are ill-conditioned. If we use a mesh with size h and the *shape functions* $\{\phi_k(\mathbf{x})\}_k$ and if furthermore $a(\,.\,,\,.\,)$ is the bilinear form associated to a weak formulation, we define A_h, the *stiffness matrix,* as

$$(A_h)_{ij} := a(\phi_j(\mathbf{x}), \phi_i(\mathbf{x}))$$

and we have that $\chi(A_h) \simeq 1/h^2$. We do not describe the *direct methods* for solving a linear system, *i.e.*, methods which produce a solution in a finite number of steps, but we prefer to consider *iterative methods,* which produce the solution as limit of a suitable sequence. The choice between direct or iterative methods can be a matter of taste, but can also be determined by the particular problem to be solved. We describe the second method, because we shall use it in the following sections regarding domain decomposition methods.

4.1.1 Iterative methods

An iterative method provides the solution of a linear system as the limit of a sequence of vectors $\mathbf{x}^m$. The calculation of such vectors involves only multiplication by A and other "less-expensive" operations. This methods are used especially when dealing with *sparse matrices*, *i.e.*, matrices such that the number of non-zero terms is of order of n.

An iterative method is generally based on the splitting of A into $A = P - N$, where P must be non singular. Given $\mathbf{x}^0 \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^m\}_{m\in\mathbb{N}}$ is generated by:

$$P\,\mathbf{x}^{m+1} = N\,\mathbf{x}^m + \mathbf{f}, \qquad m \geq 0. \tag{4.2}$$

[1] We recall that the *natural norm* $\|\,.\,\|$ is the norm induced by a vector norm $|\,.\,|$ in $\mathbb{R}^n$ as: $\|A\| = \sup_{\mathbb{R}^n \ni \mathbf{x} \neq 0} \frac{|A\mathbf{x}|}{|\mathbf{x}|}$.

The sequence $\{\mathbf{x}^m\}_{m\in\mathbb{N}}$ converges if and only if the spectral radius of $B = P^{-1}N$, the *iteration matrix,* satisfies

$$\rho(B) < 1.$$

If we define the error vector $\mathbf{e}^m := \mathbf{x}^m - \mathbf{x}$, we have that $\mathbf{e}^m = B^m\mathbf{e}^0$ and, for each $\varepsilon > 0$, there exists a *natural* norm $\|\,.\,\|$ (induced by the vector norm $|\,.\,|$) such that $\|B\| \le \rho(B) + \varepsilon$, and

$$|\mathbf{e}^m| \le \|B\|^m|\mathbf{e}^0| \le [\rho(B) + \varepsilon]^m\,|\mathbf{e}^0|.$$

This last inequality gives the convergence in the norm $|\,.\,|$, if ε is small enough. Furthermore it is clear that as $\rho(B)$ is smaller, then the convergence is quicker. The divergence of the scheme is easily proved if $\rho(B) \ge 1$. We observe that, if $\mathbf{e}^0$ is an eigenvector associated to the eigenvalue of maximum modulus, the sequence $|\mathbf{e}^m| = [\rho(B)]^m\,|\mathbf{e}^0|$ is not convergent, with respect to any vector norm $|\,.\,|$.

4.1.2 Preconditioning

We recall that an iterative method is effective when the *condition number* of the iteration matrix $P^{-1}N$ is *smaller* than that one of the original matrix A. In particular, the matrix P is called *preconditioner.* The construction of easily computable and effective preconditioner is one of the most interesting problems in the solution of linear systems by iterative methods. We now propose some classical methods.

Jacobi method

One classical method is the Jacobi method, which can be defined for matrices having non-zero diagonal elements. We write

$$A = D + E + F := \begin{pmatrix} & & 0 \\ & D_A & \\ 0 & & \end{pmatrix} + \begin{pmatrix} & & 0 \\ & 0 & \\ L_A & & \end{pmatrix} + \begin{pmatrix} & & U_A \\ & 0 & \\ 0 & & \end{pmatrix},$$

where D_A, L_A, and U_A are the diagonal of A and its lower and upper triangular parts respectively. The Jacobi method is based on the splitting $P := D$ and $N := -(E + F)$. The iteration matrix is therefore

$$B_J := -D^{-1}(E + F).$$

Gauss-Seidel

The Gauss-Seidel method is based on the splitting $P := D + E$ and $N = -F$ and is therefore well-defined for matrices with non-zero diagonal elements. The iteration matrix is

$$B_{GS} := -(D + E)^{-1}F.$$

Preconditioned Richardson method

The iteration (4.2) previously introduced can be rewritten as $P(\mathbf{x}^{m+1} - \mathbf{x}^m) = \mathbf{r}^m$, where the vector $\mathbf{r}^m := \mathbf{f} - A\,\mathbf{x}^m$ is called the *residual.* A generalization[2] can be the following

$$P(\mathbf{x}^{m+1} - \mathbf{x}^m) = \alpha\,\mathbf{r}^m.$$

[2] More general methods can be introduced if the parameter is α_m, different at each step. The method with $\alpha_m = const.$ is known as *stationary preconditioned Richardson method.*

This method can also be written as: given $\mathbf{x}^0$ (and consequently $\mathbf{r}^0 = \mathbf{f} - A\,\mathbf{x}^0$), solve for $m \geq 0$,

$$\begin{cases} P\,\mathbf{z}^m := \mathbf{r}^m, \\ \mathbf{x}^{m+1} := \mathbf{x}^m - \alpha\,\mathbf{z}^m, \\ \mathbf{r}^{m+1} := \mathbf{r}^m - \alpha\,A\,\mathbf{z}^m. \end{cases} \tag{4.3}$$

Each step requires the solution of a linear system with matrix P and a matrix-vector product involving the original matrix A.

The *preconditioned Richardson iteration matrix* is

$$R_\alpha := I - \alpha\,P^{-1}A.$$

We denote by $\eta_j \in \mathbb{C}$ the eigenvalues of $P^{-1}A$. We have the following theorem.

Theorem 4.1.2. *For any non-singular matrix P, the preconditioned Richardson method (4.3) converges if and only if*

$$|\eta_j|^2 \leq \frac{2}{\alpha}\,\mathrm{Re}\,\eta_j \qquad \forall\, j = 1, \dots, n, \tag{4.4}$$

where $\mathrm{Re}\,\eta$ denotes the real part of η.

We remark that a necessary condition for the convergence is a constant-sign of the real parts of the eigenvalues of $P^{-1}A$. We refer to the book by Golub and Van Loan [GVL89] for some sufficient conditions that ensure the convergence of the Jacobi and Gauss-Seidel methods and for the proof of Theorem 4.1.2. In the same reference it can be found the discussion relative to other modern iterative methods.

4.2 Brief introduction to domain decomposition methods

In this section we introduce some techniques and we recall some results regarding *domain decomposition methods.* In the following we shall confine to problems involving two sub-domains.

In the study of domain decomposition there are two main approaches: the one that uses overlapping domains, which takes its origin in Schwarz [Sch1869] and the other one (*substructuring*), that uses non-overlapping regions, in which the differential problem can give some additional ideas. The delicate interplay of these methods and some results towards a unified interpretation can be found in Nepomnyashchikh [Nep86] and Drya and Widlund [DW95]. We show the basic results for the Laplace operator and then we analyze the problems arising in the study of non-symmetric equations.

4.2.1 A survey of Schwarz method

We start by introducing the classical Schwarz method for the model problem of the Poisson equation:

$$Lu := -\Delta u = f \quad \text{in } D, \tag{4.5}$$

$$u = 0 \quad \text{on } \partial D. \tag{4.6}$$

It is well-known (see Nečas [Neč67] for a classical reference), that this problem can be treated successfully with a weak formulation in $V := H^1_0(D)$. By defining the symmetric, continuous and coercive bilinear form

$$a(u, v) := \int_D \nabla u \cdot \nabla v\,d\mathbf{x},$$

we can prove the existence and the uniqueness of the solution of the *variational problem*: find $u \in V$ such that

$$a(u,v) = (f,v) \qquad \forall\, v \in V. \tag{4.7}$$

In this section we denote by $|v|^2_{1,D} = \int_D |\nabla v|^2 \, d\mathbf{x}$ the norm in $H^1_0(D)$, that is induced by the scalar product defined by the bilinear form $a(\,.\,,\,.\,)$.

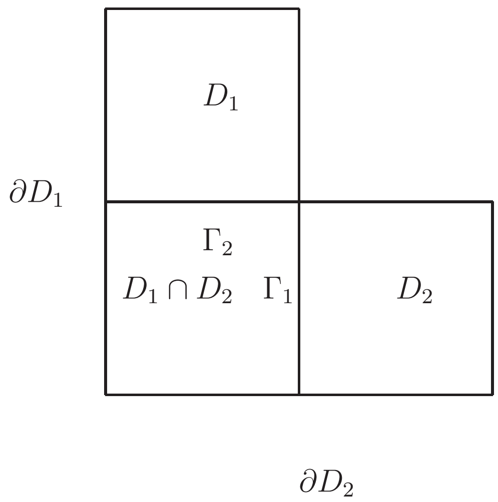
In 1869 Schwarz proposed the following iterative procedure to solve (4.5)-(4.6) in particular domains D, that are subdivided into two overlapping sub-domains D_1 and D_2.

Schwarz iterative procedure

Given u_2^0, solve for each $n \geq 1$:

$$\begin{cases} -\Delta u_1^n = f \quad \text{in } D_1, \\ u_1^n = 0 \quad \text{on } \partial D_1 \backslash \Gamma_1, \\ u_1^n = u_2^{n-1} \quad \text{on } \Gamma_1, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta u_2^n = f \quad \text{in } D_2, \\ u_2^n = 0 \quad \text{on } \partial D_2 \backslash \Gamma_2, \\ u_2^n = u_1^n \quad \text{on } \Gamma_2. \end{cases}$$

Schwarz proved that the sequence $\{u_i^n\}_{i=1,2}^{n\in\mathbb{N}}$ converges in the sup norm to $u_i := u_{|D_i}$, as $n \to +\infty$.



Note that the work by Schwarz was motivated by the construction of explicit solutions, available only for domains with particular geometry, like the $D_i's$ in the figure. A modern variational interpretation has been given by P.-L. Lions [PLL88], see also Matsokin and Nepomnyashchikh [MN85]. The fundamental result, needed to make a proper *error analysis*, is that in each step the error is projected onto a suitable subspace. We make a little change of notation (we want to follow the classical literature) to write the iteration in a single step.

Given u_2^0, solve for each $n \geq 1$:

$$\begin{cases} Lu_1^{n+1/2} = f & \text{in} \quad D_1, \\ u_1^{n+1/2} = 0 & \text{on} \quad \partial D_1 \backslash \Gamma_1, \\ u_1^{n+1/2} = u_2^n & \text{on} \quad \Gamma_1, \end{cases} \qquad \text{and} \qquad \begin{cases} Lu_2^{n+1} = f & \text{in} \quad D_2, \\ u_2^{n+1} = 0 & \text{on} \quad \partial D_2 \backslash \Gamma_2, \\ u_2^{n+1} = u_1^{n+1/2} & \text{on} \quad \Gamma_2. \end{cases}$$

By considering the weak formulation in $H_0^1(D)$, (each function belonging to $H_0^1(D_i)$ is considered, after an extension to zero on $D\backslash D_i$, as a function of $H_0^1(D)$) we easily see that the Schwarz iteration is, for $n \in \mathbb{N}$, equivalent to

$$\begin{cases} \text{find} \quad u^{n+1/2} - u^n \in H_0^1(D_1): \quad a(u^{n+1/2} - u^n, v) = (f, v) - a(u^n, v) \qquad \forall\, v \in H_0^1(D_1), \\ \text{find} \quad u^{n+1} - u^{n+1/2} \in H_0^1(D_2): \quad a(u^{n+1} - u^{n+1/2}, v) = (f, v) - a(u^{n+1/2}, v) \qquad \forall\, v \in H_0^1(D_2). \end{cases}$$

If we denote by e^n the error after n-steps, $e^n := u - u^n$, we have that it satisfies

$$a(u^{n+1/2} - u^n, v) = a(e^n, v) \qquad \forall\, v \in H_0^1(D_1)$$

and

$$a(u^{n+1} - u^{n+1/2}, v) = a(e^{n+1/2}, v) \qquad \forall\, v \in H_0^1(D_2).$$

The analysis above proves the following proposition:

Proposition 4.2.1. *In each half-step the Schwarz iteration projects the error onto $H_0^1(D_i)$, i.e. the solution is corrected (no error) in D_i. We have in fact that*

$$u^{n+1/2} - u^n = P_1\, e^n \qquad \textit{with} \quad P_1 : H_0^1(D) \to H_0^1(D_1),$$
$$u^{n+1} - u^{n+1/2} = P_2\, e^{n+1/2} \qquad \textit{with} \quad P_2 : H_0^1(D) \to H_0^1(D_2),$$

where P_i are the orthogonal projectors defined by the variational problems

$$\textit{find } P_i\, u \in H_0^1(D_i) \qquad a(P_i\, u, v) = a(u, v) \qquad \forall\, v \in H_0^1(D_i).$$

Finite dimensional interpretation

When we pass to a discretization with a finite dimensional space, as in the previous Section 1.4, we can write the iteration in matrix form. As usual, u will belong to the polynomial space $V^h \subset H_0^1(D)$, that we defined in Section 3.1.1. Furthermore A and A_i will be respectively the *stiffness matrix* relative to the bilinear form $a(\,.\,,\,.\,)$ and to its restrictions to D_i

$$a_i(u, v) := a_{|D_i}(u, v) := \int_{D_i} \nabla u \cdot \nabla v \, d\mathbf{x}.$$

In matrix form the Schwarz iteration can be written as

$$\begin{cases} u^{n+1/2} &= \quad u^n &+ \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} (f - Au^n), \\ u^{n+1} &= \quad u^{n+1/2} &+ \begin{pmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{pmatrix} (f - Au^{n+1/2}). \end{cases}$$

If we now introduce the rectangular matrices R_i, such that R_i applied to $v \in V^h$ returns the vector of coefficients defined in the interior of D_i, the iteration can be written as

$$\begin{cases} u^{n+1/2} &= \quad u^n &+ \quad R_1^T(R_1 A R_1^T)^{-1} R_1 (f - Au^n), \\ u^{n+1} &= \quad u^{n+1/2} &+ \quad R_2^T(R_2 A R_2^T)^{-1} R_2 (f - Au^{n+1/2}). \end{cases}$$

Then, by defining the matrices $B_i := R_i^T(R_i A R_i^T)^{-1} R_i$, we can use a compact notation.

MSM-Multiplicative Schwarz method

The multiplicative Schwarz method read as: solve for each $m \geq 0$

$$\begin{cases} u^{n+1/2} = u^n + B_1(f - Au^n), \\ u^{n+1} = u^{n+1/2} + B_2(f - Au^{n+1/2}). \end{cases}$$

or in a one-step version

$$u^{n+1} = u^n + (B_1 + B_2 - B_2 A B_1)(f - Au^n).$$

The particular form that we used to write the iteration points out clearly as the multiplicative Schwarz method performs a preconditioned Richardson iterations, with preconditioner

$$B := B_1 + B_2 - B_2 A B_1.$$

It can be easily seen that this preconditioner is exactly the *block Gauss-Seidel*. The blocks are relative to the unknowns of D_i for $i = 1, 2$.

ASM-Additive Schwarz Method

It is now natural to consider the *block Jacobi* preconditioner, to get the additive Schwarz method. Solve for each $m \geq 0$:

$$u^{n+1} = u^n + (B_1 + B_2)(f - Au^n).$$

The preconditioner is

$$B := B_1 + B_2.$$

We shall not investigate the properties of these *overlapping* methods. We only observe that an iterative procedure produces a preconditioner, which can be proved to be optimal, if V_h consists of piecewise linear functions. For detailed discussion, see the book by Smith, Bjørstad and Gropp [SBG96].

4.2.2 Substructuring methods

In this section we want to introduce the other approach to domain decomposition methods. We consider a domain D partitioned into two non-overlapping sub-domains $\overline{D_1}$ and $\overline{D_2}$, and we suppose that the interface $\Gamma := \overline{D_1} \cap \overline{D_2}$, which separates them, is a $(d-1)$-dimensional Lipschitz manifold. We define again $u_i := u_{|D_i}$ and by we denote by $\mathbf{n}_i$ the outward normal direction on $\partial D_i \cap \Gamma$. Moreover, we let $\mathbf{n} := \mathbf{n}_1$. It is a classical result that the Poisson problem (4.5)-(4.6) in D can be reformulated as follows:

$$\begin{cases} -\Delta u_1 = f & \text{in } D_1, \\ u_1 = 0 & \text{on } \partial D_1 \cap \partial D, \\ u_1 = u_2 & \text{on } \Gamma, \\ \dfrac{\partial u_2}{\partial \mathbf{n}} = \dfrac{\partial u_1}{\partial \mathbf{n}} & \text{on } \Gamma, \\ u_2 = 0 & \text{on } \partial D_2 \cap \partial D, \\ -\Delta u_2 = f & \text{in } D_2. \end{cases} \tag{4.8}$$

The conditions $u_1 = u_2$ and $\partial u_2/\partial \mathbf{n} = \partial u_1/\partial \mathbf{n}$ on Γ are known as *transmission conditions*. We now denote by λ the unknown value of u on Γ and, with standard notation, we define w_i, for $i = 1, 2$, by

$$\begin{cases} -\Delta w_i = f \quad \text{in } D_i, \\ u_i = 0 \quad \text{on } \partial D_i \cap \partial D, \\ u_i = \lambda \quad \text{on } \Gamma, \end{cases} \tag{4.9}$$

and we observe that $w_i = u_i^0 + \widehat{u}_i$, where u_i^0 and $\widehat{u}_i$ are defined by

$$\begin{cases} -\Delta u_i^0 = 0 \quad \text{in } D_i, \\ u_i^0 = 0 \quad \text{on } \partial D_i \cap \partial D, \\ u_i^0 = \lambda \quad \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta \widehat{u}_i = f \quad \text{in } D_i, \\ \widehat{u}_i = 0 \quad \text{on } \partial D_i \cap \partial D, \\ \widehat{u}_i = 0 \quad \text{on } \Gamma. \end{cases}$$

If we compare the problem solved by w_i and by u_i, we obtain that $w_i = u_i$ if and only if

$$\frac{\partial w_1}{\partial \mathbf{n}} = \frac{\partial w_2}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma. \tag{4.10}$$

We recall that u_i^0 is the *harmonic extension* of λ into D_i, which will be denoted by $H_i\lambda$.

Steklov-Poincaré interface equation

The problem in two domains can be written, by recalling the latter condition (4.10) involving the normal derivative of w_i, as a single interface condition. We formally define S, the *Steklov-Poincaré operator,* as:

$$S\eta := \frac{\partial H_1 \eta}{\partial \mathbf{n}} - \frac{\partial H_2 \eta}{\partial \mathbf{n}}. \tag{4.11}$$

This operator, which can be split in $S := S_1 + S_2$, with $S_i := \partial H_i \lambda / \partial \mathbf{n}_i$, was introduced[3] by Steklov and Poincaré at the end of the 19$^{\text{th}}$ century. If we define $T_i f := \widehat{u}_i$ and furthermore

$$\chi := \frac{\partial T_1 f}{\partial \mathbf{n}} - \frac{\partial T_2 f}{\partial \mathbf{n}},$$

we can write (4.10), as the following equation satisfied by λ :

$$S\lambda = \chi. \tag{4.12}$$

Variational formulation

The calculations of the previous section were only formal. To properly define the Steklov-Poincaré operators we need a weak formulation of the multi-domain Problem 4.8. In order to do this we define:

$$\begin{aligned} V_i &:= \{v_i \in H^1(D_i) : \; v_{i|\partial D \cap \partial D_i} = 0\}, \\ V_i^0 &:= H_0^1(D_i), \\ \Lambda &:= \{\eta \in H^{1/2}(\Gamma) : \; \eta = v_{|\Gamma} \text{ for some } v \in V\}. \end{aligned} \tag{4.13}$$

[3] We recall that in the Russian literature it is studied the inverse operator S^{-1} and it is called Poincaré-Steklov operator.

Remark 4.2.2. *If $\Gamma \cap \partial D = \emptyset$ then $\Lambda = H^{1/2}(\Gamma)$. Moreover, when $\Gamma \cap \partial D \neq \emptyset$ the space Λ, usually denoted by $H_{00}^{1/2}(\Gamma)$, is topologically and algebraically strictly included in $H^{1/2}(\Gamma)$, see J.-L. Lions and Magenes [LM72] for further results. Note, in particular, that the following trace inequality holds:*

$$\exists\, C_i^* > 0 \text{ such that } \quad \|v_{i|\Gamma}\|_\Lambda \leq C_i^* \|v_i\|_{1,D_i} \qquad \forall\, v_i \in V_i, \text{ for } i = 1, 2. \tag{4.14}$$

Finally, for $i = 1, 2$ denote by $\mathcal{R}_i$ any possible (continuous) operator from Λ to V_i that satisfies $(\mathcal{R}_i \eta)_{|\Gamma} = \eta$. Any such operator (we can prove that it exists) will be called an extension operator from Λ to V_i.

By using suitable test functions it can be easily proved that the Poisson equation (4.7) can be equivalently reformulated as: find $u_1 \in V_1$ and $u_2 \in V_2$ such that

$$\begin{cases} a_1(u_1, v_1) = (f, v_1)_{D_1} & \forall\, v_1 \in V_1^0, \\ u_1 = u_2 & \text{on } \Gamma, \\ a_2(u_2, v_2) = (f, v_2)_{D_2} & \forall\, v_2 \in V_2^0, \\ a_2(u_2, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{D_2} + (f, \mathcal{R}_1\mu)_{D_1} - a_1(u_1, \mathcal{R}_1\mu) & \forall\, \mu \in \Lambda, \end{cases}$$

where $\mathcal{R}_i$ denotes any possible extension operator from Λ to V_i.

We can now define properly the Steklov-Poincaré operators with a variational formulation. The Steklov-Poincaré operator S introduced in (4.11) can be characterized as an operator acting between the space of trace functions Λ and its dual Λ'. We apply the Green formula and we recall that $H_i\eta$ is the *harmonic extension* in D_i of an η belonging to Λ, for $i = 1, 2$. We have

$$\langle S\eta, \mu \rangle = \sum_{i=1}^{2} \left\langle \frac{\partial H_i \eta}{\partial \mathbf{n}_i}, \mu \right\rangle = \sum_{i=1}^{2} \int_{D_i} \nabla H_i\eta \cdot \nabla \mathcal{R}_i \mu = a_i(H_i\eta, \mathcal{R}_i\mu) \qquad \forall\, \eta, \mu \in \Lambda,$$

where $\mathcal{R}_i$ are the extension operators as in Remark 4.2.2. Hereafter $\langle\, . \,, . \,\rangle$ denotes the duality pairing between Λ' and Λ. In particular, taking $\mathcal{R}_i\mu = H_i\mu$, we obtain the following variational representation of S:

$$\langle S\eta, \mu \rangle = \sum_{i=1}^{2} a_i(H_i\eta, H_i\mu) \qquad \forall\, \eta, \mu \in \Lambda, \tag{4.15}$$

hence the operator S is *symmetric.* Moreover, from the Poincaré inequality, we have that

$$\langle S\eta, \eta \rangle = \sum_{i=1}^{2} \|\nabla H_i \eta\|_{0,D_i}^2 \geq \sum_{i=1}^{2} \frac{1}{1 + C(D_i)} |H_i\eta|_{1,D_i}^2.$$

By taking into account the trace inequality (4.14), we finally have

$$\langle S\eta, \eta \rangle \geq \alpha \|\eta\|_\Lambda^2 \qquad \forall\, \eta \in \Lambda,$$

for a suitable constant $\alpha > 0$. Therefore S is a *coercive* operator.

Proceeding in an analogous way, from (4.11) we have

$$\langle S_i\eta, \mu \rangle = \int_{D_i} \nabla H_i\eta \cdot \nabla H_i\mu = a_i(H_i\eta, H_i\mu) \qquad \forall\, \eta, \mu \in \Lambda.$$

Clearly, each S_i is symmetric and furthermore it is *coercive*:

$$\exists\, \alpha_i > 0: \quad \langle S_i\eta, \eta\rangle \geq \alpha_i \|\eta\|^2_\Lambda \qquad \forall\, \eta \in \Lambda. \tag{4.16}$$

Another relevant property of the Steklov-Poincaré operators S_i is that they are *continuous*:

$$\exists\, \beta_i > 0: \quad \langle S_i\eta, \mu\rangle \leq \beta_i \|\eta\|_\Lambda \, \|\mu\|_\Lambda \qquad \forall\, \eta, \mu \in \Lambda. \tag{4.17}$$

In fact

$$\langle S_i\eta, \mu\rangle \leq |H_i\eta|_{1,D_i} \, |H_i\mu|_{1,D_i} \qquad \forall\, \eta, \mu \in \Lambda$$

and, from well known estimates for the solution of elliptic boundary value problems (see, for example, J.-L. Lions and Magenes [LM72]), it follows that

$$\|H_i\eta\|_{1,D_i} \leq C\|\eta\|_\Lambda \qquad \forall\, \eta \in \Lambda.$$

These properties are of great interest, because they can be used to obtain a numerical solution of the Steklov-Poincaré interface Problem 4.12. Clearly, as soon as an approximation of λ is available, problem (4.9) can be reduced to the solution of two *independent* Dirichlet problems.

Finally, we can also give a variational interpretation of the right-hand side χ in (4.12). It can be expressed through the functions f and $T_i f$ as follows:

$$\begin{aligned} \langle \chi, \mu\rangle &= -\sum_{i=1}^{2} \left\langle \frac{\partial T_i}{\partial \mathbf{n}_i} f, \mu \right\rangle = \sum_{i=1}^{2} \int_{D_i} (f\, \mathcal{R}_i\mu - \nabla T_i f \cdot \nabla \mathcal{R}_i\mu) \\ &= \sum_{i=1}^{2} [(f, \mathcal{R}_i\mu)_{D_i} - a_i(T_i f, \mathcal{R}_i\mu)] \qquad \forall\, \mu \in \Lambda. \end{aligned}$$

Therefore, the Steklov-Poincaré equation (4.12) can be written in the following variational form:

$$\text{find } \lambda \in \Lambda: \quad \langle S\lambda, \mu\rangle = \langle \chi, \mu\rangle \qquad \forall\ \mu \in \Lambda. \tag{4.18}$$

From a variational point of view, the functions $u_i^0 = H_i\lambda$ and $\widehat{u}_i = T_i f$, previously introduced, are the solutions to the following problems:

$$\text{find } H_i\lambda \in V_i: \quad a_i(H_i\lambda, v_i) = 0 \qquad \forall\, v_i \in V_i^0, \quad \text{with } H_i\lambda_{|\Gamma} = \lambda.$$

and

$$\text{find } T_i f \in V_i^0: \quad a_i(T_i f, v_i) = (f, v_i) \qquad \forall\, v_i \in V_i^0.$$

One possible approach to the solution of (4.18) is to discretize the space Λ and to solve the interface equation directly. The variational formulation will use the following spaces

$$\begin{aligned} \Lambda_h &:= \{v_{h|\Gamma}: \quad v_h \in V_h\}, \\ V_{i,h} &:= \{v_{h|\Omega_i}: \quad v_h \in V_h\}, \\ V_{i,h}^0 &:= \{v_h \in V_{i,h}: \quad v_{h|\Gamma} = 0\}. \end{aligned} \tag{4.19}$$

We define the *discrete harmonic extension* $H_{i,h}\eta$ as the solution to

$$\text{find } H_{i,h}\eta_h \in V_{i,h}: \quad a_i(H_{i,h}\eta_h, v_{i,h}) = 0 \qquad \forall\, v_{i,h} \in V_{i,h}^0, \quad \text{with } H_{i,h}\eta_{h|\Gamma} = \eta_h.$$

By using the same arguments that we have shown for the continuous problem, we can define the *discrete Steklov-Poincaré operator*[4] S_h as

$$\langle S_h\eta,\mu\rangle = \sum_{i=1}^{2} a_i(H_{i,h}\eta_h, H_{i,h}\mu_h) = \sum_{i=1}^{2}\langle S_{i,h}\eta_h,\mu_h\rangle \qquad \forall\,\eta_h,\mu_h\in\Lambda_h.$$

It easily turns out that $S_{i,h}$ are both symmetric, continuous and coercive.

It should be remarked that the direct approximation of the interface problem is not useful. We are now reduced to an interface problem with much less unknowns, but the condition number still behaves badly, because $\chi(S_h)\simeq 1/h$.

4.2.3 Some iterative methods

In this section we propose some iterative methods, which can be used to solve the interface equation (4.18). The methods, we are going to introduce, are generally known as *iterative substructuring methods.* Their main feature is that they have a sub-domain counterpart, defined by solving suitable sequences of Poisson problems. In particular, we introduce the classical *Dirichlet-Neumann* and *Neumann-Neumann* methods and we prove that they furnish an optimal preconditioner.

Extension results

Before introducing the methods and the convergence results, we propose two theorems, that are the core of the convergence results. In particular, with the following two theorems we can see that the continuity and coercivity constants of the discrete Steklov-Poincaré operators do not depend on h.

Theorem 4.2.3. *Let the space Λ be defined as in (4.13) and let H_i be harmonic extension operators. Then there exist two positive constants C_1 and C_2 such that*

$$C_1\|\eta\|_\Lambda \le \|H_i\eta\|_{1,D_i} \le C_2\|\eta\|_\Lambda \qquad \forall\,\eta\in\Lambda,\quad \textit{for } i=1,2. \tag{4.20}$$

The proof is based on the application of the *trace inequality* (4.14) and classical estimates for the solution of elliptic problems, see J.-L. Lions and Magenes [LM72].

We now introduce the following theorem, see Bjørstad and Widlund [BW86], Bramble, Pasciak and Schatz [BPS86] and Marini and Quarteroni [MQ89]. It is very important because it states that, a finite dimensional level, the constants that bound the harmonic extension of a function, in terms of its trace norm, do not depend on h.

Theorem 4.2.4 (Uniform extension theorem)**.** *Let D, D_1, and D_2 be Lipschitz polygonal domains. Let the space $V_h := X_r^h \cap H_0^1(D)$. Assume that the family of triangulations $\mathcal{T}_h$ is regular and that the family of triangulations $\mathcal{M}_h$, induced by $\mathcal{T}_h$ on the interface Γ, is quasi-uniform. Then there exist two positive constants C_1 and C_2, which depend on the relative sizes of D_1 and D_2 but are independent of h, such that*

$$C_1\|\eta_h\|_\Lambda \le \|H_{i,h}\eta_h\|_{1,D_i} \le C_2\|\eta_h\|_\Lambda \qquad \forall\,\eta_h\in\Lambda_h \quad \textit{for } i=1,2. \tag{4.21}$$

[4] We recall that we can the interpret the *discrete Steklov-Poincaré operator* as the well-known *Schur complement matrix.* When the Schur matrix is explicitly computed we have the method known as substructuring. This operation is very expensive and leads to the construction of an ill-conditioned matrix. The iterative methods, we shall present, have the advantage of including a good preconditioner; furthermore, in the sub-domain iteration the Schur matrix should not be assembled explicitly.

Proof. Since η_h is the trace on the interface Γ of both $H_{1,h}\eta_h$ and $H_{2,h}\eta_h$, the trace inequality (4.14) states that there exist real positive constants C_i^* such that

$$\|\eta_h\|_\Lambda \le C_i^* \|H_{i,h}\eta_h\|_{1,D_i} \qquad \forall\, \eta_h \in \Lambda_h \quad \text{for } i = 1, 2.$$

Therefore, the left-hand inequality in (4.21) follows by choosing

$$C_1 := \min\{1/C_1^*, 1/C_2^*\}.$$

On the other hand, we have

$$\|H_{i,h}\eta_h\|_{1,D_i} \le \|H_{i,h}\eta_h - H_i\eta_h\|_{1,D_i} + \|H_i\eta_h\|_{1,D_i}.$$

From (4.20), it follows that

$$\|H_i\eta_h\|_{1,D_i} \le C_2\|\eta_h\|_\Lambda, \quad \text{for } i = 1, 2.$$

Since η_h is a piecewise-polynomial continuous function on Γ and D_i is a Lipschitz polygonal domain, the solution $H_i\eta_h$ belongs to $H^{1+s}(D_i)$ for some $s > 1/2$ (see Dauge [Dau88], Corollary 18.15). For each $r \in \mathbb{R}$ we denote by $\|\,.\,\|_{r,D_i}$ and by $\|\,.\,\|_{r,\Gamma}$, the norm of the Sobolev spaces $H^r(D_i)$ and $H^r(\Gamma)$, respectively. The following regularity estimate holds:

$$\|H_i\eta_h\|_{1+s,D_i} \le C\|\eta_h\|_{1/2+s,\Gamma}.$$

Moreover, we have the finite element error estimate

$$\|H_{i,h}\eta_h - H_i\eta_h\|_{1,D_i} \le Ch^s\|H_i\eta_h\|_{1+s,D_i},$$

which is a consequence of the *continuity* and *coercivity* of the bilinear forms $a_i(\,.\,,\,.\,)$ and of the *interpolation error* estimate (see, for example, Ciarlet [Cia78]).

We now use the following *inverse inequality*[5] (see again Ciarlet [Cia78]) to get

$$h^s\|\eta_h\|_{1/2+s,\Gamma} \le C\|\eta_h\|_\Lambda \qquad \forall\, \eta_h \in \Lambda_h,$$

with a constant C independent of h. Therefore, for a suitable constant C_2 independent of h we obtain

$$\|H_{i,h}\eta_h\|_{1,D_i} \le C_2\|\eta_h\|_\Lambda.$$

□

Remark 4.2.5. *We remark that, under the assumptions of the uniform extension theorem the discrete Steklov-Poincaré operators $S_{i,h}$ are continuous and coercive in Λ_h, uniformly with respect to h.*

[5]Recall that the quasi-uniformity of the mesh on Γ is necessary to have the *inverse inequality*

$$\|\eta_h\|_\Lambda \le Ch^{-1/2}\left(\int_\Gamma \eta_h^2\right)^{1/2}.$$

The Dirichlet-Neumann method

We start by introducing the strong formulation of the Dirichlet-Neumann method. Given λ^0, solve for each $k \geq 0$:

$$\begin{cases} -\Delta u_1^{k+1} = f \quad \text{in } D_1, \\ u_1^{k+1} = 0 \quad \text{on } \partial D_1 \cap \partial D, \\ u_1^{k+1} = \lambda^k \quad \text{on } \Gamma, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta u_2^{k+1} = f \quad \text{in } D_2, \\ u_2^{k+1} = 0 \quad \text{on } \partial D_2 \cap \partial D, \\ \dfrac{\partial u_2^{k+1}}{\partial \mathbf{n}} = \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} \quad \text{on } \Gamma, \end{cases}$$

with

$$\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k, \tag{4.22}$$

where θ is a positive acceleration parameter.

This method[6] was considered, among the others, by Bjørstad and Widlund [BW86], Bramble, Pasciak and Schatz [BPS86] and Marini and Quarteroni [MQ89].

The variational formulation of this method is the following:

$$\text{find } u_1^{k+1} \in V_1 : \quad a_1(u_1^{k+1}, v_1) = (f, v_1)_{D_1} \qquad \forall\, v_1 \in V_1^0, \tag{4.23}$$

with $u_1^{k+1} = \lambda^k$ on Γ. Then find $u_2^{k+1} \in V_2$ such that

$$\begin{cases} a_2(u_2^{k+1}, v_2) = (f, v_2)_{D_2} \qquad \forall\, v_2 \in V_2^0, \\ a_2(u_2^{k+1}, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{D_2} + (f, \mathcal{R}_1\mu)_{D_1} - a_1(u_1^{k+1}, \mathcal{R}_1\mu) \qquad \forall\, \mu \in \Lambda \end{cases} \tag{4.24}$$

and complete the scheme with (4.22). This scheme is easily interpreted as the following *preconditioned Richardson method*

$$\lambda^{k=1} = \lambda^k + \theta S_2^{-1}(\chi - S\lambda^k). \tag{4.25}$$

We now give an abstract convergence theorem concerning the Richardson iterations (4.25) for equation (4.12) preconditioned by $P_{DN} = S_2$. The convergence will be proved in the abstract way proposed in Alonso, Trotta and Valli [ATV98], even if weaker convergence results (strictly regarding symmetric problems, see Remark 4.2.8) were known till 1986. In the sequel the operators $\mathcal{S}_i$, for $i = 1, 2$, act on the Hilbert space $(X, \|\,.\,\|_X)$, and $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$.

Theorem 4.2.6. *Suppose that*

a) $\mathcal{S}_2$ *is continuous:* $\exists\, \beta_2 > 0 : \quad \langle \mathcal{S}_2\eta, \mu\rangle \leq \beta_2 \, \|\eta\|_X \, \|\mu\|_X \qquad \forall\, \eta, \mu \in X;$

b) $\mathcal{S}_2$ *is coercive:* $\exists\, \alpha_2 > 0 : \quad \langle \mathcal{S}_2\eta, \eta\rangle \geq \alpha_2 \, \|\eta\|_X^2 \qquad \forall\, \eta \in X;$

c) $\mathcal{S}_1$ *is continuous:* $\exists\, \beta_1 > 0 : \quad \langle \mathcal{S}_1\eta, \mu\rangle \leq \beta_1 \, \|\eta\|_X \, \|\mu\|_X \qquad \forall\, \eta, \mu \in X;$

d) there exists a constant $\kappa^* > 0 : \quad \langle \mathcal{S}_2\eta, \mathcal{S}_2^{-1}\mathcal{S}\eta\rangle + \langle \mathcal{S}\eta, \eta\rangle \geq \kappa^* \|\eta\|_X^2 \qquad \forall\, \eta \in X.$

[6] The same method without relaxation (that is, with $\theta = 1$) does not necessarily converge, unless special assumptions are made about D_1 and D_2, see Quarteroni and Valli [QV99] §1.

Then for any given λ^0 *in* X *and for any* θ *satisfying* $0 < \theta < \theta_{\max}$, *with*

$$\theta_{\max} := \frac{\kappa^* \alpha_2^2}{\beta_2(\beta_1 + \beta_2)^2},$$

the sequence defined by

$$\lambda^{k+1} = \lambda^k + \theta \mathcal{S}_2^{-1}(\mathcal{F} - \mathcal{S}\lambda^k)$$

converges in X *to the solution of problem*

$$\text{find } \lambda \in X : \ \langle \mathcal{S}\lambda, \mu \rangle = \langle \mathcal{F}, \mu \rangle \qquad \forall \mu \in X'. \tag{4.26}$$

Proof. We note that the operator $\mathcal{S}_2$ is invertible as a consequence of assumption $a), b)$ and of the Lax–Milgram lemma. Let us introduce the $\mathcal{S}_2$-scalar product

$$(\eta, \mu)_{\mathcal{S}_2} := \frac{1}{2}(\langle \mathcal{S}_2 \eta, \mu \rangle + \langle \mathcal{S}_2 \mu, \eta \rangle).$$

The corresponding $\mathcal{S}_2$-norm $\|\eta\|_{\mathcal{S}_2} := (\eta, \eta)_{\mathcal{S}_2}^{1/2} = \langle \mathcal{S}_2 \eta, \eta \rangle^{1/2}$ is equivalent to the norm $\|\eta\|_X$, because it satisfies the two inequalities:

$$\alpha_2 \|\eta\|_X^2 \leq \|\eta\|_{\mathcal{S}_2}^2 \leq \beta_2 \|\eta\|_X^2 \qquad \forall \eta \in X.$$

To prove the convergence of the sequence $\{\lambda^k\}_{k \in \mathbb{N}}$ it is sufficient to show that the mapping T_θ from X into itself:

$$T_\theta \eta := \eta - \theta \, \mathcal{S}_2^{-1} \mathcal{S} \eta$$

is a contraction with respect to the $\mathcal{S}_2$-norm. By assuming that $\theta \geq 0$, we have

$$\begin{aligned} \|T_\theta \eta\|_{\mathcal{S}_2}^2 &= \|\eta\|_{\mathcal{S}_2}^2 + \theta^2 \langle \mathcal{S}\eta, \mathcal{S}_2^{-1} \mathcal{S}\eta \rangle - \theta(\langle \mathcal{S}_2 \eta, \mathcal{S}_2^{-1} \mathcal{S}\eta \rangle + \langle \mathcal{S}\eta, \eta \rangle) \\ &\leq \|\eta\|_{\mathcal{S}_2}^2 + \theta^2 \frac{(\beta_1 + \beta_2)^2}{\alpha_2} \|\eta\|_X^2 - \theta \kappa^* \|\eta\|_X^2. \end{aligned}$$

By setting

$$K_\theta = 1 + \theta^2 \frac{(\beta_1 + \beta_2)^2}{\alpha_2^2} - \theta \frac{\kappa^*}{\beta_2},$$

we obtain $\|T_\theta \eta\|_{\mathcal{S}_2}^2 \leq K_\theta \|\eta\|_{\mathcal{S}_2}^2$. The bound $K_\theta < 1$ holds if $0 < \theta < \theta_{\max}$. □

Remark 4.2.7. *In Theorem 4.2.6, the upper bound* $\theta_{\max}$, *as well as the contraction constant* $K_\theta^{1/2}$, *depends only on* α_2, β_1, β_2, *and* κ^*; *consequently, the rate of convergence of the preconditioned Richardson iterative procedure in the* $\mathcal{S}_2$*-norm depends only on these parameters.*

Remark 4.2.8. *If the operator* $\mathcal{S}_2$ *is symmetric, then assumption d) reduces to the coercivity of* $\mathcal{S}$, *that is*

$$\langle \mathcal{S}\eta, \eta \rangle \geq \frac{\kappa^*}{2} \|\eta\|_X^2.$$

The corollary below follows immediately, by applying Theorem 4.2.6 with $X = H_{00}^{1/2}(\Gamma)$, $\mathcal{F} = \chi$, $\mathcal{S} = S$, and $\mathcal{S}_i = S_i$, $i = 1, 2$. In fact, by (4.16)-(4.17) we have that the Steklov-Poincaré operators satisfy all the hypotheses of Theorem 4.2.6, with constants independent of h, see Remark 4.2.5

Corollary 4.2.9. *The Dirichlet-Neumann method converges at a rate which is independent of* h.

The Neumann-Neumann method

This method was considered by Bourgat, Glowinski, Le Tallec and Vidrascu [BGLTV89], see also the paper by Agoshkov and Lebedev [AL85] for an algebraic approach.

In this case, for each $k \geq 0$ we have to solve

$$\begin{cases} -\Delta u_i^{k+1} = f \quad \text{in } D_i, \\ u_i^{k+1} = 0 \quad \text{on } \partial D_i \cap \partial D, \\ u_i^{k+1} = \lambda^k \quad \text{on } \Gamma, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta \psi_i^{k+1} = 0 \quad \text{in } D_i, \\ \psi_i^{k+1} = 0 \quad \text{on } \partial D_i \cap \partial D, \\ \dfrac{\partial \psi_i^{k+1}}{\partial \mathbf{n}} = \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} - \dfrac{\partial u_2^{k+1}}{\partial \mathbf{n}} \quad \text{on } \Gamma, \end{cases}$$

for $i = 1, 2$ with

$$\lambda^{k+1} := \lambda^k - \theta(\sigma_1 \psi_{1|\Gamma}^{k+1} - \sigma_2 \psi_{2|\Gamma}^{k+1}). \tag{4.27}$$

As before λ^0 is a given datum, $\theta > 0$ is an acceleration parameter while σ_1 and σ_2 are two positive averaging coefficients (whose introduction becomes essential only when dealing with many subdomains).

Also the Neumann-Neumann can be interpreted, after a weak formulation, as a preconditioned method to solve the interface equation (4.12). We have arrive to the following preconditioned Richardson method

$$\lambda^{k+1} = \lambda^k + \theta(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})(\mathcal{F} - S\lambda^k). \tag{4.28}$$

Remark 4.2.10. *We recall that if a linear operator $A : \Lambda \to \Lambda'$ is continuous and coercive, with continuity constant given by β and coercivity constant given by α, then its inverse $A^{-1} : \Lambda' \to \Lambda$ exists (by the Lax–Milgram lemma). Moreover, A^{-1} is continuous with continuity constant given by α^{-1} and coercive with coercivity constant given by α/β^2. Consequently, for each $\sigma_1 > 0$ and $\sigma_2 > 0$ the Neumann-Neumann preconditioner $P_{NN} := (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$ is symmetric, continuous and coercive in Λ. If denote by $\beta_{P_{NN}}$ and $\alpha_{P_{NN}}$ its continuity and coercivity constant, a straightforward computation shows that they are respectively given by*

$$\beta_{P_{NN}} := \frac{\beta_1^2 \beta_2^2}{\sigma_1 \alpha_1 \beta_2^2 + \sigma_2 \alpha_2 \beta_1^2},$$

$$\alpha_{P_{NN}} := \frac{(\sigma_1 \alpha_1 \beta_2^2 + \sigma_2 \alpha_2 \beta_1^2)\alpha_1^2 \alpha_2^2}{\beta_1^2 \beta_2^2 (\alpha_1 \sigma_2 + \alpha_2 \sigma_1)^2}.$$

We present now another abstract theorem, which concerns the preconditioned Richardson iteration, based on the preconditioner $P_{NN} = (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$, see Quarteroni and Valli [QV99]. As in Theorem 4.2.6, the operators $\mathcal{S}_i$, $i = 1, 2$ act on the Hilbert space $(X, \|\,.\,\|_X)$ and $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$.

Theorem 4.2.11. *Assume that both $\mathcal{S}_i$ are continuous and coercive; that is, for $i = 1, 2$, we have that*

a) $\exists\, \beta_i > 0 : \qquad \langle \mathcal{S}_i \eta, \mu \rangle \leq \beta_i \|\eta\|_X \|\mu\|_X \qquad \forall\, \eta, \mu \in X;$

b) $\exists\, \alpha_i > 0 : \qquad \langle \mathcal{S}_i \eta, \eta \rangle \geq \alpha_i \|\eta\|_X^2 \qquad \forall\, \eta \in X.$

Assume, moreover, that, for any choice of the averaging parameters $\sigma_1 > 0$ *and* $\sigma_2 > 0$, *the operator* $\mathcal{P}_{NN} := (\sigma_1 \mathcal{S}_1^{-1} + \sigma_2 \mathcal{S}_2^{-1})^{-1}$ *satisfies the condition*

$$c)\ \exists\, \kappa^* > 0 : \qquad \langle \mathcal{P}_{NN}\eta, \mathcal{P}_{NN}^{-1}\mathcal{S}\eta\rangle + \langle \mathcal{S}\eta, \eta\rangle \geq \kappa^* \|\eta\|_X^2 \qquad \forall\, \eta \in X.$$

Then there exists $\theta^0 > 0$ *such that for each* $\theta \in (0, \theta^0)$ *and for any given* $\lambda^0 \in X$ *the sequence*

$$\lambda^{k+1} = \lambda^k + \theta\, \mathcal{P}_{NN}^{-1}(\mathcal{F} - \mathcal{S}\lambda^k) \tag{4.29}$$

converges in X *to the solution of* (4.26).

Proof. We have already pointed out (recall Remark 4.2.10) that, due to assumptions a) and b), the operator $\mathcal{P}_{NN} : X \to X'$ is continuous and coercive. Let us denote by $\beta_{\mathcal{P}_{NN}}$ and $\alpha_{\mathcal{P}_{NN}}$ its continuity and coercivity constants. We introduce the $\mathcal{P}_{NN}$-scalar product

$$(\eta, \mu)_{\mathcal{P}_{NN}} := \frac{1}{2}(\langle \mathcal{P}_{NN}\eta, \mu\rangle + \langle \mathcal{P}_{NN}\mu, \eta\rangle),$$

and the corresponding $\mathcal{P}_{NN}$-norm $\|\eta\|_{\mathcal{P}_{NN}} := (\eta, \eta)_{\mathcal{P}_{NN}}^{1/2} = \langle \mathcal{P}_{NN}\eta, \eta\rangle^{1/2}$, which is equivalent to the norm $\|\eta\|_X$, *i.e.*,

$$\alpha_{\mathcal{P}_{NN}} \|\eta\|_X^2 \leq \|\eta\|_{\mathcal{P}_{NN}}^2 \leq \beta_{\mathcal{P}_{NN}} \|\eta\|_X^2.$$

To prove the convergence of $\{\lambda^k\}_{k\in\mathbb{N}}$, we show that the following map T_θ from X into itself

$$T_\theta \eta := \eta - \theta\, \mathcal{P}_{NN}^{-1}\mathcal{S}\eta$$

is a contraction with respect to the norm $\|\cdot\|_{\mathcal{P}_{NN}}$. By assuming that $\theta \geq 0$, for $\eta \in X$ we have

$$\begin{aligned} \|T_\theta \eta\|_{\mathcal{P}_{NN}}^2 &= \|\eta\|_{\mathcal{P}_{NN}}^2 + \theta^2 \langle \mathcal{S}\eta, \mathcal{P}_{NN}^{-1}\mathcal{S}\eta\rangle - \theta(\langle \mathcal{P}_{NN}\eta, \mathcal{P}_{NN}^{-1}\mathcal{S}\eta\rangle + \langle \mathcal{S}\eta, \eta\rangle) \\ &\leq \|\eta\|_{\mathcal{P}_{NN}}^2 + \theta^2 \langle \mathcal{S}\eta, \mathcal{P}_{NN}^{-1}\mathcal{S}\eta\rangle - \theta\kappa^* \|\eta\|_X^2. \end{aligned}$$

The operator $\mathcal{P}_{NN}^{-1} = \sigma_1 \mathcal{S}_1^{-1} + \sigma_2 \mathcal{S}_2^{-1}$ is continuous and satisfies

$$\langle \psi, \mathcal{P}_{NN}^{-1}\psi\rangle \leq \left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right) \|\psi\|_{X'}^2 \qquad \forall\, \psi \in X',$$

therefore

$$\langle \mathcal{S}\eta, {\mathcal{P}_{NN}}^{-1}\mathcal{S}\eta\rangle \leq \left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right) (\beta_1 + \beta_2)^2 \|\eta\|_X^2.$$

By setting

$$K_\theta = 1 + \theta^2 \left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right) \frac{(\beta_1 + \beta_2)^2}{\alpha_{\mathcal{P}_{NN}}} - \theta \frac{\kappa^*}{\beta_{\mathcal{P}_{NN}}},$$

we obtain

$$\|T_\theta \eta\|_{\mathcal{P}_{NN}}^2 \leq K_\theta \|\eta\|_{\mathcal{P}_{NN}}^2.$$

Finally, by setting

$$\theta^0 := \frac{\kappa^* \alpha_{\mathcal{P}_{NN}}}{\beta_{\mathcal{P}_{NN}} \left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right) (\beta_1 + \beta_2)^2},$$

we conclude that T_θ is a contraction for all $\theta \in (0, \theta^0)$. □

Remark 4.2.12. *We observe that the rate of convergence in the $\mathcal{P}_{NN}$-norm of the preconditioned Richardson iterative procedure (4.29) depends only on the constants σ_1, σ_2, α_1, α_2, β_1, β_2, and κ^*. Moreover, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric, assumption c) is equivalent to the coercivity of $\mathcal{S}$. On the other hand, this property follows directly from b). Hence, in the symmetric case assumption c) is not necessary.*

Again, by setting $X = H^{1/2}_{00}(\Gamma)$, $\mathcal{F} = \chi$, $\mathcal{S} = S$, and $\mathcal{S}_i = S_i$, for $i = 1, 2$, we have the following result, see Corollary 4.2.9

Corollary 4.2.13. *The Neumann-Neumann method converges at a rate which is independent of h.*

The Robin method

We now propose the Robin method, that was introduced and analyzed by P.-L. Lions [PLL90]. It reads as: given u_2^0, for each $k \geq 0$ solve

$$\begin{cases} -\Delta u_1^{k+1} = f \quad \text{in } D_1, \\ u_1^{k+1} = 0 \quad \text{on } \partial D_1 \cap \partial D, \\ \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} + \gamma_1 u_1^{k+1} = \dfrac{\partial u_2^k}{\partial \mathbf{n}} + \gamma_1 u_2^k \quad \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2^{k+1} = f \quad \text{in } D_2, \\ u_2^{k+1} = 0 \quad \text{on } \partial D_2 \cap \partial D, \\ \dfrac{\partial u_2^{k+1}}{\partial \mathbf{n}} - \gamma_2 u_2^{k+1} = \dfrac{\partial u_1^k}{\partial \mathbf{n}} - \gamma_2 u_1^k \quad \text{on } \Gamma, \end{cases}$$

where γ_1 and γ_2 are two nonnegative acceleration parameters such that $\gamma_1 + \gamma_2 > 0$. We now present the proof of convergence in the simplified context of $\gamma_1 = \gamma_2 = \gamma$. This reduction greatly simplifies the calculations, but leaves unaffected the main idea. Furthermore, we make the calculations in the differential context, but the same method can be used to show the convergence of the variational formulation, both at the infinite-dimensional and discrete level, see Quarteroni and Valli [QV99].

Theorem 4.2.14. *If $\gamma_1 = \gamma_2 = \gamma > 0$, then the local errors $e_i^k := u_i^k - u_{|D_i}$ converge to zero with respect to the norm $H^1(D_i)$, for $i = 1, 2$.*

Proof. The local errors satisfy

$$\begin{cases} -\Delta e_1^{k+1} = 0 \quad \text{in } D_1, \\ e_1^{k+1} = 0 \quad \text{on } \partial D_1 \cap \partial D, \\ \dfrac{\partial e_1^{k+1}}{\partial \mathbf{n}} + \gamma_1 e_1^{k+1} = \dfrac{\partial e_2^k}{\partial \mathbf{n}} + \gamma_1 e_2^k \quad \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta e_2^{k+1} = 0 \quad \text{in } D_2, \\ e_2^{k+1} = 0 \quad \text{on } \partial D_2 \cap \partial D, \\ \dfrac{\partial e_2^{k+1}}{\partial \mathbf{n}} - \gamma_2 e_2^{k+1} = \dfrac{\partial e_1^k}{\partial \mathbf{n}} - \gamma_2 e_1^k \quad \text{on } \Gamma. \end{cases}$$

We multiply the equation satisfied by e_2^{k+1} by e_2^{k+1} itself. Then we integrate by parts and we get that

$$\|\nabla e_2^{k+1}\|^2_{0,D_2} = -\int_\Gamma \frac{\partial e_2^{k+1}}{\partial \mathbf{n}} \, e_2^{k+1}.$$

By using the identity $AB = \dfrac{1}{4\gamma}[(A+\gamma B)^2 - (A-\gamma B)^2]$, we can write

$$-\int_\Gamma \frac{\partial e_2^{k+1}}{\partial \mathbf{n}} \, e_2^{k+1} = \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k+1}}{\partial \mathbf{n}} - \gamma e_2^{k+1}\right)^2 - \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k+1}}{\partial \mathbf{n}} + \gamma e_2^{k+1}\right)^2.$$

By recalling the boundary condition on Γ satisfied by e_2^{k+1}, we obtain

$$\|\nabla e_2^{k+1}\|_{0,D_2}^2 + \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k+1}}{\partial \mathbf{n}} + \gamma e_2^{k+1}\right)^2 = \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_1^{k+1}}{\partial \mathbf{n}} - \gamma e_1^{k+1}\right)^2.$$

By repeating the same argument for e_1^{k+1}, we find that

$$\|\nabla e_1^{k+1}\|_{0,D_1}^2 + \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_1^{k+1}}{\partial \mathbf{n}} - \gamma e_1^{k+1}\right)^2 = \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k}}{\partial \mathbf{n}} + \gamma e_2^{k}\right)^2.$$

Adding the last two equalities and summing over k, from $k = 0$ to $k = M - 1$, we obtain that

$$\sum_{k=1}^{M}(\|\nabla e_1^k\|_{0,D_1}^2 + \|\nabla e_2^k\|_{0,D_2}^2) + \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{M}}{\partial \mathbf{n}} + \gamma e_2^{M}\right)^2 = \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{0}}{\partial \mathbf{n}} + \gamma e_2^{0}\right)^2.$$

Consequently, the series on the left hand side is convergent and e_i^k tends to 0 in $H^1(D_i)$ for $i = 1, 2$. □

Remark 4.2.15. *In contrast with the Dirichlet-Neumann and Neumann-Neumann methods, for the Robin method we do not have estimates regarding the reduction of the error. Furthermore, we do not have any information about the rate of convergence.*

4.3 Non-symmetric problems

In this section we analyze the domain decomposition methods for non-symmetric elliptic equations. When dealing with elliptic equations with first order terms or with non-symmetric principal part, the theory becomes more difficult. In general no convergence result can be proved, if some assumption linking the "bigness" of non-symmetric part and the dimension of sub-domains are not assumed. We do not study the problem with overlapping sub-domains. For Schwarz methods regarding non-symmetric equations, we refer to Cai, Gropp and Keyes [CGK92] and Cai and Widlund [CW92]. The study of non-symmetric equation gives new additional problems, as we shall see in the following sections. Particular interface conditions should be used and new method must be introduced. We consider a non-symmetric elliptic operator of the following form:

$$Lw := -\sum_{l,j=1}^{d} \frac{\partial}{\partial x_l}\Big(a_{lj}\frac{\partial w}{\partial x_j}\Big) + \sum_{j=1}^{d}\frac{\partial(b_j w)}{\partial x_j} + a_0 w. \tag{4.30}$$

We assume that the coefficients a_{lj} are *uniformly positive-definite*:

$$\exists\, \alpha_0 > 0 : \quad \sum_{l,j=1}^{d} a_{lj}(\mathbf{x})\xi_j\xi_l \ge \alpha_0|\boldsymbol{\xi}|^2 \qquad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \quad a.e.\, \mathbf{x} \in D.$$

We shall study the homogeneous Dirichlet problem associated with the operator L. When dealing with the two-domain formulation, the interface condition on Γ are

$$\begin{cases} u_1 = u_2 & \text{on } \Gamma, \\ \dfrac{\partial u_1}{\partial \mathbf{n}_L} = \dfrac{\partial u_2}{\partial \mathbf{n}_L} & \text{on } \Gamma, \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} u_1 = u_2 & \text{on } \Gamma, \\ \dfrac{\partial u_1}{\partial \mathbf{n}_L} + \gamma u_1 = \dfrac{\partial u_2}{\partial \mathbf{n}_L} + \gamma u_2 & \text{on } \Gamma, \end{cases}$$

where

$$\frac{\partial u}{\partial \mathbf{n}_L} := \sum_{l,j=1}^{d} a_{lj} \frac{\partial u}{\partial x_j} n_l$$

is called the *co-normal derivative.*

Variational formulation

Let us assume that a_{lj}, b_j, and a_0 belong to $L^\infty(D)$, for each $l, j = 1, \dots, d$ and that $\operatorname{div} \mathbf{b}$ belongs to $L^\infty(D)$. Then we can introduce in $H^1(D)$ the continuous bilinear form, associated to L :

$$a^{\#}(w,v) := \int_D \left[\sum_{l,j=1}^{d} a_{lj} \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_l} + \left(\frac{1}{2} \operatorname{div} \mathbf{b} + a_0 \right) w\, v \right] d\mathbf{x} + \frac{1}{2} \int_D (v\, \mathbf{b} \cdot \nabla w - w\, \mathbf{b} \cdot \nabla v)\, d\mathbf{x},$$

Under the further assumption that

$$\frac{1}{2} \operatorname{div} \mathbf{b}(\mathbf{x}) + a_0(\mathbf{x}) \geq 0 \qquad a.e.\, \mathbf{x} \in D, \tag{4.31}$$

the bilinear form $a^{\#}(\cdot,\cdot)$ is coercive in $V = H_0^1(D)$. Therefore, the homogeneous Dirichlet boundary value problem: find $u \in H_0^1(D)$ such that

$$a^{\#}(u,v) = (f,v) \qquad \forall\, v \in H_0^1(D) \tag{4.32}$$

has a unique solution. By defining in V_i, $i = 1, 2$, (see page 76) the local bilinear forms

$$a_i^{\#}(w_i,v_i) := \int_{D_i} \left[\sum_{l,j=1}^{d} a_{lj} \frac{\partial w_i}{\partial x_j} \frac{\partial v_i}{\partial x_l} + \left(\frac{1}{2} \operatorname{div} \mathbf{b} + a_0 \right) w_i v_i\, d\mathbf{x} + \frac{1}{2} \int_D (v_i\, \mathbf{b} \cdot \nabla w_i - w_i\, \mathbf{b} \cdot \nabla v_i)\, d\mathbf{x},\right.$$

we can define the extension operators $E_i^{\#} : \Lambda \to V_i$ such that

$$E_i^{\#}\eta \in V_i : \quad a^{\#}(E_i^{\#}\eta, v) = 0 \qquad \forall\, v \in V_i^0 \quad \text{with} \quad E_i^{\#}\eta_{|\Lambda} = \eta.$$

With these operators, we can define the Steklov-Poincaré operators exactly as for the Laplace equation

$$\langle S_i \eta, \mu \rangle = a_i^{\#}(E_i^{\#}\eta, E_i^{\#}\mu) \qquad \forall\, \eta, \mu \in \Lambda. \tag{4.33}$$

We have easily that S_i are continuous and coercive, *i.e.*, that (4.16)-(4.17) are satisfied for suitable positive constants α_i and β_i. We can formulate an interface problem

$$\text{find } \lambda \in \Lambda : \quad \langle S\lambda, \mu \rangle = \langle \chi, \mu \rangle \qquad \forall\, \mu \in \Lambda,$$

which is equivalent to (4.32). In this problem χ is defined by

$$\langle \chi, \mu \rangle = \sum_{i=1}^{2} [(f, \mathcal{R}_i \mu)_{D_i} - a_i^{\#}(T_i^{\#} f, \mathcal{R}_i \mu)] \qquad \forall\, \mu \in \Lambda$$

and $T_i^{\#} f$ solves the following variational problem

$$\text{find } T_i^{\#} f \in V_i^0 : \quad a_i^{\#}(T_i^{\#} f, v_i) = (f, v_i) \qquad \forall\, v_i \in V_i^0, \quad \text{for } i = 1, 2.$$

Due to the choice[7] of the bilinear form $a^{\#}(\,.\,,\,.\,)$, the interface condition is:

$$\frac{\partial u_2}{\partial \mathbf{n}_L} - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}\,u_2 = \frac{\partial u_1}{\partial \mathbf{n}_L} - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}\,u_1 \qquad \text{on } \Gamma.$$

4.3.1 Results for "slightly" non-symmetric problems

We now give some convergence results for iterative methods corresponding to non-symmetric equations. We shall use the abstract theorems of the previous section and we give some condition which make the assumptions to be satisfied, see Quarteroni and Valli [QV99] §5.

Dirichlet-Neumann method

We denote by *Dirichlet-Neumann* the method that is the counterpart of (4.22)-(4.23)-(4.24), introduced for the Laplace equation. In this case we have a Dirichlet problem coupled with a Robin problem and not a Neumann one as for the Laplace equation.

We claim that if the *skew-symmetric part* of S_2 is small enough, we can prove that condition d) in Theorem 4.2.6, namely

$$\langle \mathcal{S}_2\eta, \mathcal{S}_2^{-1}\mathcal{S}\eta\rangle + \langle \mathcal{S}\eta, \eta\rangle \geq \kappa^* \|\eta\|_\Lambda^2 \qquad \forall\, \eta \in \Lambda,$$

is satisfied (for $\mathcal{S}_i = S_i$) and then the method converges. We introduce the *symmetric* and *skew-symmetric* parts of the bilinear form $a^{\#}(\cdot,\cdot)$, which are given respectively by

$$a^{\mathrm{s}}(w,v) := \int_D \left[\sum_{l,j=1}^{d} \frac{1}{2}(a_{lj} + a_{jl})\,\frac{\partial w}{\partial x_j}\,\frac{\partial v}{\partial x_l} + \left(\frac{1}{2}\operatorname{div}\mathbf{b} + a_0\right) w\,v \right] d\mathbf{x}$$

$$a^{\mathrm{ss}}(w,v) := \int_D \left[\sum_{l,j=1}^{d} \frac{1}{2}(a_{lj} - a_{jl})\,\frac{\partial w}{\partial x_j}\,\frac{\partial v}{\partial x_l} + \frac{1}{2}(v\,\mathbf{b}\cdot\nabla w - w\,\mathbf{b}\cdot\nabla v) \right] d\mathbf{x}.$$

Clearly, $a^{\#} = a^{\mathrm{s}} + a^{\mathrm{ss}}$. In a similar way we can define the local symmetric and skew-symmetric

[7] We remark that this is not the only bilinear form (and consequently weak formulation) that can be used. In the paper by Cai and Widlund [CW92] the different bilinear form

$$a_{\#}(w,v) := \int_D \left[\sum_{l,j=1}^{d} a_{lj}\,\frac{\partial w}{\partial x_j}\,\frac{\partial v}{\partial x_l} + (a_0 + \nabla\cdot\mathbf{b})wv + v\,\mathbf{b}\cdot\nabla w \right] d\mathbf{x}$$

is used for some technical reasons. This form is obtained without integrating by parts the term $\big[\nabla\cdot(\mathbf{b}\,w)\big]v$. The following bilinear form is used by *adaptive methods*

$$\hat{a}(w,v) := \int_D \left[\sum_{l,j=1}^{d} a_{lj}\,\frac{\partial w}{\partial x_j}\,\frac{\partial v}{\partial x_l} + a_0 wv - w\mathbf{b}\cdot\nabla v \right] d\mathbf{x}$$

and it is obtained integrating by parts $\big[\nabla\cdot(\mathbf{b}\,w)\big]v$, see Carlenzoli and Quarteroni [CQ95] and Gastaldi, Gastaldi and Quarteroni [GGQ96]. We believe (and numerical experiments confirm it) that in a non overlapping context our bilinear form $a^{\#}(\,.\,,\,.\,)$ (that is obtained with "1/2-integration") is more powerful, since the others can lead to non solvable sub-domain problems, see Section 4.4.

parts $a_i^{\mathrm{s}}(. , .)$ and $a_i^{\mathrm{ss}}(. , .)$, for $i = 1, 2$. By setting $\mu := S_2^{-1} S\eta$, from (4.33) we have

$$
\begin{aligned}
\langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta,\eta\rangle \quad &= \langle S_2\eta,\mu\rangle - \langle S\eta,\eta\rangle + 2\langle S\eta,\eta\rangle \\
&= \langle S_2\eta,\mu\rangle - \langle S_2\mu,\eta\rangle + 2\langle S\eta,\eta\rangle \\
&= a_2^{\mathrm{s}}(E_2^{\#}\eta, E_2^{\#}\mu) + a_2^{\mathrm{ss}}(E_2^{\#}\eta, E_2^{\#}\mu) - a_2^{\mathrm{s}}(E_2^{\#}\mu, E_2^{\#}\eta) - a_2^{\mathrm{ss}}(E_2^{\#}\mu, E_2^{\#}\eta) \\
&\quad + 2\langle S\eta,\eta\rangle \\
&= 2a_2^{\mathrm{ss}}(E_2^{\#}\eta, E_2^{\#}S_2^{-1}S\eta) + 2\langle S\eta,\eta\rangle.
\end{aligned}
$$

By denoting with α the coercivity constant of S, we get

$$\langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta,\eta\rangle \geq 2\alpha\|\eta\|_\Lambda^2 - 2|a_2^{\mathrm{ss}}(E_2^{\#}\eta, E_2^{\#}S_2^{-1}S\eta)|.$$

Therefore, we only have to show that

$$\exists\, 0 < \rho < \alpha : \quad |a_2^{\mathrm{ss}}(E_2^{\#}\eta, E_2^{\#}S_2^{-1}S\eta)| \leq \rho\,\|\eta\|_\Lambda^2. \tag{4.34}$$

By setting

$$\kappa_i^{\mathrm{ss}} := \max_{l,j} \|a_{lj} - a_{jl}\|_{L^\infty(D_i)} + \|\mathbf{b}\|_{L^\infty(D_i)} \qquad \text{for } i = 1, 2,$$

we obtain

$$
\begin{aligned}
|a_2^{\mathrm{ss}}(E_2^{\#}\eta, E_2^{\#}S_2^{-1}S\eta)| \quad &\leq C_1\kappa_2^{\mathrm{ss}}\,\|E_2^{\#}\eta\|_{1,D_2}\|E_2^{\#}S_2^{-1}S\eta\|_{1,D_2} \\
&\leq C_2\kappa_2^{\mathrm{ss}}\,\|\eta\|_\Lambda\|S_2^{-1}S\eta\|_\Lambda \\
&\leq C_3\kappa_2^{\mathrm{ss}}\,\|\eta\|_\Lambda^2.
\end{aligned}
$$

Then (4.34) follows if

$$C_3\kappa_2^{\mathrm{ss}} < \alpha, \tag{4.35}$$

which is a *smallness assumption* on the skew-symmetric part of the operator L in D_2. By applying Theorem 4.2.6 with $\mathcal{S}_i = S_i$, for $i = 1, 2$, we can infer the following result.

Corollary 4.3.1. *The Dirichlet-Neumann method converges at a rate which is independent of h, provided (4.35) is satisfied.*

Neumann-Neumann method

We consider the analogue of the Neumann-Neumann method for the Poisson equation, even if the iteration steps regarding ψ (see page 83) now involve problems with a Robin-condition on the interface Γ. We claim that we can use Theorem 4.2.11 if the non-symmetric term is small in both D_1 and D_2. In particular, this assumption is necessary to satisfy condition $c)$:

$$\exists\, k^* > 0 : \quad \langle \mathcal{P}_{NN}\eta, \mathcal{P}_{NN}^{-1}\mathcal{S}\eta\rangle + \langle \mathcal{S}\eta,\eta\rangle \geq \kappa^*\|\eta\|_\Lambda^2 \qquad \forall\, \eta \in \Lambda.$$

In this case $\mathcal{P}_{NN} = P_{NN} := (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$. In fact, by setting $\mu := P_{NN}^{-1} S\eta$, we have

$$
\begin{aligned}
\langle P_{NN}\eta, P_{NN}^{-1}S\eta\rangle + \langle S\eta,\eta\rangle \quad &= \langle P_{NN}\eta,\mu\rangle - \langle S\eta,\eta\rangle + 2\langle S\eta,\eta\rangle \\
&= \langle P_{NN}\eta,\mu\rangle - \langle P_{NN}\mu,\eta\rangle + 2\langle S\eta,\eta\rangle.
\end{aligned}
$$

Moreover, by setting $\rho_i := S_i^{-1} P_{NN}\eta$ and $\xi_i := S_i^{-1} P_{NN}\mu$ for $i = 1, 2$, we have

$$P_{NN}\eta = S_1\rho_1 = S_2\rho_2 \quad \text{and} \quad P_{NN}\mu = S_1\xi_1 = S_2\xi_2,$$

and consequently

$$\eta = P_{NN}^{-1} P_{NN}\eta = \sigma_1\rho_1 + \sigma_2\rho_2 \quad \text{and} \quad \mu = \sigma_1\xi_1 + \sigma_2\xi_2.$$

Therefore

$$\begin{aligned}
\langle P_{NN}\eta, \mu\rangle - \langle P_{NN}\mu, \eta\rangle &= \langle P_{NN}\eta, \sigma_1\xi_1 + \sigma_2\xi_2\rangle - \langle P_{NN}\mu, \sigma_1\rho_1 + \sigma_2\rho_2\rangle \\
&= \sigma_1(\langle S_1\rho_1, \xi_1\rangle - \langle S_1\xi_1, \rho_1\rangle) + \sigma_2(\langle S_2\rho_2, \xi_2\rangle - \langle S_2\xi_2, \rho_2\rangle) \\
&= 2\sum_{i=1}^{2} \sigma_i a_i^{\mathrm{ss}}(E_i^{\#}\rho_i, E_i^{\#}\xi_i) \\
&= 2\sum_{i=1}^{2} \sigma_i a_i^{\mathrm{ss}}(E_i^{\#} S_i^{-1} P_{NN}\eta, E_i^{\#} S_i^{-1} S\eta).
\end{aligned}$$

We finally obtain that condition c) in Theorem 4.2.11 is satisfied, provided that the constant κ_i^{ss} is small enough, for $i = 1, 2$. Hence, by using Theorem 4.2.11 with $\mathcal{S}_i = S_i$ for $i = 1, 2$, we obtain the following corollary.

Corollary 4.3.2. *The Neumann-Neumann method converges at a rate which is independent of h, provided κ_i^{ss}, for $i = 1, 2$, is small enough*

4.3.2 Time-harmonic Maxwell equations

The methods used in the previous section are essentially based on the fact that the operator to be studied is a small perturbation of a symmetric one. In other problems of classical mathematical physics these method are not applicable. In particular, in this section we study the *time-harmonic* Maxwell equations. This problem is interesting because leads to a complex bilinear form, that is coervice but not self-adjoint and appropriate techniques must be used. Then we show how the same analysis of domain decomposition methods applies to the real elliptic equations, to prove in a *different* and new way the results of Section 4.3.1.

We recall that the Maxwell equations read (recall the definition of **curl** in Section 1.1.1)

$$\begin{cases}
\dfrac{\partial \mathcal{D}}{\partial t} = \mathbf{curl}\,\mathcal{H} - \mathcal{J}, \\[2ex]
\dfrac{\partial \mathcal{B}}{\partial t} = -\mathbf{curl}\,\mathcal{E},
\end{cases}$$

where $\mathcal{E}$ and $\mathcal{B}$ (which are real, three-dimensional, vector-valued functions of $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$) are respectively the electric and magnetic field. Furthermore, $\mathcal{D}$ and $\mathcal{B}$ are the electric and magnetic induction and $\mathcal{J}$ is the current density. We assume the constitutive equations $\mathcal{D} = \varepsilon\,\mathcal{E}$ and $\mathcal{B} = \mu\,\mathcal{H}$, where ε is the *dielectric constant* while μ is the *magnetic permeability.* We also assume the so-called *Ohm law* $\mathcal{J} = \sigma\,\mathcal{E}$, where σ is the *electric conductivity.* The quantities $\varepsilon\,, \mu\,, \sigma$ are symmetric matrices, which depend on the spatial variable $\mathbf{x}$. We recall that ε and μ are assumed to be uniformly positive definite, *i.e.*, there exist two real positive constants C_1 and C_2 such that:

$$\sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{x})\zeta_i\zeta_j \geq C_1|\boldsymbol{\zeta}|^2 \quad \text{and} \quad \sum_{i,j=1}^{3} \mu_{ij}(\mathbf{x})\zeta_i\zeta_j \geq C_2|\boldsymbol{\zeta}|^2 \qquad \forall\,\boldsymbol{\zeta} \in \mathbb{R}^3,\ \textit{a.e.}\ \mathbf{x} \in \mathbb{R}^3.$$

We also remind that σ is positive definite in a conductor and vanishes in an isolant. For the mathematical analysis of the Maxwell equations see Dautray and J.-L. Lions [DL92] Ch. IX; for the physical meaning of the above equations see Jackson [Jac75].

If we rewrite the equations in terms of $\mathcal{E}$ and $\mathcal{H}$, the Maxwell equations become

$$\begin{cases} \varepsilon \dfrac{\partial \mathcal{E}}{\partial t} = \mathbf{curl}\,\mathcal{H} - \sigma\,\mathcal{E}, \\ \mu \dfrac{\partial \mathcal{H}}{\partial t} = -\mathbf{curl}\,\mathcal{E}. \end{cases}$$

We consider the *time-harmonic* problem by looking for solutions of the form

$$\begin{cases} \mathcal{E}(t,\mathbf{x}) = \mathrm{Re}\left[\mathbf{E}(\mathbf{x})\mathrm{e}^{i\,\alpha\,t}\right], \\ \mathcal{H}(t,\mathbf{x}) = \mathrm{Re}\left[\mathbf{H}(\mathbf{x})\mathrm{e}^{i\,\alpha\,t}\right], \end{cases}$$

where $\mathbf{E}$ and $\mathbf{H}$ are complex-valued three-dimensional vectors. Moreover, the complex number i is the *imaginary unit* and $0 \neq \alpha \in \mathbb{R}$ is a given *angular frequency.* After rewriting the Maxwell equations in terms of $\mathbf{E}$ and $\mathbf{H}$ and after substituting in the first equation the expression for $\mathbf{H}$ that is given from the second equation, we arrive to

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{E}) - \alpha^2(\varepsilon - i\alpha^{-1}\sigma)\mathbf{E} = 0.$$

We shall consider the low-frequency problem in which α is "small." We observe that for a lot of real media the term $\alpha^2\varepsilon$ is much smaller than μ^{-1} and $\alpha\sigma$. It is then physically reasonable to neglect the term $\alpha^2\varepsilon\,\mathbf{E}$ and, consequently, to study the *low-frequency time-harmonic Maxwell equations:*

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{E}) + i\alpha\sigma\,\mathbf{E} = \mathbf{0}.$$

When considering the equations in a bounded domain $D \subset \mathbb{R}^3$, the natural boundary condition is $\mathbf{n} \times \mathbf{E} = \boldsymbol{\Psi}$ on ∂D, where, as usual, $\mathbf{n}$ denotes the outward normal vector to ∂D. It is generally assumed that there exists a vector valued function $\widetilde{\mathbf{E}}$ such that $\mathbf{n} \times \widetilde{\mathbf{E}} = \boldsymbol{\Psi}$ on ∂D. By considering the new unknown $\mathbf{u} := \mathbf{E} - \widetilde{\mathbf{E}}$, we can study the following problem

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{u}) + i\alpha\sigma\,\mathbf{u} = \mathbf{f} \quad \text{in } D, \tag{4.36}$$

$$(\mathbf{n} \times \mathbf{u})_{|\partial D} = \mathbf{0} \quad \text{on } \partial D, \tag{4.37}$$

where $\mathbf{f} := \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\widetilde{\mathbf{E}}) + i\sigma\alpha\,\widetilde{\mathbf{E}}$.

Function spaces and variational formulation

When considering equations (4.36)-(4.37), it is natural to think to a variational formulation in a proper functional space. Clearly, the bilinear form associated to the above problem is given by

$$a(\mathbf{w},\mathbf{v}) := \int_D (\mu^{-1}\mathbf{curl}\,\mathbf{w} \cdot \mathbf{curl}\,\overline{\mathbf{v}} + i\alpha\sigma\,\mathbf{w} \cdot \overline{\mathbf{v}})\,d\mathbf{x}.$$

We need now a precise functional framework because the problem, for its nature, should not be treated in $H_0^1(D)$. The spaces we shall use are the following ones:

$$H(\mathbf{curl}\,; D) := \left\{ \mathbf{u} \in (L^2(D))^3 : \quad \mathbf{curl}\,\mathbf{u} \in (L^2(D))^3 \right\}$$

and

$$H_0(\mathbf{curl}\,; D) := \left\{ H(\mathbf{curl}\,; D) : \quad (\mathbf{n} \times \mathbf{u})_{|\partial D} = \mathbf{0} \right\},$$

equipped with the norm

$$\|\mathbf{u}\|^2_{H(\mathbf{curl}\,;D)} = \|\mathbf{u}\|^2_{(L^2(D))^3} + \|\mathbf{curl}\,\mathbf{u}\|^2_{(L^2(D))^3}.$$

For further properties of these spaces, see Girault and Raviart [GR86]. We shall also use, for $0 < s \in \mathbb{R}$

$$H^s(\mathbf{curl}\,; D) := \left\{ \mathbf{u} \in (H^s(D))^3 : \quad \mathbf{curl}\,\mathbf{u} \in (H^s(D))^3 \right\}$$

equipped with the norm

$$\|\mathbf{u}\|^2_{H^s(\mathbf{curl}\,;D)} = \|\mathbf{u}\|^2_{(H^s(D))^3} + \|\mathbf{curl}\,\mathbf{u}\|^2_{(H^s(D))^3}.$$

We need a knowledge of corresponding trace spaces to properly formulate the problem with two sub-domains, as in the previous sections. We define the *tangential divergence* of a vector field $\boldsymbol{\lambda}$.

Definition 4.3.3. *Given* $\boldsymbol{\lambda} \in (H^{-1/2}(\partial D))^3$ *such that* $(\boldsymbol{\lambda} \cdot \mathbf{n})_{|\partial D} = 0$, *we define its tangential divergence* $\mathrm{div}_\tau \boldsymbol{\lambda} \in H^{-3/2}(\partial D)$ *as the distribution such that*

$$\langle\langle \mathrm{div}_\tau \boldsymbol{\lambda}, \psi \rangle\rangle_{\partial D} := -\langle \boldsymbol{\lambda}, (\nabla \psi^*)_{|\partial D} \rangle_{\partial D} \qquad \forall\, \psi \in H^{3/2}(\partial D),$$

where $\psi^* \in H^2(D)$ *is any extension of* ψ *in* D. *We denote with* $\langle\langle\, . \,, \,. \rangle\rangle_{\partial D}$ *the duality between* $H^{-3/2}(\partial D)$ *and* $H^{3/2}(\partial D)$ *and with* $\langle\, . \,, \,. \rangle_{\partial D}$ *that one between* $H^{-1/2}(\partial D)$ *and* $H^{1/2}(\partial D)$

We define the Hilbert spaces $\boldsymbol{\chi}_{\partial D}$ and $\boldsymbol{\chi}_\Sigma$, where Σ is a proper subset of ∂D as:

$$\boldsymbol{\chi}_{\partial D} := \left\{ \boldsymbol{\lambda} \in (H^{-1/2}(\partial D))^3 : \quad (\boldsymbol{\lambda} \cdot \mathbf{n})_{|\partial D} = 0,\ \mathrm{div}_\tau \boldsymbol{\lambda} \in H^{-1/2}(\partial D) \right\}$$

and

$$\boldsymbol{\chi}_\Sigma := \left\{ \boldsymbol{\lambda} \in (H^{-1/2}(\Sigma))^3 : \quad (\boldsymbol{\lambda} \cdot \mathbf{n})_{|\Sigma} = 0,\ \mathrm{div}_\tau \widetilde{\boldsymbol{\lambda}} \in H^{-1/2}(\partial D) \right\},$$

equipped with the norms

$$\|\boldsymbol{\lambda}\|^2_{\boldsymbol{\chi}_{\partial D}} := \|\boldsymbol{\lambda}\|^2_{-1/2,\partial D} + \|\mathrm{div}_\tau \boldsymbol{\lambda}\|^2_{-1/2,\partial D}$$

and

$$\|\boldsymbol{\lambda}\|^2_{\boldsymbol{\chi}_\Sigma} := \|\boldsymbol{\lambda}\|^2_{-1/2,\Sigma} + \|\mathrm{div}_\tau \widetilde{\boldsymbol{\lambda}}\|^2_{-1/2,\partial D},$$

where $\widetilde{\boldsymbol{\lambda}} \in (H^{-1/2}(\partial D))^3$ is the extension of $\boldsymbol{\lambda}$ to $\mathbf{0}$ on $\partial D \backslash \Sigma$.

It is well-known, see Alonso and Valli [AV96], that $\boldsymbol{\chi}_{\partial D}$ and $\boldsymbol{\chi}_\Sigma$ are algebraically and topologically equivalent to the spaces of *tangential traces* of $H(\mathbf{curl}\,; D)$ and $H_{\partial D \backslash \Sigma}(\mathbf{curl}\,; D)$, where

$$H_{\partial D \backslash \Sigma}(\mathbf{curl}\,; D) := \left\{ \mathbf{u} \in H(\mathbf{curl}\,; D) : \quad (\mathbf{n} \times \mathbf{u})_{|\partial D \backslash \Sigma} = \mathbf{0} \right\}.$$

Furthermore there exist linear and continuous operators $\mathbf{R}_{\partial D}$ and $\mathbf{R}_\Sigma$ such that

$$\mathbf{R}_{\partial D} : \ \boldsymbol{\chi}_{\partial D} \to H(\mathbf{curl}\,; D), \quad \text{with} \quad (\mathbf{n} \times \mathbf{R}_{\partial D} \boldsymbol{\eta})_{|\partial D} = \boldsymbol{\eta}$$

and

$$\mathbf{R}_\Sigma : \ \boldsymbol{\chi}_\Sigma \to H_{\partial D\setminus\Sigma}(\mathbf{curl}\,; D), \quad \text{with} \quad (\mathbf{n}\times \mathbf{R}_\Sigma\boldsymbol{\gamma})_{|\Sigma} = \boldsymbol{\gamma}.$$

In the proof of some results we shall need additional regularity. For this reason we define, for $0 < s \in \mathbb{R}$

$$\boldsymbol{\chi}^s_{\partial D} := \left\{\boldsymbol{\lambda} \in (H^s(\partial D))^3 : \quad (\boldsymbol{\lambda}\cdot\mathbf{n})_{|\partial D} = 0, \quad \mathrm{div}_\tau \boldsymbol{\lambda} \in H^s(\partial D)\right\},$$

with norm

$$\|\boldsymbol{\lambda}\|^2_{\boldsymbol{\chi}^s_{\partial D}} := \|\boldsymbol{\lambda}\|^2_{s,\partial D} + \|\mathrm{div}_\tau\boldsymbol{\lambda}\|^2_{s,\partial D}.$$

We can now define the notion of weak solution.

Definition 4.3.4. *We say that a weak solution to* (4.36)-(4.37) *is a function* $\mathbf{u}$ *belonging to* $H_0(\mathbf{curl}\,; D)$ *such that:*

$$a(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) := \int_D \mathbf{f}\cdot\overline{\mathbf{v}}\, d\mathbf{x} \qquad \forall\, \mathbf{v} \in H_0(\mathbf{curl}\,; D).$$

We have the following result, see Leis [Lei79].

Theorem 4.3.5. *Let D be a smooth bounded open set of $\mathbb{R}^3$ and σ a positive definite matrix. Then the bilinear form $a(\,.\,,\,.\,)$ is continuous and coercive in $H_0(\mathbf{curl}\,; D)$ and, by using the Lax-Milgram lemma, we can state that there exists a unique weak solution to* (4.36)-(4.37).

Two-domain formulation

As we did for the Poisson equations, we now derive a two domain formulation. As usual, we consider the problem with the domain divided into two sub-domains and we also suppose that μ and σ are positive constants. We denote by $\mathbf{u}_j$, for $j = 1, 2$, the restriction to D_j of the solution $\mathbf{u}$. It is easy to prove that the following differential problems must be satisfied:

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{u}_1) + i\alpha\sigma\,\mathbf{u}_1 = \mathbf{f}_1 \quad \text{in } D_1, \\ (\mathbf{n}\times\mathbf{u}_1) = \mathbf{0} \quad \text{on } \partial D_1\setminus\Gamma, \\ (\mathbf{n}\times\mathbf{u}_1) = (\mathbf{n}\times\mathbf{u}_2) \quad \text{on } \Gamma, \\ (\mathbf{n}\times\mathbf{curl}\,\mathbf{u}_1) = (\mathbf{n}\times\mathbf{curl}\,\mathbf{u}_2) \quad \text{on } \Gamma, \\ (\mathbf{n}\times\mathbf{u}_2) = \mathbf{0} \quad \text{on } \partial D_2\setminus\Gamma, \\ \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{u}_2) + i\alpha\sigma\,\mathbf{u}_2 = \mathbf{f}_2 \quad \text{in } D_2. \end{cases}$$

We set now

$$\begin{aligned} &V_j := \left\{\mathbf{v}_j \in H(\mathbf{curl}\,; D_j) : \quad (\mathbf{n}\times\mathbf{v}_j)_{|\partial D_j\setminus\Gamma} = \mathbf{0}\right\} = H_0(\mathbf{curl}\,, D_j), \\ &a_j(\mathbf{w},\mathbf{v}) := \int_{D_j} (\mu^{-1}\mathbf{curl}\,\mathbf{w}_j\cdot\mathbf{curl}\,\overline{\mathbf{v}}_j + i\alpha\sigma\,\mathbf{w}_j\cdot\overline{\mathbf{v}}_j)\, d\mathbf{x}, \\ &\mathbf{L}_j(\mathbf{v}_j) := \int_{D_j} \mathbf{f}\cdot\overline{\mathbf{v}}_j\, d\mathbf{x}. \end{aligned}$$

The variational formulation of the problem with two sub-domains is then: find $(\mathbf{u}_1, \mathbf{u}_2) \in V_1 \times V_2$ such that

$$\begin{cases} a_1(\mathbf{u}_1, \mathbf{v}_1) = \mathbf{L}_1(\mathbf{v}_1) \qquad \forall\, \mathbf{v}_1 \in H_0(\mathbf{curl}\,; D_1), \\ (\mathbf{n} \times \mathbf{u}_1)_{|\Gamma} = (\mathbf{n} \times \mathbf{u}_2)_{|\Gamma}, \\ a_2(\mathbf{u}_2, \mathbf{v}_2) = \mathbf{L}_2(\mathbf{v}_2) + \mathbf{L}_1(\mathbf{R}_1(\mathbf{n} \times \mathbf{v}_2)_{|\Gamma}) - a_1(\mathbf{u}_1, \mathbf{R}_1(\mathbf{n} \times \mathbf{v}_2)_{|\Gamma}) \quad \forall\, \mathbf{v}_2 \in V_2, \end{cases}$$

where $\mathbf{R}_1 : \ \boldsymbol{\chi}_\Gamma \to V_1$ is any continuous extension operator.

Finite dimensional formulation

We now approximate the Maxwell equations (4.36)-(4.37) with the first-kind *Nédélec finite elements* N_h^k, but we could use as well the other Nédélec spaces. We recall that these spaces, introduced by Nédélec [Néd80, Néd86], are the natural ones in the finite element approximation of Maxwell equations. We suppose that $D_j \subset \mathbb{R}^3$ is a polyhedra with Lipschitz boundary and we suppose that $\{\mathcal{T}_h\}_{h>0}$ is a *regular* subdivision of D, made with tetrahedra of diameter less than h. We also suppose that each element of $\mathcal{T}_h$ intersects only D_1 or D_2 and that $\{\mathcal{T}_h\}_{h>0}$ induce on Γ a *quasi-uniform triangulation* $\mathcal{M}_h$. To construct the finite element approximant we define

$$N_{j,h}^k := \left\{ \mathbf{v}_h \in H(\mathbf{curl}\,; D_j) : \quad \mathbf{v}_{h|K} \in \mathcal{R}_k, \ \forall\, K \in \mathcal{T}_{j,h} \text{ with } \mathcal{T}_{j,h} = \mathcal{T}_h \cap D_j \right\},$$

where

$$\mathcal{R}_k := (P_{k-1})^3 \oplus \mathcal{S}_k.$$

We denoted by $P_k := P_k^3$, for $k \geq 1$, the space of polynomials with degree equal or less than k and by $\mathcal{S}_k$ the following space

$$\mathcal{S}_k := \left\{ \mathbf{p} \in (\widetilde{P}_k)^3 : \quad \mathbf{p}(\mathbf{x}) \cdot \mathbf{x} = 0 \right\},$$

where $\widetilde{P}_k \subset P_k$ is the space of homogeneous polynomials of degree k.
We also define

$$\begin{aligned} V_{j,h} &:= N_{j,h}^k \cap V_j, \\ V_{j,h}^0 &:= N_{j,h}^k \cap H_0(\mathbf{curl}\,; D_j), \\ \boldsymbol{\chi}_{\Gamma,h} &:= \left\{ (\mathbf{n} \times \mathbf{v}_{1,h})_{|\Gamma} : \quad \mathbf{v}_{1,h} \in V_{1,h} \right\} = \left\{ (\mathbf{n} \times \mathbf{v}_{2,h})_{|\Gamma} : \quad \mathbf{v}_{2,h} \in V_{2,h} \right\}. \end{aligned}$$

The finite dimensional problem is: find $(\mathbf{u}_{1,h}, \mathbf{u}_{2,h}) \in V_{1,h} \times V_{2,h}$ such that

(4.38)
$$\begin{cases} a_1(\mathbf{u}_{1,h}, \mathbf{v}_{1,h}) = \mathbf{L}_1(\mathbf{v}_{1,h}) \qquad \forall\, \mathbf{v}_{1,h} \in V_{1,h}^0, \\ (\mathbf{n} \times \mathbf{u}_{1,h})_{|\Gamma} = (\mathbf{n} \times \mathbf{u}_{2,h})_{|\Gamma}, \\ a_2(\mathbf{u}_{2,h}, \mathbf{v}_{2,h}) = \mathbf{L}_2(\mathbf{v}_{2,h}) + \mathbf{L}_1(\mathbf{R}_1(\mathbf{n} \times \mathbf{v}_{2,h})_{|\Gamma}) - a_1(\mathbf{u}_{1,h}, \mathbf{R}_1(\mathbf{n} \times \mathbf{v}_{2,h})_{|\Gamma}) \quad \forall\, \mathbf{v}_{2,h} \in V_{2,h}. \end{cases}$$

Extension and Steklov-Poincaré operators

To properly formulate the interface problem with the *Steklov-Poincaré operators* we define, for every $\boldsymbol{\gamma}_h \in \boldsymbol{\chi}_{h,\Gamma}$, the extension $\mathbf{E}_{j,\Gamma}^h \boldsymbol{\gamma}_h$ which is solution of the problem: find $\mathbf{E}_{j,\Gamma}^h \boldsymbol{\gamma}_h \in V_{j,h}$ such that

$$a_j(\mathbf{E}_{j,\Gamma}^h \boldsymbol{\gamma}_h, \mathbf{v}_{j,h}) = 0 \qquad \forall\, \mathbf{v}_{j,h} \in V_{j,h}^0, \quad \text{with} \quad (\mathbf{n} \times \mathbf{E}_{j,\Gamma}^h \boldsymbol{\gamma}_h)_{|\Gamma} = \boldsymbol{\gamma}_h.$$

We denote by $\widehat{\mathbf{u}}_{j,h} \in V^0_{j,h}$ the solution of the problem

$$\text{find } \widehat{\mathbf{u}}_{j,h} \in V^0_{j,h} : \qquad a_j(\widehat{\mathbf{u}}_{j,h}, \mathbf{v}_{j,h}) = \mathbf{L}_j(\mathbf{v}_{j,h}) \qquad \forall \mathbf{v}_{j,h} \in V^0_{j,h}.$$

The existence and uniqueness of $\mathbf{E}^h_{j,\Gamma}\gamma_h$ and $\widehat{\mathbf{u}}_{j,h}$ are a consequence of the Lax-Milgram lemma. The couple $(\mathbf{E}^h_{1,\Gamma}\boldsymbol{\lambda}_h + \widehat{\mathbf{u}}_{1,h}, \mathbf{E}^h_{2,\Gamma}\boldsymbol{\lambda}_h + \widehat{\mathbf{u}}_{2,h})$ is solution of (4.38) if and only if the following equality is satisfied

$$a_2(\mathbf{E}^h_{2,\Gamma}\boldsymbol{\lambda}_h + \widehat{\mathbf{u}}_{2,h}, \mathbf{v}_{2,h}) = L_2(\mathbf{v}_{2,h}) + L_1(\mathbf{E}^h_{1,\Gamma}(\mathbf{n} \times \mathbf{v}_{2,h})_{|\Gamma})$$
$$-a_1(\mathbf{E}^h_{1,\Gamma}\boldsymbol{\lambda}_h + \widehat{\mathbf{u}}_{1,h}, \mathbf{E}^h_{1,\Gamma}(\mathbf{n} \times \mathbf{v}_{2,h})_{|\Gamma}) \qquad \forall\, \mathbf{v}_{2,h} \in V_{2,h}.$$

By using the extension operators $\mathbf{E}^h_{j,\Gamma}$ we have that the previous equation is equivalent to

$$\begin{aligned} a_2(\mathbf{E}^h_{2,\Gamma}\boldsymbol{\lambda}_h, \mathbf{E}^h_{2,\Gamma}\boldsymbol{\mu}_h) + a_2(\widehat{\mathbf{u}}_{2,h}, \mathbf{E}^h_{2,\Gamma}\boldsymbol{\mu}_h) &= L_2(\mathbf{E}^h_{2,\Gamma}\boldsymbol{\mu}_h) + L_1(\mathbf{E}^h_{1,\Gamma}\boldsymbol{\mu}_h) \\ &-a_1(\mathbf{E}^h_{1,\Gamma}\boldsymbol{\lambda}_h, \mathbf{E}^h_{1,\Gamma}\boldsymbol{\mu}_h) - a_1(\widehat{\mathbf{u}}_{1,h}, \mathbf{E}^h_{1,\Gamma}\boldsymbol{\mu}_h) \qquad \forall\, \boldsymbol{\mu}_h \in \boldsymbol{\chi}_{\Gamma,h}. \end{aligned} \tag{4.39}$$

We can now define the *Steklov-Poincaré operators* $\{\mathcal{S}_{j,h}\}_{j=1,2}$ as follows

$$\langle\langle\, \mathcal{S}_{j,h}\gamma_h, \boldsymbol{\mu}_h \,\rangle\rangle_h := a_j(\mathbf{E}^h_{j,\Gamma}\gamma_h, \mathbf{E}^h_{j,\Gamma}\boldsymbol{\mu}_h) \qquad \forall\, \gamma_h, \boldsymbol{\mu}_h \in \boldsymbol{\chi}_{\Gamma,h},$$

where $\langle\langle\, .\, , .\,\rangle\rangle_h$ is the duality between $\boldsymbol{\chi}_{\Gamma,h}$ and $(\boldsymbol{\chi}_{\Gamma,h})'$. We also define the operator $\boldsymbol{\Phi}_h$ from $\boldsymbol{\chi}_{\Gamma,h}$ into its dual space $(\boldsymbol{\chi}_{\Gamma,h})'$ by

$$\langle\langle\, \boldsymbol{\Phi}_h, \boldsymbol{\mu}_h \,\rangle\rangle_h := L_1(\mathbf{E}^h_{1,\Gamma}\boldsymbol{\mu}_h) - a_1(\widehat{\mathbf{u}}_{1,h}, \mathbf{E}^h_{1,\Gamma}\boldsymbol{\mu}_h) + L_2(\mathbf{E}^h_{2,\Gamma}\boldsymbol{\mu}_h) - a_2(\widehat{\mathbf{u}}_{2,h}, \mathbf{E}^h_{2,\Gamma}\boldsymbol{\mu}_h) \quad \forall\, \boldsymbol{\mu}_h \in \boldsymbol{\chi}_{\Gamma,h}.$$

Interface Problem

Problem 4.38 (and consequently 4.39) can be seen as the following interface problem. In the interface spaces we have to find $\boldsymbol{\lambda}_h \in \boldsymbol{\chi}_{\Gamma,h}$ such that

$$\langle\langle\, (\mathcal{S}_{1,h} + \mathcal{S}_{2,h})\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h \,\rangle\rangle_h = \langle\langle\, \boldsymbol{\Phi}_h, \boldsymbol{\mu}_h \,\rangle\rangle_h \qquad \forall\, \boldsymbol{\mu}_h \in \boldsymbol{\chi}_{\Gamma,h}. \tag{4.40}$$

The operators $\mathcal{S}_{j,h}$ are continuous and coercive in $\boldsymbol{\chi}_{\Gamma,h}$ and, to solve problem (4.40), we use both the "Dirichlet-Neumann" and the "Neumann-Neumann" procedure[8]. We apply the Richardson method, with preconditioners that are respectively:

$$P_{h,DN} := \mathcal{S}^{-1}_{2,h} \qquad \text{and} \qquad P_{h,NN} := \left(\mathcal{S}^{-1}_{1,h} + \mathcal{S}^{-1}_{2,h}\right)^{-1}.$$

The analysis of the convergence of the Dirichlet-Neumann method has been provided by Alonso and Valli [AV99]; that one of the Neumann-Neumann method is due to Berselli [Ber99].

[8]We use the same terminology of the previous sections, even if the boundary conditions are not of Dirichlet or Neumann type. See the differential interpretation.

Dirichlet-Neumann method

We have the following iterative method: given $\boldsymbol{\lambda}_h^0 \in \boldsymbol{\chi}_{\Gamma,h}$, for each $m \geq 0$ solve

$$\boldsymbol{\lambda}_h^{m+1} = \boldsymbol{\lambda}_h^m + \theta\, \mathcal{S}_{2,h}^{-1}\left[\boldsymbol{\Phi}_h - (\mathcal{S}_{1,h} + \mathcal{S}_{2,h})\boldsymbol{\lambda}_h^m\right] = (1-\theta)\boldsymbol{\lambda}_h^m + \theta\, \mathcal{S}_{2,h}^{-1}\left[\boldsymbol{\Phi}_h - \mathcal{S}_{1,h}\boldsymbol{\lambda}_h^m\right]. \tag{4.41}$$

With an argument similar of that of the previous sections, we have that (4.41) is equivalent to the following sub-domain iteration: given $\boldsymbol{\lambda}_h^0 \in \boldsymbol{\chi}_{\Gamma,h}$, for each $m \geq 0$ find $\mathbf{u}_{1,h}^{m+1} \in V_{1,h}$ such that

$$a_1(\mathbf{u}_{1,h}^{m+1}, \mathbf{v}_{1,h}) = \mathbf{L}_1(\mathbf{v}_{1,h}) \qquad \forall\, \mathbf{v}_{1,h} \in V_{1,h}^0 \quad \text{with } (\mathbf{n} \times \mathbf{u}_{1,h}^{m+1})_{|\Gamma} = \boldsymbol{\lambda}_h^m.$$

Then find $\mathbf{u}_{2,h}^{m+1} \in V_{2,h}$ such that

$$a_2(\mathbf{u}_{2,h}^{m+1}, \mathbf{v}_{2,h}) = \mathbf{L}_2(\mathbf{v}_{2,h}) + \mathbf{L}_1(\mathbf{E}_{1,\Gamma}^h(\mathbf{n} \times \mathbf{v}_{2,h})_{|\Gamma}) - a_1(\mathbf{u}_{1,h}^{m+1}, \mathbf{E}_{1,\Gamma}^h(\mathbf{n} \times \mathbf{v}_{2,h})_{\Gamma}) \quad \forall\, \mathbf{v}_{2,h} \in V_{2,h}$$

and finally

$$\boldsymbol{\lambda}_h^{m+1} = (1-\theta)\boldsymbol{\lambda}_h^m + \theta(\mathbf{n} \times \mathbf{u}_{2,h}^{m+1})_{|\Gamma}. \tag{4.42}$$

This procedure is the finite dimensional variational formulation of the Dirichlet-Neumann iteration, relative to problem (4.36)-(4.37). In a strong formulation the continuous problem reads as: for each $m \geq 0$, find $\mathbf{u}_j^{m+1}$ such that

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{u}_1^{m+1}) + i\alpha\sigma\,\mathbf{u}_1^{m+1} = \mathbf{f}_1 & \text{in } D_1, \\ \mathbf{n} \times \mathbf{u}_1^{m+1} = \mathbf{0} & \text{on } \partial D_1 \cap \partial D, \\ \mathbf{n} \times \mathbf{u}_1^{m+1} = \boldsymbol{\lambda}_h^m & \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{u}_2^{m+1}) + i\sigma\alpha\,\mathbf{u}_2^{m+1} = \mathbf{f}_2 & \text{in } D_2, \\ \mathbf{n} \times \mathbf{u}_2^{m+1} = \mathbf{0} & \text{on } \partial D_2 \cap \partial D, \\ \mathbf{n} \times \mathbf{curl}\,\mathbf{u}_2^{m+1} = \mathbf{n} \times \mathbf{curl}\,\mathbf{u}_{1,h}^{m+1} & \text{on } \Gamma, \end{cases}$$

completed with condition (4.46).

Neumann-Neumann method

We now describe the second method of this section. We have the following iterative method. Given $\boldsymbol{\lambda}_h^0 \in \boldsymbol{\chi}_{\Gamma,h}$, for each $m \geq 0$ solve

$$\boldsymbol{\lambda}_h^{m+1} = \boldsymbol{\lambda}_h^m + \theta\,(\mathcal{S}_{1,h}^{-1} + \mathcal{S}_{2,h}^{-1})\left[\boldsymbol{\Phi}_h - (\mathcal{S}_{1,h} + \mathcal{S}_{2,h})\boldsymbol{\lambda}_h^m\right]. \tag{4.43}$$

With a standard argument, we have that (4.43) is equivalent to the following sub-domain iteration: given $\boldsymbol{\lambda}_h^0 \in \boldsymbol{\chi}_{\Gamma,h}$, for every $m \geq 0$, find $\mathbf{u}_{j,h}^{m+1}$ such that

$$a_j(\mathbf{u}_{j,h}^{m+1}, \mathbf{v}_{j,h}) = \mathbf{L}_j(\mathbf{v}_{j,h}) \qquad \forall\, \mathbf{v}_{j,h} \in V_{j,h}^0,$$

with $(\mathbf{n}\times\mathbf{u}_{j,h}^{m+1})_{|\Gamma} = \boldsymbol{\lambda}_h^m$, then find $\boldsymbol{\psi}_{1,h}^{k+1} \in V_{1,h}$ such that

$$(4.44)\qquad \begin{cases} a_1(\boldsymbol{\psi}_{1,h}^{k+1}, \mathbf{v}_{1,h}) = 0 \qquad \forall \mathbf{v}_{1,h} \in V_{1,h}^0 \\[2ex] a_1(\boldsymbol{\psi}_{1,h}^{k+1}, \mathbf{R}_1\boldsymbol{\gamma}_h) = -\mathbf{L}_1(\mathbf{R}_1\boldsymbol{\gamma}_h) - \mathbf{L}_2(\mathbf{R}_2\boldsymbol{\gamma}_h) \\ \qquad\qquad +a_1(\mathbf{u}_{1,h}^{k+1}, \mathbf{R}_1\boldsymbol{\gamma}_h) + a_2(\mathbf{u}_{2,h}^{k+1}, \mathbf{R}_2\boldsymbol{\gamma}_h) \quad \forall \boldsymbol{\gamma}_h \in \boldsymbol{\chi}_{\Gamma,h}, \end{cases}$$

then find $\boldsymbol{\psi}_{2,h}^{k+1} \in V_{2,h}$ such that

$$(4.45)\qquad \begin{cases} a_2(\boldsymbol{\psi}_{2,h}^{k+1}, \mathbf{v}_{2,h}) = 0 \qquad \forall \mathbf{v}_{2,h} \in V_{2,h}^0 \\[2ex] a_2(\boldsymbol{\psi}_{2,h}^{k+1}, \mathbf{R}_2\boldsymbol{\gamma}_h) = \mathbf{L}_1(\mathbf{R}_1\boldsymbol{\gamma}_h) + \mathbf{L}_2(\mathbf{R}_2\boldsymbol{\gamma}_h) \\ \qquad\qquad -a_1(\mathbf{u}_{1,h}^{k+1}, \mathbf{R}_1\boldsymbol{\gamma}_h) - a_2(\mathbf{u}_{2,h}^{k+1}, \mathbf{R}_2\boldsymbol{\gamma}_h) \quad \forall \boldsymbol{\gamma}_h \in \boldsymbol{\chi}_{\Gamma,h}, \end{cases}$$

with

$$(4.46)\qquad \boldsymbol{\lambda}_h^{m+1} = \boldsymbol{\lambda}_h^m - \theta\left[(\mathbf{n}\times\boldsymbol{\psi}_{1,h}^{m+1})_{|\Gamma} - (\mathbf{n}\times\boldsymbol{\psi}_{2,h}^{m+1})_{|\Gamma}\right].$$

This procedure is the variational formulation of the Neumann-Neumann iteration, relative to problem (4.36)-(4.37), which is written (in a strong formulation) as

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{u}_j^{m+1}) + i\alpha\sigma\,\mathbf{u}_j^{m+1} = \mathbf{f}_j \quad \text{in } D_j, \\[1ex] \mathbf{n}\times\mathbf{u}_j^{m+1} = \mathbf{0} \quad \text{on } \partial D_j \cap \partial D, \\[1ex] \mathbf{n}\times\mathbf{u}_j^{m+1} = \boldsymbol{\lambda}_h^m \quad \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\boldsymbol{\psi}_j^{m+1}) + i\sigma\alpha\,\boldsymbol{\psi}_j^{m+1} = \mathbf{0} \quad \text{in } D_j, \\[1ex] \mathbf{n}\times\boldsymbol{\psi}_j^{m+1} = \mathbf{0} \quad \text{on } \partial D_j \cap \partial D, \\[1ex] \mathbf{n}\times\mathbf{curl}\,\boldsymbol{\psi}_j^{m+1} = \mathbf{n}\times\mathbf{curl}\,\mathbf{u}_1^{m+1} - \mathbf{n}\times\mathbf{curl}\,\mathbf{u}_2^{m+1} \quad \text{on } \Gamma, \end{cases}$$

completed with condition (4.46).

A convergence result

We now state an abstract result. We shall use this proposition to prove the convergence, with a rate which does not depend[9] on h, of the Richardson methods (4.41) and (4.43).

Proposition 4.3.6. *Let X be a complex Hilbert space and let $\mathcal{S}_{j,h} : X_h \mapsto X_h'$ for $j = 1, 2$ be two linear operators with $X_h \subset X$ and $\dim X_h < +\infty$. Furthermore, let $\boldsymbol{\pi}_s$, $s = 1, \dots, M_h$ be a basis of X_h. We define the matrices $\mathsf{S}_{j,h}$ associated to the operators $\mathcal{S}_{j,h}$ as*

$$(\mathsf{S}_{j,h}\boldsymbol{\gamma}, \boldsymbol{\mu})_h := \langle\langle \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\mu}_h\rangle\rangle_h \quad \forall \boldsymbol{\gamma}, \boldsymbol{\mu} \in \mathbb{C}^{M_h},\ j = 1, 2,$$

[9] We observe that, in the following, the constants that bound the eigenvalues do not depend on h.

where $(\,.\,,\,.\,)_h$ *denotes the Euclidean scalar product in* $\mathbb{C}^{M_h}$*. Finally, let*

$$\boldsymbol{\gamma}_h := \sum_{s=1}^{M_h} \gamma_s\, \boldsymbol{\pi}_s \quad \textit{and} \quad \boldsymbol{\mu}_h := \sum_{s=1}^{M_h} \mu_s\, \boldsymbol{\pi}_s. \tag{4.47}$$

Let us suppose that there exist $\alpha_j, \beta_j > 0$ *for* $j = 1, 2$*, independent of* h*, such that*

$$|\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\mu}_h\,\rangle\rangle_h| \le \beta_j \|\boldsymbol{\gamma}_h\|_X \|\boldsymbol{\mu}_h\|_X \quad \forall\, \boldsymbol{\gamma}_h, \boldsymbol{\mu}_h \in X_h \quad \textit{for } j = 1, 2, \tag{4.48}$$

$$|\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h| \ge \alpha_j \|\boldsymbol{\gamma}_h\|_X^2 \quad \forall\, \boldsymbol{\gamma}_h \in X_h \quad \textit{for } j = 1, 2, \tag{4.49}$$

$$\begin{aligned} &\mathrm{Re}\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h\, \mathrm{Re}\langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h\, + \\ &\qquad + \mathrm{Im}\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h\, \mathrm{Im}\langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h \ge 0 \quad \forall\, \boldsymbol{\gamma} \in X_h, \end{aligned} \tag{4.50}$$

where

$$[j] = \begin{cases} 2 & \textit{if} \quad j = 1, \\ 1 & \textit{if} \quad j = 2. \end{cases}$$

Then if $\nu_{j,h} = \nu_{j,h}^1 + i\nu_{j,h}^2 \in \mathbb{C}$*, with* $\nu_{j,h}^i \in \mathbb{R}$ *for* $i, j = 1, 2$*, is an eigenvalue of* $\mathsf{S}_{[j],h}^{-1}\mathsf{S}_{j,h}$*, we have that*

$$\nu_{j,h}^1 \ge 0 \qquad \textit{and} \qquad |\nu_{j,h}|^2 = \frac{|\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h|^2}{\left|\langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h\right|^2} \le \left(\frac{\beta_j}{\alpha_{[j]}}\right)^2.$$

Proof. If $\nu_{j,h}$ is an eigenvalue of $\mathsf{S}_{[j],h}^{-1}\mathsf{S}_{j,h}$, then the corresponding eigenvector $0 \ne \boldsymbol{\gamma} \in \mathbb{R}^{M_h}$ satisfies

$$\mathsf{S}_{j_h}\boldsymbol{\gamma}_h = \nu_{j,h}\, \mathsf{S}_{[j],h}\boldsymbol{\gamma}$$

and consequently

$$\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h = \nu_{j,h}\, \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h,$$

where $\boldsymbol{\gamma}_h$ is defined in (4.47). Since $\nu_{j,h} = \nu_{j,h}^1 + i\,\nu_{j,h}^2$ we have

$$\mathrm{Re}\; \langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h = \nu_{j,h}^1\, \mathrm{Re}\; \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h - \nu_{j,h}^2\, \mathrm{Im}\; \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h, \tag{4.51}$$

$$\mathrm{Im}\; \langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h = \nu_{j,h}^1\, \mathrm{Im}\; \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h + \nu_{j,h}^2\, \mathrm{Re}\langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h. \tag{4.52}$$

By multiplying the two equations (4.51)-(4.52) by $\mathrm{Re}\langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h$ and by $\mathrm{Im}\langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h$, respectively and by summing up we obtain

$$\begin{aligned} \nu_{j,h}^1 &\left[\left(\mathrm{Re}\; \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h\right)^2 + \left(\mathrm{Im}\; \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h\right)^2\right] = \mathrm{Re}\; \langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h \cdot \\ &\cdot \mathrm{Re}\; \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h + \mathrm{Im}\; \langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h \mathrm{Im}\; \langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h. \end{aligned} \tag{4.53}$$

From (4.49) we obtain $|\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h| \ne 0$ and from (4.50) we get $\nu_{j,h}^1 \ge 0$.

From (4.48)-(4.49) we also obtain

$$|\nu_{j,h}|^2 = \frac{|\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h|^2}{\left|\langle\langle\, \mathcal{S}_{[j],h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h\right|^2} \le \left(\frac{\beta_j}{\alpha_{[j]}}\right)^2.$$

□

We observe that if the operators $\mathcal{S}_{j,h}$ are the Steklov-Poincaré operators associated to the low-frequency time-harmonic Maxwell equations (4.36)-(4.37), then the hypotheses of Proposition 4.3.6 are satisfied. We have that

$$\mathrm{Re}\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h = \int_{D_j} \mu^{-1}|\mathbf{curl}\,\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h|^2\, d\mathbf{x}$$

and

$$\mathrm{Im}\ \langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h = \alpha\int_{D_j} \sigma|\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h|^2\, d\mathbf{x},$$

and (4.50) holds.

By using the following trace[10] inequality, see Alonso and Valli [AV96], we have

$$\|(\mathbf{n}\times\mathbf{v})_{|\Gamma}\|^2_{\boldsymbol{\chi}_\Gamma} \le C^*\|\mathbf{v}\|^2_{H(\mathbf{curl}\,;D_j)} \qquad \text{for } j=1,2.$$

We can now prove (4.49) by observing that

$$|\langle\langle\mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \gamma_h\rangle\rangle_h| \ge C\left(\|\mathbf{E}^h_{2,\Gamma}\boldsymbol{\gamma}_h\|^2_{L^2(D_j)} + \|\mathbf{curl}\,\mathbf{E}^h_{2,\Gamma}\boldsymbol{\gamma}_h\|^2_{L^2(D_j)}\right) \ge \frac{C}{C^*}\|\boldsymbol{\gamma}_h\|^2_{\boldsymbol{\chi}_\Gamma}.$$

The last condition (4.48) is the most technical to check. This condition is based on the (uniform in h) continuity of the operator $\mathbf{E}^h_{j,\Gamma}$. We define $k_j := s_j - 1/2$ with s_j such that $H^{1+s_j}(D_j)$ gives the best regularity for the solutions of the Laplace problem (in the domain D_j) with homogeneous Dirichlet or Neumann boundary conditions and with right hand side in $L^2(D_j)$. To prove (4.48) we use the following lemmas, that are proved in Alonso and Valli [AV99]. These results are based on some regularity results by Amrouche, Bernardi, Dauge and Girault [ABDG98].

Lemma 4.3.7. *Given $\delta\in(0,k_j]$, there exist $\mathbb{R}\ni B_{1,j}>0$ for $j=1,2$, such that $\forall\boldsymbol{\gamma}\in\boldsymbol{\chi}_\Gamma$ with $\widetilde{\boldsymbol{\gamma}}\in\boldsymbol{\chi}^\delta_{\partial D_j}$ we have $\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}\in H^{1/2+\delta}(\mathbf{curl}\,;D_j)$ and*

$$\|\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}\|_{H^{1/2+\delta}(\mathbf{curl}\,;D_j)} \le B_{1,j}\|\widetilde{\boldsymbol{\gamma}}\|_{\boldsymbol{\chi}^\delta_{\partial D_j}} \qquad \textit{for } j=1,2.$$

Lemma 4.3.8. *Let $\mathcal{T}_h$ be a regular triangulation and let be given $\boldsymbol{\gamma}_h\in\boldsymbol{\chi}_{\Gamma,h}$ with $\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h$ belonging to $H^t(\mathbf{curl}\,;D_j)$ for $t\in(1/2,1)$. Then there exist $\mathbb{R}\ni B_{2,j}>0$ for $j=1,2$, independent of h, such that*

$$\|\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h - \mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)} \le B_{2,j}\,h^t\|\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H^t(\mathbf{curl}\,;D)} \qquad \forall\,\boldsymbol{\gamma}_h\in\boldsymbol{\chi}_{\Gamma,h} \quad \textit{for } j=1,2.$$

Lemma 4.3.9. *Let $\mathcal{M}_h$ be a quasi-uniform triangulation on ∂D_j induced by $\mathcal{T}_h$. Then for every $\eta\in(0,1/2)$ there exist $\mathbb{R}\ni B_{3,j}>0$ for $j=1,2$, independent of h, such that*

$$\|\widetilde{\boldsymbol{\gamma}}\|_{\boldsymbol{\chi}^\eta_{\partial D_j}} \le B_{3,j}h^{-1/2-\eta}\|\boldsymbol{\gamma}\|_{\boldsymbol{\chi}_\Gamma} \qquad \forall\,\boldsymbol{\gamma}_h\in\boldsymbol{\chi}_{\Gamma,h} \quad \textit{for } j=1,2.$$

With the above lemmas we can prove the following proposition, which concludes the verification of the hypotheses of Proposition 4.3.6.

Proposition 4.3.10. *There exist $\mathbb{R}\ni B_{4,j}>0$ for $j=1,2$, independent of h, such that*

$$|\langle\langle\, \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\mu}_h\,\rangle\rangle_h| \le B_{4,j}\|\boldsymbol{\gamma}_h\|_{\boldsymbol{\chi}_\Gamma}\|\boldsymbol{\mu}_h\|_{\boldsymbol{\chi}_\Gamma} \qquad \forall\,\boldsymbol{\gamma}_h,\boldsymbol{\mu}_h\in\boldsymbol{\chi}_{\Gamma,h} \quad \textit{for } j=1,2.$$

[10] It is the counterpart of the trace inequality (4.14) that we have seen in the $H^1(D)$ framework.

Proof. By using the definition of the Steklov-Poincaré operators, we have

$$|\langle\langle \mathcal{S}_{j,h}\boldsymbol{\gamma}_h, \boldsymbol{\mu}_h\rangle\rangle_h| \le \left| a_j(\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h, \mathbf{E}^h_{j,\Gamma}\boldsymbol{\mu}_h)\right| \le d_j \|\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)} \|\mathbf{E}^h_{j,\Gamma}\boldsymbol{\mu}_h\|_{H(\mathbf{curl}\,;D_j)},$$

where d_j is the continuity constant of $a_j(\,.\,,\,.\,)$. We prove that for $j = 1, 2$ there exist positive constants C_j, which are independent of h, such that

$$\|\mathbf{E}^h_{1,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)} \le C\|\boldsymbol{\gamma}_h\|_{\boldsymbol{\chi}_\Gamma} \qquad \forall\, \boldsymbol{\gamma}_h \in \boldsymbol{\chi}_{\Gamma,h}.$$

We have

$$\|\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)} \le \|\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h - \mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)} + \|\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)}.$$

The last term can be bounded by observing that the operator $\mathbf{E}_{j,\Gamma} : \boldsymbol{\chi}_\Gamma \to V_j$ is continuous. Since $\boldsymbol{\gamma}_h \in \boldsymbol{\chi}_{\Gamma,h}$, we have that $\widetilde{\boldsymbol{\gamma}}_h \in \boldsymbol{\chi}^\delta_{\partial D_j}$ for $0 < \delta < 1/2$. This fact holds because $\widetilde{\boldsymbol{\gamma}}_h$ and its tangential divergence are piecewise polynomials. By using Lemma 4.3.7, we have that $\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h$ belongs to $H^{1/2+\delta}(\mathbf{curl}\,; D_j)$ for each $\delta \in (0, k_j]$.

By applying the results of Lemma 4.3.8 with $t = 1/2 + \delta$, we obtain

$$\|\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h - \mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)} \le B_{2,j}h^{1/2+\delta}\|\mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H^{1/2+\delta}(\mathbf{curl}\,;D_j)} \le B_{1,j}B_{2,j}h^{1/2+\delta}\|\widetilde{\boldsymbol{\gamma}}_h\|_{\boldsymbol{\chi}^\delta_{\partial D_j}}.$$

By using Lemma 4.3.9 with $\eta = \delta$, we finally have

$$\|\mathbf{E}^h_{j,\Gamma}\boldsymbol{\gamma}_h - \mathbf{E}_{j,\Gamma}\boldsymbol{\gamma}_h\|_{H(\mathbf{curl}\,;D_j)} \le B_{1,j}B_{2,j}B_{3,j}\|\widetilde{\boldsymbol{\gamma}}_h\|_{\boldsymbol{\chi}_\Gamma}.$$

□

We can now prove the convergence of the two preconditioned Richardson methods we introduced. In particular we shall prove that *spectral radius* of the iteration matrix is less than one, or equivalently that condition (4.4) is satisfied.

Theorem 4.3.11. *There exists a positive constant C^{DN}, not depending on h, such that the preconditioned Richardson methods (4.41) (corresponding to the "Dirichlet-Neumann" method) converges for any $0 < \theta < C^{DN}$. Furthermore, there exists a positive constant C^{NN}, not depending on h, such that the preconditioned Richardson methods (4.43) (corresponding to the "Neumann-Neumann" method) converges for any $0 < \theta < C^{NN}$.*

Proof. If λ_h is an eigenvalue of $\mathsf{S}^{-1}_{2,h}(\mathsf{S}_{1,h} + \mathsf{S}_{2,h}) = I + \mathsf{S}^{-1}_{2,h}\mathsf{S}_{1,h}$ we can write $\lambda_h = 1 + \nu_{1,h}$, where $\nu_{1,h}$ is an eigenvalue of $\mathsf{S}^{-1}_{2,h}\mathsf{S}_{1,h}$. If $\boldsymbol{\gamma}$ denotes the corresponding eigenvector we have

$$\langle\langle \mathcal{S}_{1,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h = \nu\langle\langle \mathcal{S}_{2,h}\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h\rangle\rangle_h.$$

By using Proposition 4.3.6 we have that

$$2\frac{\mathrm{Re}\,\lambda_h}{|\lambda_h|^2} = 2\frac{1 + \nu^1_{1,h}}{1 + 2\nu^1_{1,h} + |\nu_{1,h}|^2} \ge \frac{1 + \nu^1_{1,h}}{1 + 2\nu^1_{1,h} + \beta_1^2/\alpha_2^2}.$$

We now observe that the real function

$$\phi(\xi) := 2\frac{1 + \xi}{1 + 2\xi + C_1^2/C_2^2},$$

defined for $\xi \geq 0$, is strictly increasing for $C_1 > C_2$, strictly decreasing for $C_2 > C_1$ and constantly equal to one for $C_1 = C_2$. Moreover, we have that

$$\phi(0) = \frac{2C_2^2}{C_1^1 + C_2^2} \qquad \text{and} \qquad \lim_{\xi \to +\infty} \phi(\xi) = 1.$$

It follows that

$$\inf_{0 \leq \xi < \infty} \phi(\xi) := C^{DN} > 0.$$

We have that, for the positive constant C^{DN}, each eigenvalue of $\mathsf{S}_{2,h}^{-1}(\mathsf{S}_{1,h} + \mathsf{S}_{2,h})$ satisfies

$$2\frac{\mathrm{Re}\,\lambda_h}{|\lambda_h|^2} \geq C^{DN}$$

and the preconditioned Richardson method (4.41) converges with a rate independent of h, for any $0 < \theta < C^{DN}$.

We pass now to the study of the second method we proposed. We make the following observation: *let be given* $A \in GL(n \times n, \mathbf{C})$ *(invertible-*$n \times n$*-complex matrix). Then the eigenvalues of* $A + A^{-1}$ *are of the form:*

$$\lambda + \frac{1}{\lambda},$$

with λ *eigenvalue of* A. This result easily follows by using the *Jordan canonical form*. We recall that there exists a non-singular $Q \in \mathrm{GL}(n \times n, \mathbf{C})$ such that $Q\,A\,Q^{-1}$ is in the *Jordan canonical form*, *i.e.*, $Q\,A\,Q^{-1}$ is a block diagonal matrix with diagonal blocks of the form

$$\begin{bmatrix} \lambda_j & 1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \lambda_j \end{bmatrix},$$

corresponding to each eigenvalue λ_j. We observe that $\left(Q\,A\,Q^{-1}\right)^{-1}$ is an upper triangular matrix and each diagonal block, relative to the eigenvalue $\mu_j = 1/\lambda_j$, is of the form

$$\begin{bmatrix} \frac{1}{\lambda_j} & * & * \\ 0 & \ddots & * \\ 0 & 0 & \frac{1}{\lambda_j} \end{bmatrix}$$

and it has the same dimension of the block relative to the eigenvalue λ_j of the matrix A. Written in the *Jordan basis* (the basis relative to the canonical form), the sum of $Q\,A\,Q^{-1}$ and its inverse $(Q\,A\,Q^{-1})^{-1}$ is an upper-triangular block-matrix with the elements $\lambda_j + 1/\lambda_j$ on the diagonal. This proves the observation.

We can now prove the convergence of the preconditioned Richardson methods (4.43). If λ_h is an eigenvalue of

$$(\mathsf{S}_{1,h}^{-1} + \mathsf{S}_{2,h}^{-1})(\mathsf{S}_{1,h} + \mathsf{S}_{2,h}) = 2\,I + \mathsf{S}_{1,h}^{-1}\mathsf{S}_{2,h} + \mathsf{S}_{2,h}^{-1}\mathsf{S}_{1,h},$$

it can be written as $2 + \nu_h$, where ν_h is eigenvalue of $\mathsf{S}_{1,h}^{-1}\mathsf{S}_{2,h} + \mathsf{S}_{2,h}^{-1}\mathsf{S}_{1,h}$. Then ν_h can be written, due to the previous observation, as

$$\nu_h := \nu_{1,h} + \nu_{2,h},$$

where $\nu_{j,h}$ is an eigenvalue of $\mathsf{S}_{[j],h}^{-1}\mathsf{S}_{j,h}$. By using the results of Proposition 4.3.6 for the eigenvalues of $\mathsf{S}_{[j],h}^{-1}\mathsf{S}_{j,h}$ we have

$$2\frac{\mathrm{Re}\,\lambda_h}{|\lambda_h|^2} = 2\frac{2+\nu_{1,h}^1+\nu_{2,h}^1}{4+4(\nu_{1,h}^1+\nu_{2,h}^1)+|\nu_h|^2} \geq 2\frac{2+\nu_{1,h}^1+\nu_{2,h}^1}{4+4(\nu_{1,h}^1+\nu_{2,h}^1)+C},$$

because

$$|\nu_h|^2 \leq (|\nu_{1,h}|+|\nu_{2,h}|)^2 \leq (\alpha_1/\beta_2+\alpha_2/\beta_1)^2 = C.$$

We easily obtain that the real function $\Phi(\xi) := 2(2+\xi)/(4+4\,\xi+C)$ is such that

$$\inf_{\xi\geq 0}\Phi(\xi) = \min\left(\frac{4}{4+C},1\right) > 0.$$

Since $\nu_{j,h}^1 \geq 0$ for $j=1,2$, we have that each eigenvalue λ_h of $(\mathsf{S}_{1,h}^{-1}+\mathsf{S}_{2,h}^{-1})(\mathsf{S}_{1,h}+\mathsf{S}_{2,h})$ satisfy

$$\frac{2\mathrm{Re}\,\lambda_h}{|\lambda_h|^2} \geq C^{NN} \qquad \forall\, s=1,\dots,M_h,$$

with

$$C^{NN} := \min\left(1,\frac{4}{4+C}\right)$$

where $C=(\alpha_1/\beta_2+\alpha_2/\beta_1)^2$. Consequently the preconditioned Richardson method (4.43) converges with a rate which does not depend on h, for every $\theta\in(0,C^{NN})$. □

An application to advection diffusion equations

The method that we introduced for the Maxwell equations, can be used to prove in an alternative way the convergence of the Dirichlet-Neumann and Neumann-Neumann methods for non-symmetric elliptic equations of Section 4.3. We consider the partial differential operator (4.30) with real coefficients. Its discretization, with the finite element method (or any other), leads to a positive-definite non-symmetric real matrix A. To study the eigenvalues of the iteration matrix for both the Dirichlet-Neumann and Neumann-Neumann problem, we complexify the problem. We consider A (denoted by $A^{\mathbb{C}}$) as a linear operator from $\mathbb{C}^n$ into itself. We easily see that, if we set $\mathbf{u} := \mathbf{u}_1 + i\,\mathbf{u}_2$ and $\mathbf{f} := \mathbf{f}_1 + i\,\mathbf{f}_2$, the real part $\mathbf{u}_1$ of the solution of $A^{\mathbb{C}}\mathbf{u} = \mathbf{f}$ is exactly the solution of the original real problem $A\mathbf{x} = \mathbf{f}_1$. This approach corresponds to a variational formulation with the *non-Hermitian* complex bilinear form

$$a^{\mathbb{C}}(w,v) := \int_D \left[\sum_{l,j=1}^{d} a_{lj}\,\frac{\partial w}{\partial x_j}\frac{\partial \overline{v}}{\partial x_l} + \left(\frac{1}{2}\,\mathrm{div}\,\mathbf{b}+a_0\right) w\,\overline{v}\right] d\mathbf{x} + \frac{1}{2}\int_D (v\,\mathbf{b}\cdot\nabla\overline{w} - w\,\mathbf{b}\cdot\nabla\overline{v})\,d\mathbf{x}.$$

Observe that the coefficients a_{lj}, b_l and a_0 are still real. To better understand the result, we suppose that $a_{lj} = a_{jl}$ and we isolate the non-symmetries of the problem in the first order terms. The real part of $a_i^{\mathbb{C}}(u,u)$ (with obvious meaning it is the restriction to D_i) is positive for $i=1,2$. Concerning the complex part, it is easy to check that it is small, provided the quantities $\|b_i\|_\infty$ are small. It holds because the expression of the imaginary part of $a_i^{\mathbb{C}}(u,u)$ is

$$\frac{1}{2}\int_{D_i}(u\,\mathbf{b}\cdot\nabla\overline{u} - \overline{u}\,\mathbf{b}\cdot\nabla u)\,d\mathbf{x} = \frac{1}{2}\int_{D_i}\left(u\,\mathbf{b}\cdot\nabla\overline{u} - \overline{u\,\mathbf{b}\cdot\nabla\overline{u}}\right) d\mathbf{x}.$$

Then easy calculation show that condition (4.50) is satisfied provided the modulus of the non-symmetric part is small enough. In this way we proved with a different method the convergence of the two classical substructuring methods, if the operator L is a "small" perturbation of a symmetric one. We also remark that the first method, which is based on a contraction argument, gives information on the specific norm with respect to convergence is achieved; the second method states that the spectral radius is smaller than one and this gives bound for every norm, at a finite dimensional level. In particular, for at least one case the norm of the iteration matrix is small than one.

4.4 Advection diffusion equations and systems

In this section we analyze again the non-symmetric operator (4.30). In particular we want to find convergence results or effective methods for equations with "big" non-symmetric terms. We propose some methods, based on coercivity, which work for equation as well for systems. We recall that this problem has been studied with some different *domain decomposition methods.* In particular we shall explain that big difficulties arise in applying the abstract contraction theorems of the previous section, because some control on the non-symmetric part is needed.

By using a terminology taken from physics, we call the non-symmetric equation associate to the operator L of (4.30) *advection-diffusion* equations. It is clear that the second order term, or diffusive term, is that one which regularizes the solution. The advection term, which corresponds to the non-symmetric part, is that one which takes into account of "hyperbolic-type" phenomena. The equation, that we consider, is the basic one to be solved in the (first step of) Chorin-Temam method (recall Section 1.4.3). We observe that the advection-diffusion equations are themselves an important numerical problem, because just in the one-dimensional case big stability problems arise, see Quarteroni and Valli [QV94], Ch. 8. In particular, by recalling the results of Chapter 1, we observe that the "bigness" of the non-symmetric part (relative to the symmetric part) depends on the *Reynolds number* R. In real industrial problems the assumption of smallness of R are highly unrealistic and the approximate solution of flows at high Reynolds numbers is one of the most challenging problems in CFD, *computational fluid dynamics*. We propose the methods introduced by Berselli and Saleri [BS99] and that one by Alonso, Trotta and Valli [ATV98], together with the analysis of a new method.

To avoid inessential calculation we assume the second order part to be symmetric and we study, for $\varepsilon > 0$, the following Dirichlet problem

$$L_\varepsilon u := -\varepsilon\,\Delta u + \sum_{j=1}^{d} \frac{\partial(b_j u)}{\partial x_j} + a_0 u = f \quad \text{in } D, \tag{4.54}$$

$$u = 0 \quad \text{on } \partial D. \tag{4.55}$$

We assume the same regularity hypotheses of the previous Section 4.3 on b_i, $i = 1, \dots, d$, and a_0. Furthermore, we assume condition (4.31), to make the problem coercive.

4.4.1 Adaptive methods

For the sake of completeness, we recall some methods for advection diffusion equations. In the first papers regarding domain decomposition methods for non-symmetric equations there where proposed some methods that should be consistent with the *hyperbolic limit:* $\varepsilon \to 0$. In the papers by Carlenzoli and Quarteroni [CQ95] and Gastaldi, Gastaldi and Quarteroni [GGQ96] there were proposed the so called *adaptive methods.* These methods are based on the observation that, to be consistent with the hyperbolic limit, a Dirichlet condition must not be imposed on *outflow boundary*, whereas Neumann interface condition has to be enforced at the same boundary. We recall that the *outflow boundary* ∂D^{out} of a domain D is defined by

$$\partial D^{\text{out}} := \{\mathbf{x} \in \partial D : \quad \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0\},$$

while the *inflow boundary* ∂D^{in} is

$$\partial D^{\text{in}} := \{\mathbf{x} \in \partial D : \quad \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}.$$

We now present two methods, without stating the convergence results, for which we refer to the papers cited above or to Quarteroni and Valli [QV99] Ch. 6 §3.

The ADN-*adaptive Dirichlet-Neumann* method reads as: given u_i^0 in D_i, for $i = 1, 2$, solve for each $k \geq 0$

$$\begin{cases} L_\varepsilon u_1^{k+1} = f \quad \text{in } D_1, \\ u_1^{k+1} = 0 \quad \text{on } \partial D \cap \partial D_1, \\ u_1^{k+1} = \lambda^k \quad \text{on } \Gamma^{\text{in}} \cup \Gamma^0, \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} = \varepsilon \dfrac{\partial u_2^k}{\partial \mathbf{n}} \quad \text{on } \Gamma^{\text{out}}, \end{cases} \quad \text{and} \quad \begin{cases} L_\varepsilon u_2^{k+1} = f \quad \text{in } D_2, \\ u_2^{k+1} = 0 \quad \text{on } \partial D \cap \partial D_2, \\ u_2^{k+1} = \mu^{k+1} \quad \text{on } \Gamma^{\text{out}}, \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial \mathbf{n}} = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} \quad \text{on } \Gamma^{\text{in}} \cup \Gamma^0, \end{cases}$$

with

$$\lambda^k := \theta' u^k_{2|\Gamma^{\text{in}} \cup \Gamma^0} + (1 - \theta') u^k_{1|\Gamma^{\text{in}} \cup \Gamma^0} \quad \text{on} \quad \Gamma^{\text{in}} \cup \Gamma^0$$

and

$$\mu^{k+1} := \theta'' u^{k+1}_{1|\Gamma^{\text{out}}} + (1 - \theta'') u^k_{2|\Gamma^{\text{out}}} \qquad \text{on} \quad \Gamma^{\text{out}},$$

where

$$\partial D^0 := \{\mathbf{x} \in \partial D : \quad \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0\}.$$

We recall that θ' and θ'' are two positive parameters, that are used to allow possible *under-relaxation* (if needed) to ensure convergence. It is also possible to introduce the ARN *adaptive Robin-Neumann* method, that is very similar: on each sub-domain iteration the Dirichlet interface condition is replaced with a Robin one. We observe that these methods need the knowledge (often non easily available) of the inflow and outflow regions. Furthermore it is very difficult to apply these methods in a vector equation, as the one we want to study.

Some very recent results on substructuring methods are that one proposed by Achdou and Nataf [AN97] and Achdou, Nataf, Le Tallec and Vidrascu [ANLTV98]. In these papers the methods used are essentially different: a Fourier analysis is provided. The authors also claim that the same method can be extended to multi-domain problems, provided the sub-domain are of very simple geometry. In particular, the analysis, that they propose, is possible only on rectangular domains, with rectangular mesh, provided some rather restrictive assumptions linking $\mathbf{b}, \varepsilon$ and h are satisfied.

4.4.2 Coercive methods

In this section we show some coercive methods for advection-diffusion equations. In particular the methods we propose easily extend to *advection-diffusion systems* of the form

$$\begin{cases} -\varepsilon\,\Delta\mathbf{u} + \displaystyle\sum_{j=1}^{d}\frac{\partial}{\partial x_j}(B^{(j)}\mathbf{u}) + A_0\mathbf{u} = \mathbf{f} \quad \text{in } D, \\ \\ \mathbf{u} = \mathbf{0} \quad \text{on } \partial D, \end{cases}$$

where $B^{(j)}$, $j = 1, \ldots, d$, and A_0 are $d \times d$ *symmetric* matrices. We assume that the coefficients of $B^{(j)}$ and A_0 belong to $L^\infty(D)$ and that the coefficients of $\sum_j \partial B^{(j)}/\partial x_j$ belong to $L^\infty(D)$. Moreover, to ensure coercivity of the problem, we require that the following matrix

$$M(\mathbf{x}) := \frac{1}{2}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}B^{(j)}(\mathbf{x}) + A_0(\mathbf{x})$$

is positive semi-definite for almost each $\mathbf{x} \in D$. We can introduce the associated bilinear form

$$a^{\#}(\mathbf{w},\mathbf{v}) := \int_D \Big[\varepsilon\nabla\mathbf{w}\cdot\nabla\mathbf{v} + (M\,\mathbf{w})\cdot\mathbf{v}\Big] + \frac{1}{2}\int_D\sum_{j=1}^{d}[(B^{(j)}\mathbf{v})\cdot\frac{\partial}{\partial x_j}\mathbf{w} - (B^{(j)}\frac{\partial}{\partial x_j}\mathbf{v})\cdot\mathbf{w}],$$

which can be used to rewrite the Dirichlet boundary value problem in the variational form

$$\text{find } \mathbf{u} \in (H_0^1(D))^d: \quad a^{\#}(\mathbf{u},\mathbf{v}) = \int_D \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in (H_0^1(D))^d,$$

and to define the Steklov-Poincaré operators to pass to an interface problem and a domain decomposition procedure.

We start by recalling some iterative methods for advection-diffusion equations. These methods are based on the introduction of symmetric preconditioners. Together with the bilinear form

$$a^{\#}(w,v) := \int_D \left[\varepsilon\nabla w\nabla v + \left(\frac{1}{2}\operatorname{div}\mathbf{b} + a_0\right)w\,v\right]d\mathbf{x} + \frac{1}{2}\int_D (v\,\mathbf{b}\cdot\nabla w - w\,\mathbf{b}\cdot\nabla v)\,d\mathbf{x},$$

associated to the scalar advection-diffusion operator, we introduce the bilinear form associated to its symmetric part

$$a^{\mathrm{s}}(w,v) := \int_D \left[\varepsilon\nabla w\nabla v + \left(\frac{1}{2}\operatorname{div}\mathbf{b} + a_0\right)w\,v\right]d\mathbf{x}.$$

We define in an obvious way the local forms corresponding to $a^{\#}(\,.\,,\,.\,)$ and $a^{\mathrm{s}}(\,.\,,\,.\,)$. Furthermore we denote by Q_i the Steklov-Poincaré operators associated to the bilinear forms $a_i^{\mathrm{s}}(\,.\,,\,.\,)$, *i.e.*,

$$\langle Q_i\eta,\mu\rangle = a_i^{\mathrm{s}}(H_i^{\mathrm{s}}\eta, H_i^{\mathrm{s}}\mu) \qquad \forall\,\eta,\mu \in \Lambda,$$

where H^{s} is the extension operator defined by the bilinear form $a_i^{\mathrm{s}}(\,.\,,\,.\,)$ through

$$\text{find } H_i^{\mathrm{s}}\lambda \in V_i: \quad a_i^{\mathrm{s}}(H_i^{\mathrm{s}}\lambda, v_i) = 0 \quad \forall\,v_i \in V_i^0, \qquad \text{with} \quad H_i^{\mathrm{s}}\lambda_{|\Lambda} = \lambda.$$

The operators Q_i are continuous and coercive, as is easily checked. By using an idea taken from the paper by Cai and Widlund [CW92] we use Q_i as preconditioners for the interface problem $S\lambda = \chi$.

A modified Dirichlet-Neumann method

In the first method we propose (that to our knowledge was not previously considered) we use $P_{MDN} := Q_2$ as preconditioner for $S = S_1 + S_2$; we are faced with the iteration

$$\lambda^{k+1} = \lambda^k + \theta Q_2^{-1}(\chi - S\lambda^k). \tag{4.56}$$

For every $k \geq 0$, we have to solve for $i = 1, 2$,

$$\begin{cases} -\varepsilon\, \Delta u_i^{k+1} + \nabla \cdot (\mathbf{b} u_i^{k+1}) + a_0 u_i^{k+1} = f \quad \text{in } D_i, \\ u_i^{k+1} = 0 \quad \text{on } \partial D_i \cap \partial D, \\ u_i^{k+1} = \lambda^k \quad \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} -\varepsilon\, \Delta \psi_2^{k+1} + \left(\frac{1}{2}\nabla \cdot \mathbf{b} + a_0\right) \psi_2^{k+1} = 0 \quad \text{in } D_2, \\ \psi_2^{k+1} = 0 \quad \text{on } \partial D_2 \cap \partial D, \\ \varepsilon \frac{\partial \psi_2^{k+1}}{\partial \mathbf{n}} = \varepsilon \frac{\partial u_1^{k+1}}{\partial \mathbf{n}} - \frac{1}{2}\mathbf{b} \cdot \mathbf{n}\, u_1^{k+1} - (\varepsilon \frac{\partial u_2^{k+1}}{\partial \mathbf{n}} - \frac{1}{2}\mathbf{b} \cdot \mathbf{n}\, u_2^{k+1}) \quad \text{on } \Gamma, \end{cases}$$

with

$$\lambda^{k+1} := \lambda^k + \theta\, \psi_{2|\Gamma}^{k+1},$$

where $\theta > 0$ is a relaxation parameter. We observe that this method corresponds to the following iteration by sub-domains. Given $\lambda_0 \in \Lambda$ and for every $k \geq 0$, find $u_i^{k+1} \in V_i$ for $i = 1, 2$

$$a_i^{\#}(u_i^{k+1}, v) = (f, v)_i \qquad \forall\, v \in V_i^0 \qquad \text{with } {u_i^{k+1}}_{|\Gamma} = \lambda^k.$$

Then solve (only in the sub-domain D_2): find $\psi_2^{k+1} \in V_2$ such that

$$a_2^{\mathrm{s}}(\psi_2^{k+1}, v) = (f, E_1 v_{|\Gamma})_1 - a_1(u_1^{k+1}, E_1^{\#} v_{|\Gamma}) + (f, E_2^{\#} v_{|\Gamma})_2 - a_2(u_2^{k+1}, E_2^{\#} v_{|\Gamma}) \quad \forall\, v \in V_2$$

and finally

$$\lambda^{k+1} := \lambda^k + \theta \psi_{2|\Gamma}^{k+1}.$$

The convergence of this method is based on the following abstract result, see Berselli and Saleri [BS99].

Theorem 4.4.1. *Let $(X, \|\,.\,\|_X)$ be a Hilbert space with dual X' and with duality pairing $\langle\, .\,, .\,\rangle$. Let $\mathcal{S}$ from X into its dual X' be a linear operator which splits as $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. Suppose that both $\mathcal{S}_i$ are continuous and coercive*

$$a)\ \exists\, \beta_i > 0: \quad \langle \mathcal{S}_i \eta, \mu \rangle \leq \beta_i \|\eta\|_X \|\mu\|_X \quad \forall\, \eta, \mu \in X \quad \text{for } i = 1, 2,$$

$$b)\ \exists\, \alpha_i > 0: \quad \langle \mathcal{S}_i \eta, \eta \rangle \geq \alpha_i \|\eta\|_X^2 \quad \forall\, \eta \in X \quad \text{for } i = 1, 2.$$

Let us suppose that $\mathcal{Q}_2$ is symmetric, continuous and coercive

$$c)\ \langle \mathcal{Q}_2 \eta, \mu \rangle = \langle \eta, \mathcal{Q}_2 \mu \rangle \quad \forall\, \eta, \mu \in X,$$

$$d)\ \exists\, \beta_2^s > 0: \quad \langle \mathcal{Q}_2 \eta, \mu \rangle \leq \beta_2^s \|\eta\| \|\mu\| \quad \forall\, \eta, \mu \in X$$

$$e)\ \exists\, \alpha_2^s > 0: \quad \langle \mathcal{Q}_2 \eta, \eta \rangle \geq \alpha_2^s \|\eta\|_X^2 \quad \forall\, \eta \in X.$$

Then there exists $\theta_{MDN} > 0$ such that, for each $\theta \in (0, \theta_{MDN})$ and for a given $\lambda^0 \in X$ and $\mathcal{F} \in X'$, the sequence $\lambda^{k+1} = \lambda^k + \theta \mathcal{Q}_2^{-1}(\mathcal{F} - \mathcal{S}\lambda^k)$ converges in X to the solution of $\mathcal{S}\lambda = \mathcal{F}$.

Proof. We only sketch out the proof, because it is similar to the one of Theorem 4.2.6. To prove the convergence we show that the map T_θ defined as $T_\theta\eta := \eta - \theta\, \mathcal{Q}_2^{-1}\mathcal{S}\eta$ is a strict contraction with respect to the norm $\| \,.\, \|_{\mathcal{Q}_2}$. We remark that, due to our choice, the preconditioner is symmetric and positive definite and induces the scalar product $\langle \eta, \mu \rangle_{\mathcal{Q}_2} = \langle \mathcal{Q}_2\eta, \mu \rangle$ and a norm $\|\eta\|^2_{\mathcal{Q}_2} = \langle \mathcal{Q}_2\eta, \eta \rangle$ which is equivalent to the norm of X :

$$\alpha_2^s \|\eta\|_X^2 \leq \|\eta\|^2_{\mathcal{Q}_2} \leq \beta_2^s \|\eta\|_X^2.$$

We calculate the $\mathcal{Q}_2$-norm of T_θ and we obtain

$$\|T_\theta\eta\|^2_{\mathcal{Q}_2} = \|\eta\|^2_{\mathcal{Q}_2} - 2\,\theta < (\mathcal{S}_1 + \mathcal{S}_2)\eta, \eta > + \theta^2\, \|\mathcal{Q}_2^{-1}(\mathcal{S}_1 + \mathcal{S}_2)\eta\|^2_{\mathcal{Q}_2}.$$

By recalling hypothesis b) we get that

$$\langle (\mathcal{S}_1 + \mathcal{S}_2)\eta, \eta \rangle \geq (\alpha_1 + \alpha_2)\|\eta\|_X^2 \geq \frac{(\alpha_1 + \alpha_2)}{\beta_2^s}\|\eta\|^2_{\mathcal{Q}_2} = C_1\|\eta\|^2_{\mathcal{Q}_2}.$$

Since $\mathcal{Q}_2^{-1}$ is continuous with continuity constant given by $1/\alpha_2^s$, we obtain that

$$\begin{aligned}
\|\mathcal{Q}_2^{-1}(\mathcal{S}_1 + \mathcal{S}_2)\eta\|^2_{\mathcal{Q}_2} &= \langle (S_1 + S_2)\eta, Q_2^{-1}(S_1 + S_2)\eta \rangle \\
&\leq \frac{(\beta_1 + \beta_2)^2}{\alpha_2^s}\|\eta\|_X^2 \\
&\leq \frac{(\beta_1 + \beta_2)^2}{(\alpha_2^s)^2}\|\eta\|^2_{Q_2} = C_2\|\eta\|^2_{\mathcal{Q}_2}
\end{aligned}$$

By collecting these inequalities we obtain

$$\|T_\theta\eta\|^2_{\mathcal{Q}_2} \leq (1 - 2\,\theta\, C_1 + \theta^2\, C_2)\|\eta\|^2_{\mathcal{Q}_2},$$

and by choosing $\theta \in (0, \theta_{MDN})$, with $\theta_{MDN} = C_1/C_2$, we have that T_θ is a strict contraction and its fixed point is the solution of the interface problem $\mathcal{S}\lambda = \mathcal{F}$. ☐

By applying this result with $\mathcal{S}_i = S_i$ and $\mathcal{Q}_i = Q_i$, for $i = 1, 2$, we obtain the following result, recall that the operators S_i and Q_i, for $i = 1, 2$, are continuous and coercive.

Corollary 4.4.2. *The modified Dirichlet-Neumann method converges at a rate which is independent of h.*

A modified Neumann-Neumann method

In this section we present the Richardson iterative method to solve the interface equation with

$$P_{MNN} := (\sigma_1 Q_1^{-1} + \sigma_2 Q_2^{-1})^{-1} \tag{4.57}$$

as preconditioner. This method was introduced and analyzed in Berselli and Saleri [BS99]. Our iteration can be described with a weak formulation as follows: given $\lambda^0 \in \Lambda$, find, for each $k \geq 0$, $u_i^{k+1} \in V_i$ such that

$$a_i(u_i^{k+1}, v) = (f, v)_i \qquad \forall\, v \in V_i^0 \quad \text{with } u_i^{k+1}{}_{|\Lambda} = \lambda^k.$$

Then find $\psi_i^{k+1} \in V_i$ such that

$$a_i^s(\psi_i^{k+1}, v) = (f, E_1^{\#} v_{|\Gamma})_1 - a_1(u_1^{k+1}, E_1^{\#} v_{|\Gamma}) + (f, E_2^{\#} v_{|\Gamma})_2 - a_2(u_2^{k+1}, E_2 v_{|\Gamma}) \forall v \in V_i$$

and finally

$$\lambda^{k+1} := \lambda^k - \theta(\sigma_1 \psi_{1|\Gamma}^k - \sigma_2 \psi_{2|\Gamma}^k).$$

Again the convergence is based on the following abstract theorem.

Theorem 4.4.3. *Let $(X, \|\,.\,\|_X)$ be a Hilbert space with dual X'. Let $\mathcal{S} : X \to X'$ be a linear operator which splits as $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. Suppose that both $\mathcal{S}_i$ are continuous and coercive*

$$a)\ \exists\, \beta_i > 0 : \quad \langle \mathcal{S}_i \eta, \mu \rangle \le \beta_i \|\eta\|_X \|\mu\|_X \quad \forall\, \eta, \mu \in X \quad \text{for } i = 1, 2,$$

$$b)\ \exists\, \alpha_i > 0 : \quad \langle \mathcal{S}_i \eta, \eta \rangle \ge \alpha_i \|\eta\|_X^2 \quad \forall\, \eta \in X \quad \text{for } i = 1, 2.$$

Let us suppose that $\mathcal{Q}_i$, for $i = 1, 2$ are both symmetric, continuous and coercive

$$c)\ \langle \mathcal{Q}_i \eta, \mu \rangle = \langle \eta, \mathcal{Q}_i \mu \rangle \quad \forall\, \eta, \mu \in X,$$

$$d)\ \exists\, \beta_i^s > 0 : \quad \langle \mathcal{Q}_i \eta, \mu \rangle \le \beta_i^s \|\eta\|_X \|\mu\|_X \quad \forall\, \eta, \mu \in X \quad \text{for } i = 1, 2,$$

$$e)\ \exists\, \alpha_i^s > 0 : \quad \langle \mathcal{Q}_i \eta, \eta \rangle \ge \alpha_i^s \|\eta\|_X^2 \quad \forall\, \eta \in X \quad \text{for } i = 1, 2.$$

Then, for any choice of the averaging parameters $\sigma_i > 0$, there exists $\theta_{MNN} > 0$ such that, for each $\theta \in (0, \theta_{MNN})$ and for a given $\lambda^0 \in X$ and $\mathcal{F} \in X'$, the sequence

$$\lambda^{k+1} = \lambda^k + \theta(\sigma_1 \mathcal{Q}_1^{-1} + \sigma_2 \mathcal{Q}_2^{-1})(\mathcal{F} - \mathcal{S}\lambda^k) \tag{4.58}$$

converges in X to the solution of $\mathcal{S}\lambda = \mathcal{F}$.

Proof. To prove the convergence, let us show that the map T_θ defined as

$$T_\theta \eta := \eta - \theta(\sigma_1 Q_1^{-1} + \sigma_2 Q_2^{-1}) Q \eta$$

is a strict contraction with respect to the norm $\|\,.\,\|_{P_{MNN}}$ below. We consider the scalar product[11] $\langle \eta, \mu \rangle_{P_{MNN}} := \langle P_{MNN} \eta, \mu \rangle$ induced by P_{MNN} and the corresponding norm $\|\eta\|_{P_{MNN}}^2 = \langle P_{MNN} \eta, \eta \rangle$, which is equivalent to the norm of X :

$$\alpha_{P_{MNN}} \|\eta\|_X^2 \le \|\eta\|_{P_{MNN}}^2 \le \beta_{P_{MNN}} \|\eta\|_X^2,$$

where $\alpha_{P_{MNN}}$ and $\beta_{P_{MNN}}$ are, respectively, the coercivity and continuity constants of the operator P_{MNN}. We calculate the P_{MNN}-norm of T_θ and we obtain

$$\|T_\theta \eta\|_{P_{MNN}}^2 = \|\eta\|_{P_{MNN}}^2 - 2\,\theta < (S_1 + S_2)\eta, \eta > +\theta^2 \, \|P_{MNN}^{-1}(S_1 + S_2)\eta\|_{P_{MNN}}^2.$$

By recalling hypothesis b) we get that

$$\langle (S_1 + S_2)\eta, \eta \rangle \ge (\alpha_1 + \alpha_2) \|\eta\|_X^2 \ge \frac{(\alpha_1 + \alpha_2)}{\beta_{P_{MNN}}} \|\eta\|_{P_{MNN}}^2 = C_1 \|\eta\|_{P_{MNN}}^2.$$

[11] Recall the results of Remark 4.2.10.

Since P_{MNN}^{-1} is continuous, with continuity constant given by $1/\alpha_{P_{MNN}}$,we obtain that

$$\begin{aligned} \|P_{MNN}^{-1}(S_1+S_2)\eta\|^2_{P_{MNN}} &= \; < (S_1+S_2)\eta, P_{MNN}^{-1}(S_1+S_2)\eta > \\ &\le \frac{(\beta_1+\beta_2)^2}{\alpha_{P_{MNN}}}\|\eta\|^2_X \\ &\le \frac{(\beta_1+\beta_2)^2}{(\alpha_{P_{MNN}})^2}\|\eta\|^2_{P_{MNN}} = C_2\|\eta\|^2_{P_{MNN}}. \end{aligned}$$

By collecting these inequalities, we obtain

$$\|T_\theta \eta\|^2_{P_{MNN}} \le (1 - 2\,\theta\, C_1 + \theta^2\, C_2)\|\eta\|^2_{P_{MNN}}$$

and by choosing $\theta \in (0, \theta_{MNN})$, with $\theta_{MNN} = C_1/C_2$, we have that T_θ is a strict contraction. □

As in the previous Corollary 4.4.2 we can infer the following.

Corollary 4.4.4. *The modified Neumann-Neumann method converges at a rate which is independent of h.*

The Neumann-Neumann method involves at each step the solution of two Dirichlet and two mixed Dirichlet-Neumann problems. In this context, due to the presence of non-zero first order terms, they are indeed Dirichlet-Robin problems. Heuristically we can see that the Dirichlet step enforces the continuity of the solution and the Neumann step the continuity of the co-normal derivative. In this way the trace and the co-normal condition on the interface are satisfied in the limit $k \to +\infty$.

In the modified Neumann-Neumann method the second step involves an approximation of the differential operator L_ε and the co-normal derivative is glued as in a transmission problem. We proved in Theorem 4.4.3 that convergence holds. From the numerical point of view, we observe that the modified method may involve less calculation than the original one since the second step involves the inversion of "better" matrices. On the other hand, an increasing of the number of iteration needed to have convergence should be expected, because we do not use the complete operator as preconditioner.

Some numerical Experiments

We recall that in the discretization of advection-dominated problems some stabilization procedure *à la* SUPG (Streamline Upwind Petrov Galerkin) is needed, because if the Péclet number is greater than one the Faedo-Galerkin method is unstable. We recall that the Péclet number is defined by

$$\mathrm{Pe} := \frac{\|\mathbf{b}\|_{\infty,\Omega}\, h}{2\varepsilon}.$$

To overcome this instability problem a standard method is the GALS (Galerkin/Least Squares) stabilization method, see Hughes, Franca and Hulbert [HFH89]. In this case the bilinear form $a^{\#}(\,.\,,\,.\,)$ must be substituted by

$$a_h^{\#}(u_h, v_h) := a^{\#}(u_h, v_h) + \sum_{K\in\mathcal{T}_h} \tau_K (L_\varepsilon u_h, L_\varepsilon v_h)_K,$$

where the constants τ_K are positive parameters and $(\,.\,,\,.\,)_K$ is the $\mathrm{L}^2(K)$ scalar product on each triangle K of the triangulation $\mathcal{T}_h$. The term related to the external force must be changed as

$$\mathcal{F}_h := (f, v_h) + \sum_{K\in\mathcal{T}_h} \tau_K (f, L_\varepsilon v_h)_K,$$

in order to have consistency. A typical result that can be obtained by comparing the modified Neumann-Neumann method with the classical Neumann-Neumann method is the following one. We set $D = (0,1)\times(0,1)$, the advective field $\mathbf{b} = (1,1)^{\mathrm{T}}$ and $a_0(\mathbf{x}) \equiv f(\mathbf{x}) \equiv 0$. The boundary condition are $u = 1$ on the side $\{x_1 = 0\}$ and vanishing on the other three sides. The computational domain is sub-divided in $D_1 := (0,1/2)\times(0,1)$ and $D_2 := (1/2,1)\times(0,1)$. In the following table we collect the number of iterations (NIT) needed to arrive at convergence that is fixed at a tolerance for L^∞-norm of the residual of 10^{-5}. We denoted by Cpu the `CPU`-time used by MATLAB™, see the conclusions for a discussion of these results. In this test case the advective field has non-vanishing

	NN				MNN				$\mathrm{Cpu}_{MNN}/\mathrm{Cpu}_{NN}$			
$\varepsilon \setminus h$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
1	2	2	2	2	7	6	6	6	0.74	0.78	0.91	1.25
10^{-1}	4	3	3	3	11	14	16	18	0.75	0.86	1.24	2.30
10^{-2}	3	3	3	4	16	24	34	59	0.78	0.99	1.95	5.78

components both in the parallel and orthogonal direction, relative to the interface. The presence of parallel components is one of the most interesting features of a test case for advection diffusion equations. Further numerical results relative to the modified Neumann-Neumann method can be found in Berselli and Saleri [BS99].

We now propose another family of preconditioners for advection-diffusion equations. The following preconditioners are labelled by "γ", which is a nonnegative parameter, used to have iterative methods for which convergence results can be proven.

γ-Dirichlet-Robin method

We present the following "iteration by sub-domain" scheme for solving (4.54)-(4.55), which will be called *γ-Dirichlet-Robin* (γ-DR), see Alonso, Trotta and Valli [ATV98]. The scheme reads: let λ^0 be given in Λ, for each $k \geq 0$ solve

$$\begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } D_1, \\ u_1^{k+1} = 0 & \text{on } \partial D_1 \cap \partial D, \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } D_2, \\ u_2^{k+1} = 0 & \text{on } \partial D_2 \cap \partial D, \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial \mathbf{n}} - \left(\dfrac{1}{2}\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_2^{k+1} = \\ \qquad = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} - \left(\dfrac{1}{2}\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_1^{k+1} & \text{on } \Gamma, \end{cases}$$

with

$$\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma,$$

where $\theta \neq 0$ is a relaxation parameter introduced to accelerate convergence. Here $\gamma = \gamma(\mathbf{x})$ is a given function belonging to $L^\infty(\Gamma)$, satisfying $\gamma(\mathbf{x}) \geq 0$ for almost each $\mathbf{x} \in \Gamma$; the rate of convergence of the method is in principle dependent on the choice of this function. The iterative scheme reads, in a variational formulation, as: find $u_1^{k+1} \in V_1$ such that

$$a_1^\#(u_1^{k+1}, v_1) = \int_{D_1} f v_1 \qquad \forall\, v_1 \in V_1^0(D_1), \quad \text{with} \quad u_{1|\Gamma}^{k+1} = \lambda^k. \tag{4.59}$$

Then find $u_2^{k+1} \in V_2$ such that

$$\begin{aligned} a_2^\#(u_2^{k+1}, v_2) + \int_\Gamma \gamma u_{2|\Gamma}^{k+1} v_{2|\Gamma} &= \int_{D_2} f v_2 + \int_{D_1} f \mathcal{R}_1 v_{2|\Gamma} - \\ &- a_1^\#(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \int_\Gamma \gamma u_{1|\Gamma}^{k+1} v_{2|\Gamma} \quad \forall\, v_2 \in V_2, \end{aligned} \tag{4.60}$$

where $\mathcal{R}_i$ denotes again any extension operator from Λ to V_i. Finally set

$$\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma. \tag{4.61}$$

The problem for u_1 is coercive[12] in $H_0^1(D_1)$, whereas the problem for u_2 is coercive in V_2, for any $\gamma \geq 0$. Hence the iterative scheme is correctly defined.

Remark 4.4.5. *We remark that this method is different from the ADN scheme proposed by Gastaldi, Gastaldi and Quarteroni [GGQ96]. Here the Dirichlet boundary condition is imposed on the whole interface Γ, no matter if it is an inflow or an outflow boundary. If the flow has always the same direction on Γ, say $\mathbf{b} \cdot \mathbf{n} < 0$ on Γ, by choosing $\gamma = -\frac{1}{2}\mathbf{b} \cdot \mathbf{n}$, we recover the ADN scheme.*

It is possible to propose a modified algorithm and hereafter $\gamma(\mathbf{x}) = \gamma \geq 0$. By setting $((\eta, \mu))_\Lambda$ the scalar product in the trace space $\Lambda = H_{00}^{1/2}(\Gamma)$, we solve, in the second step, the following problem: find $u_2^{k+1} \in V_2$ such that

$$\begin{aligned} a_2^\#(u_2^{k+1}, v_2) + \gamma((u_{2|\Gamma}^{k+1}, v_{2|\Gamma}))_\Lambda &= \int_{D_2} f v_2 + \int_{D_1} f \mathcal{R}_1 v_{2|\Gamma} \\ &- a_1^\#(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \gamma((u_{1|\Gamma}^{k+1}, v_{2|\Gamma}))_\Lambda \qquad \forall\, v_2 \in V_2. \end{aligned} \tag{4.62}$$

Remark 4.4.6. *We remark that this latter scheme* (4.59)-(4.62)-(4.61) *is more difficult to implement than* (4.59)-(4.60)-(4.61)*, but it has better convergence properties.*

To prove the convergence of the latter scheme we apply the results of Theorem 4.2.6. In this problem the Steklov-Poincaré operators (S_i, for $i = 1, 2$) must be defined in a somewhat different way.

For each $\eta, \mu \in \Lambda$, we define the Steklov-Poincaré operators $S_{i,\gamma} : \Lambda \to \Lambda'$ as

$$\langle S_{1,\gamma}\eta, \mu\rangle = a_1^\#(E_1^\#\eta, E_1^\#\mu) - \gamma((\eta, \mu))_\Lambda, \tag{4.63}$$

$$\langle S_{2,\gamma}\eta, \mu\rangle = a_2^\#(E_2^\#\eta, E_2^\#\mu) + \gamma((\eta, \mu))_\Lambda, \tag{4.64}$$

[12] Recall that is due do the appropriate choice of the bilinear form.

and set, as usual, $S = S_{1,\gamma} + S_{2,\gamma}$. The reader can compare the definition (4.63)-(4.64) above, with the standard one (4.33), we previously used. It is clear, as we shall see in a while, that this definition makes the coercivity constant of $S_{2,\gamma}$ bigger than the corredponding one of the "standard"operator S_2 defined in (4.33).

By recalling that $\tilde{k}_i\|\eta\|_\Lambda \le \|E_i^{\#}\eta\|_{i,D_i} \le k_i\|\eta\|_\Lambda$, we get that the operator $S_{1,\gamma}$ turns out to be continuous as

$$\langle S_1\eta, \mu\rangle \le (\beta_1^{\#}k_1^2 + \gamma)\|\eta\|_\Lambda\|\mu\|_\Lambda.$$

Moreover, for each $\gamma \ge 0$, $S_{2,\gamma}$ is continuous and coercive, as

$$\langle S_{2,\gamma}\eta, \mu\rangle \le (\beta_2^{\#}k_2^2 + \gamma)\|\eta\|_\Lambda\|\mu\|_\Lambda$$

and

$$\langle S_{2,\gamma}\eta, \eta\rangle \ge (\alpha_2^{\#}\tilde{k}_2^2 + \gamma)\|\eta\|_\Lambda^2.$$

It is easily shown that the iteration operator for the γ-DR scheme is given by $T_\theta := I - \theta S_{2,\gamma}^{-1}S$. In fact, the γ-DR scheme is a preconditioned Richardson method with preconditioner given by $P := S_{2,\gamma}$. To apply the abstract convergence Theorem 4.2.6 we have to check that condition d)

$$\text{there exists a constant } \kappa^* > 0 : \qquad \langle \mathcal{S}_2\eta, \mathcal{S}_2^{-1}\mathcal{S}\eta\rangle + \langle \mathcal{S}\eta, \eta\rangle \ge \kappa^*\|\eta\|_X^2 \qquad \forall\, \eta \in X$$

is satisfied if $\mathcal{S}_i = S_{i,\gamma}$ and $X = \Lambda$; the other conditions are trivially satisfied. We have the following result.

Theorem 4.4.7. *There exists $\gamma^* \ge 0$ and $\theta_{\gamma-DR} > 0$, such that, for each $\gamma \ge \gamma^*$ and for each λ^0 in Λ, the γ-Dirichlet-Robin iterative scheme (4.59)-(4.62)-(4.61) is convergent in Λ, provided that θ belongs to $(0, \theta_{\gamma-DR})$.*

First we observe that the operators $S_{i,\gamma}$ are continuous, with continuity constants given by $\beta_i^{\#}k_i^2 + \gamma$, while $S_{2,\gamma}$ is coercive, with coercivity constant given by $\alpha_2 = \alpha_2^{\#}\tilde{k}_2^2 + \gamma$. Then

$$\begin{aligned}\langle S_{2,\gamma}\eta, S_{2,\gamma}^{-1}S\eta\rangle + \langle S\eta, \eta\rangle &= 2\langle S\eta, \eta\rangle + \langle S_{2,\gamma}\eta, S_{2,\gamma}^{-1}S\eta\rangle - \langle S\eta, \eta\rangle \\ &\ge 2\langle S\eta, \eta\rangle - |\langle S_{2,\gamma}\eta, S_{2,\gamma}^{-1}S\eta\rangle - \langle S\eta, \eta\rangle|\end{aligned}$$

and, by using the same techniques used in the study of convergence of the Dirichlet-Neumann method of Section 4.3.1, we get that

$$|\langle S_{2,\gamma}\eta, S_{2,\gamma}^{-1}S\eta\rangle - \langle S\eta, \eta\rangle| \le 2\|\mathbf{b}\|_{(L^\infty(D_2))^d}k_2^2\|\eta\|_\Lambda\|S_{2,\gamma}^{-1}S\eta\|_\Lambda.$$

By setting

$$\beta := \beta_1^{\#}k_1^2 + \beta_2^{\#}k_2^2 \qquad \text{and} \qquad \alpha := \alpha_1^{\#}\tilde{k}_1^2 + \alpha_2^{\#}\tilde{k}_2^2,$$

(they are the continuity and coercivity constant of S, respectively) we have that

$$\begin{aligned}\langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta, \eta\rangle &\ge 2\alpha\|\eta\|_\Lambda^2 - 2\|\mathbf{b}\|_{(L^\infty(D_2))^d}k_2^2\tfrac{\beta}{\alpha_2}\|\eta\|_\Lambda^2 \\ &= 2\left(\alpha - \|\mathbf{b}\|_{(L^\infty(D_2))^d}k_2^2\tfrac{\beta}{\alpha_2}\right)\|\eta\|_\Lambda^2,\end{aligned}$$

and the assumption d) is satisfied if

$$\kappa^* := 2\left(\alpha - \|\mathbf{b}\|_{(L^\infty(D_2))^d}k_2^2\frac{\beta}{\alpha_2}\right) > 0.$$

By recalling the definition of α_2, to get convergence it is sufficient to take

$$\gamma \geq \gamma^* := \begin{cases} 0 & \text{for } \alpha_2^{\#} \tilde{k}_2^2 > \|\mathbf{b}\|_{(L^\infty(D_2))^d} k_2^2 \dfrac{\beta}{\alpha}, \\ > \|\mathbf{b}\|_{(L^\infty(D_2))^d} k_2^2 \dfrac{\beta}{\alpha} - \alpha_2^{\#} \tilde{k}_2^2 & \text{for } \alpha_2^{\#} \tilde{k}_2^2 \leq \|\mathbf{b}\|_{(L^\infty(D_2))^d} k_2^2 \dfrac{\beta}{\alpha}. \end{cases}$$

The convergence of the method can be proved also at a finite dimensional level. We observe that the convergence depends only on the constants $\beta_i^{\#}, \alpha_2^{\#}, k_i$ and $\tilde{k}_i$. If we consider a finite element approximation all the constant except the k_i are independent of the mesh size h.

Remark 4.4.8. *In the finite dimensional case, the convergence of the first γ-DR iterative scheme (4.59)-(4.60)-(4.61) can be proved by a similar argument. In fact, for discrete functions all the norms are equivalent, hence there exists a constant $\kappa_h > 0$ such that*

$$\kappa_h \|\eta_h\|_\Lambda^2 \leq \|\eta_h\|_{0,\Gamma}^2 \quad \forall \eta_h \in \Lambda_h,$$

where $\|\cdot\|_{0,\Gamma}$ denotes the norm in $L^2(\Gamma)$. By using this estimate, we only have to substitute the coercivity constant $\alpha_2 := \alpha_2^{\#} \tilde{k}_2^2 + \gamma$ of S_2 with

$$\alpha_{2,h} := \alpha_2^{\#} \tilde{k}_2^2 + \gamma \kappa_h \ ,$$

and convergence is achieved for $\inf_D \gamma \geq \gamma_h^* := \gamma^*/\kappa_h$. In this case we are not in a condition to prove that the iterative procedure introduces an optimal preconditioner. However, the numerical results show that also in this case the rate of convergence is independent of h, see Alonso, Trotta and Valli [ATV98], Section 6. Furthermore, the convergence result in Theorem 4.4.7 holds only for γ sufficiently large. Numerical experiments show that the γ-DR iterative scheme indeed converges for any $\gamma \geq 0$ and in particular for $\gamma = 0$.

γ-Robin-Robin method

A variant of the Robin method is the so called γ-Robin-Robin method defined as follows: given λ^0 in $L^2(\Gamma)$, for each $k \geq 0$ solve

$$\begin{cases} L_\varepsilon u_1^{k+1} = f \quad \text{in } D_1, \\ u_1^{k+1} = 0 \quad \text{on } \partial D_1 \cap \partial D, \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} - \left(\dfrac{1}{2} \mathbf{b} \cdot \mathbf{n} - \gamma\right) u_1^{k+1} = \lambda^k \quad \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} L_\varepsilon u_2^{k+1} = f \quad \text{in } D_2, \\ u_2^{k+1} = 0 \quad \text{on } \partial D_2 \cap \partial D, \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial \mathbf{n}} - \left(\dfrac{1}{2} \mathbf{b} \cdot \mathbf{n} + \gamma\right) u_2^{k+1} = \\ = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial \mathbf{n}} - \left(\dfrac{1}{2} \mathbf{b} \cdot \mathbf{n} + \gamma\right) u_1^{k+1} \text{ on } \Gamma, \end{cases}$$

where

$$\lambda^{k+1} := \varepsilon \frac{\partial u_2^{k+1}}{\partial \mathbf{n}} - \left(\frac{1}{2} \mathbf{b} \cdot \mathbf{n} - \gamma\right) u_2^{k+1} \quad \text{on } \Gamma$$

and $\gamma = \gamma(\mathbf{x})$ is a given function in $L^\infty(\Gamma)$ satisfying $\gamma(\mathbf{x}) \geq \hat{\gamma} > 0$ for almost each $\mathbf{x} \in \Gamma$. The convergence of this method is proved in Section 4 of Alonso, Trotta and Valli [ATV98]. By using a technique very similar to that one of Theorem 4.2.14 it follows the following result.

Theorem 4.4.9. *Assume either that D is a Lipschitz polygonal domain or that $\partial D \in C^2$. Suppose moreover that $\mathbf{b}_{|\Gamma} \in (L^\infty(\Gamma))^d$. For each $\lambda^0 \in L^2(\Gamma)$ and for each $i = 1, 2$, the sequences u_i^k converge in $H^1(D_i)$ to $u_{|D_i}$, which is the restriction of the solution u of the homogeneous Dirichlet problem* (4.54)-(4.55).

A modified γ-Robin-Robin method

We now introduce a different $\gamma-$Robin-Robin method that, to the author knowledge, was not previously considered. A convergence result will be proved. The modified γ-Robin-Robin method is very similar to the γ-Robin-Robin method. The main difference is that in the variational formulation a term involving the $H^{1/2}_{00}(\Gamma)$-scalar product is added, instead of the $L^2(\Gamma)$ term of the γ-Robin-Robin method. This method is very similar to the Neumann-Neumann, previously analyzed. The iteration reads as find $u_i^{k+1} \in V_i$ such that

$$a_i^{\#}(u_i^{k+1}, v_1) = \int_{D_i} f v_i \qquad \forall\, v_i \in V_i^0 \quad \text{with } {u^{k+1}}_{i|\Gamma} = \lambda^k, \tag{4.65}$$

then find $\psi_1^{k+1} \in V_1$ such that

$$\begin{cases} a_1^{\#}(\psi_1^{k+1}, v_1) + \gamma\,((\psi_{1|\Gamma}^{k+1}, v_{1|\Gamma}))_\Lambda = 0 \qquad \forall v_1 \in V_1^0 \\ a_1^{\#}(\psi_1^{k+1}, \mathcal{R}_1\mu) + \gamma\,((\psi_{1|\Gamma}^{k+1}, \mu))_\Lambda = -\int_{D_1} f\mathcal{R}_1\mu\, d\mathbf{x} - \int_{D_2} f\mathcal{R}_2\mu\, d\mathbf{x} \\ \qquad\qquad + a_1^{\#}(u_1^{k+1}, \mathcal{R}_1\mu) + a_2^{\#}(u_2^{k+1}, \mathcal{R}_2\mu) \quad \forall \mu \in \Lambda, \end{cases} \tag{4.66}$$

then find $\psi_2^{k+1} \in V_2$ such that

$$\begin{cases} a_2^{\#}(\psi_2^{k+1}, v_2) + \gamma\,((\psi_{2|\Gamma}^{k+1}, v_{2|\Gamma}))_\Lambda = 0 \qquad \forall v_2 \in V_2^0 \\ a_2^{\#}(\psi_2^{k+1}, \mathcal{R}_2\mu) + \gamma\,((\psi_{2|\Gamma}^{k+1}, \mu))_\Lambda = \int_{D_1} f\mathcal{R}_1\mu\, d\mathbf{x} + \int_{D_2} f\mathcal{R}_2\mu\, d\mathbf{x} \\ \qquad\qquad - a_1^{\#}(u_1^{k+1}, \mathcal{R}_1\mu) - a_2^{\#}(u_2^{k+1}, \mathcal{R}_2\mu) \quad \forall \mu \in \Lambda \end{cases} \tag{4.67}$$

and finally

$$\lambda^{k+1} = \lambda^k + \theta(\sigma_1\psi_{1|\Gamma}^{k+1} - \sigma_2\psi_{2|\Gamma}^{k+1}). \tag{4.68}$$

We suppose that γ is a positive constant and to pass to an interface problem we define the operators $\widehat{S}_{i,\gamma}$, for $i = 1, 2$, and from Λ into Λ', as

$$\langle \widehat{S}_{i,\gamma}\eta, \mu\rangle := \langle S_i\eta, \mu\rangle + \gamma((\eta, \mu)).$$

Remark 4.4.10. *We observe that the operator $\widehat{S}_{2,\gamma}$ equals the operator $S_{2,\gamma}$ introduced in the γ-DR method, while $\widehat{S}_{1,\gamma}$ is different from $S_{1,\gamma}$; compare the definition of the "hatted" operators with* (4.63)-(4.64)*. We observe that in this case $\widehat{S}_{1,\gamma} + \widehat{S}_{2,\gamma} \neq S$.*

We also observe that the operators $\widehat{S}_{i,\gamma}$ are continuos and coercive with constants related to that ones of S_i (recall that they are α_i and β_i) as follows

$$\langle \widehat{S}_{i,\gamma}\eta, \mu\rangle \leq (\beta_i + \gamma)\|\eta\|_X\|\mu\|_X \quad \forall\, \eta, \mu \in X, \tag{4.69}$$

$$\langle \widehat{S}_{i,\gamma}\eta, \eta\rangle \geq (\alpha_i + \gamma)\|\eta\|_X^2 \quad \forall\, \eta \in X, \tag{4.70}$$

We introduce the γ-Robin-Robin preconditioner $P_{\gamma-mRR}$ given by

$$P_{\gamma-mRR} := \left(\sigma_1 S_{1,\gamma}^{-1} + \sigma_2 S_{2,\gamma}^{-1}\right)^{-1}$$

and the previous iteration by subdomain (4.65)-(4.66)-(4.67) -(4.68) can be described by the following Richardson preconditioned iteration

$$\lambda^{k+1} = \lambda^k + \theta(\sigma_1 \widehat{S}_{1,\gamma}^{-1} + \sigma_2 \widehat{S}_{2,\gamma}^{-1})(\chi - S\lambda^k). \tag{4.71}$$

To avoid ugly formulas we set $P := P_{\gamma-mRR}$ and we have the following convergence theorem.

Theorem 4.4.11. *There exists $\gamma^* > 0$ and $\theta_{\gamma-MRR} > 0$ such that, for each $\gamma \geq \gamma^*$ and for each λ^0 in Λ, the modified γ-Robin-Robin iterative scheme (4.71) is convergent in Λ, provided that θ belongs to $(0, \theta_{\gamma-MRR})$.*

Proof. To prove the convergence we shall use Theorem 4.2.11 and in particular we have only to check that condition c) holds, because the $\widehat{S}_{i,\gamma}$'s are continuous and coercive and consequently P inherits these properties. By setting $\rho_i := \widehat{S}_{i,\gamma}^{-1}P\eta$ and $\xi_i := \widehat{S}_{i,\gamma}^{-1}P\mu$, for $i = 1, 2$, we have $P\eta = \widehat{S}_{1,\gamma}\,\rho_1 = \widehat{S}_{2,\gamma}\,\rho_2$, $P\mu = \widehat{S}_{1,\gamma}\,\xi_1 = \widehat{S}_{2,\gamma}\,\xi_2$,

$$\eta = P^{-1}P\eta = \sigma_1\rho_1 + \sigma_2\rho_2$$

and similarly

$$\mu = \sigma_1\xi_1 + \sigma_2\xi_2.$$

Therefore

$$\begin{aligned}
\langle P\eta, \mu\rangle - \langle P\mu, \eta\rangle &= \langle P\eta, \sigma_1\xi_1 + \sigma_2\xi_2\rangle - \langle P\mu, \sigma_1\rho_1 + \sigma_2\rho_2\rangle \\
&= \sigma_1(\langle \widehat{S}_{1,\gamma}\rho_1, \xi_1\rangle - \langle \widehat{S}_{1,\gamma}\xi_1, \rho_1\rangle) + \sigma_2(\langle \widehat{S}_{2,\gamma}\rho_2, \xi_2\rangle - \langle \widehat{S}_{2,\gamma}\xi_2, \rho_2\rangle) \\
&= 2\sum_{i=1}^{2} \sigma_i a_i^{\mathrm{ss}}(E_i^{\#}\rho_i, E_i^{\#}\xi_i) \\
&= 2\sum_{i=1}^{2} \sigma_i a_i^{\mathrm{ss}}(E_i^{\#}S_{i,\gamma}^{-1}P\eta, E_i^{\#}S_{i,\gamma}^{-1}P\mu).
\end{aligned}$$

Furthermore, by recalling Remark 4.2.10, the continuity constant β_P of the preconditioner is given by

$$\beta_\gamma := \frac{(\beta_1+\gamma)^2(\beta_2+\gamma)^2}{\sigma_1(\alpha_1+\gamma)(\beta_1+\gamma)^2 + \sigma_2(\alpha_2+\gamma)(\beta_2+\gamma)^2}. \tag{4.72}$$

Hereafter we suppose that $\alpha_1 = \alpha_2 := \widehat{\alpha}$ and $\beta_1 = \beta_2 := \widehat{\beta}$, to simplify the calculations, but the results still work without this assumption, as can be easily seen. We have, since $P\mu = S\eta$

$$2\sum_{i=1}^{2}\left|\sigma_i a_i^{\rm ss}(E_i^{\#}S_{i,\gamma}^{-1}P\eta, E_i^{\#}S_{i,\gamma}^{-1}S\eta)\right| \leq 2\|\mathbf{b}\|_{(L^\infty(D))^d}\sum_{i=1}^{2}\|S_{i,\gamma}^{-1}P\eta\|_\Lambda\|S_{i,\gamma}^{-1}S\eta\|_\Lambda,$$

and finally, by recalling that the continuity constant of $\widehat{S}_{i,\gamma}^{-1}$ is given by $(\widehat{\alpha}+\gamma)^{-1}$ and by using (4.72),we have that

$$\beta^2_{S^{-1}_{i,\gamma}}\beta_P\beta_S = \frac{1}{(\widehat{\alpha}+\gamma)^2}\frac{(\widehat{\beta}+\gamma)^4\widehat{\beta}}{(\widehat{\alpha}+\gamma)(\widehat{\beta}+\gamma)^2} = \frac{\widehat{\beta}(\widehat{\beta}+\gamma)^2}{(\widehat{\alpha}+\gamma)^3}.$$

We get that,

$$\sum_{i=1}^{2}\left|\sigma_i a_i^{\rm ss}(E_i^{\#}S_{i,\gamma}^{-1}P\eta, E_i^{\#}S_{i,\gamma}^{-1}P\eta)\right| \leq \frac{2\widehat{C}(\widehat{\alpha},\widehat{\beta})\,\|\mathbf{b}\|_{(L^\infty(D))^d}}{\widehat{\alpha}+\gamma}\|\eta\|^2_\Lambda,$$

with $\widehat{C}(\widehat{\alpha},\widehat{\beta}) := \max\left\{\widehat{\beta}, \widehat{\beta}^3/\widehat{\alpha}^2\right\}$. We observe that, if γ is big enough, the condition c) of Theorem 4.2.11 is satisfied and the modified γ-Robin-Robin method converges. □

Remarks on the numerical implementation

In this section we proposed some domain decomposition methods to solve advection-diffusion equations, when the advective term is "dominant" on the diffusive one. We showed some convergence results and some of these methods have been tested numerically; unfortunately, it is not known a method which enjoys, at the same time, of good theoretical and numerical properties.

We proposed some methods for which we have convergence theorems (the "modified" ones and the "γ methods"), but we can not prove such a result for the simplest method, *i.e.*, the classical Neumann-Neumann. The "modified methods" (MDN and MNN, defined respectively by (4.56) and (4.57)) are based on symmetric preconditioners. They have optimal theoretical convergence results, but the numerical *scenario* concerning the MNN method is the following: the numerical computations show its convergence and a certain independence on the mesh parameter h (on the other hand, the MDN method has not been implemented yet). We observe that the number of iterations seems to increase if the "viscosity" ε (or better the Péclet number) becomes small, with respect to the other parameters of the problem. The bound on the number of iterations (independent of h) is asymptotic and in our experiments we may not have reached the upper bound. In practical computations, the number of iterations for the classical Neumann-Neumann method (*i.e.*, if $P_{NN} := (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$) is lower, but we recall again that no convergence theory exists for this method, in the context of advection-dominated equations. If we see the `CPU`-time, then we have that, for moderate advective term (or big diffusion), the modified method MNN requires less elementary operations, even if the number of iterations increases of a factor up to 5. This is the main advantage of the "modified" methods and, even if they converge for each convective term, they can be successfully used in problems that are not advection-dominated, see Berselli and Saleri [BS99].

On the other hand, the γ-DR and γ-RR methods show convergence for suitable γ (see Theorem 4.4.7 and Theorem 4.4.9), which may depend on the advective field. We also recall (see Remark 4.4.8) that the γ-DR method converges numerically for any $\gamma \geq 0$, even if the proof works only for γ big enough. Observe, in particular, that the γ-DR method converge numerically even if

$\gamma = 0$. In this last case the γ-DR method reduces to a Richardson method with S_2 as preconditioner; this preconditioned iterative method is *formally* identical to the Dirichlet-Neumann method used for the Laplace operator, but in this case the interface conditions are of Dirichlet and Robin type. Furthermore, for several numerical examples, the γ-RR method converges for $\gamma > 0$ and the convergence is independent of h, even if the abstract convergence results are not strong enough to give such an independence. For the numerical results cited above see Alonso, Trotta and Valli [ATV98] § 6.

Regarding the last method, the "modified" γ-RR method (4.71), we have not numerical results except for the special case $\gamma = 0$. In this case the "modified" γ-RR method reduces to the classical Neumann-Neumann (recall that the name was used in analogy with the Poisson equation, because the interface preconditioner $P_{NN} := (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$ is formally the same, but a more appropriate name should be Robin-Robin) and numerical tests were proposed to compare this method with the "modified" one. We note that the "modified" 0-Robin-Robin can be used with large advection coefficient and different mesh sizes and the rate of convergence results essentially the same. We expect that the "modified" γ-RR method works well (from the numerical point of view) even for $\gamma > 0$ in the range of applicability of Theorem 4.4.11.

In conclusion, we proposed different methods for which the theory is completely satisfactory. All these methods are approximations of the Dirichlet-Neumann and Neumann-Neumann methods, for which we have very good numerical results, but no convergence results in general situations; precisely, the known convergence results for the Dirichlet-Neumann and Neumann-Neumann methods work only with rather restrictive assumptions on the advective field or on the geometry and size of the subdomains and not for a general advection-dominated problem.

Bibliography

[ANLTV98] Y. Achdou, F. Nataf, P. Le Tallec and M. Vidrascu, *A domain decomposition preconditioner for an advection diffusion problem* Tech. Report, 1998.

[AN97] Y. Achdou and F. Nataf, *A Robin-Robin preconditioner for an advection-diffusion problem*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 11, 1211-1216.

[Ada75] R.A. Adams, *Sobolev spaces*, Academic Press, 1975, Pure and Applied Mathematics, Vol. 65.

[AL85] V.I. Agoshkov and V.I. Lebedev, *Poincaré-Steklov operators and methods of partition of the domain in variational problems*, Computational processes and systems, No. 2 (Moscow), "Nauka", Moscow, 1985, pp. 173–227.

[ATV98] A. Alonso, R.L. Trotta, and A. Valli, *Coercive domain decomposition algorithms for advection-diffusion equations and systems*, J. Comput. Appl. Math. **96** (1998), no. 1, 51–76.

[AV96] A. Alonso and A. Valli, *Some remarks on the characterization of the space of tangential traces of* $H(\mathrm{rot};\Omega)$ *and the construction of an extension operator*, Manuscripta Math. **89** (1996), no. 2, 159–178.

[AV99] A. Alonso and A. Valli, *An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations*, Math. Comp. **68** (1999), no. 226, 607–631.

[ABDG98] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional non-smooth domains*, Math. Methods Appl. Sci. **21** (1998), no. 9, 823–864.

[BV92] A.V. Babin and M.I. Višhik, *Attractors of evolution equations*, North-Holland Publishing Co., Amsterdam, 1992. Translated and revised by Babin from the 1989 Russian original: Аттракторы Йеволюционных уравнениуЮ, "Наука."

[Bar96] G.I. Barenblatt, *Scaling, self-similarity, and intermediate asymptotics*, Cambridge University Press, Cambridge, 1996, with a foreword by Ya. B. Zeldovich. Revised edition of the russian: Подобие, автомоделност, промежуточная асимптотика.

[BdV87] H. Beirão da Veiga, *Existence and asymptotic behavior for strong solutions of the Navier–Stokes equations in the whole space*, Indiana Univ. Math. J. **36** (1987), no. 1, 149–166.

[BdV97a] H. Beirão da Veiga, *A new approach to the L^2-regularity theorems for linear stationary nonhomogeneous Stokes systems*, Portugal. Math. **54** (1997), no. 3, 271–286.

[BdV97b] H. Beirão da Veiga, *Concerning the regularity of the solutions to the Navier-Stokes equations via the truncation method. I*, Differential Integral Equations **10** (1997), no. 6, 1149–1156.

[BdV98] H. Beirão da Veiga, *Concerning the regularity of the solutions to the Navier-Stokes equations via the truncation method. II*, Équations aux dérivées partielles et applications (Éd. Sci. Méd. Elsevier, Paris), Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998, pp. 127–138.

[BdV99] H. Beirão da Veiga, *Regarding the effect of the pressure on the regularity of the solutions to the Navier-Stokes Equations.*, J. Math. Fluid Mech. (1999), in press.

[BT73] A. Bensoussan and R. Temam, *Équations stochastiques du type Navier-Stokes*, J. Functional Analysis **13** (1973), 195–222.

[Berk63] R. Berker, *Intégration des équations du mouvement d'un fluide visqueux incompressible*, Handbuch der Physik, Bd. VIII/2 (Berlin), Springer, Berlin, 1963, pp. 1–384.

[Ber99] L.C. Berselli, *A Neumann-Neumann preconditioner for low-frequency time-harmonic Maxwell equations*, Tech. Report 2.327.1158, Dept. of Mathematics, Pisa University, 1999, see also Quarteroni and Valli: *Domain decomposition methods for Partial Differential Equations.* Oxford Science Publications (1999), pp. 130-132.

[Ber99] L.C. Berselli, *Sufficient conditions for the regularity of the solutions of the Navier-Stokes equations*, Math. Meth. Appl. Sci. **22** (1999), 1079–1085.

[BF99] L.C. Berselli and F. Flandoli, *Remarks on determining projections for stochastic dissipative equations*, Discrete Contin. Dynam. Systems **5** (1999), no. 1, 197–214.

[BS99] L.C. Berselli and F. Saleri, *New substructuring domain decomposition methods for advection-diffusion equations*, J. Comput. Appl. Math. **116** (1999), in press.

[Bil97] P. Billingsley, *Probability and measure.*, 3rd ed., John Wiley & Sons, 1997.

[BW86] P.E. Bjørstad and O. B. Widlund, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, SIAM J. Numer. Anal. **23** (1986), no. 6, 1097–1120.

[BGLTV89] J.F. Bourgat, R. Glowinski, P. Le Tallec, and M. Vidrascu, *Variational formulation and algorithm for trace operator in domain decomposition calculations*, Domain decomposition methods (Los Angeles, CA, 1988) (Philadelphia, PA), SIAM, Philadelphia, PA, 1989, pp. 3–16.

[BPS86] J.H. Bramble, J.E. Pasciak, and A.H. Schatz, *An iterative method for elliptic problems on regions partitioned into substructures*, Math. Comp. **46** (1986), no. 174, 361–369.

[BF91] F. Brezzi and M. Fortin *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.

[CGK92] X.-C. Cai, W.D. Gropp and D.E. Keyes. *A comparison of some domain decomposition algorithms for nonsymmetric elliptic problems*, Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991) SIAM, Philadelphia, PA, 1992, pp. 224–235.

[CW92] X.-C. Cai and O.B. Widlund, *Domain decomposition algorithms for indefinite elliptic problems,* SIAM J. Sci. Statistic. Comput. **13** (1992), no. 1, 243–258.

[CQ95] C. Carlenzoli and A. Quarteroni, *Adaptive domain decomposition methods for advection-diffusion problems*, Modeling, mesh generation, and adaptive numerical methods for partial differential equations (Minneapolis, MN, 1993) (I. Babuška et al., ed.), IMA Vol. Math. Appl., vol. 75, Springer, New York, 1995, pp. 165–186.

[Cat61] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Sem. Mat. Univ. Padova **31** (1961), 308–340.

[Cho67] A.J. Chorin, *The numerical solution of the Navier-Stokes equations for an incompressible fluid*, Bull. Amer. Math. Soc. **73** (1967), 928–931.

[Cho68] A.J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp. **22** (1968), 745–762.

[Cho94] A.J. Chorin, *Vorticity and turbulence*, Springer-Verlag, New York, 1994.

[CM93] A.J. Chorin and J.E. Marsden, *A mathematical introduction to fluid mechanics*, third ed., Springer-Verlag, New York, 1993.

[Cia78] P.G. Ciarlet, *The finite element method for elliptic problems*, North-Holland Publishing Co., Amsterdam, 1978, Studies in Mathematics and its Applications, Vol. 4.

[Clé75] Ph. Clément, *Approximation by finite element functions using local regularization*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér. **9** (1975), no. R-2, 77–84.

[CF88] P. Constantin and C. Foiaş, *Navier-Stokes equations*, University of Chicago Press, Chicago, IL, 1988.

[CDF97] H. Crauel, A. Debussche, and F. Flandoli, *Random attractors*, J. Dynam. Differential Equations **9** (1997), no. 2, 307–341.

[CF94] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields **100** (1994), no. 3, 365–393.

[Cru89] A.B. Cruzeiro, *Solutions et mesures invariantes pour des équations d'évolution stochastiques du type Navier-Stokes*, Exposition. Math. **7** (1989), no. 1, 73–82.

[DPZ92] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, 1992.

[DPZ96] G. Da Prato and J. Zabczyk, *Ergodicity for infinite-dimensional systems*, Cambridge University Press, Cambridge, 1996.

[Dau88] M. Dauge, *Elliptic boundary value problems on corner domains*, Springer-Verlag, Berlin, 1988.

[DL88] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 2,* Functional and variational methods. Springer-Verlag, Berlin, 1988

[DL92] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 3,* Spectral theory and applications. Springer-Verlag, Berlin, 1992

[DG57] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

[DW95] M. Dryja and O. B. Widlund, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Comm. Pure Appl. Math. **48** (1995), no. 2, 121–155.

[DL76] G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics*, Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[Eul1755] L. Euler, *Opera omnia* Series Secunda **12** (1755), 274–361.

[Fla96] F. Flandoli, *Stochastic differential equations in fluid dynamics*, Rend. Sem. Mat. Fis. Milano **66** (1996), 121–148.

[FL99] F. Flandoli and J. A. Langa, *Determining modes for dissipative random dynamical systems*, Stochastics Stochastics Rep. **66** (1999), no. 1–2, 1–25.

[FP67] C. Foiaş and G. Prodi, *Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension* 2, Rend. Sem. Mat. Univ. Padova **39** (1967), 1–34.

[FT84] C. Foiaş and R. Temam, *Determination of the solutions of the Navier–Stokes equations by a set of nodal values*, Math. Comp.**43** (1984), no. 167, 117–133.

[FT91] C. Foiaş and E. S. Titi, *Determining nodes, finite difference schemes and inertial manifolds*, Nonlinearity **4** (1991), no. 1, 135–153.

[GGQ96] F. Gastaldi, L. Gastaldi, and A. Quarteroni, *Adaptive domain decomposition methods for advection dominated equations*, East-West J. Numer. Math. **4** (1996), no. 3, 165–206.

[GM85] Y. Giga and T. Miyakawa, *Solutions in L_r of the Navier–Stokes initial value problem*, Arch. Rational Mech. Anal. **89** (1985), no. 3, 267–281.

[GR86] V. Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer-Verlag, Berlin, 1986, Theory and algorithms.

[GVL89] G.H. Golub and C.F. Van Loan, *Matrix computations*, second ed., Johns Hopkins University Press, Baltimore, MD, 1989.

[Hey80] J.G. Heywood, *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J. **29** (1980), no 5, pagg 639–681.

[Hey90] J.G. Heywood, *Open problems in the theory of the Navier-Stokes equations for viscous incompressible flow*, The Navier-Stokes equations (Oberwolfach, 1988) (Berlin), Springer, Berlin, 1990, pp. 1–22

[HR82] J.G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier–Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal. **19** (1982), no. 2, 275–311.

[HT97] M.J. Holst and E.S. Titi, *Determining projections and functionals for weak solutions of the Navier-Stokes equations*, Recent developments in optimization theory and nonlinear analysis (Jerusalem, 1995) (Providence, RI), Amer. Math. Soc., Providence, RI, 1997, pp. 125–138.

[Hop51] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213–231.

[HFH89] T.J.R. Hughes, L.P. Franca, and G.M. Hulbert, *A new finite element formulation for computational fluid dynamics. VIII. The Galerkin/least-squares method for advective-diffusive equations*, Comput. Methods Appl. Mech. Engrg. **73** (1989), no. 2, 173–189.

[IK66] E. Isaacson and H. B. Keller, *Analysis of numerical methods*, John Wiley & Sons Inc., New York, 1975.

[Jac75] J.D. Jackson, *Classical electrodynamics*, second ed., John Wiley & Sons Inc., New York, 1975.

[JT92] D.A. Jones and E.S. Titi, *Determining finite volume elements for the* 2*D Navier-Stokes equations*, Phys. D **60** (1992), no. 1-4, 165–174, Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991).

[JT93] D.A. Jones and E.S. Titi, *Upper bounds on the number of determining modes, nodes, and volume elements for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), no. 3, 875–887.

[Kan68] S. Kaniel, *A sufficient condition for smoothness of solutions of Navier-Stokes equations*, Israel J. Math. **6** (1968), 354–358 (1969).

[KL57] A.A. Kiselev and O.A. Ladyženskaya, *On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid*, Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 655–680.

[Koz84] A.N. Kozhevnikov, *The operator of the linearized stationary Navier-Stokes problem*, Mat. Sb. (N.S.) **125(167)** (1984), no. 1(9), 3–18.

[Lad66] O.A. Ladyženskaja, *On the classical nature of generalized solutions of general nonstationary Navier-Stokes equations*, Trudy Mat. Inst. Steklov. **92** (1966), 100–115.

[Lad69] O.A. Ladyžhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach Science Publishers, New York, 1969, Second English edition, revised and enlarged. Translated by R. A. Silverman and J. Chu from the Russian: Математические вопросы динамики вязкоЮ несжимаемоЮ жидкости. Mathematics and its Applications, Vol. 2.

[Lad72] O.A. Ladyženskaya, *The dynamical system that is generated by the Navier-Stokes equations*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27** (1972), 91–115, Boundary value problems of mathematical physics and related questions in the theory of functions, 6.

[LSU67] O.A. Ladyženskaya, V.A. Solonnikov, and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Providence, R.I., 1967. Translated by S. Smith from the Russian original: Линеиные и квазилинеи ные уравнения параболического типа, "наука." Translations of Mathematical Monographs, Vol. 23.

[Lam93] H. Lamb, *Hydrodynamics, reprint of the 1932 sixth edition*, Cambridge University Press, Cambridge, 1993, With a foreword by R.A. Caflisch

[LL59] L.D. Landau and E.M. Lifshitz, *Fluid mechanics*, Pergamon Press, London, 1959. Translated from the Russian by J.B. Sykes and W.H. Reid Course of Theoretical Physics, Vol. 6.

[Lei79] R. Leis, *Exterior boundary-value problems in mathematical physics*, Trends in applications of pure mathematics to mechanics, Vol. II (Second Sympos., Kozubnik, 1977) (Boston, Mass.), Monographs Stud. Math., vol. 5, Pitman, Boston, Mass., 1979, pp. 187–203.

[Ler33] J. Leray, *Étude de diverses équations intégrales non linéaires et de quèlques problemes que pose l'hydrodynamique*, J. Math. Pures. Appl. **12** (1933), 1–82.

[Ler34a] J. Leray, *Essai sur le mouvement plans d'un liquide visqueux que limitent des parois*, J. Math. Pures. Appl. **13** (1934), 331–418.

[Ler34b] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.

[JLL69] J.-L. Lions, *Quelque méthodes de résolution des problemès aux limitex non–lineaires*, Dunod Gauthier–Villars, 1969.

[LM72] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[PLL88] P.-L. Lions, *On the Schwarz alternating method. I*, First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987) (Philadelphia, PA), SIAM, Philadelphia, PA, 1988, pp. 1–42.

[PLL90] P.-L. Lions, *On the Schwarz alternating method. III. A variant for nonoverlapping subdomains*, Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989), SIAM, Philadelphia, PA, 1990, pp. 202–223.

[PLL96] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 1*, The Clarendon Press Oxford University Press, New York, 1996, Incompressible models, Oxford Science Publications.

[Lor63] E.N. Lorenz, *Deterministic nonperiodic flow*, J. Atmospheric Sci. **20** (1963), 130–141.

[MQ89] L.D. Marini and A. Quarteroni, *A relaxation procedure for domain decomposition methods using finite elements*, Numer. Math. **55** (1989), no. 5, 575–598.

[MN85] A. M. Matsokin and S. V. Nepomnyashchikh, *The Schwarz alternation method in a subspace* [Russian], Izv. Vyssh. Uchebn. Zaved. Mat. **1985**, no. 10, 61–66, 85.

[Miy81] T. Miyakawa, *On the initial value problem for the Navier–Stokes equations in L^p spaces*, Hiroshima Math. J. **11** (1981), no. 1, 9–20.

[Nav1822] C.L.M.H. Navier, *Mémoire sur les lois du mouvement des fluides* Mém. Acad. Sci. Inst. France **6** (1822), 375–394.

[Neč67] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris, 1967.

[Néd80] J.-C. Nédélec, *Mixed finite elements in* $\mathbf{R}^3$, Numer. Math. **35** (1980), no. 3, 315–341.

[Néd86] J.-C. Nédélec, *A new family of mixed finite elements in* $\mathbf{R}^3$, Numer. Math. **50** (1986), no. 1, 57–81.

[Nep86] S. V. Nepomnyashchikh, *Domain decomposition and schwarz methods in a subspace for the approximate solution of elliptic boundary value problems*, Ph.D. Thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, 1986.

[Pro59] G. Prodi, *Un teorema di unicità per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl. (4) **48** (1959), 173–182.

[QV94] A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations*, Springer-Verlag, Berlin, 1994.

[QV99] A. Quarteroni and A. Valli, *Domain decomposition methods for partial differential equations*, Oxford Sci. Publications, 1999.

[SBG96] B.F. Smith, P.E. Bjørstad, and W.D. Gropp, *Domain decomposition*, Cambridge University Press, Cambridge, 1996.

[Sch1869] H.A. Schwarz, *Über einige Abbildungsdufgaben*, J. Reine Angew. Math., **70** (1869), 105–120.

[Ser62] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187–195.

[Ser63] J. Serrin, *The initial value problem for the Navier-Stokes equations* in Nonlinear Problems (Proc. Sympos., Madison, Wis.) R.E. Langer ed., Univ. of Wisconsin Press, Madison, Wis. (1963), 69–98.

[Soh83] H. Sohr, *Zur Regularitätstheorie der instationären Gleichungen von Navier–Stokes*, Math. Z. **184** (1983), no. 3, 359–375.

[SZ90] L.R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), no. 190, 483–493.

[SZ92] L.R. Scott and S. Zhang, *Higher-dimensional nonnested multigrid methods*, Math. Comp. **58** (1992), no. 198, 457–466.

[Sta65] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), no. fasc. 1, 189–258.

[Sto1849] G.G. Stokes, *On the theories of internal friction of fluids in motion and of the equilibrium and motion of elastic solids*, Trans. Camb. Phil. Soc. **8** (1849), 207–319.

[Tem69] R. Temam, *Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires. II*, Arch. Rational Mech. Anal. **33** (1969), 377–385.

[Tem77] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland Publishing Co., Amsterdam, 1977, Studies in Mathematics and its Applications, Vol. 2.

[Tem82] R. Temam, *Behaviour at time* $t = 0$ *of the solutions of semilinear evolution equations*, J. Differential Equations, **43** (1982), no. 1, 73–92.

[Tem97] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, second ed., Springer-Verlag, New York, 1997.

[VF88] M.I. Višik and A.V. Fursikov, *Mathematical problems of statistical hydromechanics*, Kluwer, Amsterdam, 1988. Translated and revised from the 1980 Russian original: Математические задачи статистическоуи гидромеханики, "Наука."

[vW80] W. von Wahl, *Regularity questions for the Navier-Stokes equations*, Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979) (Berlin), Springer, Berlin, 1980, pp. 539–542.

[Wig90] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, Springer-Verlag, New York, 1990.

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