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Keeping Meaning in Proportion: The Multiplication Table as a Case of Pedagogical Bridging Tools

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Dor Abrahamson

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ABSTRACT

Keeping Meaning in Proportion:

The Multiplication Table as a Case of Pedagogical Bridging Tools

Dor Abrahamson

The broad agenda of this dissertation is the design of mathematics curricula for elementary and middle-school students to build meaning for and fluency with mathematical concepts. The paradigm of the work is design research, which combines empirical and theoretical studies. The empirical substance of this dissertation is a set of studies that were implementations of an experimental curricular unit for 5^{th}-grade students learning the domain of ratio and proportion. The theoretical component of this paper is the *apprehending zone* (Fuson & Abrahamson, 2004a), a Piagetian–Vygotskiian conceptual–practical model of design, teaching, and learning in mathematics classrooms. The design positions the multiplication table as the central representation for ratio and proportion. Students working with the multiplication table and related representations ground in situations of repeated adding their multiplicative understandings and solution procedures for the domain. Essential to the design are opportunities for students to build connections or 'links' between the mathematical representations and word problems depicting real-world situations. The multiplication table is a *bridging tool*, a pedagogical artifact designed to support students' construction of domain-relevant situational–representational links. The apprehending-zone model draws on constructivist, social–constructivist, psychological, psycholinguistic, and philosophical resources. The model highlights the need for students to link situations and representations reciprocally into cohesive, coherent, and fluent activity structures. This model, which informed the design and evolved with it, also frames the analysis of data from implementing this design. Several *learning issues*, challenges inherent in some of the links students must build in this design, are identified. The dissertation presents evidence of student learning, demonstrates student variation in constructing the mathematical representations, and then tracks day-by-day interactions in one classroom implementation to show students moving from difficulty with to understanding of the identified learning issues of the design. This work contributes an innovative design for the domain of ratio and proportion as well as an original model that generalizes to other domains and may inform the practice of mathematics-education researchers as well as teachers of mathematics.

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For Gabi, who's been so patient

POEM WITH RHYTHMS

The hand between the candle and the wall
Grows large on the wall.

The mind between this light or that and space,
(This man in a room with an image of the world,
That woman waiting for the man she loves,)
Grows large against space.

There the man sees the image clearly at last.
There the woman receives her lover into her heart
And weeps on his breast, though he never comes.

It must be that the hand
Has a will to grow larger on the wall,
To grow larger and heavier and stronger than
The wall; and that the mind
Turns to its own figurations and declares,
"This image, this love, I compose myself

Of these. In these, I come forth outwardly.
In these, I wear a vital cleanliness,
Not as in air, bright-blue-resembling air,
But as in the powerful mirror of my wish and will."

Wallace Stevens,
from Parts of a World.

Stevens, W. (1982). *The collected poems of Wallace Stevens*. New York: Vintage books. (Original work published 1942)

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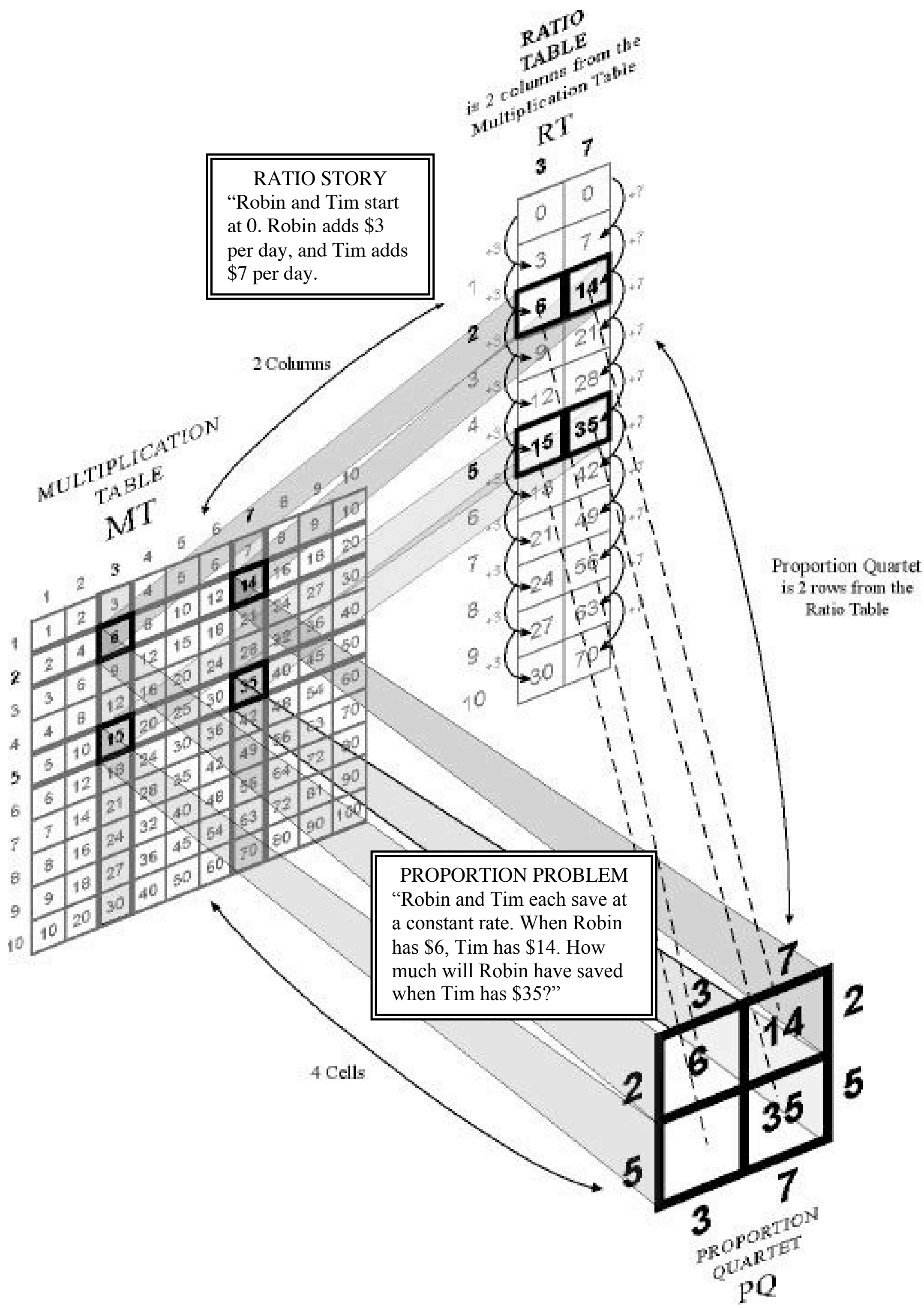
Keeping Meaning in Proportion: The Multiplication Table as a Case of Pedagogical Bridging Tools

Introduction

This dissertation is on research in mathematics education. The broad agenda of this research is the design of mathematics curricula for elementary and middle-school students to build meaning for and fluency with mathematical concepts (Fuson, 2001; Fuson et al., 2000). The paradigm of the work is design research, which combines empirical and theoretical studies. The empirical substance of this dissertation is a set of studies that were implementations of an experimental curricular unit designed by the author for 5th-grade students learning the mathematics domain of ratio and proportion. The concomitant theoretical component of this dissertation is the *apprehending zone* (Fuson & Abrahamson, 2004a), a conceptual–practical model of design, teaching, and learning in mathematics classrooms. This domain-general model could develop through a domain-specific design because I diagnose much of students' well-documented challenges in the specific domain of ratio and proportion as characterizing challenges they face in other sub-domains of mathematics.

The design positions the multiplication table as the central representation for ratio and proportion. Through working with the multiplication table and related representations (see Figure 1, next pages) students ground a multiplicative understanding and solution procedures for ratio and proportion in situations of repeated-adding. Essential to the design are opportunities for students to build connections—'links'—between the design elements, which are mathematical representations and word problems depicting

Figure 1. Mathematical representations in the ratio-and-proportion design space. In the experimental unit, students re-instrumentalize the multiplication table (MT) as a model of proportion situations. The ratio table (RT) is 2 columns embedded in and emanating out of the MT. The *proportion quartet* (PQ) is 4 products and their respective row/column-number factors embedded in and emanating from the MT or 2 rows from the RT.


RATIO TABLE
is 2 columns from the Multiplication Table
RT
RATIO STORY
"Robin and Tim start at 0. Robin adds $3 per day, and Tim adds $7 per day.
2 Columns
MULTIPLICATION TABLE
MT
Proportion Quartet
is 2 rows from the Ratio Table
PROPORTION PROBLEM
"Robin and Tim each save at a constant rate. When Robin has $6, Tim has $14. How much will Robin have saved when Tim has $35?"
4 Cells
PROPORTION QUARTET
PQ

real-world situations.

The apprehending-zone model foregrounds the process of students linking contexts and representations reciprocally into cohesive, coherent, and fluent action models for the domain (see Figure 2, next page). This model, which co-evolved with the design, frames the analysis of student learning in classroom implementations of a design. To measure student learning, student fluency with a target domain is operationalized as critical links that students construct between the design elements through participating in classroom activities and discussion. I identify several *learning issues* that are challenges inherent in links students must build in this design. I present evidence of student learning, demonstrate student variation in constructing the mathematical representations, and then track day-by-day interactions in one classroom implementation to show students' moving after some difficulty toward understanding of the identified learning issues of the design. The dissertation builds the claim that student conceptual learning is the process of design- and teacher-mediated interlinking of classroom artifacts into personal action models and procedures for the target domain. This work contributes an innovative design for teaching the domain of ratio and proportion as well as an original model that generalizes to other domains and may inform the practice of mathematics-education researchers as well as teachers of mathematics.

The following section, Design Research as a Research Paradigm, explains the nature of the study reported in this dissertation and essentially how design and theory are mutually nurturing in the process of creating classroom materials and activities. The section The Domain of Ratio and Proportion introduces the design rationale through

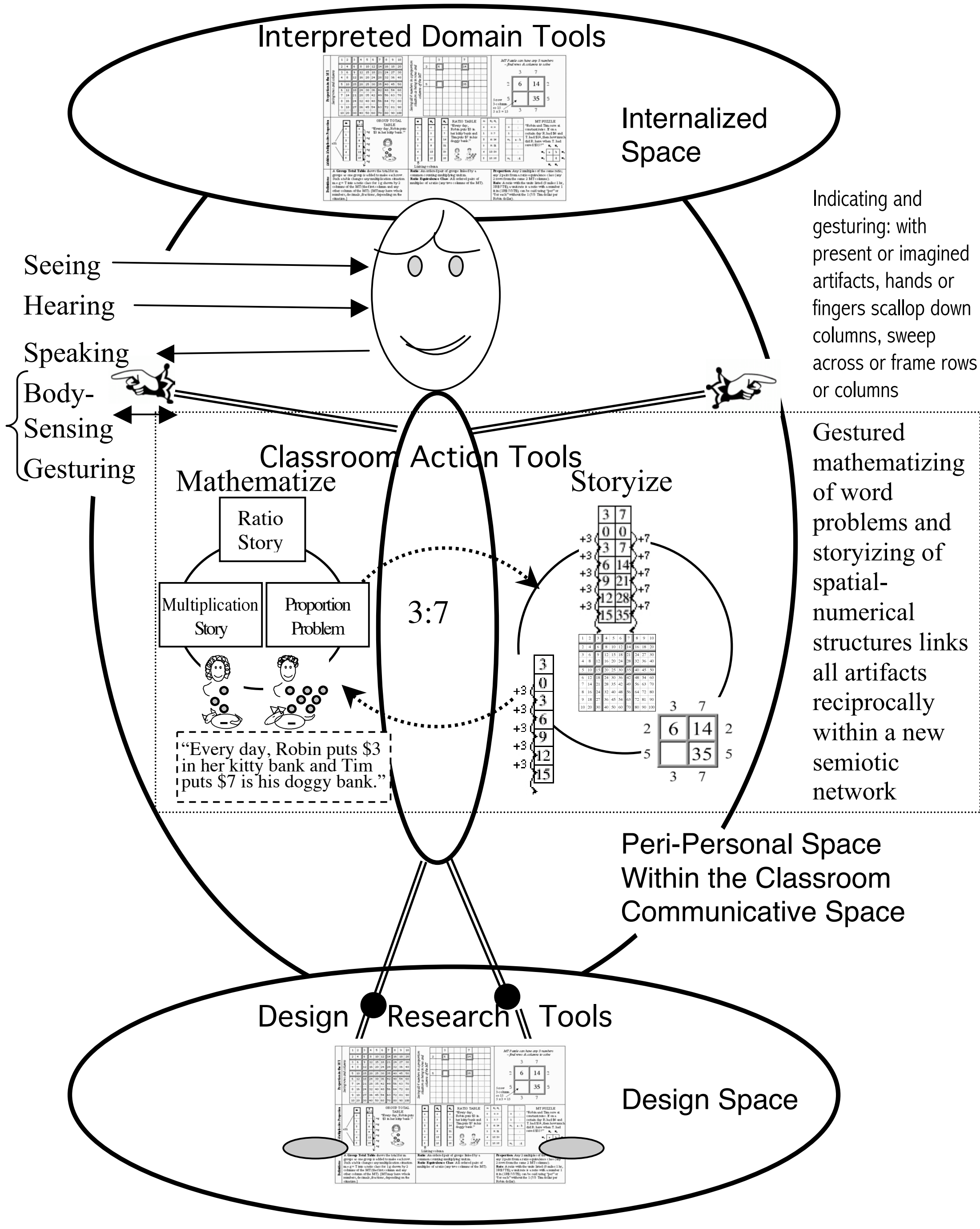


Figure 2. The body-based teaching–learning *apprehending zone* model: The time–space of problem solving and cultural communicating.

comparison to previous approaches to the domain of ratio and proportion. The Apprehending-Zone section explains the cognitive, constructivist, socio-constructivist, and philosophical sources of the model that evolved with the design and then further explains the model itself. The Design section details the mathematical tools and activities of the design experiment. The Learning Issues section further explains the rationale of measuring student learning by identifying and quantifying student mastery of the links between the design elements. Also, the section details and defines categories of learning issues that are pivotal to learning the domain through this design. The dissertation continues with a methodology section and ends with reporting and discussing the results from implementing the design and then discussing conclusions and implications of the study for the design, teaching, and research in mathematics education.

Design Research as a Research Paradigm

Design studies in education contribute to the creation of innovative design and development of theory (Collins, Joseph, & Bielaczyc, 2004; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Theorizing, designing, implementing, and analyzing findings are all mutually informing processes. These processes are cyclical, iterative, and reciprocal—a confluence of modification and refinement. Initial analysis and articulation of design problems is couched in terms of some theoretical position and set of hypotheses. Potential solutions to this design problem inform the design of materials and activities for intervention. The on-going stream of data in each study-based intervention challenges the initial theory and assumptions, which, once improved, inform improvement in subsequent design towards further implementation, and so on. A design study is complete when results from implementing the design product are satisfactory in terms of responding to the design problem, and once the concomitant theory establishes a coherent vantage point for further design research in other domains.

This dissertation focuses on the penultimate set of classroom interventions that led to the current curricular unit for ratio and proportion and on the apprehending zone, the design-, teaching-, and learning model that evolved with these design studies. The design of the unit and follow-up mini-units evolved over several studies (see the following papers that report the evolution of the design based on intervention studies: Abrahamson, 2002a, 2002b, 2002c, 2003a, 2003b; Abrahamson & Cigan, 2003; see also Appendix A). Our participants ranged in: (a) age—from 7 to 12; (b) background—from backgrounds of poverty and minority status (African–American free-lunch students in a summer supplementary-schooling program), to upper-middle class suburban; (c) settings—from individual interviews, through small-group pilot studies, to classroom studies; and (d) interaction methods for eliciting student understanding—from semi-structured construction tasks with minimal intervening, through semi-clinical interviews, to classroom teaching in which the author gradually stepped down from teaching to consulting and observing, and the teachers progressively took on leadership of activities and discussion. Each study stemmed from questions regarding student difficulty in the domain that arose in previous studies and design attempts to address those learning difficulties; each informed the goals and development of materials for the successive study. So the activities were progressively developed and ordered into lesson plans, including numerical items that would best forge continuity between students' entry understandings and age-appropriate learning objectives within the domain. The evolving aspects of these designs were stimulated by observing and analyzing student work and classroom discussion in each part of the design as it was implemented.

The Domain of Ratio and Proportion

A study of students learning a mathematical topic is always a study of students learning the topic *within a specific design.* So in order to understand the problematics of a domain, student learning must be deconstructed in terms of the designs that facilitated that learning. This section begins by discussing the design problem inherent in students' learning ratio and proportion and how the current design responds to the problem. Next, I discuss previous responses to the design problem so as to foreground unique aspects of the current design's rationale. I explain how previous design contributions can be embraced and enhanced by providing learning tools that help students link complementary action models of the domain. Specifically, I will suggest that the multiplication table affords student linking between complementary action models of multiplication: the repeated-adding or additive–multiplicative model and the factor*factor→product or multiplicative model (a later section will couch this linking affordance of the multiplication table as exemplifying spatial–numerical bridging tools).

Student Learning in the Domain of Ratio and Proportion

The challenge. The domain of ratio and proportion is a major learning challenge for elementary and middle-school students and remains so into later schooling (e.g., Clark, Berenson, & Cavey, 2003; Hart, 1989; Lamon, 1999). And yet, of the various rational-number constructs (e.g., part–to-whole *a/b* fractions, decimal fractions, measure, quotient, arithmetic of fractions), the domain of ratio and proportion has received relatively less attention from the mathematics-education community (but see the work of Confrey, 1995; Moss, 2000; Abrahamson, 2000). The paucity in design-research work in this domain resonates with the generally cursory attention the domain receives in elementary- and middle-school curricula, which isolate ratio-and-proportion units as disjoint from fractions.

In typical incorrect performance in the domain or ratio and proportion, students appear to ignore the multiplicativity inherent in proportion situations, using inappropriate additive strategies instead (e.g., Behr, Harel, Post, & Lesh, 1993). For instance, students may respond to a word-problem describing some unknown-value proportion, e.g., 6:14 = 15:?, with the answer 6:14 = 15:23 (the difference is 8 in each ratio, or 9 between matching numbers in each ratio). Such inappropriate use of additive strategies characterizes student strategies in other rational-number concepts, such as fractions (see Nunes & Bryant, 1996; Hughes, 1986; Kieren, 1988, 1992; Streefland, 1984, 1991).

It is not the case that addition and subtraction are always inappropriate operations in dealing with rational numbers. For instance, the following proportional progression incorporates addition: 3:7 = (3+3): (7+7) = (3+3+3): (7+7+7) = etc. Because additive procedures are sometimes appropriate and sometimes inappropriate for dealing with rational numbers, one criterion for indexing student fluency with the domain is whether or not students are using additive strategies appropriately in their solution procedures across different numerical cases.

A proportion is an equivalence between two ratios, and the predominant symbolical format of proportion is $m:n = s:t$. Two ratios are equivalent if and only if one ratio number pair is a multiple of the other, $a:b = ka:kb$, where a and b can either be integer numbers or not. To simplify the following discussion, we will consider cases where a and b are integers. For instance, we will discuss proportions such as 3:7 = 15:35. This simplification is also tuned to 5^{th}-grade students' fluency with rational numbers that is not initially conducive to dealing with proportions involving fractions. Note that k, the multiplicative relation between the two ratios, can either be an integer or a non-integer. In the proportion 3:7 = 15:35, k is the integer 5. In the proportion 6:14 = 15:35 k is not an integer—it is 2.5. Previous work in the domain has shown that students use inappropriate additive strategies in solving proportions in which k is a non-integer much more often than in cases where k is an integer (so $3:7 = x:35$ is easier than $6:14 = x:35$; e.g., Behr, Harel, Post, & Lesh, 1993). These studies can be interpreted as suggesting that students use inappropriate additive strategies because these students lack coherent conceptual structures; that such conceptual structures could support the use of appropriate solution procedures for numerical cases that do not readily afford multiplicative solutions.

Operations and meanings in ratio and proportion: The problem of unlinked meanings. Most classroom designs for ratio and proportion (e.g., Charles, Dossey, Leinwand, Seeley, & Vonder Embse, 1998) begin by interpreting proportion as a relation between two ratios in a proportional progression (but see a discussion, below, of J. Confrey's work). For example, the equivalence 3:7 = 15:35 is supported because 15:35 is an instance in the proportional progression beginning with 3:7, thus: 3:7, 6:14, 9:21, 12:28, 15:35. These proportional progressions are often presented in a vertical or horizontal ratio table with each successive multiple of the ratio unit below or to the right of its predecessor, respectively. Generating the succession of consecutive multiples of an *a:b* ratio is tantamount to repeatedly adding on the *a:b* ratio unit. That is, we begin from 3 and 7 and then keep adding on, respectively, "+3" and "+7," over and over: 3:7 = (3+3): (7+7) = (3+3+3): (7+7+7) = etc. Note the distinction between 'multiples' of the *a:b* ratio and the repeated-adding action model for determining these multiples. This operational distinction is elucidated in terms of situations, as follows.

The predominant grounding (i.e. sense making, concretizing) of proportional progression in classroom designs for the domain is as some real-world situation involving a pair of meaningfully related quantities and the iterated adding on of these quantities. For instance, *a:b* may signify a recipe that relates the number of teaspoons of coffee to the number of teaspoons of sugar for preparing a cup of coffee. Thus, *1a:1b, 2a:2b, 3a:3b,* etc.—multiples of the ratio—or *a:b, (a+a):(b+b), (a+a+a):(b+b+b)* etc.—repeated addings of the ratio—signify the respective numbers of teaspoons of coffee and of sugar needed for making 1, 2, 3, etc. cups of coffee that all have the same taste. For proportions in which one ratio is an integer multiple of the other ratio, students learn to use multiplication to relate directly between the two ratios, e.g., 3:7 = 15:35 because 15 is 5x3 and 35 is 5x7. For proportions in which one ratio is not an integer multiple of the other, e.g., 6:14 = 15:35, students learn to relate the two ratios through their common-multiple ratio, 3:7. Note that whereas most designs ground the meaning of proportion in repeated-adding procedures, students are encouraged to use multiplicative solution procedures that shortcut repeated-adding. Yet students often fail to use multiplicative solution procedures and instead use inappropriate additive procedures. It could be that previous designs do not sufficiently link multiplicative solution procedures to the repeated-adding grounding of proportion.

Linking additive-multiplicative and multiplicative meanings of ratio and proportion. Both interpretations of proportion—as a succession of multiples and as a chain of adding on the basic ratio—may be useful for students' understanding of and fluency with the domain. I maintain that students need to understand how these two interpretations or models of proportion are related. The rationale of the design presented in this dissertation (Figure 1) is that the multiplication table (MT) can help students ground multiplicative solution procedures both for integer- and non-integer-multiples proportions in repeated-adding proportional progression.

Any single column in the MT can be thought of as an arithmetic sequence representing a *multiplication story*, some situation in which a constant value is added repeatedly, beginning from zero. For instance, the 3-column could represent the accumulating daily savings of Robin, who adds $3 per day to her savings (+3, +3, +3, +3, +3, etc. totaling at 3, 6, 9, 12, 15, etc.). The 1-column can help track how many times '3' was added. For example, in the 5-row of the MT the value in the 3-column is 15. That means that by the end of Day 5, Robin has saved $15. Similarly, the 7-column can represent the daily totals of Tim, who is saving $7 per day. In Tim's multiplication story, the 5-row shows that by the end of Day 5 Tim has saved $35. If Robin and Tim each begin saving from $0 on the same day, the 1-column constitutes a link between Robin's and Tim's stories—each value in the 1-column, e.g., 5, anchors a comparison between Robin's and Tim's respective totals, e.g., 15 and 35 (15:35). So any two columns in the MT can form a ratio table that is a row-by-row proportional progression of multiples of a basic ratio, e.g., 3:7 (see ratio table and ratio story, Figure 1, top center). The repeated adding of groups of $3 and $7, respectively, totals at 3:7, 6:14, 9:21, 12:28, 15:35, etc. Linking these two repeated-adding multiplication stories into a single *ratio story* is one of several learning issues in this design. That is, some students who understand Robin's multiplication story and Tim's multiplication story cannot initially coordinate these two stories into a ratio story.

Also, any four values in the MT at the intersection of two columns with two rows constitute a proportion, and this proportion and its MT row-and-column factors can form a separate 2-by-2 mini-MT that I call a *proportion quartet* (see Figure 1, bottom right). The proportion quartet foregrounds and affords the factor*factor→product MT shortcut of the repeated-adding model of multiplication (see proportion problem, Figure 1, bottom and center). Also, the proportion quartet relates as two rows out of a ratio table comprising the same MT columns (Figure 1, dashed lines on right). This link between the ratio table and the proportion quartet may help students ground the factor*factor→product action models of multiplication in the ratio table's repeated-adding model.

Each of these three representations, the multiplication table (MT), the ratio table (RT), and the proportion quartet (PQ), is a tool for solving numerical and situation-based problems involving proportion. Because the design introduces three mathematical representations, and not just the RT as in most other designs, a major challenge in the design is for students to understand how these representations are all related and how they relate to the situations that give them meaning. The design concern with students' building domain-specific mathematical meaning through relating between representations and situations informed the general apprehending-zone theoretical model of how students learn mathematics.

I have introduced the additive–multiplicative (repeated adding) and multiplicative (factor*factor→product) models of multiplication. These models structure analyses of students' sense making and solution procedures afforded by different design elements (situational contexts and symbolical representations). Specifically, these models will now frame an examination of two designs for the domain of ratio and proportion.

Models of Multiplication in Designs for Ratio and Proportion

Mathematically informed individuals use multiplication and division to solve proportion problems, especially problems with numbers that do not readily afford mental calculation, e.g., $56:63 = x:72$. So the domain of ratio and proportion may appear "purely" multiplicative, and addition-and-subtraction operations might appear as irrelevant, unsophisticated, or even impediments to gaining mastery of the domain. Yet constructivist mathematics educators seek means to foster students' assimilating new mathematical ideas into existing understandings and making these new ideas meaningful, i.e., realistic, grounded, and fitting the learner's conception of the world (Freudenthal, 1981; von Glasersfeld, 1990). So an issue at stake in constructivist designs for ratio and proportion has been how to foster students' meaningful assimilation of the domain into their robust understanding of addition and subtraction without oversimplifying this new domain or sacrificing its mathematical integrity in making it more learnable.

At the heart of previous designs for ratio and proportion is the designer's conceptions of the nature of the multiplication operation—what it means to multiply a quantity, how multiplication operations look in real-world situations, what are ratio-and-proportion real-world situations, and how to help students apply their understanding of multiplication in learning the new domain. In this section I first look at a couple of previous designs for ratio and proportion. I focus on the underlying interpretation of the multiplication operation in these designs, whether as inherent in the design tools or as explicit assumptions in the design rationale. My choice of these designs was informed by their oppositional design rationales. Juxtaposing these rationales will foreground what I will then frame as a learning gap in ratio and proportion. Second, I present a theoretical analysis of the multiplication operation, situations, and solution procedures, the design implications of my position, and an overview of my design's rationale.

Kaput and West's design. In a study conducted by Kaput and West (1994), students interacted with a computer program. The screen was split into a situation field with iconic representations of groups of objects, such as dogs and bones, and a more formal mathematical field in which there was a ratio table with numerical entries. The situational and numerical representations were interlinked through the computer program so that icons of objects in ratio-and-proportion situations (e.g., groups of dogs and groups of bones) were linked to their corresponding entries in ratio-table rows. Also, both the icons and the tables were manipulatable and the manipulation of each affected the other.

In a variety of activities, students: (a) added or subtracted icons from the situation and observed the affect of this manipulation on the appearance of rows and numbers in the ratio table; (b) moved up and down existing rows in the ratio table and observed the automatic adding or subtracting of icons from the situation field; and (c) manipulated numbers in the table, observed the automatic adding or subtracting of icons from the situation field, and received feedback on their numerical entries. The design was for students to ground the repeated-addition numerical model of proportional progression in their actions of inserting and deleting grouped objects.

Confrey's design. Confrey's design-research work (1995, 1998; Scarano & Confrey, 1996) constitutes a major contribution to the study of students' learning ratio and proportion. Confrey designed and implemented a three-year intervention that followed one class from Grade 3 through Grade 5. For this experimental curriculum, "ratio and proportion were assumed to be intimately connected to multiplication and division" (1996, p. 3). Confrey regards the repeated-addition model of multiplication (Fischbein, Deli, Nello, & Marino, 1985) as potentially hindering the development of multiplicative reasoning that she sees as qualitatively different from additive reasoning (for related views see also Thompson & Saldahana, 2003; Stroup, 2004; for an ethnography-based distinction between models of multiplication, see Urton, 1997). Confrey introduces the 'splitting' model of multiplication as actions of doubling, folding, sharing, and magnifying that generate from a single object multiple parts or multiple copies. This model is posed as an alternative to the repeated-addition model of multiplication—an alternative she argues can better prepare students to work in 'multiplicative space.' Confrey integrated splitting activities into her design for ratio and proportion to support students moving into multiplicative space.

In Confrey's implementation, 1st and 3rd-grade students practiced sorting shapes on the basis of geometrical similarity. They performed fair sharing of manipulatives, with some students spontaneously constructing arrays to support the creation of equivalent groups. Arrays, e.g., of 4-by-32 chips, helped students, later in the program, to relate across numerically-identical situations involving different models of dividing—students saw that 128 divided by 4 resulted in 32 whether the situation involved dividing into 4 equal parts ('partitive') or into groups of 4 objects each ('quotative').

Confrey introduced the idea of 'unit ratio' (1:a or a:1) in 4th grade as the invariance across different numerical cases in a multiplicative relationship that exists between two situated quantifiable dimensions, where one of these quantities is 1. These dimensions may be tied one to the other intrinsically, logically, or by some constant rule, for example, the number of horses to the number of horse legs (1:4) or the number of dinner guests to the number of chopsticks (1:2). A related idea to 'unit ratio' was that of 'ratio unit.' The idea of 'ratio unit' (a:b) was introduced through students' work on maintaining qualitative equivalence between sets of mixed quantities (e.g., maintaining the same "taste" between different liquid mixtures). Confrey did not ask students to start from 0:0 and then "build up" (repeatedly add) *a:b* pairs so as to maintain the equivalence. Confrey asked students to begin from larger quantity-pairs and then determine the 'littlest [a:b] recipes' (1998, p. 44), "the smallest whole number equivalence for a given proportion" (Confrey, 1995, p. 13). In working on such problems, students used ratio tables and then abbreviated forms of these tables that included only two rows of the ratio table ("ratio box").[1] Students practiced finding missing values in proportions. The students determined the multiplication and division operations that would turn one of a known pair of values into the other, and then they applied these operations to the other pair with the unknown value. The curriculum also included students' exploring representations of proportion on the Cartesian plane (proportionate ordered pairs, e.g. (2,3), (4, 6), (6, 9), etc., form a straight line projecting from the origin). Finally, students applied their understanding of ratios in thinking of fractions, decimals, and percentages as based on part-to-whole relations.

Whereas Confrey's view is stimulating, and certainly has important implications for later work with exponents (see Confrey, 1994), I do not regard her view as necessarily constituting an argument for the advantage of a curriculum that downplays repeated addition as a model of multiplication and hence of ratio and proportion. I now explain my position and its implications for the design.

Multiplicative Operations, Situational Models, and Numerical Solution Procedures

In order to gain mastery of a mathematical domain, students need to practice, develop, and integrate situational models, numerical solution procedures, and fluency with recurring numerical operation patterns (the basic multiplications and divisions, such as the multiplication table). In Fuson and Abrahamson (2004b) we analyze rate, ratio, and proportion situations to demonstrate in these domains the theoretical dissociation

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[1] There are structural similarities between Confrey's 'ratio box' and my 'proportion quartet,' yet the proportion quartet also includes the multiplication-table row-and-column numbers, and these are foregrounded as solution tools.

between, on the one hand, action-based models that may undergird students' understanding of situations involving multiplicative relations and, on the other hand, students' number-based actions in solving such problem situations. For instance, the following two multiplicative situations may evoke different action-based models but would be solved with the same number-based actions: (a) a mummy-dog that has a litter of 3 puppies every year over 5 consecutive years; and (b) 5 mummy-dogs that on a single day each gives birth to a litter of 3 puppies. The single-mum situation suggests a repeated-addition model in which the 3-puppy group is iterated sequentially a total of 5 times (3, 6, 9, 12, 15). The many-mums situation suggests a 'from-one→many' model in which the five 3-puppy groups emerge simultaneously (5→15). Yet for both situational cases, a student who does not know the product of 5 and 3 will use a repeated-addition numerical solution procedure: the student will determine the total number of puppies by counting on the 3-puppy unit 5 times (3, 6, 9, 12, 15).

Students' fluency with recurring numerical cases of multiplication may hide difficulty with or even lack of mental models for interpreting mathematical situations. For instance, a student who knows that 5 * 3 = 15 and therefore need not count up to determine the product may nevertheless be challenged when asked to ground that knowing in a situation, say as 5 batches of 3 puppies. That is, an instantaneous response to a numerical multiplication does not necessarily indicate a student's fluency with a 'multiplicative space' that ostensibly transcends addition-and-subtraction numerical operations. In fact, I contend that misinterpreting students' fluency with multiplication numerical operations as indicating a mastery of multiplicative situations or concepts may impede these students' developing these multiplicative concepts. This contention informs the following assumption of my design rationale: in order to ground fluency with multiplication in multiplicative situations, students need support in construing multiplicative situations as affording repeated-adding numerical solutions. Such scaffolding, I maintain, is crucial for fostering students' understanding of multiplication and, moreover, of ratio-and-proportion situations, in which *two* multiplicative processes need to be coordinated. Specifically, such scaffolding could afford students' situation models that structure their solving with understanding complex numerical cases of ratio and proportion, such as $56{:}63 = x{:}72$.

Proportional equivalence as constant change. Similar to Confrey, I see an *a:b* ratio as conveying mathematically that aspect of a situation involving variation along two dimensions which lends it qualitative consistency across different quantitative instances. This qualitative consistency can be, for example, the unchanging taste of *a*:*b* mixtures as I change both *a* and *b*, the sameness in shapes that are geometrically similar, such as 3-by-6 and 2-by-4 rectangles (or the identity of a shape as seen from different distances), or the unvarying orientation of a sloped graph that connects coordinate points. But consistency across proportional relations need not be evoked only by the qualitative *identity* (e.g., same taste) across the numerical instances. Consistency in the *change* from one state to the other, e.g., the action of increasing the number of bones by 2 for every 3 dogs, may also afford a sense of the invariance of ratio. In that, I regard the build-up situational model as a powerful learning support towards understanding the numerical relatedness of proportional ratios. This consistency in +*a:*+*b* change can be lodged in a narrative, e.g., about +3 dogs that are linked to +2 bones in some situation, say a situation in which every 3 additional dogs that enter a dog fair receive 2 bones to share. Such a repeated-adding interpretation of numerical proportional progression sits well with all 5th-grade students' comfort with the addition operation. It scales up the ratio unit, e.g., 3:2, to form successive multiples of the ratio unit, e.g., 3:2, 6:4, 9:6, 12:8, etc., while maintaining a sense of consistency that supports an understanding of equivalence.

Mind the gap. The responsibility of mathematics educators is to foster students' grounding new ideas in existing understanding. I identify a possible learning gap between additive and multiplicative concepts and argue that foregrounding repeated-addition as the psychological underpinnings of multiplicative situations promotes continuity in students' mathematical development, both in performing operations with understanding and, as a result, in their affect towards learning multiplicative solution procedures. We have suggested the term *additive–multiplicative* to describe students' grounding multiplicative concepts in repeated-addition situation models and numerical procedures (see Fuson, Kalchman, Abrahamson, & Izsák , 2002; Abrahamson, 2002a). That is, I believe that students need time to move on from addition to multiplication and that critical to this moving on is students' understanding of how numerical multiplication shortcuts numerical repeated addition. So in designing the experimental units, I anticipated that introducing ratio as "build-up" situations would not hinder students understanding, but would undergird a robust model of the domain.

A Design for Ratio and Proportion That Combines the Best of Previous Designs

Ratio tables constitute a major mathematical representation in my design (see Figure 1) as in the Confrey (1995) and Kaput and West (1994) designs. Also, similar to Confrey (1995), I use a smaller 2-by-2 table (the proportion quartet) that is derived from the ratio table, though the proportion quartet also includes the factors of the ratio numbers. However, unlike all previous designs, I foreground the multiplication table (MT) as the central mathematical representation of the domain that supports students' moving from additive–multiplicative to multiplicative understanding of ratio and proportion. The MT is the defining source of the structure and use of the ratio table (RT) and the proportion quartet (PQ). Specifically, the MT is the source both of the additive–multiplicative repeated-addition build-up context that is associated primarily with the RT and of the multiplicative solution tools for unknown-value proportions that are associated with the PQ.

I reserve the name 'ratio' for horizontal and not vertical relations between numbers in tables, and thus use ratio tables in which the ratios are rows. Also, I reserve the name 'fraction' for vertical part-to-whole number pair relations. Such a differentiation in the formats of the symbolical representations can potentially support students' conceptual differentiation between ratios and fractions. At the same time, I create opportunities for students to explore relations between ratios and fractions (see, in Figure 1, how the four values in the PQ can be seen either as two equivalent ratios, 6:14 = 15:35, or as two equivalent fractions, 6/15 = 14/35). That is, the PQ is a mathematical tool that was designed so as to afford students' linking among a range of multiplicative concepts. The idea of mathematics-education linking tools, or spatial–numerical bridging artifacts, is the focus of the following theoretical section.

Theoretical Sources of the Apprehending-Zone Model of Design, Teaching, and Learning

This section explains the apprehending-zone model. The apprehending zone (AZ, Fuson & Abrahamson, 2004a; see Figure 2) is a model of student learning in mathematics classrooms. The model foregrounds students' building understanding for a domain through interacting with a design's learning tools. An understanding of a domain is: (a) embedded in a designed curricular unit (Figure 2, on bottom); (b) mediated as a cultural artifact through student participation in designed activities that are modeled by the teacher and higher-achieving participating classmates (Figure 2, middle); and (c) becomes individual students' action tools for identifying, organizing, and solving domain-specific situations (Figure 2, on top). I now situate the AZ within the contributing literature by discussing cognitive, philosophical, constructivist, and social-constructivist perspectives on learning with objects (tools, artifacts). To focus the discussion of these perspectives, I begin by describing a special class of teaching-and-learning tools that I call *spatial–numerical pedagogical bridging artifacts*. These artifacts help students link informal schemes for real-world situations and formal understandings of mathematical constructs that are targeted in a design. The design discussed in this dissertation attempted to foster students' seeing and using (= instrumentalizing) the multiplication table as a spatial–numerical bridging artifact for the domain of ratio and proportion. Thus, this section will position the multiplication table as a domain-pivotal spatial–numerical tool designed to bear and build individual and interpersonal meaning for the target domain of ratio and proportion.

Spatial–Numerical Pedagogical Bridging Artifacts

This dissertation discusses the multiplication table as a case of pedagogical bridging tools, specifically as a case of spatial–numerical bridging tools. In the design that is the focus of this study, classroom activities around the multiplication table (MT) facilitate students' linking their informal and formal understandings pertaining to the target domain of ratio and proportion. The MT serves as a platform for linking repeated-adding and factor*factor→product models of the multiplication operation in the context of ratio-and-proportion situations. I make no claims of precedence in creating learning tools of this general type. Designers and teachers of mathematics create these tools regularly. Rather, I am submitting a perspective on these designed tools as a unique class of pedagogical artifacts. Elucidating a general rationale for designing these artifacts may inform the work of designers of mathematics education. Specifically, understanding how these artifacts foster students' building meaning for and fluency with the target domain of ratio and proportion may, in turn, foster designers' sensitivity to possible learning gaps that students experience in other domains and guide design responses to these gaps. So my focus on this class of bridging artifacts is an attempt to bring together a theory of learning and a theory of design.

Spatial–numerical pedagogical bridging artifacts are mathematical representations that help students bridge between spatial and numerical media in which students' mathematical understanding is grounded and instantiated. Everyday situations are inherently spatial because agents occupy space and act over time. Formal mathematical knowledge is numerical, or more broadly symbolical, because it is expressed in culturally agreed signs—signs that in historical and day-to-day practical use refer to agents and objects in the world and relations between them.

Mathematical symbolical structures, such as equations (e.g., 6:14 = 15:35) *are* spatial only in the limited sense that the symbols occupy space on a page and are written and read in a particular conventional orientation, e.g., left to right. Conversely, groups of objects in the world *are* numerical in the narrow sense that we can estimate and compare between quantities just by perceiving them, e.g., we can attend to 14 objects as being more numerous than are 6 objects. We can count objects, too, and counting itself is a bridging context for integrating quantitative and spatial facets of cognition into psychological constructs (*central conceptual structures*) that include cultural referents, such as the verbal utterances "one, two, three, etc." and their numerical counterparts "1, 2, 3, etc." (Case and Okamoto, 1996).[2]

Mathematical notation, such as equations or graphs, captures and communicates numerical relations that potentially can be understood in terms of quantities in the world and the relations between them. However, for many students, and certainly for challenging mathematical concepts, the gap between numerical and everyday spatial representations is too wide to ford. For instance, it is difficult just by using implicit knowledge of situations in the world to understand in what sense 6:14 is equal to 15:35.

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[2] The idea of bridging artifacts in *design* owes much to the work of Robbie Case and his co-authors, yet I do not necessarily subscribe to all aspects of their neo-Piagetian *theory*.

Representations that link spatial and numerical aspects of a math domain afford opportunities for many students to bridge the spatial–numerical conceptual gap. These artifacts constitute a platform or arena where students can link concise-yet-opaque symbols to implicit understandings of everyday situations that occur in space and time. Through these classroom activities, students leverage, organize, and discuss in the classroom forum their implicit understandings to make sense of mathematical symbols.[3]

The thrust of this dissertation is that whereas spatial–numerical bridging artifacts contain the critical elements and relations for understanding mathematical concepts, these elements and relations are visible only to those who have learned, through individual multi-step learning paths, to see these critical aspects in the bridging artifact. Mathematical literacy requires adjusting the way we think both about the world and about inscribed diagrams and notations and linking these new views. The multiplication table and other spatial–numerical artifacts that were used in the intervention reported in this dissertation have the potential to empower students' intuitions for proportion because they afford procedures that are *spatially* attuned to students' intuitions. *Numerically,* bridging artifacts may bear a degree of mathematical sophistication that initially conflicts with students' intuitions (hence the learning issues of the designed domain) yet ultimately organizes students' work and supports their limited fluency with relevant operations. For instance, a row-by-row scalloping with the hands down the MT columns in narrating a repeated-adding ratio story affords students' grounding the mathematical operation in a

[3] A technology-facilitated example of spatial–numerical bridging artifacts are well-designed computer-based learning environments, such as *microworlds* (see Papert, 1980; Edwards, 1993).

spatial continuity. At the same time, the values that students see in that column, e.g., 3, 6, 9, 12, etc., may conflict with the quantitative element of the story, +3, +3, +3, +3, etc. Thus, students' conceptual development in mathematics in a constructivist classroom involves accommodating implicit ways of understanding situations so as to attune with *epistemic forms* (Collins & Ferguson, 1993) embedded in cultural instruments. This process is reciprocal, as I now discuss.

Tools, Instruments, Artifacts, and Learning

Bridging tools are designed classroom objects that become, through teacher-facilitated individual and collaborative activities, students' vehicles of reasoning about or a conceptual lens for real-world situations that students come to see as classes of some mathematical idea. In analyzing expert abacus manipulators' "*mental abacus*," Stigler—echoing Vygotsky—concludes that "perhaps the most powerful tools a culture can provide to the developing child will come in the form of specialized mental representations that are passed down through education" (1984, p. 175). I will be examining an example of a tool that is not only passed down *through* education, but is designed specifically *for* education.

By '*bridging* tools' I am referring to conceptual facilitators that help students both understand situations in terms of mathematical notation and, conversely, make sense of mathematical notation in terms of some situational schema. By 'bridging *tools'* I am drawing from many theoretical perspectives from diverse yet related disciplines that I have found pertinent for designing and for analyzing teachers' teaching and students' learning of mathematics. The following resources constitute the intellectual underpinnings of the apprehending-zone model and specifically the design for ratio and proportion.

The broader socio–cultural perspective (e.g., Vygotsky, 1962, 1978; Cole, 1996; Stetsenko, 2002; Ernest, 1988) examines a community's ways of using material or conceptual objects, even familiar objects that come to be used in novel ways, so as to inform novices' operative understanding of a new domain (see Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997, on the *emergent perspective* that explains how individual and social classroom practices are co-constituted; see Mariotti, 2002, on *instrumentalization*; see D. R. Olson, 1994, on how literacy modified human thinking). In that sense, the MT, instrumentalized by students participating in the design, is at the juncture of personal and interpersonal construction of meaning for the domain of ratio and proportion.

Greeno's (1998) construct of classroom *semiotic networks* includes a web of coordination between participants, objects, and goal-oriented ways of applying these objects (and see Lemke, 2003, for a semiotic analysis of mathematical notation). A goal of the design was that students come to build around the mathematical representation a semiotic network that is linked to their broader mathematical understandings and supports their reasoning and communicating about situated cases of ratio and proportion.

Anthropological studies of human–tool interaction in the work place (e.g., Hutchins, 1986, 1995) have demonstrated the inextricability of tools from human cognition-in-action. Goodwin (1994), who looked in particular at the mediation of knowledge from experts to novices in the workplace, has coined the term *professional* *vision* — "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 33). The mediation of professional vision includes the expert's purposeful highlighting and coding for the novices perceptually available information as well as discourse related to specific ways of constructing the meaning of objects, actions, and situations in the perceptual field. Applying the construct of professional vision to mathematics education, one can see an important role of mathematics educators as helping students gain mathematical insight into situational contexts embedded in classroom learning supports. So learning to use professional vision or mathematical vision is a crucial aspect of students' instrumentalizing design elements in building the classroom domain-specific semiotic network.

Psycholinguistic theory (e.g., McNeill, 1992; McNeill & Duncan, 2002; Goldin-Meadow, 2003) expands the Vygotskiian perspective of adults' modeling knowledge so as to include, beyond verbal cues, all inter- and intra-personal aspects of communicating (*thinking-for-speaking*). These aspects include the prosody on which the verbal message rides (e.g., intonation, inflections, stresses, and pauses), gesturing, and facial expressions. All these aspects communicate and hence model subtle aspects of attending to contextualized objects, whether these objects are directly perceived or imaged. This perspective broadens the analysis of how teachers model and students adopt culturally-shared modes of interacting with artifacts (see Abrahamson, 2004, on *embodied spatial articulation*). My appreciating the importance of gesturing in fostering students' mathematical vision of the designed representations directed my attention to students' gesturing in analyzing classroom data. A study that focuses on students' verbal participation and does not foreground their gesturing is nevertheless implicitly informed by the gesturing, because students' utterances per se (audio data) are often difficult to understand without the visual element (video data). Likewise, for the students in the classroom, the visual–dynamic information from speaking students is important in making sense of the discourse.

In the background of the AZ design-research model of students' learning mathematical ideas through interacting with objects is the philosophical perspective of phenomenology. The agenda of Husserlian–Heideggerian phenomenologists (Heidegger, 1962) and their critics (e.g., Merleau–Ponty, 1968; Levinas, 1974) has been to develop viable terminology for describing perception/body-based reasoning in action. Thus, this agenda contributes to our understanding of how students are able to build complex action-based understandings of the world upon their sensory–motoric interacting with tools and people (see also Varela, Thompson, & Rosch, 1991, on cognition as *embodied action*; see Vergnaud, 1983, 1994 on *theorems-in-action*; see Freudenthal, 1981, 1986, on phenomenology in *realistic mathematical education*; see Nemirovsky, Noble, Ramos–Oliveira, & DiMattia, 2003, on the *symbolic body*—the motorics of developing mathematical and scientific competencies; see Piaget, 1952, 1971, on schemes).[4]

A related resource that describes the role of body-based activities in mathematics cognition is Urton's (1997) *grammar of space–time*. With this ethnomethodology-of-

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[4] "The hand cannot be thought as a thing, a being, even less an object. The hand thinks before being thought; it is thought, a thought, thinking" (Derrida, 1987, p. 171).

mathematics construct, Urton demonstrates a case of tacit cultural schematic images. These images are grounded in objects that are focal to a culture's practices. For instance, a many-leafed stem of corn serves the Quechua-speaking people of the Andes to ground communications involving numeracy *whether or not the stem is physically present*. That is, the shared image serves as a cultural metaphor that is evoked with speech and gesture and supports reasoning in communication. I believe that the spatial–numerical bridging artifacts that we introduce into the classroom microculture foster shared images that ground the discourse of the microculture about the domain of ratio and proportion (Abrahamson, 2004).

A study of students' learning through interacting with tools and with images of these tools that students gesture to even in the absence of these tools is also informed by certain agenda of theories of *kinesthetic image schemas* (Johnson, 1987, 2003; Lakoff & Nuñez, 2000). Particularly relevant is the search for perceptual–conceptual primitives that may underlie cognitive processes and the role of language in mediating these images. The role of images in human reasoning has been a focus of cognitive science, too. Some theoreticians believe that percepts are not only useful but all-encompassing elements of reasoning. Barsalou (1999) maintains that "abstract concepts are perceptual, being grounded in temporally extended simulations of external and internal events" (p. 603). Parallel efforts to understand the role of images in students' learning mathematics can be seen in contemporary models of mathematics cognition. For example, there are models that emphasize the centrality of students' building and using images to make sense of mathematical activities and to construct new understandings, e.g., the Pirie–Kieren *dynamical theory* (Kieren, Pirie, & Gordon Calvert, 1999; see also Goldin, 1987). In a similar vein, there are calls from mathematicians and logicians to affirm pictorial forms of reasoning as credible and sometimes sufficient tools of proof (e.g., Brown, 1997,[5] see also Davis, 1993, on *visual theorems*; Barwise & Etchemendy, 1991, on *visual information*; and Arnheim, 1969, *visual thinking*; see Polanyi, 1958, and Whewell, 1989, for a philosophical perspective on *tacit knowledge* and *intuition*)

I find of particular promise an agenda that would explore relations that can be established between students' implicit forms of reasoning and formal cultural forms and heuristics in mathematics and science. On the one hand, there is the study of *enabling constraints* (Gallistel & Gelman, 1992; Gelman & Williams, 1998), the idea that human perception itself can be thought of as a set of specialized tools that has evolved to support selective attention and mental construction of objects in the environment (see also Gigerenzer, 1998, on *ecological intelligence*). From this perspective, mathematics-education practitioners may be supported by research that informs our understanding of what is easy and what is difficult for students to understand when they work with visual aids (see Abrahamson, 2002b). On the other hand, there are *epistemic forms* and *epistemic games* (Collins and Ferguson, 1993), "structures and strategies to guide inquiry" (see also Norman, 1991, on *cognitive artifacts*). For example, tabular structures (rows and columns) support exploration into properties of a class of phenomena. From the latter perspective, teachers model practices of inquiry into real-world situations by

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[5] "As telescopes help the unaided eye, so some diagrams are instruments (rather than representations) which help the unaided mind's eye" (Brown, 1997, p. 174)

foregrounding the affordances and constraints of these problem-solving tools and situations. An example of these tools for inquiry in mathematics are spatial–numerical artifacts. The promise I see is in studying relations between enabling constraints and epistemic forms so as to inform the design of spatial–numerical pedagogical artifacts that bridge between these personal and cultural resources (see Abrahamson, 2004).

Students who have not yet learned the topic of ratio and proportion nevertheless have certain implicit sense-making intuitions that could inform their learning proportionality (e.g., Resnick, 1992; Abrahamson, 2002b). Moreover, by participating in constructivist learning environments that embrace and foster this informal knowledge, students can assimilate the professional instruments of proportion. That is, I believe that students can learn a mathematical topic through learning to operate instruments in modeling and solving situations that they identify as affording these instruments. Yet, in order for students to identify the applicability of mathematical tools to a problem at hand, these tools themselves need to resonate with students' intuitions of these problem situations that they are modeling. This is why bridging artifacts are necessary—students initially incur difficulty in appreciating the affordances of the given mathematical tools for the solution of problems, because the tools do not seem directly related to the problem at hand. The vision and challenge of constructivist education (e.g., von Glasersfeld, 1987) has been to foster learning environments in which students can leverage their intuitive knowledge in learning to use and communicate about and through mathematical instruments that are introduced into their classroom (see also Whewell, 1989; Wilensky, 1997). These instruments may be historical or newly designed.

Historical instruments are designed to be assimilated into students' implicit sense making—the instruments are *epistemically ergonomic*—simply by virtue of these instruments having evolved over time by humans who had the same or similar cultural objectives and sense-making intuitions. Likewise, recently-designed mathematical instruments that resonate with humans' implicit sense-making intuitions can afford better learning, and it is through iterated design-based teaching–learning studies that we modify these historically young instruments so as to make them as epistemically ergonomic as possible. These recently-designed pedagogical mathematical instruments that I am calling spatial–numerical bridging artifacts should comply with two complementary objectives. These artifacts should support classroom discourse that evokes spatial, temporal, and quantitative features of students' intuitive reasoning coming from their lived experience as cognitive beings in the world. At the same time, these artifacts should incorporate numerical and other symbolical–syntactical features of formal mathematical inscriptions that support students' developing as mathematics problem solvers.

The goal of education reform and in particular the constructivist mathematics classroom is to foster students' learning with understanding. I see students' coming to understand a mathematical concept as their being able to use relevant mathematical artifacts so as to leverage available information towards making informed inferences. So learning a mathematical concept involves progressively mastering the skill of choosing, constructing, applying, operating, and interpreting relevant mathematical instruments in modeling situations that one might say potentially 'carry' or 'reflect' the concept.

As the above discussion suggests, conceptual understanding and material and symbolic tools are intertwined and inform each other in the enactment of goal-oriented procedures. On the one hand, problem-solving procedural skills incorporate culturally- and personally-attested and valued representations. Reciprocally, structural elements of these representations can be heuristic in directing attention to crucial quantities and quantitative relations between elements in problem situations. So learning to use these representation–tools can empower students: (a) The tools afford and shape intuitive models for complex situations; (b) they constitute classroom objects that focus discussion, in which intuitions are explicated as mathematical knowledge couched in shared vocabulary; and (c) learning to use the tools can support mathematically mature understandings that are grounded both in the symbolical representations and in the situations. So I maintain that it is warranted to introduce cultural tools in a constructivist classroom—that attentively preparing and timing the provision of such tools need not preclude students' experiencing reinvention and discovery (see also Montessori, 1986). On the contrary, these tools empower students by affording and explicating intuitive models for complex situations towards mathematically mature understandings that are grounded both in the symbolical formats and in the situations. Also, the representational–situational perspective on the learning of mathematics—a perspective that foregrounds the centrality of students' mastery of tools—may constitute a sufficient analytic structure for looking at students' understanding of a domain. Finally, this perspective may be useful in couching mathematical understanding in terms of classroom activities (see Abarhamson & Wilensky, 2004, for further elucidation of 'bridging tools').

Coming from this "tools" perspective, this dissertation examines a design-research study in the domain of ratio and proportion in which 5th-grade students used new mathematical representations as tools for representing and solving real-world situations conveyed in word problems. I will present and analyze data suggesting that students assimilate domain constructs embedded in mathematical representations by learning to use these representations as tools for participating in classroom personal activities. Moreover, students assimilate domain constructs by understanding and linking between the representations through participating in classroom *inter*personal interactions with their teacher and peers. Also, teachers orchestrating discussions in a classroom community can foster student learning that varies in individual path and pace towards mastery of complex mathematical domains. Thus, teachers integrate constructivist and social-constructivist pedagogical philosophies. I perceive our role as design researchers in making available for the teacher and students tools that afford such meaningful learning of mathematics.

I believe that design should help identify, foster, and sustain students' use of intuition in their operating the designed mathematical instruments of the domain so as to make this use meaningful. For instance, as students learn to reason about proportion, the design helps keep meaning in proportion. Also, the apprehending-zone theoretical perspective on mathematical meaning, which focuses on students learning to use instruments, sharpens the usually murky notion of mathematical 'meaning,' and thus this perspective helps keep 'meaning' in proportion.

The Apprehending-Zone Model

In creating the ratio-and-proportion unit, the cognitive and social-constructivist theoretical resources *implicitly* informed the design of mathematical representations and activities around these representations. Later, towards the data analysis, this theory-in-practice gradually coalesced into a model of design, teaching and learning that focuses on students' learning to use mathematical tools.

We sought a perspective that would use the classroom mathematical representations and situations and students' activities and discourse around them as lenses on students' learning of a domain. The *apprehending zone* model (Fuson & Abrahamson, 2004a; see Figure 2) is our attempt to strike a balance between the dual concomitant aspects of our practice—on the one hand, the creative design and work with material objects that students use individually and communicate about with their teacher and peers, and, on the other hand, the intellectual resources that inform our work. So in creating the apprehending-zone model, we focused on mathematical representations and students' building understandings of how these representations are used and how the representations relate one to the other. Such focusing: (a) affords descriptive language that is grounded in students' and teachers' classroom experiences (the classroom space–time reality) and is thus amenable to analyzing classroom data (student utterances and actions); (b) facilitates coherence of our perspective for a broad and diverse range of practitioners in education and education research; and (c) is informed by our viewing mathematics as a social practice that is assimilated by individuals working with and learning to use mathematical tools in collaboration with teachers and peers.

The *apprehending zone* (AZ, see Figure 2) models student learning as developing mastery of a domain's mathematical tools. Initially, such mastery is embedded in aspects of the design only as a potentiality of learning, because students have not built any meaning for these tools (Figure 2, bottom). The potentiality of the design is actuated through student learning, when students study the designed unit through designed classroom activities. Students learn to use the tools needed to link representations (Figure 2, center, to the right of the student), link situations (Figure 2, center, to the left of the student), and link representations and situations (Figure 2, center, arrows across from right to left—*mathematizing* the situations—and from left to right—*storyizing* the mathematical representations). Classroom activities foster linking actions that ground solution procedures in situated action models. Thus, student participation in teacher-facilitated interactions with designed tools mediates the design-researcher's own model of the domain: this model is embedded in the design (Figure 2, bottom), is acted out in the classroom individual and interpersonal time–space (Figure 2, center), and is assimilated by individual students into action schemes (Figure 2, top). So students apprehend mathematics by interacting with concrete designed objects (a Piagetian constructivist perspective) and these interactions are modeled by informed adults and student leaders (a Vigotskiian socio–cultural perspective).[6] Our use of 'zone' captures the attention of the Vygotskiian framework to interpersonal sensitivities that inform teacher and student adaptive communicating.

[6] I use the Piagetian–Vygostian antinomy as a rhetorical dichotomy, even though their work is by no means orthogonal (e.g., see Cole & Wertsch, 2002) but rather overlapping and complementary.

According to the AZ model, students' learning process is dynamic, complex, and personal: during an intervention, for each student the various mathematical constructs are at different points of emerging from the design and these emerging understandings feed back into the learning process in accommodating other constructs. So each student follows a different learning path within the design, but with the help of an effective teacher, students in the classroom as a whole build sufficiently shared understandings to permit meaningful communication. The AZ model illustrates a single student but the horizontal axis implies the presence of and interactions with others. These other participants are the students and the teacher within the classroom space (what Brousseau, 1986, calls the *milieu*). In the periphery of the student's milieu are interactions among researchers and teachers in building the design space. All these interactions are pivotal for individual students' understanding of the design, because the teacher models the use of the designed mathematical artifacts. Also, classroom discourse invites students' verbalizing and formalizing their implicit interpretations towards creating classroom taken-as-shared mathematical constructs that students can use in the larger community (Cobb & Bauersfeld, 1995; Ernest, 1988). So the flow of information in the interpersonal plane enhances the learning in the personal plane and vice versa, and both planes are locked in a single complex dynamic mechanism of classroom learning centered on actions and interactions with the mathematical tools.

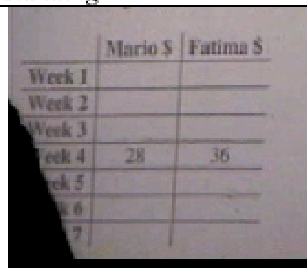
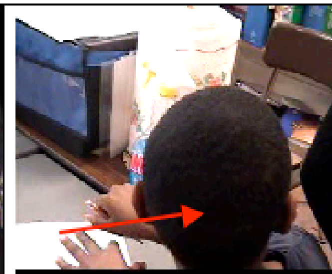
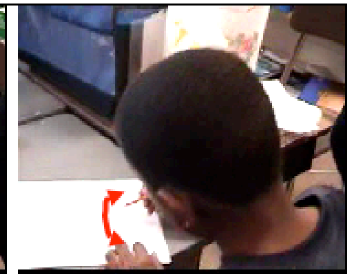
The Piagetian and Vygotskiian perspectives are intertwined in the AZ model. When applied to classroom data, these complementary perspectives foreground different aspects of individual and classroom learning. In this dissertation, I foreground the

Piagetian perspective of the AZ model. In applying the Piagetian perspective, I view individual-student learning opportunities as instances of students attempting to link design aspects that are embedded in the mathematical representations and situations. In Abrahamson (2004) I foreground the Vygotskiian perspective of the AZ model. In applying the Vygotskiian perspective, I focus on teacher- and student-mediated understandings of the mathematical representations. In particular, I apply gesture-based analytic lenses to show how teachers mediate their mathematical seeing-in-using of the classroom representations-as-tools. Also, I demonstrate students performing *embodied spatial articulation*. Embodied spatial articulation is an individual's design-facilitated negotiation between personal and cultural resources pertaining to the visuo-spatiality of mathematical situations and representations. The personal resources are proto-mathematical action-based images, e.g., a student who rapidly gyrates his hand while speaking about repeatedly adding a constant unit. The cultural resources are the appropriate seeing-in-using of classroom spatial–numerical artifacts, e.g., the same student scalloping his hand down a column of the MT (even though the MT is not physically present). Having explained the theoretical use of the AZ framework, I will now look at its complementary function in design.

The AZ model is a template for design in mathematics education, and a "filled" AZ model is a blueprint for classroom activities. The model both outlines the vectors of prospective classroom activities to be designed—back and forth between representations and situations—and creates placeholders for tools that will facilitate these activities. Specifically, where such tools do not exist, the designer should create bridging tools that will help students link reciprocally between the situations, between the representations, and between the situations and the representations.

From the AZ Piagetian perspective, the objective of the design reported in this dissertation was to foster individual students' linking of the situational and numerical aspects of the domain of ratio and proportion into an informed mastery of the designed classroom tools. I mean by *linking*, students' using of a design element with a new understanding that emanates from and is reconciled and aligned with an understanding of a previous element. Also, 'linking' describes students' sorting out connections between two or more action models and/or solution procedures afforded by a single bridging tool. The AZ model informs the analysis of classroom data in terms of students' learning issues. *Learning issues* are students' challenges in linking together the domain's elements, some challenges of which are specific to the design. So learning-with-understanding a mathematical domain is analogous to assembling an elaborate mechanism with some parts needing smoothing so as to fit one into the other. For instance, students need to link the PQ and the MT, but some students do not readily see/use the PQ as a mini-MT; that is, they do not attend to the multiplicative structure of the PQ. Another example is students linking repeated-adding situations with columns in the MT. Some students confuse between the repeated addend, e.g., +$3, + $3, +$3, etc. and the running totals $3, $6, $9, $12, etc. So the data analysis in this study reflects my viewing students' tackling the learning issues of the designed domain as successes and confusions in assimilating the design's situational and numerical elements into each other.

Understanding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[Ms W. projects the next problem on the overhead screen: "Mario and Fatima save their money on a regular basis. Fill in the numbers that are missing in this table." The representation is a labeled ratio table with missing values.]	**Saul**: You see what times 4 equ… gives you… [points from '4' to '28' and '36'] This would give you the first week,	**Saul**: …and then you keep on going [drives pencil in the air, down along column while looking away from table and up at Ms. W.]	**Saul**: 'What times 4 equals 36,' 'what times 4 equals 28.'

Difficulty

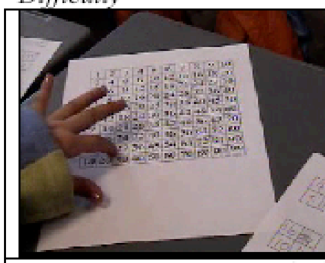
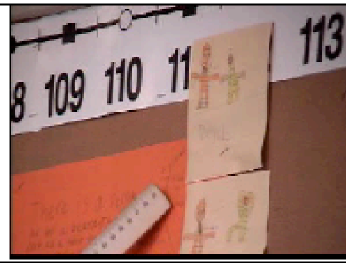
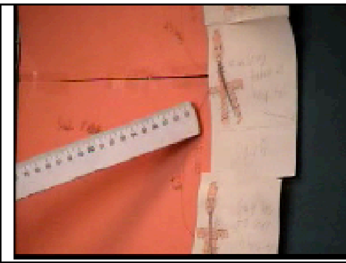
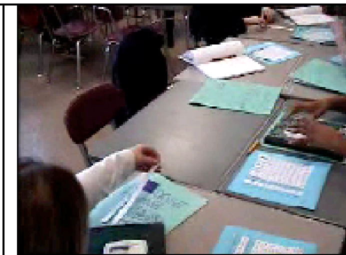
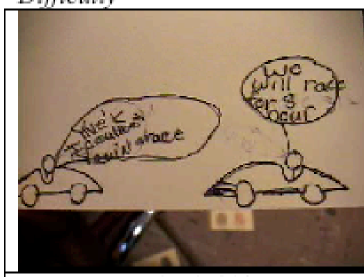
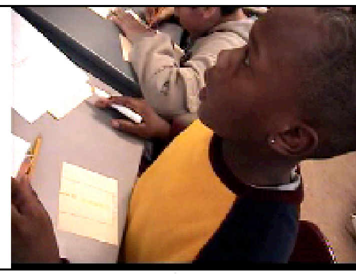
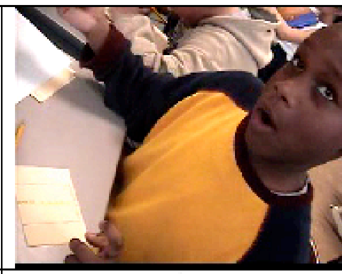
	7 [21 | 35] 7 3 [27 | 9] 3	3 (top) 7 [21 | 35] 7 [27 |] 3 (bottom)	3, 5 (top) 7 [21 | 35] 7 [27 |] 3, 5 (bottom)	3, 5 (top) 7 [21 | 35] 7 9 [27 |] 9 3, 5 (bottom)
<Day6_T2_29:19>		<30:20>	<30:26>	<30:33>
[Margarita has completed an unknown-value PQ with '9' (should be 45). Also, she has not used factors. She is copying from the MT that is on her desk directly into the PQ on her worksheet. Dor asks her to explain her solution] **Margarita**: 9… 9 times 3 is 27, so you put the 3 here [to left and right of the PQ bottom row] and you put the 7 here [top row] **Dor**: You know what, let's do this one together. Erase the 9 and we'll work together. [Following, D. and M. move between the MT and the PQ, and M. writes numbers into the PQ]		**Dor**: Ok, so you found that 7 is a common multiple of 21 and 35. Ok, so what column is 21 in? **M**: The 7… oh, no, in the 3. **D**: So put a 3 here [on top]…and immediately put a 3 across [below] because it's in the same column.	**Dor**: Good. 35—what column is it in? **M**: 5 [makes '5' w/ LH five fingers splayed]. **D**: ok, so put a 5… and immediately put a '5' across. [M. writes in the 5's] Alright!	**D**: 27, what… see, 3 times what? It has to be the… **M**: 9. **D**: Right, and immediately put it across. **D**: You see, so now you have an empty cell here. Kind of '9*5 is 'what'? **M**: 9*9? **D**: Look, you wrote it: 5… **M:** Ahh, 45 [writes 45].

Figure 3. Examples of student turn taking showing either understanding or difficulty with the learning issues
Figure 3a. Multiplicative Structure and Use of the Representations

Understanding

			
<Day5_H1_53:12>	<53:17>	<53:21>	<53:43>
Fernando: Ok, this comic is about the man and his beard. As you can see here his beard started out as 3 inches right *here*,	and then each day it keeps growing… [scallops his pointer down from top picture along penciled arc with the label "+3"] it keeps growing by 3…	by 3 more inches every *day*. [note beard getting longer] **Ms W.**: [to class] What are the columns you need to create a rate table for this situation?	[class responds that they need 2 columns, "1" and "3," that are labeled "Days" and "Inches." Students work with the MT and its column cutouts]

Difficulty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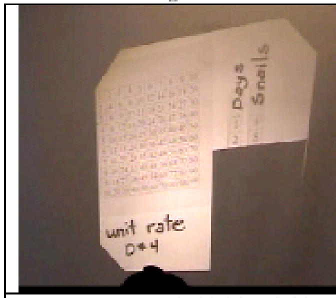
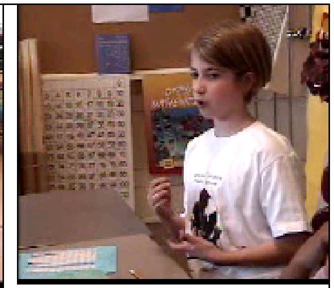
[Students are composing ratio-stories and preparing filmstrip columns that illustrate these stories. M'Buto's ratio story: "Bob was a race car driver so was M'Buto. One day Bob said I bet you in a race. M'Buto said we will race for 8 hours. Bob's car was 3 miles an hour and M'Buto's car was 5 miles an hour. Was Bob right?"]	[D. asks if that's the same as saying that "Every hour, Bob's car goes 3 miles, etc.," and M. says it's the same. D. asks how many pictures there will be in the ratio-story filmstrip that M. is preparing, and M. says there will be 3: beginning, middle, and end. D. asks for more details, and M. looks up to the MT poster and says that there will be a picture after 1 hour.]	**D**: Oh, super, so '1, 2, 3, 4,…'—like that? **M**: Yeah. [thinks a moment— that's not what he had said—he had only promised *three* pictures, regardless of the hours or the miles in the story] No—wait a moment—[looks up at the MT, looks back at Dor] then I'll show it after 1 hour again.	**D**: Oh, so you're like a reporter, and you're taking a photograph every hour. **M**: mm'hmm. **D**: Excellent.

Figure 3b. Rows/Columns are Repeated-Addition Sequences

Understanding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Ms W: [reads] "Big Bird collects snails. Every day, he adds 4 snails into his terrarium. How many snails does Big Bird have after 1 day?" [Students use their MT and cutout columns to model the situation then discuss the situation, labels and vocabulary]	**Ms W**: How many snails does Big Bird collect on the second day? **Class**: 8…. 4…. **Odelia**: 4 more, because you said *on the second day how many did he collect,* and he collects 4 each day…so on the second day he would have 4 more, which would equal 8 in total. **Ms W**: Right.	**Dor:** I'm just confused—Odelia told us that every day he collected 4, but I see here that opposite the 2 there's an 8, so, how does that work out? **Ms W**: Odelia?	**O**: she asked How many snails he collected on the 2nd day, and he collects 4 snails every day, so on the second day he would have collected 4 *more*, and not 8—he would have collected 8 in total.

Difficulty and then Understanding

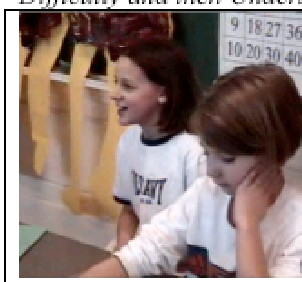
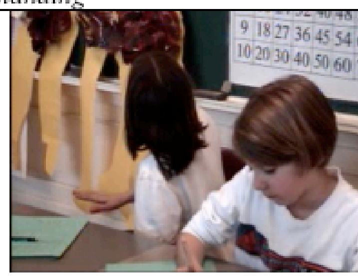
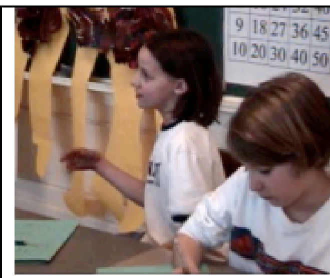
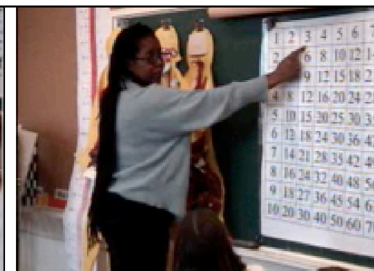
			
<Day4_H1_4:35>	<4:41>	<4:57>	<5:09>
Violet: Duffy Duck was going one day to Porky Pig's house,	**Violet:** and, uhhm, [turns around to look at the MT poster; turns back] one hour… in one hour he walked 3 miles […] So the first mile… the first hour…hnnn... Duffy Duck walked 3 miles…	**Violet:** In the second hour [turns around to look at the MT poster; turns back] Duffy Duck walked 6 miles, in the third hour Duffy Duck walked 9 miles…	**Ms. W**: That would be a total of 18 miles. [scallops down from 3 to 6, waits for V.'s response, then scallops to 9, 12, etc.] **Violet**: In the first hour he walked 3 miles, in the second hour he walked 3 *more* miles…

Figure 3c. Repeated Addends vs. Totals

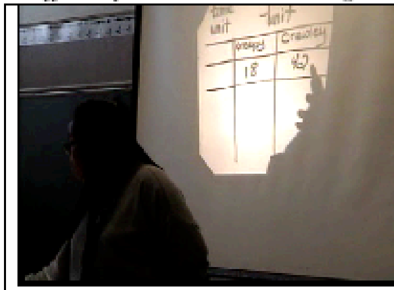
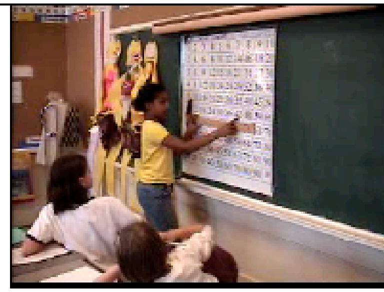
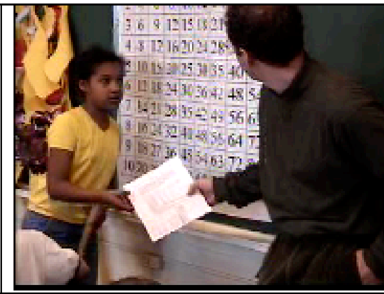
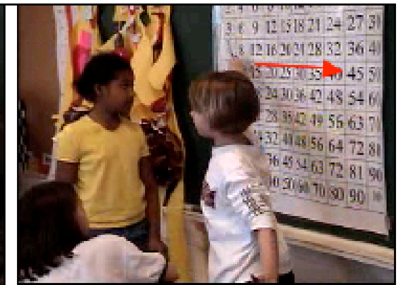
			
<Day4_H1_41:08>	<41:37>	<42:15>	<44:28>
[Students are discussing the following problem: "A pair of lizards—Creepy and Crawley—walk down the side of the Sears Tower. When Creepy has walked 18 floors down, Crawley walks 42 floors down. What could their walking rates be?" That is, students are asked to reduce 18:42 to a smaller ratio] **Ms W**: In some unit of time, Creepy walked 18 and Crawley walked 42. How are we going to find out how many time-units this all happened at?	**Alice**: I know the way: you can look on the chart and you can go to 18 and 42, and since they're across from each other… you could go up and see what column they're in; and 42's in 7, and 18 's in the 3 column…[…] Crawley's rate would be 7 and Creepy's rate would be 3.	**Dor**: So, tell me, Alice, how did you know to go to *this* 18 and *this* 42? ['18' and '42' each appear twice on the MT] **Alice**: I just tried it—I used 'guess and check' […] I looked at *this* 18 and it didn't have a 42, so I looked at *this* one and it had a 42. **Ms W.**: How did you know they had to be next to each other? **Alice**: I donno. **Ms W**: You're not sure, but you knew that they had to be next to each other? **Alice**: yeah. **Ms W**: Ok, just like in our table…	**Ms W**: Can anybody support or explain why the 18 and 42 have to be in the same row? **Odelia**: Because it's in the same amount of time, so that this [points to 1-column] will be the 'time,' sort of, so that's how you could tell [gestures across the 6-row, beginning from '6' in the 1-column and towards the right, covering 18 and stopping at 42] **Ms W**: They have to be next to each other because they represent the same period of time. **Odelia**: yeah.

Figure 3d. Rows/Columns in Stories

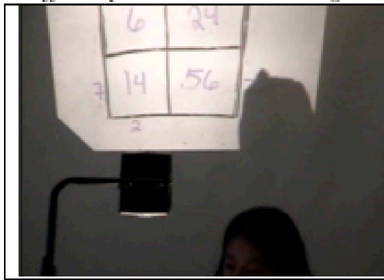
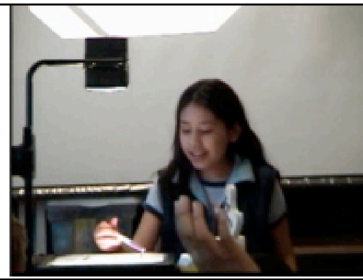
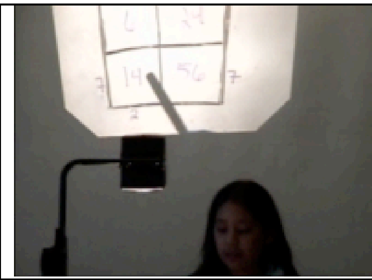
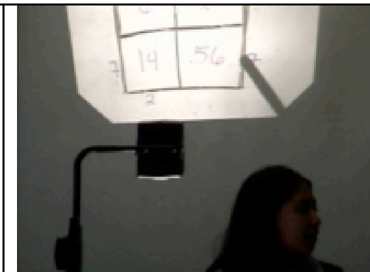
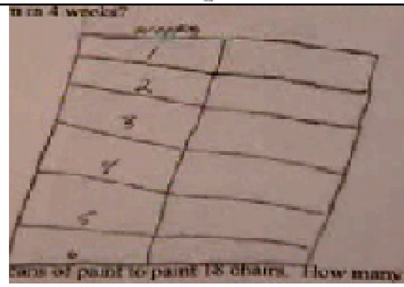
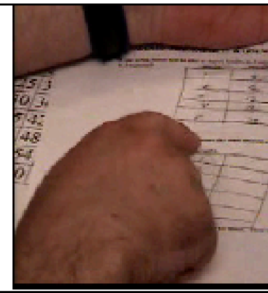
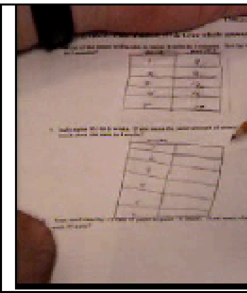
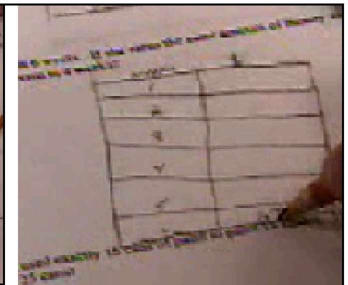
			
<Day5_T3_10:03>	<10:17>	<10:28>	<10:38>
[Students have been taking turns leading a discussion about common factors in PQ rows and columns. M'Buto had written in '2' both above the 6 and below the 14 and stated that 6 and 14 are in the 2-column of the MT and that 2 is the common factor of 6 and 14. It's Margarita's turn at the overhead projector, and her job is to determine, write in, and state the common factor of 14 and 56. She has written a 7 to the left of 14 as the complementary factor to '2' that was already below the 14, and she is now writing the '7' to the right of the 56, on the opposite side of the PQ]	[To all appearances, M. has understood that the 7 is the row number of both 14 and 56; that 7, which is in the factor column, is the *common* row number that links 14 and 56. It is not clear whether M. can articulate this linking-column idea mathematically or if she has just procedurally copied the 7 across, according to the practice that has been established in the classroom.] **Ms W**: Alright, state it, M.— what did you write? **M**: Uhhm, 7 times 2 is 14… **Ms W**: Ok, but now we're only going to talk about the 7, remember? We're stating it as a common factor.	**M**: 7 is a factor of 14, and… [3 sec pause]	**M:** 7 is a factor of 56. **Ms W:** Right, so 7 is a *common factor* of… **M**: …of 14 and 56.

Figure 3e. Linking Column for the Two Sequences

Difficulty and Understanding

			
<Day6_H1_38:28>	<38:30>	<38:32>	<39:03>
[Students are working individually on the following word problem: "Judy earns $63 in 6 weeks. If she earns the same amount of money each week, how much does she earn in 4 weeks?" (Kaput & West, 1994). Albertino has created a table. He has labeled the left column 'Weeks.' He has called Dor to help him.]	**Dor:** [gesturing down the 'Weeks' column] so <u>this</u> is the 'weeks' and what is this column? [gestures down the right column]	**Albertino:** [gesturing to top of right column] Ahh, money? **D**: Ok. So make a dollar sign. [Al. labels "$"]	**D:** Ok, so where should we put the fact that in 6 weeks she had $63? **Al**: Here [points w/ pencil, then writes in '63' in appropriate cell in the 'Week6' row]

Difficulty and Understanding

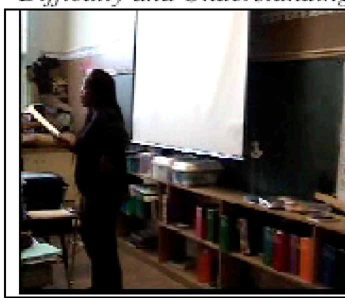
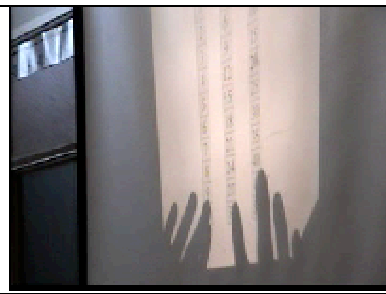
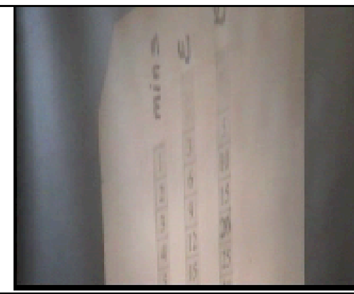
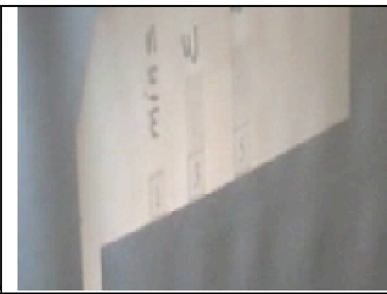
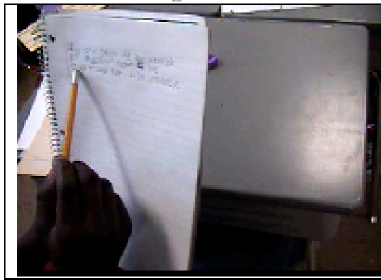
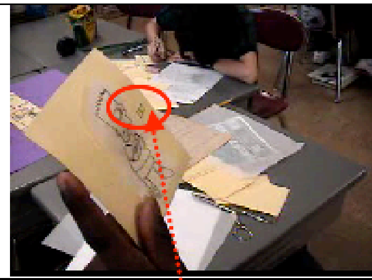
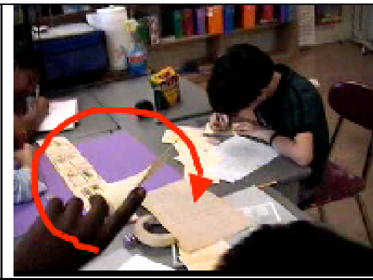
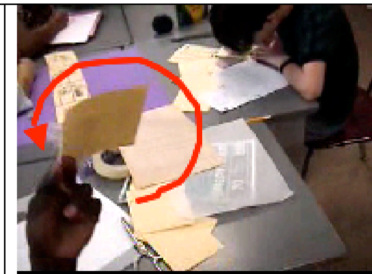
			
<Day7_H1_2:17>	<5:21>	<5:29>	<5:32>
Ms W: Ernie & Bert have a hotdog-eating competition. Every minute Ernie eats 3 hotdogs and Burt eats 5 hotdogs. At the given rate, when will Ernie catch up? Is this a rate problem—can we solve this w/ a RT? **Voices**: Yes. **Ms W**: Why?	**Fernando**: Because it's "change over time." **Ms W**: Which columns do I need for my RT? **Class**: 1...3…5. **Ms W:** [selects the 1-, 3-, and 5- cutout columns, places them on the projector] Do I have a RT? **Class**: No. **Ms W**: Why not? **Class**: You don't have labels. **Ms W**: Because we must… **Voices**: …label the table.	**Ms W**: [labeling] This represents… **Class**: Days…no, minutes! **Ms W**: This represents… **Class**: Ernie. **Ms W**: This represents… **Class**: Bert **Ms W**: Ok. [covers all but the top row] Here's my rate in the 1st day. [gradually exposing row after row]	**Ms W.:** Have they caught up yet? **Class**: No. **Ms W**: Are they closer? **Voices**: No. **Ms W**: [further revealing rows] Have they caught up? **Class**: No. **Ms W:** Are they closer? **Class**: No. **Ms W**: [chuckling] Have they caught up? **Class**: No. **Ms W**: Are they closer? **Class**: No [focused] no… no… no… no…no…no

Figure 3f. Labeling ("Table Manners")

Understanding

			
<Day3_H1_3:33>	<later…23:17>	<23:20>	<23:21>
Moses: [LD student. Reads his ratio story towards creating a filmstrip] "One day Bart at work for AJ Car Store. He makes $5 an hour. Lisa makes $3 an hour…"	**Moses**: I put zero so it will be at zero dollars. They didn't make money yet. And the next day,	Bert's going to have 5, uhhhm 5 [jerks LH forward] dollars and [twists paper clockwise]	Lisa's going to have 3 [quick anticlockwise twist] dollars.

Difficulty

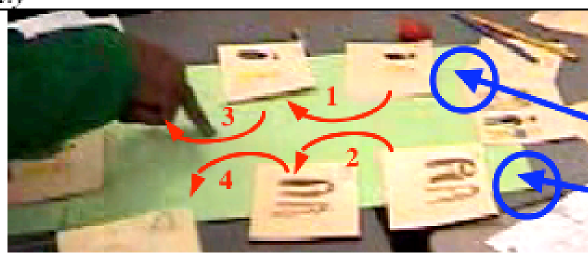
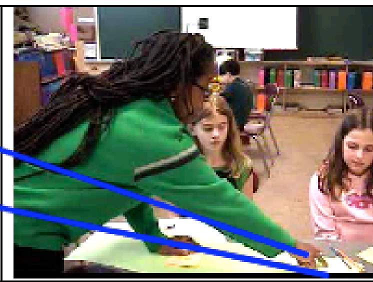
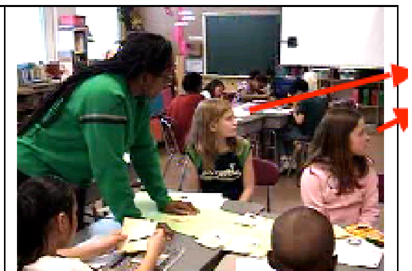
		
<Day 7_H1_25:12zoom_in>	<25:26>	<25:31>
[Ms W. discusses with students Kay's filmstrip ratio-story. The story is about a hair-growing competition between girls and boys. Form day to day, beards and hair grow longer, each by some constant increment. Numerals 1 – 4 show order of Ms W.'s gesturing, as she explains how to insert arcs with the constant addend marked in them. Note, at the top of Kay's poster is the "zero moment," before the hair-growing competition had even started.]	**Ms W**: [pointing above the boys column] Because in fact, these two points—on our chart—what would these two points be? [Ms W. compares picture columns to MT columns; prompts students to see this.]	[Students look back at the MT] **Kay**: [returns gaze to Ms W.] 1? **Ms W**: 1? Do we start at 1? **K**: Zero. **Ms W**: Zero. And in some ways, this space represents zero. [points again above picture columns in poster]

Figure 3g. Zero Starting Point

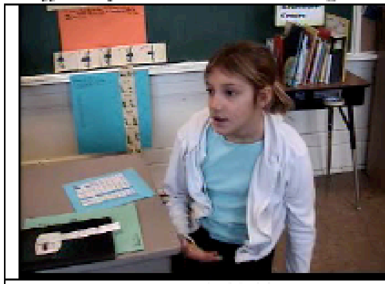
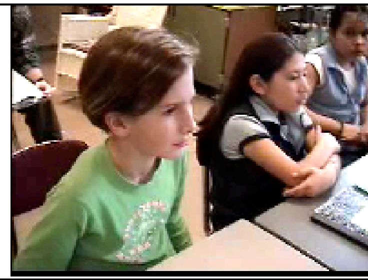
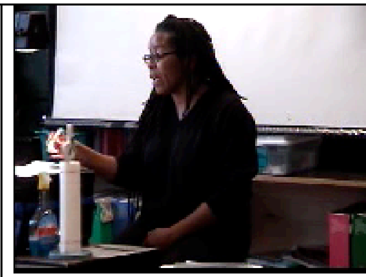
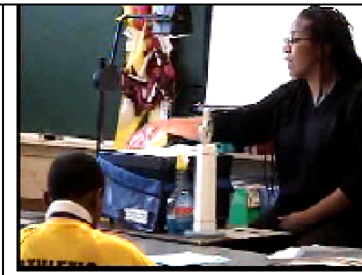
			
<Day5_H1_23:29>	<24:10>	<24:18>	<24:43>
[Dor had asked if '+4, +8, +12, +16…' is a rate] **Jay**: It's not a rate because it's not…*consecutive*…number per day [Class searches for a word. Jay suggests 'skip counting']	**Ms W**: We're not really skip counting we're just saying '4 +4 +4…' **Molly**: 'Consistent.' [Ms W. suggests 'constant']	**Ms W**: …so right now we have a constant… **Class:** Problem? Number? Rate? Unit? Unit-rate! **Voice**: Constant unit rate **Ms W**: *Constant* unit rate; and that's what's going to help us to know that we're not going to have…	[running down MT 4-column on overhead slide] …4 one time, *8* the next time, 12 the next time, 16 the next time, 'n 20 the next time. The question becomes "What's our rate?" and in our case "What's our unit-rate?"

Figure 3h. Vocabulary

Figure 3 (see preceding pages) shows examples of classroom discussion that are couched in terms of the designed domain's learning issues. Although my commentary on the excerpts focuses on individual students, the interactions in the data will suggest the complementarity and interdependence of the personal and interpersonal dimensions or axes of classroom learning. These examples both illustrate the classroom action tools of the design (see Figure 2, center) and ground the rationale of the data analysis that will be further explained in the Learning Issues section and detailed in the methodology section.

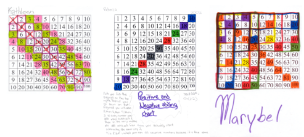
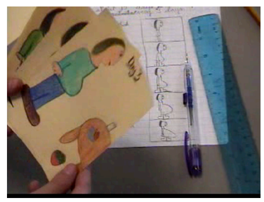
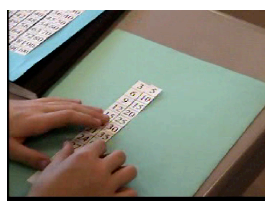
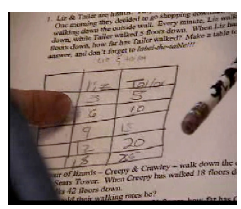
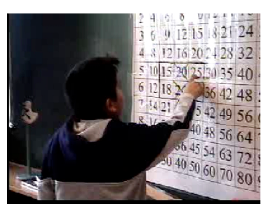
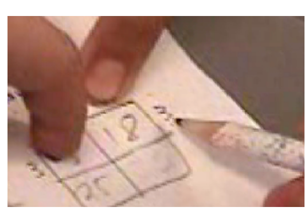
Although design research may produce data that help understand cognitive and social aspects of students' behavior, it is ultimately about developing as-supportive-as-possible materials and activities for students who are learning a new domain in mathematics. We now turn to the design for ratio and proportion that was the focus of this study.

Design

In this section, I detail stages in the ratio-and-proportion design. Table 1 (see below) overviews a day-by-day map of the design activities according to the classroom implementation that is the focus of this study. The three focal mathematical representations that will be discussed in this section are the multiplication table (MT), the ratio table (RT), and the proportion quartet (PQ; see also Figure 1). Any implementation is by nature a variation on the idealized plan of the design. I chose to explain the design through its implementation and not in its idealized form, both because this implementation was close enough to the design plan and so as to prepare the reader for the subsequent analysis that focused on this implementation.

Table 1

Classroom Activities in the Implementation of the Ratio-and-Proportion Design

Classroom Time in 60-Minute Periods	Name	Classroom Example	Activity
1.5	MT Patterns		Students find mathematical patterns in the MT and mark and explain them to their classmates.
1.5	Filmstrip		Students invent repeated-adding situations (e.g., a man getting fatter every day by 3 inches), draw series of pictures to show the running totals and constant addends, and connect the stories to MT columns.
Total of 8	Solving verbal and numerical problems		Progressively sophisticated representations are introduced and linked. Students constantly review representations.
2	MT cutout columns (strips)		Students select MT columns to create a ratio table, determine the unknown values on this table, and report solutions. Some problems state situations and others just give numbers.
2	Ratio Table (RT): 2 columns from the MT		Students build and label ratio tables, insert given values, determine unknown values, and state, present, and discuss their solutions. Discussions focus on linking the RT to the MT and inventing shortcuts.
2	Multiplication Table (MT): personal & classroom copies		Students locate on the MT three given values from a numerical or story problem, e.g., with 15:18 = x:30, to determine the unknown value, e.g., 25, that completes the rectangular constellation.
2	Proportion Quartet (PQ): a rectangle in the MT or 2 RT-rows		Students build and label a PQ (a 2-by-2 mini-MT), insert given values, and use common factors, multiplication, and division to determine and report the unknown value.

Lessons

The design's objective is for students to use with understanding the designed representations in solving numerical and word problems in the domain of ratio and proportion. The overarching design framework for accomplishing this objective was is for students to learn progressively sophisticated techniques for modeling and solving situations of proportionality. The numerical cases embedded in the word problems and numerical items (the PQ "puzzles") were chosen so as to foster a classroom learning trajectory that would promote students' assimilation of the new representations: students would expand from the MT to the RT and from the MT and RT to the PQ, and the numerical cases would encourage students to take advantage of the affordances of the new mathematical representations.

The lesson framework that was to facilitate the implementation of the design goals within this learning trajectory and, specifically, the pedagogical principle for keeping as many students on board as much of the time as possible was to transition between individual work and classroom conversations, with possibly more than a single such transition per day.

Re-instrumentalizing the Multiplication Table in Teaching and Learning Ratio and Proportion

The multiplication table ('multiplication chart,' 'times table') is the central mathematical representation of the design studied in this dissertation. The multiplication table (MT) was historically designed (see Ifrah, 2000, p. 154) and is used primarily to display the multiplication products (but see Tierney, 1985, and Sherin & Fuson, 2004, on MT patterns in teaching multiplication and division). In order to determine the product of two factors, e.g., of 2 and 3, one locates these two numbers along the left column and top row, and then, possibly by sliding two fingers across and down, respectively, one locates the product value at the intersection of row and column, respectively.[7]

However, the MT can also potentially foster an understanding of ratio and proportion that is based on a repeated-addition model of multiplication. The following sections track the sequence of classroom activities. The design begins with inquiry into patterns in the MT, then focuses on repeated-adding patterns and grounds these patterns in real-world situations. The ratio table organizes students' representing and solving word problems using additive–multiplicative solution procedures, and then the proportion quartet affords multiplicative solution procedures that are linked to the ratio table.

MT Patterns

The design begins with students exploring spatial–numerical patterns in the multiplication table (rows, columns, diagonals, etc.) on the first and second days of the intervention (Table 1, top row). Similar to the ratio table and proportion quartet that can be discerned in the MT (see Figure 1), these are spatial–numerical patterns that emerge from the cross-product structure of the MT and are not the normative interpretations or uses of the MT. For instance, students discovered that values in MT diagonals can be interpreted as numerical sequences that follow patterned increase/decrease rules. For example, students determined the rule for the diagonal 10-18-24-28-30-30-28-24-18-10

[7] Providing these numbers are within the range and resolution of an available MT.

(beginning on the ‘10’ at the bottom of the 1-column and rising up and right; see Table 1, top row, the middle of the three MTs in students’ work). The rule is “add 8, add 6, add 4, add 2, add 0, add –2, add –4, add –6, add -8” (the increments constantly decrease by 2). Students each searched for, discovered, marked, analyzed, and then shared and discussed with their classmates patterns in the MT (see Abrahamson & Cigan, 2003, for some more patterns 5th-grade students found).

The several objectives of the MT-pattern activity are to: (a) enable the teacher to diagnose students’ understanding of multiplicative relations by observing what patterns they find novel; (b) encourage students to construct new associations with a familiar artifact; (c) revisit and build the classroom vocabulary that is necessary for the intervention (e.g., rows, columns, across, down, factor, product, multiple); (d) begin to develop students’ seeing the MT in new ways (e.g., repeated adding down to make columns); and (e) to set the tone for an engaging intervention that encourages individual ownership of and perspectives on the artifacts, while participating in a community of discourse that seeks to develop taken-as-shared perspectives on the artifacts.

Filmstrips: Pictorial Ratio Tables

On Day 3 of our intervention in the research focus-classroom, students were asked to concentrate on the repeated-adding sequence in columns and compose a story that is modeled by one, two, or three of these columns. That is, students were asked to invent, write out, and draw accompanying pictures that convey a situation in which some agent or two agents started from zero quantity along some dimension, e.g., money as measured in dollars, and then each accumulated their respective substance by their respective constant increments over a shared and fixed time unit (see Table 1, second row from the top). For instance, the 3 and 7 columns might inspire the composition of a story about a man and a woman who are having a hair-growing competition, with the man's beard growing down at a rate of 3 mm a day and the woman's hair growing down at a rate of 7 mm a day. So the respective lengths in millimeters of the man and woman's hair is 3:7, 6:14, 9:21, 12:28, etc.

A filmstrip consists of a vertical concatenation of comics-like drawings, numbers, and words showing the protagonist(s) after each time unit. For example, the first drawing in the hair-growing filmstrip may show the man and woman just before the competition began (the 0:0 point), the next drawing may show the man with 3 mm of beard standing by the woman with 7 mm of hair (3:7), the next drawing may show the man with 6 mm of beard standing by the woman with 14 mm of hair (6:14), and so on. This vertical sequence of paired pictures is the comics-version of the ratio table. We call this mathematical–pictorial artifact a 'filmstrip' to emphasize the consistent increment along the time dimension from picture to picture (just as celluloid filmstrips capture frames at a uniform pace).

Students were asked to include in the pictures numbers and units representing the accumulations, e.g., "6 mm" and "14 mm," and the time signatures corresponding to these quantities, e.g., "Day 2." Note that the time-signatures, e.g., Day 1, Day 2, Day 3, etc., correspond to the values in the 1-column of the MT, e.g., 1, 2, 3, etc. So a story that involves a single protagonist tracks two sequences, e.g., Day1:3mm, Day2:6mm, Day3:9mm, etc., and a story that involves 2 protagonists tracks three sequences, e.g., Day1:3mm:7mm, Day2:6mm:14mm, Day3:9mm:21mm, etc. Students were also asked to draw arc-shaped connectors between every two successive pictures running down their filmstrip and mark by these connectors the constant addend. For example, "+3" may connect the Day-1 beard to the Day-2 beard below it on the left of the filmstrip, and "+7" may connect the Day-1 hair to the Day-2 hair below it on the right of the filmstrip.

The objective of students' constructing filmstrips is for students to: (a) practice the repeated-adding interpretation of numerical sequences running down MT columns and in particular articulate the constant addend, e.g., +3, that is implied by the repeated-adding interpretation of the column but does not appear explicitly within the column (there are no '+3' symbols and values in MT columns); (b) revisit and ground multiplication as repeated adding in a meaningful context that is relatively simple and may be preparatory for more complex contexts in subsequent activities with the RT and PQ (see Table 1; note similarity between the vertical filmstrip and the columns in the other representations that appear in Table 1, lower down); (c) practice coordinating two or three repeated adding sequences and in particular attending to the horizontal numerical relations, e.g., 1:3:7, 2:6:14, 3:9:21, 4:12:28, etc., and not only to the vertical 'count-by-*x*' sequences, e.g., 1, 2, 3, 4, etc., 3, 6, 9, 12, etc., and 7, 14, 21, 28, etc.—the horizontal axis is where the ratio symbols and vocabulary lives, e.g., '3mm-per-day rate of growth' or '3-to-7 ratio'; (d) actively disembed two or three columns out of the MT and transition towards constructing a numerical RT within a meaningful and engaging context; (e) appropriate the design by creating elaborate and aesthetically invested artifacts that are then displayed in the public domain (hung on the classroom walls); (f) use their filmstrips to practice presenting and discussing ratio stories in front of the classroom; (g) produce a variety of ratio stories that become, through students' presenting and discussing their stories and modeling these stories with the MT and RT, part of the classroom's shared repertory of ratio situations that will serve in future discussions of proportion; and (h) transition in classroom tone from the MT-pattern activity that involved drawing towards working with situations, tables, numbers, and symbols in subsequent days of the unit.

MT Cutout Columns

Just as the filmstrip was designed to support students' linking ratio stories to the RT, so the cutout columns (Table 1, third row from top) were designed to support students linking the MT to the RT (Table 1, fourth row from top). An MT that is cut into a set of 10 columns (the 1-column, 2-column, 3-column, etc.) allows for students to quickly assemble a 2- or 3-columned RT in response to a word problem. For instance, when the teacher reads out a problem involving a 3:5 ratio, students can select the 1-, 3-, and 5- columns and place them alongside each other with the tops and bottoms of the columns aligned. The objective of the cutout columns are for students to: (a) focus on the MT columns that are critical to the problem at hand without possible distraction from the other columns that are not entirely relevant or absolutely necessary for representing the problem; (b) develop flexibility in thinking about and representing the problem by shuffling the order of the columns, e.g., 3:5 becomes 5:3, so as to appreciate that the proportional situational relations remain intact even though their inherent order coming from the MT has been violated; and (c) address many situations (the rapid assembly of the RT saves classroom time).

Ratio Table and Numerical Cases

Beginning on the 4th day of the intervention and continuing through to the 10th day, students solved word problems that they read in worksheets or the teacher read out or other students composed and shared with the classroom. On Days 4 and 5 (but not beyond) the worksheets scaffolded students' work through ready-made mathematical representations, which students completed according to the situational context. For instance, one problem stated that Jerry and Elaine save at $3-per-day and $5-per-day, respectively, and included a ratio table with missing values that students completed in order to determine how much money Jerry and Elaine saved after 7 days.

The given information in these problems initially included some redundant information to help students become familiar with the representations. For instance, in the Mario-and-Fatima different-multiple money-saving story that followed the Jerry-and-Elaine story, a column on the left listed the time frame—Week 1, Week 2, Week 3, etc.—and Mario and Fatima's respective savings on Week 4 was given—$28 and $36. Students had to determine the savings on the other weeks. Because the numbers were inserted into the 4th row of an RT that had missing values and because an RT includes, in its default form, all the increments beginning from 0, the time frames (Week 1, Week 2, etc.) were redundant—they could be interpreted by counting the rows and inserting these values. Next, these structural scaffolds were faded out, and eventually students had to construct these tables on their own.

Later in the intervention, students could use any method they chose to solve unknown-value proportion problems (see Table 6, the four bottom rows, for a variety of representations students used to solve problems). I expected that as students learned to use progressively sophisticated mathematical representations —MT, RT, and PQ—they would prefer to use these representations, especially seeing as each representation requires less construction than its predecessor. The teacher's task was to ensure that students are using the progressively sophisticated representations with understanding coming from the previous representations and not just performing rote procedures. I see great potential value in classroom discussions in eliciting students' difficulties in understanding the new mathematical representations and in creating opportunities for students to connect to and between the representations.

An objective of the design was for students to develop robust schemes using integer-number cases and that these schemes would support their later addressing more complex numerical cases. So the numerical cases in the problem situations that students worked on using the designed mathematical representations gradually increased in complexity, but were designed so as to remain within the zone of the majority of students in this 5th-grade classroom. Thus, items could be solved without involving any arithmetic with fractions. At the same time, I did not discourage students' using fractions unless I judged that these students had gone down solution paths that would ultimately detract from rather than contribute to their assimilating the core ideas of the domain.

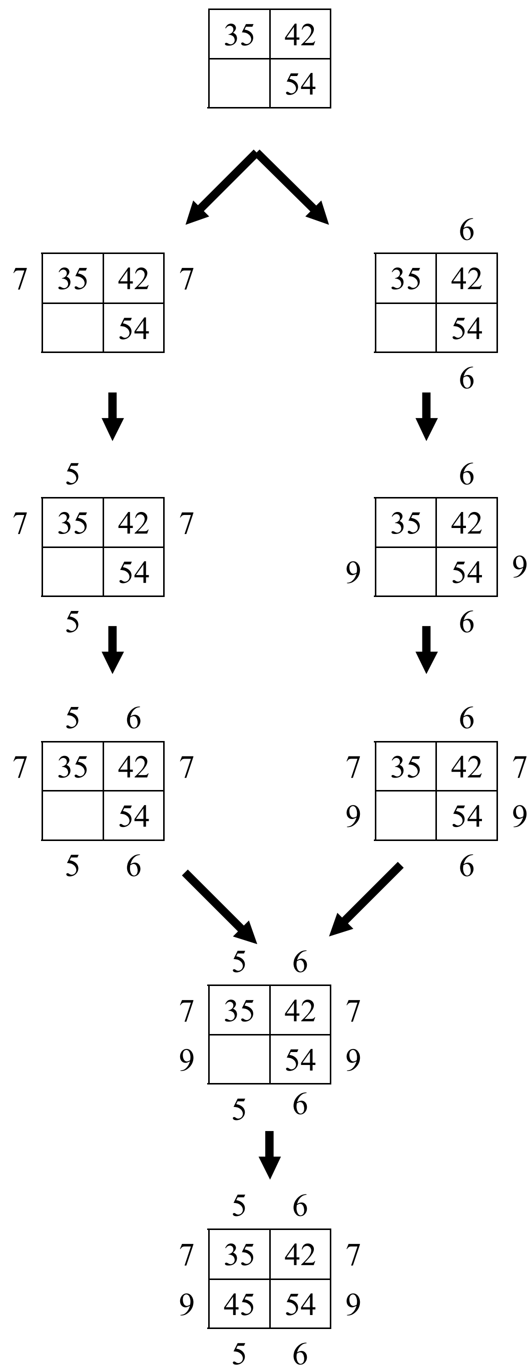
The numerical cases began from simple multiples, such as the Jerry-and-Elaine money-saving story that stated the basic ratio 3:7, and continued with different-multiple cases, where the basic ratio is not given and the ratios in the proportion are not multiples of each other, e.g., 6:14 and 15:*x*. The designed increasing complexity in numerical cases in the problem situations was expected to encourage students to adopt the more sophisticated mathematical representations. For instance, to solve with a RT a situation involving the proportion 6:14 = 15:*x*, students need to reconstruct all the rows of the RT going back to 3:7 and then forwards to 15:35. Yet, students need to do so even before they know how many rows are necessary in order to complete the RT successfully. Moreover, some numerical cases, such as 30:33 = 40: *x*, would involve the construction of numerous RT rows. A PQ solution method, on the other hand, is uniform in the number of necessary steps across all numerical cases, as we now demonstrate.

The Proportion Quartet

To understand the PQ method for solving unknown-value proportion problems, the reader is encouraged to create a PQ in solving the MT problem 35:42 = *x*:54 (we will focus at this point on the numerical solution without the situation context). Build a 2-by-2 table (a PQ) and insert the problem values as follows: 35 in the top-left cell, 42 in the top-right cell, and 54 in the bottom-right cell (see Figure 4, next page). Note that we have left the bottom-left cell vacant for the unknown value. At this point, the question is either in what MT row the 35 and 42 could both be (mark the row number or common factor outside of the PQ both to the left of the 35 and to the right of the 42) or in what MT column the 42 and 54 could both be (mark the column number or common factor outside of the PQ both above the 42 and to below the 54). So there are two possible gambits. Irrespective of which of these operations you have chosen as your first, you can now continue solving the PQ “puzzle” by determining further the row/column numbers or

Figure 4. Examples of solution paths to the problem 35:42 = x:54 using the proportion quartet (PQ) (four products out of the MT and these products' row-and-column numbers in the MT). The problem can be given as an equation, as a word problem, or directly in the PQ representation. The problem is solved by determining MT row-and-column numbers (common factors) of number pairs that are in the same row or column, respectively. There are at least six different solution paths to each proportion problem involving four integers. In this problem, the factors could be found in any of the following orders: (a) 7, 6, 5, 9; (b) 7, 6, 9, 5; (c) 7, 5, 6, 9; (d) 6, 7, 5, 9; (e) 6, 7, 9, 5; and (f) 6, 9, 7, 5.

Given problem: $35:42 = x:54$



What MT row are 35 and 42 in?

What MT column are 42 and 54 in?

What MT column is 35 in, given that it is in the 7-row?

What MT row is 54 in, given that it is in the 6-column?

What MT column is 42 in, given that it is in the 7-row? Does this check with the 54 that is in the same column?

What MT row is 42 in, given that it is in the 6-column? Does this check with the 35 that is in the same row?

What MT row is 54 in, given that it is in the 6-column?

What MT column is 35 in, given that it is in the 7-row?

Determine product of 9 and 5.

Determine product of 9 and 5.

factors. For instance, if your first step was to establish that 35 and 42 are in the 7-row (see Figure 4, the left-hand solution path), then the next step could be to look at the 35 and determine which MT column it could be in *given that it is in the 7-row.* That is, the question is what is the complementary factor of 35, given that we have 7. There are at least 6 different solution paths for each PQ, and the number of solution paths that include only integer numbers depends on how many different common factors are in the rows and column. In the case of the present problem, the factors could be found in any of the following orders: (a) 7, 6, 5, 9; (b) 7, 6, 9, 5; (c) 7, 5, 6, 9; (d) 6, 7, 5, 9; (e) 6, 7, 9, 5; and (f) 6, 9, 7, 5. Regardless of the order in which you have determined the factors, you will establish both factors—9 and 5—of the unknown value that is the product of these two factors (row and column numbers). So the factors 9 and 5 are flanking the bottom-left empty cell, and the last step is to establish the product of these two factors and write it into that cell.

Note that the problem 35:42 = x: 54 can be solved directly on an MT even without thinking about row and column numbers but by searching and locating the rectangle that the 3 values configure together with the unknown value that is found to be 45 (see Table 1, second row from the bottom, for an example of a student solving the unknown-value proportion problem 15:18 = x:30). Because each of these three given values appears twice on the MT, locating an unknown value involves choosing a combination that allows for creating a constellation of three values on the MT that would be completed into a rectangle by the fourth value that is unknown. In that sense, solving the problem with a PQ demanded more understanding because the unknown value had to be determined using multiplicative knowledge and procedures and not just through a visual–spatial matching task. The design challenge is to encourage students to use more sophisticated representations (to prefer the PQ over the MT) even though the more sophisticated representations may appear to involve more work (see Table 1, bottom row; note that the student had to construct the proportion quartet). We hoped that one contributor to students' preferring to work with the PQ would be in the satisfaction of mastering the solution of the PQ "puzzles." Also, not all problems can be solved on the MT. For example, the problem 30:33 = 40:*x* does not appear on a 10-by-10 MT. We expected some students to extend the MT for some problems—adding columns and rows as necessary—and hoped that the awkwardness of this procedure would encourage students to use the more concise methods associated with the RT and PQ.

A design goal was that as students learned to use new mathematical representations in solving problems, different students would be using different mathematical representations, and students would discuss their choices of representations in presenting their solutions. That way, students would have opportunities to see alternative solutions for a single problem and would thus hopefully translate and connect between the representations, e.g., between the MT and RT, between the RT and PQ, and between the MT and PQ.

Numerical PQs

On Days 6 and 7 of the intervention in our focus classroom, students solved PQs out of any word-problem context. The objective of this activity was for students to develop fluency with the procedures of using the PQ representation so that operating the

PQ would not constitute a hindrance in solving word problems with the PQ. A possibly challenging aspect of the PQ method for 5th-grade students who are not yet comfortable with manipulating fractions is that in solving a problem a student may go down a solution path that deviates from the MT and thus introduces fraction factors. For instance, in the 35:42 = *x*: 54 problem, if a student found 2 as the common factor of 42 and 54, then the student would then need to determine the quotient of 35 divided by 21 in order to complete the solution. All our PQ problems allowed for integer solutions. Thus, students who went down a fraction path were first asked to attempt to complete the problem, but if they could not then they were told to try to find larger common factors.

Scrambled MT

A major objective of our design is to foster students' learning general procedures for addressing and solving ratio-and-proportion situations. The MT constitutes the primary model and playground for students who are learning proportion, and the PQ captures the multiplicative factor*factor→product mechanism of the MT. Whereas the proportions that can be modeled with the MT are constrained to the order of values in the MT, students need to learn to reason more flexibly about proportions than is afforded by the structure of the MT. Note that in the MT values grow both towards the right and towards the bottom. The MT is normally an immutable artifact—any deviation from its rigid sequencing of rows and columns would hinder using it easily to find the product of two factors. However, for the purposes of our design, we have developed a version of the MT for which we shuffled the order of the rows and then shuffled the order of the columns (see Appendix B). Interestingly, this MT keeps intact all the products at the intersection of their factors' row and column. Also, in this scrambled MT, any 4 cells that configure a rectangle are still a proportion as in the usual MT. When the MT is scrambled thus and moreover if many of the products are omitted, we receive a giant puzzle. Solving this puzzle consists of determining all the rows and columns of the scrambled MT. The objective of students' solving this puzzle was for them to become familiar with the fact that the multiplicative structure of the PQ is independent of whether or not the greater values are in the bottom row and right-hand-side column. Also, the activity constituted further practice in working with the PQ that can be seen as a subset of this puzzle. Finally, solving the scrambled MT puzzle was to constitute an engaging challenge. These experiences then facilitated students' setting up PQs in any numerical order given in a problem

The "Eye-Trick"—Grounding Proportion in Geometrical Similitude

In the "eye trick" optical illusion (Abrahmson, 2002b), students determine whether two cutout rectangles are geometrically similar by holding the rectangles up, one in each hand, shutting one eye, and moving the shapes independently nearer and further from their open eye. For geometrically similar rectangles, students find a point where the two rectangles appear identical. This illusion is a potentially powerful perceptual–experiential grounding for students' understanding of proportional equivalence because it gives one meaning for why or in what sense, 6:14 is *equal* or *equivalent* to 15:35 (a different sense as compared to the repeated-addition ratio story). Students can make a RT by measuring and pairing the width and height of successive rectangles that are "the same only different sizes" and see two rows in the RT as measures of 6-by-14 cm and 15-by-35 cm similar rectangles. The aim of this activity is for students to: (a) work with ratio tables made from measures of rectangles that connote a different and complementary sense of the proportional equivalence in ratio equivalence classes, providing a different sense of consistency as compared to linked and situated repeated adding; (b) plot proportionately-equivalent width–height ordered pairs as linear-function coordinates on the Cartesian plane, e.g., (3; 7), (6; 14), (9; 21), etc.; and (c) observe, mark, and extend the line made by these points to anticipate and then create new rectangles, e.g., of width 12 cm and height 28 cm, that students then cut out and confirm through the "eye trick" as indeed "the same but larger/smaller" as compared to the other rectangles in that set. The eye-trick geometrical-similitude mini-unit was fully implemented in one of our two study classrooms and is an integral part of the current version of our ratio-and-proportion curricular unit.

Situational Meanings in Ratio and Proportion

A design goal was for students to instrumentalize the MT as a modeling tool for viewing multiplicative situations as coming from a set of linked repeated-addition real-world situations. In doing so, we anticipated that the situations would constitute grounding contexts for the mathematical vocabulary, notations, and procedures of the ratio-and-proportion concept. Reciprocally, we anticipated that students would develop a schema for the class of situations that are proportional—a situational schema that would support their identifying, modeling, and solving situations of this class. Following the intervention in one of our two classrooms, I became concerned that whereas students are constructing such a mathematical class, this class was insufficiently distinct from other classes. I responded with the following complementary design.

Three months following the main 10-day ratio-and-proportion intervention, I implemented in one of the classrooms a 3-day mini-unit on situational meanings in ratio and proportion. Our objectives in designing and implementing this mini-unit were, on the one hand, to evaluate students' comfort in working on a variety of problem types and, on the other hand, to understand how to guide teachers in leading discussions around issues of modeling. By 'issues of modeling' I mean students analyzing and evaluating the fit of the mathematical representations (MT, RT, and PQ) vis-à-vis each new word-problem situation they encounter. In this mini-unit, students solved word problems that stated 3 values and asked for a 4th unknown value, but each problem demanded analysis of the situation and its corresponding mathematical model, representation, and procedure. These were not necessarily ratio-and-proportion problems, and some problems actually called for the additive reasoning that often undermines students' success in ratio number problems. Also, some problems gave redundant or insufficient information.

The learning objectives of this mini-unit were for students to expand on their existing mathematical understanding to: (a) read word problems critically; (b) carefully address and explicate their assumptions underlying their choices of a solution strategy for each problem; (c) realize that the correctness of a solution to a problem is always contingent on appropriate modeling of the situation; (d) move between alternative models of situations and realize that these may give different 'answers'; (e) feel confident to leave a problem unsolved, stating that it cannot be solved due to insufficient information, and not due to their own incapacity; (d) realize that real-world situations do not necessarily fall neatly into textbook formats and that nevertheless we can use at least partial quantitative bearing and inferring mechanisms in addressing these situations logically–mathematically (coping with fuzzy situations); (f) appreciate the value of suspending automatic responses ("number plugging"); and (g) experience a learning environment that values an alternative view of mathematics and habits of mathematical and scientific practice—an environment wherein arithmetic is helpful yet the least important aspect of the work.

Worksheets guided students through first responding whether enough information was given in the problem, then stating what type of problem it was, and finally solving the problem numerically. For example, this problem was given on the second day of the mini-unit as part of several homework items (see Appendices C3 and C6 for additional problems of this type and other types that were used in this miniunit):

> Mr. Munchkin owns a donut bakery downtown. His donut-making machine is pretty good. Usually, out of every 9 donuts, only 2 are not absolutely perfect. He sells these for less. One day, he baked 63 donuts and 17 were not perfect. Is this a usual day? A better day than usual? A worse day than usual?
>
> - Is there enough given information to solve this problem?
>
> - What type of problem is this? For instance, is this a ratio problem? How can we know?
>
> - Solve the problem.

I asked the teacher to encourage students to discuss and debate their individual solutions by using the classroom mathematical representations they had practiced using during the ratio-and-proportion unit. The objective was for students to 'debug' their difficulties within a supportive social forum, articulate the adequacy or inadequacy of various mathematical models and representations in addressing specific problems, and establish the limitations of individual and classroom practices and values that do not embrace critical mathematical modeling.

The classroom activities were designed to help students link between the design elements that we introduced into the classroom space. We now turn to discussing student learning in terms of developing these links.

Learning Issues

Learning Issues in Design and Data Analysis

The practice of design research necessitates special models and data-analysis tools that can succinctly process heterogeneous, rich, and voluminous classroom data. These data need to be couched in terms of the core elements of design and classroom implementation: the tools that the designers introduce into the classroom, the mathematical constructs embedded in interacting with these tools, and students' written and verbalized understandings and difficulties with the design and with the domain. I have discussed the Fuson–Abrahamson apprehending-zone model that foregrounds students' linking design elements towards an understanding of a mathematical domain. This section elaborates on students' learning issues inherent in linking within this design.

I will also explain the evolution and objectives of the representation tool I developed to display the transcriptions and photo images of the data.

The learning issues are aspects of students' attending to, using, and linking the design elements—mathematical representations and situations—in solving and discussing problems. These aspects of student behavior are pivotal for students to successfully construct the domain of ratio and proportion according to our design. The learning issues are not quite couched as 'learning objectives' that a teacher may plan for an activity or discuss with students in summarizing a lesson—they are more like emphases in teacher-guide documents, such as "make sure students see that…etc." So the learning issues are not necessarily foregrounded in classroom discourse as target competencies. Nevertheless, these issues are manifest in students' participation in the activities in terms of students' construction of the mathematical representations and verbal and gestural referring to them. An attentive teacher is constantly evaluating whether or not participating students manifest understanding of the learning issues and responds accordingly. So the learning issues can be seen as key points in the instantiation of the domain's *conceptual field* (Vergnaud, 1983) in our design. Within the intricate, variable, dynamic, and dense mass of classroom interactions with the designed activities, the learning issues are the theorems-in-action "beacons" that I have identified as crucial connectors. These connectors facilitate students' integrating their burgeoning understandings of the domain and constructing the classroom semiotic network for the domain through the twinned lenses of the situations and representations of this specific design.

From the perspective of analyzing classroom data, the learning issues serve as indices of students' assimilating the design elements into a web of meaning and fluency. Evaluating a design according to students' assimilating the core learning issues of a domain as well as according to student achievement on target items illuminates learning as well as performance outcomes. Such a perspective gives traction on students' moving from difficulty to understanding in the domain. How these difficulties are overcome through teaching is central to design research.

My characterization of the design's specific learning issues was not an a priori decision. I had quite well-defined objectives for the design in terms of student achievement, but it is only through implementing and then analyzing the design that I could progressively articulate the learning issues at a level of specificity that was sufficient for the final coding of the data. So the categories evolved over repeated iterations of coding, refining, and discussing, in attempting to capture the domain's core learning issues and conceptual elements as reflected in both the design (mathematical representations and activities) and in students written and verbally expressed ideas.

In the data analysis of student verbal participation, I code for *learning-issue points* (*LIPs*). I couch student turn taking in terms of the LIPs in each turn, and in developing the LIPs coding system I intended for the system to allow me to exhaust students' learning-related verbal participation. Again, I wish to emphasize that students do not necessarily articulate the learning issues per se, but these learning issues are at play in the psychological background of student utterances. For instance, students do not necessarily make a statement about the multiplicative structure of the design's mathematical representations, but students may describe their use of the RT in ways that evidence to us that they see that the RT bears the same multiplicative structure as the MT. So our data analysis is designed to capture, track, and detect patterns in student learning in terms of building an individual and classroom semiotic network between representations—a network that is in consonance with the situations' contextual substrate. This semiotic network is the web of understandings the design attempted to foster in and between students. So the developing of such a classroom web over the implementation constitutes an important criterion for evaluating the effectiveness of the design in fostering students' understanding of the target domain of ratio and proportion.

Finally, note that the learning issues are not associated exclusively with any one of the 3 main mathematical referents, the MT, RT, and PQ; neither are they associated exclusively with any particular design day. Figure 3 is my attempt to convey through selected implementation episodes how the learning issues of the design feature and apply across the range of the design's mathematical representations and classroom media of expression. That is, I wish to emphasize that each learning issue may occur in students' working with each of the mathematical representations. Also, I wish to demonstrate how the story context that is embedded in word problems undergirds students' attending to, making sense of, and linking to different representations. This linking, I believe, fosters students' progressively interpreting the representations in line with the design and integrating conceptual aspects of the target domain of ratio and proportion. Following, I will delineate and articulate the categories of the domain learning issues.

Learning Issue Categories

Table 2

Learning Issues of the Ratio-and-Proportion Design Grouped by Type of Reasoning

Learning Issue	Definition
Qualitative Attending to the Representations as Mathematical Stories	
Rows/Columns in Stories.	Columns are categories of quantities in stories with the top values being the constant addends, and values in the same row are contemporaneous in the story.
Additive–Multiplicative	
Columns (or Rows) Are Repeated Addition Sequences	Attending to, parsing, constructing, and articulating multiplication stories or ratio stories as MT columns. Columns begin either at 0 or at the column number and then iterate the constant addend (the column number) without repeating or skipping rows.
Repeated Addends Versus Totals	Interpreting the sequence of values running down MT columns as running totals in multiplication stories or ratio stories and specifically distinguishing between these running totals and the constant addend (the column number).
Multiplicative Structure and Use of the Representations Multiplication Table, Ratio Table, and MT Puzzle	These grid-structured representations have rows and columns with uniform cell sizes, and each number is the product of the left row number and top column number. A 10-by-10 Multiplication Table can be modified by reordering (to make Scrambled Multiplication Tables or Proportion Quartets) and by extending rows and columns.
Ratio-Table	
Linking Column for the 2 Sequences	The number of iterations so far in the linked Ratio-Story columns can be represented in a separate left-most column, but it may be implicit rather than physically present (as in standard 2-column ratio tables).
General	
Zero Starting Point	Multiplication and ratio stories have a starting point at zero from which the repeated addend is iterated (in some versions of the mathematical representations this zero moment is omitted and the table begins with the addend that will be repeatedly added).
Vocabulary	Using new and familiar terms for the representations (e.g., row, column), the situations that are grounded in these representations (e.g., miles-an-hour, growth unit, per), and the “pure” mathematics of the domain (e.g., multiple, common factor, rate, ratio, proportion).
Labeling (“Table Manners”)	Labeling columns or rows with the kinds of quantities they represent in the story situation.

The learning issues capture what I believe students must see and understand in using the mathematical representations in the design, so as to learn the topic of ratio and proportion. This list of eight domain learning-issues (see Table 2, next page) came from my domain analysis, evolved through close attending to students' and teachers' responses in iterative implementations of the design, and was corroborated through numerous discussions in the design-research team over a year of analyzing the data. This list and definitions of the design's learning issues informed our analyzing the data. We located and coded each student's verbal turn-taking over all classroom interactions according to the learning-issue points (see previous section). I also coded the mathematical representation(s) the student was referring to and whether those utterances reflected understanding of or difficulty with the learning issue. In Figure 3, I present the name and definition of each learning issue and contextualize it in examples from classroom interactions. Figure 1 and Table 1 may be helpful as maps for locating the excerpts in Figure 3 within the design's mathematical representations and general plan. The excerpts in Figure 3 are only from Days 3 through 7 of the 10-day implementation that is featured in this study, because on Day 3 of that implementation we began introducing new representations and new uses of familiar representations and by Day 7 all the learning issues of the design had been addressed through activities and discussions. In the remaining three days students worked on extensions, such as part-part-whole proportional relations, that are not the focus of this study.[8]

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[8] Figure 3 features *transcliptions* (transcriptions + clips) of the data. I found that transcriptions alone were insufficient for the data analysis of students using the

Multiplicative structure and use of the representations MT, RT, and PQ. These grid-structured representations have rows and columns with uniform cell sizes, and each number is the product of the left row number and top column number. A 10-by-10 MT can be modified by reordering (to make a scrambled MT or proportion quartet) and by extending rows and columns. In Figure 3, Saul is working on an unknown-value RT. He is interpreting each value in the RT columns (the 28 and the 36) as a product of the value in the factor row (4) and some yet undetermined value at the top of each column. Saul is working with an RT and is linking the RT to the MT (I code this as RT/MT). Margarita is working on a PQ. She is not coordinating between rows and columns. Her initial explanation suggests that she does not understand the PQ as a cross-product form—she has not yet linked the PQ to the MT (PQ/MT).

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mathematical representations and that verbal descriptions of students' physical behavior were still insufficient. So I clipped out freeze frames from the mpg files and interpolated them into the text documents above students' utterances, resulting in thousands of transcliptions. The transcliptions format allowed me to attend to students gesturing (see Abrahamson, 2004) and insert graphic annotation, such as arrows that indicate the direction of a gesture, and diagrams of mathematical representations students were creating and discussing. I found the optimal transcliption format to be a 7-by-4 Microsoft Word table, on 'landscape-view' page setup, which creates 2 triple-rows, with a single empty row in between them. The top of each triple row contains 4 clips, each edited to the height of 1.6'' (with locked aspect ratio). Under each clip is its time-and-camera capture, e.g., <Day2_H2_13:52>, and below that, in a variety of fonts and font color, size, and styles (to permit highlighting of different features), the transcribed utterances and my notes. I thickly annotated the data so as to be able to communicate my evolving ideas within the design-research team and specifically so as to sensitize us, in our iterative analysis of the data, to the emerging constructs we were developing in understanding students' participating in the intervention and their difficulties and learning paths in the domain. For Figure 3, I attenuated the number of photographs and the annotation beneath them so as to create a uniform format that has adequate contextual information.

Rows/columns are repeated addition sequences. Attending to, parsing, constructing, and articulating single- or dual-protagonist ratio stories as MT columns. Columns begin either at 0 or at the column number and then iterate the constant addend (the column number) without repeating or skipping rows. In Figure 3b, Fernando describes the constant addends (3 inches more every day)—he is linking the filmstrip and a RT (Film/RT). On an earlier day, M'Buto initially has difficulty understanding the task of creating the filmstrip—he does not realize that the illustrated ratio-story must follow rigorously each iteration of the time unit.

Repeated addends vs. totals. Interpreting the sequence of values running down MT columns as running totals in multiplication stories or ratio stories and specifically distinguishing between these running totals and the constant addend (the column number). Figure 3c demonstrates how some students initially confuse the values in the RT or MT columns and the implied constant addends. Odelia grounds her distinction between addends and totals in a story about Big Bird who collects 4 snails every day. Violet confuses between totals and addends, and Ms Winningham[9] (the 5th-grade teacher) supports Violet in grounding a distinction between these constructs in an MT column that is interpreted as representing Duffy Ducks's total mileage.

Rows/columns in stories. Columns are categories of quantities with the top values being the constant addends, and values in the same row are contemporaneous in the story with the linking-column row-number being the common number of iterations of the constant addends. Figure 3d focuses on rows in stories. The example begins at the point

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[9] Ms Noreen Winningham agreed to be identified by name.

where a student had already suggested that 18 and 42, which represent contemporaneous events in the story, should be in the same row. The insight that the values are in the same row is critical for understanding how to use the MT columns in modeling the situation. The ensuing discussion demonstrates that Alice and Odelia both know *that* the values should be in the same row, yet only Odelia knows *why* the values should be in the same row. Odelia uses the 1-column to warrant the co-rowness of the values—her explanation implies that she interprets the story as an equal number of iterations of two different constant addends. This is the idea of the 'linking column,' as follows.

Linking column for the two sequences. The number of iterations so far in the linked ratio-story columns can be represented in a separate left-most column, but it may be implicit rather than physically present (as in standard 2-column ratio tables). In Figure 3e this learning issue is demonstrated not with an RT, where the repeated addition is perhaps more salient, but with a PQ, in order to emphasize that the learning issues apply to all the mathematical representations in the design. Margarita does not initially understand that the values 14 and 56 by necessity share one factor because they are in the same MT row and that 7 is the row number.

Labeling ("table manners"). Labeling columns or rows that represent categories of quantities in the situation to facilitate correct use of the representations. In Figure 3f we see how once Albertino had labeled the right-hand column of his RT, he could continue using the RT in solving the word problem. Similarly, once students had established the correct nominal titles for each of the transparent MT cutout columns projected on the overhead screen, they could proceed to solve the problem.

Zero starting point. Multiplication and ratio stories have a starting point at zero from which the repeated addend is iterated (in some versions of the mathematical representations this zero moment is omitted and the table begins with the addend that will be repeatedly added). In Figure 3g, Moses volunteers that his ratio story will begin at the zero moment, though he does not map his story onto the MT. Ms W. works with students who have initial difficulty in mapping their filmstrip ratio-story onto the MT.

Vocabulary. Using new and familiar terms for the representations (e.g., row, column), the situations that are grounded in these representations (e.g., miles-an-hour, growth unit, per), and the “pure” mathematics of the domain (e.g., multiple, common factor, rate, ratio, proportion). Figure 3h demonstrates how vocabulary may serve directly in creating a common understanding of the mathematical representations. In particular, the term ‘constant rate’ is grounded in a story-based interpretation of an MT column as a sequence of totals created through iterating a constant quantity (the original situation was Big Bird collecting 4 snails a day).

With the above explanation of the learning-issues categories, I conclude the introduction of this dissertation. The learning issues will constitute one of the foci of the Method section, where I explain how I coded for learning issues that occurred in students’ utterances.

Method

Participants

Classrooms

Two very heterogeneous 5th-grade urban/suburban classrooms (*n* = 19; *n* = 20, with 20% African–American, 20% Latino, 25% ESL, 40% of students on free lunch, and some students from families with post-college levels of education). The classes had not studied ratio and proportion or participated in the designed activities using our innovative artifacts prior to the study. The teachers were a male White teacher, Mr. Chris Cigan, in his 2nd year as a teacher, and a very experienced African–American teacher, Ms. Noreen Winningham, who was completing her doctoral studies in reform education (the teachers have asked that their real names be used). In each classroom, 2 students were chosen by the teacher to participate in the after-school tutoring sessions on the basis of their need for special help and availability at that time. The tutoring sessions helped us understand better the difficulties that the less-advantaged students were experiencing in our design and afforded us an opportunity to contribute to our participants. During these tutoring sessions, I worked with the students on the same items used during the classroom lessons as well as on similar problems. I carefully observed these students' construction of and use of the mathematical representations and asked them many questions so as to scaffold their work, pinpoint any misunderstandings, and elicit their spontaneous language for refereeing to aspects of the representations and the activities.

Profile of Focus-Classroom Teacher

The more intensive analysis in this dissertation focuses on data collected in Ms. Winningham's classroom, because the design includes students' verbal participation, and Ms. Winningham was the more experienced of our two participating teachers in leading classroom discussions. This profile is based on observing Ms. Winningham's classroom throughout the intervention and later in the video data, on field notes of those observations, on conversations with Ms. Winningham before and after classes, and on a transcribed 2-hour interview with her after the conclusion of the ratio-and-proportion unit.

With a background in drama and over twenty years of teaching experience and as a graduate student in education with a special interest in reform education, Ms. Winningham runs what she calls a 'constructivist classroom' (much in line with the work of Paul Cobb and his collaborators, e.g., Cobb & Bauersfeld, 1995, with which she is familiar). Ms. Winningham believes that the teacher's role is to guide students building their own meaning for existing mathematical–cultural tools through discussions in which students articulate and negotiate their burgeoning understandings. Ms. Winningham repeatedly tells students that they are a community of mathematicians, encourages students to articulate their ideas using accurate mathematical vocabulary, and regularly uses terminology such as 'given and implied information,' 'assumption,' 'postulate,' and 'support' in her conversations with the students. This phraseology resonates well with the reform-mathematics curriculum (Bell et al. 1998) used throughout Ms. Winningham's school district.

Ms. Winningham encourages all students to express a diversity of perspectives on the material being taught, capitalizes on input from the entire range of mathematical competency in her classroom, and is vigilant in her attempts to include as many students as possible in the classroom zone of focus, engagement, and participation. Doing so is a challenge in her very heterogeneous classroom. Ms. Winningham patiently finds ways to include students who, for various reasons (e.g., LD, ESL, speech impediments), are challenged in expressing their ideas verbally. At the same time, Ms. Winningham stimulates the very high-achieving students to pursue their own ideas without excluding other students. For instance, when one of the higher-achieving students communicates an insight that may go beyond what many other students are prepared to assimilate, Ms. Winningham elicits some input from the classroom to "sense the zone" and then either invites the student to explain the idea further so as to include more students in the conversation or tells the student, "I see where you are going. I really like that thought, but that's not where we're all going right now. Why don't you capture that thought, hold on to it, and write a postulate in your journal?"

Ms. Winningham's teaching practices, and in particular her treating mathematical concepts as alive and evolving topics in the classroom forum, were well suited both for our design and for our design-research needs. Our design required students to negotiate and create classroom taken-as-shared meanings for familiar and new mathematical representations. Our design research required that the participating–observing design-researcher (the author) occasionally consult with the teacher briefly during class time on how to respond to students' unanticipated insights about the domain and their suggestions for innovative vocabulary to describe the mathematical constructs they were discussing. So at the levels both of the students and of the designers, Ms. Winningham's classroom was tolerant towards and supportive of understanding-in-the-making, and this support and skill contributed to the learning of the students and of the designers.

In the design-research classroom, students' inventions are authentic contributions to the iterative design work. In fact, the linked nature of the design research, and in particular our great interest in students' contributions, seemed to stimulate students to make many mathematical insights into patterns and connections in the mathematical representations.

Materials and Procedure

Lessons

Students were seated in groups, which was the regular practice of these classrooms. Lessons began with students solving *work-alone* worksheets that were a data-collection device (see Appendix C for a compilation of a work-alone and a homework assignment, including items that were not used in the implementations described in this dissertation). The individual working time allowed for the teacher and researcher to attend to students individually so as to probe for students' individual understanding and difficulties. These elicited issues were then shared with the classroom through presentations and discussion. Discussion focused on items from these work-alones, from homework, or additional items, which the teacher read out and projected. Students were encouraged to lead classroom discussion by using the classroom mathematical representations that mirrored the students' own representations: (a) a large laminated classroom poster of the multiplication table that was fastened to the blackboard mirrored students' individual multiplication tables; (b) "transparencies" (transparent printouts) of worksheets helped focus students on the overhead projection of the problems they had just been working on individually and were now discussing and presenting (including word problems, ratio tables with missing values, MT cutout columns, numerical PQ problems without a situation context, and the scrambled MT puzzles); and (c) pointers that facilitated students' pointing to values on the MT poster. These pointers were long wooden rulers or bi-pronged compasses for pointing to two numbers at once and moving up and down two multiplication-table columns simultaneously.

In each classroom, the implementation lasted a total of 10 days (for totals of 10.5 hours with a 63-minute average daily duration in Mr. Cigan's classroom and 14.5 hours with a 85-minute average in Ms. Winningham's classroom). In Mr. Cigan's classroom 7 tutoring sessions lasted on average 54 minutes per session, and in Ms. Winningham's classroom 5 tutoring sessions lasted on average 41 minutes per session. Also, in Mr. Cigan's classroom we conducted a 3-day mini-unit on geometrical similitude (56-minutes per day average) and a 3-day mini-unit on percentage (63-minute per day average), and in Ms. Winningham's classroom we conducted a 3-day mini-unit on situational meanings for ratio-and-proportion vis-à-vis other 3-given-values word problems (modeling; 1.5-hour per day average).

Before each lesson the author and teacher consulted for 5 minutes to coordinate specific objectives and skim the materials, and at the end of each day they debriefed for 5-to-10 minutes during which the author took notes on the teacher's comments and concerns about the following day(s) of the design. Each day the author debriefed with the design team to select and improve on the available materials so as to keep on track with the learning objectives vis-à-vis the day-by-day data concerning students' difficulties. Several students were absent during the implementation, but no student was absent more than a single lesson. When a student returned the following day, we asked other students to explain what we had done on the previous day, both so as to help that student and to elicit other students' understandings. In Mr. Cigan's classroom the author taught the first few lessons essentially alone, with the teacher present and helping students with individual work and helping the author with classroom management. On later days the teacher progressively led more of the discussion and initiated interactive responses to students' difficulties. In Ms. Winningham's classroom the teacher taught the entire unit essentially alone, with the author briefly highlighting some points, initiating discussion topics by asking students a question, consulting with the teacher, and eliciting student understandings and difficulties during individual work.

Data Collection

We collected data both of students' written work and their verbal participation in the classroom activities. Written work from work-alone classroom assignments—a set of items students solved individually at the beginning of each day—affords monitoring all students' learning, but can only elicit understandings that have matured to the point that students can articulate them in inscribed representations, such as language and other shapes and symbols. Verbal participation in whole-group interactions cannot provide a continuous sampling of all students, because the teacher's goals when leading conversation create the distribution of speaking students. However, verbal participation constitutes a possibly more sensitive source for measuring participating-student understanding, as compared to written work, because speaking students can use alternative modes of communicating in explaining their use of the design tools and in conveying thinking that is not fluent understanding. Students can gesture toward mathematical representations in the classroom and incorporate these gestures into syntactically incomplete utterances that they could not have written, and they can be helped by other students and by the teacher to articulate their points more clearly (and ESL students needed special help).

Video Data

Two video cameras, one on a tripod and one carried by a videographer, filmed all classroom and tutoring sessions. When students worked individually, often the author also carried and operated the camera that had been on the tripod, so as to maximize coverage. Our video footage comes from 29 days of interventions, ranging from 54 to 123 minutes (mean = 71, *SD* = 19), 10 tutoring sessions of about 50 minutes each, and interviews with several high-achieving students (total of 1.5 hours) for a total of just over 45 hours, about a third of which contains data on different students from the second camera (so total data time is roughly 60 hours).

Written and Miscellaneous Data

Our written data include field notes (on lesson plans and on daily debriefing with teachers and the design team), student work (in work-alones at the beginning of each day, home-work, pre/post-tests, feedback questionnaires from all students, journal entries from about a third of the students, tutor-sessions with 4 students, and interview and written feedback from high-achieving students), and teacher feedback (audio recording and transcribed teacher interview and input from professional-development workshops with half of the district's grade-5 teachers).

Pre/Post-Test

In preparing the ratio-and-proportion pre/post-test (see Table 3, next pages), we searched for items from previous studies that would afford a basis for evaluating the efficacy of our design in supporting students to correct solutions. We looked at several large-scale studies (TIMSS, 1995; NAEP, 1990; Stigler, Lee, & Stevenson, 1990) and other research studies (Kaput & West, 1994; Vanhille & Baroody, 2002), and chose items that reflected a range in: (a) numerical cases (relatively small numbers, e.g., 12:16 = 15:?, versus larger numbers, e.g., 9:108 = 36:?), keeping within the ability scope of 5^{th}-grade students so as to allow them to show understanding without being overwhelmed by calculation; (b) complexity (a ratio and its integer multiple, e.g., 7:13 = ?: 52, versus different multiples of a ratio, e.g., 8:14 = 12:? are both integer multiples of 4:7); (c) format (word problems or ratio format); and (d) extension items (fraction format and percentage). In preparing the posttests for our mini-units on percentage and on non-ratio-and-proportion word problems, we created new items because we did not find adequate comparison items that suited the design.

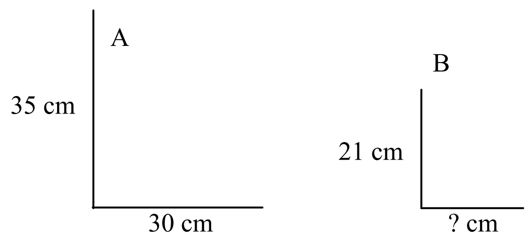
The posttest included 3 items with higher-complexity proportions (Table 3, the 3 top items) that presented situations with the following proportions: (a) 12:16 = 15: ?; (b) 8:14 = 12:?; and (c) 15:21 = 50:?. We termed these 3 difficult problems '*critical,'*

Table 3
Posttest Items

Items	Comparative Studies
Critical Items	
a. In a field trip, 12 people eat 16 boxes of food. How many boxes of food would 15 people eat?	45 Grade 6
b. To bake donuts, Jerome needs exactly 8 cups of flour to make 14 donuts. How many donuts can he make with 12 cups of flour?	na
c. The Boston Park Committee is building parks. They found out that 15 maple trees can shade 21 picnic tables when they built the Raymond Street Park. On Charles Street, they will make a bigger park and can afford to buy 50 maple trees. How many picnic tables can be shaded at the new park?	na
Numbers beyond the 10 * 10 MT	
d. If the ratio 7 to 13 is the same as the ratio *x* to 52, what is the value of *x*? Multiple choice: 7 / 13 / 28 / 364	69 Grade 8
e. If 2/25 = N/500, then N = ; Multiple choice: 10 / 20 / 30 / 40 / 50	48 Grade 8
f. Every 9 hours the central hearing system in school uses 108 liters of oil. How many liters of oil does the system use in 36 hours?	82 Grade 6
g. If there are 300 calories in 100g of a certain food, how many calories are there in a 30g portion of this food? Multiple choice: 90 / 100 / 900 / 9000	69 Grade 8
Numerical Proportion	
h. Fill in the missing number: 3:10 = ___:100	03 Grade 5

Posttest Transfer Items: Similar Shapes and Percent

i. The two sides of Figure A are 35 cm high and 30 cm long. Figure B is the same shape but smaller. If one side of Figure B is 21 cm high, how long is the other side? — na



j. What percent of 175 is 7?
Multiple choice: 4% / 12.25% / 25% / 40% — 49 Grade 12

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k. In Mr. Roberts' class there are 30 children. 18 of them are girls. What percent of the children in Mr. Roberts' class are girls? — na

l. Entrance to Arizona's National Park costs $35. On Coyote Day it will cost only 40% of that. How much will the entrance cost on Coyote Day? — na

m. After a long diet, Fat Cat weighed only 54 lbs. That was 90% of his normal weight. How much did Fat Cat weigh before the diet? — na

n. Simon keeps 250 pet spiders in his living room. 70 of these spiders love spinach. What percent of Simon's pet spiders love spinach? — na

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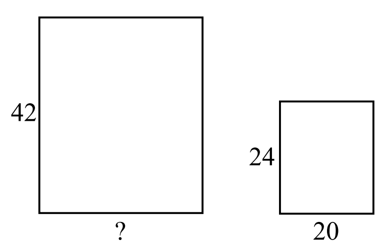
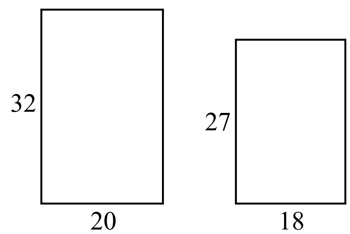
o. Jose used the "eye-trick" illusion and found that these two shapes are the same, only different in size. He measured them on a grid, but forgot to mark one of the widths. Can you complete it? — na



p. Look at these two shapes and their measured heights and widths. If you used the “eye-trick” on these two shapes, do you think they would appear exactly the same?[c] — na



Our own items for a test given after a 3-day miniunit on proportion vs. non-proportion problems

na

q. Two items:

Foxy is a champion computer-game expert. Dr. Labby is doing research on Foxy. Dr. Labby found that Foxy needs 25 missiles to shoot down 45 aliens. If Foxy has only 10 missiles left, how many aliens can he shoot down? Explain your answer.

In Wisconsin a cow named Sheila gave 2 gallons of milk on Tuesday, then 3 gallons of milk on Wednesday. In [our own town (not in WI)] Mr. A. saw 5 videos last weekend. How many videos will Mr. A. see next weekend?

Notes: Comparison items were: a. Vanhille & Baroody (2002); b. Kaput & West (1994); c. Kaput & West (1994); d. TIMSS (1995); e. NAEP (1990); f. Vergnaud (1983); g. TIMSS (1995); h. Stigler, Lee, & Stevenson (1990); i. Kaput & West (1994); j. NAEP (1990). No post-test data is available for Kaput and West (1994) study.
The rationale for choosing these particular items as well as some of the new terminology in this table (e.g., the ‘eye trick’) will be clarified in this dissertation.

because they could reveal students' facility in dealing with several-step solution methods that students could learn in the design. In particular, critical items could possibly capture the pre-to-post learning of the middle- and high-achieving students, who may solve simpler proportion problems successfully by using spontaneous methods that would prove inadequate for the more complex proportions.

The Transcliptions

I fully transcribed the 8 first days from the intervention in Ms. Winningham's class as well as the two tutoring sessions that followed the days on which the proportion quartets were introduced and practiced (Days 5 and 6) because these were particularly challenging days (see also footnote 7 on page 51 for technical information regarding the preparation of transcliptions).

Coding Student Verbal Participation

Rationale and Data

In this study I wished to focus on students' learning of the domain's core ideas in order to understand the advantages and challenges inherent in implementing this design for ratio and proportion in 5th-grade classrooms. The analysis of the classroom data was aimed to reveal and articulate students' assimilation of the designed domain's learning issues in the form of patterns that may emerge in students' day-by-day verbal–gestural participation in activities involving the mathematical representations (see Table 2).

I focused on the implementation in Ms. Winningham's classroom and initially coded for intervention Days 1 through 8 (out of 10 days), because on Days 9 and 10 advanced ideas were introduced. Examples of such advanced ideas are: (a) situations that give one part and a total from which the other part must be found (e.g., "Sally collects red and blue pens. If she has 8 red pens and altogether has 20 pens, what is the ratio between Sally's red pens and her blue pens?"); (b) attending to confusing elements in word problems, e.g., when a ratio is reported in a certain order but then this order is reversed in the question. Later, I decided to drop Day 8 because by that day all the core ideas had been discussed and about a third of that day was dedicated to discussing non-core ideas (e.g., students presented and discussed MT board-games they had invented as a homework elective assignment). The utterances I studied were the exhaustive body of learning-related utterances in the classroom data, including classroom discussion, teacher–student conversations during individual work, and discussion within student groups. Including teacher–student conversations does not tip the total number of LIPs (Learning Issue Points) towards the lower students, because the teacher regularly visited many students and attended to the ideas of both lower and higher-achieving students.

For Days 1 through 7, I analyzed all of students' learning-related verbal–gestural participation, i.e., their utterances and the head-and-hand movements that sometimes help to clarify which object they are referring to, such as when they use the article 'that' or 'it' while turning their head and possibly pointing towards the MT poster on the classroom wall. My unit of analysis is a student's participation *turn*. A turn, as in 'turn taking,' is a student's verbal interaction with the teacher or with a classmate. A turn may be as short as a single word, e.g., in responding to a question directed to that student by the teacher, or as long as several minutes, such as when the student is presenting an idea to the classroom. Within each participation turn, I identified the LIPs (Learning-Issue Points). A student may raise a single or several different LIPs in a turn, and the number of LIPs in a turn can be relatively independent of the duration of the turn or the number of spoken words in that turn. Also, student LIPs could reflect correct or incorrect use of these aspects. Any incorrect LIP resulted in its being coded as incorrect even if other aspects were used correctly. If the teacher or classmates requested clarification and then the student restated essentially the same idea, I did not code the LIP in this successive response, unless the speaking student elaborated the reply into a new idea, in which case I coded the LIP(s) in that new idea as well.

LIP Coding Dimensions

I coded LIPs along ten dimensions: (1) learning issue (which one of the eight learning issues; see Table 2); (2) classroom mathematical representation supporting the discussion (see 'LIP referents,' below); (3) any other mathematical representation that informed students' discussion of the focal representation; (4) whether the LIP evidenced understanding of or difficulty with the learning issue; (5) whether the coded LIP was in the utterance of an individual student or of a group (*choral* participation; see below in this section); (6) intervention day; (7) time from beginning of lesson; (8) duration of turn if it were longer than half a minute; (9) student pre-test rank ordering; and (10) student gender. The first round of coding resulted in two tables, one of all utterances that evidenced understanding and the other of utterances that evidenced difficulty. These tables served me in preparing all subsequent tables that each revealed different patterns in the coded data because they highlight different coded dimensions.

For each LIP, I coded the math representations to which the student referred explicitly, and which if any additional representation the student was apparently informed by or linking to in the LIP. For example, a student may be pointing to the MT and highlighting a PQ structure within it (MT/PQ) or discussing a particular PQ as 2 rows out of a RT (PQ/RT). So there is a total of six ordered combinations of focal and linked representations (MT/RT, RT/MT, MT/PQ, PQ/MT, RT/PQ, PQ/RT).

I supplemented these six combinations with four more that depict student linking the MT and PQ without attending to the factors of the PQ. First, a student may locate and discuss 4 PQ values (e.g., 6, 14, 15, 35) on the MT without alluding to the row or column numbers (3, 7, 2, and 5). I see such linking as a transitional stage towards seeing and using proportionality. That is, the distinct rectangular spatiality of proportion in our design may play a role in students' developing the semiotic network of proportion, and in particular a class of mathematical objects called 'proportion.' Students' understanding of the PQ as just a rectangle on the MT (coded as MT/PQ-rect or PQ/MT-rect) may be manifest in their working with the MT to complete an unknown-value proportion, e.g., 6:14 = ?:35. Second, on Days 5, 6, and 7, several students discussed fractions that they saw in the MT or PQ. Because fractions are closely related to our design and because I am interested in examining how the design may foster students' linking and integrating the multiplicative conceptual field that includes fractions, I coded the referents of these utterances as MT/Fractions or PQ/Fractions, respectively.

Individual vs. "Choral" Utterances

In early rounds of data analysis, I identified a unique type of classroom interaction that involved the teacher and multiple students. Such classroom "choral" interacting was elicited by the teacher addressing a question to the whole class, usually animatedly or in a singsong voice. Students called out single-word or short-phrase responses in unison.

Percentage of Student Utterances Coded

All utterances related to learning issues or MT-patterns were coded. Interrater agreement on the learning-issue categories reflected in student utterances was 88%. The other coder was a computer-sciences graduate student with a background as a mathematics schoolteacher, and who was working on a mathematics-education project that involved implementations in middle school classrooms and analysis of classroom data. He coded a total of roughly a fifth of the classroom utterances in chunks that were sampled from four different intervention days. Disagreements were resolved by discussion between the coders. Discussing these disagreements helped us further articulate the learning-issue categories and the rationale of our coding.

Having created the two tables of raw code, we analyzed and compiled the tables by aggregating the information in the cells according to different coding dimensions so as to locate aspects of the design that were particularly difficult for students to understand.

Written Work

In analyzing students' written work, I looked at work-alone problems given at the beginning of Days 5 through 9. Day 5 was the first day that the PQ representation was introduced to students, and during Day 9 the class began addressing learning issues that are not core to this design (part-part-total ratio-and-proportion word problems). Also, I focused on the first problems students worked on individually on each day—before the class discussed these problems—in order to evaluate the understanding with which students came into class on that day. The video footage from one or two portable cameras proved valuable in evaluating students understanding because we can see and therefore analyze all students' written work before they made any corrections during or after the class discussion.

Results and Discussion

In this section, I present and discuss findings from students' responses to the posttest, from analyses of students' written work on the daily work-alone tasks, and from their verbal participation in classroom activities.

Written Work as Evidence of Student Learning

Posttests

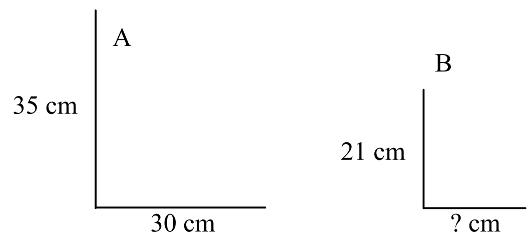
Table 4 (see next pages) presents students' scores on a post-intervention test as well as scores and grades of students who solved these items in previous studies. On comparison items, our 5^{th}-grade students outperformed students from previous studies (mean of our performance advantage = 26%, *SD* =33, across all ratio-and-proportion comparison items) who were from 1 to *3* grades more advanced on 5 of the 6 items. (I acknowledge that conclusions from these results should be tempered by various unknowns regarding previous studies, such as the proximity of the tests following the interventions.) The mean of 79% over all items indicates that the design puts the topic of ratio and proportion in the 5^{th}-grade learning zone. The *critical* items (with different multiples of the basic ratio) showed a particularly high performance (87%). Transfer items showed considerable transfer, and all of the mini-unit extensions showed good levels of learning.

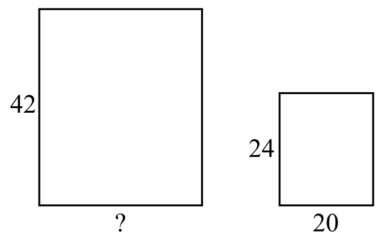
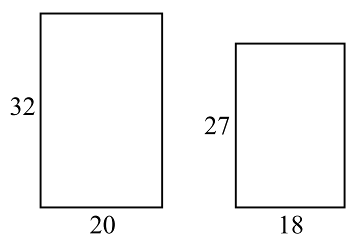
I sought a more detailed view of student learning by plotting the data as classroom achievement distributions and by using several scoring dimensions for partially correct responses of students on the lower end of this distribution, as follows. Out of students' 117 total posttest responses to the 3 critical items in both classrooms, only 7 errors of the

Table 4

Students' Mean Percentages of Correct Responses on Posttest Comparison Items

Items	Our Grade 5 n = 19	Our Grade 5 n = 20	Comparative Studies
Critical Items			
a. In a field trip, 12 people eat 16 boxes of food. How many boxes of food would 15 people eat?	72	85	45 Grade 6
b. To bake donuts, Jerome needs exactly 8 cups of flour to make 14 donuts. How many donuts can he make with 12 cups of flour?	83	85	na
c. The Boston Park Committee is building parks. They found out that 15 maple trees can shade 21 picnic tables when they built the Raymond Street Park. On Charles Street, they will make a bigger park and can afford to buy 50 maple trees. How many picnic tables can be shaded at the new park?	89	85	na
Numbers beyond the 10 * 10 MT			
d. If the ratio 7 to 13 is the same as the ratio x to 52, what is the value of x? Multiple choice: 7 / 13 / 28 / 364	78	90	69 Grade 8
e. If 2/25 = N/500, then N = ; Multiple choice: 10 / 20 / 30 / 40 / 50	56	50	48 Grade 8
f. Every 9 hours the central hearing system in school uses 108 liters of oil. How many liters of oil does the system use in 36 hours?	83	70	82 Grade 6
g. If there are 300 calories in 100g of a certain food, how many calories are there in a 30g portion of this food? Multiple choice: 90 / 100 / 900 / 9000	72	90	69 Grade 8
Numerical Proportion			
h. Fill in the missing number: 3:10 = ___:100	94	85	03 Grade 5
Mean over the six items that have comparisons scores	76	78	53

Posttest Transfer Items: Similar Shapes and Percent			
i. The two sides of Figure A are 35 cm high and 30 cm long. Figure B is the same shape but smaller. If one side of Figure B is 21 cm high, how long is the other side? 	78	60[a]	na
k. What percent of 175 is 7? Multiple choice: 4% / 12.25% / 25% / 40%	44	40[b]	49 Grade 12
Our own items from a test given after a 2-day miniunit on percent problems for the first class and no work on percent by the second class[b]			
k. In Mr. Roberts' class there are 30 children. 18 of them are girls. What percent of the children in Mr. Roberts' class are girls?	81	30[b]	na
l. Entrance to Arizona's National Park costs $35. On Coyote Day it will cost only 40% of that. How much will the entrance cost on Coyote Day?	81	35[b]	na
m. After a long diet, Fat Cat weighed only 54 lbs. That was 90% of his normal weight. How much did Fat Cat weigh before the diet?	81	45[b]	na
n. Simon keeps 250 pet spiders in his living room. 70 of these spiders love spinach. What percent of Simon's pet spiders love spinach?	56	16[b]	na

Our own items from a test given after a 2-day miniunit on geometrical-similarity			
o. Jose used the “eye-trick” illusion and found that these two shapes are the same, only different in size.[c] He measured them on a grid, but forgot to mark one of the widths. Can you complete it? 	79	na	na
p. Look at these two shapes and their measured heights and widths. If you used the “eye-trick” on these two shapes, do you think they would appear exactly the same?[c] 	79	na	na

Our own items from a test given after a 3-day miniunit on proportion vs. non-proportion problems

q. Two items[d]: Foxy is a champion computer-game expert. Dr. Labby is doing research on Foxy. Dr. Labby found that Foxy needs 25 missiles to shoot down 45 aliens. If Foxy has only 10 missiles left, how many aliens can he shoot down? Explain your answer. In Wisconsin a cow named Sheila gave 2 gallons of milk on Tuesday, then 3 gallons of milk on Wednesday. In [our own town (not in WI)] Mr. A. saw 5 videos last weekend. How many videos will Mr. A. see next weekend?	na	68	na

Note: Comparison items were: a. Vanhille & Baroody (2002); b. Kaput & West (1994); c. Kaput & West (1994); d. TIMSS (1995); e. NAEP (1990); f. Vergnaud (1983); g. TIMSS (1995); h. Stigler, Lee, & Stevenson (1990); i. Kaput & West (1994); j. NAEP (1990). No post-test data is available for Kaput and West (1994) study.

[a]Only 10 minutes were devoted in this classroom to the study of proportional measurements of geometrically similar shapes due to time constraints.

[b]Percentage was not taught to these students due to class scheduling constraints. These items were given at the end of the posttest as transfer items.

[c]On the "eye-trick," see the design section in the introduction of this paper.

[d]Students were scored only if they solved both answers correctly, so this is a conservative score. For more information on this miniunit, see the design section in the introduction of this paper.

na means not available.

same-difference ('additive reasoning') strategy. Students' additive reasoning has often been implicated as the prime cause of their performance failure in rational-number tests, yet our students used additive reasoning overall only 6% (7/117) of the time in attempting to solve ratio-and-proportion word problems. This stands in a striking juxtaposition to U.S. students' usual misunderstanding of this multiplicative domain.

Table 5
Mean Percentages of Students' Correct and Partially Correct Pretest and Posttest Responses on 3 Non-Multiples Items[a]

Number of Students in Test Achievement Groups	Pretest	Posttest	
		Answer Correct	Correct Strategy
Top Quarter			
n = 2	100	100	100
n = 8	33	100	100
Middle Half			
n = 20	0	100	100
Bottom Quarter			
n = 5	0	53	93
n = 4	0	0	58
Total			
n = 39	12	84	95

[a]Items involving proportions that cannot be expressed as integer within- or between-ratio multiplicative-relation equivalences, e.g., in 12:16 = 15:20 the multiplicative relations are 1 1/3 (within-ratio) and 1 1/4 (between-ratios), yet the two ratios are integer multiples of the same integer basic-ratio pair—they are both multiples of 3:4—so the pairs of proportion values, 12 and 16 and 15 and 20, are the products of the 3:4 ratio (the 3-product and the 5-product).
Note. The 3 items of this class type were:
a. In a field trip, 12 people eat 16 boxes of food. How many boxes of food would 15 people eat? (Baroody & Vanhille, 2002)
b. To bake donuts, Jerome needs exactly 8 cups of flour to make 14 donuts. How many donuts can he make with 12 cups of flour? (Kaput & West, 1994)
c. The Boston Park Committee is building parks. They found out that 15 maple trees can shade 21 picnic tables when they built the Raymond Street Park. On Charles Street, they will make a bigger park and can afford to buy 50 maple trees. How many picnic tables can be shade at the new park? (Kaput & West, 1994)

Table 5 (see previous page) groups the 39 5th-grade students according to their pretest achievement. Nine of the 39 students did not respond correctly to all 3 critical items. Only 4 of these 9 bottom students committed one or more same-difference type errors, and only 1 of these 4 committed this error on all 3 critical items. So all of the 5 partially correct students had partially correct solutions, and 3 of the 4 totally incorrect students had partially correct solutions. Even the least-advanced student was trying to set up a RT (a correct approach). Thus, all 9 students were some place in their path to multiplicative problem solving, as I now elaborate.

Further analysis of these 9 bottom-quarter students' incorrect responses on the 3 posttest critical items reveals that 6 of the 7 incorrect responses by the 5 students who solved at least one item correctly were partially correct: 2 were failed attempts to establish the multiplicative relations between correctly-positioned PQ products (the relation was non-integer), 2 were due to incorrect labeling of PQ columns and/or rows, 1 was misinserting values into a correctly-labeled PQ, 1 was a calculation error in a correctly-factored PQ (3*7 = 18). Two responses showed the incorrect same-difference strategy (12:16 = 15:19). Of the 12 incorrect responses committed by the 4 students who did not solve a single item correctly: 7 reflected correct strategies: 1 was a failed attempt to establish an integer multiplicative relation between correctly-positioned PQ products, 1 was due to mislabeling a PQ, 2 were calculation errors in correctly-factored PQs, 3 were failed attempts to create an RT (all committed by the LD student); and 5 reflected incorrect same-difference errors, 3 in a RT-like representation, and 2 in a correctly-constructed PQ.

Using and Coordinating Mathematical Representations

Table 6
Number of Students Using Different Mathematical Representations in Solving Non-Multiples Word Problems Incorrectly and Correctly

	Intervention Days							
	5		6		8[a]		9	
Representation	I	C	I	C	I	C	I	C
RT	5	6	6	9	1	-	-	-
RT–PQ	2	-	-	3	-	-	-	3
PQ	1	4	-	1	3	15	2	14
Total[b]	8	10	6	13	4	15	2	17

Note. In the data from each day, we looked at the first problem that students worked on individually during the lesson. Students in the RT–PQ row are those whose work on that item is a mixture of ratio-table and proportion-quartet strategies: (a) they started a RT, erased it, and used a PQ (or vice versa); (b) their representation is some combination of a RT and a PQ; or (c) they used both representations. During classroom work (but not on the pre- and post-test) students were allowed to consult the MT, albeit we encouraged students to work without it. In preparing this table, we scored the following word problems:
Day 5: "Two flower buds peeped out of the ground on the same morning—a daffodil and a petunia. After some days, the daffodil was 12cm tall and the petunia was 21cm tall. When the petunia is 35cm, how tall will the daffodil be?" [20] (Our word problem.)
Day 6: " A car of the future will be able to travel 8 miles in 2 minutes. How far will it travel in 5 minutes?" [20 miles] (Kaput & West, 1994)
Day 8: "During a very long movie, Lynnie ate 72 bowls of popcorn in 80 hours. At this rate, how many bowls will Lynnie eat in 100 hrs?" [90 bowls] (Problem composed by a student as a homework assignment given on the previous afternoon.)
Day 9: "George earns $28 in 16 minutes. How long must he work to earn $42?" [24 minutes] (Our word problem.)
[a]On Day 7 students practiced solving many PQ problems that were numerical only (there was no situation to be considered).
[b]The total deviates from 20 due to absences of different students on different days.

Table 6 (see above) addresses students' learning to use the design's mathematical representations MT, RT, and PQ while individually solving unknown-value ratio-and-proportion word problems (these data are from Ms. Winningham's classroom). The 'Total' row in Table 6 reveals a steady increase in the number of students who solved the work-alone items correctly. Note that these items (see Table 6, footnote) are all of comparable complexity, and so the number of students solving these problems correctly reflects a gradual mastery by the class of the non-multiple items (two different multiples of a basic ratio). This table also reveals students' migration from using primarily the RT to using the PQ, with some students bridging over by using either both or hybrid representations.[10] Students' migration from the RT towards the PQ reflects the order in which these representations were introduced to the class. This table also demonstrates that students migrated on different days towards the multiplicative PQ tool.

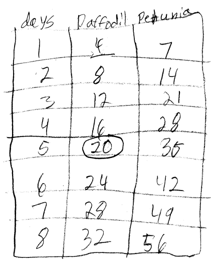
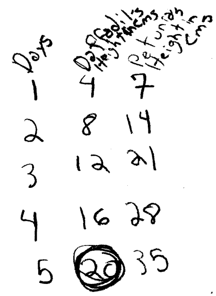
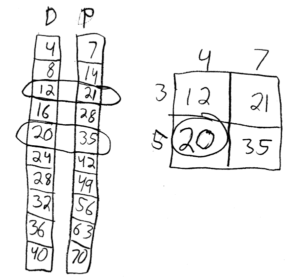
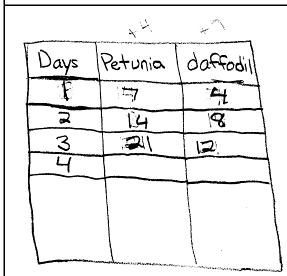
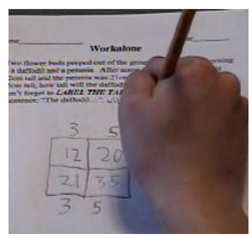
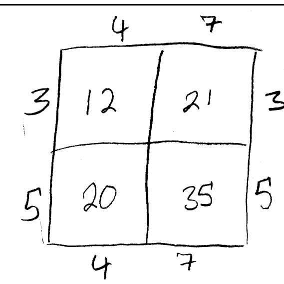
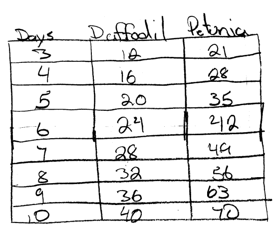
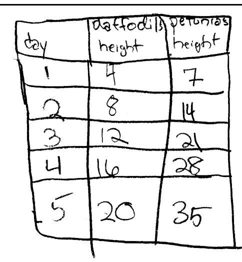
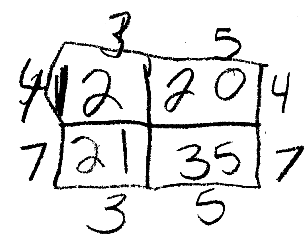
Variation in Solution Strategies

Figure 5 (see next page) shows nine examples of student work in solving a problem that was given as a classroom work-alone assignment on Day 5. The examples were chosen so as to represent the range in students' use of mathematical representations. The figure indicates that not only were students learning at different paces but how they varied in their solution strategies. Some students used a RT and others used a PQ to solve the same item, and one student (Item C) used both a RT and a PQ.

Students' written explanations as well as our video data of work-in-the-making shed light on how the classroom mathematical representations scaffolded students' learning in our design. Items E and I show students who are copying their PQ—the products inside and the factors outside—from the MT. Items C and G show students who used 10 rows (as in the MT) where only 4 were necessary. Note in Item G that the student

[10]M., a high-achieving and highly-verbal participating student, persevered with using the RT long after her performance on and discussion of the PQ had evidenced that she had mastered the PQ format. In a future study we will focus on the very-high-achieving students whose patterns of adopting our formats differed qualitatively from the other students and may inform the design of our curricular materials as a short unit for middle-school students who have already worked in the domain.

Figure 5. Variation in student solution representations and accompanying verbatim written responses in solving individually the Day 5 in-classroom word problem, “Two flower buds peeped out of the ground on the same morning—a daffodil and a petunia. After some days, the daffodil was 12cm tall and the petunia was 21cm tall. When the petunia is 35cm, how tall will the daffodil be?” Students’ work suggests a classroom bootstrapping—each student at their personal pace and along their personal path—the familiar structure and function of the multiplication table that was available for their use in developing an understanding of additive–multiplicative properties of situated ratio and proportion. Although the proportion-quartet (PQ) representation was taught after the rate table (RT) in the design, students using the PQ do not necessarily evidence deeper understanding (e.g., compare B., an RT solution and full explanation, to I., a PQ copied out of an MT).

 A. The daffodil was 20 cm long when the petunia was 35 long	 B. ~~If the two flowers grow at rates of~~ If a petunia grows at a rate of seven cms per day and the daffodil grows at a rate of four cms per day, when the petunia is thirty-five cms, the daffodil will be twenty cms tall.	 C. The daffodil will be 20 cm tall.
 D.[a]	 E.[b]	 F. The daffodil will be taller than the petunia.
 G. The daffodil will be 20 cm tall.	 H. When the petunia was 35 cm tall the daffodil was 20 cm tall.	 I. I looked for 12 and 21 that was underneath it and 35 that bigh it and when it is I look on the side. 20 cm.

[a]This student's incomplete table was included to demonstrate students' flexibility in column order in the RT representation (e.g., compare to Item A.) as well as in the PQ format (e.g., compare Items C. and F.). On the posttest (1 week later), she correctly solved all three *critical* items (items with proportional ratios that are related by a non-integer multiple).

[b]Picture E. has been included to demonstrate both our access to work-in-the-making, and specifically to show that such access informed us of students' strategies: this student apparently consulted the MT rather than factoring the PQ.

did not even have the '4' and '7' addends at the top of her table but appears to have begun copying from Row 3 of the MT in which the given ratio 12:21 appears. One week later, all four of these students used a PQ in solving correctly all posttest critical items, even though the MT was not available during the posttest.

Figure 6 (see next page) demonstrates variations in students' strategies in solving an item that was given as a homework assignment. As compared to the range of strategies students used in individually solving classroom items, their range of strategies in solving homework items was larger. This is probably because some students who received help at home used strategies that were not taught in class (e.g., the bottom-left example), and some students who did not receive help at home used naïve strategies that do not take advantage of the mathematical representations available in the classroom (e.g., the top-left example and the example immediately below it are essentially iconic). These iconic strategies suggest that the RT and PQ tabular forms, which segregate quantitative categories into separate columns, may resonate with naïve techniques of tackling situations involving linked repeated-adding (e.g., +3 for the kitten, +5 for the mummy cat)[11]. However, that students reverted to such iconic strategies after they had used mathematical representations indicates, on the one hand, the robustness and resilience of naïve strategies, and, on the other hand, the importance of the classroom learning-support network, including the mathematical representations and the teacher and classmates, to support students assimilating the representations into their naïve forms.

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[11] On the pretests, too, 6 more students (out of 39) used pictorial strategies at least once in attempting to solve the critical items

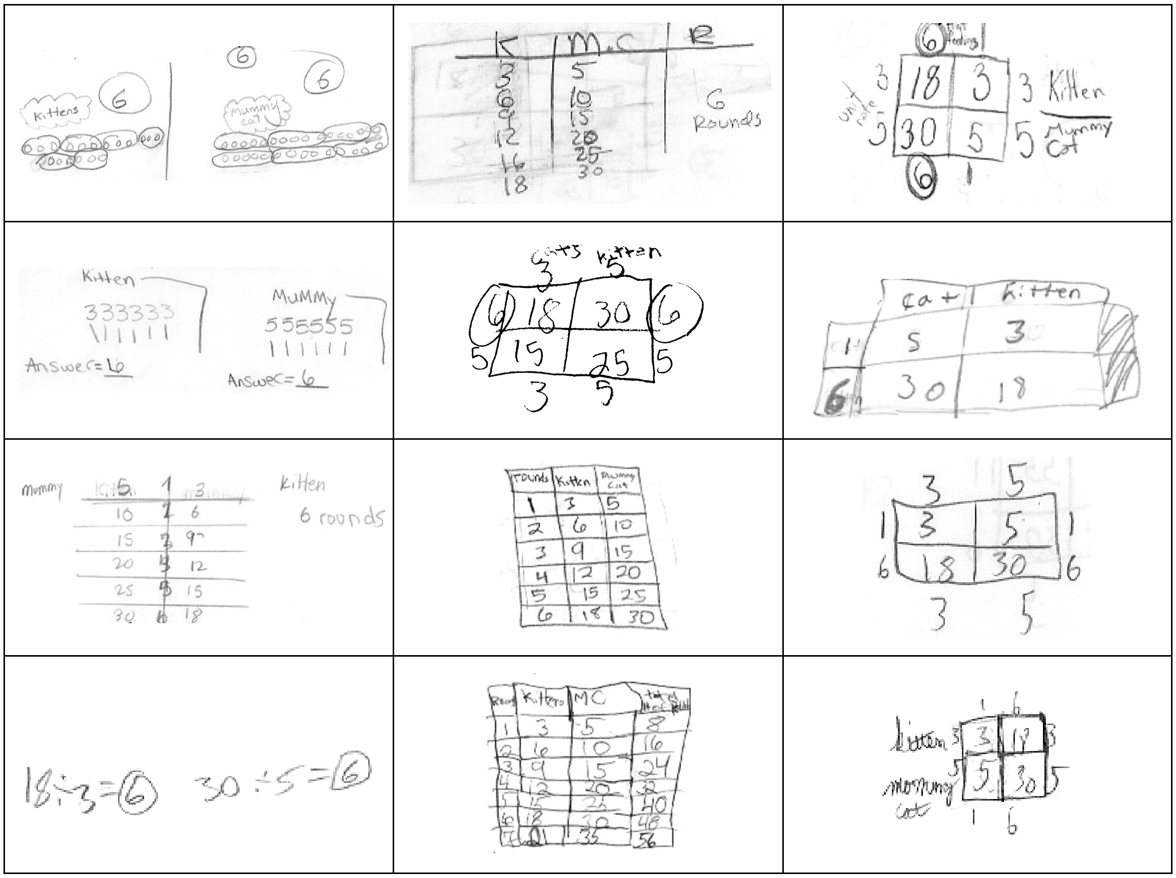

Figure 6. Variation in students use of representations in responding correctly to the Day-8 homework problem, “When Tom feeds his cats, he shares out the pellets of cat food like this: 3 for the kitten and 5 for the mummy cat, 3 for the kitten and 5 for the mummy cat, 3 for the kitten and 5 for the mummy cat, and so on. If the cats have 18 pellets (kitten) and 30 pellets (mummy cat), how many rounds did Tom feed them?” The left-column work is not from our focus classroom but is included here to demonstrate the full range of students’ techniques. (The bottom item may have been informed by parental intervention.) The central column shows rate tables and transitioning into proportion quartets. The second-from-top item shows a student who included the 15:25 kitten-and-cat quantities in Round 5 to make a PQ, as the basic ratio 3:5 was used as the column labels. The bottom item, a response to the subsequent clause, “If Tom has 56 pellets left, how many rounds can he feed his cats?,” has been included to show a student spontaneously extending the ratio table to include a ‘total column,’ that is also in the MT. The right column shows a variety of proportion quartets (note the different positions of the 3 and 5). This variety may indicate students’ growing comfort with setting up a PQ in orders that deviate from the MT order (students had been practicing representing ratios with MT cutout columns and rows).

Verbal Participation as Evidence of Students Learning

Analysis of students' verbal participation in the first 7 days of the intervention in Ms. Winningham's classroom, and in particular students' utterances showing understanding and those showing difficulty, enabled monitoring students' day-by-day increasing understanding of the representations and helped to pinpoint conceptual and design areas of difficulty. The following sections discuss learning patterns revealed through analyzing these utterances along several coded dimensions. Students were distributed uniformly by gender across all these analyses, so data were collapsed for all tables.

Increased Fluency With Mathematical Representations

Table 7 (see next page) shows students' total day-by-day verbal-participation LIPs and presents ratios between LIPs showing understanding and all LIPs. (For definition of 'LIPs,' the Learning Issue Points coded in students' verbal turn taking, see the methods section of this dissertation). The overall intervention ratio, 0.66, indicates that much of the time student utterances were showing understanding and that during the intervention, the teacher was successful in probing and eliciting from students their misunderstandings. Such elicitation was the result of closely attending to students' ideas and not only to whether or not they were responding correctly. Also, the discussion seems to have enabled students to understand the material by overcoming their individual difficulties with particular learning issues and to help students with similar difficulties.

Beginning from Day 3, when the mathematical representations began replacing the pictorial representations, student participation ranged between 0.93 (Day 3) and 0.52

Table 7

Totals of Student Learning-Issue Points Showing Understandings and Ratios of These to All Learning-Issue Points by Learning Issue and Intervention Day

Understanding	Intervention days							Totals
	1	2	3	4	5	6	7	
Multiplicative Structure and Use of the Reps.	12/12 1.00	8/8 1.00	12/14 0.86	15/19 0.79	31/35 0.89	14/24 0.58	25/29 0.86	117/141 0.83
Rows/Columns are Repeated Addition Sequences	9/9 1.00	0/1 0.00	12/21 0.57	5/9 0.56	6/6 1.00	1/13 0.08	-	33/59 0.56
Repeated Addends vs. Totals	0/1 0.00	3/6 0.50	0/5 0.00	2/4 0.50	1/2 0.50	-	-	6/18 0.33
Rows/Columns in Stories	-	8/8 1.00	9/14 0.64	13/23 0.57	8/9 0.89	5/9 0.56	4/7 0.57	47/70 0.67
Linking Column for the 2 Sequences	-	1/1 1.00	2/2 1.00	5/7 0.71	3/4 0.75	-	-	11/14 0.79
Labeling (“Table manners”)	-	-	-	1/1 1.00	1/6 0.17	3/9 0.33	0/2 0.00	5/18 0.28
Zero Starting Point	-	1/1 1.00	3/6 0.50	-	2/2 1.00	0/1 0.00	-	6/10 0.60
Vocabulary	-	1/1 1.00	1/2 0.50	0/2 0.00	6/13 0.46	0/1 0.00	0/2 0.00	8/21 0.38
Totals	21/22 1.00	21/26 0.81	39/64 0.61	41/69 0.59	57/77 0.74	23/56 0.41	29/41 0.71	233/351 0.66
LIPs per minutes[a]	.35	.38	.93	.84	.70	.52	.57	.61

Note: Several tables in this section analyze the totals of student utterances showing understanding and showing difficulty. There are slight discrepancies between these tables due to several LIPs that were difficult to code.
[a]Total number of LIPs (learning-issue points) on that intervention day divided by the duration of the lesson that day, e.g., on Day 1 of the intervention there were on average a total of 0.35 LIPs per minute. LIPs are not distributed uniformly across each lesson, but this metric affords a comparison *between* days.

(Day 6) LIPs/minute. The total participation indices, taken together with the ratios of understanding to total LIPs reveal the particularly difficult days, as follows.

Of particular interest are values within Table 7 that show both a low understanding ratio and considerable participation, i.e., these particular learning issues were challenging and were prominent in student discussion, as follows. Students' difficulties with the learning issue "Rows/Columns are Repeated-Addition Sequences" on Day 6 reflects their difficulty in building tables for problems where the ratio is not given but must be determined. For instance, in a word problem involving the proportion 2:8 = 5:? the basic ratio 1:4 is not given. Several students had difficulty creating an RT with repeated-addition values—even students who, on other tasks, could build the tables given the basic ratio and could readily interpret a pre-completed table as linked repeated-addition sequences. Instead of attempting to work backwards, e.g., from 2:8 to 1:4, and then forwards again to 5:20, some students wrote the given ratio, e.g., 2:8, at the top of their table and then had trouble completing the table. Other students did work backwards but were not consistent in their repeated adding down each column.

This learning issue was later no longer problematic, because students learned to determine the basic ratio by finding the common factors of the given ratio and stating the basic ratio as the other two factors (e.g., 2 is the common factor of 2:8, so the basic ratio is 1:4). In fact, Day 6 can be seen as pivotal in students migrating from the RT to the PQ. The PQ is inherently multiplicative and not additive, and students already using the PQ did not evidence difficulty with this learning issue (even those students who did evidence this learning issue when working on the RT). Thus, students' Day-6 challenges with the repeated-addition aspect of the RT can possibly be seen as motivating students to migrate to the PQ, because the students saw that the PQ circumvented calculation and construction challenges that they were experiencing with the RT. In hindsight, I realize that the classroom needed more support in using the RT in such situations and that for the lowest students the teaching–learning passage from the RT to the RT-with-missing-values (a scaffolding representation) and eventually to the PQ (that has only 2 rows) may have been too swift to have fostered a full understanding of how the multiplicative PQ-solution is grounded in the repeated-adding model of ratio-and-proportion. In the current version of the design, students have more opportunities to create RTs, even after they have learned to use the PQ, and link them to PQs and to stories.

Students' difficulty with the learning issue 'Rows/Columns in Stories' on Day 4 involved understanding that contemporaneous events are represented in the same row of the RT or MT, as in the following problem: "Creepy and Crawley are lizards. They live on the roof of the Sears Tower. One day, they decided to go shopping, so they started walking down the outside wall. When Creepy had come down 28 floors, Crawley had come down 49 floors. What could their walking rates be?" (see also Figure 3d, for a numerical variant on this problem). Students were comfortable with labeling the column categories, e.g., 'Creepy' and 'Crawley,' but in word problems such as this, where no specific time unit is designated, students had difficulty assigning a conceptual category that would tie the events as contemporaneous or articulating mathematically why the values, e.g., 28 and 49, should be in the same row, as the following exchange demonstrates:

Fernando: Because they're both going at the same rate.
Ms. Winningham: They're… no they're not going at the same rate. They have both traveled for…
Fernando joins: …the same amount of time-unit.
Ms. Winningham: Absolutely, so they have to be in the same row.

Several students confronted the same learning issue on Day 5, when they were creating an RT. In generating their own tables, these students had trouble interpreting the text as indicating that the values of concurrent totals should be in the same rows. Of the 4 Day-6 instances of students showing difficulty with this learning issue, 2 instances involved an ESL student, 1 the LD student, and 1 a student who had been absent on the previous day.

Students' difficulty with the learning issue 'Repeated Addends vs. Totals' peaked on Days 2 and 3 of the intervention. On those days, students first learned to interpret MT rows and columns as corresponding to stories and pictures. Note in Table 7 that in the row below—learning issue 'Rows/Columns in Stories'—it appears that students were generally understanding *that* the MT rows and columns were being used to model the ratio stories. Comparison between these two rows in Table 7 implicates students' difficulty with this activity in understanding *how* the MT models the ratio stories. In particular, their difficulty was in reconciling between the repeated addend in the stories, e.g., +$3 every day, with the increasing values in the MT column that they were using to tell the story, e.g., 3, 6, 9, 12, etc., as the following transcription demonstrates:

Violet: Duffy Duck was going one day to Porky Pig's house, and, uhhm, one hour… in one hour he walked 3 miles […] So the first mile… the first hour…hnnn... Duffy Duck walked 3 miles. In the second hour Duffy Duck walked 6 miles, in the third hour Duffy Duck walked 9 miles…
Ms. Winningham: That would be a total of 18 miles.

> Violet: In the first hour he walked 3 miles, in the second hour he walked 3 *more* miles….

We responded to students' difficulty with interpreting the MT columns as running totals and not as stand-alone values with the following design element: students connected consecutive values going down the MT story column with small arcs—")"—and wrote next to each one of them the constant addend, e.g., "+3," so that they could see both the constant addend and the running totals (see Figure 1 for this design element in a RT). Students used these interpolated symbols to support their telling repeated-adding stories, such as about their filmstrips. Also, this design element helped students see the top value in the column, e.g., 3, as being the first total in that column because they wrote a '0' above it and connected down to the 3 with an arc and a "+3" next to it. Finally, this element allowed the teacher to highlight the constancy of the addend by asking students whether or not non-constant addends, e.g., +3, +5, +4, etc., would still model a rate story. On the later days of the intervention we faded out this scaffold. Table 7 indicates that this learning issue was discussed less and less after Day 3 of the intervention and was no longer a topic of discussion by Day 6.

Despite our beseeching the students on a daily basis to "label the table," and although this rhyming epigram became a classroom slogan, many students persisted in refraining from writing labels for the columns and rows in their tables. Several students began using labels only once they had personally experienced solving a problem incorrectly due to the absence of labels. I hypothesize that students' resistance to labeling is due to their previous experience with arithmetical problems in which they never got lost or confused. Up to studying ratio and proportion, most problems these students had encountered gave 2 values and not 3, or gave 3 values that had all to be added up or multiplied. So until having personally experienced the woes of not labeling, students appeared to have taken labels to be a superfluous "school" practice that did not really bear on or improve their chances of successfully solving the problems. Thus, I concluded that students' initial resistance to the use of labels stemmed from their not appreciating the utility of labeling. In responding to students' difficulty with this learning issue, I found classroom discussions useful in students witnessing firsthand that unlabeled tables may undermine solution processes. For instance, the moment a presenting student looked away from an unlabeled table on the overhead projector to look at the word-problem text on their worksheet or at another classroom mathematical representation, they forgot either the names of the characters in the story or which column corresponded with which character, and the ensuing confusion led to realizing that labels could have helped. I believe that students coming to appreciate the effectiveness of combining lexical labels in mathematical work is pivotal for their coping with complex data and mathematical representations, such as graphs in middle school. Therefore, I see promise in leveraging our design in instilling good "table manners" in 5th-grade students.

Finally, student challenges with assimilating new vocabulary stemmed both from the many new terms and syntactic forms (e.g., 'acorns per day') that this design introduced and from the necessary explorativeness of this design that created more terms than a usual implementation of such a unit would. For instance, our encouraging students to invent their own descriptions of 'rate' created a plethora of terms, e.g., 'growing number,' 'growth unit,' 'constant plus number,' etc., which the students then confused. However, some confusions, e.g., 'row' and 'column,' were core to any design for ratio and proportion. This dissertation does not analyze the special challenges that ESL students faced.

Student LIPs by Achievement Group

Table 8
Totals of Student Learning-Issue Points Showing Understandings and Ratios of These to All Learning-Issue Points by Achievement Group[a] *and Intervention Day*

Number of Students in Test Achievement Groups	Students' Average Daily Participation[b]	Intervention Days 1	2	3	4	5	6	7	Totals Mean[c]
Top									
n = 1	2.6	3/3 1.00	0/0 0.00	1/1 1.00	0/0 0.00	6/7 0.86	2/2 1.00	4/5 0.80	16/18 0.89
n = 6	2.6	8/8 1.00	9/9 1.00	14/22 0.64	12/18 0.67	19/24 0.79	7/15 0.47	13/15 0.87	82/111 0.74
Middle Group									
n = 9	2.9	7/8 0.88	12/13 0.92	22/36 0.61	15/27 0.56	38/50 0.76	10/28 0.36	12/19 0.63	116/181 0.64
Bottom									
n = 2	1.1	-	-	1/2 0.50	0/3 0.00	2/3 0.67	3/7 0.43	0/1 0.00	6/16 0.38
n = 2	1.4	3/3 1.00	1/1 1.00	1/3 0.33	3/4 0.75	3/4 0.75	1/3 0.33	1/1 1.00	13/19 0.68

[a]Total number of learning-issue points was 351 (233 showing understanding and 118 showing difficulty).
[b]Total number of learning-issue points showing either understanding or difficulty over the entire 7 days per student (variation may occur due to rounding). For example, a Middle-Half student made on averge 2.9 daily learning-issue points.
[c]Average intervention ratio between total learning-issue points showing understanding and all learning-issue points.

Table 8 (see above) presents daily ratios between students' LIPs showing understanding and all LIPs by achievement group. Looking at the second column of this table, Students' Average Daily Participation, we see that students' daily utterances in the two top groups and in the middle-half group were of the same order of magnitude (2.6, 2.6, and 2.9 per student per day). I attribute this uniform distribution to Ms. Winningham's experience in leading classroom discussions. That the bottom quarter spoke less is attributable, on the one hand, to these students' inferior fluency with the material and associated lower self-confidence, but also to 3 of these students being ESL students, who volunteered less often (the remaining student is LD). The middle half of the class was particularly active on Day 5, in consolidating the RT as MT columns that are both additive and multiplicative and later during the introduction of the PQ. I interpret this group's enthusiasm in working with the PQ as stemming from their sense of empowerment—students volunteered to come up to the large classroom MT, locate PQ rectangles on the MT, and factor these PQs on the overhead projector. Their engagement with solving purely-numerical PQs and using the PQ to solve unknown-value proportion problems continued throughout the design. The following day (Day 6), the middle-half students' total of 18 utterances showing difficulty were exclusively related to the RT and not to the PQ (the second-from-top group of students did not fare much better). As I suggested earlier, the middle-half students' greater comfort with the PQ as compared to the RT does not necessarily mean that the RT is an unsuitable representation for solving ratio-and-proportion problems. Indeed, a student could learn to use the PQ without having learned to use the RT at all. However, students' difficulty with and eventual success in addressing the additive–multiplicative structure of the RT indicates the importance of the RT in grounding students' understanding of ratio and proportion in their previous mathematical understanding.

Student Use of Linking

Table 9
Student Use of Linking: The Number of Student Learning-Issue Points Showing Understanding and the Average Student Rank[a] of Speaker by Intervention Day and Linked Domai433n Referents

		Intervention Days						
Ref.	Linked to	1	2	3	4	5	6	7
MT								
	-	21(8.8)	18(9.3)	3(11)	2(5.3)	-	1(9)	-
	Pictures	-	1(4)	5(8.6)	-	-	-	-
	RT	-	-	14(10.7)	18(9.9)	1(7.5)	1(4)	-
	PQ-rect	-	-	-	-	1(10)	4(13.7)	8(13.3)
	PQ	-	-	-	-	-	1(14)	9(8.0)
	Related Domain:							
	Fractions	-	-	-	-	-	-	7(4.7)
RT								
	-	-	2(7.2)	2(12.5)	10(8.9)	4(9.3)	3(16.3)	3(10)
	MT	-	-	2(12.5)	-	23(11.1)	5(10)	-
	Pictures	-	-	11(10.1)	-	2(13.5)	-	-
	PQ	-	-	-	-	-	1(1)	-
PQ								
	-	-	-	-	-	-	1(9.9)	1(11)
	MT-rect	-	-	-	-	9(10.9)	2(14)	-
	MT	-	-	-	-	16(9.5)	1(12)	3(14.7)
	RT	-	-	3(9.3)	-	2(1)	1(13)	2(2)
	Related Domain:							
	Fractions	-	-	-	-	-	1(4)	3(2.3)

Note. MT-rect and PQ-rect refer to students speaking about the PQ as a rectangular constellation of only 4 values in the MT without any apparent understanding of the multiplicative structure of the PQ, i.e., they do not refer to the rows and columns of this rectangle as the factors and/or to the PQ values as the products.
[a]Students (n = 20) are ranked from 1 through 20 according to their achievement on the pretest of the ratio-and-proportion unit, with 1 being the top performer.

As in the analysis of students' written work, in analyzing student utterances I examined students' coordinating between the mathematical representations. Table 9 (see above) presents the numbers of student LIPs showing understanding according both to the mathematical representation that they were speaking about (MT, RT, or PQ) and any other representation that their utterances and gestures suggested they were seeing in the focal representation. Also, this table tracks the average student ID of all students in each table cell. Thus, "21(8.8)" in the top-left cell means that 21 students whose ID average was 8.8 (between 1, top-achieving student, and 20, lowest-achieving student) spoke about the MT without making any links to other referents. That there are no links is anticipated since on the first day of the intervention students worked with the MT only. The overall sweep of this table from the top-left corner and down to the bottom-right corner shows the general design of the unit—MT, RT, and then PQ—and students continuing to discuss previously-introduced representations after new representations have been introduced. The classroom interaction examples in Figure 3 show the kinds of links made. As the intervention progressed, an increasingly larger part of student LIPs showing understanding were LIPs that linked between representations; the types of links, too, grew from day to day. That students connected the representations to fractions was due both to their having found fraction-equivalence patterns in the MT on earlier days—patterns that they drew into the domain of proportion on later days—and due to their seeing relations between ratio equivalences (proportions) and fraction equivalences in the PQ representation. The higher-achieving students made these connections.

Table 10

Ratios Between the Number of Student Learning-Issue Points Showing Understanding and All Learning-Issue Points by Intervention Day and Domain Referent With Any Linked Domain Referent

Ref.	Link	Intervention Days 1	2	3	4	5	6	7	Totals Mean
MT									
	-	21/21 1.00	18/18 1.00	3/3 1.00	2/2 1.00	-	1/1 1.00	-	45/45 1.00
	Pictures	-	1/1 1.00	5/5 1.00	-	-	-	-	6/6 1.00
	RT	0/1 0.00	-	14/18 0.78	18/24 0.75	1/2 0.50	1/3 0.33	-	35/48 0.73
	PQ-rect	-	-	-	-	1/1 1.00	4/4 1.00	8/8 1.00	13/13 1.00
	PQ	-	-	0/1 0.00	-	-	1/2 0.50	9/14 0.64	10/17 0.59
	Related Domain: Fractions	-	-	-	-	-	-	7/8 0.88	7/8 0.88
RT									
	-	-	2/3 0.67	2/2 1.00	10/25 0.40	4/10 0.40	3/29 0.10	3/3 1.00	24/72 0.33
	MT	-	-	2/2 1.00	0/5 0.00	24/24 1.00	-	-	26/31 0.84
	Pictures	-	-	11/11 1.00	-	2/2 1.00	-	-	13/13 1.00
	PQ	-	-	-	-	-	1/1 1.00	-	1/1 1.00
PQ									
	-	-	-	0/1 0.00	-	0/3 0.00	1/5 0.20	1/5 0.20	2/14 0.14
	MT-rect	-	-	-	-	9/9 1.00	-	-	9/9 1.00
	MT	-	-	-	-	16/24 0.67	3/3 1.00	3/3 1.00	22/30 0.73
	RT	-	-	3/6 0.50	-	2/3 0.67	2/2 1.00	2/2 1.00	9/13 0.69
	Related Domain: Fractions	-	-	-	-	1/1 1.00	3/3 1.00	3/3 1.00	7/7 1.00

*Students (*n* = 20) are ranked from 1 through 20 according to their achievement on the pretest of the ratio-and-proportion unit, with 1 being the top performer.

Table 10 (see previous page) presents ratios of students' LIPs showing understanding and all LIPs according both to the mathematical representation being discussed and any other representations that the utterances and gestures suggested the student was seeing or referring to in the focal representation. The sharp shift in the focus of student utterances from Day 6 to Day 7 between the RT and the PQ reflects these students' learning to use the PQ. The generally low ratios in the row of utterances referring to the RT without links reflects the middle-half students' struggle to construct correctly RT representations in the solution of word problems.

Table 11
Number of Student Learning-Issue Points Showing Understanding and Showing Difficulty According to Whether or Not the Domain Referent Was Linked to Another Referent

	Focal Referent		
Utterance	Linked	Unlinked	Total
Understanding	159	74	233
Difficulty	40	38	78
Total	199	112	311

$*p < 0.01$; X^2 $(1, N = 311) = 7.3$

Table 11 (see above) shows student LIPs according to whether or not the LIP showed understanding and whether or not they were linked to other domain referents. Students' tendency to link referents in their LIPs is related to their tendency to show understanding (X^2 $(1, N = 311) = 7.3$, $p < 0.01$). A linked LIP is more likely to show understanding as compared to an unlinked LIP. It is important to stress that students are equally distributed in these 4 cells by achievement, so the achievement average of students in each cell is not different. That is, it is *not* the case that most understanding was manifested by high-achieving students and most difficulty by lower achieving students, and it is *not* the case that most linked representations were referred to by high-achieving students and most unlinked representations by lower-achieving students. This is because in the voluntary component of students' verbal participation as well as in the individually-tailored scaffolding questions from the teacher, students spoke and ventured from within their own understanding zone. So higher-achieving students showed difficulty with content that was probably over the head of lower-achieving students, and lower-achieving students showed understandings with content that was well understood by higher-achieving students.

Relationships Between Additive–Multiplicative vs. Multiplicative Learning Issues and Representations

The design was for the ratio table to foreground the additive–multiplicative action model of multiplication (repeated adding) and for the proportion quartet to foreground the multiplicative action model (factor*factor→product). Table 12 (see on next page) charts the LIPs according to whether they were additive–multiplicative or multiplicative and according to their referent mathematical representation. The learning issues associated with additive–multiplicative reasoning are 'Rows/Columns are Repeated-Addition Sequences' and 'Repeated Addends vs. Totals.' The learning issue associated with multiplicative reasoning is 'Multiplicative Structure and Use of Representations.' Table 12 relates these additive–multiplicative and multiplicative learning issues with the RT and PQ uses over interventions days 3 through 7. Students were more likely to address additive–multiplicative issues of the domain when working with the RT and multiplicative issues when working with the PQ ($p < 0.001$; X^2 (1, N = 179) = 53.5). During Days 3 and 4, before the PQ had been introduced as a specialized tool for multiplicative reasoning, it could be that the RT had not yet become a specialized tool for additive–multiplicative reasoning, so including Days 3 and 4 in this table might bias the results. When data from Days 3 and 4 are eliminated from this table, the statistic becomes stronger ($p < 0.001$; X^2 (1, N = 122) = 61.3).

Table 12
Number of Student Learning-Issue Points by Additive–Multiplicative or Multiplicative Learning Issue and Focal or Linked Domain Referent on Days 3 Through 7

	Focal or Linked Domain Referent		
Learning Issue	Additive–Multiplicative RT	Multiplicative PQ	Total
Additive–Multiplicative	59	6	65
Multiplicative	39	75	114
Total	98	81	179

Note: The additive–multiplicative learning issues are: (a) Rows/Columns are Repeated-Addition Sequences; and (b) Repeated Addends versus Totals. The multiplicative learning issue is Multiplicative Structure and Use of Representations.
*$p < 0.001$; X^2 (1, N = 179) = 53.5

This result supports the design working-hypothesis that both the RT and the PQ would be necessary and complementary in teaching the additive–multiplicative and multiplicative aspects of ratio and proportion, and that each representation differentially supports thinking about its associated domain aspect. Further analysis shows that until the PQ had been introduced, students were equally likely to use the MT to speak about additive–multiplicative or multiplicative learning issues. However, once the PQ had been introduced, students used the MT to refer exclusively to multiplicative learning issues. It is not clear to what extent the latter finding simply reflects our design progression from the RT to the PQ or whether the repeated-adding aspect of the domain, that was strongly associated with the RT interpretation of the MT, had become sufficiently assimilated as an implicit classroom construct and no longer needed to be discussed. It could also be that the growing complexity of the numerical cases students addressed in solving word problems was such that students found the PQ more effective. In hindsight, it could be that at various checkpoints along the later days of the design the class should have examined RT solutions to word problems that students were solving with a PQ so as to strengthen student grounding in and fluency with the additive–multiplicative model of the domain.

Student Individual vs. Choral Utterances

In the video footage of the first seven intervention days in Ms. Winningham's classroom, I found a total of 351 learning-related utterances by identified individual students (233 showing understanding and 118 showing difficulty). In addition, there were another approximately 259 learning-related utterances (187 showing understanding and 72 showing difficulty) by unidentified individuals and groups of students. Table 13 (see next page) presents a comparison between students *choral* and individual verbal participation. Differences between daily ratios of students' utterances showing understanding and those showing difficulty vary between choral and individual responding on Days 4 and 7 of the intervention. Contextual analysis of the choral responses indicated that the teacher elicited students' choral responses so as to maximize participation and feelings of community, assess where the class as a whole was, consolidate ideas that had been agreed upon, and induce a sense of drama that animated the classroom. Analysis of the distribution of utterances over the duration of the lesson does not show bursts of individual utterances following choral utterances. Rather, this practice possibly helped keep students focused, such as at the end of Day 5, which was a particularly long day (110 minutes). On Day 1, students did not need any stirring; neither were there new ideas to present or any consolidated ideas. On Day 7, a substantial part of the lesson was led by students and not by the teacher, and so the teacher had less opportunity to elicit choral participation.

Table 13
Ratios Between "Choral" Learning-Issue Points Showing Understanding and All "Choral" Learning-Issue Point by Intervention Day

		Intervention Day							
Utterance	Totals	1	2	3	4	5	6	7	Avrg.
Choral	187/259 0.72	0/0 0	37/45 0.82	14/19 0.74	31/31 1.00	82/105 0.78	10/16 0.63	13/43 0.30	.71
Individual	233/351 0.65	21/22 1.00	21/26 0.81	39/64 0.61	41/69 0.59	57/77 0.74	23/56 0.41	29/41 0.70	.79

Note. Choral utterances scored as showing difficulty if one or more voices uttered a wrong answer. This is a very conservative measure.

Classroom Mathematical Connectivity

Figure 7 shows the distribution of student LIPs by prime referent and any linked referent on two intervention days (Days 5 and 7). The figure backgrounds individual speakers and foregrounds the meandering of the classroom conversation between the mathematical representations. In preparing this figure, I chose two 40-minute periods

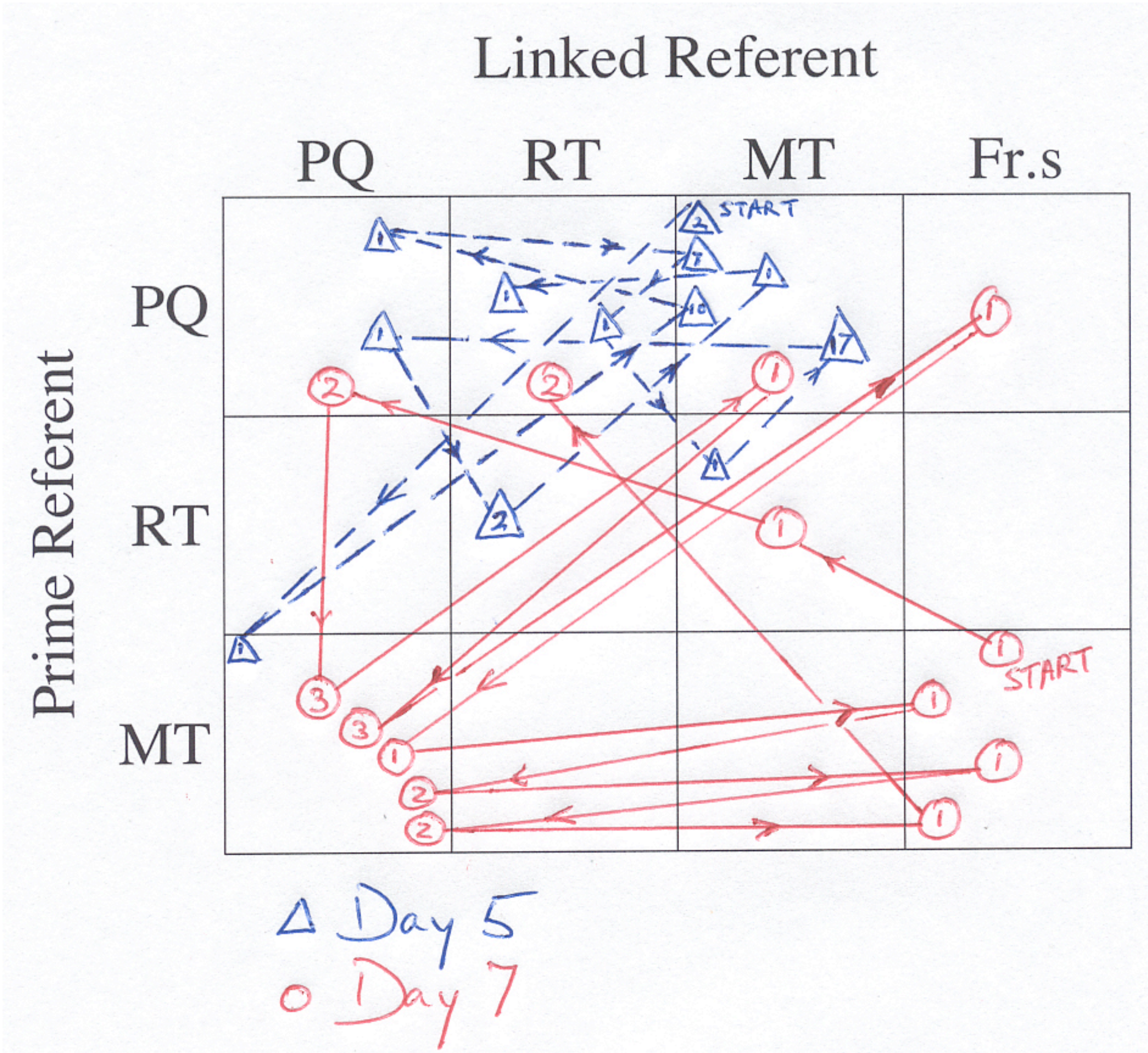

Figure 7. Referent-space classroom conversation connectivity. Each broken line running between *referent-link nodes* (circular nodes for Day 5, triangular nodes for Day 7) tracks a 40-minute classroom period as it moves between regions (cells in the referent space). The value in each node shows the number of consecutive student learning-issue points in each region. Computing properties of conversations that are indexed by the number of nodes, the number of learning-issue points in each node, and the number of regions out of all regions in the space affords metric of the conversations' connectivity. Comparing connectivity on different implementation days affords monitoring students learning, and specifically their growing fluency with seeing-in-using the mathematical tools (representations in action) and understanding their structure and functionality. The domain's constructs are initially embedded in designed artifacts but emerge through classroom activities that foster students instrumentalizing and linking the representations and thus assimilating the mathematical ideas.
MT—Multiplication Table; RT—Ratio Table; PQ—Proportion Quartet; Fr.s—Fractions.

that were characterized by dense student participation that included multiple students coming up to the front of the classroom and using classroom referents (the large MT and the overhead projector) to communicate their ideas. Because student conversation is topically concatenated—that is, each student responds to the previous speaker and triggers the next speaker, following a classroom continuous conversation indexes *classroom* fluency with a domain. Also, comparison between representational fluency on different intervention days brings out any trends in classroom fluency building. In order to probe for any such development, I selected intervention days and periods when all three design domain-referents—MT, RT, and PQ—had been introduced.

The table in Figure 7 plots LIPs as intersections of their focal referent and any linked referent. So each table cell (*referent–link region*) contains *linked–referent nodes* in the classroom participation-space, and the values in the cells represent how many *consecutive* LIPs dwelled in that node. For instance, the Day-7 conversation, beginning at the "START" in the bottom-right MT–Fraction region and represented with circular nodes, began with a single LIP (see the "1" in the circular "START" node), continued with another single LIP in the RT–MT region, 2 LIPs in the PQ-unlinked region, 3 LIPs in the MT–PQ region, etc. I find this representation potentially illuminating of subtle mechanisms and changes in classroom verbal participation in a learning environment that is rich with mathematical supports that need to be coordinated in building an understanding of the domain. I am only beginning to explore this data-analysis tool and believe that it affords inferences that could be informed by the knowledge and representation-technologies of networks studies, e.g., the calculation of traffic between regions. The following analyses are presented both to understand students' development in this intervention and to illustrate the use of this tool. For a design that attempts to foster students' linking actions between classroom tools, an index of fluency as seen in data helps identify classroom activities and settings that are conducive to such linking.

Note that although both samples are of the same duration (40 min.), the total number of LIPs is 47 for Day 5 and 22 for Day 7. This is because on Day 7 individual students presented their ideas over longer periods as compared to Day 5 (because on Day 7, students' ideas were more complex, involving nascent insights into new connections students were discerning between the mathematical representations). Note, too, that although there are more than double the number of LIPs on Day 5 (47) as compared to Day 7 (22), on Day 5 student discussion fell into 6 different referent-link regions (cells) and on Day 7 it fell into 7 different referent–link regions, with LIPs on both days sharing 4 common regions. An analysis that looked only at the number of regions per day would fail to find that on Day 5 the LIPs-per-region ratio (45/6 = 7.6) was larger than on Day 7 (22/7 = 3.1). The LIPs-per-region ratio reflects the density of the conversation—it inversely reflects the variability in the distribution of the conversation—but this metric does not reflect the conversation's connectivity. That is, it reveals conversation foci within each day and, given that we have taken equal durations on each day, it affords comparison between numbers of LIPs per region on these days. However, this metric cannot show how the conversation moved *between* the regions (see final paragraphs of this section).

The referent-network representation affords direct visual overviews of the *linked-referent space* of students' conversation (the space contains all regions). First, it is immediately evident that students' conversation dwelled in different regions of this space, with Day-5 LIPs located around the PQ as a prime referent (41/45 = .9 of the total utterances in the sample period) and especially around the MT as linked referent (37/45 = .8 of the total LIPs in the sample period or 37/41 = .9 of the utterances with PQ as the prime referent). The Day-7 LIPs are located around the MT as prime referent (15/22 = .7) and in particular around the PQ as linked referent (11/22 = .5 of the total utterances in that period, or 11/15 = .7 of the LIPs with MT as the prime referent). The difference in location of LIP clusters in the linked-referent space on Day 5 and Day 7 results from the novelty of the PQ on Day 5. Students focused on PQs that the teacher and then they created on the overhead projector. The PQ had to be extracted from the MT and isolated in order that students clearly see and use its factor*factor→product multiplicative structure—an insight that expanded on merely seeing the PQ as a rectangular constellation on the MT (a seeing that often misses the multiplicative relations). By Day 7, students were familiar enough with PQ structure and function so as to be able to discuss it while working on the MT and, specifically, without needing to create a PQ each time they wished to refer to a PQ in the MT. Note that on both days the majority of the LIPs dealt with the PQ as either prime or linked referent (42/45 on Day 5 and 16/ 22 on Day 7). That students had moved to focusing on the MT as prime referent reflects not so much a change in the referent of the conversation as much as in the mathematical level of the conversation: it suggests that students had developed a mathematical vision of the

MT—they were comfortable to look at the MT, yet pull back and speak multiplicatively about the PQs that they saw within the MT.

The three linked-referent regions that the Day-5 and Day-7 conversations do not share are RT (unlinked), MT/Fractions, and Fractions/PQ. The first of these appears on Day 5 (2 LIPs) and the fraction-related discussion appears exclusively on Day 7 (5 LIPs).[12] Conversation about fractions was led by the higher-achieving students—the student ID average in all fraction-related utterances is 4—and these students' statements suggest that the PQ-in-MT was helping them in negotiating between ratio and fraction equivalences and thus re-grounding their understanding of fraction equivalencies: these students were seeing within the MT the PQ structure and alternation interpreting it as equivalent ratios or as equivalent fractions, and the three highest-achieving students of the classroom debated what this alternation meant in terms of the word-problem situation. What had been two equivalent ratios between cans of paint and painted chairs (15 cans :25 chairs = 18 cans:30 chairs) became two equivalent ratios between cans and cans and between chairs and chairs (15cans:18cans = 25chairs:30chairs). These three students using the MT as a conceptual springboard or explorative arena for struggling between mathematical insights and mathematical models of situations—a spontaneous and voluntary inquiry—typifies many student discussions in our mini-unit on ratio-and-proportion situational meanings, three months later (see below).

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[12] The current 2-dimensional referent-link space representation is somewhat inadequate in that it is sometimes difficult to differentiate referent from content of complex LIPs. For instance, 'Fractions' is used here to designate both students' gesturing to fraction equivalences that they see in the PQ that is in the MT and to students' writing out these equivalences.

The number of nodes in proportion to the total number of LIPs in a period reveals the dynamics of the referent connectivity within the conversation, and in particular shows how the LIPs are distributed *over time* across the conversation. Note that a conversation's dynamics is orthogonal to the variability in its distribution: a conversation could move rapidly between only two regions. I suggest the term *connectivity* as a metric that captures both the rapidness in which a conversation moves between regions in the linked-referent space (Nodes/LIPs) and the proportion of regions within that space that the conversation visits (Regions Visited / Total Regions in the Referent-Link Space). Connectivity is a product of these two metrics. On Day 5 the connectivity of the conversation was .14 and on Day 7 the connectivity of the conversation was .37. Seeing as students led the conversations during both of these periods, I interpret the higher connectivity of the discussion on Day 7 as compared to Day 5 as suggesting that the classroom as a whole was becoming more fluent in the designed domain of ratio and proportion.

Summary of Analysis of Student Verbal Participation

In summary, students at all achievement levels progressed both in terms of the number of mathematical representations they understood and the connections they built between these representations. In learning the RT and PQ representations, students began by discerning the representations as patterns in the MT, then worked directly on these representations and built connections back to the MT. Students' existing familiarity with the multiplicative use of the MT and new understandings of the repeated-adding interpretation of the MT columns informed student understanding of the additive–multiplicative and multiplicative aspects of the new representations, with the RT being interpreted primarily as additive–multiplicative and the PQ almost exclusively as multiplicative. Once familiar with the new representations, students could use the MT both to speak about the new representations and to connect between these representations and to other mathematical constructs, e.g., fractions. In building these connections, students were alternately interpreting the MT in several ways that are complementary in structuring an understanding of the domain according to our design. Students who were higher-achieving led conversations about new connections they were making between the representations, and other students were distributed at different points on the same learning path.

3-Day Mini-Unit on Situational Meaning in Ratio and Proportion

This section presents finding from the mini-unit we carried out in Ms. Winningham's classroom three months following the main ratio-and-proportion intervention.

Students attempting to differentiate proportion from non-proportion word problems engaged in animated debates in which 2/3 of the students participated at least once about the relation between real-world situations and mathematical representations and algorithms. Using the classroom mathematical representations and gestures, the top achieving-half of the class debated whether or not situations that they themselves invented were cases of proportion. For example, one student suggested a situation in which 2 basketball teams each double their score on successive matches between them: 1st game: 12 & 15; 2nd game: 24 & 30; 3rd game: 48 & 60). The score pairs—(12, 15), (24, 30), (48, 60)—constitute a proportional progression that students represented on an extended MT and tabulated with the RT and PQ representations, but the number triads that include the ordinal value of the game—(1, 12, 15), (2, 24, 30) (*3*, 48, 60)—created a dilemma: the situation had been established as bearing proportional relations, but the game numbers (1, 2, 3) could not be used to enumerate the linked repeated adding because in game #3 there were 4 iterations of the 12:15 ratio. Students' effortful discussion in addressing and solving this dilemma was essentially about the semiotics and epistemology of mathematics and specifically about the tension between mathematical representations and the situations they model. We interpreted the debate as students confronting, understanding, and possibly transcending the procedural scaffold inherent in the mathematical representations they had been using in building an understanding of the domain. Whereas students' work and discussion was promising, I judged that positioning the modeling activities as an isolated unit at the end of the design does not optimize its inherent potential for student learning, as I now explain.

The main message of the mini-unit was that students must be careful in choosing solutions to situations. Some students were frustrated that using their newly developed skills from the ratio-and-proportion unit—skills that had proved infallible and had earned them peer esteem and prestige—was now considered an insufficient and possibly incorrect response. Thus, the mini-unit caused some students to doubt their own skills without giving them adequate time to assimilate the new demand to account mathematically for their choices of strategy. I believe that in order for a design that dwells on modeling issues to be truly equitable, it must involve an earlier and broader treatment in which modeling is an integral element. That way students have an opportunity both to build their procedural skills in solving proportion problems and to build their critical mathematical reasoning.

I concluded that students' critical mathematical modeling of situations and analyzing mathematical representations in discussing solution methods is a highly desirable learning objective that is attainable through embedding this mini-unit's materials and activities back into the existing ratio-and-proportion design as a theme running through all the days (see also Lave, 1992, for a related critique of the common classroom usage of word problems; see Verschaffel, Greer, & De Corte, 2000). The new iteration of the design includes a range of multiplicative and other types of situations from the beginning.

Conclusions and Implications for Design, Teaching, and Research in Mathematics Education

Strengths, Limitations, and Tradeoffs of the Design

Strengths of the Design

The multiplication table (MT) constituted a central, powerful, and engaging learning tool that, on the one hand, encompassed and embodied the whole design, because students learned to interpret and connect in the MT the additive–multiplicative and multiplicative focal aspects of the domain. The MT also appears to have broadened the classroom learning zone, because all students could refer to and use the MT to solve word problems with a level of understanding that was paced throughout the intervention according to their individual achievement. Students' generally high achievement on the posttest and high level of participation across the range of mathematical achievement suggest that this design consolidates students' fluency with addition–subtraction and multiplication–division, links these kinds of understandings, and enables students to solve ratio-and-proportion problems with understanding. Specifically, the presence of a large multiplication table in the classroom as a readily accessible and vivid referent affords the teacher and students a mathematical hub for connecting between addition, subtraction, multiplication, division, rate, ratio and proportion, fraction equivalencies, and percentage. Students opting to solve proportion problems without the multiplication table even when it was available and both teachers' anecdotal reports on increased classroom fluency with multiplying and dividing after the intervention, suggest that an added value of students' participation in this unit is practice in multiplying and factoring.

Perhaps more importantly for teachers and for designers of elementary- and middle-school curricula, our results further suggest that students need not be fluent in rational-number constructs and operations, e.g., fractions and addition/subtraction and multiplication/division of fractions, in order to study ratio and proportion and ground a sense of the domain in multiplicative conceptual–situational models. There is an entire class of unknown-value proportion problems—non-multiple problems—that can be solved without using any non-integer numbers; by using the PQ mathematical representation and knowing multiplication and division relations. In fact, students can understand these problems and how to solve them, even if they are not initially fluent in the basic multiplications, but have a multiplication table at hand. Moreover, students' spontaneously connecting the content to their 5th-grade understanding of fractions (especially fraction equivalencies), their successfully solving geometrical-similitude and percentage problems (see Table 4), and their performance in the extension items on the posttest, all suggest that the additive–multiplicative model of ratio and proportion supports students' assimilating related models and constructs.[13]

Furthermore, the design is suitable for working with a wide range of students. This study focused on a 10-day implementation of the design with a total of 14.5 hours of classroom time that is equivalent to 14.5 60-minute periods or 17.5 50-minute periods. During the last 3 of these 10 days we extended the activities beyond the core content.

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[13] Whereas I have presented data of the higher-achieving students discussing fraction equivalencies in the context proportion, many students found equivalent fractions as patterns in the MT and, later in the unit, in PQs.

Based on analyzing the trajectory patterns of student learning, it seems likely that the bottom quarter of the classroom could have improved their achievement if we had restricted our teaching to the core concepts and certainly if the implementation had been longer.

Given the suitability of the design in fostering 5^{th}-grade students' basic understanding of the target domain and given that the design affords opportunities to revisit and strengthen students' earlier mathematical understandings, I would cast the implementation of this ratio-and-proportion unit as a Grade 5 or Grade 6 unit that structures a ratio-and-proportion integrated perspective. I have described the design as building from students' understanding of multiplication towards their understanding ratio and proportion. Reciprocally, the design positions multiplication as a case of ratio and proportion. This perspective expands on and structures factor*factor→product multiplication (a 3-value model, e.g., 5 groups of 3 units each makes 15 units) as a case of ratio and proportion that introduces a 4^{th} value, so 1:3 = 5:15.

Limitations of the Design

In evaluating the limitations of the design I have in mind not a research-participant teacher who interacts with a design team before and during an implementation, but a teacher who is first studying the unit package that includes a teacher guide and student work- and assessment sheets. So one challenge of the design is in communicating the design to teachers. Also, the complexity of this design, and specifically its demand that students revisit, consolidate, and leverage deep understandings of arithmetic towards seeing-in-using the multiplication table in solving problems in a new domain, calls for dedicated teachers. These teachers would need to be committed to restructure their own model of the domain as plotted on the designed representations. Also, the teachers would possibly need to study the domain deeply for the first time, to practice eliciting students' difficulties, and to develop the skills of moving flexibly between the new representations and leading math talk.

Design Tradeoffs

In developing the current version of our design, I attempted to balance between fostering opportunities for students to build procedural fluency with solving ratio-and-proportion problems and sensitivity for whether or not these skills and associated mathematical representations applied to given situations. Ultimately, I would like for students to be able to discern exemplars from non-exemplars and to solve each situation with appropriate mathematical tools. Further research is needed to establish whether our current design achieves these learning objectives.

Learning as Building Links Between Mathematical Tools

The design plan and data analysis were concerned with students' linking mathematical representations, real-world situations, and between the representations and the situations. The term 'linking' is both a theoretical construct that is informed by my embracing constructivist and Vygotskiian pedagogy and a design term that informs the creation of materials and activities. I use the same term to convey students' cognitive-developmental processes as well as classroom activities so as to reflect needs of design research: the term 'link' is broad enough to agree with many useful philosophical and theoretical contributions in the literature and practical enough to inform the daily practice of mathematics design and teaching (see Abrahamson & Wilensky, 2004, for an application of the bridging-tool design approach to the domain of probability and statistics). I prefer speaking of 'linking' (the verb) and not 'link' (the noun), because I see student understanding as an activity-driven way of attending to phenomena (seeing X as Y) and using objects (e.g., instrumentalizing the multiplication table in solving ratio-and-proportion word problems). Students build connections between existing understandings—they do not build stand-alone mental objects. Linking is one way that students assimilate new objects through activities that support detecting in these objects affordances (Gibson, 1977) of familiar structures and operations. Seeing columns of the multiplication table as making a ratio table (MT/RT) and seeing the ratio table as coming from a multiplication table (RT/MT) are different experiences even though they connect the same two objects. Student linking is a bi-directional and reciprocal association between ways of seeing-in-using mathematical tools, with each tool constituting an activity context for interpreting the other tool. It is in sorting out this negotiation—often through classroom discussion of what students are seeing and how they are seeing and using it—that students gesture and verbalize mathematical 'understandings' (constructing equivalence classes, assimilating, and accommodating, Piaget, 1952; detecting family resemblances; Wittgenstein, 1958; Noble, Nemirovsky, Wright, & Tierney, 2001; creating classroom semiotic networks, Greeno, 1998; grounding or concretizing the 'abstract,' Wilensky, 1991).

Constructivism and Mathematical Tools

Some mathematics-education scholars may object that students' engaging of mathematical tools without initial understanding goes counter to constructivist principles and is counter productive. Yet, toddlers practice the use of language even before they can compose and enunciate complete sentences. So whereas I strongly embrace the fundamental tenets of constructivism, I maintain that the reality of heterogeneous classrooms with broad learning zones, the common 1:25 or even 1:35 teacher-to-students ratio, the brevity of class periods, and the pressures currently built into most curricula do not make it feasible to tailor a learning progression that caters individually to each student's idiosyncratic moment-to-moment grounding of the domain in deep understanding. Learning a mathematical domain is a complex task involving the development both of deep understandings and procedural fluency with mathematical tools—a fluency that may be an initially superficial practice for some students. So during an intervention, and in particular an implementation of a complex design involving several mathematical representations, many ideas and procedures potentially form mental images of or from classroom discourse and visual-tool use, and each student builds their own learning progression from these images. Through classroom discussion, some of these nebulous conceptual–procedural images are clarified as students articulate and reflect on the meaning of their procedures and the mathematical–situational assumptions that go into choosing representations and solution strategies. These classroom interactions interface and, within a pragmatic framework, reconcile the constructivist and social-constructivist agenda because students develop and consolidate personal meaning for cultural mathematical tools, whose use the teacher and more-advanced students model and mediate.

When a new artifact is introduced into a mathematics classroom, learning a mathematical concept with understanding involves: (a) constructing a set of domain-specific modeling criteria, i.e., rules for verifying sufficient fit of the new problem-solving tools to perceived situations (or, vice versa, that the situations afford the tools); (b) practicing the formation and procedural use of these tools; and (c) developing vocabulary for communicating methods of modeling and of executing procedures. These processes are contemporaneous and interwoven from a bird's-eye view of a design, but within the classroom teaching–learning time, each process is usually given some special space and attention. From the apprehending-zone perspective, process *c*, developing vocabulary, is lodged in the *mathematizing* (modeling situations mathematically) and *storyizing* (inventing narratives about properties of and patterns in mathematical artifacts). Process *a*, modeling, is lodged in the students' stepping back from reciprocal activities of mathematizing and storyizing and critiquing their own mathematizing, and thus both gaining mastery over the domain through sharing heuristics and developing constructs for thinking about the practice of mathematics.

Students' critiquing their modeling through examining the 'fitness' of artifacts to situations is essential for learning with understanding. Thus, apprenticeship models by which students can master a domain through emulating experts' use of tools are inadequate as models for the design, teaching, and learning of mathematical concepts—students who use tools well do not necessarily understand the tools and therefore may encounter insurmountable difficulty in adjusting the tools or their use of the tools when the students encounter unfamiliar task demands.

Designed mathematical artifacts need not be flashy—students can engage in imaginative and profound discourse about simple aesthetical objects, such as the multiplication table, and the austerity of these objects may invite personalization, discovery of conceptual connections, and generalization (see also Uttal & DeLoache, 1997). The multiplication table is what Seymour Papert calls ‘an object to think with.’ The richness of the multiplication table affords a self-consistent mathematical microworld in which a classroom can explore new numerical relations, patterns, and consistencies, and this discursive exploration constitutes much of students’ necessary learning. Also, the rigorous schematic structure of mathematical concepts embedded in spatial–numerical artifacts, such as proportion as linked repeated-addition pairs scalloping down the MT-columns, builds classroom images that carry and communicate the mathematical concepts (see Abrahamson, 2004). Thus, mathematical ideas, and certainly elementary-school mathematical ideas, continue to live in and can be traced back to the artifacts that support the teaching, learning, and use of these ideas. So carefully chosen tools can help in establishing and keeping meaning in mathematical concepts, e.g., the multiplication table, ratio table, and proportion quartet help in keeping meaning in ‘proportion.’ My current understanding and definition of elementary-school mathematical meanings as “taken-as-shared spatial–dynamic images arising and interpreted within a classroom semiotic network of tools, situational contexts, vocabulary, participants, activities, and procedures” is perhaps complex, but I find that such a definition can help demystify or concretize mathematical meaning—it keeps ‘meaning,’ if you will, in proportion.

References

Abrahamson, D. (2000). *The effect of peer-argumentation on cognitive development: The case of acquiring the concept of fractions*. Unpublished Masters thesis, Tel Aviv University, Israel.

Abrahamson, D. (2001). Microworlds Ratio: A computer-based microworld for the teaching and learning of ratio and proportion. Unpublished design specification, Northwestern University, Evanston, IL.

Abrahamson, D. (2002a). Optical illusions and computation formats: Supporting Chandra's learning path to advanced ratio and proportion word problems. Unpublished manuscript, Northwestern University, Evanston, IL.

Abrahamson, D. (2002b). When "the same" is the same as different differences: Aliya reconciles her perceptual judgment of proportional equivalence with her additive computation skills. In D. Mewborn, P. Sztajn, E. White, H. Wiegel, R. Bryant, and K. Nooney (Eds.), *Proceedings of the Twenty Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA, October 26 – 29, 2002: Vol. 4* (pp. 1658–1661). Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Abrahamson, D. (2002c). *The multiplication table as an 'object to think with': A constructionist project with a phenomenological twist*. Unpublished manuscript, Northwestern University, Evanston, IL.

Abrahamson, D. (2003a). Text talk, body talk, table talk: A design of ratio and proportion as classroom parallel events. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Annual Meeting of the International Group for the Psychology of Mathematics Education, Honolulu, Hawaii, 2003* (Vol. 2, pp. 1 – 8). Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Abrahamson, D. (2003b). *A situational–representational didactic design for fostering conceptual understanding of mathematical content: The case of ratio and proportion*. Poster presented at the annual meeting of the American Educational Research Association, Chicago, IL, April 21 – 25, 2003.

Abrahamson, D. (2004). *Handing down mathematics: Embodied spatial articulation in the design, teaching, and learning of intensive quantities.* In Doug McDougall (Ed.), *Proceedings of the Twenty Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Toronto, Canada, October 21 – 24, 2004*. (Manuscript in press)

Abrahamson, D. & Cigan, C. (2003). A design for ratio and proportion. *Mathematics Teaching in the Middle School, 8(9),* 493 – 501. National Council of Teachers of Mathematics: Reston, Virginia.

Abrahamson, D. & Wilensky, U. (2004). S.A.M.P.L.E.R.: Statistics As Multi-Participant Learning-Environment Resource—Learning axes and bridging tools in a technology-based design for statistics. In U. Wilensky (Chair) & S. Papert (Discussant) *Networking and complexifying the science classroom: Students simulating and making sense of complex systems using the HubNet networked architecture*. The annual meeting of the American Educational Research Association, San Diego, CA, April 12 – 16, 2004.

Arnheim, R. (1969). *Visual Thinking*. CA: UC Press.

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and brain sciences*, *22,* 577 – 660.

Barwise, J., & Etchemendy, J. (1991). Visual information and valid reasoning. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics*. USA: Mathematical Association of America.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1993). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). NY: Macmillan.

Bell et al. (1998). *Everyday mathematics.* Chicago, IL: Everyday Learning Corporation.

Brousseau, G. (1986). *Fondaments et méthodes de la didactique des mathématiques.* Thèse d'Etat, Université de Bordeaux-I. (Ed: IREM de Bordeaux).

Brown, J. R. (1997). Pictures and proofs. *British Journal for the Philosophy of Science, 48,* 161–180.

Case, R. & Okamoto, Y. (1996). The role of central conceptual structures in the development of children's thought. *Monographs of the society for research in child development: Serial No. 246, Vol. 61, Nos. 1 – 2*. Chicago: University of Chicago Press.

Charles, R. I., Dossey, J. A., Leinwand, S. J., Seeley, C. L., & Vonder Embse, C. B. (1998). *Scott Forseman – Addison Wesley middle school math: Course 3.* Menlo Park, CA: Addison Wesley Longman.

Clark, M. R., Berenson, S. B., & Cavey, L. O. (2003). A comparison of rations and fractions and heir roles as tools in proportions reasoning. *Journal of Mathematical Behavior, 22,* 297 – 317.

Cobb, P. & Bauersfeld, H. (Eds.) (1995). *The Emergence of mathematical meaning: Interaction in Classroom Cultures.* Hillsdale, NJ: Lawrence Erlbaum.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher 32(1),* 9 – 13.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner & J. A. Whitson (Eds.), Situated cognition: Social, semiotic and psychological perspectives (p. 151–233). Mahwah, NJ: Lawrence Erlbaum Associates.

Cole, M. (1996). *Cultural psychology: A once and future discipline.* Cambridge, MA: Harvard University Press.

Cole, M. & Wertsch, J. V. (2002). *Beyond the individual–social antimony in discussions of Piaget and Vygotsky*. Retrieved January 1, 2002, from http://www.massey.ac.nz/~alock//virtual/colevyg.htm

Collins, A. & Ferguson, W. (1993). Epistemic forms and epistemic games: Structures and strategies to guide inquiry. *Educational Psychologist, 28(1), 25 – 42*.

Collins, A. Joseph, D. & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. In J. Campione (Ed.) *Journal of the Learning Sciences: Volume in honor of Ann Brown*. (Manuscript in press)

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291 – 330). Albany, NY: SUNY.

Confrey, J. (1995). Student voice in examining "splitting" as an approach to ratio, proportions and fractions. In L. Meira and D. Carraher (Eds.), *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education: Vol. 1* (pp. 3-29). Recife, Brazil: Universidade Federal de Pernambuco.

Confrey, J. (1998). Building mathematical structure within a conjecture driven teaching experiment on splitting. *Proceedings of the Twentieth Annual Conference of the North American Chapter of the Psychology of Mathematics Education*, (pp. 39-48). Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Davis, P. J. (1993). Visual theorems. *Educational Studies in Mathematics, 24*, 333 – 44.

Derrida, J. (1987). Geshlect II: Heidegger's hand. In J. Sallis (Ed.), *Deconstruction and philosophy: The texts of Jacques Derrida* (pp.161–196). Chicago: University of Chicago Press.

Edwards, L. (1993). Microworlds as representations. In A.A. diSessa, C. Hoyles, & R. Noss (Eds.), *Computers and exploratory learning* (p. 127 – 154). New York: Springer–Verlag.

Ernest, P. (1988). *Social constructivism as a philosophy of mathematics*. Albany, NY: SUNY Press.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16(1)*, 3-17.

Freudenthal, H. (1981). Major problems of mathematics education. *Educational Studies in Mathematics, 12,* 133 – 150.

Freudenthal, H. (1986). *Didactical phenomenology of mathematical structure*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Fuson, K. C. (2001). Toward a coherent network of learning goals for pre-K-grade 5. In Speiser, S., Maher, C. A., & Walter, C. N. (Eds.), *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2*. (pp. 734 – 5). Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Fuson, K. C. & Abrahamson, D. (2004a). Understanding ratio and proportion as an example of the Apprehending Zone and Conceptual-Phase problem-solving models In J. Campbell (Ed.) *Handbook of mathematical cognition* (pp. XX-XX). (Manuscript in press).

Fuson, K. C. & Abrahamson, D. (2004b). *A mathematical and situational analysis of ratio and proportion: A domain analysis towards designing, teaching, and initial learning of the domain with understanding.* (Manuscript in preparation)

Fuson, K. C., De La Cruz, Y., Smith, S. B., Lo Cicero, A., Hudson, K., Ron, P., & Steeby, R. S. (2000). Blending the best of the twentieth century to achieve a mathematics equity pedagogy in the twenty-first century. In M. J. Burke (Ed.), *Learning mathematics for a new century* (pp. 197 – 212). Reston, VA: NCTM.

Fuson, K. C., Kalchman, M., Abrahamson, D., & Izsák, A. (2002, April). *Bridging the addition–multiplication learning gap: Teaching studies in four multiplicative domains.* Symposium conducted at the annual meeting of the American Educational Research Association, New Orleans, LA.

Gallistel, C. R. & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44,* 43 – 74.

Gelman, R. & Williams, E. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesis. In D. Kuhn and R. Siegler, (Eds.). *Cognition, perception and language*. Vol. 2. Handbook of Child Psychology (Fifth Ed), (pp. 575 – 630). W. Damon, Editor-in-Chief; New York: John Wiley and Sons.

Gibson, J. (1977). The Theory of Affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting and knowing: Toward an ecological psychology* (pp. 67-82). Hillsdale, NJ: Lawrence Erlbaum Associates.

Gigerenzer, G. (1998). Ecological intelligence: An adaptation for frequencies. In D. D. Cummins & C. Allen (Eds.), *The evolution of mind* (pp. 9-29). Oxford: Oxford University Press.

Goldin, G. A. (1987). Levels of language in mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 59--65). Hillsdale, NJ: Lawrence Erlbaum Associates.

Goldin-Meadow, S. (2003). *Hearing gestures: How our hands help us think.* Cambridge, MA: Harvard University Press. (Manuscript in press)

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96(3),* 603 – 633.

Greeno, J. G. (1998). The situativity of knowing, learning, and research. *American Psychologist*, *53(1)*, 5 – 26.

Hart, K. (1989). Ratio and Proportion. In J. Hiebert & M. Behr (Eds), *Number concepts and operations in the middle grades.* Reston, VA: National Council of Teachers of Mathematics.

Heidegger, M. (1962). *Being and time* (J. Macquarrie and E. Robinson, Trans.). NY: Harper & Row. (Original work published 1927)

Hughes, M. (1986). *Children and number*. Oxford: Blackwell.

Hutchins, E. (1995). *Cognition in the wild*. Cambridge, MA: M.I.T. Press.

Hutchins, E. & Palen, L. (1998). Constructing meaning from space, gesture and speech. In L. B. Resnick, R. Saljo, C. Pontecorvo, & B. Burge (Eds.), *Discourse, tools, and reasoning: Situated cognition and technologically supported environments* (pp. 23 – 40). Heidelberg, Germany: Springer-Verlag.

Ifrah, G. (2000). *The universal history of numbers: From prehistory to the invention of the computer* (D. Bellos, E. F. Harding, S. Wood, and I. Monk, Trans.). Toronto: John Wiley and Sons. (Original work published 1981)

International Study Center, Lynch School of Education, Boston College. Third International and Science Study (TIMSS 1995*). Study Instruments and Procedures*. Retrieved November 21, 200, from timss.bc.edu/timss1995i/t95_study.html

Johnson, M. (1987). The body in the mind: The bodily basis of meaning, imagination, and reason. Chicago University Press.

Johnson, M. (2003). *Imagination incarnate* [videotaped by Dor Abrahamson]. Lecture given at the Mary and Leigh Block Museum at Northwestern University, Program in the Study of Imagination. April 17, 2003.

Kaput, J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 237-287). Albany, NY: SUNY.

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, VA: National Council of Teachers of Mathematics.

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, VA: National Council of Teachers of Mathematics.

Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge: Implications for curriculum and instruction. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323 – 72). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kieren, T.E., Pirie, S.E.B., & Gordon Calvert, L. (1999). Growing minds, growing mathematical understanding: Mathematical understanding, abstraction and interaction. In L. Burton (Ed.), *Learning mathematics, from hierarchies to networks* (pp. 209 – 231). Falmer Press, London.

Lakoff, G. & Nuñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* Basic Books.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers.* Mahwah, NJ: Lawrence Erlbaum Associates.

Lave, G. (1992). Word problems: A microcosm of theories of learning. In P. Light & G. Butterworth (Eds.) *Context and cognition: Ways of learning and knowing* (pp. 74 – 92). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lemke, J.L. (2003). Mathematics in the middle: Measure, picture, gesture, sign, and word. In M. Anderson, V. Cifarelli, A. Saenz-Ludlow, & A. Vile (Eds.), *Semiotic perspectives on mathematics education*. Erlbaum. (Manuscript in press)

Levinas, E. (1998). *Otherwise than being, or Beyond essence* (A. Lingis, Trans.). PA: Duquesne University Press. (Original work published 1974)

Mariotti, M. A. (2002). The influence of technological advances on students' mathematics learning. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 695 – 723). Mahwah, NJ: Lawrence Erlbaum Associates.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought.* University of Chicago Press.

McNeill, D. & Duncan, S. D. (2000). Growth points in thinking-for-speaking. In D. McNeill (Ed.), *Language and gesture* (pp. 141—161). New York: Cambridge University Press.

Merleau–Ponty, M. (1968). *The visible and the invisible* (C. Lefort, Ed., A. Lingis, Trans). Evanston, IL: Northwestern University Press. (Original work published 1964)

Montessori, M. (1986). *Discovery of the child*, (M.J. Costelloe, S. J., Trans.). Ballantine Books. (Original work published 1962)

Moss, J. (2000). *Deepening children's understanding of rational numbers: A developmental model and two experimental studies*. Unpublished doctoral thesis, University of Toronto.

National Center for Education Statistics. *National Assessment of Education Progress*. Retrieved November 21, 2001, from http://nces.ed.gov/nationsreportcard/itmrls/pickone.asp

Nemirovsky, R., Noble, T., Ramos–Oliveira, D., & DiMattia, C. (2003). *The symbolic body.* Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, April 21 – 25, 2003.

Noble, T., Nemirovksy, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education. 32*(1), 85-108.

Norman, D. A. (1991). Cognitive artifacts. In J. M. Carroll (Ed.), *Designing interaction: Psychology at the human-computer interface* (pp. 17-38). New York: Cambridge University Press.

Norman, D. (1988). *The design of everyday things*. New York: Doubleday.

Olson, D. R. (1994). *The world on paper*. Cambridge UK: Cambridge University Press.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373 – 429) . Hillsdale, NJ: Lawrence Erlbaum.

Nunes, T. & Bryant, P. (1996). *Children doing mathematics.* Conrwall, UK: T.J. Press Ltd.

Papert, S. (1980). *Mindstorms*. NY: Basic Books.

Piaget, J. (1952). *The child's conception of number*. London, England: Routledge and Kegan.

Piaget, J. (1971). *Biology and knowledge: An essay on the relations between organic regulations and cognitive processes*. Chicago, IL: The University of Chicago Press.

Polanyi, M. (1958). *Personal knowledge: Towards a post-critical philosophy.* Chicago: The University of Chicago Press.

Scarano, G. H. & Confrey, J. (1996). *Results from a three-year longitudinal teaching experiment designed to investigate splitting, ratio and proportion*. Paper presented at the Annual Meeting of the American Educational Research Association, New York, April, 2006.

Sherin, B. & Fuson, K. C. (2004). *Multiplication strategies and the appropriation of computational resources*. (Manuscript under review)

Stetsenko, A. (2002). Commentary: Sociocultural activity as a unit of analysis: How Vygotsky and Piaget converge in empirical research on collaborative cognition. In D. J. Bearison & B. Dorval (Eds.), *Collaborative cognition: Children negotiating ways of knowing* (pp. 123 – 135). Westport, CN: Ablex publishing.

Stigler, J. W. (1984). “Mental abacus”: The effect of abacus training on Chinese children's mental calculation. *Cognitive Psychology, 16,* 145 – 176.

Stigler, J. W., Lee, S., & Stevenson, H. W. (1990). *Mathematical knowledge of Japanese, Chinese, and American elementary school children*. Reston, VA: National Council of Teachers of Mathematics.

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long-term learning process (towards a theory) Part1: Reflections on a teaching experiment. *Educational Studies in Mathematics, 15,* 328-340.

Streefland, L. (1991). *Fractions in realistic mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Stroup, W. M. (2002). Understanding qualitative calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning,* 7(2), 167 – 215.

Thompson, P. & Saldahana, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95 – 114). National Council of Teachers of Mathematics: Reston, VA.

Tierney, C. (1985). Patterns in the multiplication table. *Arithmetic Teacher 32,* 36 – 40.

Urton, G. (1997). *The social life of numbers: A Quechua ontology of numbers and philosophy of arithmetic*. University of Texas Press.

Uttal, D. H. & DeLoache, J. S. (1997). Manipulatives as symbols. *Journal of Applied Developmental Psychology, 18*, 37-54.

Vanhille, L. S. & Baroody, A. J. (2002). Fraction instruction that fosters multiplicative reasoning. In B. H. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions.* (pp. 224-236). Reston, VA: NCTM

Varela, F. J., Thompson, E., and Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 127 – 174). New York: Academic Press.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41-59). Albany, NY: SUNY Press.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse, The Netherlands: Swets & Zeitlinger.

von Glasersfeld, E. (1987). Learning as a constructive activity. In C. Janvier, (Ed.), *Problems of representation in the teaching and learning of mathematics* (p. 3 – 18). Hillsdale, NJ: Lawrence Erlbaum Associates.

von Glasersfeld, E. (1990). Environment and communication. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education* (pp. 357 – 376). Hillsdale, NJ: Lawrence Erlbaum, Publishers.

Vygotsky, L. S. (1962). *Thought and language*. Cambridge MA: M.I.T. Press

Vygotsky, L. S. (1978). In M. Cole, V. John-Steiner, S. Scribner, & E. Souberman (Eds.), *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.

Whewell, W. (1989). *Theory of scientific method* (R. E. Butts, Ed.). Indianapolis, IN: Hackett Publishing Company. (Original work published 1837)

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics education. In I. Harel & S. Papert (Eds.), *Constructionism.* Norwood (pp. 193 – 204), NJ: Ablex Publishing Corporation.

Wilensky, U. (1997). What is normal anyway? Therapy for epistemological anxiety. In R. Noss (Ed.), Computational environments in mathematics education [Special issue]. *Educational Studies in Mathematics, 33(2)*, 171 – 202.

Wittgenstein, L. (1958). *Philosophical investigations*. NJ: Prentice Hall. (Original work published 1953)

Appendices

Appendix A—Data Map of 2001 – 2002 Ratio-and-Proportion Design-Research Studies

<table>
<tr><th>Study / Content</th><th>AG</th><th>SE</th><th>Constructionist</th><th>Pilot</th><th>5th-grade (Mr. CC)</th><th>5th-grade (Ms. NW)</th><th>5th-grade (Ms. SF)</th><th>Teachers</th></tr>
<tr><td>n</td><td>1</td><td>1</td><td>3 – 5</td><td>4 – 5</td><td>19</td><td>20</td><td>2 * 25</td><td>-</td></tr>
<tr><td>Repeated Adding (Coins)</td><td>-</td><td>video</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr>
<tr><td>Geometrical Similitude (“Eye Trick”)</td><td colspan="2">Student work, video footage from 3 * 1 hr (each student) semi-clinical interviews</td><td>Microworlds Ratio</td><td rowspan="7">Student work (work-alones) and video from 15 group days and 1 focus-kid day (Marcus, %)</td><td>Student work, posttest, and Video from 2 days of eye-trick w/ vertical pairs, rectangles, coordinate graphs.</td><td>Video from 10 minutes classroom work; posttest items</td><td>-</td><td>-</td></tr>
<tr><td rowspan="4">MT (finding patterns, using MT, and building RT and PQ to solve unknown-value numerical and verbal proportion problems)</td><td rowspan="4">-</td><td rowspan="4">-</td><td rowspan="4">-</td><td colspan="2">Field notes (lesson plans, daily brief/debrief to teachers and design team), student work (work-alones, home-work, pre/post-tests, feedback questionnaires, tutor-sessions, pre-test from comparison schools in the same school district, high-achiever feedback), teacher feedback (PD workshops), curricular material (produced by project PI)</td><td rowspan="4">Video footage (2 hrs. from Day 6 and from Day 7)</td><td rowspan="4">Doc.s created towards teachers think-tank</td></tr>
<tr><td>-</td><td>Journal entries, transcribed teacher interv.</td></tr>
<tr><td colspan="2">Video footage (10 days of class and tutor sessions)</td></tr>
<tr><td>High-students interview (video, trans), open house (video), longitudinal tutoring (AP)</td><td></td></tr>
<tr><td>Cross Multiplication</td><td>-</td><td>-</td><td>-</td><td>Student work and Video from 1 day: analogies in PQ; factors inside; tutoring sessions w/ AP</td><td>-</td><td>-</td><td>-</td></tr>
<tr><td>Percentage</td><td>-</td><td>-</td><td>-</td><td>Student work and video from 3 days: a, b, and c unknown in a:b = c:100 word problems; posttests</td><td>Posttest scores</td><td>-</td><td>-</td></tr>
<tr><td>Non-MT</td><td>-</td><td>-</td><td>-</td><td colspan="2">Discussion of sibling problem</td><td>St. work, video from 3-day mini unit</td><td>-</td><td>-</td></tr>
<tr><td>Other</td><td>-</td><td>-</td><td>Re-invent MT</td><td>-</td><td>Proportion fortune teller</td><td>Invent MT game (Day 8, St work, video 1/2 hr. discussion</td><td>-</td><td>-</td></tr>
<tr><td>Paper</td><td colspan="2">Term papers for Dr. BS; PME-NA02</td><td>Microworlds Ratio software and deign specifications (2001); Independent study for Dr. UW (2002c)</td><td>AERA02; PME-NA proposal</td><td>NCTM–MTMS, 2002; AERA, 2003</td><td>PME27; Handbook of Mathematics Cognition chapter; AERA, 2004; PME28; PME-NA26</td><td>-</td><td>-</td></tr>
</table>

Appendix B—Scrambled Multiplication Table with Missing Values (Fill-In Activity Material)

						42				63
								30		
	7									
					8					
				6					15	
		4								
							72			
	35									45

Note. Students are asked to fill in the missing products and are encouraged to use the narrow columns (on the left) and row (on top) to write the common factors they determine for two or more numbers in each row or column, respectively. Proportion quartets still are 4 values at the corners of any rectangle on this MT, even though both the rows and the columns have been shuffled. Understanding this constancy may foster a deeper understanding of the multiplicative structure of the MT and through that an understanding of proportion and of cross-product multiplication. Also, this activity provides opportunities for practicing multiplying, dividing, and determining common factors.

Appendix C—Compilation of Classroom "Work-alones," Homework Assignments, Discussion Problems, Tutoring and Interview Items, and Material Designed Towards Teacher Guides and Extension Units

Appendix C1—"Work-alones" and Homework From the Ratio-and-Proportion Unit

Name_____________ Date_________

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Show and explain some <u>new</u> things you know about the multiplication table.

Note: Spaces and fonts in this and most worksheets in this appendix have been condensed for reading convenience. Students received these documents printed at font size 14 and had ample space to work on all problems.

Name________________ Date_________

Workalone

In the following quartets, fill in the missing products and factors

6	8
	16

12	36
	42

	20
28	35

56	
63	81

How did you solve these? Can you think of a useful strategy that you would like to share with your friends?

____

____

____

Name_____________ Date_________

Homework

In the following quartets, fill in the missing products and factors

12	28
	42

49	63
56	

	18
28	63

32	
36	54

Big challenge: Invent a story that goes with on of your quartets. Copy the quartet on the back of this page and write your story next to the quartet.

Jumbo challenge: Invent a board game where the multiplication table is the board. Best inventions will be designed professionally.

Name________________ Date_________

Workalone

1. Every week, Jerry puts $3 in his doggy-bank, and Elaine puts $5 in her kitty-bank. Mark in the table how much money each person saves up as the weeks go by.

	Jerry $	**Elaine $**
Week 1		
Week 2		
Week 3		
Week 4		
Week 5		
Week 6		
Week 7		

2. Mario and Fatima save their money, too, on a regular basis. Fill in the missing numbers in this table.

	Mario $	**Fatima $**
Week 1		
Week 2		
Week 3		
Week 4	28	36
Week 5		
Week 6		
Week 7		

a. Here is how much Gabi and Matou saved. Fill in the table.

	Gabi $	**Matou $**
Week 3	21	
Week 5		40

b. Same old story, different people, different numbers…

	Adam $	**Zohar $**
Week		25
Week	24	30

c. How much do Wood and Jordan save every week? Can you think of more than one correct answer?

	Wood $	**Jordan $**
Week	12,000	18,000

6. Henry and Gillaume decide to grow beards. After several weeks have passed Henry's beard is 32mm long and Gillaume's beard is 48mm long. When Henry's beard is 40mm long, how long will Gillaume's beard be? Draw a table like in the previous questions. Don't forget to ***LABEL THE TABLE !!!***

7. Is this Workalone in anyway like what we've done so far?
   - ❑ No, this is totally different
   - ❑ Yes, but I don't know how. Leave me alone!
   - ❑ Yes, and this is how: ______________________________

Name_______________ Date________

Workalone

Two flower buds peeped out of the ground on the same morning – a daffodil and a petunia. After some days, the daffodil was 12cm tall and the petunia was 21cm tall. When the petunia is 35cm tall, how tall will the daffodil be?
Don't forget to ***LABEL THE TABLE!!!*** Write your answer as a sentence: "The daffodil…"

Name________________   Date_________

Homework

Using our new thinking & computation tool (Proportion)

Organize your work: Don't forget to… ***LABEL THE TABLE !!!***

1. One day, two birds began collecting twigs to build their nests. After some hours, the bluebird had 28 twigs and the robin had 42 twigs. Later, the bluebird had 36 twigs. How many twigs did the robin have?

2. Some time ago, Chip and Dale the squirrels decided to collect acorns. Now, Chip has 54 acorns and Dale has 30 acorns. Chip wants to collect 81 acorns. When he gets there, how many will Dale have?

3. To make a jug of super-duper lemonade, you need 18 tablespoons of lemon juice and 30 teaspoons of sugar. If you have only 20 teaspoons of sugar left, how much lemon juice should you use to make super-duper lemonade?

4. In a show that "Dance with Eagles" put on there were 12 boys and 18 girls. The kids were divided into groups. In all of the groups there was the same number of boys, and in all of the groups there was the same number of girls.
   a. How many children were there in each group?
   b. Could this question have more than a single answer?
   c. Look at the Multiplication Table. Mark the "Boys" column, the "Girls" column, and the column where the answer(s) appear(s). How could we call this column?

Name_____________ Date________

Homework

Using our new thinking tool (Proportion)

Don't forget to… ***LABEL THE TABLE !!!***

1. When Tom feeds his cats, he shares out the pellets of cat food like this: 3 for the kitten and 5 for the mummy cat, 3 for the kitten and 5 for the mummy cat, 3 for the kitten and 5 for the mummy cat, and so on.
    a. If the cats have 18 pellets (kitten) and 30 pellets (mummy cat), how many rounds did Tom feed them?

    b. If there are exactly 56 pellets left in the box, how many rounds can Tom feed his cats?

2. When the kitten grew, Tom changed his rule for feeding the kitten and its mummy. You'll have to find the new rule. What could the rule be if after some rounds the kitten had 20 pellets and the mummy cat had 25?

Name_______________ Date________

Workalone

***LABEL THE TABLE !!!* & Give whole answers.**

1. A car of the future will be able to travel 8 miles in 2 minutes. How far will it travel in 5 minutes?

2. Judy earns $63 in 6 weeks. If she earns the same amount of money each week, how much does she earn in 4 weeks?

3. Joan used exactly 15 cans of paint to paint 18 chairs. How many chairs can she paint with 25 cans?

Name____________ Date________

Homework

Using our new thinking tool (Proportion)

Don't forget to… ***LABEL THE TABLE !!!***

1. At exactly the same moment, Juan opened the tap in the bath, and Marge opened the tap in the sink. After some moments, there were $35cm^3$ of water in the bath and $63cm^3$ of water in the sink.
   a. When the sink gets to $81cm^3$, how full will the bath be?

   b. When the bath gets to $55cm^3$, how full will the sink be?

   c. When will the amount of water in the bath and the sink together be $140cm^3$?

2. A green fern and a brown fern (a kind of plant) were planted on the roof of a skyscraper and started growing down at steady rates. After many years, the green fern had grown 48 floors down and the brown fern had grown 18 floors down.
   a. What is the growing-down ratio of the plants? (There may be more than one correct answer to this question. Pick one with the smallest numbers and stick to it throughout this question).

   b. When the brown fern has grown 63 floors down, how far down will the green fern have grown?

   c. After how many years will their total length be 121 floors?

Name________________ Date_________

Workalone

***LABEL THE TABLE !!!* & Give whole answers.**

1. To make Italian dressing, you need 4 parts vinegar for 9 parts oil.
   a. How much oil do you need for 828 ounces of vinegar?

   b. How much vinegar do you need for 143 ounces of Italian dressing?

2. Evan collects orange and purple marbles. The ratio between his orange and purple marbles is 3:5.
   a. If he has 72 marbles all together, how many orange marbles does he have?

   b. If he has 45 purple marbles, how many orange marbles does he have?

Name____________ Date________

Homework

Oh, yes, and don't forget to… ***LABEL THE TABLE !!!***

1. A large restaurant sets tables by putting 7 pieces of silverware and 4 pieces of china on each place-mat. If it used 392 pieces of silverware in its table settings last night, how many pieces of china did it use?

2. In Sara's aquarium there are 15 fish. If there are 12 parrot-fish and the rest are zebra-fish, what is the ratio between the parrot fish and the zebra fish?

3. Nelly has 100 blocks. She wants to build two towers of ratio 3:4 in height. How many blocks will she have <u>left over</u>? [Think, kids, think – there's something new here! – Don't jump to hasty conclusions!]

4. Jo-Jo's Juice bar makes the famous "Frooty-Smoothie." Dr. Jenny Juicer bought a keg of "Frooty-Smoothie" and took it to her lab. She found that the Frooty-Smoothie was a mixture of 20 cups orange juice and 35 cups strawberry juice. That evening, Dr. Jenny Juicer had some friends over. She poured 16 cups of orange juice into a big bowl. How many cups of strawberry juice should she put it to get Jo-Jo's "Frooty-Smoothy"?

5. We have spoken of running competitions, buildings, flowers, squirrels, cans of paint and chairs, doggies & kitties, and many more cases of "this-for-that". Try to come up with your own math story that has a "this-for-that" situation in it. It can be wacky or "far out"! Solve it with a Proportion Quartet, and explain your solution. Three of the most original math-stories will become part of the next Workalone (you'll be a famous published author!).

Name_______________ Date________

Workalone

If you don't ***LABEL THE TABLE*** you'll get triple homework

- The ratio between the weight of a baby carp and a father carp is 3 to 7. If the father carp weighs 28 oz., how much does the baby weigh?
- If I have 20 blue marbles and 25 red marbles, what is the ratio of blue to red marbles?
- It is said that the average life span of a human is 72 years. If 9 of these are spent eating, how many years has a 16-year old spent eating already?
  A. 6 years
  B. 5 years
  C. 4 years
  D. 3 years
  E. 2 years
- A printing press takes exactly 12 minutes to print 14 dictionaries. How many dictionaries can it print in 30 minutes?
- Every evening, Cuba the wonderful dog eats 28 pellets of food. If 21 of the pellets are beef and the rest are turkey, what is the ratio between the turkey and beef pellets that Cuba the amazing mongrel devours for dinner?
- A men's clothing store sells "Gregory" dress socks in gift boxes. Every gift box has 2 pairs of solid socks and 3 pairs of striped socks. If the store sold 270 pairs of solid socks last week, how many pairs of striped socks were sold?
- 18:30 = __ :75

Name____________ Date________

Homework

LABEL THE TABLE or you'll be court-martialed at 5 AM

1. In Juanita's bookshelf there are altogether 55 books: some are fiction and some are reference books. If 40 of the books on Juanita's bookshelf are reference books, what is the ratio between Juanita's reference books and her fiction books?

2. Every morning, Santa Claus feeds his 20 reindeer 24 lollipops. For Christmas, he will receive an <u>extra</u> 15 reindeer. How many lollipops will Santa have to feed <u>all</u> of his reindeer together?

3. Joachim, the great film-critic says: "For every 2 *cool* movies I see, there are always another 5 that '*suck'* big-time!". How many lousy movies does Joachim have to suffer in order to enjoy 10 good movies? (Joachim also says: "Don't forget to label the table like Mr. A. keeps reminding you")

Name_______________ Date_______________

The Last Workalone

☺ ☺ ☺ ☺ ☺ ☺ ☺ ***LABEL THE TABLE*** ☺ ☺ ☺ ☺ ☺ ☠ ☺

or the Spring-break will be postponed to August

Towards Easter, Jack and Jill decided to spend a day collecting money for the homeless in Evanston. Standing at the corner of Sherman and Church one chilly morning, they collected $72 in just 56 minutes. When had they collected $45?

Reflection:

a. Is Jack and Jill's story a proportion situation? If so, how do you know this?

____

____

b. Does the Proportion Quartet system make sense to you? Explain your feeling.

____

____

Name_______________ Date________

Workalone

If you don't ***LABEL THE TABLE*** you'll join the turtle

1. Two spaceships, Endeavor and Liberty, leave Mars at the same moment on their way to Earth. When Endeavor has traveled 28 miles, Liberty has traveled 63 miles. How far will Endeavor have traveled when Liberty travels 81 miles?

2. Ariel has 48 gerbils and 72 hamsters. What is the ratio between his hamsters and gerbils?

3. I have 20 pens. 12 are blue and the rest are red. What is the ratio between my blue and red pens?

Name___________ Date________

Homework

LABEL THE TABLE

…or you'll have to eat a peanut-butter & mustard sandwich

1. In Mrs. Marina's class there are 36 pupils. If there are 20 girls in class, what is the ratio between boys and girls in Mrs. Marina's class?

2. Tell a ratio story for 14:56.

3. In Frosty's Snowman Company, for every 30 tons of snow used in production they need 45 carrots. How much snow would they use with 189 carrots?

Name_______________ Date______________

Workalone [mixed situation types]

a. A red jet flew east from Chicago to Boston. Later, a blue jet also flew east from Chicago to Boston at exactly the same speed. When the red jet had gone 300 miles the blue jet had gone 200 miles. When the red jet had gone 600 miles, how far had the blue jet gone? Explain your answer

b. Anna has a yellow Ford toy car, and James has a purple Toyota toy car. They have a race between the two cars on the classroom floor. They each set their cars to go as fast as possible, put them side by side, and say "Ready, Set…Go!!!" When the yellow Ford has gone 12 tiles along the floor, the purple Toyota goes 15 tiles. Where will the yellow Ford be when the purple Toyota has gone 25 tiles? Explain your answer.

c. Jenny had 5 cousins. Then, last year 4 new baby cousins were born. Randy had 7 cousins. How many new cousins of Randy were born last year? Explain your answer.

d. Invent a math-question for the class. DON'T tell the class what kind of situation your story is, but give *just* enough clues so they can figure that out themselves.

Name_________________ Date______________

Homework

1. Mr. Ronald travels a lot, so he doesn't bother to set his watch when he goes to different time zones. When he was in New York last week, his friend Mr. Pickle said: "Look, your watch shows 3 PM and mine shows 5 PM." When Mr. Ronald's watch shows 6 PM, what will Mr. Pickle's show? Explain your answer

2. A contractor is building two identical towers. The strong blue team is building Tower A and the weak red team is building Tower B. Each team works at a steady pace. They begin on May 1st. When the blue team has completed 8 stories in Tower A, the red team has completed only 6 stories in Tower B. How many stories will the red team complete when the blue team has completed 12 stories? Explain your answer.

3. Danny and Marta each get a fixed allowance every week. On New Year's Eve they both decide to save their money for their summer vacation. When the vacation arrived, Danny had $45.00 and Marta had $63.00. If at spring break Danny had $30.00, how much did Marta have at spring break? Explain your answer.

4. Stan and Maya are traveling by "L" to the Loop. At the same moment, they get on the same train, but Stan goes to the very front, and Maya goes to the very back. The train leaves the station. When Stan is 50 yards away from the station, Maya is only 20 yards away. When Stan is 100 yards away from the station, how far will Maya be? Explain your solution.

5. Foxy is a champion computer-game expert. Dr. Labby is doing research on Foxy. Dr. Labby found that Foxy needs 25 missiles to shoot down 45 aliens. If Foxy has only 10 missiles left, how many aliens can he shoot down? Explain your answer.

6. In Wisconsin a cow named Sheila gave 2 gallons of milk on Tuesday, then 3 gallons of milk on Wednesday. In Evanston Mr. A. saw 5 videos last weekend. How many videos will Mr. A. see next weekend?

7. Jose and Tanya are sliding down a mountain on their sleds. They begin together, but Tanya's sled is smoother so she goes faster. When she has gone 1/2 of the way down, Jose has gone only a 1/4 of the way down. Where will Tanya be when Jose has gone 1/2 of the way down? Explain your answer.

Name __________ Date_________

Challenging Problems

1. John and Fernando each began collecting baseball cards on the same day. Every day John bought the same number of cards. Every day, Fernando bought the same number of cards (a different number from John). After a certain number of days, John had 28 cards and Fernando had 42 cards. One day, later, they decided to become partners, and on that day they had 75 cards together. How many cards did Fernando put in? [31 cards]

2. Sally and Nima are each building a snowman. Every minute Sally's snowman grows by 7 inches, and Nima's snowman grows by 11 inches. When will the total height of both snowman <u>together</u> be 216 inches? [12 min]

3. Bob and Anita are having a race along a 108 yard running course. They both run at a steady pace, but Bob ran only 84 yards when Anita got to the end line. How much had Bob run at the exact moment when Anita was <u>leading</u> by 8 yards? [28 yrds]

4. Jenny loves very sweet juice. Her friend says to her: "You have a choice of drinking a single glass from either of two different jugs. In one jug I have mixed 3 cups of strawberry juice with 5 cups of water, in the second jug I have mixed 5 cups of strawberry juice with 8 cups of water." Which jug should Jenny choose? [5/8 is sweeter than 3/5]

5. Salim is filling his bathtub. He puts the plug in place, opens both the cold water and the hot water and lets them run. After one minute there are 5 liters of water in the bath. After 6 minutes the hot-water tap has poured alone 18 liters into the bath. How much has the cold-water tap poured in? [12 liters]

6. Silly Fred tried to fill his backyard swimming pool. Only, he didn't notice that the pump was emptying the pool as the tap was filling it… After an hour, there were only 5 m^3 of water in the pool. Later that day, he was shocked to read on the water meter that he had poured 495 m^3 into the pool, but there were only 55 m^3 in the pool! How many m^3 of water does the pump empty in 7 hours? [280 m^3]

Appendix C2—Rate and Ratio (Rates) Situation Bank

The following collection of situations is to be presented to students in the context of learning to interpret the MT as a collection of "rate-columns". Students are to construct for each situation an appropriate table. Note that we want students to construct an understanding of rate. Therefore, it is not sufficient that students build a column and fill in the cells by skip counting. The teacher should accompany the students' activity with questions related to the rate situation, as we shall now demonstrate.

The rate situation…

"*Big Bird collects snails. Every day he adds 4 snails into his terrarium*"

…could be accompanied by the following questions:

1. How many snails does Big Bird have after 1 day? [4]
2. How many (more or new) snails does Big Bird collect on the second day? [note, 4 and not 8!]
3. How many (more or new) snails does Big Bird collect on the fifth day? [4, not 20]
4. On which day will Big Bird have (altogether) 32 snails? [8]
5. At the end of 3 <u>weeks</u> of collecting snails, will Big Bird have 100 snails? If not, how many more days would he need to keep collecting?
6. If we said that Big Bird collects at a rate of 8 snails per 2 days, would that be slower? -faster? -the same?

The ratio (rates) situation…

Ernie and Bert have a hotdog eating competition. Every minute, Ernie eats 3 hotdogs and Bert eats 5 hotdogs.

…could be accompanied by the following questions:

1. How many hotdogs will each guy eat in the first minute?
2. How many hotdogs will each guy eat in the second minute?
3. Who will eat more hotdogs in the third minute? In the fourth minute?
4. At this rate, when will Ernie catch up? Will he ever?
5. When Ernie has eaten 21 hotdogs, how many hotdogs has Bert eaten?
6. When Bert has eaten 25 hotdogs, how many has Ernie eaten?
7. When will Bert have eaten 33 hotdogs? How many will Ernie have eaten at that moment?
8. When will Bert lead by 16 hotdogs?

<u>Rate (1-column + single column)</u>

- ♦ Big bird collects snails. Every day he adds 4 snails into his terrarium.
- ♦ Farmer Stanley is driving his tractor down the field. He ploughs exactly 7 rows per hour
- ♦ A big silver hot-air balloon is rising up from the school basketball field. It goes up 9 feet every second
- ♦ Georgette loves spaghetti. She eats very slowly. She slurps up 6 noodles per minute

- Mr. Hernandez was completely bald. One day, a single hair started growing on his head. Every day it grew 5 mm.
- Grover is snowboarding down a hill. He goes down 10 yards a second.
- Sandy loves crossword puzzles. She can solve 8 clues every minute.

<u>Rates [Ratio] (1-column + two columns)</u>

- Ernie and Bert have a hotdog eating competition. Every minute, Ernie eats 3 hotdogs and Bert eats 5 hotdogs
- A palm tree and a cedar tree start growing at the same moment. Every month, the palm tree grows 2 feet and the pine tree grows 7 feet
- Flipper the Dolphin and Salty the Seal compete to see who can bounce a ball faster on his nose. When Flipper has bounced the ball 35 times, Salty has bounced it 56 times. When Flipper has bounced the ball 45 times, how many times has Salty bounced it? [72]
- Two ants – Anthony and Antenna – walk up a mountain side. When Anthony has gone 45 meter Antenna has gone 72 meters. Earlier, when Antenna walked 32 meters, how far did Antenna walk? [20]
- Jennifer and Suzanne are sisters. Jennifer is older than Suzanne. Every week, Jennifer and Suzanne get allowance. After some weeks, Jennifer has received $48 and Suzanne has received $36. How much will Jennifer have received when Suzanne has received $54? [72]
- Two hens – Pecker and Cackle – live in the coop. Each hen lays a fixed number of eggs per week. After some weeks, Pecker has laid 40 eggs and Cackle has lain 45 eggs. When Cackle has laid 81 eggs, how many eggs has Pecker laid? [72]
- George earns $28 in 16 minutes. How long must he work to earn $42? [24]
- Mean Ol' Daddy, the famous rapster, can say 99 words in 55 seconds. How many seconds does it take him to say 54 words? [30]
- Little Tanya and her big brother Albert are having a race. When Tanya has run 63 yards, Albert has run 81 yards. How far did Tanya run when Albert was at the 27 yard mark? [21]
- How much do Wood and Jordan save every week? [6,000:9,000 OR 4,000:6,000, etc] Can you think of more than one correct answer? [yes]

	Wood $	**Jordan $**
Week	12,000	18,000

- Gil and Ben decide to grow beards. After several weeks have passed Gil's beard is 32mm long and Ben's beard is 48mm long. When Gil's beard is 40mm long, how long will Ben's beard be? Draw a table like in the previous questions. [60; note that it really helps here to reduce the ratio to the smallest possible numbers]

Multiple Rates (1-column + three or more columns)

- Three kids are building snow castles. Greg's snowman grows 3 inches per minute. Yoko's snowman grows 7 inches per minute. Jamal's snowman grows 9 inches per minute. When Yoko's snowman is 42 inches tall how much taller will Jamal's snowman be as compared with Greg's snowman? [36]
- Mrs. Washington and her kids – Johanna, Mike and Abe – have just moved to Evanston. They have no friends here at all! But people in Evanston are very nice to them. Every day, Mrs. Washington makes 2 new friends, Johanna makes 5 new friends, Mike makes 3 new friends and Abe makes 7 new friends. How many friends will each person have after a week? When Abe has 63 friends, how many friends will the rest of the family have <u>altogether</u>? [90]
- A team of five basketball players is doing very well in the championship. Every game, Adam shoots 20 points, Bob shoots 25 points, Collin shoots 30 points, Dan shoots 35 points and Erlich shoots 40 points. After how many games will Collin shoots120? How many points will each of his teammates shoot at that time?

Numbers Only

1. 32 : 48 = __ : 84 [72]
2. __ : 35 = 32 : 40 [28]

Part-Whole

1. Tony has 16 friends. Six of them speak Spanish. What is the ratio between Tony's friends who do NOT speak Spanish and those who do? [10:6, so 5:3. This is an opportunity to discuss the importance of *order* in ratio. We must be extremely careful with our answers. Would 3:5 be correct here? If not, then why? What question would make 3:5 the correct answer?]
2. Harry works in a zoo. It is his responsibility to make sure all animals get anti-flu injections. There are 99 animals in the zoo. 63 of them have got injections. What is the ratio between the animals that have not got injections and those that have? How should Harry go about calculating this? (Harry doesn't have an MT…) [36 : 63, so 4: 7. Once again, <u>not</u> 7:4. Discuss method to ensure we never get the wrong order. How about some kind of label and checking carefully against the question to see what "they" are asking]

Total PQ

1. Karen and Bob are putting together a pony puzzle. Each kid is working at a steady pace. At a certain moment Karen has put in 24 pieces Bob has put in 32. When they have put in 63 pieces <u>together</u> how many pieces has each put in? [27 and 36 – it's all on the MT!!]

<u>Challenges</u>

1. Can you think of a way to use and label a PQ so as to teach a kid how to make fractions into percentages? For instance
   - Use a PQ to find out what 3/4 is in %
   - What percent is 24 out of 60?
2. What is the ratio between 40 and 72? You should always keep reducing until you get the smallest (whole) numbers you can. Some of you could probably go on into fractions, but we're not doing that now. [5 : 9]

Name ___________ Date____________

Group Work

1. Baker Jones sells his famous Chocolate Delight cake for $25.00. But, if it's your birthday, you pay only 80% of the full price. Today is Suzy's birthday, and she has just bought a Chocolate Delight cake from Baker Jones. How much did she pay?

2. Ludmila Smetana, the Czech Women's Basketball champion, scored 56 points in a single game. 42 of these points were from the field. What is the percent of Ludmila's field points out of all her points?

Appendix C3—Selected Classroom Problems From the Miniunit on Situational Meanings

Name_________________ Date______________

Workalone

1. In an average basketball game, Roberta throws the ball 50 times and gets 30 of those balls into the basket. If the game had an extension and Roberta threw another 10 balls, how many more do you expect she got in?
   Is there enough given information to solve this problem?
   What type of problem is? For instance, is this a ratio problem? How can we know?
   Solve the problem.

2. Two express elevators leave the ground floor of a skyscraper on their way up to the 100th floor (no stops on the way). They both move at the same speed, but elevator A left before elevator B, because its doors closed faster. When elevator A passes by the 15th floor, elevator B passes by the 10th floor. When elevator A gets to the top, where will elevator B be?

   Is there enough given information to solve this problem?
   What type of problem is? For instance, is this a ratio problem? How can we know?
   Solve the problem.

Appendix C4—Selected Problems From the Percentage Extension Unit

Name__________________ Date_____________

Group Work

1. In Mr. A.'s front-yard there is an olive tree and a palm tree. The olive tree is 12 feet tall, and the palm tree is 15 feet tall. What percent (%) is the olive tree as compared to the palm tree?

	O	P
feet		
%		100

Answer: The olive tree is ____% of the palm tree.

2. Grover eats 18 cookies a day. But that is only 20% of what the Cookie Monster eats every day. How many cookies does the Cookie Monster eat every day?

	G	CM
cookies		
%		100

3. Asha is 20 months old. Ivan is 90% as old as Asha. How old is Ivan?

	100

4. Milwaukee is 140 miles away from Dan's house. If you travel 45% of the way from Dan's house to Milwaukee you get to his favorite beach. , How far is the beach from Dan's house?

5. Chip has already eaten 320 of the 400 acorns he collected for winter. What percent of his acorns has Chip eaten?

6. 25% of Sharon's friends speak Spanish. That is, 11 of Sharon friends speak Spanish.
   a. How many friends does Sharon have?
   b. How many <u>don't</u> speak Spanish?
   c. What is the percent of Sharon's friends who <u>don't</u> speak Spanish?

Name________________ Date___________

Homework

1. Mr. Bushy's beard is 32cm long. Mr. Grizzly's beard is 80cm long! What percent (%) is Mr. Bushy's beard as compared to Mr. Grizzly's beard?

2. Cuba the amazing dog is chasing Matou the fat cat. Lucky for Matou, he runs faster! When Matou has run 120 yards, Cuba has run only 75% of that. How far has Cuba run?

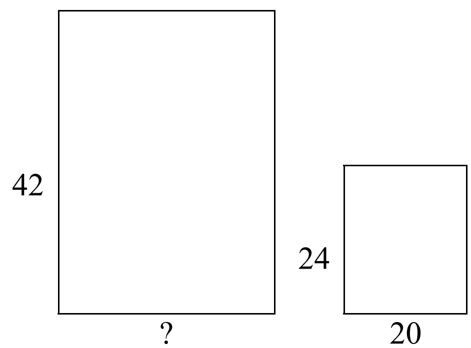
3. If Emma has seen the movie "The Mummy" 4 times and that is only 80% of the number of times Yoko has seen it, then how many times has Yoko seen it?

Appendix C4—Selected Problems From the Geometrical-Similitude Extension Miniunit

Name____________________ Date_______________

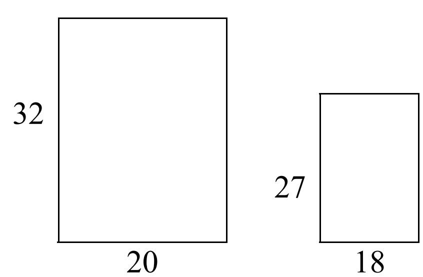
Workalone

1) Jose used the "eye-trick" illusion and found that these two shapes are the same, only different in size. He measured them on a grid, but forgot to mark one of the widths. Can you complete it?



Explain why you used this strategy. Why did you think it should work?

2) Look at these two shapes and their measured heights and widths. If you used the "eye-trick" on these two shapes, do you think they would appear exactly the same?



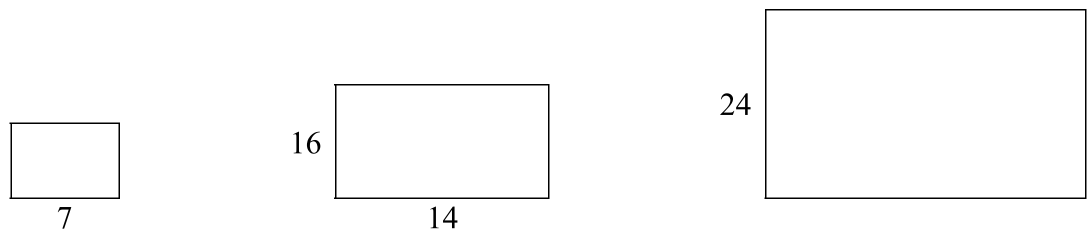
a. Prove and explain your answer.

b. For each shape, find a matching shape of height 24. What is the width of each shape?

Name____________________ Date_______________

Homework

1. Complete the measurements in this ratio-match set. [earlier terminology for proportional progression]



2. The World Bank Tower (⇑) is 96 floors high. The Bonita Center (Δ) is 84 floors high. Aaron photographed these buildings for his sky-scraper poster. In his poster, the Bonita Center is 35 inches high. How high is the World Bank Tower in Aaron's poster?

World Bank Tower Bonita Center Aaron's Poster

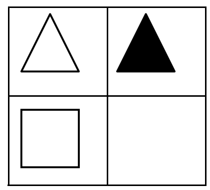
Appendix C4—Design Sketches Toward Classroom and Tutor Work on Cross Multiplication

Cross Multiplication (CM) and Analogy in the Proportion Quartet

[Find the factors and the unknown value for each of these special PQs]

10	15
14	

Numerical



Forms

uncle	aunt
father	

Lexical

Fig. 1. Numerical (unfactored-inside), Forms, and Lexical Mystery Quartets

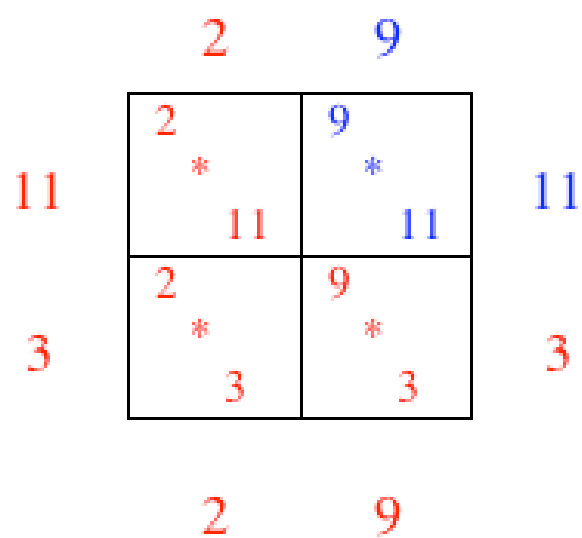
3 * 7	2 * 7
3 * 5	

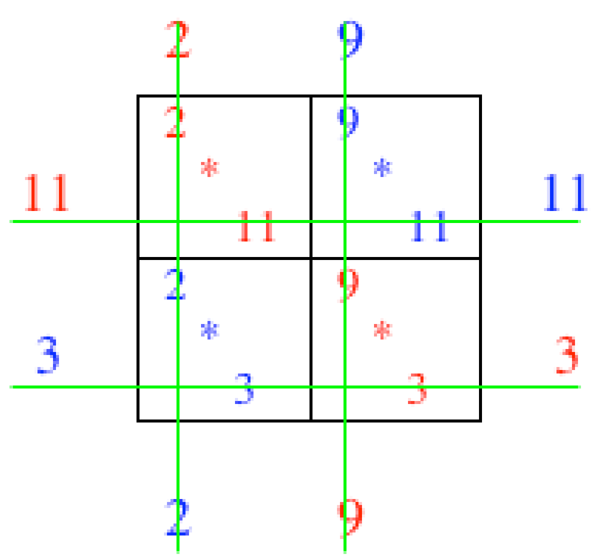
Fig 2. Factored-Inside PQ

My Dog	Your Dog
My Cat	

Fig. 3. Compound-Lexical

Proportion Quartet and Cross Multiplication: Sketches for Design





Appendix C5—Design Sketches Towards an Extension of the Unit to Functions

Building Connections From Ratio and Proportion to Functions: Sketches for Design

1. Summer is here again. Chip and Dale are, once again, collecting acorns. Chip has no acorns at all. But Dale has 1 acorn left over from the winter. Every day, Chip collects 3 acorns, and Dale collects 5 acorns. Let's look at how many acorns Chip and Dale have from day to day.

		Chip 3	Dale : 5	
Day 0		0	**1**	
Day 1	+3 (	3	5 + **1**	)+5
Day 2	+3 (	6	10 + **1**	)+5
Day 3	+3 (	9	15 + **1**	)+5
Day 4	+3 (	12	20 + **1**	)+5
Day 5	+3 (	15	25 + **1**	)+5
Day 6	+3 (	18	30 + **1**	)+5
Day 7	+3 (	21	35 + **1**	)+5
Day 8	+3 (	24	40 + **1**	)+5
Day 9	+3 (	27	45 + **1**	)+5
Day 10	+3 (	30	50 + **1**	)+5

	Chip 3	Dale : 5
Day 0	0	1
Day 1	3	6
Day 2	6	11
Day 3	9	16
Day 4	12	21
Day 5	15	26
Day 6	18	31
Day 7	21	36
Day 8	24	41
Day 9	27	46
Day 10	30	51

When Chip has 21 acorns, how many acorns will Dale have?

* 7

Chip 3	Dale : 5
0	**+1**
21	

* 7 +1

2. Two flower bulbs, a red-tulip bulb and a yellow-daffodil bulb, were planted in pots, and put side by side. The pot with the tulip bulb was put on the floor. The pot with the daffodil bulb was put on a stone 3 inches above the floor. The flowers budded on the same day. Every day, the tulip grows 2 inches, and the daffodil grows 7 inches.

		Tulip	: Daffodil	
Day 0		0	+ **3**	
Day 1	+ (			)+
Day 2	+ (			)+
Day 3	+ (			)+
Day 4	+ (			)+
Day 5	+ (			)+
Day 6	+ (			)+
Day 7	+ (			)+
Day 8	+ (			)+
Day 9	+ (			)+
Day 10	+ (			)+

	Tulip	: Daffodil
Day 0	0	+ **3**
Day 1		
Day 2		
Day 3		
Day 4		
Day 5		
Day 6		
Day 7		
Day 8		
Day 9		
Day 10		

When the tulip is 16 inches tall, how high will the daffodil get?

	Tulip	Daff.	
	2	: 7	
* _	0	+_	* _ +_

3. Jill and Mr. Cigan are having another running competition. This time, Jill gives Mr. Cigan a 14 yard head-start. Every second Jill runs 5 yards and Mr. Cigan runs 3 yards.

    a. Build a table to show their progress. Be careful when you Label-the-Table. The tables might look a bit different this time.

    b. What have you discovered about this competition? Who will win? Explain!

    c. When Jill will have run 100 yards, where will Mr. Cigan be? What will the difference between them be then?

Appendix C6—Assorted Ratio-and-Proportion Problems Not Used in Dissertation Implementations

Array Multiplication Stories

1. Sammy works in a nursery. He is planting trees in straight rows. In every row he puts 5 trees and then begins a new row. Slowly, he is planting a little forest!
    a. What will the shape of the forest be after 5 days? [a square]
    b. How many trees in Sammy's forest after 7 days? [35]
    c. Jane is also planting a little forest. In her forest the rows are 9 trees long. If Sammy and Jane each start planting their little forests on the same day, who will have more trees in their forest after 4 days? [Jane] How many more? [16 trees]

2. Tylor the tiler is decorating a bathroom with many tiny tiles. Today he is working on a bit of a wall that should have 8 * 70 tiles. It's takes him a minute to put a single line of 8 tiles in place.
    a. How many tiles will Tylor put in 5 minutes? [40]
    b. How many tiles will Tylor put in 9 minutes? [72]
    c. How many tiles will Tylor put in a quarter of an hour? [120]
    d. How long will it take Tylor to complete today's job?? [70 min, or 1 hour and 10 minutes]

3. Greta is a gardener. She works at Soldier Field. The football field is 160 feet wide and 360 feet long. Greta's job is to mow the stadium lawn. She drives a huge lawn-moving machine that can mow 8 feet across. Greta drives the width of the field. It takes Greta exactly 1 minute to complete a width and turn around for the next width.
    a. How much time will it take Greta to mow the whole field? [45 min]
    b. If Greta's machine mowed 16 feet across, would she mow the whole field in more time or in less time? How long would it take her? [22.5 min]
    c. Did you use all the information in this question? If not, what not? [160 f. wide]

4. Peggy, Karen, and Edi works at a free-range chicken farm. All day, they fill trays with big eggs on a conveyor belt that has room for 6 eggs across. They work together and with both hands and it takes them only 1 second to fill a row.
    a. How many rows can they fill in 3 seconds? [18]
    b. How many rows can they fill in 9 seconds? [54]
    c. How many rows can they fill in 1 minute? [360]

5. Constantia likes drawing patterns on her checkered sketchpad. She works very carefully. Her new project is a pattern that is 9 squares wide and 14 squares long. Every day, Constantia fills in a single row.
    a. How many squares are there in the whole pattern? [126 sq]
    b. How many squares will Constantia fill in 5 days? [45 sq]
    c. After a week of work, how many rows and how many squares will Constantia fill in? [7 rows; 63 sq]

Non-Multiplication-Table Stories [Not Used in Dissertation Implementations]

1. Maggie is buying vegetables at the farmers' market. She chose 6 tomatoes and 9 broccoli flowers. Then she chose 8 carrots. How many sticks of celery do you think Maggie chose at the farmers' market? [insufficient information]

2. *[Compare to this M T story]* Maggie is buying vegetables at the farmers' market to make vegetable soup. The recipe for this soup calls for 6 tomatoes and 9 sticks of broccoli. But Maggie wants to make a lot of soup. She buys 8 tomatoes. How many broccoli flowers should she buy? [12]

3. Alice is a mail carrier. Today she is delivering letters on Maple St. She delivered 6 letters to the house on #10, Maple St. How many letters do you think she may deliver to the house on #12, Maple St.? [insufficient information]

4. Alice is a mail carrier. Today she is delivering letters on Maple St. She has letters for people living in houses #4 and #6. If she has letters for house #20, can you guess what other house she may have letters for? Explain your answer. [insufficient information; perhaps some sense that the street only has one side]

5. Ted lives in Alabama. He is preparing for the Montgomery Marathon. Ted ran 10 miles last Saturday and 12 miles last Sunday. Jen lives in Alaska. She is preparing for the Juno Triathlon. She swam 1 mile on Tuesday. When Jen trains next week on Thursday, how far would you expect her to swim? Explain your answer. [insufficient information; perhaps some sense of general trend, so 1.2]

6. Denise sells balloons at the county fairs. On Labor Day she sold 10 red balloons and 12 blue balloons. If she sold 5 Micky Mouse balloons, how many Donald Duck balloons do you think Denise sold? [insufficient information; unless these total at 22]

7. In the town library there are 15 books in French and 20 books in German. If there are 10 books in Dutch, how many books are there in Spanish? [insufficient information]

8. Every day, Mark and Mindy watch Nickelodeon together, but Mindy missed some of the first episodes. When Mark had seen 7, Mindy had seen 4. When Mark saw 14, how many episodes had Mindy seen? [Fixed difference with head start; 11]

Appendix C7: Selected Pages Toward the Teacher Guide

Rate: "Every turn – the same we earn"

How much you add every time.
How much the number grows every time.

Every day, Robin puts $3 into her kitty-bank:
She is adding $3 to her savings every day, so she is saving $3 per / Day.
The growing number is 3.

The rule is: *every* **1** *day*………$**3**

Day		$	
+1		**+3**	
Day *0*	*0**3 =	0	+3
Day *1*	*1**3 =	3	+3
Day *2*	*2**3 =	6	+3
Day *3*	*3**3 =	9	+3
Day *4*	*4**3 =	12	+3
Day *5*	*5**3 =	15	+3
Day *6*	*6**3 =	18	+3
Day *7*	*7**3 =	21	+3
Day *8*	*8**3 =	24	+3
Day *9*	*9**3 =	27	+3
Day *10*	*10**3=	30	

Rate: "Every turn – the same we earn"

How much you add every time.
How much the number grows every time.

Every day, Tim puts $5 into his doggy-bank:
He is adding $5 to his savings every day, so he is saving $5 per / Day.
The growing number is 5.

The rule is: *every* **1** *day*………$**5**

Day		$	
+1		**+5**	
Day *0*	*0**5 =	0	+5
Day *1*	*1**5 =	5	+5
Day *2*	*2**5 =	10	+5
Day *3*	*3**5 =	15	+5
Day *4*	*4**5 =	20	+5
Day *5*	*5**5 =	25	+5
Day *6*	*6**5 =	30	+5
Day *7*	*7**5 =	35	+5
Day *8*	*8**5 =	40	+5
Day *9*	*9**5 =	45	+5
Day *10*	*10**5=	50	

Pro-Portion: **[Material for Teacher Guide]**
The experts' portion of the Multiplication Table:
Your new strategy for solving tough problems!

It's easy. All you have to do is copy the correct numbers from Robin and Tims' Ratio Table into the Proportion Quartets below.

If there is no label, don't forget to…

LABEL THE TABLE!!!!!

Do you know the folding technique?
You can fold the Ratio Table to get Proportion Quartets. It's cool. It can help you.
Try it!

Mystery Proportion Quartet Challenge: Can you solve the quartets <u>without</u> looking at the Ratio Table? That's for real *pro*'s!! If you have a good strategy, please share it with the class.

	Rob$	Tim$
	3 **:**	**5**
Day 0	0	0
Day 1	3	5
Day 2	6	10
Day 3	9	15
Day 4	12	20
Day 5	15	25
Day 6	18	30
Day 7	21	35
Day 8	24	40
Day 9	27	45
Day 10	30	50

	R$ 3	T$ 5
D 1	3	5
D 2	6	

	R$ 3	T$ 5
D 1	3	5
D 3		15

	R$ 3	T$ 5
D 2		
D 4	12	20

	R$	T$
D 3	9	15
	30	

	R$	T$
D 1	3	5
		45

	R$	T$
2		10
	21	
	3	

	3	
		20
D 6		
		5

	3	5	
2			2
7			7
	3	5	

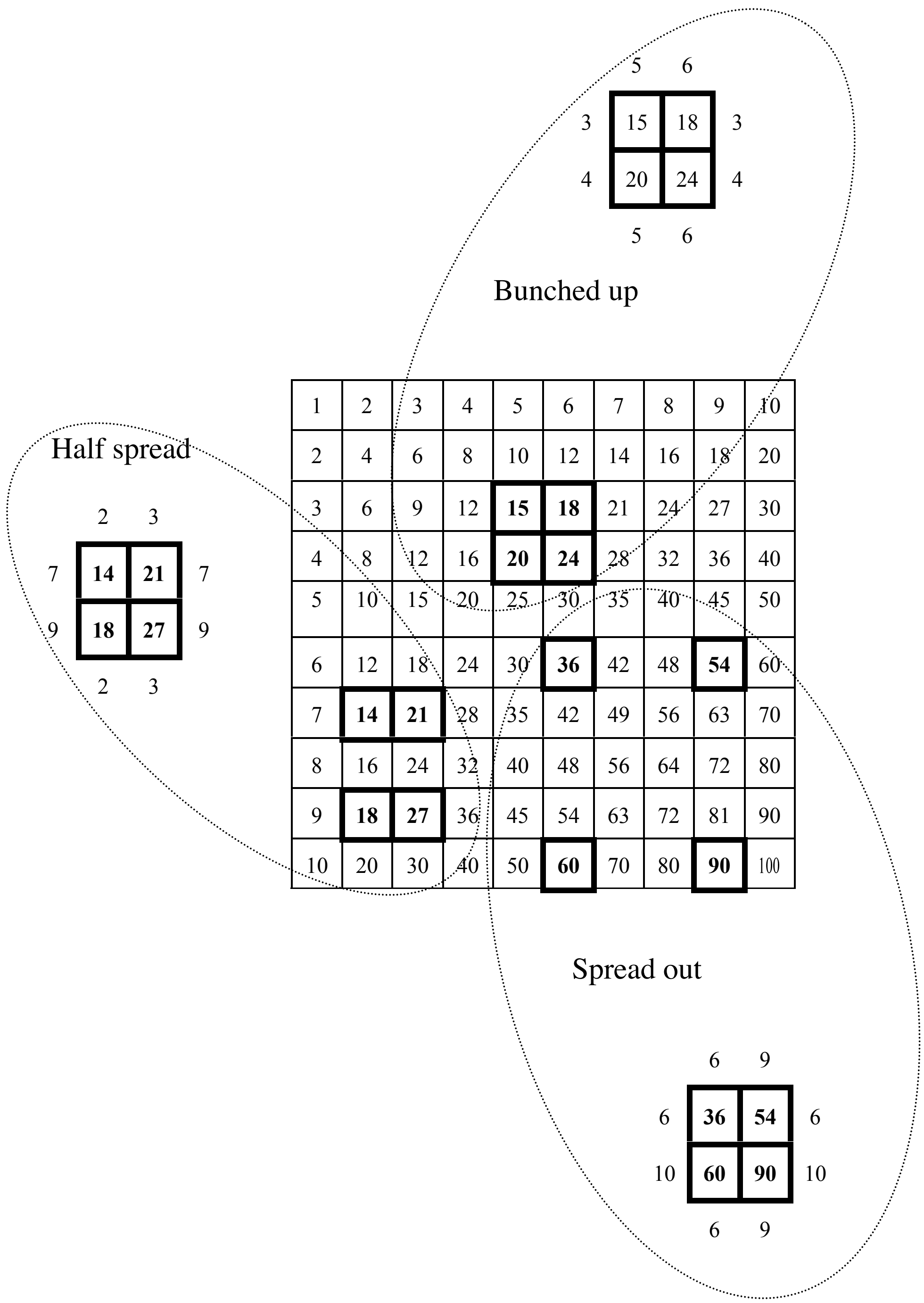
	3		
2			
			8
		5	

	12	20
	27	45

	3	5
		40
D 12		

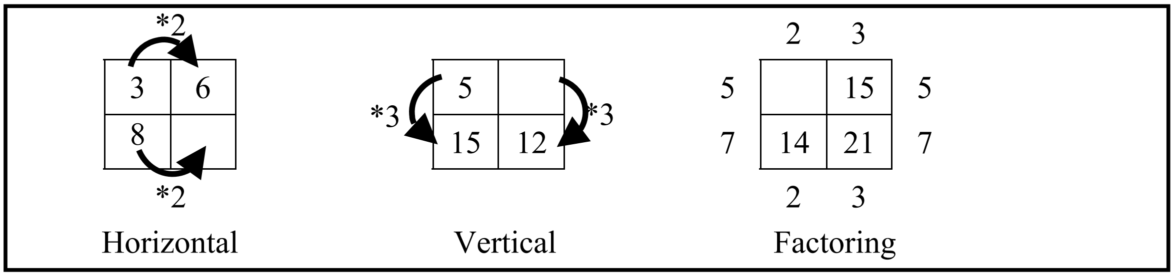
☺	Lee$ 4	Pat$ 9	
2			2
7			7
	4	9	

Proportion Quartets in the Multiplication Table [Material for Teacher Guide]



1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	**15**	**18**	21	24	27	30
4	8	12	16	**20**	**24**	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	**36**	42	48	**54**	60
7	**14**	**21**	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	**18**	**27**	36	45	54	63	72	81	90
10	20	30	40	50	**60**	70	80	**90**	100

Typse of Solutions and Practice Proportion Quartets [Material for Teacher Guide]



18	24
	8

55	22
20	

	20
8	32

45	54
	48

	36
9	54

6	8
30	

18	54
	12

	22
9	11

	90
63	70

12	18
14	

54	72
	40

35	25
56	

	32
6	48

24	39
	26

9	
63	42

Appendix C8: Proportion Quartet "Fortune Teller" [Briefly Implemented in a Tutor Session]