FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 1 - FLUID FLOW THEORY

S.A.E. No. 1

1. Describe the principle of operation of the following types of viscometers.
a. Redwood Viscometers.
b. British Standard 188 glass U tube viscometer.
c. British Standard 188 Falling Sphere Viscometer.
d. Any form of Rotational Viscometer

The solutions are contained in part 1 of the tutorial.

S.A.E. No. 2

1. Oil flows in a pipe 80 mm bore diameter with a mean velocity of 0.4 m/s. The density is 890 kg/m^3 and the viscosity is 0.075 Ns/m^2. Show that the flow is laminar and hence deduce the pressure loss per metre length.

$$R_e = \frac{\rho u d}{\mu} = \frac{890 \text{ x } 0.4 \text{ x } 0.08}{0.075} = 379.7$$

Since this is less than 2000 flow is laminar so Poiseuille's equation applies.

$$\Delta p = \frac{32\mu 2\mu}{d^2} = \frac{32 \text{ x } 0.075 \text{ x } 1 \text{ x } 0.4}{0.08^2} = 150 \text{ Pa}$$

2. Oil flows in a pipe 100 mm bore diameter with a Reynolds' Number of 500. The density is 800 kg/m^3. Calculate the velocity of a streamline at a radius of 40 mm. The viscosity μ = 0.08 Ns/m^2.

$$R_e = 500 = \frac{\rho u_m d}{\mu}$$

$$u_m = \frac{500\mu}{\rho d} = \frac{500 \text{ x } 0.08}{800 \text{ x } 0.1} = 0.5 \text{ m/s}$$

Since R_e is less than 2000 flow is laminar so Poiseuille's equation applies.

$$\Delta p = \frac{32\mu 2\mu}{d^2} = \frac{32 \text{ x } 0.08 \text{ x } L \text{ x } 0.5}{0.1^2} = 128L \text{ Pa}$$

$$u = \frac{\Delta p\left(R^2 - r^2\right)}{4L\mu} = \frac{128L\left(0.05^2 - 0.04^2\right)}{4L \text{ x } 0.08} = 0.36 \text{ m/s}$$

3. A liquid of dynamic viscosity 5 x 10^{-3} Ns/m^2 flows through a capillary of diameter 3.0 mm under a pressure gradient of 1800 N/m^3. Evaluate the volumetric flow rate, the mean velocity, the centre line velocity and the radial position at which the velocity is equal to the mean velocity.

$$\frac{\Delta P}{L} = 1800 = \frac{32\,\mu\, u_m}{d^2} \qquad u_m = 0.10125 \text{ m/s}$$

$u_{max} = 2\, u_m = 0.2025$ m/s

$$u = 0.10125 = \frac{\Delta p\left(R^2 - r^2\right)}{4L\mu} = \frac{1800\left(0.0015^2 - r^2\right)}{4 \text{ x } 0.005} \qquad r = 0.0010107 \text{ m or } 1.0107 \text{ mm}$$

4.

a. Explain the term Stokes flow and terminal velocity.

b. Show that a spherical particle with Stokes flow has a terminal velocity given by
$u = d^2 g(\rho_s - \rho_f)/18\mu$
Go on to show that $C_D=24/R_e$

c. For spherical particles, a useful empirical formula relating the drag coefficient and the Reynold's number is

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$

Given ρ_f = 1000 kg/m^3, μ= 1 cP and ρ_s= 2630 kg/m^3 determine the maximum size of spherical particles that will be lifted upwards by a vertical stream of water moving at 1 m/s.

d. If the water velocity is reduced to 0.5 m/s, show that particles with a diameter of less than 5.95 mm will fall downwards.

a) For R_e<0.2 the flow is called Stokes flow and Stokes showed that R = 3πd μ u hence

R = W = volume x density difference x gravity

$$R = W = \frac{\pi d^3 g(\rho_s - \rho_f)}{6} = 3\pi d\,\mu\,u$$

ρ_s = density of the sphere material ρ_f = density of fluid d = sphere diameter

$$u = \frac{\pi d^3 g(\rho_s - \rho_f)}{18\,\pi\,d\,\mu} = \frac{d^2 g(\rho_s - \rho_f)}{18\,\mu}$$

b) C_D = R/(projected area x $\rho u^2/2$) $C_D = \dfrac{\pi d^3 g(\rho_s - \rho_f)}{(\rho\,u^2/2)6\,\pi\,d^2/4} = \dfrac{4dg(\rho_s - \rho_f)}{3\rho\,u^2}$

$$C_D = \frac{4 \text{ x } 9.81\text{x } (1630 - 998)d}{3\text{x } 998 \text{ x } u^2} = 21.389\frac{d}{u^2}$$

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4 = 21.389\frac{d}{u^2}$$

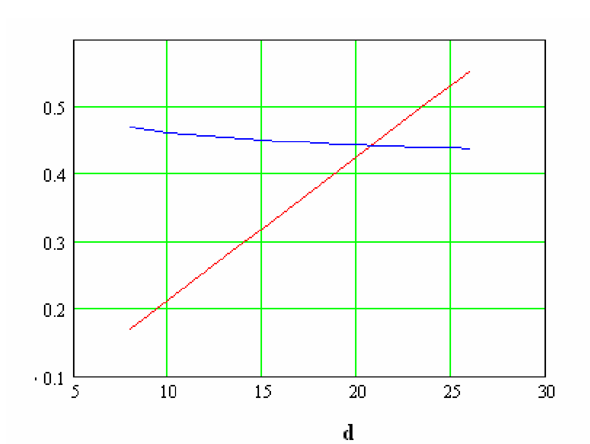
$$21.389\frac{d}{u^2} - \frac{24}{R_e} - \frac{6}{1+\sqrt{R_e}} = 0.4 \qquad \text{let } 21.389\frac{d}{u^2} - \frac{24}{R_e} - \frac{6}{1+\sqrt{R_e}} = x$$

Re = ρud/μ = 998 x 1 x d/0.89 x 10^{-3} = 1.1213 x 10^6 d

Make a table

D	0.001	0.003	0.01	0.02	0.03
Re	1121.3	3363.9	11213	22426	33639
x	-0.174	-0.045	0.156	0.387	0.608

Plot and find that when d = 0.0205 m (20.5 mm) x = 0.4



c) u = 0.5m/s d = 5.95mm

Re = ρud/μ = 998 x 0.5 x 0.00595/0.89 x 10^{-3} = 3336

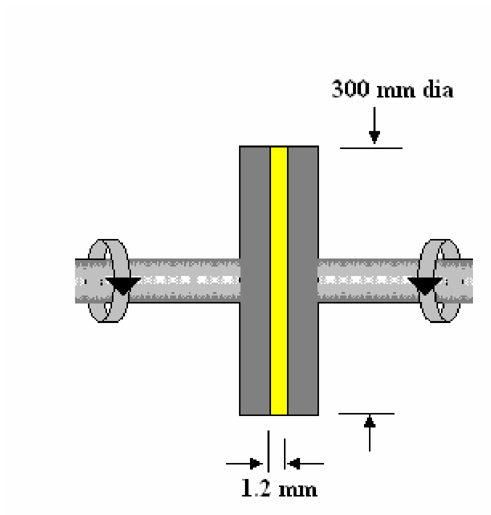
$$C_D = 21.389\frac{d}{u^2} = 0.509$$

$$C_D = \frac{24}{3336} + \frac{6}{1+\sqrt{3336}} + 0.4 = 0.509$$

Since C_D is the same, larger ones will fall.

5. Similar to Q5 1998

A simple fluid coupling consists of two parallel round discs of radius R separated by a a gap h. One disc is connected to the input shaft and rotates at ω_1 rad/s. The other disc is connected to the output shaft and rotates at ω_2 rad/s. The discs are separated by oil of dynamic viscosity μ and it may be assumed that the velocity gradient is linear at all radii.



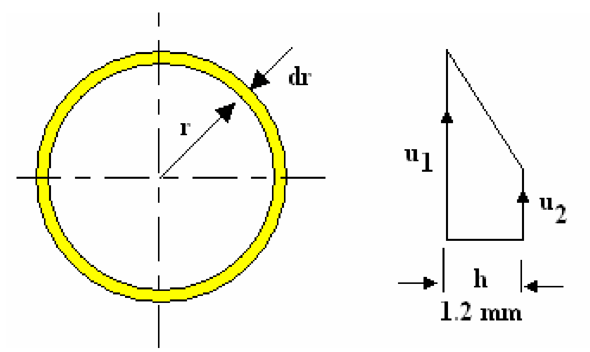
Show that the Torque at the input shaft is given by

$$T = \frac{\pi D^4 \mu(\omega_1 - \omega_2)}{32h}$$

The input shaft rotates at 900 rev/min and transmits 500W of power. Calculate the output speed, torque and power. (747 rev/min, 5.3 Nm and 414 W)

Show by application of max/min theory that the output speed is half the input speed when maximum output power is obtained.

SOLUTION



Assume the velocity varies linearly from u_1 to u_2 over the gap at any radius. Gap is h = 1.2 mm

T = μ du/dy = μ (u_1 - u_2)/h

For an elementary ring radius r and width dr the shear force is

Force = τ dA = τ 2πr dr

$$dF = \mu\frac{u_1 - u_2}{h} x 2\pi r \ dr$$

Torque due to this force is $$dT = rdF = \mu\frac{u_1 - u_2}{h} x\ 2\pi\ r^2\ dr$$

Substitute u = ωr $$dT = rdF = \mu\frac{(\omega_1 - \omega_2)}{h} x\ 2\pi\ r^3\ dr$$

Integrate $$T = \mu\frac{(\omega_1 - \omega_2)}{h} x\ 2\pi\int_0^R r^3\ dr = \mu\frac{(\omega_1 - \omega_2)}{h} x\ 2\pi\frac{R^4}{4}$$

Rearrange and substitute R = D/2 $$T = \mu\frac{(\omega_1 - \omega_2)}{h} x\ \pi\frac{D^4}{32}$$

Put D = 0.3 m, μ = 0.5 N s/m^2, h = 0.012 m $T = 0.5\frac{(\omega_1 - \omega_2)}{0.012} \text{ x } \pi\frac{0.3^4}{32} = 0.33(\omega_1 - \omega_2)$

N = 900 rev/min P = 500 W Power = 2πNT/60 $T = \frac{60P}{2\pi N} = \frac{60 \text{ x } 500}{2\pi \text{ x } 900} = 5.305 \text{ Nm}$

The torque input and output must be the same. $\omega_1 = 2\pi N_1/60 = 94.25$ rad/s

$5.305 = 0.33(94.25_1 - \omega_2)$ hence $\omega_2 = 78.22$ rad/s and $N_2 = 747$ rev/min

$P_2 = 2\pi N_2 T/60 = \omega_2 T = 78.22 \text{ x } 5.305 = 414$ W (Power out)

For maximum power output $dp_2/d\omega_2 = 0$ $P_2 = \omega_2 T = 0.33(\omega_1\omega_2 - \omega_2^2)$

Differentiate $\frac{dP_2}{d\omega_2} = 0.33(\omega_1 - 2\omega_2)$

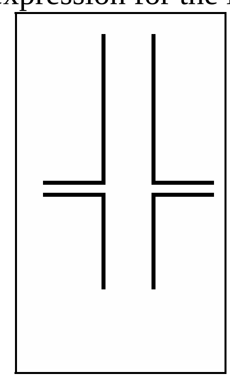
Equate to zero and it follows that for maximum power output $\omega_1 = 2\,\omega_2$

And it follows $N_1 = 2\,N_2$ so $N_2 = 450$ rev/min

6. Show that for fully developed laminar flow of a fluid of viscosity μ between horizontal parallel plates a distance h apart, the mean velocity u_m is related to the pressure gradient dp/dx by

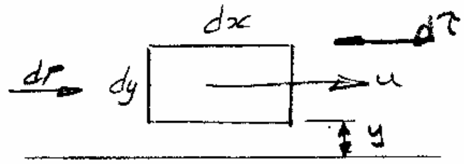
$u_m = -(h^2/12\mu)(dp/dx)$

A flanged pipe joint of internal diameter d_i containing viscous fluid of viscosity μ at gauge pressure p. The flange has an outer diameter d_o and is imperfectly tightened so that there is a narrow gap of thickness h. Obtain an expression for the leakage rate of the fluid through the flange.



Note that this is a radial flow problem and B in the notes becomes 2πr and dp/dx becomes -dp/dr. An integration between inner and outer radii will be required to give flow rate Q in terms of pressure drop p.

The answer is $Q = (2\pi h^3 p/12\mu)/\{\ln(d_o/d_i)\}$



$d\tau$ acts on dx dp acts dy

Balancing forces $d\tau\, dx = dp\, dy$

$$dp/dx = d\tau/dy$$

But $\tau = \mu \frac{du}{dy}$ for Newtonian fluids

$$dp/dx = \mu \; d\left(du/dy\right)/dy$$

Assume dp/dx is constant

Integrate $\left(\frac{dp}{dx}\right) y = \mu \frac{du}{dy} + A$

Integrate $\left(\frac{dp}{dx}\right) \frac{y^2}{2} = \mu u + Ay + B$ —— (1)

Boundary conditions $y = 0$ @ $u = 0$ $\therefore B = 0$

$y = h$ $u = 0$ (upper solid surface)

$$\left(\frac{dp}{dx}\right) \frac{h^2}{2} = \mu(0) + Ah \qquad A = \left(\frac{dp}{dx}\right) \frac{h}{2}$$

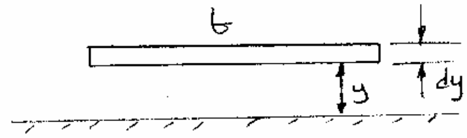
Substitute into (1)

$$\left(\frac{dp}{dx}\right) \frac{y^2}{2} = \mu u + \left(\frac{dp}{dx}\right) \frac{h}{2} y$$

Rearrange

$$u = \left(\frac{dp}{dx}\right) \frac{1}{2\mu} \left[y^2 - h y \right]$$

CONSIDER FLOW THROUGH A SMALL RECTANGULAR SLIT. (OUT OF PAGE)



$$d\varphi = u\, b\, dy = b\left(\frac{dP}{dx}\right)\frac{1}{2\mu}\left[y^2 - hy\right]dy$$

INTEGRATE

$$\varphi = b\left(\frac{dP}{dx}\right)\frac{1}{2\mu}\int_0^h (y^2 - hy)\, dy$$

$$\varphi = b\left(\frac{dP}{dx}\right)\frac{1}{2\mu}\left[y^3/3 - hy^2/2\right]_0^h$$

$$\varphi = b\left(\frac{dP}{dx}\right)\frac{1}{2\mu}\left[h^3/3 - h^3/2\right]$$

$$\varphi = -b\left(\frac{dP}{dx}\right)\frac{h^3}{12\mu}$$

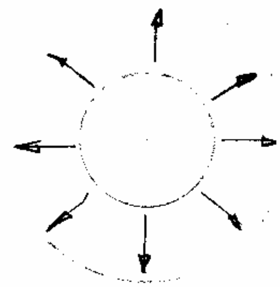
Mean velocity $= \varphi/A \qquad A = bh$

$$u_m = \frac{-b\left(\frac{dP}{dx}\right)\frac{h^3}{12\mu}}{bh} = -\left(\frac{dP}{dx}\right)\frac{h^2}{12\mu}$$

PART B

$$Q = -b\left(\frac{dp}{dx}\right)\frac{h^3}{12\mu}$$

Flow between flanges is radial



b = circumference = $2\pi r$

x = radius = r

h = GAP

$$Q - 2\pi r \frac{dp}{dr} \frac{h^3}{12\mu}$$

$$\frac{dr}{r} = -\frac{2\pi h^3}{12\mu Q} dp$$

Q is constant at all radii

$$\int_r^R \frac{dr}{r} = \frac{-2\pi h^3}{12\mu Q} \int_p^0 dp$$

$$\ln r/R = -\frac{2\pi h^3}{12\mu Q} p$$

$$Q = -\frac{2\pi h^3 p}{12\mu \ln r/R} = +\frac{2\pi h^3 p}{12\mu \ln(R/r)}$$

$$Q = \frac{2\pi h^3 p}{12\mu \ln(d_o/d_i)}$$

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ASSIGNMENT 3

1. A pipe is 25 km long and 80 mm bore diameter. The mean surface roughness is 0.03 mm. It carries oil of density 825 kg/m^3 at a rate of 10 kg/s. The dynamic viscosity is 0.025 N s/m^2.

   Determine the friction coefficient using the Moody Chart and calculate the friction head. (Ans. 3075 m.)

   $Q = m/\rho = 10/825 = 0.01212\ m^3/s$
   $u_m = Q/A = 0.01212/(\pi \times 0.04^2) = 2.411$ m/s
   $Re = \rho ud/\mu = 825 \times 2.4114 \times 0.08/0.025 = 6366$
   $k/D = 0.03/80 = 0.000375$
   From the Moody chart $C_f = 0.0083$
   $h_f = 4\ C_f\ L\ u^2/2gd = 4 \times 0.0083 \times 25000 \times 2.4114^2/(2 \times 9.91 \times 0.08) = 3075$ m

2. Water flows in a pipe at 0.015 m^3/s. The pipe is 50 mm bore diameter. The pressure drop is 13 420 Pa per metre length. The density is 1000 kg/m^3 and the dynamic viscosity is 0.001 N s/m^2.

   Determine
   i. the wall shear stress (167.75 Pa)
   ii. the dynamic pressure (29180 Pa).
   iii. the friction coefficient (0.00575)
   iv. the mean surface roughness (0.0875 mm)

   $\tau_o = \Delta p\ D/4L = 13420 \times 0.05/4 = 167.75$ Pa

   $u_m = Q/A = 0.015/(\pi \times 0.025^2) = 7.64$ m/s

   Dynamic Pressure = $\rho u^2/2 = 1000 \times 7.64^2/2 = 29180$ Pa

   $C_f = \tau_o$/Dyn Press = 167.75/29180 = 0.00575

   From the Moody Chart we can deduce that $\varepsilon = 0.0017 = k/D$ $\quad k = 0.0017 \times 50 = 0.085$ mm

3. Explain briefly what is meant by fully developed laminar flow. The velocity u at any radius r in fully developed laminar flow through a straight horizontal pipe of internal radius r_o is given by

$$u = (1/4\mu)(r_o^2 - r^2)dp/dx$$

dp/dx is the pressure gradient in the direction of flow and μ is the dynamic viscosity.
Show that the pressure drop over a length L is given by the following formula.

$$\Delta p = 32\mu L u_m/D^2$$

The wall skin friction coefficient is defined as $C_f = 2\tau_o/(\rho u_m^2)$.
Show that $C_f = 16/R_e$ where $R_e = \rho u_m D/\mu$ and ρ is the density, u_m is the mean velocity and τ_o is the wall shear stress.

3) THE BOUNDARY LAYER IS ENTIRELY LAMINAR AND CONSTANT THICKNESS

$$u = \frac{1}{4\mu}(r_o^2 - r^2)\frac{dp}{dx} \qquad \text{Assume } \frac{dp}{dx} = \frac{\Delta p}{L}$$

$$d\varphi = u \times 2\pi r\, dr \qquad \varphi = \frac{1}{4\mu}\frac{\Delta p}{L} \times 2\pi \int_0^{r_o} (r r_o^2 - r^3)\, dr$$

$$\varphi = \frac{\Delta p \pi}{2\mu L}\left[\frac{r^2 r_o^2}{2} - \frac{r^4}{4}\right]_0^{r_o} = \frac{\Delta p \pi}{2\mu L}\left[\frac{r_o^4}{2} - \frac{r_o^4}{4}\right]$$

$$\varphi = \frac{\Delta p \pi}{2\mu L}\frac{r_o^4}{4} = \frac{\Delta p \pi\, r_o^4}{8\mu L} \qquad r_o = D/2$$

$$\varphi = \frac{\Delta p \pi D^4}{128\mu L} \qquad \varphi = u_m \times A = u_m \frac{\pi D^2}{4}$$

$$u_m = \frac{\Delta p D^2}{32\mu L} \qquad \Delta p = \frac{32\mu L u_m}{D^2}$$

$$C_f = \frac{2\tau_o}{\rho u_m^2} \qquad \tau_o \times \pi D L = \Delta p \times \frac{\pi D^2}{4}$$

$$\tau_o = \frac{\Delta p}{L}\frac{D}{4}$$

$$C_f = \frac{2\Delta p D}{\rho u_m^2 4L}$$

$$C_f = \frac{\Delta p D}{2\rho u_m^2 L} = \frac{\Delta p D}{2\rho L u_m} \times \frac{32\mu L}{\Delta p D^2} = \frac{32\mu}{2\rho D u_m}$$

$$C_f = \frac{16\mu}{\rho u_m D} = \frac{16}{R_e}$$

3. Oil with viscosity 2 x 10^{-2} Ns/m^2 and density 850 kg/m^3 is pumped along a straight horizontal pipe with a flow rate of 5 dm^3/s. The static pressure difference between two tapping points 10 m apart is 80 N/m^2. Assuming laminar flow determine the following.

i. The pipe diameter.
ii. The Reynolds number.

Comment on the validity of the assumption that the flow is laminar

4.

$\mu = 2\times10^{-2}$ Ns/m^2

$\rho = 850$ kg/m^3

$Q = 5$ dm^3/s

$\Delta p = 80$ N/m^2

$L = 10$ m

POISEUILLES EQUATION

$$\Delta p = \frac{32\mu L u_m}{D^2}$$

$$D^2 = \frac{32\mu L u_m}{\Delta p} = \frac{32\times 2\times10^{-2}\times 10\times u_m}{80}$$

$$D^2 = 0.08\,u_m \qquad u_m = \frac{4Q}{\pi D^2} = \frac{4\times 5\times10^{-3}}{\pi D^2}$$

$$D^2 = \frac{.08\times.006366}{D^2} \qquad u_m = \frac{.006366}{D^2}$$

$$D^4 = 509\times10^{-6} \qquad D = 0.150\text{ m}$$

$$Re = \frac{\rho u D}{\mu} = \frac{850\times u_m\times .15}{2\times10^{-2}}$$

$$u_m = .006366/.15^2 = .2829\text{ m/s}$$

$$Re = \frac{850\times.2829\times.15}{2\times10^{-2}} = 1803.7$$

$Re < 2000$

JUST LAMINAR

ASSIGNMENT 4

1. Research has shown that tomato ketchup has the following viscous properties at 25^oC.

Consistency coefficient $K = 18.7$ Pa s^n
Power $n = 0.27$
Shear yield stress = 32 Pa

Calculate the apparent viscosity when the rate of shear is 1, 10, 100 and 1000 s^{-1} and conclude on the effect of the shear rate on the apparent viscosity.

This fluid should obey the Herchel-Bulkeley equation so $\mu_{app} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1} = \frac{32}{\dot{\gamma}} + 18.7\dot{\gamma}^{0.27-1}$

put $\gamma = 1$ and $\mu_{app} = 50.7$
put $\gamma = 10$ and $\mu_{app} = 6.682$
put $\gamma = 100$ and $\mu_{app} = 0.968$
put $\gamma = 1000$ and $\mu_{app} = 0.153$
The apparent viscosity reduces as the shear rate increases.

2. A Bingham plastic fluid has a viscosity of 0.05 N s/m^2 and yield stress of 0.6 N/m^2. It flows in a tube 15 mm bore diameter and 3 m long.

(i) Evaluate the minimum pressure drop required to produce flow.

The actual pressure drop is twice the minimum value. Sketch the velocity profile and calculate the following.

(ii) The radius of the solid core.
(iii) The velocity of the core.
(iv) The volumetric flow rate.

$\tau = \tau_Y + \mu\frac{du}{dy}$ The minimum value of τ is τ_y

Balancing forces on the plug τ_y x $2\pi rL = \Delta p\pi r^2$

$\Delta p = \tau_Y \frac{2L}{r}$ and the minimum Δp- is at r = R $\Delta p = 0.6\frac{2 \text{ x } 3}{0.0075} = 480 \text{ Pa}$

b $\Delta p = 2 \text{ x } 480 = 960$ Pa From the force balance $\Delta p = \tau_Y \frac{2L}{r}$

$$r = \tau_Y \frac{2L}{\Delta p} = 0.6\frac{2 \text{ x } 3}{960} = 0.00375 \text{ m or } 3.75 \text{ mm}$$

The profile is follows Poiseuille's equation

$$u = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right) = \frac{960}{4 \text{ x } 0.05 \text{ x } 3}\left(0.0075^2 - 0.00375^2\right) = 0.0675 \text{ m/s}$$

Flow rate of plug =Au = $\pi(0.00375^2)$x0.0675 = 2.982 x 10^{-6} m^3/s

$$dQ = u\,(2\pi r\,dr) = \frac{\Delta p(2\pi r)}{4\mu L}\left(R^2 - r^2\right)$$

$$Q = \int_r^R \frac{\Delta p(2\pi)}{4\mu L}\left(rR^2 - r^3\right) \qquad Q = \frac{\Delta p(2\pi)}{4\mu L}\left[\frac{R^2 r^2}{2} - \frac{r^4}{2}\right]_r^R$$

$$Q = \frac{\Delta p(2\pi)}{4\mu L}\left\{\left[\frac{R^4}{2} - \frac{R^4}{4}\right] - \left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]\right\}$$

$$Q = \frac{\Delta p\,\pi}{2\mu L}\left[\frac{R^4}{4} - \frac{R^2 r^2}{2} - \frac{r^4}{4}\right]$$

$$Q = \frac{960\,\pi}{2\,x 0.05\,x\,3}\left[\frac{0.0075^4}{4} - \frac{0.0075^2\ x\ 0.00375^2}{2} - \frac{0.00375^4}{4}\right] = 4.473x10^{-6}\,m^3/s$$

Total Q = (4.473 + 2.982) x 10^{-6} = 7.46 x 10^{-6} m^3/s

3. A non-Newtonian fluid is modelled by the equation $\tau = K\left(\frac{du}{dr}\right)^n$ where n = 0.8 and K = 0.05 N $s^{0.8}/m^2$. It flows through a tube 6 mm bore diameter under the influence of a pressure drop of 6400 N/m^2 per metre length. Obtain an expression for the velocity profile and evaluate the following.

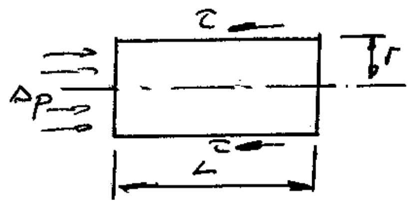
(i) The centre line velocity. (0.953 m/s)

(ii) The mean velocity. (0.5 m/s)

$$\tau = K\left(\frac{du}{dy}\right)^n$$

TUBE ϕ6mm $\quad \frac{\Delta p}{L} = 6400$ Pa/m

CYLINDER OF FLUID



SHEAR FORCE $F_s = \tau \times$ SURFACE AREA

$= \tau \times 2\pi r L$

PRESSURE FORCE $= \Delta p \times \pi r^2$

TOTAL IS ZERO $\quad \Delta p\, \pi r^2 + \tau\, 2\pi r L = 0$

$$\frac{\Delta p r}{2L} = \tau = -K\left(\frac{du}{dy}\right)^n \quad \text{note } dy = -dr$$

$$\frac{\Delta p\, r}{2LK} = \left(\frac{du}{dr}\right)^n$$

$$du = \left(\frac{\Delta p\, r}{2LK}\right)^{\frac{1}{n}} \times dr \qquad \text{INTEGRATE}$$

$$u = \int \left(\frac{\Delta p\, r}{2LK}\right)^{1/n} dr = \left(\frac{\Delta p}{2LK}\right)^{1/n} \int_R^r r^{1/n}\, dr$$

$$u = \left(\frac{\Delta p}{2LK}\right)^{1/n} \left[\frac{r^{1+1/n}}{1+1/n}\right]_R^r = \left(\frac{\Delta p}{2LK}\right)^{1/n} \left[\frac{nR^{\frac{n+1}{n}}}{n+1} - \frac{n r^{\frac{n+1}{n}}}{n+1}\right]$$

AT THE CENTRE LINE $r = 0$

$$u = \left(\frac{\Delta P}{2LK}\right)^{1/n} \frac{n}{n+1} \left[R^{\frac{n+1}{n}} - 0 \right]$$

$\frac{\Delta P}{L} = 6400 \quad R = .003\,m \quad K = \mu = 0.05\,Ns/m^2$

$n = 0.8$

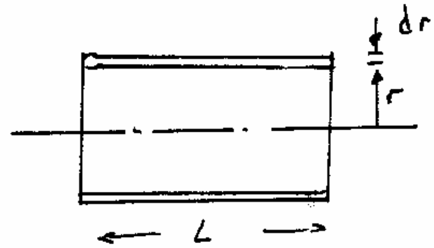
$$u = \frac{6400}{2 \times 0.05} \times \frac{0.8}{1.8} \left[0.003^{1.8/.8} \right]$$

$$u = 1.0179 \times 10^6 \times \frac{.8}{1.8} \times .003^{2.25}$$

$$u = 452.4 \times 10^3 \times .003^{2.25}$$

$$u = 0.953\ m/s$$

FLOW RATE



CONSIDER A THIN WALL CYLINDER MOVING AT VELOCITY u

FLOW $= d\varphi =$ CROSS SECTIONAL AREA $\times\ u$

$= 2\pi r\,dr\ u$

$$d\varphi = \left(\frac{\Delta P}{2LK}\right)^{1/n} \frac{n}{n+1}\ 2\pi \left[R^{\frac{n+1}{n}} r - r^{1+1+1/n} \right] dr$$

TOTAL FLOW

$$\varphi = \left(\frac{\Delta P}{2LK}\right)^{1/n} \frac{n}{n+1}\ 2\pi \int_0^R \left[R^{\frac{n+1}{n}} r - r^{2+1/n} \right] dr$$

$$Q = \left(\frac{\Delta p}{2Lk}\right)^{1/n} \frac{n}{n+1} \quad 2\pi \left[\frac{R^{1+1/n} r^2}{2} - \frac{r^{2+1/n+1}}{2+1/n+1} \right]_0^R$$

$$Q = \left(\frac{\Delta p}{2Lk}\right)^{1/n} \frac{n}{n+1} \quad 2\pi \left[\frac{R^{3+1/n}}{2} - \frac{R^{3+1/n}}{3+1/n} \right]$$

$$Q = \left(\frac{\Delta p}{2Lk}\right)^{1/n} \frac{n}{n+1} \quad 2\pi \quad R^{3+1/n} \left[\frac{1}{2} - \frac{1}{3+1/n} \right]$$

$$Q = \left(\frac{\Delta p}{2Lk}\right)^{1/n} \frac{n}{n+1} \quad 2\pi \left[\frac{n+1}{6n+2} \right] R^{3+1/n}$$

$$Q = \left(\frac{\Delta p}{2Lk}\right)^{1/n} \pi R^{3+1/n} \left(\frac{n}{3n+1}\right)$$

$$Q = \left(\frac{6400}{2 \times .05}\right)^{1/0.8} \pi \times 0.003^{3+1/.8} \times \frac{0.8}{(3 \times 0.8 + 1)}$$

$$Q = 64000^{1.25} \times \pi \times .003^{4.25} \times \frac{1}{4.25}$$

$$Q = 14.26 \times 10^{-6} \ m^3/s$$

MEAN VELOCITY

$$u_m = Q/A = \frac{14.26 \times 10^{-6}}{\pi \times .003^2} = \underline{\underline{0.504 \ m/s}}$$

NOTE IF $n = 1$ ALL EQUATIONS BECOME THE SAME AS FOR NEWTONIAN FLOW

ADDITIONAL PROOF

$$u = \left(\frac{\Delta P}{2L\mu}\right)^{1/n} \left[\frac{n R^{1+1/n}}{n+1} - \frac{n r^{1+1/n}}{n+1}\right]$$

If $n = 1$ (Newtonian)

$$u = \frac{\Delta P}{2L\mu} \times \frac{1}{2} \left[R^2 - r^2\right]$$

$$dQ = u \times 2\pi r \, dr$$

$$= \left(\frac{\Delta P}{2L\mu}\right)^{1/n} \left(\frac{n}{n+1}\right) \left[R^{1+1/n} - r^{1+1/n}\right] \times 2\pi r \, dr$$

$$Q = \left(\frac{\Delta P}{2L\mu}\right)^{1/n} \frac{2\pi n}{(n+1)} \int_0^R \left(R^{1+1/n} r - r^{2+1/n}\right) dr$$

$$= \left(\frac{\Delta P}{2L\mu}\right)^{1/n} \frac{2\pi n}{n+1} \left[R^{1+1/n} \frac{r^2}{2} - \frac{r^{3+1/n}}{3+1/n}\right]_0^R$$

$$= \left(\frac{\Delta P}{2L\mu}\right)^{1/n} \frac{2\pi n}{n+1} \left[\frac{R^{3+1/n}}{2} - \frac{R^{3+1/n}}{3+1/n}\right]$$

$$= \left(\frac{\Delta P}{2L\mu}\right)^{1/n} \frac{2\pi n}{n+1} R^{3+1/n} \left[\frac{1}{2} - \frac{1}{3+1/n}\right]$$

If $n = 1$ $$Q = \frac{\Delta P}{2L\mu} \times \frac{2\pi}{2} R^4 \left[\frac{1}{2} - \frac{1}{4}\right]$$

$$Q = \frac{\Delta P \, \pi R^4}{2L\mu} \times \frac{1}{4} \quad \text{— Correct}$$

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 2 – APPLICATIONS OF BERNOULLI

SELF ASSESSMENT EXERCISE 1

1. A pipe 100 mm bore diameter carries oil of density 900 kg/m^3 at a rate of 4 kg/s. The pipe reduces to 60 mm bore diameter and rises 120 m in altitude. The pressure at this point is atmospheric (zero gauge). Assuming no frictional losses, determine:
   i. The volume/s (4.44 dm^3/s)
   ii. The velocity at each section (0.566 m/s and 1.57 m/s)
   iii. The pressure at the lower end. (1.06 MPa)

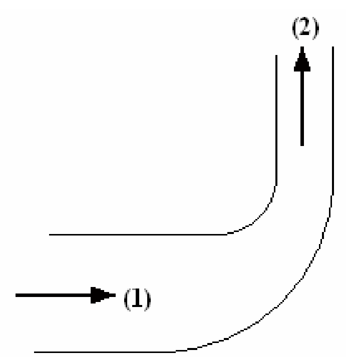
   $Q = m/\rho = 4/900 = 0.00444$ m^3/s
   $u_1 = Q/A_1 = 0.00444/(\pi \times 0.05^2) = 0.456$ m/s $\quad u_2 = Q/A_2 = 0.00444/(\pi \times 0.03^2) = 1.57$ m/s
   $h_1 + z_1 + u_1^2/2g = h_2 + z_2 + u_2^2/2g \qquad h_2 = 0 \quad z_1 = 0$
   $h_1 + 0 + 0.566^2/2g = 0 + 120 + 1.57^2/2g$
   $h_1 = 120.1$ m $\qquad p = \rho g h = 900 \times 9.81 \times 120.1 = 1060$ kPa

2. A pipe 120 mm bore diameter carries water with a head of 3 m. The pipe descends 12 m in altitude and reduces to 80 mm bore diameter. The pressure head at this point is 13 m. The density is 1000 kg/m^3. Assuming no losses, determine

   i. The velocity in the small pipe (7 m/s) ii. The volume flow rate. (35 dm^3/s)

   $h_1 + z_1 + u_1^2/2g = h_2 + z_2 + u_2^2/2g \qquad 3 + 12 + u_1^2/2g = 13 + 0 + u_2^2/2g$
   $2 = (u_2^2 - u_1^2)/2g \qquad (u_2^2 - u_1^2) = 39.24$
   $u_1 A_1 = Q = u_2 A_2 \qquad u_1 = u_2 (80/120)^2 = 0.444 u_2$
   $39.24 = u_2^2 - (0.444 u_2)^2 = 0.802 u_2^2 \qquad u_2 = 6.99$ m/s $\qquad u_1 = 3.1$ m/s
   $Q = u_2 A_2 = 6.99 \times \pi \times 0.04^2 = 0.035$ m^3/s or 35 dm^3/s

3. A horizontal nozzle reduces from 100 mm bore diameter at inlet to 50 mm at exit. It carries liquid of density 1000 kg/m^3 at a rate of 0.05 m^3/s. The pressure at the wide end is 500 kPa (gauge). Calculate the pressure at the narrow end neglecting friction. (196 kPa)

   $A_1 = \pi D_1^2/4 = \pi(0.1)^2/4 = 7.854 \times 10^{-3}$ m^2
   $A_2 = \pi D_2^2/4 = \pi(0.05)^2/4 = 1.9635 \times 10^{-3}$ m^2
   $u_1 = Q/A_1 = 0.05/7.854 \times 10^{-3} = 6.366$ m/s
   $u_2 = Q/A_2 = 0.05/1.9635 \times 10^{-3} = 25.46$ m/s
   $p_1 + \rho u_1^2/2 = p_2 + \rho u_2^2/2$
   $500 \times 10^3 + 100 \times (6.366)^2/2 = p_2 + 1000 \times (25.46)^2/2 \qquad p_2 = 196$ kPa

4. A pipe carries oil of density 800 kg/m^3. At a given point (1) the pipe has a bore area of 0.005 m^2 and the oil flows with a mean velocity of 4 m/s with a gauge pressure of 800 kPa. Point (2) is further along the pipe and there the bore area is 0.002 m^2 and the level is 50 m above point (1). Calculate the pressure at this point (2). Neglect friction. (374 kPa)

   $800 \times 10^3 + 800 \times 4^2/2 + 0 = p_2 + 800\ 10^2/2 + 800 \times 9.81 \times 50$
   $p_2 = 374$ kPa

5. A horizontal nozzle has an inlet velocity u_1 and an outlet velocity u_2 and discharges into the atmosphere. Show that the velocity at exit is given by the following formulae.

$u_2 = \{2\Delta p/\rho + u_1^2\}^{½}$ and $u_2 = \{2g\Delta h + u_1^2\}^{½}$

$p_1 + \rho u_1^2/2 + \rho g z_1 = p_2 + \rho u_2^2/2 + \rho g z_2$ $z_1 = z_2$

$p_1 + \rho u_1^2/2 = p_2 + \rho u_2^2/2$

$p_1 - p_2 = (\rho/2)(u_2^2 - u_1^2)$ $2(p_1 - p_2)/\rho = (u_2^2 - u_1^2)$

$u_2 = \sqrt{(2\Delta p/\rho + u_1^2)}$

Substitute $p = \rho g h$ and $u_2 = \sqrt{\{2g\Delta h + u_1^2\}^{½}}$

SELF ASSESSMENT EXERCISE 2

1. A pipe carries oil at a mean velocity of 6 m/s. The pipe is 5 km long and 1.5 m diameter. The surface roughness is 0.8 mm. The density is 890 kg/m^3 and the dynamic viscosity is 0.014 N s/m^2. Determine the friction coefficient from the Moody chart and go on to calculate the friction head h_f.

L = 5000 m d = 1.5 m k = 0.08 mm ρ = 890 kg/m^3 μ = 0.014 Ns/m^2 u = 6 m/s

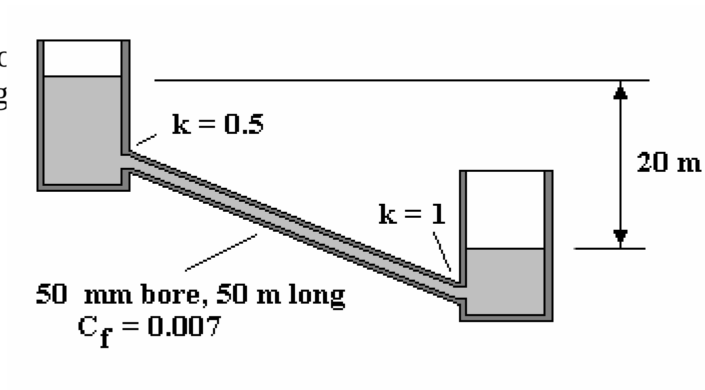
$\varepsilon = k/D0.8/1500 = 533 \times 10^{-6}$

$R_e = \rho u D/\mu = 890 \times 6 \times 1.5/0.014 = 572 \times 10^3$

From the Moody Chart $C_f = 0.0045$

$h_f = 4\,C_f\,Lu^2/(2\,g\,d) = 110$ m

2. The diagram shows a tank draining into anc pressure is both zero on the surface on a larg the diagram. (Ans. 7.16 dm^3/s)



$h_1 + z_1 + u_1^2/2g = h_2 + z_2 + u_2^2/2g$

$0 + z_1 + 0 = 0 + 0 + 0 + h_L$

$h_L = 20$

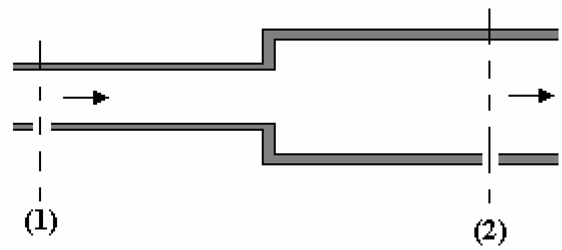
$20 = 4\,C_f\,Lu^2/(2\,g\,d)$ + minor losses

$20 = \{4 \times 0.007 \times 50u^2/(2 \times 9.81 \times 0.05)\} + 0.5\,u^2/(2 \times 9.81) + u^2/(2 \times 9.81) = 29.5\,u^2/(2 \times 9.81)$

$u = 20(2 \times 9.81)/\,29.5 = 3.65$ m/s

$A = 0.00196$ m^2 $Q = Au = 0.00196 \times 3.65 = 0.00716$ m^3/s or 7.16 dm^3/s

3. Water flows through the sudden pipe expansion shown below at a flow rate of 3 dm^3/s. Upstream of the expansion the pipe diameter is 25 mm and downstream the diameter is 40 mm. There are pressure tappings at section (1), about half a diameter upstream, and at section (2), about 5 diameters downstream. At section (1) the gauge pressure is 0.3 bar.



Evaluate the following.

(i) The gauge pressure at section (2) (0.387 bar)

(ii) The total force exerted by the fluid on the expansion. (-23 N)

$u_1 = Q/A_1 = 0.003/(\pi \text{ x } 0.0125^2) = 6.11$ m/s $\quad u_2 = Q/A_2 = 0.003/(\pi \text{ x } 0.02^2) = 2.387$ m/s

h_L (sudden expansion) $= (u_1^2 - u_2^2)/2g = 0.7067$ m

$u_1^2/2g + h_1 = u_2^2/2g + h_2 + h_L$

$h_1 - h_2 = 2.387^2/2g – 6.11^2/2 + 0.7067 = -0.9065$

$p_1 - p_2 = \rho g(h_1 - h_2) = 997 \text{ x } 9.81 \text{ x } (-0.9065) = -8866$ kPa

$p_1 = 0.3$ bar $\quad p_2 = 0.3886$ bar

$p_1 A_1 - + \rho Q u_1 = p_2 A_2 - + \rho Q u_2 + F$

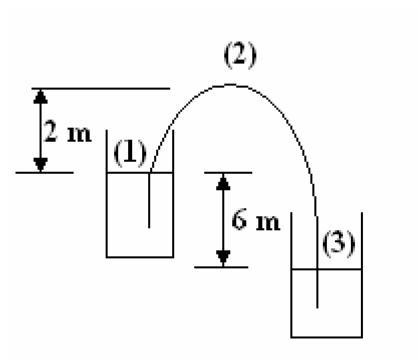
$0.3 \text{ x}10^5 \text{x } 0.491 \text{ x}10^{-3} + 997 \text{ x } 0.003 \text{ x } 6.11 = 0.38866 \text{ x}10^5 \text{ x } 1.257 \text{ x}10^{-3} + 997 \text{ x } 0.003 \text{ x } 2.387 + F$

$F = -23$ N

If smooth $h_L = 0 \quad h_1 - h_2 = -1.613$ and $p_2 = 0.45778$ bar

4. A tank of water empties by gravity through a siphon into a lower tank. The difference in levels is 6 m and the highest point of the siphon is 2 m above the top surface level. The length of pipe from the inlet to the highest point is 3 m. The pipe has a bore of 30 mm and length 11 m. The friction coefficient for the pipe is 0.006.The inlet loss coefficient K is 0.6.

Calculate the volume flow rate and the pressure at the highest point in the pipe.



Total length = 11 m $C_f = 0.006$

Bernoulli between (1) and (3)

$h_1 + u_1^2/2g + z_1 = h_3 + u_3^2/2g + z_3 + h_L$

$0 + 6 + 0 = 0 + 0 + 0 + h_L$ $h_L = 6$

h_L= Inlet + Exit + pipe

$6 = 0.6\, u^2/2g + u^2/2g + (4 \times 0.006 \times 11/0.03)\, u^2/2g$

$6 = 0.6\, u^2/2g + u^2/2g + 8.8\, u^2/2g = 10.4\, u^2/2g$

$u = 3.364$ m/s

$Q = A\,u = (\pi \times 0.03^2/4) \times 3.364$ $Q = 0.002378\ m^3/s$

Bernoulli between (1) and (2)

$h_1 + u_1^2/2g + z_1 = h_2 + u_2^2/2g + z_2 + h_L$

$0 + 0 + 0 = h_2 + 2 + u^2/2g + h_L$

h_L= Inlet + pipe $= 0.6\, u^2/2g + (3/11) \times 8.8\, u^2/2g$

$h_L = 0.6 \times 3.364^2/2g + (3/11) \times 8.8 \times 3.364/2g$

$h_L = 1.73$ m

$0 = h_2 + 2 + 3.364^2/2g + 1.73$

$h_2 = -4.31$ m

5. (Q5 1989)

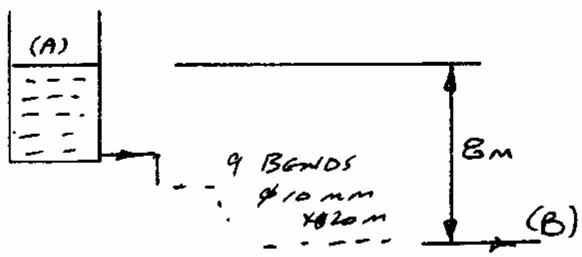
A domestic water supply consists of a large tank with a loss free-inlet to a 10 mm diameter pipe of length 20 m, that contains 9 right angles bends. The pipe discharges to atmosphere 8.0 m below the free surface level of the water in the tank.

Evaluate the flow rate of water assuming that there is a loss of 0.75 velocity heads in each bend and that friction in the pipe is given by the Blasius equation $C_f=0.079(Re)^{-0.25}$

The dynamic viscosity is 0.89×10^{-3} and the density is 997 kg/m^3.

($0.118\ dm^3/s$).

Q5 (Q5 1989)



$C_f = 0.079\, Re^{-.25}$

$Re = \rho u D/\mu$

$\rho = 997\ kg/m^3$

$\mu = 0.89 \times 10^{-3}\ Ns/m^2$

$Re = 997 \times u \times 0.01 / 0.89\times10^{-3} = 11202\, u$

$C_f = 0.079\,(11202u)^{-.25} = 7.679\times10^{-3}\, u^{-.25}$

$h_f = \frac{4 C_f}{2g}\frac{L u^2}{d} = \frac{4 \times 7.679\times10^{-3}\, u^{-.25} \times 20\, u^2}{2g \times 0.01}$

$h_f = 3.1311\, u^{1.75}$

LOSS IN BENDS $= 9 \times \frac{.75 u^2}{2g} = 0.344\, u^2$

BERNOULLI (A) → (B)

$h_A + z_A + \frac{u_A^2}{2g} = h_B + z_B + \frac{u_B^2}{2g} + h_L$

$0 + 8 + 0 = 0 + 0 + \frac{u^2}{2g} + .344u^2 + 3.1311u^{1.75}$

$8 = .395u^2 + 3.1311u^{1.75}$

SOLVE BY GUESSING OR NEWTON'S METHOD

$u = 1.5\ m/s$

$Q = Au$

$Q = 117.8\times10^{-6}\ m^3/s$

6. A pump A whose characteristics are given in table 1, is used to pump water from an open tank through 40 m of 70 mm diameter pipe of friction factor C_f=0.005 to another open tank in which the surface level of the water is 5.0 m above that in the supply tank.

Determine the flow rate when the pump is operated at 1450 rev/min. (7.8 dm^3/s)

It is desired to increase the flow rate and 3 possibilities are under investigation.

(i) To install a second identical pump in series with pump A.

(ii) To install a second identical pump in parallel with pump A.

(iii) To increase the speed of the pump by 10%.

Predict the flow rate that would occur in each of these situations.

Head-Flow Characteristics of pump A when operating at 1450 rev/min

Head/m	**9.75**	**8.83**	**7.73**	**6.90**	**5.50**	**3.83**
Flow Rate/(l/s)	**4.73**	**6.22**	**7.57**	**8.36**	**9.55**	**10.75**

Q6 (Q10 1989)

PIPE 40m × ϕ70 mm $C_f = 0.005$

$$h_f = \frac{4 \times 0.005 \times 40 u^2}{2g \times 0.07} = 0.582\, u^2$$

HEAD REQUIRED = h = LIFT + h_f + EXIT LOSS

$$h = 5 + 0.582u^2 + \frac{u^2}{2g} = 5 + 0.632u^2$$

PLOT PUMP HEAD H AGAINST Q

PLOT SYSTEM HEAD h AGAINST Q

FIND Q WHERE H = h

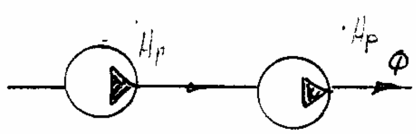
$h = 5 + 42672.2\,Q^2$

TABLE

H	9.75	8.83	7.73	6.9	5.5	3.83	m
Q	4.73	6.22	7.57	8.36	9.55	10.75	dm^3/s
u	1.23	1.62	1.97	2.17	2.48	2.79	m/s
h	5.88	6.52	7.25	7.75	8.89	9.93	m

FROM THE GRAPH A MATCHING POINT IS
$Q = 7.8\ dm^3/s$ $h = 7.4m$

WITH TWO PUMPS IN SERIES, EACH PUMP HAS SAME Q BUT HALF VALUE OF h

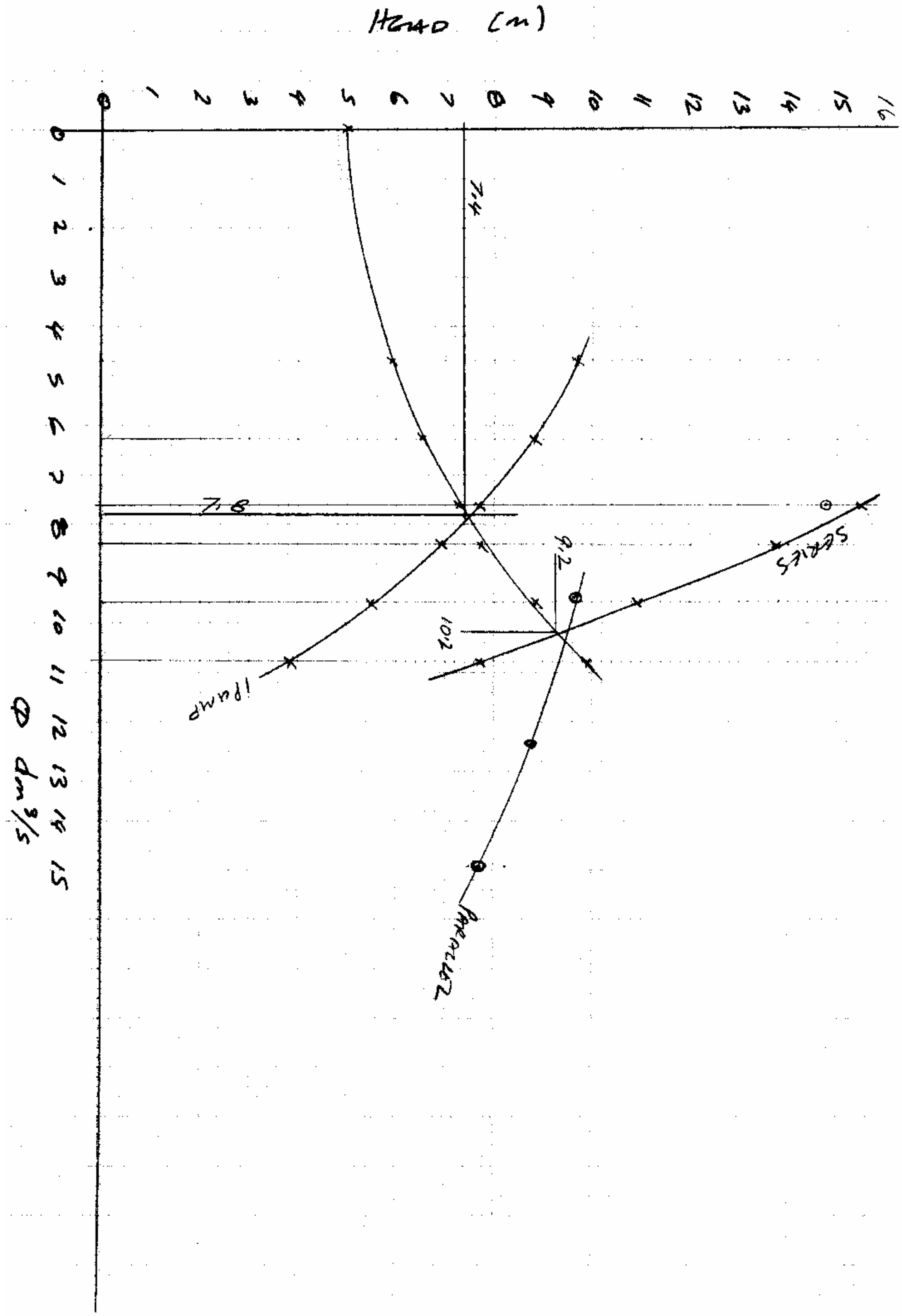


PLOT $2H_p - Q$
$h - Q$ ← REMAINS SAME

MATCHING POINT IS $h = 9.8m$ $Q = 10.3\ dm^3/s$
∴ FLOW IS INCREASED

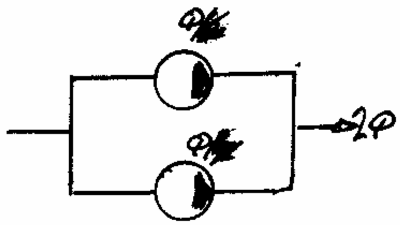
WITH TWO PUMPS IN PARALLEL, FLOW IS HALVED BUT SAME h

PLOT $H - Q$ MATCHING POINT IS
$h - Q/2$ $h = 9.6m$ $Q = 10\ dm^3/s$



HEAD (m)
Q dm³/s
7.4
7.8
9.2
10.2
PUMP
SERIES
PARALLEL

PUMPS IN PARALLEL



SAME H – Q
FOR PUMP PLOT 2Q – H

H	9.75	8.83	7.73	6.9	5.5	3.83
2Q	9.46	12.44	15.14	16.72	19.1	21.5

FROM GRAPH, THE MATCHING POINT IS ALMOST THE SAME AS FOR SERIES PUMPS.

SPEED CONTROL

$$Q_2 = Q_1 \left(N_1 / N_2 \right)^2$$

FOR 10% INCREASE

$N_2 = 1.1 N_1$

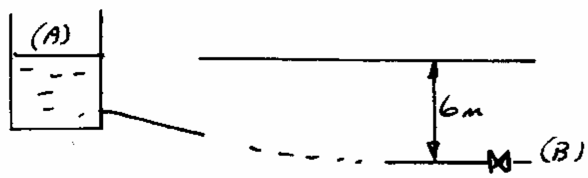
$Q_2 = .826\, Q_1$

REDUCED FLOW SAME HEAD

7. A steel pipe of 0.075 m inside diameter and length 120 m is connected to a large reservoir. Water is discharged to atmosphere through a gate valve at the free end, which is 6 m below the surface level in the reservoir. There are four right angle bends in the pipe line. Find the rate of discharge when the valve is fully open. (ans. 8.3 dm^3/s).The kinematic viscosity of the water may be taken to be 1.14 x 10^{-6} m^2/s. Use a value of the friction factor C_f taken from table 2 which gives C_f as a function of the Reynolds number Re and allow for other losses as follows.
   at entry to the pipe 0.5 velocity heads.
   at each right angle bend 0.9 velocity heads.
   for a fully open gate valve 0.2 velocity heads.

Re x 10^5	0.987	1.184	1.382
C_f	0.00448	0.00432	0.00419

Q7 (Q12 1985)



BERNOULLI (A) → (B)

$$h_A + z_A + \frac{u_A^2}{2g} = h_B + z_B + \frac{u_B^2}{2g} + h_L$$

$$0 + 6 + 0 = 0 + 0 + \frac{u_B^2}{2g} + h_L$$

$$h_L = 6 - \frac{u_B^2}{2g}$$

$$h_L = \frac{4 C_f L u^2}{2g d} + \frac{0.5 u^2}{2g} + \frac{4 \times .9 u^2}{2g} + \frac{.2 u^2}{2g}$$

pipe, entry, Bends, gate

$$h_L = \frac{u^2}{2g}\left\{\frac{4 C_f L}{d} + 4.3\right\} = 6 - \frac{u_B^2}{2g}$$

$$6 = \frac{u^2}{2g}\left\{\frac{4 C_f L}{d} + 5.3\right\} = \frac{u^2}{2g}\left\{\frac{4 C_f \times 120}{0.075} + 5.3\right\}$$

$$6 = \frac{u^2}{2g}\left\{6400 C_f + 5.3\right\}$$

$$117.72 = u^2\left\{6400 C_f + 5.3\right\}$$

$$Re = \frac{u d}{\nu} = \frac{u \times 0.075}{1.14 \times 10^{-6}} = 65789.5\, u$$

$$u = \frac{Re}{65789.5}$$

$$117.72 = \frac{R_e^2}{4.3283\times10^9}\left\{6400 C_f + 5.3\right\}$$

$$509.522\times10^9 = R_e^2\left\{6400 C_f + 5.3\right\}$$

SOLVE BY TRIAL & ERROR OR PLOTTING DATA
A SUITABLE GRAPH

A PAIR OF POINTS THAT FIT ARE

$R_e = 1.24\times10^5 \quad C_f = 0.00428$

$1.24\times10^5 = 65789.5\,u$

$u = 1.885$ m/s

DISCHARGE $Q = Au = \frac{\pi \times .075^2}{4} \times 1.885$

$Q = 8.33$ dm^3/s

8. (i) Sketch diagrams showing the relationship between Reynolds number, Re, and friction factor, C_f , for the head lost when oil flows through pipes of varying degrees of roughness. Discuss the importance of the information given in the diagrams when specifying the pipework for a particular system.

(ii) The connection between the supply tank and the suction side of a pump consists of 0.4 m of horizontal pipe , a gate valve one elbow of equivalent pipe length 0.7 m and a vertical pipe down to the tank.

If the diameter of the pipes is 25 mm and the flow rate is 30 l/min, estimate the maximum distance at which the supply tank may be placed below the pump inlet in order that the pressure there is no less than 0.8 bar absolute. (Ans. 1.78 m)

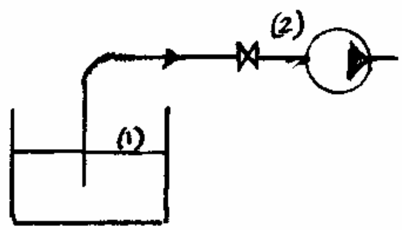
The fluid has kinematic viscosity 40×10^{-6} m^2/s and density 870 kg/m^3.

Assume
(a) for laminar flow C_f =16/(Re) and for turbulent flow C_f = 0.08/(Re)$^{0.25}$.

(b) head loss due to friction is $4C_f V^2L/2gD$ and due to fittings is $KV^2/2g$.

where K=0.72 for an elbow and K=0.25 for a gate valve.

What would be a suitable diameter for the delivery pipe ?



$\nu = 40\times10^{-6}\ m^2/s$

$Q = 30/60 = 0.5\ dm^3/s$

$u = Q/A = \dfrac{0.5\times10^{-3}}{\pi\times0.0125^2} = 1.0186\ m/s$

$Re = \dfrac{\rho u D}{\mu} = \dfrac{uD}{\nu} = \dfrac{1.0186\times0.025}{40\times10^{-6}}$

$Re = 636.6$

LAMINAR FLOW $C_f = 16/Re = 0.0251$

BERNOULLI

$h_1 + z_1 + \dfrac{u_1^2}{2g} = h_2 + z_2 + \dfrac{u_2^2}{2g} + loss$

$h_1 + 0 + 0 = h_2 + z_2 + \dfrac{u_2^2}{2g} + loss$

$h_1 = p_1/\rho g = \dfrac{1.013\times10^5}{997\times9.81} = 11.869\,m$

$h_2 = p_2/\rho g = \dfrac{0.8\times10^5}{997\times9.81} = 9.373\,m$

LOSS = h_f + BEND + GATE + INLET

$h_f = \dfrac{4C_f L u^2}{2gd}$ $\quad L = .4 + .7 + z$ $\quad L = 1.10 + z$

$h_f = \dfrac{4\times.0251\times(1.1+z)\times1.0186^2}{2g\times0.025} = 0.2124(1.1+z)$

$h_L = h_f + \dfrac{.72\times1.0186^2}{2g} + \dfrac{.25u^2}{2g} + ?$ NO DATA FOR INLET IGNORE

$h_L = h_f + 0.0381 + 0.0132 = h_f + 0.05129$

$h_L = .2124(1.1 + z) + 0.05129$

$h_L = 0.2124z + 0.2336 + 0.05129$

$h_L = 0.2124Z + 0.285$

$h_1 = h_2 + z_2 + \frac{u_2^2}{2g} + h_L \qquad z_2 \equiv z$

$11.869 = 9.373 + Z + \frac{1.0186^2}{2g} + .2124Z + .285$

$11.869 = 9.373 + Z + 0.0523 + .2124Z + .285$

$2.159 = Z + .2124Z$

$2.159 = 1.2124Z$

$Z = 1.78\,m$

CONVENTION $D_1 = 3/4\, D_2$

D_1 = SUCTION PIPE
D_2 = DELIVERY PIPE

HELPS PREVENT CAVITATION

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 2 – APPLICATIONS OF BERNOULLI

SELF ASSESSMENT EXERCISE 3

Take the density of water to be 997 kg/m^3 throughout unless otherwise stated.

1. A Venturi meter is 50 mm bore diameter at inlet and 10 mm bore diameter at the throat. Oil of density 900 kg/m^3 flows through it and a differential pressure head of 80 mm is produced. Given C_d = 0.92, determine the flow rate in kg/s.

$$Q = C_d A_1 \sqrt{\frac{2\Delta p}{\rho(r^2 - 1)}} \qquad r = A_1/A_2 = 25 \quad \Delta p = \rho g \Delta h = 900 \text{ x } 9.81 \text{ x } 0.08 = 706.3 \text{ x } 10^3 \text{ Pa}$$

$$Q = \frac{0.92 \text{ x } \pi \text{ x } 0.05^2}{4}\sqrt{\frac{2 \text{ x } 706300}{900(25^2 - 1)}} = 909.59 \text{ x } 10^{-6} \text{ m}^3/\text{s} \qquad m = \rho Q = 0.0815 \text{ kg/s}$$

2. A Venturi meter is 60 mm bore diameter at inlet and 20 mm bore diameter at the throat. Water of density 1000 kg/m^3 flows through it and a differential pressure head of 150 mm is produced. Given C_d = 0.95, determine the flow rate in dm^3/s.

$$Q = C_d A_1 \sqrt{\frac{2\rho g \Delta h}{\rho(r^2 - 1)}} \qquad r = 9$$

$$Q = \frac{0.95 \text{ x } \pi \text{ x } 0.06^2}{4}\sqrt{\frac{2 \text{ x } 1000 \text{ x } 9.81 \text{ x } 0.15}{1000(9^2 - 1)}} = 515 \text{ x } 10^{-6} \text{ m}^3/\text{s or } 0.515 \text{ dm3/s}$$

3. Calculate the differential pressure expected from a Venturi meter when the flow rate is 2 dm^3/s of water. The area ratio is 4 and C_d is 0.94. The inlet c.s.a. is 900 mm^2.

$$Q = 0.002 = C_d A_1 \sqrt{\frac{2\Delta p}{\rho(r^2 - 1)}} \qquad r = 4$$

$$0.002 = 0.94 \text{ x } 900\text{x}10^{-6}\sqrt{\frac{2\Delta p}{1000(4^2 - 1)}} \qquad 2.3641 = \sqrt{\frac{\Delta p}{7500}} \qquad 5.589 = \frac{\Delta p}{7500}$$

Δp = 41916 Pa

4. Calculate the mass flow rate of water through a Venturi meter when the differential pressure is 980 Pa given C_d = 0.93, the area ratio is 5 and the inlet c.s.a. is 1000 mm^2.

r = 5

$$m = \rho C_d A_1 \sqrt{\frac{2\Delta p}{\rho(r^2 - 1)}} = 1000 \text{ x } 0.93 \text{ x } 1000 \text{ x } 10^{-6}\sqrt{\frac{2 \text{ x } 980}{1000(5^2 - 1)}} = 0.2658 \text{ kg/s}$$

5. Calculate the flow rate of water through an orifice meter with an area ratio of 4 given C_d is 0.62, the pipe area is 900 mm^2 and the d.p. is 586 Pa. (ans. 0.156 dm^3/3).

r = 4

$$Q = C_d A_1 \sqrt{\frac{2\Delta p}{\rho(r^2 - 1)}} = 900 \text{ x } 10^{-6} \text{ x } 0.62\sqrt{\frac{2 \text{ x } 586}{1000(4^2 - 1)}} = 155.9 \text{ x } 10^{-6} \text{ m}^3/\text{s}$$

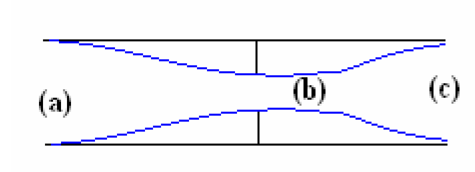
6. Water flows at a mass flow rate 0f 0.8 kg/s through a pipe of diameter 30 mm fitted with a 15 mm diameter sharp edged orifice.

There are pressure tappings (a) 60 mm upstream of the orifice, (b) 15 mm downstream of the orifice and (c) 150 mm downstream of the orifice, recording pressure p_a, p_b and p_c respectively. Assuming a contraction coefficient 0f 0.68, evaluate

(i) the pressure difference (p_a - p_b) and hence the discharge coefficient.
(ii)the pressure difference (p_b - p_c) and hence the diffuser efficiency.
(iii) the net force on the orifice plate.

d_o = 15 mm d_j = jet diameter $Cc = 0.68 = (A_b/A_o) = (d_b/15)^2$ d_b = 12.37 mm

No Friction between (a) and (b)
so   Cv = 1.0   Cd = Cc Cv = Cc



$$m = \rho A_o C_d \sqrt{\frac{2\Delta p}{\rho\left(1 - Cc^2\beta^4\right)}} \qquad \beta = 15/30 = 0.5$$

$$0.8 = 997\,\frac{\pi \times 0.015^2}{4} \times 0.68 \sqrt{\frac{2\Delta p}{997\left(1 - 0.68^2 \times 0.5^4\right)}}$$

$$6.677 = \sqrt{\frac{\Delta p}{484}} \qquad \Delta p = p_a - p_b = 21581\,\text{Pa}$$

Note the same answer may be obtained by applying Bernoulli's equation between (a) and (b)
Now apply Bernoulli's equation between (b) and (c)

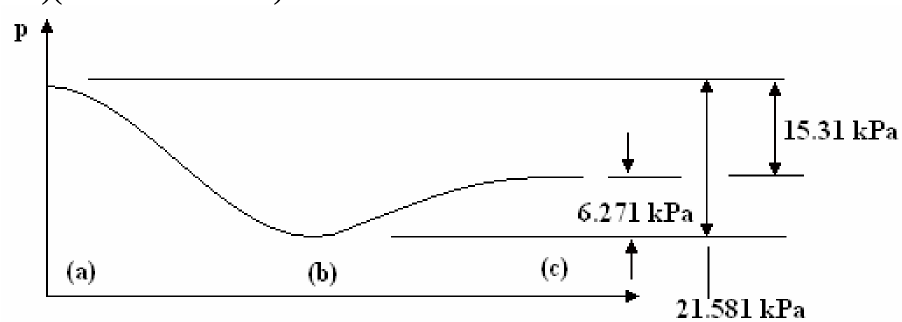
$p_b + \rho u_b^2/2 = p_c + \rho u_c^2/2 + \text{loss}$ $\text{loss} = \rho (u_b - u_c)^2/2$

$$u_b = \frac{m}{\rho A_b} = \frac{0.8}{997 \times \pi \times 0.01237^2/4} = 6.677 \text{ m/s}$$

$$u_c = \frac{m}{\rho A_c} = \frac{0.8}{997 \times \pi \times 0.03^2/4} = 1.135 \text{ m/s}$$

loss = 997 (6.677 – 1.135)^2/2 = 15311 Pa

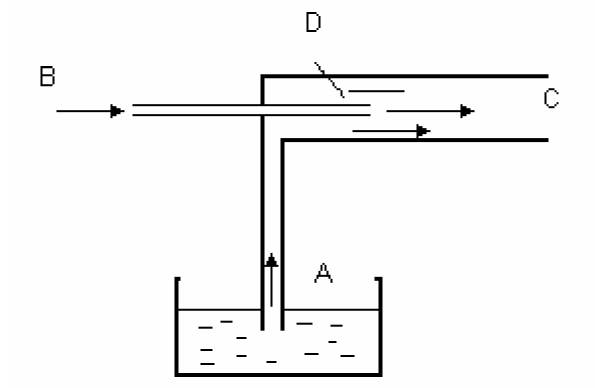
$p_c - p_b = (997/2)(6.677^2 - 1.135^2) - 15311 = 6271$ Pa



η = 15.31/21.581 = 71% Energy recovered = 6.27/21.58 = 29%

Force = $\pi \times 0.03^2/4 \times 15310$ = 10.8 N (on the control section)

7. The figure shows an ejector (or jet pump) which extracts 2×10^{-3} m^3/s of water from tank A which is situated 2.0 m below the centre-line of the ejector. The diameter of the outer pipe of the ejector is 40 mm and water is supplied from a reservoir to the thin-walled inner pipe which is of diameter 20 mm. The ejector discharges to atmosphere at section C.



Evaluate the pressure p at section D, just downstream of the end of pipe B, the velocity in pipe B and the required height of the free water level in the reservoir supplying pipe B. (-21.8 kPa gauge, 12.9 m/s, 6.3 m).

It may be assumed that both supply pipes are loss free.

$A_B = \pi \times 0.02^2/4 = 314.2 \times 10^{-6}\ m^2$ $\qquad A_C = \pi \times 0.04^2/4 = 1256 \times 10^{-6}\ m^2$

$A_D = A_C - A_B = 942.48 \times 10^{-6}\ m^2$ $\qquad u_D = Q_D/A_D = 0.002 \times 10^{-6}/0.94248 \times 10^{-6} = 2.122$ m/s

Apply Bernoulli from A to D $\qquad h_A + \frac{u_A^2}{2g} + z_A = h_D + \frac{u_D^2}{2g} + z_D$

$$h_D = -\frac{u_D^2}{2g} - z_D = -\frac{2.122^2}{2g} - 2 = -2.23\,\text{m} \qquad p_D = \rho g h_D = -21.8\ \text{kPa}$$

Next apply the conservation of momentum between the points where B and D join and the exit at C. This results in the following.

$$Q_B^2\left\{\frac{1}{A_B} - \frac{1}{A_C}\right\} - \frac{2Q_BQ_D}{A_C} + \frac{p_BA_C}{\rho} + Q_D^2\left\{\frac{1}{A_D} - \frac{1}{A_C}\right\} = 0$$

$$a = \left\{\frac{1}{A_B} - \frac{1}{A_C}\right\} = \left\{\frac{10^6}{314.2} - \frac{10^6}{1256}\right\} = 2386$$

$$b = \frac{2Q_D}{A_C} = \frac{2 \times 2\text{x}10^{-3}}{1.256 \times 10^{-3}} = 3.1847$$

$$c = \frac{p_BA_C}{\rho} + Q_D^2\left\{\frac{1}{A_D} - \frac{1}{A_C}\right\} = \frac{-21800 \times 1256 \times 10^{-6}}{1000} + \left(2 \times 10^{-3}\right)^2\left\{\frac{10^6}{942.48} - \frac{10^6}{1256}\right\}$$

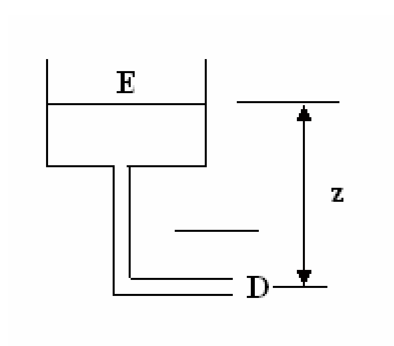
$$c = -27.38\text{x}10^{-3} + 1.06 \times 10^{-3} = -26.32 \times 10^{-3}$$

$$aQ_B^2 + bQ_B + c = 0 \qquad Q_B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$Q_B = \frac{-3.1847 \pm \sqrt{3.1847^2 + 4 \times 2386 \times 0.02632}}{2 \times 2386}$$

$$Q_B = \frac{-3.1847 \pm 16.17}{2 \times 2386} = -0.00272 \text{ or } 0.00405\,m^3/s$$

$u_B = Q_B/A_B = 12.922$ m/s



Apply Bernoulli between E and point D

$z = h_B + u_B^2/2g = 6.282$ m

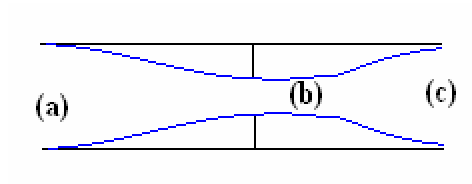
8. Discuss the use of orifice plates and venturi-meters for the measurement of flow rates in pipes.

Water flows with a mean velocity of 0.6 m/s in a 50 mm diameter pipe fitted with a sharp edged orifice of diameter 30 mm. Assuming the contraction coefficient is 0.64, find the pressure difference between tappings at the vena contracta and a few diameters upstream of the orifice, and hence evaluate the discharge coefficient.
Estimate also the overall pressure loss caused by the orifice plate.
It may be assumed that there is no loss of energy upstream of the vena contracta.

d_o = 30 mm d_j = jet diameter = d_b Cc = 0.64 = $(A_b/A_o) = (d_b/30)^2$ d_b = 24 mm

$Q = 0.6 \times \pi \times 0.05^2/4 = 0.001178$ m^3/s

No Friction between (a) and (b)
so   Cv = 1.0   Cd = Cc Cv = Cc



$$Q = A_o C_d \sqrt{\frac{2\Delta p}{\rho(1 - Cc^2\beta^4)}} \qquad \beta = 30/50 = 0.6$$

$$0.001178 = \frac{\pi \times 0.03^2}{4} \times 0.64 \sqrt{\frac{2\Delta p}{997(1 - 0.64^2 \times 0.6^4)}}$$

$$2.06 = \sqrt{\frac{\Delta p}{472}} \qquad \Delta p = p_a - p_b = 3200 \text{ Pa}$$

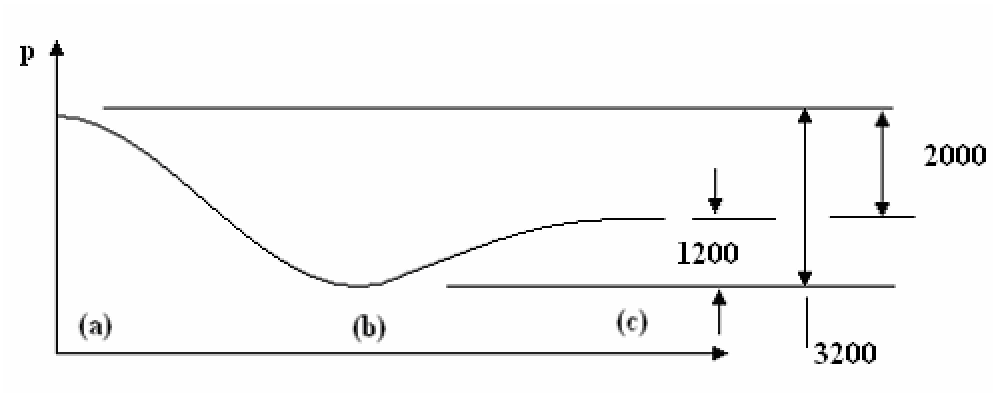
Note the same answer may be obtained by applying Bernoulli's equation between (a) and (b)
Now apply Bernoulli's equation between (b) and (c)

$p_b + \rho u_b^2/2 = p_c + \rho u_c^2/2 + \text{loss}$ $\text{loss} = \rho (u_b - u_c)^2/2$

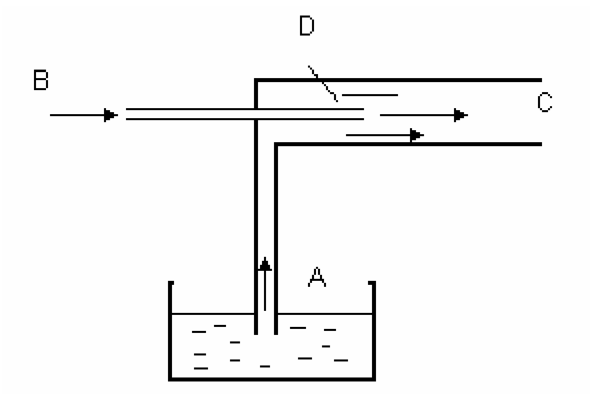
$$u_b = \frac{Q}{A_b} = \frac{0.001178}{\pi \times 0.024^2/4} = 2.6 \text{ m/s}$$

$$u_c = \frac{q}{A_c} = \frac{0.001178}{\pi \times 0.035^2/4} = 0.6 \text{ m/s}$$

$\text{loss} = 997 (2.6 - 0.6)^2/2 = 2000$ Pa



9. The figure shows an ejector pump BDC designed to lift 2 x 10^{-3} m^3/s of water from an open tank A, 3.0 m below the level of the centre-line of the pump. The pump discharges to atmosphere at C.



The diameter of thin-walled inner pipe 12 mm and the internal diameter of the outer pipe of the is 25 mm. Assuming that there is no energy loss in pipe AD and there is no shear stress on the wall of pipe DC, calculate the pressure at point D and the required velocity of the water in pipe BD.

Derive all the equations used and state your assumptions.

$A_B = \pi$ x $0.012^2/4 = 113.1$ x 10^{-6} m^2 $\quad$ $A_C = \pi$ x $0.025^2/4 = 491$ x 10^{-6} m^2

$A_D = A_C - A_B = 377.8$ x 10^{-6} m^2 $\quad$ $u_D = Q_D/A_D = 0.002$ x $10^{-6}/377.8$ x $10^{-6} = 5.294$ m/s

Apply Bernoulli from A to D $\quad$ $h_A + \frac{u_A^2}{2g} + z_A = h_D + \frac{u_D^2}{2g} + z_D$

$h_D = -\frac{u_D^2}{2g} - z_D = -\frac{5.294^2}{2g} - 3 = -4.429\text{ m}$ $\quad$ $p_D = \rho g h_D = -43.4$ kPa

Next apply the conservation of momentum between the points where B and D join and the exit at C. This results in the following.

$$Q_B^2\left\{\frac{1}{A_B} - \frac{1}{A_C}\right\} - \frac{2Q_BQ_D}{A_C} + \frac{p_BA_C}{\rho} + Q_D^2\left\{\frac{1}{A_D} - \frac{1}{A_C}\right\} = 0$$

$$a = \left\{\frac{1}{A_B} - \frac{1}{A_C}\right\} = \left\{\frac{10^6}{113.1} - \frac{10^6}{491}\right\} = 6805$$

$$b = \frac{-2Q_D}{A_C} = \frac{2\text{ x }2\text{x}10^{-3}}{491\text{ x }10^{-6}} = -8.149$$

$$c = \frac{p_BA_C}{\rho} + Q_D^2\left\{\frac{1}{A_D} - \frac{1}{A_C}\right\} = \frac{-43400\text{ x }491\text{ x }10^{-6}}{1000} + \left(2\text{ x }10^{-3}\right)^2\left\{\frac{10^6}{377.8} - \frac{10^6}{491}\right\}$$

$$c = -18.886\text{ x }10^{-3}$$

$$aQ_B^2 + bQ_B + c = 0 \qquad Q_B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$Q_B = \frac{8.149 \pm \sqrt{8.149^2 + 4\text{ x }6805\text{ x }0.0188}}{2\text{ x }6805}$$

$$Q_B = -0.001172\text{ or }0.002369\text{ m}^3/\text{s}$$

$u_B = Q_B/A_B = 20.95$ m/s

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 3 – BOUNDARY LAYERS

SELF ASSESSMENT EXERCISE 1

1. A smooth thin plate 5 m long and 1 m wide is placed in an air stream moving at 3 m/s with its length parallel with the flow. Calculate the drag force on each side of the plate. The density of the air is 1.2 kg/m^3 and the kinematic viscosity is 1.6 x 10^{-5} m^2/s.

$R_{ex} = u L/\nu = 3 \times 5/1.6 \times 10^{-5} = 937.5 \times 10^3$
$C_{DF} = 0.074\ R_{ex}^{-1/5} = 4.729 \times 10^{-3}$
Dynamic Pressure = $\rho u_o^2/2 = 1.2 \times 3^2/2 = 5.4$ Pa
$\tau_w = C_{DF}$ x dyn press = 0.0255 Pa
$R = \tau_w \times A = 0.0255 \times 5 = 0.128$ N

2. A pipe bore diameter D and length L has fully developed laminar flow throughout the entire length with a centre line velocity u_o. Given that the drag coefficient is given as $C_{Df} = 16/Re$ where $Re = \frac{\rho u_o D}{\mu}$, show that the drag force on the inside of the pipe is given as $R=8\pi\mu u_o L$ and hence the pressure loss in the pipe due to skin friction is
$p_L = 32\mu u_o L/D^2$

$C_{DF} = 16/R_e$
$R = \tau_w \times \rho u_o^2/2 = C_{DF} \times (\rho u_o^2/2) \times A$
$R = (16/R_e)(\rho u_o^2/2)\ A$
$R = (16\mu/\rho u_o D)(\rho u_o^2/2)\ \pi DL$
$R = (16\ \mu\ u_o\ \pi\ L/2) = 8\ \pi\ \mu\ u_o\ L$
$p_L = R/A = 8\ \pi\ \mu\ u_o\ L\ /(\pi D^2/4) = 32\ \mu\ u_o\ L/D^2$

SELF ASSESSMENT EXERCISE No. 2

1. Calculate the drag force for a cylindrical chimney 0.9 m diameter and 50 m tall in a wind blowing at 30 m/s given that the drag coefficient is 0.8.
The density of the air is 1.2 kg/m^3.

$C_D = 0.8 = 2R/(\rho u^2 A)$ $\quad R = 0.8\ (\rho u^2/2)A = 0.8\ (1.2 \times 30^2/2)(50 \times 0.9) = 19440$ N

2 Using the graph (fig.1.12) to find the drag coefficient, determine the drag force per metre length acting on an overhead power line 30 mm diameter when the wind blows at 8 m/s. The density of air may be taken as 1.25 kg/m^3 and the kinematic viscosity as 1.5 x 10^{-5} m^2/s. (1.8 N).

$R_e = u\ d/\nu = 8 \times 0.03/1.5 \times 10^{-5} = 16 \times 10^3$
From the graph
$C_D = 1.5$

$R = C_D\ (\rho u_o^2/2)A = 1.5\ (1.25 \times 8^2/2)(0.03 \times 1) = 1.8$ N

SELF ASSESSMENT EXERCISE No. 3

1. a. Explain the term Stokes flow and terminal velocity.
   b. Show that the terminal velocity of a spherical particle with Stokes flow is given by the formulae = $d^2g(\rho_s - \rho_f)/18\mu$. Go on to show that $C_D=24/R_e$

Stokes flow –for ideal fluid - no separation - Re <0.2
R = Buoyant weight = $(\pi d^3/6)g(\rho_s - \rho_f) = 3\pi d \mu u_t$
$u_t = d^2 g(\rho_s - \rho_f)/18\mu$ $R = C_D(\rho u_t^2/2)(\pi d^2/4)$ $C_D = 26\mu/(\rho u_t d) = 24/R_e$

2. Calculate the largest diameter sphere that can be lifted upwards by a vertical flow of water moving at 1 m/s. The sphere is made of glass with a density of 2630 kg/m^3. The water has a density of 998 kg/m^3 and a dynamic viscosity of 1 cP.

$C_D = (2/\rho u^2 A)R = \{(2 \times 4)/(\rho u^2 \pi d^2)\}(\pi d^3/6)g(\rho_s - \rho_f) = 21.38\, d$
Try Newton Flow first
D = 0.44/21.38 = 0.206 m R_e =(998 x 1 x 0.0206)/0.001 = 20530 therefore this is valid.

3. Using the same data for the sphere and water as in Q2, calculate the diameter of the largest sphere that can be lifted upwards by a vertical flow of water moving at 0.5 m/s. (5.95 mm).

$C_D = 85.52\, d$ Try Newton Flow
D = 0.44/85.52 = 0.0051 R_e =(998 x 0.5 x 0.0051)/0.001 = 2567 therefore this is valid.

4. Using the graph (fig. 1.12) to obtain the drag coefficient of a sphere, determine the drag on a totally immersed sphere 0.2 m diameter moving at 0.3 m/s in sea water. The density of the water is 1025 kg/m^3 and the dynamic viscosity is 1.05 x 10^{-3} Ns/m^2.

$R_e = (\rho u d/\mu) = (1025 \times 0.3 \times 0.2/1.05 \times 10^{-3}) = 58.57 \times 10^3$ From the graph $C_D = 0.45$
$R = C_D(\rho u^2/2)A = 0.45(1025 \times 0.3^2/2)(\pi \times 0.2^2/4) = 0.65$ N

5. A glass sphere of diameter 1.5 mm and density 2 500 kg/m^3 is allowed to fall through water under the action of gravity. The density of the water is 1000 kg/m^3 and the dynamic viscosity is 1 cP.
   Calculate the terminal velocity assuming the drag coefficient is $C_D = 24 R_e^{-1}(1+ 0.15R_e^{0.687})$

C_D = F/(Area x Dynamic Pressure)

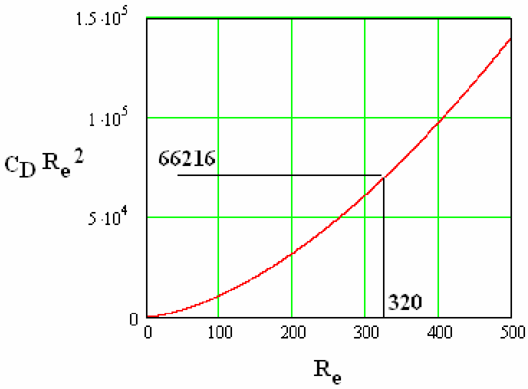
$$R = \frac{\pi d^3 g(\rho_s - \rho_f)}{6} \qquad C_D = \frac{\pi d^3 g(\rho_s - \rho_f)}{(\pi d^2/4)(\rho u^2/2)} \qquad C_D = \frac{4d^3 g(\rho_s - \rho_f)}{3\rho_f u^2 d^2}$$

Arrange the formula into the form $C_D R_e^2$ as follows.

$$C_D = \frac{4d^3 g(\rho_s - \rho_f)}{3\rho_f u^2 d^2} \times \frac{\rho_f \mu^2}{\rho_f \mu^2} = \frac{4d^3 g(\rho_s - \rho_f)\rho_f}{3\mu^2} \times \frac{\mu^2}{\rho_f^2 u^2 d^2} = \frac{4d^3 g(\rho_s - \rho_f)\rho_f}{3\mu^2} \times \frac{1}{R_e^2}$$

$$C_D R_e^2 = \frac{4d^3 g(\rho_s - \rho_f)\rho_f}{3\mu^2}$$ and evaluating this we get 66217

From $C_D = \frac{24}{R_e}\left[1 + 0.15 R_e^{0.687}\right]$ we may solve by plotting $C_D R_e^2$ against Re
From the graph R_e = 320 hence
$u = R_e \mu/\rho d = 0.215$ m/s



6. A glass sphere of density 2 690 kg/m^3 falls freely through water. Find the terminal velocity for a 4 mm diameter sphere and a 0.4 mm diameter sphere. The drag coefficient is

$$C_D = 8F/\{\pi d^2 \rho u^2\}$$

This coefficient is related to the Reynolds number as shown for low values of R_e.

R_e	15	20	25	30	35
C_D	3.14	2.61	2.33	2.04	1.87

The density and viscosity of the water is 997 kg/m^3 and 0.89 x 10^{-3} N s/m^2.
For Re >1000 C_D = 0.44

For a 4 mm sphere we might guess from the question that R_e is greater than 1000 and hence C_D = 0.44

$$R = \frac{\pi d^3 g(\rho_s - \rho_f)}{6} \qquad C_D = \frac{\pi d^3 g(\rho_s - \rho_f)}{(\pi\ d^2/4)(\rho\, u^2/2)} \qquad C_D = \frac{4d^3 g(\rho_s - \rho_f)}{3\rho_f\ u^2 d^2}$$

$$u = \sqrt{\frac{4d^3 g(\rho_s - \rho_f)}{3\rho_f\ C_D d^2}}$$ Putting in values ρ = 997 μ = 0.0089 d = 0.004 C_D = 0.44

u = 0.45 m/s Check R_e = ρud/μ = 2013 so this is valid

For the 0.4 mm sphere we might guess from the question that $C_D = \frac{8F}{\pi d^2 \rho u^2}$

$$C_D = \frac{8F}{\pi d^2 \rho u^2} x \frac{\rho\mu^2}{\rho\mu^2} = \frac{8F\rho}{\pi\,\mu^2} x \frac{\mu^2}{\rho^2 u^2 d^2} = \frac{8F\rho}{\pi\,\mu^2} x \frac{1}{R_e^2}$$

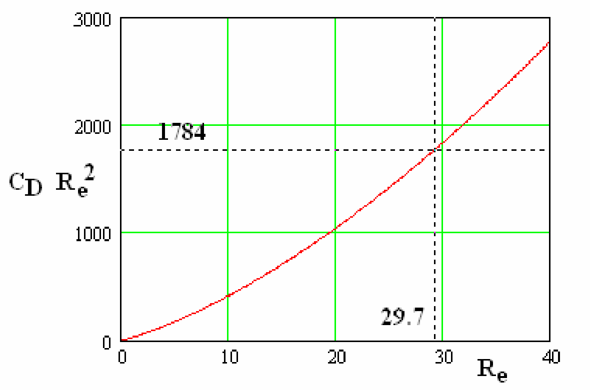
$$C_D R_e^2 = \frac{8F\rho}{\pi\,\mu^2} = 3.205x10^9 F$$

$$R = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$ (The buoyant weight)

For a 0.4 mm sphere F = 556.55 x 10^{-9} N

$$C_D R_e^2 = 3.205x10^9 F = 1784$$

Plot graph for the 0.4 mm sphere
The 0.4 mm sphere fits the table
u = R_e μ/ρd = 0.066 m/s



7. A glass sphere of diameter 1.5 mm and density 2 500 kg/m^3 is allowed to fall through water under the action of gravity. Find the terminal velocity assuming the drag coefficient is
$C_D = 24\ R_e^{-1}(1 + 0.15R_e^{0.687})$

$$R = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$ (The buoyant weight)

$$R = \frac{\pi x\ 0.0015^3 x\ 9.81(2500 - 997)}{6} = 26.056\ x\ 10^{-6}\ N$$

$$C_D = \frac{8F}{\pi d^2 \rho u^2} = \frac{8\ x\ 26x10^{-6}}{\pi\ (0.0015)^2 x\ 997\ x\ u^2} = \frac{29.578\ x\ 10^3}{u^2}$$

$$C_D = \frac{29.578\ x\ 10^3}{u^2} = \frac{24}{R_e}\left[1 + 0.15 R_e^{0.687}\right]$$

$$\frac{29.578\ x\ 10^3}{u^2} = \frac{24\ x\ 0.00089}{997\ u\ (0.0015)}\left[1 + 0.15\left(\frac{997\ u\ (0.0015)}{0.00089}\right)^{0.687}\right]$$

$$2.0709 = u\left[1 + 24.657u^{0.687}\right] = u + 24.657u^{1.687}$$

Solve for u and u = 0.215 m/s (plotting might be the best way)

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 3 – BOUNDARY LAYERS

SELF ASSESSMENT EXERCISE 4

1. The BL over a plate is described by $u/u_1=\sin(\pi y/2\delta)$. Show that the momentum thickness is 0.137δ.

$$\theta=\int_0^\delta\left[\frac{u}{u_1}\right]\left[1-\frac{u}{u_1}\right]dy=\int_0^\delta\left[\frac{u}{u_1}-\left(\frac{u}{u_1}\right)^2\right]dy=\int_0^\delta\left[\sin\left\{\frac{\pi y}{2\delta}\right\}-\left(\sin\left\{\frac{\pi y}{2\delta}\right\}\right)^2\right]dy$$

We need the trig identity $\sin^2 A = ½ - ½ \cos 2A$

$$\theta=\int_0^\delta\left[\sin\left\{\frac{\pi y}{2\delta}\right\}-\frac{1}{2}+\frac{1}{2}\left(\cos\left\{\frac{\pi y}{2\delta}\right\}\right)\right]dy$$

$$\theta=\left[-\frac{2\delta}{\pi}\cos\left\{\frac{\pi y}{2\delta}\right\}-\frac{y}{2}+\frac{\delta}{2\pi}\sin\left\{\frac{\pi y}{2\delta}\right\}\right]_0^\delta$$

$$\theta=\left[-0-\frac{\delta}{2}+0\right]-\left[-\frac{2\delta}{\pi}-0+0\right]=0.137\delta$$

2. The velocity profile in a laminar boundary layer on a flat plate is to be modelled by the cubic expression $u/u_1=a_0+a_1y + a_2y^2+a^3y^3$
where u is the velocity a distance y from the wall and u_1 is the main stream velocity.

Explain why a_0 and a_2 are zero and evaluate the constants a_1 and a_3 in terms of the boundary layer thickness δ.

Define the momentum thickness θ and show that it equals $39\delta/280$
Hence evaluate the constant A in the expression $\delta/x = A (R_{ex})^{-0.5}$
where x is the distance from the leading edge of the plate. It may be assumed without proof that the friction factor $C_f = 2\, d\theta/dx$

At y = 0, u = 0 so it follows that $a_o = 0$

$d^2u/dy^2 = 0$ @ y = 0 so $a_2 = 0$. Show for yourself that this is so.

The law is reduced to $u/u_l = a_ly + a_3y^3$
At $y = \delta$, $u = u_1$ so $1 =a_1\delta + 3a_3 \delta^2$
hence $a_l =(1-a_3\delta^3)/\delta$

Now differentiate and $du/dy = u_l (a_1 + 3a_3y^2)$
at $y = \delta$, du/dy is zero so $0 =a_1+ 3a_3 \delta^2$ so $a_1 = -3a_3\delta^2$

Hence by equating $a_1 = 3/2\delta$ and $a_3 = -1/2\delta^3$

Now we can write the velocity distribution as $u/u_1=3y/2\delta - (y/\delta)^3/2$

and $du/dy = u_1 \{3/2\delta + 3y^2/2\delta^3\}$

If we let $y/\delta = \eta$ $u/u_1=\{3\eta/2 + (\eta)^3/2\}$

The momentum thickness is

$$\theta = \int_0^{\delta}\left[\frac{u}{u_1}\right]\left[1-\frac{u}{u_1}\right]dy \text{ but } dy = \delta d\eta$$

$$\theta = \int_0^{1}\left[\frac{3\eta}{2}-\frac{\eta^3}{2}\right]\left[1-\frac{3\eta}{2}-\frac{\eta^3}{2}\right]d\eta$$

Integrating gives: $$\theta = \delta\left[\frac{3\eta^2}{4}-\frac{\eta^4}{8}-\frac{9\eta^3}{12}-\frac{\eta^7}{28}+\frac{3\eta^5}{10}\right]$$

Between the limits $\eta = 0$ and $\eta = 1$ this evaluates to $\theta = 39\delta/280$

Now must first go back to the basic relationship. $du/dy = u_1 \{3/2\delta + 3y^2/2\delta^3\}$

At the wall where $y = 0$ the shear stress is

$$\tau_o = \mu\, du/dy = \mu\, u_l \{3/2\delta + 3y^2/2\delta^3\} = (\mu\, u_l/\delta)\, \delta\, [(3/2\delta) + 3y^2/2\delta^3]$$

Putting $y/\delta = \eta$ we get

$$\tau_o = (\mu\, u_l/\delta)\, \delta\, [(3/2\delta) + 3\delta^2/2\delta]$$

$$\tau_o = (\mu\, u_l/\delta)\, [(3/2) + 3\delta^2/2]$$

at the wall $\eta = 0$ $$\tau_o = (\mu\, u_l/\delta)\,(3/2) \text{...... ....(2.1)}$$

The friction coefficient C_f is always defined as

$$C_f = 2\tau_o/(\rho\, u_2^2) \text{............. (2.2)}$$

It has been shown elsewhere that $C_f = 2d\theta/dx$. The student should search out this information from test books.

Putting $\theta = 39\delta/280$ (from the last example) then

$$C_f = 2d\theta/dx = (2 \times 39/280)\, d\delta/dx \quad \text{..............(2.3)}$$

Equating (2.2) and (2.3) gives

$$\tau_o = (\rho u1^2)(39/280) d\delta/dx \text{ ....(2.4)}$$

Equating (2.1) and (2.4) gives

$$(\rho u_1^2)(39/280) d\delta/dx = (\rho u/\delta)(3/2)$$

Hence $$(3 \times 280)/(2 \times 39)(\mu\, dx)/\rho u) = \delta d\delta$$

Integrating $$10.77(\mu x/\rho u_l) = \delta^2/2 + C$$

Since $\delta = 0$ at $x = 0$ (the leading edge of the plate) then $C=0$

Hence $$\delta = \{21.54\, \mu x/\rho u_l\}^{1/2}$$

Dividing both sides by x gives $\delta/x = 4.64(\mu/\rho u_l x)^{-1/2} = 4.64 Re^{-1/2}$

NB $R_{ex} = \rho u_l x/\mu$. and is based on length from the leading edge.

3.(a) The velocity profile in a laminar boundary layer is sometimes expressed in the formula

$$\frac{u}{u_1} = a_0 + a_1\frac{y}{\delta} + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4$$

where u_1 is the velocity outside the boundary layer and δ is the boundary layer thickness. Evaluate the coefficients a_0 to a_4 for the case when the pressure gradient along the surface is zero.

(b) Assuming a velocity profile $u/u_1=2(y/\delta) - (y/\delta)^2$ obtain an expression for the mass and momentum fluxes within the boundary layer and hence determine the displacement and momentum thickness.

Part A

$$\frac{u}{u_1} = a_0 + a_1\frac{y}{\delta} + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4$$

Boundary conditions

Where y = 0, u = 0 hence $a_0 = 0$

Where $y = \delta$, $u = u_1$ $\quad 1 = a_1 + a_2 + a_3 + a_4$ ....(A)

Differentiate with respect to y

$$\frac{1}{u_1}\frac{du}{dy} = \frac{a_1}{\delta} + 2a_2\left(\frac{y}{\delta^2}\right) + 3a_3\left(\frac{y^2}{\delta^3}\right) + 4a_4\left(\frac{y^3}{\delta^4}\right)$$

Where $y = \delta$, du/dy = 0

$$0 = \frac{a_1}{\delta} + 2a_2\left(\frac{1}{\delta}\right) + 3a_3\left(\frac{1}{\delta}\right) + 4a_4\left(\frac{1}{\delta}\right)$$

$0 = a_1 + 2a_2 + 3a_3 + 4a_4$ ....(B)

Differentiate a second time.

$$\frac{1}{u_1}\frac{d^2u}{dy^2} = 2a_2\left(\frac{1}{\delta^2}\right) + 6a_3\left(\frac{y}{\delta^3}\right) + 12a_4\left(\frac{y^2}{\delta^4}\right)$$

Where y = 0 , $d^2u/dy^2 = 0$ hence $0 = 2a_2\left(\frac{1}{\delta^2}\right)$ Hence $a_2 = 0$

(A) becomes $\quad 1 = a_1 + a_3 + a_4$

(B) becomes $\quad 0 = a_1 + 3a_3 + 4a_4$

Subtract $\quad 1 = 0 - 2a_3 - 3a_4$ ....(C)

The second differential becomes

$$\frac{1}{u_1}\frac{d^2u}{dy^2} = 6a_3\left(\frac{y}{\delta^3}\right) + 12a_4\left(\frac{y^2}{\delta^4}\right)$$

Where $y = \delta$, $d^2u/dy^2 = 0$

$$0 = 6a_3\left(\frac{y}{\delta^3}\right) + 12a_4\left(\frac{y^2}{\delta^4}\right) \; 6a_3\left(\frac{1}{\delta^2}\right) + 12a_4\left(\frac{1}{\delta^2}\right) = 6a_3 + 12a_4 \text{ ....(D)}$$

Divide through by 3 $\quad 0 = 2a_3 + 4a_4$ ....(E)

Add (C) and (E) $\quad a_4 = 1$

Substitute into (E) $\quad 0 = 2a_3 + 4 \quad a_3 = -2$

Substitute into (A) $\quad 1 = a_1 - 2 + 1 \quad a_1 = 2$

Hence

$$\frac{u}{u_1} = 2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + 2\left(\frac{y}{\delta}\right)^4$$

PART B

$\frac{u}{u_1} = 2\eta - \eta^2$ $\delta^* = \int_0^\delta \left[1 - \frac{u}{u_1}\right] dy$ $\delta^* = \delta \int_0^1 \left[1 - 2\eta + \eta^2\right] d\eta$

$\delta^* = \delta\left[\eta - \eta^2 + \frac{\eta^3}{3}\right]_0^1$ $\delta^* = \delta\left[1 - 1 + \frac{1}{3}\right]_0^1 = \frac{\delta}{3}$

$\theta = \int_0^\delta \left[\frac{u}{u_1}\right]\left[1 - \frac{u}{u_1}\right] dy = \int_0^\delta \left[2\eta - \eta^2\right]\left[1 - 2\eta + \eta^2\right] dy$

$\theta = \delta \int_0^1 \left(2\eta - 5\eta^2 + 4\eta^3 - d\eta^4\right) d\eta$ $\theta = \delta\left[\eta^2 - 5\eta^2/3 + \eta^4 - \eta^5/5\right]_0^1$

$\theta = \delta\left[1 - 5/3 + 1 - 1/5\right]$ $\theta = 2\delta/15$

4. When a fluid flows over a flat surface and the flow is laminar, the boundary layer profile may be represented by the equation

$u/u_1 = 2(\eta) - (\eta)^2$ where $\eta = y/\delta$

y is the height within the layer and δ is the thickness of the layer. u is the velocity within the layer and u_1 is the velocity of the main stream.

Show that this distribution satisfies the boundary conditions for the layer.
Show that the thickness of the layer varies with distance (x) from the leading edge by the equation

$\delta = 5.48x(R_{ex})^{-0.5}$

It may be assumed that $\tau_o = \rho u_1^2\, d\theta/dx$

Where y = 0, u = 0 $\eta = y/\delta = 0$ so the condition is satisfied.

Where y = δ, $u = u_1$ $\eta = 1$ $u/u_1 = 2(\eta) - (\eta)^2 = 1$ so the condition is satisfied.

Where y = δ, du/dy = 0 $\frac{1}{u_1}\frac{du}{dy} = \frac{2}{\delta} + 2\left(\frac{y}{\delta^2}\right) = \frac{4}{\delta}$

Where y = 0, $d_2u/dy^2 = 0$ $\frac{1}{u_1}\frac{d^2u}{dy^2} = \left(\frac{2}{\delta^2}\right)$

The last two are apparently not satisfactory conditions.

Starting with $\frac{du}{dy} = u_1\left\{\frac{2}{\delta} + \frac{2y}{\delta^2}\right\}$

At the wall where y =0 the shear stress is

$\tau_o = \mu\, du/dy = \mu u_1 \{2/\delta + 2y/\delta^2\} = (\mu u_1/\delta)\ [2 + 2y/\delta]$

Putting y/δ= η we get $\tau_o = (\mu u_1/\delta)\ \delta\ [2 + 2\eta]$

at the wall η=0 $\tau_o = (2\mu u_1/\delta)$……….........(1)

Putting $\theta = 2\delta/15$ (last example) then $\tau_o = (\rho u_1^2)\, d\theta/dx = (\rho u_1^2)(2/15)d\delta/dx$ ....(2)

Equating (1) and (2) $(\rho u_1^2)(2/15)d\delta/dx = (2\mu u_1/\delta)$

Hence $15(\mu/\rho u_1)dx = \delta\, d\delta$

Integrating $15(\mu x/\rho u_1) = \delta^2/2 + C$

Since δ = 0 at x = 0 (the leading edge of the plate) then C=0

Hence $\delta^2 = 30(\mu x/\rho u_1)$ $\delta = 5.478(\mu x/\rho u_1) = 5.478R_{ex}^{-½}$

5. Define the terms *displacement thickness* δ^* and *momentum thickness* θ.
Find the ratio of these quantities to the boundary layer thickness δ if the velocity profile within the boundary layer is given by $u/u_1=\sin(\pi y/2\delta)$
Show, by means of a momentum balance, that the variation of the boundary layer thickness δ with distance (x) from the leading edge is given by $\delta = 4.8(R_{ex})^{-0.5}$
It may be assumed that $\tau_o = \rho u_1^2\, d\theta/dx$
Estimate the boundary layer thickness at the trailing edge of a plane surface of length 0.1 m when air at STP is flowing parallel to it with a free stream velocity u_1 of 0.8 m/s. It may be assumed without proof that the friction factor C_f is given by $C_f = 2\, d\theta/dx$

N.B. standard data $\mu = 1.71 \times 10^{-5}$ N s/m^2. $\rho = 1.29$ kg/m^3.

DISPLACEMENT THICKNESS δ*
The flow rate within a boundary layer is less than that for a uniform flow of velocity u_1. If we had a uniform flow equal to that in the boundary layer, the surface would have to be displaced a distance δ* in order to produce the reduction. This distance is called the displacement thickness.

MOMENTUM THICKNESS θ
The momentum in a flow with a BL present is less than that in a uniform flow of the same thickness. The momentum in a uniform layer at velocity u_1 and height h is $\rho h u_1^2$. When a BL exists this is reduced by $\rho u_1^2\theta$. Where θ is the thickness of the uniform layer that contains the equivalent to the reduction.

$$\theta = \int_0^\delta \left[\frac{u}{u_1}\right]\left[1-\frac{u}{u_1}\right]dy = \int_0^\delta \left[\frac{u}{u_1}-\left(\frac{u}{u_1}\right)^2\right]dy = \int_0^\delta \left[\sin\left\{\frac{\pi y}{2\delta}\right\}-\left(\sin\left\{\frac{\pi y}{2\delta}\right\}\right)^2\right]dy$$

We need the trig identity $\sin^2 A = ½ - ½ \cos 2A$

$$\theta = \int_0^\delta \left[\sin\left\{\frac{\pi y}{2\delta}\right\}-\frac{1}{2}+\frac{1}{2}\left(\cos\left\{\frac{\pi y}{2\delta}\right\}\right)\right]dy \qquad \theta = \left[-\frac{2\delta}{\pi}\cos\left\{\frac{\pi y}{2\delta}\right\}-\frac{y}{2}+\frac{\delta}{2\pi}\sin\left\{\frac{\pi y}{2\delta}\right\}\right]_0^\delta$$

$$\theta = \left[-0-\frac{\delta}{2}+0\right]-\left[-\frac{2\delta}{\pi}-0+0\right] = 0.137\delta$$

$\tau_o = \mu\, du/dy = \mu u_1 \sin(\pi y/2\delta) = \mu u_1(\pi/2\delta)\cos(\pi y/2\delta)$

At the wall y = $\tau_o = \mu u_1 (\pi/2\delta)$ ....(1)
$C_f = 2\tau_o/\rho u_1^2$ ....(2)
$C_f = 2\, d\theta/dx$ and $\theta = 0.137\delta$ $C_f=2\, d(0.137\delta)/dx = 0.274\delta\, d\delta/dx$ ....(3)

Equate (2) and (3) $2\tau_o/\rho u_1^2 = 0.274\delta\, d\delta/dx$
$\tau_o = \rho u_1^2(0.137\delta)\, d\delta/dx$............... (4)

Equate (1) and (4) $\mu u_1 (\pi/2\delta) = \rho u_1^2(0.137\delta) d\delta/dx$
$\mu\pi x/(0.274\rho u_1^2) = \delta^2/2\ C$ but where x =0, δ = 0 so C = 0
$\delta/x = \{(2\pi)/0.274)\}^{½} R_{ex}^{-½} = 4.8 R_{ex}^{-½}$
x = 0.1 m u = 0.8 m/s $\rho = 1.29$ kg/m^3 $\mu = 1.7I \times 10^{-5}$ (from fluids tables)
$R_{ex} = (1.29)(0.8)(0.1)/1.71 \times 10^{-5} = 6035$
$\delta/0.1 = 4.8(6035)^{-½}$ δ = 0.006 m

Extra ...
$\tau_o = \mu u_1\pi/2\delta$ $C_f = 2\tau_o\, \rho u_1^2\ 2(\mu u_l\, \pi/2\delta) / \rho u_1^2 = \mu\pi x/(\rho u_1 \delta x) = \pi x/ R_{ex}\delta$
$\delta/x = 4.8 R_{ex}^{-½}$ $C_f = (\pi / R_{ex}) (R_{ex}^{½} /4.8) = 0.65 R_{ex}^{-½}$

6. In a laminar flow of a fluid over a flat plate with zero pressure gradient an approximation to the velocity profile is $u/u_1=(3/2)(\eta) - (1/2)(\eta)^3$

$\eta = y/\delta$ and u is the velocity at a distance y from the plate and u_1 is the mainstream velocity. δ is the boundary layer thickness.

Discuss whether this profile satisfies appropriate boundary conditions.

Show that the local skin-friction coefficient C_f is related to the Reynolds' number (Re_x) based on distance x from the leading edge by the expression $C_f = A (Re_x)^{-0.5}$

and evaluate the constant A.

It may be assumed without proof that $C_f = 2\, d\theta/dx$

and that θ is the integral of $(u/u_1)(1 - u/u_1)dy$ between the limits 0 and δ

This is the same as Q2 whence $\theta = 39\delta/280$ $C_f = 2\, d\theta/dx = (78/280)d\delta/dx$

$\tau_o = \rho u^2\pi/2\delta$

$C_f = 2\, \tau_o/\rho u^2 = (\rho\, u_1\, \pi\, x)/\delta\, \rho\, u_1^2 x) = (\pi\, x /\delta)\, R_{ex}^{½}$

$\delta/x = 4.64\, R_{ex}^{½}$

$C_f = \pi/(4.64\, R_{ex}^{½})\, x\, (1/ R_{ex}) = 0.65\, R_{ex}^{½}$

SELF ASSESSMENT EXERCISE No. 5

1. Under what circumstances is the velocity profile in a pipe adequately represented by the 1/7 th power law $u/u_1=(y/R)^{1/7}$ where u is the velocity at distance y from the wall, R is the pipe radius and u_1 is the centre-line velocity ?

The table shows the measured velocity profile in a pipe radius 30 mm. Show that these data satisfy the 1/7 th power law and hence evaluate

(i) the centre-line velocity
(ii) the mean velocity u_m
(iii) the distance from the wall at which the velocity equals u_m.

1.0	2.0	5.0	10.0	15.0	20.0	y (mm)
1.54	1.70	1.94	2.14	2.26	2.36	u (m/s)

Limitations are that the flow must be turbulent, with $R_e > 10^7$ and the velocity gradient must be the same at the junction between laminar sub layer and the turbulent layer.
$u/u_1 = (y/R)^{1/7}$ a = radius = 30 mm

Evaluate at various values of y

y	1	2	5	10	15	20	mm
u	1540	1700	1940	2140	2260	2350	mm/s
u_l	2503	2503	2506	2504	2495	2500	mm/s

Since u_l is constant the law is true. Take u_l = 2502 mm/s

$$Q = 2\pi\int_0^R (R-y)\left(\frac{y}{R}\right)^{1/7} = 2\,\pi \text{ x } 2502\int_0^R \left(R^{6/7}y^{1/7} - y^{8/7}R^{-1/7}\right)dy$$

$$Q = 2\pi \text{ x } 2502\left[\frac{49R^2}{120}\right]$$

Mean velocity $u_m = Q/A = \dfrac{2\pi \text{ x } 2502}{\pi R^2}\left[\dfrac{49R^2}{120}\right] = 2403$

$2043/2502 = (y/30)^{1/7}$ y = 7.261 mm. Note this fits with $u_m = (49/60)\ u_l$ and if this was the starting point the question would be simple.

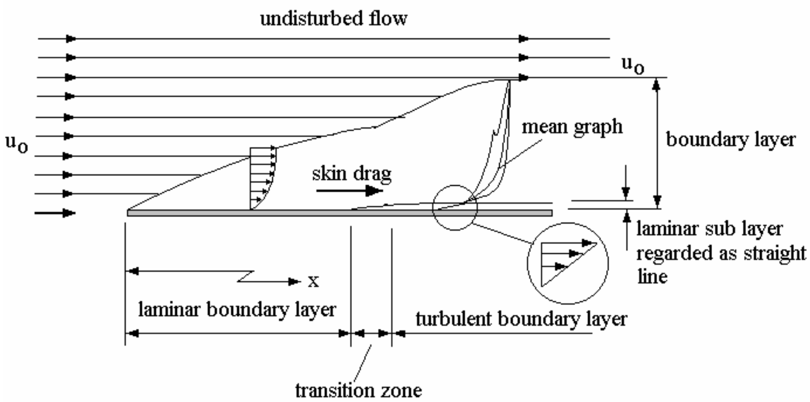
2.

(a) Discuss the limitations of the 1/7th power law $u/u_1=(y/R)^{1/7}$ for the velocity profile in a circular pipe of radius R, indicating the range of Reynolds numbers for which this law is applicable.
(b) Show that the mean velocity is given by $49u_1/60$.

(c) Water flows at a volumetric flow rate of 1.1 x 10^{-3} m^3/s in a tube of diameter 25 mm. Calculate the centre-line velocity and the distance from the wall at which the velocity is equal to the mean velocity.

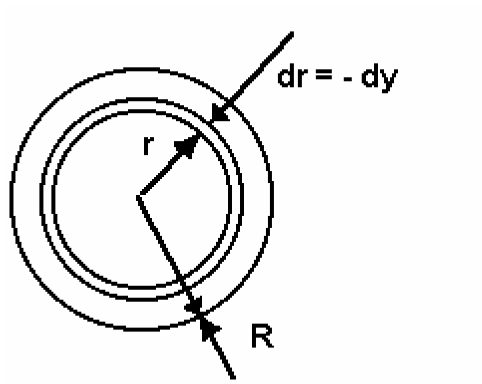
(d) Assuming that $C_f=0.079(Re)^{-0.25}$ evaluate the wall shear stress and hence estimate the laminar sub-layer thickness.

$\mu = 0.89 \times 10^{-3}$ N s/m^2. $\rho = 998$ kg/m^3.



Limitations are that the flow must be turbulent with $R_e>10^7$ and the velocity gradient must be the same at the junction between the laminar sub layer and the turbulent level.

Flow through an elementary cylinder. For a pipe, the B.L. extends to the centre so δ = radius = R. Consider an elementary ring of flow.



The velocity through the ring is u.
The volume flow rate through the ring is $2\pi rudr$
The volume flow rate in the pipe is $Q = 2\pi\int rudr$

Since δ = R then $u = u_1(y/R)^{1/7}$

also $r = R-y$

$$Q = 2\pi\int (R-y)udr = 2\pi\int u_1R^{-1/7} (R-y)y^{1/7}dy$$

$$Q =2\pi u_1R^{-1/7}[Ry^{1/7}-y^{8/7}]$$

$$Q =2\pi u_1R^{-1/7}[(7/8)Ry^{8/7}- (7/15)y^{15/7}]$$

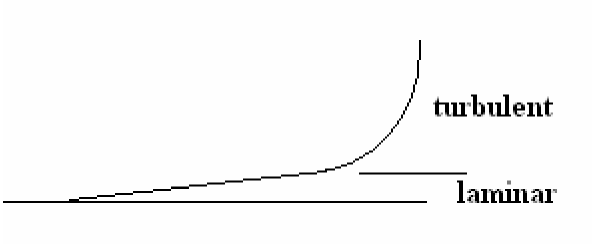
$$Q =(49/60)\pi R^2u_1.$$

The mean velocity is defined by $u_m=Q/\pi R^2$ hence $u_m=(49/60)u_1$

$Q = 1.1 \times 10^{-3}$ m^3/s R = 0.025/2 = 0.0125 m u_{mean} = 2.241 m/s u_1 = 2.744 m/s

$u = u_1 (y/R)^{1/7}$ y = 3.0285 m when $u = u_{mean}$

At the junction, the gradients are the same.



Laminar sub layer $\tau_o = \mu\, du/dy$

$R_e = \rho u D/\mu = 2.241 \times 0.025 \times 998/0.89 \times 10^{-3} = 62820$
$C_f = 0.005 = 2\tau_o/\rho u^2$ $\tau_o = 0.005 \times 998 \times 2.241^2/(2 \times 0.005) = 12.5\ N/m^2$

For the turbulent layer $\tau_o = \mu\, du/dy$

$$12.5 = 0.89x10^{-3}\frac{d\left\{u_1\left(\frac{y}{R}\right)^{1/7}\right\}}{dy}$$

$$y^{-6/7} = \frac{7 \text{ x } 14045 \text{ x } 0.0125^{1/7}}{2.744} = 52.196 \text{ x } 10^{-6}$$

$y = 10.09 \times 10^{-6}$ m

Some text uses the following method.

$y = 5\mu/\rho u^*$ where $u^* = \surd(\tau_o/\rho) = \surd(12.7/998) = 0/112$
In this case $y = 39.81 \times 10^{-6}$ m (the thickness of the laminar sub-layer)

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 4 – FLOW THROUGH POROUS PASSAGES

SELF ASSESSMENT EXERCISE No.1

Q.1
Outline briefly the derivation of the Carman-Kozeny equation.

$$\frac{dp}{dl} = -\frac{180\,\mu\,u(1-\varepsilon)^2}{d_s^2\varepsilon^3}$$

dp/dl is the pressure gradient, μ is the fluid viscosity, u is the superficial velocity, d_s is the particle diameter and ε is the void fraction.

A cartridge filter consists of an annular piece of material of length 150 mm and internal diameter and external diameters 10 mm and 20 mm. Water at 25°C flows radially inwards under the influence of a pressure difference of 0.1 bar. Determine the volumetric flow rate. (21.53 cm^3/s)

For the filter material take d = 0.05 mm and ε = 0.35.
μ=0.89 x 10^{-3} N s/m^2 and ρ= 997 kg/m^3.

The solution for part 1 is as given in the tutorial.

For radial flow we change dl to dr $\quad \frac{dp}{dr} = -\frac{180\,\mu\;u(1-\varepsilon)^2}{d_s^2\varepsilon^3}$

The surface area of the annulus is 2πrL L = 0.15 m and d = 0.05 x 10^{-3} m
The velocity is u = Q/2πrL

$$\frac{dp}{dr} = -\frac{180\,\mu\;(1-\varepsilon)^2}{d_s^2\varepsilon^3} \text{ x } \frac{Q}{2\,\pi\,r\,L}$$

$$\frac{dp}{dr} = -\frac{180 \text{ x } 0.89 \text{ x}10^{-3}\;(1-0.35)^2}{(0.05 \text{ x } 10^{-3})^2\,0.35^3} \text{ x } \frac{Q}{2\,\pi\;\,r \text{ x } 0.15}$$

$$dp = -670 \text{ x } 10^{-6}\,Q\,\frac{dr}{r}$$

Integrate $\quad \Delta p = -670 \text{ x } 10^{-6}\,Q\,\ln\left(\frac{r_1}{r_2}\right) = -670 \text{ x } 10^{-6}\,Q\,\ln(2)$

Δp = -0.1 x 10^5 = -464.4 x 10^6 Q
Q = 21.53 x 10^{-6} m^3/s

Q.2

(a) Discuss the assumptions leading to the equation of horizontal viscous flow through a packed bed

$$\frac{dp}{dL} = -\frac{180\mu u(1-\varepsilon)^2}{d_s^2 \varepsilon^3}$$

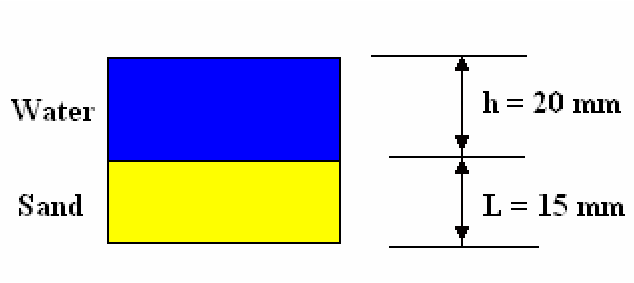
Δp is the pressure drop across a bed of depth L, void fraction ε and effective particle diameter d. u is the approach velocity and μ is the viscosity of the fluid.

(b) Water percolates downwards through a sand filter of thickness 15 mm, consisting of sand grains of effective diameter 0.3 mm and void fraction 0.45. The depth of the effectively stagnant clear water above the filter is 20 mm and the pressure at the base of the filter is atmospheric. Calculate the volumetric flow rate per m^2 of filter. (2.2 dm^3/s)

(Note the density and viscosity of water are given in the instructions on all exams papers)

μ=0.89 x 10^{-3} N s/m^2 and ρ= 997 kg/m^3

Part (a) is as stated in the tutorial.



$\Delta p = \rho\, g\, h$ = 997 x 9.81 x 0.02 = 195.61 Pa

$$\frac{dp}{dL} = -\frac{195.61}{0.015} - \frac{180 \times 0.89 \times 10^{-3} u(1-0.45)^2}{(0.3 \times 10^{-3})^2 0.45^3}$$

u = 0.0022 m/s

Q = u A = 0.022 m^3/s per unit area

Q3.

Oil is extracted from a horizontal oil-bearing stratum of thickness 15 m into a vertical bore hole of radius 0.18 m. Find the rate of extraction of the oil if the pressure in the bore-hole is 250 bar and the pressure 300 m from the bore hole is 350 bar.

Take d= 0.05 mm, ε= 0.30 and μ = 5.0 x 10^{-3} N s/m^2.

$$\frac{dp}{dr} = -\frac{180\,\mu\;u(1-\varepsilon)^2}{d_s^2\varepsilon^3}$$

$$\frac{dp}{dr} = -\frac{180\,\mu\;(1-\varepsilon)^2}{d_s^2\varepsilon^3} \text{ x } \frac{Q}{2\,\pi\,r\,L}$$

The surface area of the annulus is 2πrL L = 15 m and d = 0.05 x 10^{-3} m

The velocity is u = Q/2πrL

$$\frac{dp}{dr} = -\frac{180\,\mu\;(1-\varepsilon)^2}{d_s^2\varepsilon^3} \text{ x } \frac{Q}{2\,\pi\,r\,L}$$

$$\frac{dp}{dr} = -\frac{180 \text{ x } 5 \text{ x } 10^{-3}\;(1-0.3)^2}{(0.05 \text{ x } 10^{-3})^2\,0.3^3} \text{ x } \frac{Q}{2\,\pi\;\;r \text{ x } 15}$$

$$dp = -69.32 \text{ x } 10^6\,Q\,\frac{dr}{r}$$

Integrate $\Delta p = -69.32 \text{ x } 10^6\,Q\,\ln\left(\frac{r_1}{r_2}\right) = -69.32 \text{ x } 10^6\,Q\,\ln\left(\frac{300}{0.18}\right) = 514.3 \text{ x } 10^6\,Q$

Δp = 250 – 350 = -100bar

Q = 100 x 10^5/514.3 x 10^6 = 0.01944 m^3/s

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 5 – POTENTIAL FLOW

SELF ASSESSMENT EXERCISE No.1

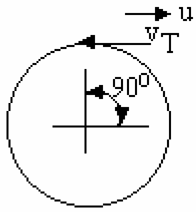
1.a. Show that the potential function $\phi = A(r + B/r)\cos\theta$ represents the flow of an ideal fluid around a long cylinder. Evaluate the constants A and B if the cylinder is 40 mm radius and the velocity of the main flow is 3 m/s.

b. Obtain expressions for the tangential and radial velocities and hence the stream function ψ.
c. Evaluate the largest velocity in the directions parallel and perpendicular to the flow direction. (6 m/s for tangential velocity)
d. A small neutrally buoyant particle is released into the stream at r = 100 mm and $\theta = 150^o$. Determine the distance at the closest approach to the cylinder. (66.18 mm)

Part (a) is as given in the tutorial. Normally this equation is given as $\phi = (A\, r + B/r)\cos\theta$ but both are the same but the constants represent different values.

Part (b)
The values of the constants depend upon the quadrant selected to solve the boundary conditions. This is because the sign of the tangential velocity and radial velocity are different in each quadrant. Which ever one is used, the final result is the same. Let us select the quadrant from 90^o to 180^o.
At a large distance from the cylinder and at the 90^o position the velocity is from left to right so at this point $v_T = -u$. From equation 4 we have

$$v_T = \frac{d\varphi}{rd\theta} \qquad \varphi = A\left\{r + \frac{B}{r}\right\}\cos\theta$$



$$v_T = -\frac{A}{r}\left\{r + \frac{B}{r}\right\}\sin\theta = -A\left\{1 + \frac{B}{r^2}\right\}\sin\theta$$

$$v_R = \frac{d\varphi}{dr} = A\left(1 - \frac{B}{r^2}\right)\cos\theta$$

Putting r = infinity and $\theta = 90^o$ and remembering that $+v_T$ is anticlockwise +u is left to right, we have $v_T = -3$ m/s. $B/r^2 \to 0$

$$v_T = -A\left\{1 + \frac{B}{r^2}\right\}\sin\theta = -3 = -A\{1 + 0\} \qquad \text{Hence} \quad A = 3$$

At angle 180^o with r = 0.04, $v_R = 0$

$$v_R = A\left(1 - \frac{B}{r^2}\right)\cos\theta = 0 = 3\left(1 - \frac{B}{0.04^2}\right)(-1) \qquad \text{Hence} \quad B = 0.0016$$

$$v_T = -3\left\{1 + \frac{0.0016}{r^2}\right\}\sin\theta \qquad v_R = 3\left(1 - \frac{0.0016}{r^2}\right)\cos\theta$$

$$d\psi = v_R\, r\, d\theta = 3\left(1 - \frac{0.0016}{r^2}\right) r\cos\theta\, d\theta = 3\left(r - \frac{0.0016}{r}\right)\cos\theta\, d\theta \qquad \psi = 3\left(r - \frac{0.0016}{r}\right)\sin\theta$$

Part (c) The maximum velocity is 2u = 6 m/s (proof is in the tutorial)
Part (d) R = 0.1 m $\theta = 150^o$

$$\psi = 3\left(r - \frac{0.0016}{r}\right)\sin\theta = 3\left(0.1 - \frac{0.0016}{0.1}\right)\sin 150 = 3\left(0.1 - \frac{0.0016}{0.1}\right)\sin 150 = 0.244$$

The closest approach is at $\theta = 90^o$

$$\psi = 0.244 = 3\left(r - \frac{0.0016}{r}\right)\sin 90 \qquad 0.0813r = \left(r^2 - 0.0016\right)$$

$r^2 - 0.0813\, r - 0.0016 = 0$ solve the quadratic and r = 0.098 m or 98 mm

2.a. Show that the potential function $\phi = (Ar + B/r)\cos\theta$ gives the flow of an ideal fluid around a cylinder. Determine the constants A and B if the velocity of the main stream is u and the cylinder is radius R.

b. Find the pressure distribution around the cylinder expressed in the form $(p - p')/(\rho u^2/2)$ as a function of angle.

c. Sketch the relationship derived above and compare it with the actual pressure profiles that occur up to a Reynolds number of 5×10^5.

Part (a) is in the tutorial.

Part (b) $\quad v_T = \frac{d\varphi}{rd\theta} \qquad v_T = -\frac{1}{r}\left\{\frac{B}{r} + Ar\right\}\sin\theta \qquad v_T = -\left\{\frac{B}{r^2} + A\right\}\sin\theta$

Putting r = infinity and $\theta = 90^o$ and remembering that $+v_T$ is anticlockwise +u is left to right, we have

$$v_T = -u = -\left\{\frac{B}{r^2} + A\right\}\sin\theta = -\{0 + A\}x1$$

Hence $v_T = -A = -u$ so A = u as expected from earlier work.

At angle 180^o with r = R, the velocity is only radial in directions and is zero because it is arrested.

From equation 3 we have $\qquad v_R = \frac{d\varphi}{dr} = \left(-\frac{B}{r^2} + A\right)\cos\theta$

Putting r = R and $v_R = 0$ and $\theta = 180$ we have

$$0 = \left(-\frac{B}{R^2} + A\right)(-1) = \left(\frac{B}{R^2} - A\right)$$

Put A = u $\qquad 0 = \frac{B}{R^2} - u \qquad B = uR^2$

Substituting for $B = uR^2$ and A = u we have

$$\varphi = \left\{\frac{B}{r} + Ar\right\}\cos\theta = \left\{\frac{uR^2}{r} + ur\right\}\cos\theta$$

At the surface of the cylinder r = R the velocity potential is

$$\varphi = \{uR + uR\}\cos\theta = 2uR\cos\theta$$

The tangential velocity on the surface of the cylinder is then

$$v_T = \frac{d\varphi}{rd\theta} = -\left\{\frac{B}{r^2} + A\right\}\sin\theta \qquad v_T = -\left\{\frac{uR^2}{r^2} + u\right\}\sin\theta \qquad v_T = -2u\sin\theta$$

This is a maximum at $\theta = 90^o$ where the streamlines are closest together so the maximum velocity is 2u on the top and bottom of the cylinder.

The velocity of the main stream flow is u and the pressure is p'. When it flows over the surface of the cylinder the pressure is p because of the change in velocity. The pressure change is p - p'.

The dynamic pressure for a stream is defined as $\rho u^2/2$

The pressure distribution is usually shown in the dimensionless form

$$2(p - p')/(\rho u^2)$$

For an infinitely long cylinder placed in a stream of mean velocity u we apply Bernoulli's equation between a point well away from the stream and a point on the surface. At the surface the velocity is entirely tangential so :

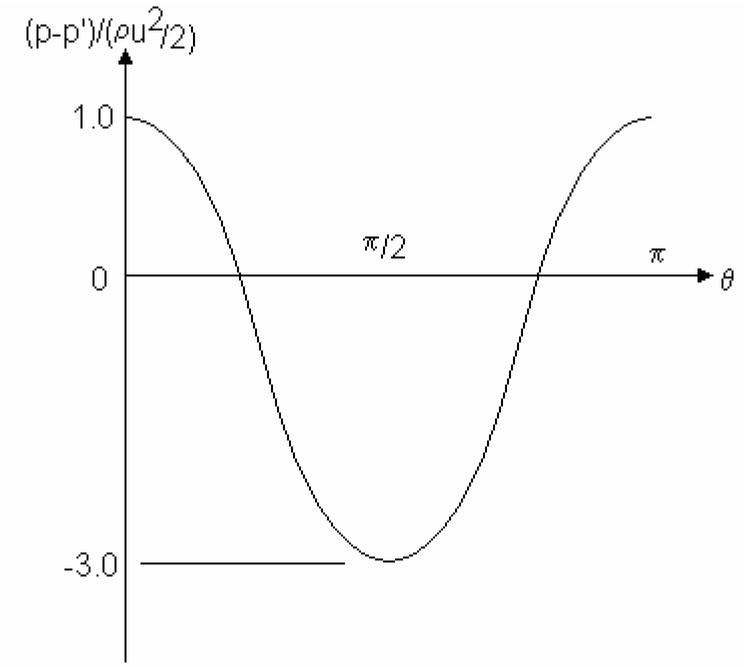
$$p' + \rho u^2/2 = p + \rho v_T^2/2$$

From the work previous this becomes

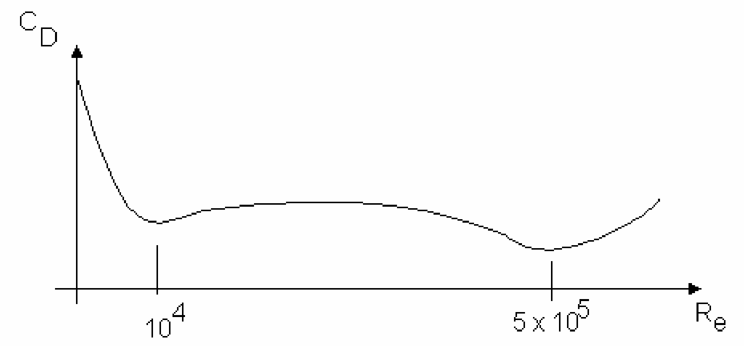
$$p' + \rho u^2/2 = p + \rho(2u\sin\theta)^2/2$$

$$p - p' = \rho u^2/2 - (\rho/2)(4u^2 \sin^2\theta) = (\rho u^2/2)(1 - 4\sin^2\theta)$$
$$(p - p')/(\rho u^2/2) = 1 - 4 \sin^2\theta$$

If this function is plotted against angle we find that the distribution has a maximum value of 1.0 at the front and back, and a minimum value of -3 at the sides.



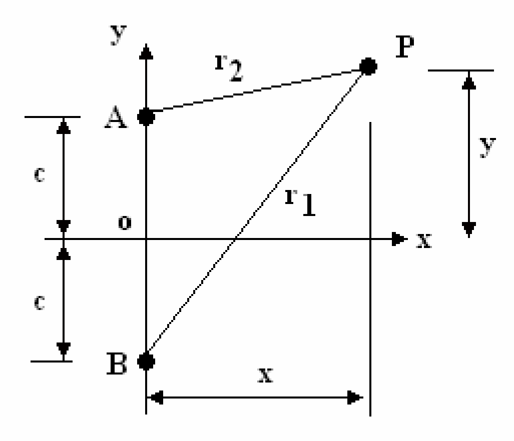
Research shows that the drag coefficient reduces with increased stream velocity and then remains constant when the boundary layer achieves separation. If the mainstream velocity is further increased, turbulent flow sets in around the cylinder and this produces a marked drop in the drag. This is shown below on the graph of C_D against Reynolds's number. The point where the sudden drop occurs is at a critical value of Reynolds's number of 5×10^5.



3.Show that in the region y>0 the potential function $\phi = a \ln[x^2 + (y-c)^2] + a \ln[x^2 + (y+c)^2]$ gives the 2 dimensional flow pattern associated with a source distance c above a solid flat plane at y=0.

b. Obtain expressions for the velocity adjacent to the plane at y = 0. Find the pressure distribution along this plane.

c. Derive an expression for the stream function ϕ.



The key to this problem is knowing that two identical sources of strength m equal distance above and below the origin produces the pattern required.

$\phi_A = -(m/2\pi)\ln r_2$

$\phi_B = -(m/2\pi)\ln r_1$ Now use pythagoras

$r_2 = \{x^2 + (y-c)^2\}^{1/2}$ $r_1 = \{x^2 + (y+c)^2\}^{1/2}$

$\phi_p = \phi_A + \phi_B$

$$\varphi_p = -\frac{m}{2\pi}\left[\ln\left\{x^2 + (y+c)^2\right\}^{1/2} + \ln\left\{x^2 + (y+c)^2\right\}^{1/2}\right]$$

$$\varphi_p = -\frac{m}{4\pi}\left[\ln\left\{x^2 + (y+c)^2\right\} + \ln\left\{x^2 + (y+c)^2\right\}\right]$$

$$\varphi_p = a\left[\ln\left\{x^2 + (y+c)^2\right\} + \ln\left\{x^2 + (y+c)^2\right\}\right] \quad a = -m/2\pi$$

At y = 0 $\phi = 2a\ln(x^2 + c^2)$

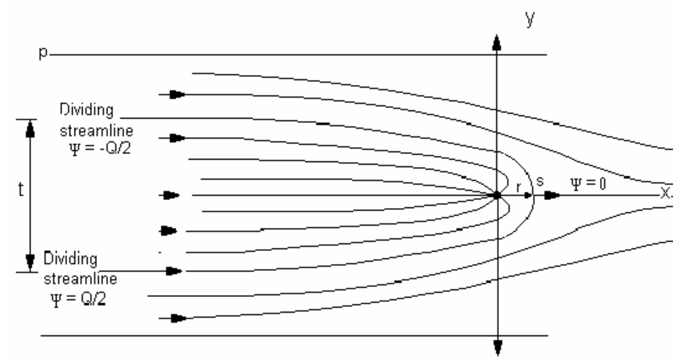
v = $-d\phi/dy = 0$ at all values of x so it is the same as an impervious plane. u = $-d\phi/dx = -\frac{4ax}{x^2 + c^2}$

At very large values of x, u = 0 and p = p_o

Apply Bernoulli and $p_o = p + \rho u^2/2 = p_o + \frac{\rho}{2}\left(\frac{4ax}{x^2 + c^2}\right)^2$

4. A uniform flow has a sink placed in it at the origin of the Cartesian co-ordinates. The stream function of the uniform flow and sink are

$\psi_1 = Uy$ and $\psi_2 = B\theta$



Write out the combined stream function in Cartesian co-ordinates.

Given U=0.001 m/s and B= -0.04 $m^3/3$ per m thickness, derive the velocity potential.

Determine the width of the flux into the sink at a large distance upstream.

$\psi_1 = u\,y$ $\psi_2 = B\theta$ $\psi = u\,y + B\theta$ $B = Q/2\pi$ for a sink

$$-\frac{d\psi}{r d\theta} = -\frac{ur\cos\theta + B}{r} \qquad d\phi = \left(-u\cos\theta + \frac{B}{r}\right)dr$$

$$\phi = (-u\,r\cos\theta + B\ln r) \qquad \phi = (-u\,x + B\ln r) \qquad \phi = \left(-u\,x + B\ln\left\{x^2 + y^2\right\}^{1/2}\right)$$

u = 0.001 B = -0.04

$$\phi = \left(-0.001x + 0.04\ln\left\{x^2 + y^2\right\}^{1/2}\right) \qquad \phi = \left(-0.001x + 0.02\ln\left\{x^2 + y^2\right\}\right)$$

Q=ut t = Q/u Q = 2π x 0.04 u = 0.001 t = (2π x 0.04)/0.001 = 80π metres

SELF ASSESSMENT EXERCISE No.2

1. Define the following terms.

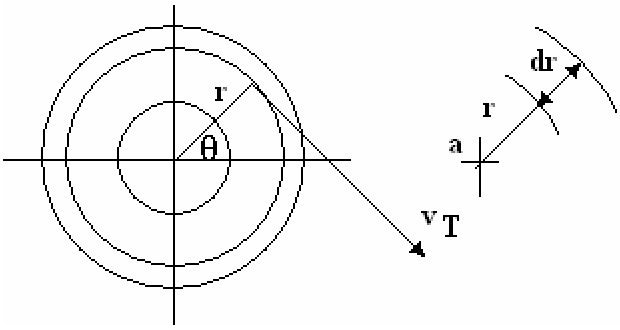
Stream function.
Velocity potential function.
Streamline
Stream tube
Circulation
Vorticity.

All these definitions are in the tutorial.

2.A free vortex of with circulation $K = 2\pi v_T R$ is placed in a uniform flow of velocity u.

Derive the stream function and velocity potential for the combined flow.

The circulation is 7 m^2/s and it is placed in a uniform flow of 3 m/s in the x direction. Calculate the pressure difference between a point at x = 0.5 and y = 0.5.
The density of the fluid is 1000 kg/m^3.
(Ans. 6695 Pascal)



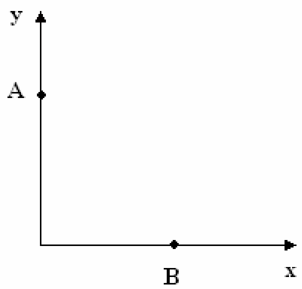
Free Vortex $v_T = k/2\pi r$ $\quad d\psi = v_T\, dr = (k/2\pi r)\, dr$

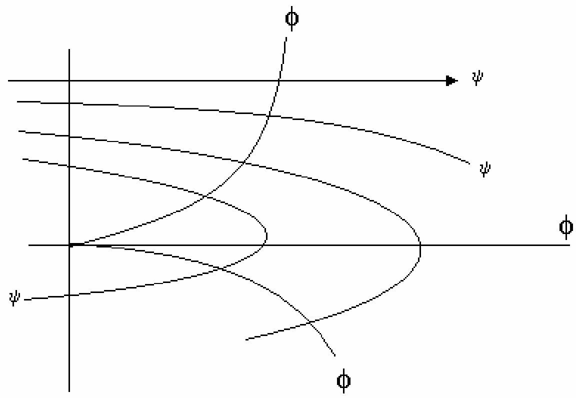
$$\psi = \int_a^r \frac{k}{2\pi\pi} dr = \frac{k}{2\pi} \ln\left(\frac{r}{a}\right) \qquad \phi = \int_0^\theta v_T r\, d\theta = \int_0^\theta \frac{k}{2\pi} r d\theta = \frac{k}{2\pi}\theta$$

Uniform flow
$\psi = -u\,y = -\,u\,r\sin\theta$ $\qquad \phi = u\,r\cos\theta$

Combined Flow

$$\psi = \frac{k}{2\pi}\ln\left(\frac{r}{a}\right) - u\,r\sin\theta \qquad \phi = \frac{k}{2\pi}\theta + u\,r\cos\theta$$





$v_\theta = \sqrt{(v_T^2 + v_R^2)}$ $k = 7$ $u = 3$

$v_R = d\phi/dr = u \cos\theta$ $v_T = d\psi/dr = k/2\pi r - u \sin\theta$

Point A $v_T = 7/(2\ \pi\ 0.5) - 3 \sin 90^o = -0.7718$ m/s

$v_R = 0$ $v_\theta = \sqrt{(v_T^2 + v_R^2)} = 0.7718$ m/s

Pont B $\theta = 0^o$ $v_R = u \cos\theta = 3$ $v_T = 7/(2\ \pi\ 0.5) - 3 \sin 0^o = 2.228$ m/s

$v_\theta = \sqrt{(v_T^2 + v_R^2)} = 3.74$ m/s

Bernoulli between stream and A $p = p_A + \rho\ v_\theta^{\ 2}/2$

$p_A - p_\theta = (\rho/2)(v_{\theta B}^2 - v_{\theta A}^2) = (1000/2)(3.74^2 - 0.7718^2) = 6695$ Pa

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 6 – DIMENSIONAL ANALYSIS

SELF ASSESSMENT EXERCISE No. 1

1. It is observed that the velocity 'v' of a liquid leaving a nozzle depends upon the pressure drop 'p' and the density 'ρ'.

Show that the relationship between them is of the form $v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$

$v = C\{p^a \rho^b\}$ $[v] = LT^{-1}$ $[p] = ML^{-1}T^{-2}$ $[\rho] = ML^{-3}$

$M^0L^1T^{-1} = (ML^{-1}T^{-2})^a(ML^{-3})^b$

(T) $-1 = -2a$ $a = ½$
(M) $0 = a + b$ $b = -½$

$v = C\{p^{1/2} \rho^{-1/2}\}$ $v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$

2. It is observed that the speed of a sound in 'a' in a liquid depends upon the density 'ρ' and the bulk modulus 'K'.

Show that the relationship between them is $a = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$

$a = C\{K^a \rho^b\}$ $[a] = LT^{-1}$ $[K] = ML^{-1}T^{-2}$ $[\rho] = ML^{-3}$

$M^0L^1T^{-1} = (ML^{-1}T^{-2})^a(ML^{-3})^b$

(T) $-1 = -2a$ $a = ½$
(M) $0 = a + b$ $b = -½$

$v = C\{K^{1/2} \rho^{-1/2}\}$ $v = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$

3. It is observed that the frequency of oscillation of a guitar string 'f' depends upon the mass 'm', the length 'l' and tension 'F'.

Show that the relationship between them is $f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$

$f = C\{F^a m^b l^c\}$

$[f] = T^{-1}$ $[F] = MLT^{-2}$ $[m] = M$ $[l] = L$

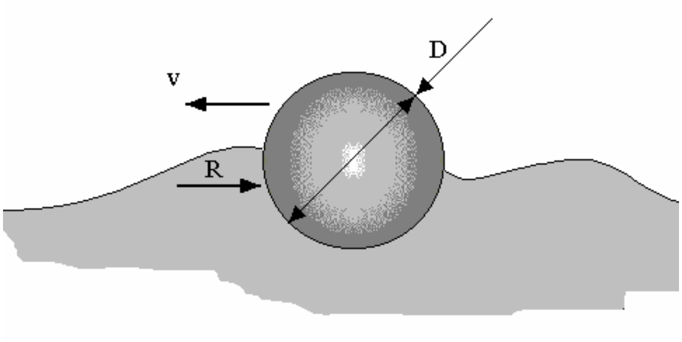
$T^{-1} = (MLT^{-2})^a(M)^b(L)^c$

(T) $-1 = -2a$ $a = ½$
(M) $0 = a + b$ $b = -½$
(L) $0 = a + c$ $c = -½$

$f = C\{F^{1/2} m^{-1/2} l^{-1/2}\}$ $f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$

SELF ASSESSMENT EXERCISE No.2

1. The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' on the surface of a liquid is due to the density 'ρ' and the surface waves produced by the acceleration of gravity 'g'. Show that the dimensionless equation linking these quantities is N_e = function(F_r)



F_r is the Froude number and is given by

$$F_r = \sqrt{\frac{v^2}{gD}}$$

R = function (D v ρ g) = C $D^a\ v^b\ \rho^c\ g^d$

There are 3 dimensions and 5 quantities so there will be 5 –3 = 2 dimensionless numbers. Identify that the one dimensionless group will be formed with R and the other with K.

Π_1 is the group formed between g and D v ρ

Π_2 is the group formed between R and D v ρ

$g = \Pi_2 D^a\ v^b\ \rho^c$	$R = \Pi_1 D^a\ v^b\ \rho^c$
$[g] = L T^{--2}$	$[R] = MLT^{-2}$
$[D] = L$	$[D] = L$
$[v] = LT^{-1}$	$[v] = LT^{-1}$
$[\rho] = ML^{-3}$	$[\rho] = ML^{-3}$
$LT^{--2} = L^a (LT^{-1})^b\ (ML^{-3})^c$	$MLT^{-2} = L^a (LT^{-1})^b\ (ML^{-3})^c$
$LT^{--2} = L^{a+b-3c}\ M^c\ T^{-b}$	$ML^1T^{-2} = L^{a+b-3c}\ M^c\ T^{-b}$
Time -2 = -b b = 2	**Time -2 = -b b = 2**
Mass c = 0	**Mass c = 1**
Length 1 = a + b -3c	**Length 1 = a + b -3c**
1 = a + 2 – 0 a = -1	**1 = a + 2 – 3 a = 2**
$g = \Pi_2\ D^1\ v^2\ \rho^0$	$R = \Pi_1\ D^2\ v^2\ \rho^1$
$\Pi_2 = \frac{gD}{v^2} = F_r^{-2}$	$\Pi_1 = \frac{R}{\rho v^2 D^2} = Ne$

$\Pi_1 = \phi\Pi_2$ Ne = $\phi(F_r)$

2. The Torque 'T' required to rotate a disc in a viscous fluid depends upon the diameter 'D' , the speed of rotation 'N' the density 'ρ' and the dynamic viscosity 'μ'. Show that the dimensionless equation linking these quantities is $\{T D^{-5} N^{-2}\ \rho^{-1}\}$ = function $\{\rho N D^2\ \mu^{-1}\}$

Use the other method here. Identify d as the unknown power.

$T = f(D N \rho \mu) = C D^a N^b \rho^c \mu^d$

$ML^2T^{-2} = (L)^a (T^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^d$

(T)	-2 = -b - d	b = 2 – d
(M)	1 = c + d	c = 1 - d
(L)	2 = a - 3c – d = a – 3(1 - d) – d	a = 5 – 2d

$T = C D^{5-2d} N^{2-d} \rho^{1-d} \mu^d = C D^5 N^2 \rho (D^{-2} N^{-1} \rho^{-1} \mu^1)^d$

$T D^{-5} N^{-2} \rho^{-1} = f(D^{-2} N^{-1} \rho^{-1} \mu^1)$

$$\frac{T}{D^5N^2\rho} = f\left(\frac{D^2N\rho}{\mu}\right)$$

SELF ASSESSMENT EXERCISE No.3

1.The resistance to motion 'R' of a sphere travelling through a fluid which is both viscous and compressible, depends upon the diameter 'D' , the velocity 'v' , the density 'ρ' , the dynamic viscosity 'μ' and the bulk modulus 'K'. Show that the complete relationship between these quantities is :

N_e = function{R_e } {M_a}

where $N_e = R\ \rho^{-1}\ v^{-2}\ D^{-2}$ $R_e = \rho\ v\ D\ \mu^{-1}$ $M_a = v/a$ and $a = (k/\rho)^{0.5}$

This may be solved with Buckingham's method but the traditional method is given here.

R = function $(D\ v\ \rho\ \mu\ K) = C\ D^a\ v^b\ \rho^c\ \mu^d\ K^e$

First write out the MLT dimensions.

$[R] = ML^1T^{-2}$
$[D] = L$
$[v] = LT^{-1}$
$[\rho] = ML^{-3}$
$[\mu] = ML^{-1}T^{-1}$
$[K] = ML^{-1}T^{--2}$

$ML^1T^{-2} = L^a\ (LT^{-1})^b\ (ML^{-3})^c(ML^{-1}T^{-1})^d\ (ML^{-1}T^{--2})^e$
$ML^1T^{-2} = L^{a+b-3c-d-e}\ M^{c+d+e}\ T^{-b-d-2e}$

Viscosity and Bulk Modulus are the quantities which causes resistance so the unsolved indexes are d and e.

TIME $-2 = -b - d - 2e$ hence $b = 2 - d - 2e$ is as far as we can resolve b

MASS $1 = c + d + e$ hence $c = 1 - d - e$

LENGTH $1 = a + b - 3c - d - e$ $1 = a + (2 - d - e) - 3(1 - d - e) - d - e$
$1 = a - 1 - d$ $a = 2 - d$

Next put these back into the original formula.

$$R = C\ D^{2-d}\ v^{2-d-2e}\ \rho^{1-d-e}\ \mu^d\ K^e$$
$$R = C\ D^2\ v^2\ \rho^1\ (D^{-1}\ v^{-1}\rho^{-1}\mu)^d\ (v^{-2}\rho^{-1}K)^e$$
$$\frac{R}{\rho v^2 D^2} = \left(\frac{\mu}{\rho v D}\right)^d \left(\frac{K}{\rho v^2}\right)^e$$
$$N_e = f(R_e)(M_a)$$

SELF ASSESSMENT EXERCISE No.4

1.(a) The viscous torque produced on a disc rotating in a liquid depends upon the characteristic dimension D, the speed of rotation N, the density ρ and the dynamic viscosity μ. Show that :

$$\{T/(\rho N^2 D^5)\} = f(\rho N D^2/\mu)$$

(b) In order to predict the torque on a disc 0.5 m diameter which rotates in oil at 200 rev/min, a model is made to a scale of 1/5. The model is rotated in water. Calculate the speed of rotation for the model which produces dynamic similarity.
For the oil the density is 750 kg/m^3 and the dynamic viscosity is 0.2 Ns/m^2.
For water the density is 1000 kg/m^3 and the dynamic viscosity is 0.001 Ns/m^2.
(c) When the model is tested at 18.75 rev/min the torque was 0.02 Nm. Predict the torque on the full size disc at 200 rev/min.

Part (a) is the same as in SAE 2 whence $\frac{T}{D^5N^2\rho} = f\left(\frac{D^2N\rho}{\mu}\right)$

Part (b) For dynamic similarity

$$\left(\frac{D^2N\rho}{\mu}\right)_{model} = \left(\frac{D^2N\rho}{\mu}\right)_{object} \qquad \left(\frac{(0.2D)^2 N_m \text{ x } 1000}{0.001}\right)_{model} = \left(\frac{D^2\, 200 \text{ x } 750}{0.2}\right)_{object}$$

$$N_m = 18.75 \text{ rev/min}$$

$$\left(\frac{T}{D^5N^2\rho}\right)_{model} = \left(\frac{T}{D^5N^2\rho}\right)_{objectl} \qquad T_o = \left(\frac{T_m D_o^5 N_o^2 \rho_o}{D_m^5 N_m^2 \rho_m}\right) = \left(\frac{0.02 \text{ x } 5^5 200^2 \text{ x } 750}{1^5 \text{ x } 18.75^2 \text{ x } 1000}\right)$$

$$T_o = 5333 \text{ N}$$

2. The resistance to motion of a submarine due to viscous resistance is given by :

$$\frac{R}{\rho v^2 D^2} = f\left(\frac{\rho v D}{\mu}\right)^d$$ where D is the characteristic dimension.

The submarine moves at 8 m/s through sea water. In order to predict its resistance, a model is made to a scale of 1/100 and tested in fresh water. Determine the velocity at which the model should be tested. (690.7 m/s)
The density of sea water is 1036 kg/m^3
The density of fresh water is 1000 kg/m^3
The viscosity of sea water is 0.0012 N s/m^2.
The viscosity of fresh water is 0.001 N s/m^2.
When run at the calculated speed, the model resistance was 200 N. Predict the resistance of the submarine. (278 N).

$$\left(\frac{\rho v D}{\mu}\right)_m = \left(\frac{\rho v D}{\mu}\right)_o \qquad \left(\frac{1000 \text{ x } v_m \text{ x } D_m}{0.001}\right) = \left(\frac{1036 \text{ x } 8 \text{ x } D}{100 \text{ x } 0.0012}\right)$$

$$v_m = 690.7 \text{ m/s}$$

$$\left(\frac{R}{\rho v^2 D^2}\right)_m = \left(\frac{R}{\rho v^2 D^2}\right)_m \qquad \left(\frac{200}{1036 \text{ x } 8^2 (D_m/100)^2}\right) = \left(\frac{R_o}{1000 \text{ x } 690.7^2 D^2}\right)$$

$$R_o = 278 \text{ N}$$

3. The resistance of an aeroplane is due to, viscosity and compressibility of the fluid. Show that:

$$\left(\frac{R}{\rho v^2 D^2}\right) = f(M_a)(R_e)$$

An aeroplane is to fly at an altitude of 30 km at Mach 2.0. A model is to be made to a suitable scale and tested at a suitable velocity at ground level. Determine the velocity of the model that gives dynamic similarity for the Mach number and then using this velocity determine the scale which makes dynamic similarity in the Reynolds number. (680.6 m/s and 1/61.86)

The properties of air are

sea level	a= 340.3 m/s	μ= 1.7897 x 10^{-5}	ρ = 1.225 kg/m^3
30 km	a= 301.7 m/s	μ= 1.4745 x 10^{-5}	ρ = 0.0184 kg/m^3

When built and tested at the correct speed, the resistance of the model was 50 N. Predict the resistance of the aeroplane.

For dynamic similarity $(M_a)_m = (M_a)_o$ $\quad$ $(v/a)_m = (v/a)_o = 2$ $\quad$ $v_m = 340.3 = 680.6$ m/s

and $(R_e)_m = (R_e)_o$

$$\left(\frac{\rho v D}{\mu}\right)_m = \left(\frac{\rho v D}{\mu}\right)_o \qquad \left(\frac{D_o}{D}_m\right) = \left(\frac{\rho_m v_m \mu_o}{\rho_o v_o \mu_m}\right) = \frac{1.225 \times 680.6 \times 1.4745 \times 10^{-5}}{0.0184 \times 603.4 \times 1.7897 \times 10^{-5}} = 61.87$$

and $$\left(\frac{R}{\rho v^2 D^2}\right)_o = \left(\frac{R}{\rho v^2 D^2}\right)_m \qquad R_o = \left(\frac{R_m\ \rho_o\ v_o^2 D_o^2}{\rho_m\ v_m^2 D_m^2}\right) = \frac{50 \times 0.0184 \times 602.4^2}{1.225 \times 680.6^2} \times 61.87^2$$

$$R_o = 2259 \text{ N}$$

4. The force on a body of length 3 m placed in an air stream at 1 bar and moving at 60 m/s is to be found by testing a scale model. The model is 0.3 m long and placed in high pressure air moving at 30 m/s. Assuming the same temperature and viscosity, determine the air pressure which produced dynamic similarity.

The force on the model is found to be 500 N. Predict the force on the actual body.

The relevant equation is $\left(\frac{F}{\rho v^2 D^2}\right) = f(R_e)$

For dynamic similarity $(R_e)_m = (R_e)_o$ $\qquad \left(\frac{\rho v D}{\mu}\right)_m = \left(\frac{\rho v D}{\mu}\right)_o \qquad \mu_o = \mu_m$

$$(\rho v D)_m = (\rho v D)_o$$

For a gas $p = \rho RT$ $\quad$ The temperature T is constant so $\rho \propto p = c\,p$

$p_m \times 30 \times 0.3 = p_o \times 60 \times 3 = 1\text{bar} \times 60 \times 3$ $\qquad$ $p_m = 20$ bar

$$\left(\frac{F}{\rho v^2 D^2}\right)_m = \left(\frac{F}{\rho v^2 D^2}\right)_o \qquad \left(\frac{500}{c\,p_m \times 30^2 \times 0.3^2}\right) = \left(\frac{F_o}{c\,p_o \times 60^2 \times 3^2}\right)$$

$$F_o = 10\ 000\text{N}$$

5. Show by dimensional analysis that the velocity profile near the wall of a pipe containing turbulent flow is of the form $u^+ = f(y^+)$

$u^+ = u(\rho/\tau_o)^{1/2}$ and $y^+ = y(\rho\tau_o)^{1/2}/\mu$

When water flows through a smooth walled pipe 60 mm bore diameter at 0.8 m/s, the velocity profile is $u^+ = 2.5\ln(y^+) + 5.5$

Find the velocity 10 mm from the wall.

The friction coefficient is $C_f = 0.079\, R_e^{-0.25}$.

This is best solved by Buckingham's Pi method.

$$\tau_o = \phi(y\ u\ \rho\ \mu)$$

Form a dimensionless group with μ and leave out u that means the group $\left(y^{x_1}u^{y_1}\rho^{z_1}\mu^1\right)$ has no dimensions

$$M^0L^0T^0 = (L)^{x_1}\left(LT^{-1}\right)^{y_1}\left(ML^{-3}\right)^{z_1}\left(ML^{-1}T^{-1}\right)^1$$

Time	$0 = 1 - 2z_1$	$z_1 - ½$
Mass	$0 = y_1 + z_1 + 1$	$y_1 = -½$
Length	$0 = x_1 - 3y_1 - z_1 - 1$	$x_1 = -1$

The group is $\mu\, y^{-1}\, \rho^{-½}\, \tau_o^{-½}$ or $\dfrac{\mu}{y(\rho\tau_o)^{1/2}}$

Form a dimensionless group with u and leave out μ that means the group $\left(y^{x_2}u^{y_2}\rho^{z_2}u^1\right)$ has no dimensions

$$M^0L^0T^0 = (L)^{x_2}\left(LT^{-1}\right)^{y_2}\left(ML^{-3}\right)^{z_2}\left(LT^{-1}\right)^1$$

Time	$0 = 2z_2 - 1$	$z_2 - ½$
Mass	$0 = y_2 + z_2$	$y_2 = ½$
Length	$0 = x_2 - 3y_2 - z_2 + 1$	$x_2 = 0$

The group is $u\, y^0\, \rho^{½}\, \tau_o^{-½}$ or $u\left(\dfrac{\rho}{\tau_o}\right)^{1/2}$

Hence $$u\left(\frac{\rho}{\tau_o}\right)^{1/2} = f\left(\frac{\mu}{y(\rho\tau_o)^{1/2}}\right) \quad \text{or} \quad u^+ = f(y^+)$$

$C_f = 2\tau_o/(\rho u^2)$= Wall Shear Stress/Dynamic Pressure

$C_f = 0.079\, R_e^{-0.25}$ $\quad 0.079\,(\rho\, u\, D/\mu)^{-0.25} = 5.1879 \times 10^{-3}$

$5.1879 \times 10^{-3} = 2\tau_o/(997 \times 0.8^2)$ $\quad \tau_o = 1.655$ Pa

$u^+ = 2.5 \ln y^+ + 5.5$ $\quad u(\rho/\tau_o)^{1/2} = 2.5 \ln \{(y/\mu)(\rho\tau_o)^{1/}\} + 5.5$

$u(997/1.6551)^{1/2} = 2.5 \ln \{(0.01/0.89\text{x}10^{-3})(1.655 \times 997)^{1/2}\} + 5.5$

24.543 u = 2.5 ln 451.42 + 5.5

u = 0.85 m/s

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 7 – FLUID FORCES

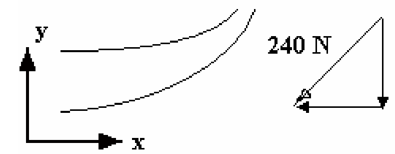
SELF ASSESSMENT EXERCISE No. 1

1. A pipe bends through an angle of 90° in the vertical plane. At the inlet it has a cross sectional area of 0.003 m^2 and a gauge pressure of 500 kPa. At exit it has an area of 0.001 m^2 and a gauge pressure of 200 kPa.
Calculate the vertical and horizontal forces due to the pressure only.

Fh = 500 000 x 0.003 = 1500 N→   Fv = 200 000 x 0.001 = 200 N↓

2. A pipe bends through an angle of 45° in the vertical plane. At the inlet it has a cross sectional area of 0.002 m^2 and a gauge pressure of 800 kPa. At exit it has an area of 0.0008 m^2 and a gauge pressure of 300 kPa.
Calculate the vertical and horizontal forces due to the pressure only.

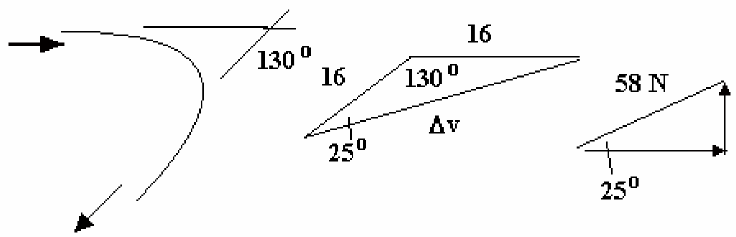
Fp1 = 800 000 x 0.002 = 1600 N   Fpx1 = 1600 N   Fpy1 = 0
Fp2 = 300 000 x 0.0008 = 240 N



Fpy2 = 240 sin 45° = 169.7 N   Fpx2 = 240 cos 45°= 169.7 N

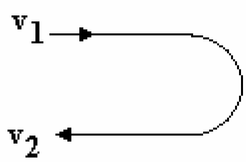
Totals F_h =1600 – 169.7 = 1430 N F_v = 0 - 169.7 = -169.7 N

3. Calculate the momentum force acting on a bend of 130° that carries 2 kg/s of water at 16m/s velocity.
Determine the vertical and horizontal components.



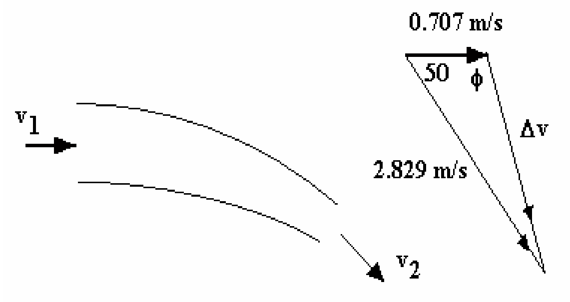
Δv = 16 sin 130/sin 25 = 29 m/s   F = m Δv = 2 x 29 = 58 N
Fv = 58 sin 25 = 24.5 N   Fh = 58 cos 25 = 52.57 N

4. Calculate the momentum force on a 180° bend that carries 5 kg/s of water. The pipe is 50 mm bore diameter throughout. The density is 1000 kg/m^3.



v_1 = Q/A = m/ρA = 5/(1000 x π x 0.025^2) =2.546 m/s
v_2 = -2.546 m/s
Δv = 2.546 – (-2.546) = 5.093 m/s   F = m Δv = 5 x 5.093 = 25.25 N

5. A horizontal pipe bend reduces from 300 mm bore diameter at inlet to 150 mm diameter at outlet. The bend is swept through 50° from its initial direction.
The flow rate is 0.05 m^3/s and the density is 1000 kg/m^3. Calculate the momentum force on the bend and resolve it into two perpendicular directions relative to the initial direction.



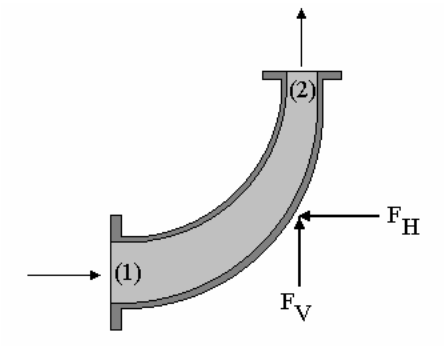
$$\Delta v = \sqrt{(2.82\sin 50)^2 + (2.825\sin 50 - 0.707)^2} = 2.4359 m/s$$

$$\varphi = \tan^{-1}\left(\frac{2.825\sin 50}{2.82\cos 50 - .707}\right) = 62.8^o$$

F = m Δv = 50 x 2.43 = 121.5 N
Fv = 121.5 sin 6.84 = 108.1 N   Fh = 121.5 cos 62.84 = 55.46 N

SELF ASSESSMENT EXERCISE No. 2

Assume the density of water is 1000 kg/m^3 throughout.
1. A pipe bends through 90° from its initial direction as shown in fig.13. The pipe reduces in diameter such that the velocity at point (2) is 1.5 times the velocity at point (1). The pipe is 200 mm diameter at point (1) and the static pressure is 100 kPa. The volume flow rate is 0.2 m^3/s. Assume there is no friction. Calculate the following.
a) The static pressure at (2).
b) The velocity at (2).
c) The horizontal and vertical forces on the bend F_H and F_V.
d) The total resultant force on the bend.



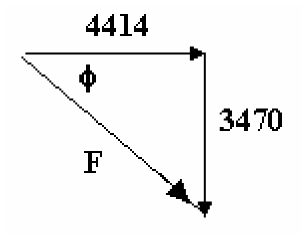
$u_2 = 1.5\, u_1$ $D_1 = 200$ mm $p_1 = 100$ kPa $Q = 0.02\ m^3/s$
$m = 200$ kg/s $A_1 = \pi D_1^2/4 = 0.0314\ m^2$
$u_1 = Q/A_1 = 6.37$ m/s $u_2 = 1.5\, u_1 = 9.55$ m/s

Bernoulli $p_1 + \rho u_1^2/2 = p_2 + \rho u_2^2/2$
Gauge pressures assumed.
$100\,000 + 1000 \times 6.37^2/2 = p_2 + 1000 \times 9.55^2/2$
$p_2 = 74.59$ kPa $A_2 = Q/u_2 = 0.0209\ m^2$

$F_{p1} = p_1 A_1 = 3140\ N \rightarrow$ $F_{p2} = p_2 A_2 = 1560\ N \downarrow$

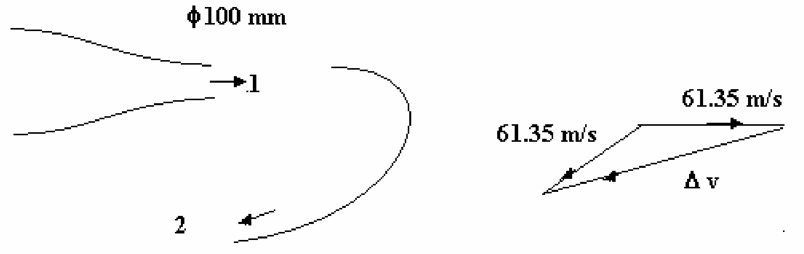
$F_{m1} = m\Delta v$ (hor) = 200 (0 – 6.37) = -1274 N on water and 1274 N on bend →
$F_{m2} = m\Delta v$ (vert) = 200 (9.55 - 0) = 1910 N on water and -1910 N on bend ↓
Total horizontal force on bend = 3140 + 1274 = 4414→
Total vertical force on bend = 1560↓ + 1910 = 3470 N ↓

$F = \sqrt{(4414^2 + 3470^2)} = 561$ N $\phi = \tan^{-1}(3470/4414) = 38.1^o$



2. A nozzle produces a jet of water. The gauge pressure behind the nozzle is 2 MPa. The exit diameter is 100 mm. The coefficient of velocity is 0.97 and there is no contraction of the jet. The approach velocity is negligible. The jet of water is deflected 165º from its initial direction by a stationary vane. Calculate the resultant force on the nozzle and on the vane due to momentum changes only.

$c_v = 0.97$ $\Delta p = 2$ MPa $\rho = 1000$ kg/m^3



$v_1 = c_v \sqrt{(2\Delta p/\rho)} = 0.97\sqrt{(2 \times 2 \times 10^6/1000)} = 61.35$ m/s

$m = \rho A_1 v_1 = 1000 \times \pi \times 0.1^2/4 \times 61.35 = 481.8$ kg/s

Force on Nozzle = m Δv = 481.8 x (61.35 – 0) = 29.56 kN

Force on vane = m Δv $\Delta v = 61.35\sqrt{\{2(1 - \cos 165^o)\}} = 121.6$ m/s

Force on vane = m Δv = 481.8 x 121.6 = 58.6 kN

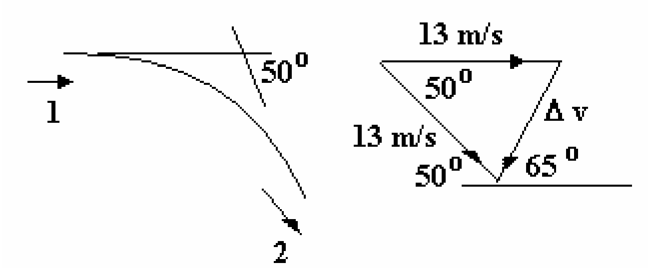
3. A stationary vane deflects 5 kg/s of water 50º from its initial direction. The jet velocity is 13 m/s. Draw the vector diagram to scale showing the velocity change. Deduce by either scaling or calculation the change in velocity and go on to calculate the force on the vane in the original direction of the jet.

$v_1 = 13$ m/s m = 5 kg/s

$\Delta v = 13 \sin 50^o/\sin 65^o = 10.99$ m/s

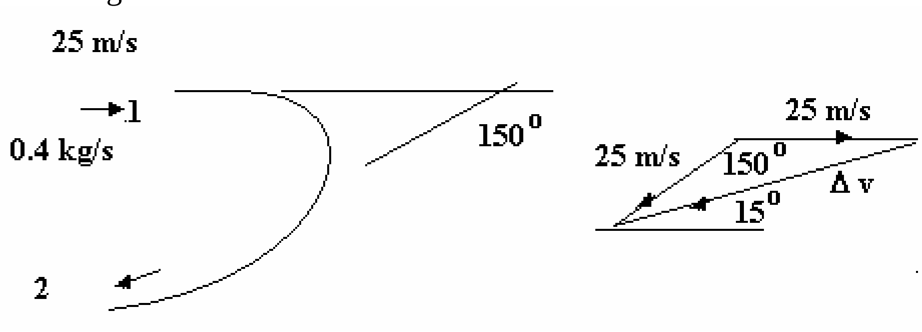
Force on vane = m Δv = 5 x 10.99= 54.9 N

Horizontal component is F$\cos 65^o$ = 23.2 N



4. A jet of water travelling with a velocity of 25 m/s and flow rate 0.4 kg/s is deflected 150º from its initial direction by a stationary vane. Calculate the force on the vane acting parallel to and perpendicular to the initial direction.

$v_1 = 25$ m/s m = 0.4 kg/s



$\Delta v = 25 \sqrt{\{2(1 - \cos 150^o)\}} = 48.3$ m/s F = m Δv = 0.4 x 48.3 = 19.32 N

$F_v = 19.32 \sin 15^o = 5$ N

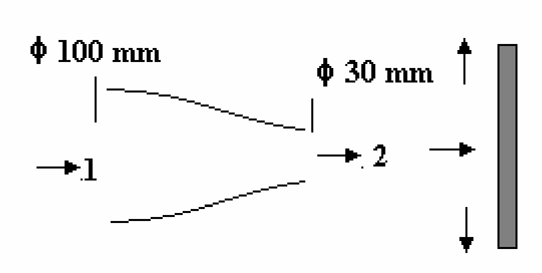
$F_h = 19.32 \cos 15^o = 18.66$ N

5. A jet of water discharges from a nozzle 30 mm diameter with a flow rate of 15 dm^3/s into the atmosphere. The inlet to the nozzle is 100 mm diameter. There is no friction nor contraction of the jet. Calculate the following.

i. the jet velocity. ii. the gauge pressure at inlet. iii. the force on the nozzle.

The jet strikes a flat stationary plate normal to it. Determine the force on the plate.

$Q = 0.015\ m^3/s$ $\rho = 1000\ kg/m^3$ $m = 15\ kg/s$



$A_1 = \pi \times 0.1^2/4 = 0.00785\ m^2$
$v_1 = Q/A_1 = 0.015 \div 0.00785 = 1.901\ m/s$
$A_2 = \pi \times 0.03^2/4 = 0.0007068\ m^2$
$v_2 = Q/A_2 = 0.015 \div 0.0007068 = 21.22\ m/s$

Bernoulli $p_1 + \rho v_1^2/2 = p_2 + \rho v_2^2/2$

Gauge pressures assumed.

$p_1 + 1000 \times 1.901^2/2 = 0 + 1000 \times 21.22^2/2$
$p_1 = 223.2\ kPa$

Force on nozzle $= (p_1A_1 - p_2A_2) + m(v_2 - v_1)$ v_1 is approximately zero.
$= (223.2 \times 10^3 \times 0.00785 - 0) + 15(21.22 - 0) = 2039\ N \leftarrow$

Force on Plate = $m\,\Delta v$ Δv in horizontal direction is 21.22

Force on Plate = $15 \times 21.22 = 311.8\ N \rightarrow$

Some common sense is needed determining the directions.

SELF ASSESSMENT EXERCISE No.3

1. A vane moving at 30 m/s has a deflection angle of 90°. The water jet moves at 50 m/s with a flow of 2.5 kg/s. Calculate the diagram power assuming that all the mass strikes the vane.

$\rho = 100\ kg/m^3$ $m = 2.5\ kg/s$ $u = 30\ m/s$ $v = 50\ m/s$

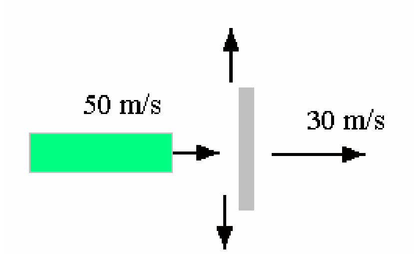
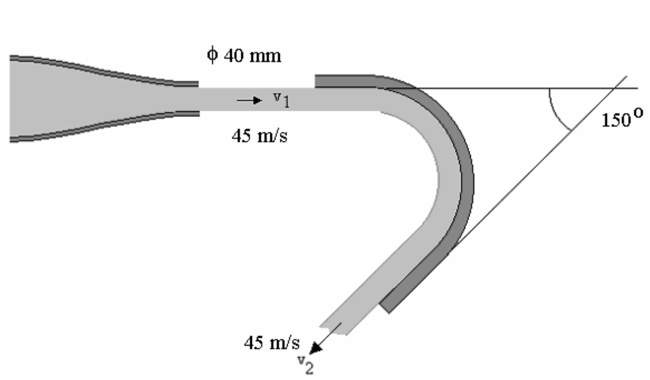
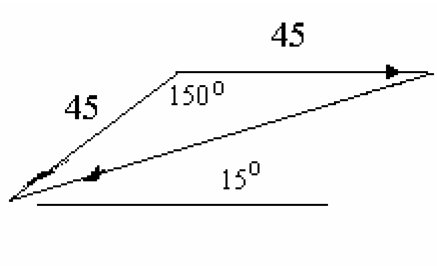


Diagram Power = $mu(v - u) = 2.5 \times 30\,(50 - 30) = 1500$ Watts

2. Figure 10 shows a jet of water 40 mm diameter flowing at 45 m/s onto a curved fixed vane. The deflection angle is 150°. There is no friction. Determine the magnitude and direction of the resultant force on the vane.

The vane is allowed to move away from the nozzle in the same direction as the jet at a velocity of 18 m/s. Draw the vector diagram for the velocity at exit from the vane and determine the magnitude and direction of the velocity at exit from the vane.



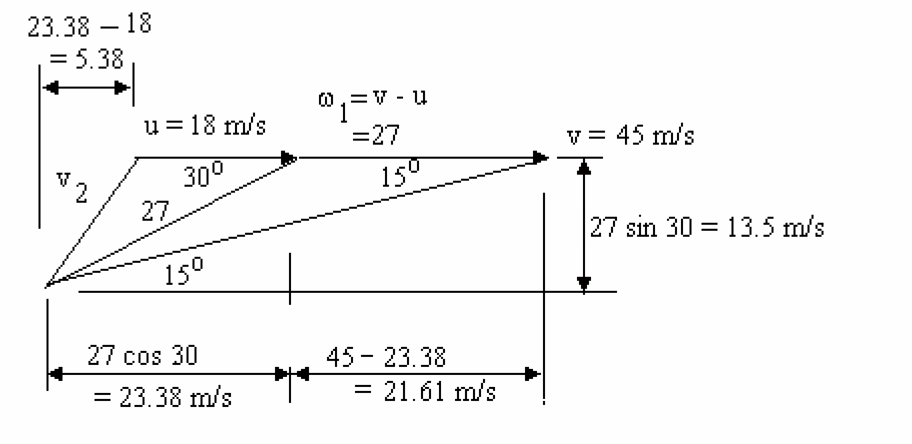


STATIONARY VANE

$\Delta v = 45\sqrt{\{2(1-\cos 150^o)\}} = 86.93$ m/s $m = \rho A v = 1000 \times \pi \times 0.04^2/4 \times 45 = 56.54$ kg/s

$F = m\,\Delta v = 4916$ N

MOVING VANE



The relative velocity at exit is $\omega_2 = 27$ m/s

The absolute velocity $v_2 = \sqrt{(13.5^2 + 5.38^2)} = 14.53$ m/s

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 8A –TURBINES

SELF ASSESSMENT EXERCISE No. 1

1. The buckets of a Pelton wheel revolve on a mean diameter of 1.5 m at 1500 rev/min. The jet velocity is 1.8 times the bucket velocity. Calculate the water flow rate required to produce a power output of 2MW. The mechanical efficiency is 80% and the blade friction coefficient is 0.97. The deflection angle is 165°.

$D = 1.5$ m $N = 1500$ rev/min $v = 1.8\,u$ $\eta = 80\%$ $k = 0.97$ $\theta = 165^o$

Diagram Power = 2MW/0.8 = 2.5 MW
$u = \pi ND/60 = 117.8$ m/s
$v = 1.8 \times 117.8 = 212$ m/s
$DP = m\,u\,(v-u)\,(1-k\cos\theta) = 2.5 \times 10^6$
$m \times 117.8\,(94.24)(1-0.97\cos 165) = 2.5 \times 10^6$
$m = 2.5 \times 10^6/21503 = 116.26$ kg/s

2. Calculate the diagram power for a Pelton Wheel 2m mean diameter revolving at 3000 rev/min with a deflection angle of 170° under the action of two nozzles , each supplying 10 kg/s of water with a velocity twice the bucket velocity. The blade friction coefficient is 0.98.

If the coefficient of velocity is 0.97, calculate the pressure behind the nozzles.
(Ans 209.8 MPa)

$D = 2$m $N = 3000$ rev/min $\theta = 170^o$ $v = 2u$ $k = 0.98$ $c_v = 0.97$ $m = 2 \times 10 = 20$ kg/s
$u = \pi ND/60 = 314.16$ m/s
$DP = m\,u\,(v-u)\,(1-k\cos\theta) = 20 \times 314.16 \times 314.16(1-0.98\cos 170^o) = 3.879$ MW
$v = c_v \sqrt{2\Delta p/\rho}$
$\Delta p = (314.16 \times 2/0.97)^2 \times 1000/2 = 209.8$ MPa

3. A Pelton Wheel is 1.7 m mean diameter and runs at maximum power. It is supplied from two nozzles. The gauge pressure head behind each nozzle is 180 metres of water. Other data for the wheel is :

Coefficient of Discharge C_d= 0.99
Coefficient of velocity C_v= 0.995
Deflection angle = 165°.
Blade friction coefficient = 0.98
Mechanical efficiency = 87%
Nozzle diameters = 30 mm

Calculate the following.

i. The jet velocity (59.13 m/s)
ii. The mass flow rate (41.586 kg/s)
iii The water power (73.432 kW)
iv. The diagram power (70.759 kW)
v. The diagram efficiency (96.36%)
vi. The overall efficiency (83.8%)
vii. The wheel speed in rev/min (332 rev/min)

$D = 1.7$ m $\Delta H = 180$ m $c_d = 0.99$ $c_v = 0.995$ $c_c = c_d/c_v = 0.995$ $\rho = 1000$ kg/m^3

Power is maximum so v = 2u   2 nozzles

$v = c_v \sqrt{2g\,\Delta H} = 0.995\sqrt{(2g \times 180)} = 59.13$ m/s
$m = c_c\,\rho\,A\,v = 0.995 \times 1000\,(\pi \times 0.03^2/4) \times 59.13 = 41.587$ kg/s per nozzle
Water Power = $mg\,\Delta H = 41.587 \times 9.81 \times 180 = 73.43$ kW per nozzle.
$u = v/2 = 29.565$ m/s
Diagram Power = $m\,u\,(v-u)\,(1- k\cos\theta)$
DP = $41.587 \times 29.565\,(29.565)(1 - 0.98\cos165) = 70.76$ kW per nozzle
$\eta_d = 70.76/73.43 = 83.8\%$
Mechanical Power = $70.76 \times 87\% = 61.56$ kW per nozzle.
$\eta_{oa} = 61.56/73.43 = 83.8\%$
$N = 60u/\pi D = 29.565 \times 60/(\pi \times 1.7) = 332.1$ rev/min

4. Explain the significance and use of 'specific speed $Ns = \dfrac{NP^{1/2}}{\rho^{1/2}(gH)^{5/4}}$

Calculate the specific speed of a Pelton Wheel given the following.

d = nozzle diameter. D = Wheel diameter.
u = optimum blade speed = $0.46\,v_1$ v_1 = jet speed.
$\eta = 88\%$ C_V = coefficient of velocity = 0.98

$v_j = c_v\sqrt{2gH} = 0.98\sqrt{2gH} = 4.34H^{1/2}$ $u = 0.46\,v_j = \pi ND/60$

$$N = \frac{0.46\,v_j \times 60}{\pi D} = \frac{0.46 \times 4.34H^{1/2} \times 60}{\pi D} = 38.128\frac{H^{1/2}}{D}$$

$Q = A_j v_j = (\pi d^2/4) \times 4.34\,H^{½} = 3.41\,H^{½}\,d^2$

$P = \eta\,m\,g\,H = \eta \times \rho g\,Q\,H = 0.88 \times 1000 \times 9.81 \times 3.41 \times H^{½}\,d^2 = 29438\,H^{½}\,d^2\,H$

$$Ns = 38.128\frac{H^{1/2}}{D} \times \frac{\left(28438\,H^{3/2}d^2\right)^{1/2}}{\rho^{1/2}(gH)^{5/4}} = \frac{38.128 \times 28438^{1/2}}{1000^{1/2}9.81^{5/4}} \times \frac{H^{1/2}H^{3/4}d}{D\,H^{5/4}} = 11.9\frac{d}{D}$$

5. A turbine is to run at 150 rev/min under a head difference of 22 m and an expected flow rate of 85 m^3/s.

A scale model is made and tested with a flow rate of 0.1 m^3/s and a head difference of 5 m. Determine the scale and speed of the model in order to obtain valid results.

When tested at the speed calculated, the power was 4.5 kW. Predict the power and efficiency of the full size turbine.

$N_1 = 150$ rev/min $Q_1 = 85\ m^3/s$ $\Delta H_1 = 22$ m
$Q_2 = 0.1\ m^3/s$ $\Delta H_2 = 55$ m

For similarity of Head Coefficient we have

$$\frac{\Delta H_1}{N_1^2 D_1^2} = \frac{\Delta H_2}{N_2^2 D_2^2} \qquad \frac{D_2^2}{D_1^2} = \frac{5x150^2}{22N_2^2} = \frac{5114}{N_2^2} \qquad \frac{D_2}{D_1} = \sqrt{\frac{5114}{N_2^2}} = \frac{71.51}{N_2}$$

For similarity of Flow Coefficient we have

$$\frac{Q_1}{N_1 D_1^3} = \frac{Q_2}{N_2 D_2^3} \qquad \frac{D_2^3}{D_1^3} = \frac{0.1x150}{85N_2} = \frac{0.176}{N_2} \qquad \frac{D_2}{D_1} = \sqrt[3]{\frac{71.51}{N_2}} = \frac{0.560}{N_2^{1/2}}$$

Equate $\dfrac{D_2}{D_1} = \dfrac{71.51}{N_2} = \dfrac{0.560}{N_2^{1/2}}$ $N_2^{2/3} = \dfrac{71.51}{0.56}$ $N_2 = 1443$ rev/min

$$\frac{D_2}{D_1} = 0.0496$$

Note if we use $\frac{N_1 Q_1^{1/2}}{H_1^{3/4}} = \frac{N_2 Q_2^{1/2}}{H_2^{3/4}}$ we get the same result.

Power Coefficient

$$\frac{P_1}{\rho N_1^3 D_1^5} = \frac{P_2}{\rho N_2^3 D_2^5} \qquad P_1 = \frac{P_2\left(\rho N_1^3 D_1^5\right)}{\rho N_2^3 D_2^5} = \frac{P_2 N_1^3 D_1^5}{N_2^3 D_2^5} = \frac{4.5 \times 150^3 \times \left(\frac{1}{0.05}\right)^5}{1443^3} = 16.2\text{MW}$$

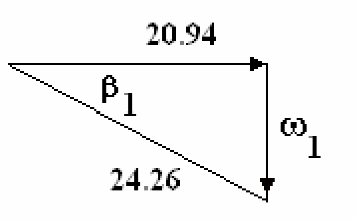
Water Power = mgΔH = (85 x 1000) x 9.81 x 22 = 18.3 MW

H = 16.2/18.3 = 88%

SELF ASSESSMENT EXERCISE No.2

1. The following data is for a Francis Wheel
   Radial velocity is constant
   No whirl at exit.
   Flow rate=0.4 m^3/s D_1=0.4 m D_2=0.15 m k =0.95 α_1=90^o N=1000 rev/min
   Head at inlet = 56 m head at entry to rotor = 26 m head at exit = 0 m
   Entry is shock less.

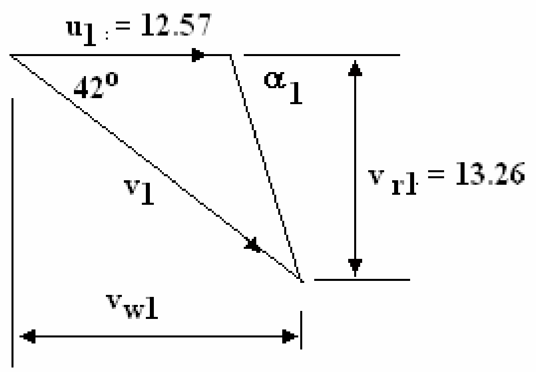
Calculate i. the inlet velocity v_1 (24.26 m/s)
ii. the guide vane angle (30.3^o)
iii. the vane height at inlet and outlet (27.3 mm, 72.9 mm)
iv. the diagram power (175.4 MW)
v. the hydraulic efficiency (80%)



$v_1 = (2gh)^{½} = \{2 \times 9.81 \times (56 – 26)\}^{½} = 24.26$ m/s
$u_1 = \pi ND/60 = \pi \times 1000 \times 0.4/60 = 20.94$ m/s
$\beta_1 = \cos^{-1}(20.94/24.26) = 30.3^o$
$\omega_1 = v_{r1} = 12.25$ m/s
$Q = 0.4 = \pi D t k v_r$
$t_1 = 0.4/(\pi \times 0.4 \times 0.95 \times 12.25) = 0.0273$ m
$t_2 = 0.4/(\pi \times 0.15 \times 0.95 \times 12.25) = 0.0729$ m
$v_{w1} = 20.94$ $v_{w2} = 0$
$P = mu_1 v_{w1} = 400 \times 20.94 \times 20.94 = 174.4$ kW
Water Power = m g H = 400 x 9.81 x 56 = 219.7 kW
$\eta = 174.4/219.7 = 80\%$

2. A radial flow turbine has a rotor 400 mm diameter and runs at 600 rev/min. The vanes are 30 mm high at the outer edge. The vanes are inclined at 42^o to the tangent to the inner edge. The flow rate is 0.5 m^3/s and leaves the rotor radially. Determine

   i. the inlet velocity as it leaves the guide vanes. (19.81 m/s)
   ii. the inlet vane angle. (80.8^o)
   iii. the power developed. (92.5 kW)

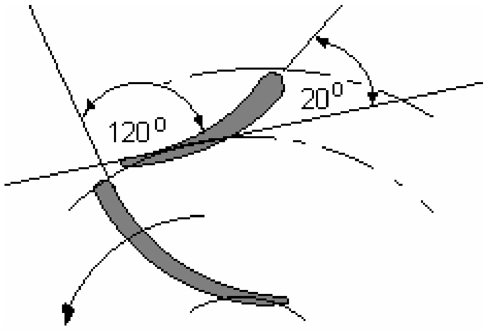


Radial Flow Turbine Inlet is the outer edge. $u_1 = \pi ND/60 = \pi \times 600 \times 0.4/60 = 12.57$ m/s
$v_{r1} = Q/\pi Dt = 0.5/(\pi \times 0.4 \times 0.03) = 13.26$ m/s

$13.26/v_{w1} = \tan 42^o$
$v_{w1} = 14.72$ m/s
$v_1 = (13.26^2 + 14.72^2)^{½} = 19.81$ m/s

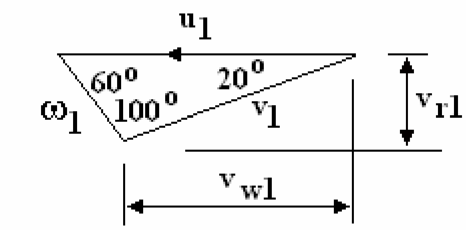
$13.26/(14.72 - 12.57) = \tan \alpha_1$
$\alpha_1 = 80.8^o$
$v_{w2} = 0$
$DP = mu\ \ = v_{w1}$
$DP = 500 \times 12.57 \times 14.72 = 92.5$ kW

3. The runner (rotor) of a Francis turbine has a blade configuration as shown. The outer diameter is 0.45 m and the inner diameter is 0.3 m. The vanes are 62.5 mm high at inlet and 100 mm at outlet. The supply head is 18 m and the losses in the guide vanes and runner are equivalent to 0.36 m. The water exhausts from the middle at atmospheric pressure. Entry is shock less and there is no whirl at exit. Neglecting the blade thickness, determine :



i. The speed of rotation.
ii. The flow rate.
iii. The output power given a mechanical efficiency of 90%.
iv. The overall efficiency.
v. The outlet vane angle.

INLET
Useful head is $18 - 0.36 = 17.64$ m

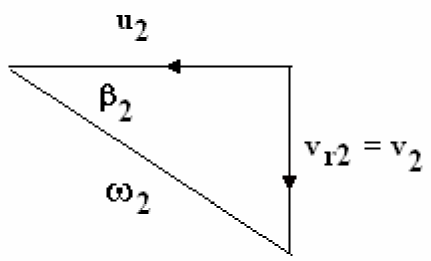


$m\, u_1\, v_{w1} = m\, u_2\, v_{w2}$
$u_1\, v_{w1} = u_2\, v_{w2}$

$(u_1\, v_{w1}/g) = \Delta H = 17.64$

sine rule $(v_1/\sin 60) = (u_1/\sin 100)$
$v_1 = 0.879\, u_1$
$(v_{r1}/ v_1) = \sin 20$ $\qquad v_1 = 2.923\, v_{r1}$
Equate $\qquad 0.879\, u_1 = 2.923\, v_{r1} \qquad v_{r1} = 0.3\, u_1$
$v_{w1} = v_{r1}/\tan 20 = 0.824\, u_1$
$17.64 = u_1 \times 0.824\, u_1 /g \qquad u_1^2 = 210 \quad u_1 = 14.5$ m/s
$v_{r1} = 0.3\, u_1 = 4.35$ m/s

EXIT



$u = \pi N D \quad N = u_1 / \pi D_1 = u_2 / \pi D_2$
$u_2 = u_1 D_1 / D_2 = 14.4 \times 300/450 = 9.67$ m/s
$N = u_1/ \pi D_1 = 14.5 \times 60/(\pi \times 0.45) = 615$ rev/min
$v_r = Q/\pi D h$
$v_{r1} = 4.35 = Q/\pi D_1 h_1 = Q/(\pi \times 0.45 \times 0.0625)$
$Q = 0.384\ m^3/s$
$v_{r2} = Q/\pi D_2 h_2 = Q/(\pi \times 0.3 \times 0.1) = 10.61\, Q = 4.08$ m/s
$4.08/9.67 = \tan \beta_2$
$\beta_2 = 22.8^o$

$P = m\, g\, \Delta H = 384 \times 9.81 \times 17.64 = 66.45$ kW

Output Power = $66.45 \times 90\% = 59.8$ kW

Overall efficiency = $59800/(m\, g\ \ \Delta H) = 58805/(384 \times 9.81 \times 18) = 88.2$ %

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 8B – CENTRIFUGAL PUMPS

SELF ASSESSMENT EXERCISE No. 1

1. A centrifugal pump must produce a head of 15 m with a flow rate of 40 dm^3/s and shaft speed of 725 rev/min. The pump must be geometrically similar to either pump A or pump B whose characteristics are shown in the table below.

Which of the two designs will give the highest efficiency and what impeller diameter should be used ?

Pump A D = 0.25 m N = 1 000 rev/min

Q (dm^3/s)	8	11	15	19	
H (m)		8.1	7.9	7.3	6.1
η%	48	55	62	56	

Pump B D = 0.55 m N = 900 rev/min

Q (dm^3/s)	6	8	9	11	
H (m)		42	36	33	27
η%	55	65	66	58	

$$Ns = \frac{NQ^{1/2}}{H^{3/4}} = \frac{725 \text{ x } 0.04^{1/2}}{15^{3/4}} = 19$$

PUMP A

Q (m^3/s)	0.008	0.011
H (m)	8.1	7.9
η%	48	55
Ns	18.6	22.26

Q (m^3/s)	0.06	0.008	0.009
H (m)	42	36	33
η%	55	65	66
Ns	13.36	17.32	19.6

Pump B gives the greater efficiency when Ns = 19

Drawing a graph or interpolating we find Q = 0.085 m^3/s H = 34.5 m , η = 65.5% when Ns=19

$$\frac{Q_1}{N_1 D_1^3} = \frac{Q_2}{N_2 D_2^3} \qquad \frac{0.04}{725 D_1^3} = \frac{0.085}{900 \text{ x } 0.55^3} \qquad D = 0.46 \text{ m}$$

or using the head coefficient

$$\frac{\Delta H_1}{N_1^2 D_1^2} = \frac{\Delta H_2}{N_2^2 D_2^2} \qquad \frac{0.15}{725^2 D_1^2} = \frac{34.5}{19002^2 \text{x} 0.55^2} \qquad D = 0.45 \text{ m}$$

Take the mean D = 0.455 m for the new pump

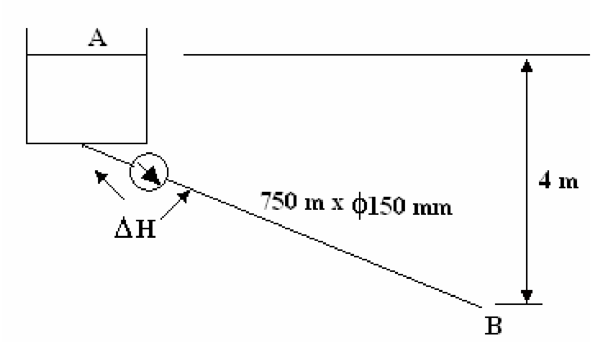
To commence pumping $\sqrt{2gH} = \pi ND/60$ $\qquad D = \frac{60\sqrt{2g \text{ x } 15}}{\pi \text{ x } 725}$

Hence D = 0.452 m This seems to give the right answer more simply.

2. Define the Head and flow Coefficients for a pump.

Oil is pumped through a pipe 750 m long and 0.15 bore diameter. The outlet is 4 m below the oil level in the supply tank. The pump has an impeller diameter of 508 mm which runs at 600 rev/min. Calculate the flow rate of oil and the power consumed by the pump. It may be assumed $C_f=0.079(Re)^{-0.25}$. The density of the oil is 950 kg/m^3 and the dynamic viscosity is 5 x 10^{-3} N s/m^2. The data for a geometrically similar pump is shown below. D = 0.552 m N = 900 rev/min

Q (m^3/min)	0	1.14	2.27	3.41	4.55	5.68	6.86
H (m)	34.1	37.2	39.9	40.5	38.1	32.9	25.9
η%	0	22	41	56	67	72	65



ΔH = Head Added to the system by the pump.
This may be put into Bernoulli's equation.

$h_A + z_A + u_A^2/2g + \Delta H = h_B + z_B + u_B^2/2g + h_L$

$0 + 4 + 0 + \Delta H = 0 + 0 + u_B^2/2g + h_L$

$4 + \Delta H = u_B^2/2g + h_L$

$u_B = Q/A = Q/(\pi \times 0.075^2) = 56.588\ Q$

$h_L = 4\ C_f\ Lu^2/2gD$ $\quad R_e = \rho u D/\mu = 950 \times (56.588\ Q)0.15/0.005 = 1612758Q$

$C_f = 0.079\ R_e^{-0.25} = 0.079\ (1612758Q)^{-0.25} = 0.002217\ Q^{-0.25}$

$h_L = 4\ (0.002217\ Q^{-0.25}) \times 750\ (56.588\ Q)^2/(2g \times 0.15) = 7236\ Q^{1.75}$

$4 + \Delta H = (56.588\ Q)^2/2g + 7236\ Q^{1.75}$

$\Delta H = 163.2\ Q^2 + 7236\ Q^{1.75} - 4$

If Q is given in m^3/min this becomes

$\Delta H = 0.0453\ Q^2 + 5.594\ Q^{1.75} - 4$ ow create a table for the system.

Q (m^3/min)	0	1	2	3	4	5
ΔH (m)	-4	1.64	15	34.7	60	90.6

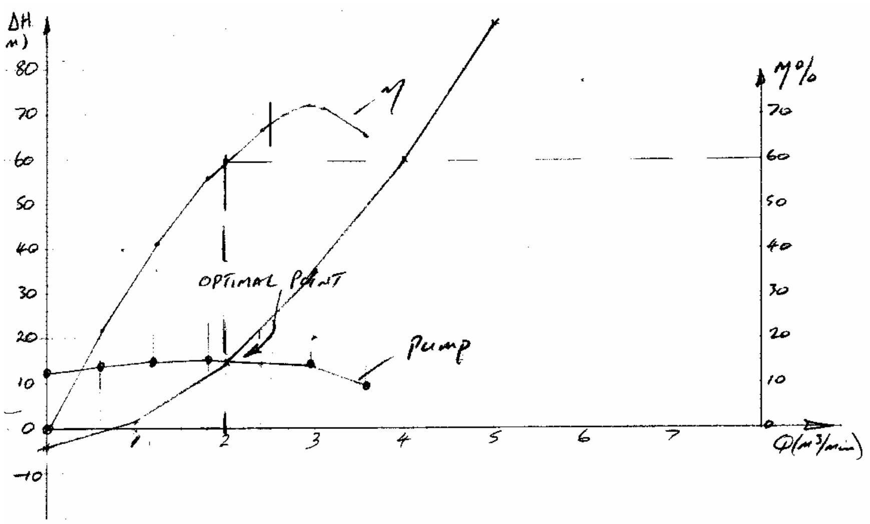
Now compile a table for a similar pump using $Q_2 = \dfrac{N_2 D_2^3 Q_1}{N_1 D_1^3}$ $\quad \Delta H_2 = \dfrac{N_2^2 D_2^2 \Delta H_1}{N_1^2 D_1^2}$

$N_2 = 600$ $D_2 = 508$ $N_1 = 900$ $D_1 = 552$ produces $Q_2 = 0.52 Q_1$ $\Delta H_2 = 0.38\Delta.\ _1$

Q_1 (m^3/min)	0	1.14	2.27	3.41	4.55	5.68	6.86
ΔH_1 (m)	34.1	37.2	39.9	40.5	38.1	32.9	25.9
η%	0	22	41	56	67	72	65
Q_2 (m^3/min)	0	0.59	1.18	1.8	2.4	2.9	3,6
ΔH_1 (m)	12.83	14	15	15.2	14.3	12.4	9.8

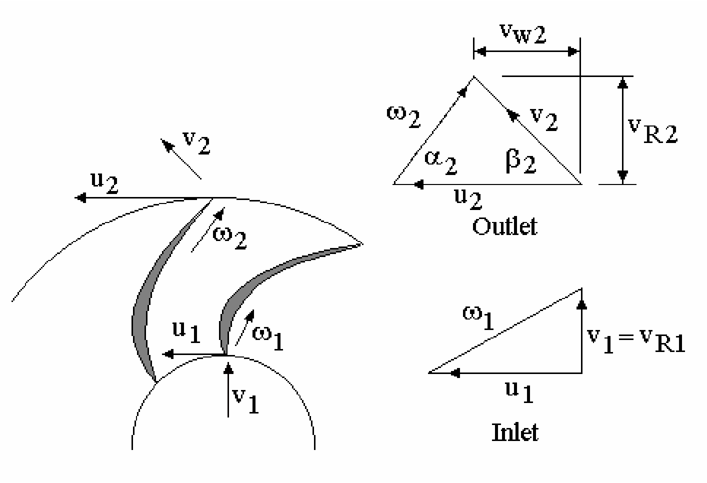
Plotting ΔH for the system against Q and η produces a matching point ΔH = 15 m Q = 2 (m^3/min) and η = 59%

P = mgΔH/η = 2 x (950/60) x 9.81 x 15/0.59 = 7.89 kW



SELF ASSESSMENT EXERCISE No. 2

1. The rotor of a centrifugal pump is 100 mm diameter and runs at 1 450 rev/min. It is 10 mm deep at the outer edge and swept back at 30°. The inlet flow is radial. the vanes take up 10% of the outlet area. 25% of the outlet velocity head is lost in the volute chamber. Estimate the shut off head and developed head when 8 dm^3/s is pumped. (5.87 m and 1.89 m)



$v_{R2} = Q/A_2$
$= 0.008/(\pi \times 0.1 \times 0.01 \times 0.9) = 2.829$ m/s

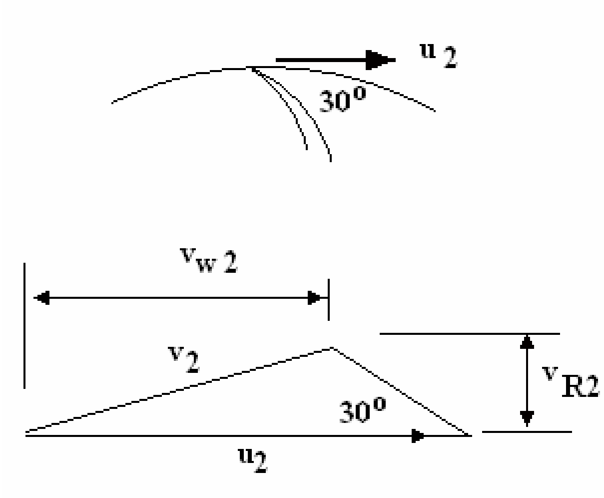
OUTLET
$u_2 = \pi N D_2$
$= \pi \times (450/60) \times 0.1 = 7.592$ m/s
$v_{w2} = 7.592 - 2.829/\tan 30^o = 2.692$ m/s
$v_2 = (2.692^2 + 2.829^2)^{½} = 3.905$ m/s
Kinetic Head = $v_2^2/2g$
$= 3.905^2/2g = 0.777$ m
Loss in chamber = 25% x 0.777 = 0.194 m
Manometric Head = $u_2\, v_{w2}/g$
= 7.592 x 2.692/9.81 = 2.08 m
Developed Head = 2.08 – 0.194 = 1.89 m

$\Delta h = u_2\, v_{w2}/g = (u_2 - Q/A_2 \tan\alpha_2)$
When there is no flow Q = 0 so $\Delta h = u_2\, v_{w2}/g - u_2 = (7.592/9.81) \times 7.592 = 5.875$ m

2. The rotor of a centrifugal pump is 170 mm diameter and runs at 1 450 rev/min. It is 15 mm deep at the outer edge and swept back at 30°. The inlet flow is radial. the vanes take up 10% of the outlet area. 65% of the outlet velocity head is lost in the volute chamber. The pump delivers 15 dm^3/s of water.

Calculate

i. The head produced. (9.23 m)
ii. The efficiency. (75.4%)
iii. The power consumed. (1.8 kW)



$u_2 = \pi N D_2 = \pi \times (1450/60) \times 0.17 = 12.906$ m/s
$v_{R2} = Q/A_2$
$v_{R2} = 0.015/(\pi \times 0.17 \times 0.015 \times 0.9) = 2.08$ m/s
$v_{w1} = 0$

OUTLET
$v_{w2} = 12.906 - 2.08/\tan 30^o = 9.3$ m/s
$v_2 = (9.3^2 + 2.08^2)^{½} = 9.53$ m/s
Kinetic Head = $v_2^2/2g = 9.53^2/2g = 4.628$ m
Head Recovered = 35% x 4.628 = 1.62 m
Head Loss = 3 m
Manometric Head = $u_2\, v_{w2}/g$
= 12.906 x 9.3/9.81 = 12.23 m
Developed Head 12.23 – 3= 9.23 m

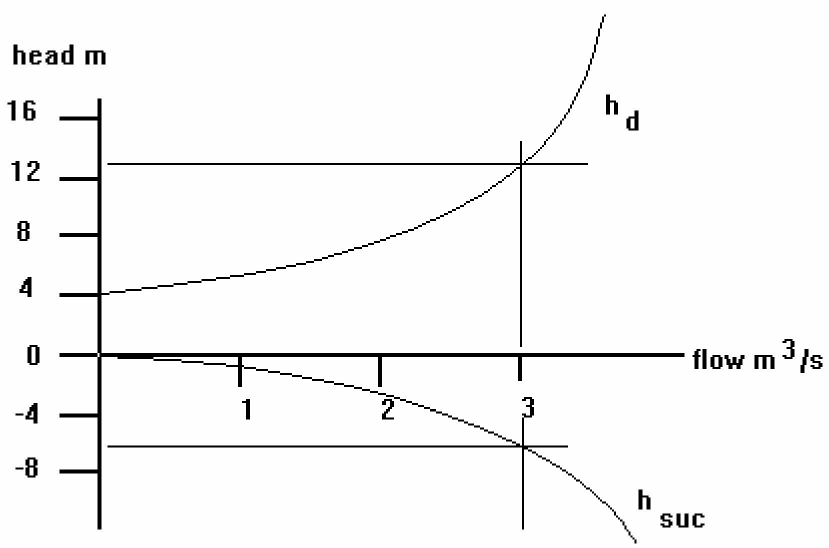
$\eta_{man} = 9.23/12.23 = 75.3\%$

DP = m $u_2\, v_{w2}$ = 15 x 12.906 x 9.3 = 1.8 kW
WP = m g Δh = 15 x 9.81 x 9.23 = 1.358 kW
η = 1.358/1.8 = 75.4%

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 8C – PUMPED PIPED SYSTEMS

SELF ASSESSMENT EXERCISE No. 1

1. A pump has a suction pipe and a delivery pipe. The head required to pass water through them varies with flow rate as shown.



The pump must deliver 3 m^3/s at 2 000 rev/min. Determine the specific speed.

The vapour pressure is 0.025 bar and atmospheric pressure is 1.025 bar. Calculate the NPSH and the cavitation parameter.

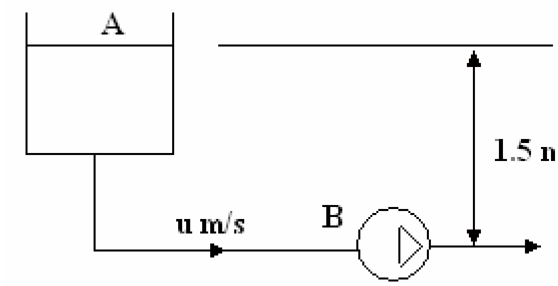
From the graph at 3 m/s h_d = 13 m h_s = -6 m
NPSH = {1.025 x 10^5/(9.81 x 1000) – 6 } – 0.025 x 10^5/(9.81 x 1000)
NPSH = 4.448 – 0.2548 = 4.19 m
σ = NPSH/h_d = 4.19/13 = 0.323
Specific speed Ns = $NQ^{½} / H^{3/4}$ = 2000 x $3^{½}/19^{3/4}$ = 380.6

2. Define the term "Net Positive Suction Head" and explain its significance in pump operation.

1.2 kg/s of acetone is to be pumped from a tank at 1 bar pressure. The acetone is at 40°C and the pump is 1.5 m below the surface. The suction pipe is 25 mm bore diameter. Calculate the NPSH at the pump inlet.

Losses in the suction pipe are equal to three velocity heads.
The vapour pressure of acetone is 55 kPa. The density is 780 kg/m^3.

The Net Positive Suction Head is the amount by which the absolute pressure on the suction side is larger than the vapour pressure (saturation pressure) of the liquid.



$u = m/\rho A = 1.2/(780 \times \pi \times 0.025^2) = 3.134$ m/s

$h_A + z_A + u_A^2/2g = h_B + z_B + u_B^2/2g + h_L$

$h_L = 3\ u_B^2/2g$
$0 + 1.5 + 0 = h_B + 0 + 3.134^2/2g + 3\ u_B^2/2g$
$h_B = 1.5 - 4\ u_B^2/2g = -0.5$ m gauge

Atmospheric pressure = 1.0 bar $\rho = 780$ kg/m^3
Convert to pressure head $h = p/\rho g = 13.06$ m
Absolute head at pump = 13.06 – 0.5 = 12.56 m
Vapour pressure head = 55 x 10^3 /ρg = 7.19 m
NPSH = 12.56 – 7.19 = 5.37 m

3. A centrifugal pump delivers fluid from one vessel to another distant vessel. The flow is controlled with a valve. Sketch and justify appropriate positions for the pump and valve when the fluid is a) a liquid and b) a gas.

(a) Minimum suction is required to avoid cavitation so put the valve on the pump outlet and this will also keep the pump primed when closed. The pump should be as close to the tank as possible.

(b) With gas cavitation is not a problem but for minimal friction the velocity must be kept low. If the gas is kept under pressure by putting the valve at the end of the pipe, it will be more dense and so the velocity will be lower for any given mass flow rate. The pump should be close to the supply tank.

SELF ASSESSMENT EXERCISE No. 2

The density of water is 1000 kg/m^3 and the bulk modulus is 4 GPa throughout.

1. A pipe 50 m long carries water at 1.5 m/s. Calculate the pressure rise produced when
   a) the valve is closed uniformly in 3 seconds.
   b) when it is shut suddenly.

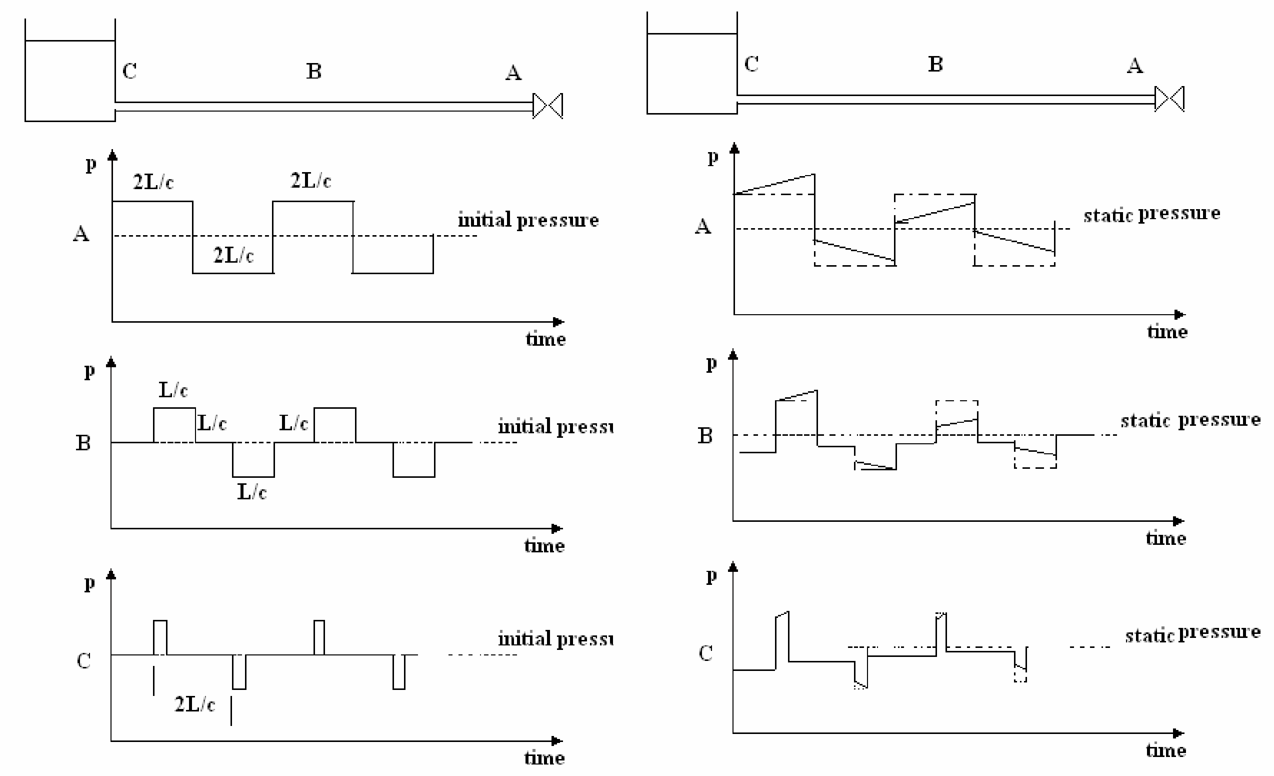
(a) $\Delta p = \rho Lu/t$ = 1000x 50 x1.5/3 = 25 kPa
(b) $\Delta p = u(K\rho)^{0.5}$ = 1.5 x (4 x 10^9 x 1000)$^{0.5}$ = 3 MPa

2. A pipe 2000 m long carries water at 0.8 m/s. A valve is closed. Calculate the pressure rise when
   a) it is closed uniformly in 10 seconds. a)
   b) it is suddenly closed.

(a) $\Delta p = \rho Lu/t$ = 1000 x 2000 x 0.8/10 = 160 kPa
(b) $\Delta p = u(K\rho)^{0.5}$ = 0.8 x (4 x 10^9 x 1000)$^{0.5}$ = 1.6 MPa

SELF ASSESSMENT EXERCISE No. 3

1. Derive the water hammer equation for a long elastic pipe carrying water from a large upstream reservoir with a constant water level to a lower downstream reservoir. Flow is controlled by a valve at the downstream end.
Sketch the variation in pressure with time for both ends and the middle of the pipe following sudden closure of the valve. Sketch these variations for when friction is negligible and for when both friction and cavitation occur.
Assuming the effective bulk modulus is given by $K' = \{(D/tE) + 1/K\}^{-1}$ and that the maximum stress in the pipe is σ, derive a formula for the maximum allowable discharge.
Part (a) is given in the tutorial. Part (b) below – no friction on left.



When cavitation occurs the minimum pressure is the vapour pressure so the bottom part of the cycle will be at this pressure.
Part (c)

For a thin walled cylinder $\sigma = \frac{pD}{2t}$ $\qquad p = \frac{2t\sigma}{D}$ $\qquad u = Q/A = 4Q/\pi D^2$

$$\Delta p = u\sqrt{\frac{\rho}{\left\{\frac{D}{2tE} + \frac{1}{K}\right\}}} \qquad \frac{2t\sigma}{D} = \frac{4Q}{\pi D^2}\sqrt{\frac{\rho}{\left\{\frac{D}{2tE} + \frac{1}{K}\right\}}} \qquad Q = \frac{\sigma \pi D t}{2}\sqrt{\frac{\left\{\frac{D}{2tE} + \frac{1}{K}\right\}}{\rho}}$$

2a. Explain the purpose and features of a surge tank used to protect hydroelectric installations.

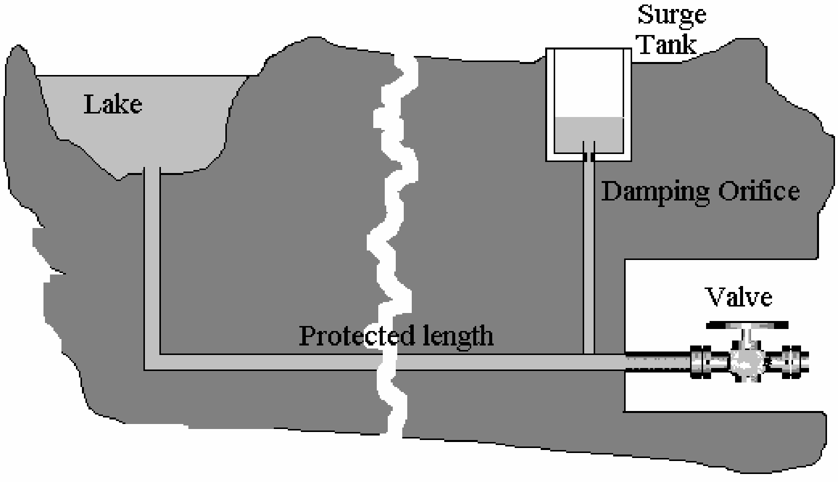
b. Derive an expression for the amplitude of oscillation of the water surface in a surge tank of cross sectional area A_T connected to a pipe of cross sectional area A_p and length L following a complete stoppage of the flow. The normal mean velocity in the pipe is u_o and friction may be ignored.

The general solution to the standard second order differential equation

$$\frac{d^2z}{dt^2} + m^2 z = c^2 \text{ is } z = E\sin(mt) + F\cos(mt) + \frac{c^2}{m^2}$$

Part (a)

On hydroelectric schemes or large pumped systems, a surge tank is used. This is an elevated reservoir attached as close to the equipment needing protection as possible. When the valve is closed, the large quantity of water in the main system is diverted upwards into the surge tank. The pressure surge is converted into a raised level and hence potential energy. The level drops again as the surge passes and an oscillatory trend sets in with the water level rising and falling. A damping orifice in the pipe to the surge tank will help to dissipate the energy as friction and the oscillation dies away quickly.



Part (b)

Mean velocity in surge tank $u_T = \frac{dz}{dt} = \frac{Q}{A_T}$ $\quad Q = A_T \frac{dz}{dt}$

Mean velocity in the pipe $u_p = \frac{Q}{A_p}$ $\quad$ Substitute for Q $\quad u_p = \frac{dz}{dt}\frac{A_T}{A_p}$ ....(1)

The diversion of the flow into the surge tank raises the level by z. This produces an increased pressure at the junction point of $\Delta p = \rho g z$

The pressure force produced $F = A_p \Delta p = A_p \Delta g z$

The inertia force required to decelerate the water in the pipe is

F = mass x deceleration = - mass x acceleration = $-\rho A_p L\, du/dt$

Equating forces we have the following.

$A_p \rho g z = -\rho A_p L \frac{du}{dt} \qquad gz = -L\frac{du}{dt} \qquad z = -\frac{L}{g}\frac{du}{dt}$..................(2)

Putting (1) into (2) we get

$z = -\frac{L}{g}\frac{A_T}{A_p}\frac{d^2z}{dt^2} \qquad \frac{d^2z}{dt^2} = -\frac{gA_p}{LA_T}z$ ....(3)

By definition this is simple harmonic motion since the displacement z is directly proportional to the acceleration and opposite in sense. It follows that the frequency of the resulting oscillation is

$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}\frac{A_p}{A_T}}$ The periodic time will be $T = 1/f$

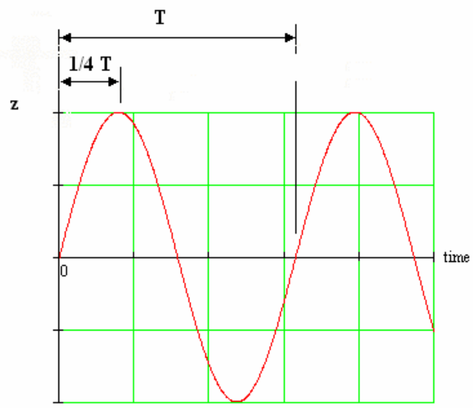
The amplitude and periodic time are referred to as the APO (amplitude and period of oscillation).

Equation (3) maybe re-written as follows.

$$\frac{d^2z}{dt^2} = -\frac{gA_p}{LA_T}z = -\omega^2 z$$

$$\frac{1}{\omega^2}\frac{d^2z}{dt^2} + z = 0$$

$$\frac{d^2z}{dt^2} + \omega^2 z = 0$$



The standard solution to this equation is $z = z_o \sin(\omega t)$

z_o is the amplitude, that is, the amount by which the height in the tank will move up and down from the mean level. The following is a direct way of finding the amplitude.

$$\text{The mean change in height} = \frac{z_o}{2}$$

$$\text{The weight of water entering the surge tank} = \rho g A_T z_o$$

$$\text{The potential energy stored in the tank} = \rho g A_T z_o \frac{z_o}{2} = \rho g A_T \frac{z_o^2}{2}$$

$$\text{The kinetic energy lost} = \text{Mass x } \frac{u^2}{2} = \rho L A_p \frac{u^2}{2}$$

Equate the energies. $\rho L A_p \frac{u^2}{2} = \rho g A_T \frac{z_o^2}{2}$ $\qquad z_o = u_o \sqrt{\frac{LA_p}{gA_T}}$

The equation for the motion in full is $z = u_o \sqrt{\frac{LA_p}{gA_T}} \sin(\omega t)$

The peak of the surge occurs at T/4 seconds from the disturbance.

3.a. A hydroelectric turbine is supplied with 0.76 m^3/s of water from a dam with the level 51 m above the inlet valve. The pipe is 0.5 m bore diameter and 650 m long.
Calculate the pressure at inlet to the turbine given that the head loss in the pipe is 8.1 m.
Calculate the maximum pressure on the inlet valve if it is closed suddenly. The speed of sound in the pipe is 1200 m/s.

b. The pipe is protected by a surge tank positioned close to the inlet valve.
Calculate the maximum change in level in the surge tank when the valve is closed suddenly (ignore friction).
Calculate the periodic time of the resulting oscillation.

$A = \pi \times 0.5^2/4 = 0.1963\ m^2$ $\quad {}^{u} = Q/A = 0.76/0.1963 = 3.87$ m/s

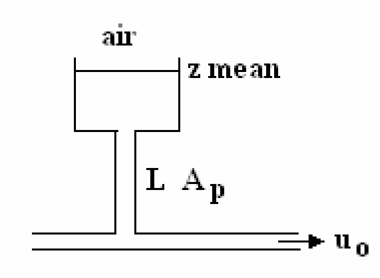
$h_A + z_A + u_A^2/2g = h_B + z_B + u_B^2/2g + h_L$
$0 + 189 + 0 = h_B + 138 + 3.87^2/2g + 8.4$
$h_B = 41.83$ m $\quad p_B = \rho g\, h_B = 0.41 \times 10^6$ Pa

Sudden closure $\quad \Delta p = \rho\, u\, a'$ $\quad a' = 1200$ m/s $\quad \Delta p = 998 \times 3.87 \times 1200 = 4.635 \times 10^6$ Pa

The maximum pressure is 0.41 + 4.635 = 5.045 MPa

This will occur at T/4 seconds

Part (b)



$u_o = Q/A_p \quad dz/dt = Q/A_T \qquad u_o = (A_T/A_p)dz/dt$ ....(1)

$\Delta p = \rho\, g\, z \qquad \Delta F = A_p\, \rho\, g\, z$

This force decelerates the fluid and the mass decelerated is $m = \rho\, A_p\, L \qquad \Delta F = m\, a$
Acceleration is $-du_o/dt$
$A_p\, \rho\, g\, z = -\rho\, A_p\, L\, du_o/dt$
$g\, z = -\, L\, du_o/dt$
$z = -\,(L/g)\, du_o/dt$ ....(2)

Put (1) into (2) $\quad z = -\dfrac{L}{g}\dfrac{A_T}{A_p}\dfrac{d^2z}{dt^2}$ ....(3)

Simple Harmonic Motion so $\omega^2 = \dfrac{L}{g}\dfrac{A_T}{A_p}$ and the amplitude is $u_o\left\{\dfrac{A_p}{A_T}\dfrac{L}{g}\right\}^{1/2}$

$A_T = \pi\, 4^2/4 = 12.566\ m^2$
$\Delta p = 4.635 \times 10^6$ Pa $\quad \Delta h = \Delta p/\rho g = 473.4$ m

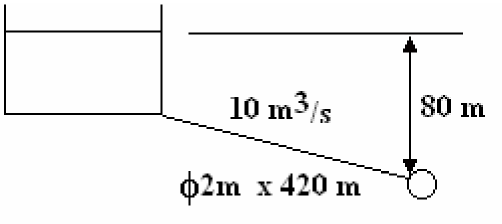
Amplitude $= 3.87\left\{\dfrac{0.1963}{12.566} \times \dfrac{650}{9.81}\right\}^{1/2} = 3.987$ m

4. A pipe 2 m bore diameter and 420 m long supplies water from a dam to a turbine. The turbine is located 80 m below the dam level. The pipe friction coefficient f is 0.01 ($f = 4C_f$).

Calculate the pressure at inlet to the turbine when 10 m^3/s of water is supplied.

Calculate the pressure that would result on the inlet valve if it was closed suddenly. The speed of sound in the pipe is 1432 m/s.

Calculate the fastest time the valve could be closed normally if the pressure rise must not exceed 0.772 MPa).



$A = \pi\, 2^2/4 = 3.142\ m^2$ $\qquad$ $u = Q/A = 10/3.142 = 3.18$ m/s

Sudden closure $\quad \Delta p = \rho\ u\ a' = 998 \times 3.18 \times 1432 = 4.55$ MPa
Gradual closure $\quad \Delta p = \rho Lu/t = 998 \times 420 \times 3.18/t = 1.333/t$ MPa
$h_B = 80 - h_L$
Loss in pipe $= 4C_f\, Lu^2/2gD = f\, Lu^2/2gD = 0.01 \times 420 \times 3.18^2/(2 \times 9.81 \times 2) = 1.082$ m
$h_B = 78.92$ m
$p = \rho g\, h_B = 0.772$ MPa
To avoid cavitation Δp is about 0.772 MPa

$T = 1.333/0.772 = 1.72$ seconds

5.
a) Sketch the main features of a high-head hydro-electric scheme.

b) Deduce from Newton's laws the amplitude and period of oscillation (APO) in a cylindrical surge tank after a sudden stoppage of flow to the turbine. Assume there is no friction.

c) State the approximate effect of friction on the oscillation.

d) An orifice of one half the tunnel diameter is added in the surge pipe near to the junction with the tunnel. What effect does this have on the APO ?

All the answers to this question are contained in the tutorial.

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL9 – COMPRESSIBLE FLOW

SELF ASSESSMENT EXERCISE No. 1

1. Calculate the specific entropy change when a perfect gas undergoes a reversible isothermal expansion from 500 kPa to 100 kPa. R = 287 J/kg K.

T is constant so $\Delta s = mR\ln(p_1/p_2) = 1 \times 287 \times \ln(5/1) = 462$ J/kg K

2. Calculate the total entropy change when 2 kg of perfect gas is compressed reversibly and isothermally from 9 dm^3 to 1 dm^3. R=300 J/kg K.

$\Delta s = mR\ln(V_2/V_1) = 1 \times 300 \times \ln(1/9) = 470$ J/kg K

3. Calculate the change in entropy when 2.5 kg of perfect gas is heated from 20°C to 100°C at constant volume. Take c_v= 780 J/kg K (Answer 470 J/K)

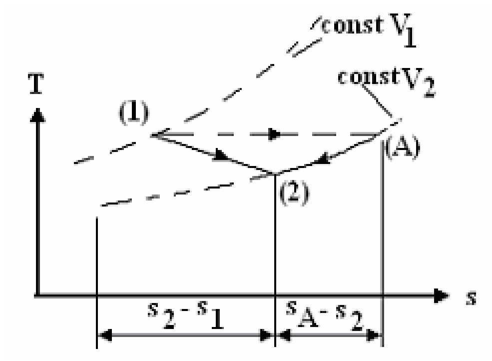
$\Delta s = m\, c_v \ln(T_2/T_1) = 2.5 \times 780 \times \ln(373/293) = -1318$ J/kg K

4. Calculate the total entropy change when 5 kg of gas is expanded at constant pressure from 30°C to 200°C. R = 300 J/kg K c_v= 800 J/kg K (Answer 2.45 kJ/K)

$\Delta s = m\, c_p \ln(T_2/T_1)$ $\quad c_p = R + c_v = 1100$ J/kg K

$\Delta s = = 5 \times 1100 \times \ln(473/303) = 2450$ J/kg K

5. Derive the formula for the specific change in entropy during a polytropic process using a constant volume process from (A) to (2).



$s_2 - s_1 = (s_A - s_1) - (s_A - s_2)$

$s_2 - s_1 = (s_A - s_1) + (s_2 - s_A)$

For the constant temperature process

$(s_A - s_1) = R\ln(p_1/p_A)$

For the constant volume process

$(s_2 - s_A) = (c_v/R)\ln(T_2/T_A)$

Hence

$$\Delta s = R\ln\frac{p_1}{p_A} + C_p\ln\frac{T_2}{T_A} + s_2 - s_1 \quad T_A = T_1$$

Then $\quad \Delta s = s_2 - s_1 = \Delta s = R\ln\left(\frac{p_1}{p_A}\right) + c_v\ln\left(\frac{T_2}{T_A}\right)$

Divide through by R $\quad \Delta s/R = \ln\left(\frac{p_1}{p_A}\right) + \frac{c_v}{R}\ln\left(\frac{T_2}{T_A}\right)$

From the relationship between c_p, c_v, R and γ we have $c_p/R = \gamma/(\gamma-1)$

From the gas laws we have $p_A/T_A = p_2/T_2$ $\quad p_A = p_2 T_A/T_2 = p_2 T_1/T_2$

Hence

$$\frac{\Delta s}{R} = \ln\left(\frac{p_1}{p_2}\right) + \frac{1}{\gamma-1}\ln\left(\frac{T_2}{T_1}\right) = \ln\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)^{1+\frac{1}{\gamma-1}} = \ln\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

6. A perfect gas is expanded from 5 bar to 1 bar by the law $pV^{1.6} = C$. The initial temperature is 200°C. Calculate the change in specific entropy.

R = 287 J/kg K $\gamma = 1.4$.

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{1-1/n} = 473\left(1/5\right)^{1-1/1.6} = 258.7\ \text{K}$$

$$\Delta s = R\ln\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = 287\ \ln(5)\left(\frac{258.7}{473}\right)^{\frac{1.4}{0.4}} = -144\ \text{J/K}$$

7. A perfect gas is expanded reversibly and adiabatically from 5 bar to 1 bar by the law $pV^{\gamma} = C$. The initial temperature is 200°C. Calculate the change in specific entropy using the formula for a polytropic process. R = 287 J/kg K $\gamma = 1.4$.

$$T_2 = 473\left(1/5\right)^{1-1/1.4} = 298.6\ \text{K}$$

$$\Delta s = R\ln\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = 287\ \ln(5)\left(\frac{298.6}{473}\right)^{\frac{1.4}{0.4}} = 0$$

SELF ASSESSMENT EXERCISE No. 2

Take γ = 1.4 and R = 283 J/kg K in all the following questions.

1. An aeroplane flies at Mach 0.8 in air at 15º C and 100 kPa pressure. Calculate the stagnation pressure and temperature. (Answers 324.9 K and 152.4 kPa)

$$\frac{\Delta T}{T} = M^2 \frac{k-1}{2} = 0.8^2 \frac{1.4}{2} = 0.128 \quad \Delta T = 0.128 \text{ x } 288 = 36.86 \text{ K}$$

$$\frac{p_2}{p_1} = \left(M^2 \frac{k-1}{2} + 1\right)^{\frac{k}{k-1}} = 1.128^{3.5} = 1.5243 \qquad p_2 = 100 \text{ x } 1.5243 = 152.43 \text{ kPa}$$

2. Repeat problem 1 if the aeroplane flies at Mach 2.

$$\frac{\Delta T}{T} = M^2 \frac{k-1}{2} = 2^2 \frac{1.4}{2} = 0.8 \qquad \Delta T = 0.8 \text{ x } 288 = 230.4 \text{ K}$$

T_2 = 288 + 230.4 = 518.4 K

$$\frac{p_2}{p_1} = \left(M^2 \frac{k-1}{2} + 1\right)^{\frac{k}{k-1}} = 1.8^{3.5} = 7.824 \qquad p_2 = 100 \text{ x } 7.824 = 782.4 \text{ kPa}$$

3. The pressure on the leading edges of an aircraft is 4.52 kPa more than the surrounding atmosphere. The aeroplane flies at an altitude of 5 000 metres. Calculate the speed of the aeroplane.(Answer 109.186 m/s)

From fluids tables, find that a = 320.5 m/s p_1 = 54.05 kPa γ = 1.4

$$\frac{p_2}{p_1} = \frac{58.57}{54.05} = 1.0836 = \left(M^2 \frac{k-1}{2} + 1\right)^{\frac{k}{k-1}}$$

$$1.0836 = \left(M^2 \frac{1.4-1}{2} + 1\right)^{\frac{1.4}{1.4-1}} = \left(0.2\,M^2 + 1\right)^{3.5}$$

1.0232 = 0.2 M^2 + 1 M = 0.3407 = v/a v = 109.2 m/s

4. An air compressor delivers air with a stagnation temperature 5 K above the ambient temperature. Determine the velocity of the air. (Answer 100.2 m/s)

$$\frac{\Delta T}{T_1} = \frac{v_1^2(k-1)}{2\gamma R T_1} \qquad \Delta T = \frac{v_1^2(1.4-1)}{2 \text{ x } 1.4 \text{ x } 287} = 5\,\text{K} \qquad v_1 = 100.2 \text{ m/s}$$

SELF ASSESSMENT EXERCISE No. 3

1. A Venturi Meter must pass 300g/s of air. The inlet pressure is 2 bar and the inlet temperature is 120°C. Ignoring the inlet velocity, determine the throat area. Take C_d as 0.97.
Take $\gamma = 1.4$ and R = 287 J/kg K (assume choked flow)

$$m = C_d A_2 \sqrt{\left[\frac{2\gamma}{\gamma - 1}\right] p_1 \rho_1 \left\{ (r_c)^{\frac{2}{\gamma}} - (r_c)^{1+\frac{1}{\gamma}} \right\}} \qquad r_c = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{2}{2.4}\right)^{3.5} = 0.528$$

$\rho_1 = p_1/RT_1 = 2 \times 10^5/(287 \times 393) = 1.773\ \text{kg/m}^3$

$$0.3 = 0.97 A_2 \sqrt{7 \times 2 \times 10^5 \times 1.773 \left\{ (0.528)^{1.428} - (0.528)^{1.714} \right\}} = 0.97 A_2 \sqrt{166307}$$

$A_2 = 758 \times 10^{-6}\ \text{m}^2$ and the diameter = 31.07 mm

2. Repeat problem 1 given that the inlet is 60 mm diameter and the inlet velocity must not be neglected.

$$m = C_d A_2 \sqrt{\frac{\left[\frac{2\gamma}{\gamma - 1}\right] p_1 \rho_1 \left\{ (r_c)^{\frac{2}{\gamma}} - (r_c)^{1+\frac{1}{\gamma}} \right\}}{1 - \left(\frac{A_2}{A_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{2/\gamma}}} \qquad 0.3 = C_d A_2 \sqrt{\frac{166307}{1 - \left(\frac{A_2}{A_1}\right)^2 (0.4)}}$$

$1 - (A_2/A_1)^2 \times 0.4 = 1738882 A_2^2$ $\qquad$ $A_1^2 = (\pi \times 0.06^2/4)^2 = 7.99 \times 10^{-6}\ \text{m}^2$

$1 - 50062\, A_2^2 = 1738882 A_2^2$ $\qquad$ $A_2^2 = 559 \times 10^{-9}$ $\qquad$ $A_2 = 747.6 \times 10^{-6}\ \text{m}^2$

The diameter is 30.8 mm. Neglecting the inlet velocity made very little difference.

3. A nozzle must pass 0.5 kg/s of steam with inlet conditions of 10 bar and 400°C. Calculate the throat diameter that causes choking at this condition. The density of the steam at inlet is 3.263 kg/m^3. Take γ for steam as 1.3 and C_d as 0.98.

$$m = C_d A_2 \sqrt{\left[\frac{2\gamma}{\gamma - 1}\right] p_1 \rho_1 \left\{ (r_c)^{\frac{2}{\gamma}} - (r_c)^{1+\frac{1}{\gamma}} \right\}} \qquad r_c = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{2}{2.3}\right)^{4.33} = 0.5457$$

$$0.5 = 0.98 A_2 \sqrt{8.667 \times 3.2626 \times 10 \times 10^5 \left\{ (0.5457)^{1.538} - (0.5457)^{1.538} \right\}} = 0.98 A_2 \sqrt{1.4526 \times 10^6}$$

$A_2 = 423 \times 10^{-6}\ \text{m}^2$ and the diameter =23.2 mm

4. A Venturi Meter has a throat area of 500 mm^2. Steam flows through it, and the inlet pressure is 7 bar and the throat pressure is 5 bar. The inlet temperature is 400°C. Calculate the flow rate. The density of the steam at inlet is 2.274 kg/m^3. Take $\gamma = 1.3$. R = 462 J/kg K. $C_d = 0.97$.
From the steam tables $v_1 = 0.4397\ \text{m}^3/\text{kg}$ so $\rho_1 = 1/0.4397 = 2.274\ \text{kg/m}^3$

$$m = C_d A_2 \sqrt{\left[\frac{2\gamma}{\gamma - 1}\right] p_1 \rho_1 \left\{ (r_c)^{\frac{2}{\gamma}} - (r_c)^{1+\frac{1}{\gamma}} \right\}}$$

$$m = 0.97 \times 500 \times 10^{-6} \sqrt{\left[\frac{2 \times 1.3}{1.3 - 1}\right] 7 \times 10^5 \times 2.274 \left\{ (5/7)^{1.538} - (5/7)^{1.764} \right\}}$$

$m = 485 \times 10^{-6} \times 783$ $\qquad$ $m = 0.379\ \text{kg/s}$

5. A pitot tube is pointed into an air stream which has an ambient pressure of 100 kPa and temperature of 20ºC. The pressure rise measured is 23 kPa. Calculate the air velocity. Take γ = 1.4 and R = 287 J/kg K.

$$\frac{p_2}{p_1} = \frac{123}{100} = 1.23 = \left(M^2\frac{\gamma - 1}{2} + 1\right)^{\frac{\gamma}{\gamma - 1}} \qquad 1.23 = \left(0.2M^2 + 1\right)^{3.5}$$

$1.0609 = 0.2M^2 + 1$ $\quad 0.0609 = 0.2M^2$ $\quad$ M 0.5519

$a = \gamma RT^{1/2} = (1.4 \times 287 \times 293)^{1/2} = 343.1$ m/s

$v = 0.5519 \times 343.1 = 189.4$ m/s

6. A fast moving stream of gas has a temperature of 25ºC. A thermometer is placed into it in front of a small barrier to record the stagnation temperature. The stagnation temperature is 28ºC. Calculate the velocity of the gas. Take γ= 1.5 and R = 300 J/kg K. (Answer 73.5 m/s)

$\Delta T = 3$ K $\quad \Delta T/T_1 = v^2/\gamma RT$ $\qquad c_p = \gamma R/(\gamma-1)$

$\Delta T = 3 = v^2/2\, c_p = v^2(\gamma - 1)/(2\, \gamma R) = v^2(1.5 - 1)/(2 \times 1.5 \times 300)$

$v^2 = 5400$ $\quad v = 73.48$ m/s

FLUID MECHANICS D203
SAE SOLUTIONS TUTORIAL 9 – COMPRESSIBLE FLOW

SELF ASSESSMENT EXERCISE No. 4

1. Air discharges from a pipe into the atmosphere through an orifice. The stagnation pressure and temperature immediately upstream of the orifice is 10 bar and 287 K at all times.

Determine the diameter of the orifice which regulates the flow rate to 0.03 kg/s.
Determine the diameter of the orifice which regulates the flow rate to 0.0675 kg/s.

Atmospheric pressure is 1 bar, the flow is isentropic and the air should be treated as a perfect gas. The following formulae are given to you.

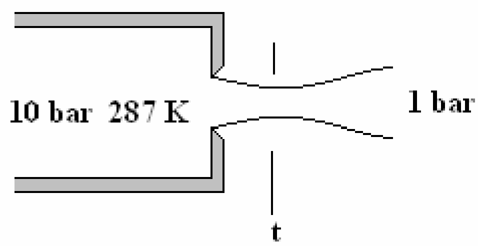
$T_o = T\{1 + M^2(\gamma-1)/2\}$ $\qquad$ $p_1/p_2 = (T_1/T_2)^{\gamma/(\gamma-1)}$

The relationship between areas for the flow of air through a convergent- divergent nozzle is given by $\quad A/A^* = (1/M)\{(M^2 + 5)/6\}^3$

where A and A* are cross sectional areas at which the Mach Numbers are M and 1.0 respectively.

Determine the ratio of exit to throat areas of the nozzle when the Mach number is 2.44 at exit.

Confirm that an exit Mach number of 0.24 also gives the same area ratio.



Pressure ratio is 10/1 so it is clearly choked.

$$T_t = \frac{T_o}{1 + 0.2M^2} \qquad M = 1 \text{ at throat}$$

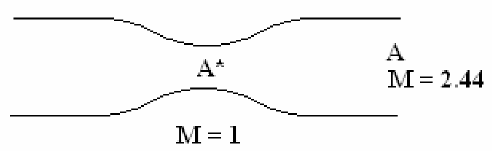
$$T_t = \frac{287}{1.2} = 239\text{ K}$$

$$\frac{p_o}{p_t} = \left(\frac{287}{239}\right)^{3.5} = 1.893 \qquad p_t = 10/1.893 = 5.28 \text{ bar}$$

$\rho_t = p/RT = 5.28 \times 10^5/(287 \times 239) = 7.7 \text{ kg/m}^3$ $\qquad$ $a = (\gamma R T)^{1/2} = 310$ m/s

$m = 0.03 = \rho A a = 77 A \times 310 \quad A = 12.57 \times 10^{-6} \text{ m}^2 \quad \text{Diameter} = \sqrt{(4A/\pi)} = 4 \text{ mm}$
$m = 0.0675 = \rho A a = 77 A \times 310 \quad A = 28.278 \times 10^{-6} \text{ m}^2 \quad \text{Diameter} = \sqrt{(4A/\pi)} = 6 \text{ mm}$

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{M^2+5}{6}\right)^3 = \frac{1}{2.44}\left(\frac{2.44^2+5}{6}\right)^3 = 2.49$$



With M = 0.24

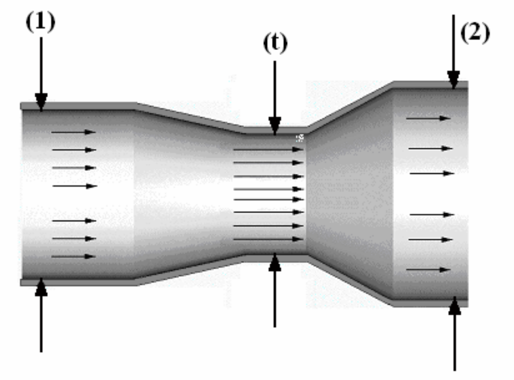
$$\frac{A}{A^*} = \frac{1}{2.4}\left(\frac{2.4^2+5}{6}\right)^3 = 2.49$$

Hence this is a correct solution and this is the theoretical result when m = 0.24 at inlet and 1.0 at the throat.

2. Air discharges from a vessel in which the stagnation temperature and pressure are 350 K and 1.3 bar into the atmosphere through a convergent-divergent nozzle. The throat area is 1 x 10^{-3} m^2. The exit area is 1.2 x 10^{-3} m^2. Assuming isentropic flow and no friction and starting with the equations

$a = (\gamma RT)^{1/2}$ $\qquad$ $C_p T_o = C_p T + v_2/2$ $\qquad$ $p\rho^{-\gamma}$ = constant

Determine the mass flow rate through the nozzle , the pressure at the throat and the exit velocity.



$T_t/T_1 = (p_t/p_1)^{(\gamma-1)/\gamma}$ hence T_t = 291.7 K

p_t = 0.528 p_1 = 0.686 bar if chocked.

$a_t = (\gamma R T_t)^{½}$ = 342.35 m/s

$\rho_t = p_t/RT_t$ = 0.686 x 10^5 /(287 x 291.7) = 0.82 kg/m^3

m = ρ A a = 0.82 x 1 x 10^{-3} x 342 = 0.28 kg/s

$T_2/T_0 = \{1 + M^2(\gamma -1)/2\}$ hence T_o = 350 K

$T_2 = T_o (p_2/p_t)^{(\gamma-1)/\gamma} = 350(1.013/1.3)^{(\gamma-1)/\gamma}$ = 325.9 K

$\rho_2 = p_2/RT_2$ = 1.013 x 10^5 /(287 x 325.9) = 1.083 kg/m^3

$c_2 = m/(\rho_2 A_2)$ = 0.28/(1.083 x 1.2 x 10^{-3}) = 215.4 m/s

3. Show that the velocity of sound in a perfect gas is given by $\qquad$ $a = (\gamma RT)^{½}$

Show that the relationship between stagnation pressure, pressure and Mach number for the isentropic flow of a perfect gas is $\qquad$ $p_o/p = \{1 + (\gamma-1)M^2/2\}^{\gamma/(\gamma-1)}$

It may be assumed that $ds = C_p\, d(l_n v) + C_v\, d(l_n p)$ $\qquad$ where v is the specific volume.

A convergent - divergent nozzle is to be designed to produce a Mach number of 3 when the absolute pressure is 1 bar. Calculate the required supply pressure and the ratio of the throat and exit areas.

The solution for part 1 requires the derivations contained in the tutorial.

$$p_o = p\left(M^2\frac{\gamma-1}{2}+1\right)^{\frac{\gamma}{\gamma-1}} = 1\,x\left(3^2\frac{0.4}{2}+1\right)^{3.5} = 36.73\,\text{bar}$$

For chocked flow p_t = 0.528 p_o = 19.39 bar

$c_t = 1\,M_t = (\gamma R T_t)^{½}$ $\qquad$ $c_e = 3\,M_e = 3(\gamma R T_e)^{½}$ $\qquad$ $m = \rho_t A_t c_t = \rho_e A_e c_e$

$$\frac{A_t}{A_e} = \frac{\rho_e c_e}{\rho_t c_t} \qquad \frac{\rho_e}{\rho_t} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} \qquad \frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} x\,3\,x\frac{\sqrt{\gamma RT_e}}{\sqrt{\gamma RT_t}} \qquad \frac{T_e}{T_t} = \left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}$$

$$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} x\,3\,x\sqrt{\left(\frac{p_e}{p_t}\right)^{1-1/\gamma}} = 3\left(\frac{p_e}{p_t}\right)^{\frac{1+\gamma}{2\gamma}} = 3\left(\frac{1}{19.39}\right)^{2.4/2.8} = 0.236$$

SELF ASSESSMENT EXERCISE No. 5

1. An air storage vessel contains air at 6.5 bar and 15°C. Air is supplied from the vessel to a machine through a pipe 90 m long and 50 mm diameter. The flow rate is 2.25 m^3/min at the pipe inlet. The friction coefficient C_f is 0.005. Neglecting kinetic energy, calculate the pressure at the machine assuming isothermal flow.

m = pV/RT = 6.5 x 10^5 x 2.25/(60 x 288 x 287) = 0.295 kg/s

$(1-p_2^2/p_1^2) = (64\ m^2RT\ C_f\ L)/(\pi^2\ D^5\ p_1^2)$

$$1-\left(\frac{p_2}{6.5}\right)^2 = \frac{64 \text{ x } 0.295^2 \text{ x } 287 \text{ x } 288 \text{ x } 0.005 \text{ x } 90}{\pi^2 \text{x } 0.05^5 \text{ x } \left(6.5 \text{ x } 10^5\right)^2} = 0.1589$$

$$1-0.1589 = 0.841 = \left(\frac{p_2}{6.5}\right)^2$$

$p_2^2 - 35.5 \quad p_2 = 5.96$ bar

SELF ASSESSMENT EXERCISE No. 6

1. A natural gas pipeline is 1000 m long and 100 mm bore diameter. It carries 0.7 kg/s of gas at a constant temperature of 0° C. The viscosity is 10.3 x 10^{-6} N s/m^2 and the gas constant R = 519.6 J/kg K. The outlet pressure is 105 kPa. Calculate the inlet pressure. Using the Blazius formula to find f. (Answer 357 kPa.)

$$R_e = \frac{4m}{\pi\mu D} = \frac{4 \text{ x } 0.7}{\pi \text{ x } 10.3 \text{ x } 10^{-6} \text{ x } 0.1} = 865 \text{ x } 10^3$$

$$C_f = 0.079 R_e^{-0.25} = 0.00259$$

$$1-\left(\frac{p_2}{p_1}\right)^2 = \frac{64 \text{ x } 0.7^2 \text{ x } 519.6 \text{ x } 273 \text{ x } 0.00269 \text{ x } 1000}{\pi^2 \text{x } 0.1^5 \text{ x } (p_1)^2}$$

$$1 = \frac{116.7 \text{ x } 10^9 + 11.025 \text{ x } 10^9}{(p_1)^2} \qquad p_1 = 357 \text{ kPa}$$

2. A pipeline is 20 km long and 500 mm bore diameter. 3 kg/s of natural gas must be pumped through it at a constant temperature of 20°C. The outlet pressure is 200 kPa. Calculate the inlet pressure using the same gas constants as Q.1.

$$R_e = \frac{4m}{\pi\mu D} = \frac{4 \text{ x } 3}{\pi \text{ x } 10.3 \text{ x } 10^{-6} \text{ x } 0.5} = 741693$$

$$C_f = 0.079 R_e^{-0.25} = 0.00269$$

$$1-\left(\frac{200 \text{ x } 10^3}{p_1}\right)^2 = \frac{64 \text{ x } 3^2 \text{ x } 519.6 \text{ x } 293 \text{ x } 0.00269 \text{ x } 20000}{\pi^2 \text{x } 0.5^5 \text{ x } (p_1)^2} = \frac{15.296 \text{ x } 10^9}{p_1^2}$$

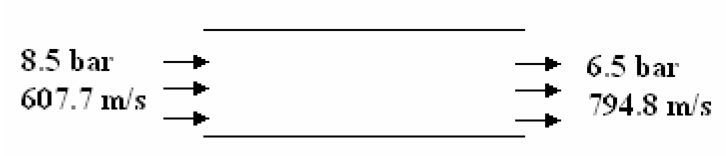
$p_1^2 = 15.296 \text{ x} 10^9 + 40 \text{ x } 10^9$

p_1= 235 kPa

3. Air flows at a mass flow rate of 9.0 kg/s isothermally at 300 K through a straight rough duct of constant cross sectional area of 1.5 x 10^{-3} m^2. At end A the pressure is 6.5 bar and at end B it is 8.5 bar. Determine

a. the velocities at each end. (Answers 794.8m/s and 607.7 m/s)
b. the force on the duct. (Answer 1 380 N)
c. the rate of heat transfer through the walls. (Answer 1.18 MJ)
d. the entropy change due to heat transfer. (Answer 3.935 KJ/k)
e. the total entropy change. (Answer 0.693 kJ/K)
It may be assumed that $ds = C_p\, dT/T + R\, dp/p$

$v_2 = mRT/p_2A = 9 \times 287 \times 300/(6.5 \times 10^5 \times 1.5 \times 10^{-3}) = 794.8$ m/s
$v_1 = mRT/p_1A = 9 \times 287 \times 300/(8.5 \times 10^5 \times 1.5 \times 10^{-3}) = 607.7$ m/s



$p_1A_1 + m\, v_1 = p_2A_2 + m\, v_2 + F$
$F = 1.5 \times 10^{-3}\,(2 \times 10^5) + 9\,(607.76 - 794.48) = 300 - 1680 = -1380$ N
The force to accelerate the gas is greater than the pressure force.
$\Phi + P = mc_p\Delta T + (m/2)(v_2^2 - v_1^2)$ $\Delta T = 0$ $P = 0$
$\Phi = c_p\Delta T + (m/2)(v_2^2 - v_1^2)$
$\Phi = 0 + (9/2)(794.8^2 - 607.7^2) = 1.18$ MJ
$\Phi = \int T\, ds = T\,\Delta s$ $\Delta s = \Phi/T = 1180/300 = 3.935$ kJ/k
$\Delta s = mR\ln(p_1/p_2) = 9 \times 287 \ln(8.5/6.5) = 693$ J/K

4. A gas flows along a pipe of diameter D at a rate of m kg/s.

Show that the pressure gradient is $-\dfrac{dp}{dL} = \dfrac{32C_f\, m^2RT}{\pi^2 p\, D^5}$

Methane gas is passed through a pipe 500 mm diameter and 40 km long at 13 kg/s. The supply pressure is 11 bar. The flow is isothermal at 15°C. Given that the molecular mass is 16 kg/kmol and the friction coefficient C_f is 0.005 determine

a. the exit pressure.
b. the inlet and exit velocities.
c. the rate of heat transfer to the gas.
d. the entropy change resulting from the heat transfer.
e. the total entropy change calculated from the formula $ds = C_p \ln(T_2/T_1) - R\ln(p_2/p_1)$

The derivation is given in the tutorial.

$$R = \frac{R_o}{\tilde{N}} = \frac{8314.4}{16} = 520 \text{ J/kgK} \quad -\int_o^{p_2} p\,dp = \frac{32C_f\, m^2RT}{\pi^2D^5}\int_0^L dL \quad -\left(\frac{p_2^2 - p_1^2}{2}\right) = \frac{32C_f\, m^2RTL}{\pi^2D^5}$$

$$-\left(p_2^2 - p_1^2\right) = \frac{64C_f\, m^2RTL}{\pi^2D^5} \qquad -\left[p_2^2 - \left(11 \times 10^5\right)^2\right] = \frac{64 \times 0.005 \times 13^2 \times 520 \times 288 \times 40000}{\pi^2 \times 0.5^5}$$

$$-\left[p_2^2 - \left(11 \times 10^5\right)^2\right] = 1.05 \times 10^{12} \qquad \left[\left(11 \times 10^5\right)^2\right] - 1.05 \times 10^{12} = p_2^2 \qquad p_2 = 3.99 \text{ bar}$$

$v_1 = mRT_1/p_1A_1 = 13 \times 520 \times 288/(11 \times 10^5 \times \pi \times 0.25^2) = 9.014$ m/s
$v_2 = mRT_2/p_2A_2 = 13 \times 520 \times 288/(3.99 \times 10^5 \times \pi \times 0.25^2) = 24.85$ m/s
$\Phi + P = mc_p\Delta T + (m/2)(v_2^2 - v_1^2)$ $\Delta T = 0$ $P = 0$
$\Phi = c_p\Delta T + (m/2)(v_2^2 - v_1^2)$
$\Phi = 0 + (13/2)(24.85^2 - 9.014^2) = 3.484$ kW
$\Delta s = \Phi/T = 3484/288 = 12.09$ J/k
$\Delta s = -R\ln(p_2/p_1) = -520 \ln(3.99/11) = 526$ J/K

SELF ASSESSMENT EXERCISE No. 7

1. Write down the equations representing the conservation of mass, energy and momentum across a normal shock wave.
Carbon dioxide gas enters a normal shock wave at 300 K and 1.5 bar with a velocity of 450 m/s. Calculate the pressure, temperature and velocity after the shock wave. The molecular mass is 44 kg/kmol and the adiabatic index is 1.3.

R = 8314/44 = 188.95 J/kg K $a_1 = \sqrt{(\gamma R T_1)} = \sqrt{(1.3 \times 188.95 \times 300)} = 271.5$ m/s
$M_1 = v_1/a_1 = 450/271.5 = 1.6577$

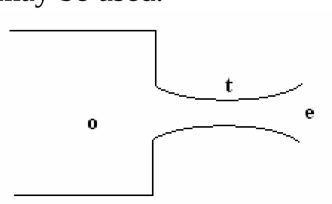
$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma M_1^2}{\gamma - 1} - 1} = \frac{1.6577^2 + 2/0.3}{(2 \text{ x } 1.3 \text{ x } 1.657^2) - 1} = 0.4126 \qquad M_2 = 0.643$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{(1 + \gamma M_2^2)} \qquad \frac{p_2}{p_1} = \frac{1 + 1.3 \text{ x } 1.6577^2}{(1 + 1.3 \text{ x } 0.643^2)} = 2.976 \qquad p_2 = 1.5 \text{ x } 10^5 \text{ x } 2.976 = 446 \text{ kPa}$$

$$\frac{T_2}{T_1} = \frac{1 + (\gamma - 1)\frac{M_1^2}{2}}{1 + (\gamma - 1)\frac{M_2^2}{2}} \qquad \frac{T_2}{T_1} = \frac{1 + (0.3)\frac{1.6577^2}{2}}{1 + (0.3)\frac{0.643^2}{2}} = 1.329 \qquad T_2 = 300 \text{ x } 1.329 = 398.8 \text{ K}$$

$a_2 = \sqrt{(\gamma R T_2)} = \sqrt{(1.3 \times 188.95 \times 398.8)} = 313$ m/s $v = a_2 M_2 = 201$ m/s

2. Air discharges from a large container through a convergent - divergent nozzle into another large container at 1 bar. the exit mach number is 2.0. Determine the pressure in the container and at the throat.
When the pressure is increased in the outlet container to 6 bar, a normal shock wave occurs in the divergent section of the nozzle. Sketch the variation of pressure, stagnation pressure, stagnation temperature and Mach number through the nozzle.
Assume isentropic flow except through the shock. The following equations may be used.



Energy Balance from o to e o is the stagnation condition $u_o = 0$
$\gamma/(\gamma-1)RT_o + 0 = \gamma/(\gamma-1)RT_e + u_e^2/2$ $u_e = 2\sqrt{(\gamma R T_e)}$
$3.5 RT_o + 0 = 3.5RT_e + 2\gamma RT_e$

$$\frac{p_o}{p_e} = \left(\frac{T_o}{T_e}\right)^{\frac{\gamma}{\gamma-1}} \quad \frac{T_o}{T_e} = \left(\frac{p_o}{p_e}\right)^{\frac{\gamma-1}{\gamma}} \quad 3.5RT_e\left(\frac{p_o}{p_e}\right)^{\frac{\gamma-1}{\gamma}} = 3.5RT_e + 2\gamma\gamma R_e$$

$$\left(\frac{p_o}{p_e}\right)^{\frac{\gamma-1}{\gamma}} = \frac{(3.5 + 2.8)}{3.5} = 1.8 \quad p_e = 1 \text{ bar} \qquad p_o = 1.8^{(1/0.286)} = 7.82 \text{ bar}$$

The throat is chocked $\left(\frac{p_t}{p_o}\right) = \left(\frac{2}{\gamma - 1}\right)^{\frac{\gamma}{\gamma-1}} = 0.528$