# The multiplication tables for $F_7$ and $F_4$

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This example gives the multiplication table and the addition table for a field with seven elements. It turns out that any two finite fields with the same number of elements are isomorphic, so this is the only field with seven elements. Writing down these two tables completely specifies the field, since a field is determined by its multiplication and addition.

The first is the multiplication table. Note that it is symmetric across the diagonal, which reflects the fact that multiplication is commutative. Also notice that, other than the entries corresponding to multiplication by 0, every row and column contains each element exactly once. This is because multiplication by a nonzero number is a bijection: multiplying by any nonzero $a \in F$ has an inverse given by multiplying by $a^{-1}$. Thus, each row and each column (other than those corresponding to 0) is some permutation of the numbers $\{0, 1, 2, 3, 4, 5, 6\}$.

| $\cdot$ | **0** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| **2** | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| **3** | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| **4** | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| **5** | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| **6** | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

This table allows us to read off the multiplicative inverse for a given number. For example, the multiplicative inverse of 4 is 2 and the multiplicative inverse of 6 is 6. (This has to be the case since, by the addition table below, we have $6 = -1$ in $F_7$.)

The next table is the addition table for $F_7$. Again, since addition is commutative, this table is symmetric across the diagonal. Notice that in the addition table every row and every column is a permutation of $\{0, 1, 2, 3, 4, 5, 6\}$.

| $+$ | **0** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| **2** | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| **3** | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| **4** | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| **5** | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| **6** | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

If you've seen modular arithematic before, you might recognise the addition and multiplication tables for $F_7$ as those of the integers modulo 7. That is, $F_7 \cong \mathbb{Z}/7\mathbb{Z}$. On the other hand, here are the addition and multiplication tables for the field with four elements, $F_4$:

| $+$ | $\boldsymbol{0}$ | $\boldsymbol{1}$ | $\boldsymbol{x}$ | $\boldsymbol{x+1}$ |
|---|---|---|---|---|
| $\boldsymbol{0}$ | $0$ | $1$ | $x$ | $x+1$ |
| $\boldsymbol{1}$ | $1$ | $0$ | $x+1$ | $x$ |
| $\boldsymbol{x}$ | $x$ | $x+1$ | $0$ | $1$ |
| $\boldsymbol{x+1}$ | $x+1$ | $x$ | $1$ | $0$ |

| $\cdot$ | $\boldsymbol{0}$ | $\boldsymbol{1}$ | $\boldsymbol{x}$ | $\boldsymbol{x+1}$ |
|---|---|---|---|---|
| $\boldsymbol{0}$ | $0$ | $0$ | $0$ | $0$ |
| $\boldsymbol{1}$ | $0$ | $1$ | $x$ | $x+1$ |
| $\boldsymbol{x}$ | $0$ | $x$ | $x+1$ | $1$ |
| $\boldsymbol{x+1}$ | $0$ | $x+1$ | $1$ | $x$ |

Note that every element of $F_4$ appears on the diagonal of the multiplication table; that is, every element of $F_4$ is a square. The tables for $F_4$ are not the same as the addition and multiplication tables in $\mathbb{Z}/4\mathbb{Z}$:

| $+$ | **0** | **1** | **2** | **3** |
|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 |
| **1** | 1 | 2 | 3 | 0 |
| **2** | 2 | 3 | 0 | 1 |
| **3** | 3 | 0 | 1 | 2 |

| $\cdot$ | **0** | **1** | **2** | **3** |
|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 |
| **2** | 0 | 2 | 0 | 2 |
| **3** | 0 | 3 | 2 | 1 |

So $F_4$ is not $\mathbb{Z}/4\mathbb{Z}$; in fact, $\mathbb{Z}/4\mathbb{Z}$ is not a field. This can be seen from its multiplication table, since multiplication by 2 does not give a permutation of $\{0, 1, 2, 3\}$: 0 appears twice in the column corresponding to 2, and 1 never appears. This means that 2 does not have a multiplicative inverse in $\mathbb{Z}/4\mathbb{Z}$.