Some questions of control in fluid mechanics

Olivier Glass

Abstract The goal of these lecture notes is to present some techniques of non-linear control of PDEs, in the context of fluid mechanics. We will consider the problem of controllability of two different models, namely the Euler equation for perfect incompressible fluids, and the one-dimensional isentropic Euler equation for compressible fluids. The standard techniques used to deal with the Cauchy problem for these two models are of rather different nature, despite the fact that the models are close. As we will see, this difference will also appear when constructing solutions of the controllability problem; however a common technique (or point of view) will be used in both cases. This technique, introduced by J.-M. Coron as the *return method*, is a way to exploit the nonlinearity of the equation for control purposes. Hence we will see its application in two rather different types of PDEs.

The plan of these notes is the following. In a first part, we recall in a very basic way some types of questions that can be raised in PDE control (in a non-exhaustive way). In a second part, we expose results concerning the controllability of the incompressible Euler equation. In a third part, we show how the techniques used to prove the controllability of the incompressible Euler equation can be used to prove some other controllability properties for this equation, namely the so-called Lagrangian controllability. In a fourth and last part, we consider the controllability of the isentropic Euler equation.

1 Introduction

In this first section, we give a short and elementary presentation of some questions in control theory as a general introduction before getting to some specific control problems in fluid mechanics.

Olivier Glass, Ceremade, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France e-mail: glass@ceremade.dauphine.fr

1.1 Control systems

We start with the definition of the basic object studied in control theory.

Definition 1. A *control system* is an evolution equation (an ODE or a PDE) depending on a parameter u, that we will write in a formal way as follows:

$$\dot{y} = f(t, y, u), \tag{CS}$$

where $t \in [0,T]$ is the time and

- $y : [0,T] \to \mathscr{Y}$ is the unknown, called the *state* of the system,
- $u : [0,T] \to \mathscr{U}$ is the parameter called the *control*, that one can choose as a function of the time.

Of course, above, $\dot{y}$ stand for the time derivative of y.
The two standard examples that we have in mind with this definition are the following:

- The state $y(t)$ belongs to $\mathbb{R}^n$ (or to some finite-dimensional manifold), the control $u(t)$ to $\mathbb{R}^m$ (or again to some other finite-dimensional manifold), and equation (CS) is an ODE,
- Both the state $y(t)$ and the control $u(t)$ belong to some functional spaces, and (CS) is a PDE (so f is typically a differential operator acting on y).

The general question accompanying this definition is the following: how can one use the control to make the system fulfill some purpose that has been prescribed in advance? Before giving precise mathematical definitions corresponding to this general problem, let us give some examples of control systems.

1.2 Examples

To fix the ideas, we give examples of control systems both of finite and infinite dimensional type. In these lecture notes, we will be more interested in infinite-dimensional systems governed by PDEs.

1. Finite dimensional linear autonomous control systems. Here (CS) is as follows:

$$\dot{y} = Ay + Bu,$$

where the state $y \in \mathscr{Y} = \mathbb{R}^n$, the control $u \in \mathscr{U} = \mathbb{R}^m$, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are fixed matrices.

2. Driftless control-affine systems. Here:

$$\dot{y} = \sum_{i=1}^{m} u_i f_i(y),$$

where the state $y \in \mathbb{R}^n$, the control $u \in \mathbb{R}^m$, and $f_1, \ldots, f_m$ are smooth vector fields on $\mathbb{R}^n$.

Let us now give some examples of infinite-dimensional control systems, connected to fluid mechanics. There are different classical ways to consider the action of a control on a distributed system governed by a PDE.

3. Internal control of a PDE: the Navier-Stokes case. Consider Ω a smooth bounded domain in $\mathbb{R}^n$, and a nonempty open set $\omega \subset \Omega$, see Figure 1.

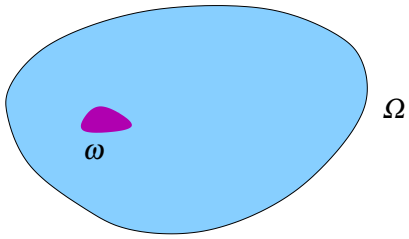

Fig. 1 Internal control

Here we consider an evolution PDE on Ω, e.g. the incompressible Navier-Stokes equations, the control acting as a source term located in ω:

$$\begin{cases} \partial_t v + (v \cdot \nabla) v - \Delta v + \nabla p = \mathbf{1}_\omega u \ \text{ in } \ [0,T] \times \Omega, \\ \text{div } v = 0 \ \text{ in } \ [0,T] \times \Omega, \\ v = 0 \ \text{ on } \ [0,T] \times \partial\Omega. \end{cases}$$

Above, $v : \Omega \to \mathbb{R}^n$ is the velocity field, $p : \Omega \to \mathbb{R}$ is the pressure field. As well-known this equation describes the evolution of the velocity field of an incompressible, viscous fluid. Note that p is not a real unknown of the equation; as a matter of fact, the whole system could be reformulated without it. Here:

- the state is the velocity field v for instance taken in $L^2(\Omega;\mathbb{R}^n)$ (or a subspace in $L^2(\Omega;\mathbb{R}^n)$ in order to take div $v = 0$ and the boundary conditions into account),
- the control is the localized force $u = u(t,x)$, which we compel to be supported in ω, belonging for instance in $L^2(\omega;\mathbb{R}^n)$.

4. Boundary control of the Navier-Stokes equation. Consider $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, and a non empty open part of the boundary $\Sigma \subset \partial\Omega$, see Figure 2.

Consider the Navier-Stokes equations, the control acting as a boundary condition located in Σ:

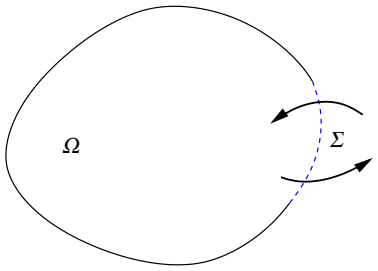

Fig. 2 Boundary control

$$\begin{cases} \partial_t v + (v\cdot\nabla)v - \nu\Delta v + \nabla p = 0 \ \text{ in } \ [0,T]\times\Omega, \\ \text{div } v = 0 \ \text{ in } \ [0,T]\times\Omega, \\ v = \mathbf{1}_\Sigma(x)u(t,x) \ \text{ on } \ [0,T]\times\partial\Omega,. \end{cases}$$

Here:

- The state is the velocity field v, for instance in $L^2(\Omega;\mathbb{R}^n)$,
- the control is the localized boundary term $u = u(t,x)$.

5. Control by lower modes. One may also consider finite-dimensional control in the context of PDEs, for instance:

$$\begin{cases} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = \sum_{i=1}^m u_i(t)e_i(x) \ \text{ in } \ [0,T]\times\Omega, \\ \text{div } v = 0 \ \text{ in } \ [0,T]\times\Omega, \\ v = 0 \ \text{ on } \ [0,T]\times\partial\Omega, \end{cases}$$

where:

- the state is again the velocity field v,
- the control is $(u_1,\dots,u_m)\in\mathbb{R}^m$.

6. Many other possibilities. Let us underline that there are many other natural possibilities: u appearing in the coefficients, through an internal/a boundary operator, ...

1.3 Examples of control problems

As explained above, the goal of control theory is to understand how one can use the control function in order to influence the dynamics of the system in a prescribed way. This general problem can take different forms and yield different mathematical problems. We list several of these questions below.

1. Optimal control. One looks for a control that minimizes some cost function, e.g.

$$J(u) = \|y(T;u) - \bar{y}\|^2 + \|u\|^2,$$

where $\bar{y}$ is some target and $y(T;u)$ is the state reached by the system at time T, starting from y_0 and with control u. The problem is to determine if such a control exists, is unique, and to try to characterize it.

2. Exact controllability. The question is the following: given two times $T_0 < T_1$, and y_0, y_1 two possible states of the system, does there exist $u : [T_0, T_1] \to \mathscr{U}$ such that

$$y_{|t=T_0} = y_0,\ \dot{y} = f(y,u) \implies y(T_1) = y_1?$$

In other words, is it possible to find, for each y_0 and y_1, a control which drives the system from the initial state y_0 to the target y_1?

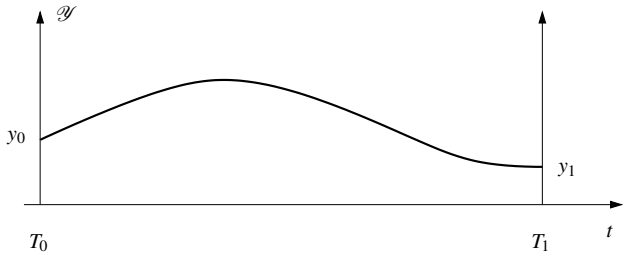

Fig. 3 Exact controllability

Remark 1. For autonomous systems, this notion depends on $T_1 - T_0$ rather than on both T_0 and T_1.

3. Approximate controllability. The problem of approximate controllability is a relaxed version of the exact controllability. Instead of requiring that the state of the system exactly reaches the target, one may wonder if, at least, one can get arbitrarily close to the target. Mathematically speaking, this can be written as follows.

Given $T_0 < T_1$, y_0 and y_1 two possible states of the system **and** $\varepsilon > 0$, does there exist $u : [0,T] \to \mathscr{U}$ such that

$$y_{|t=0} = y_{T_0},\ \dot{y} = f(y,u) \implies \|y(T_1) - y_1\| \leq \varepsilon?$$

Needless to say, the problem highly depends on the choice of the norm.

4. Null controllability. We suppose that $\mathscr{Y}$ is a vector space. Given $T_0 < T_1$, y_0 an initial state of the system, does there exist $u : [0,T] \to \mathscr{U}$ such that

$$y_{|t=T_0} = y_0,\ \dot{y} = f(y,u) \implies y(T_1) = 0?$$

Typically: can one use the control to put the fluid to rest?

5. Controllability to trajectories. Given $T_0 < T_1$, $y_0 \in \mathscr{Y}$ and $\overline{y} : [T_0, T_1] \to \mathscr{Y}$ a given trajectory of the system (corresponding to a control $\overline{u} : [T_0, T_1] \to \mathscr{U}$), does there exist $u : [T_0, T_1] \to \mathscr{U}$ such that

$$y_{|t=T_0} = y_0,\ \dot{y} = f(y,u) \implies y(T_1) = \overline{y}(T_1)?$$

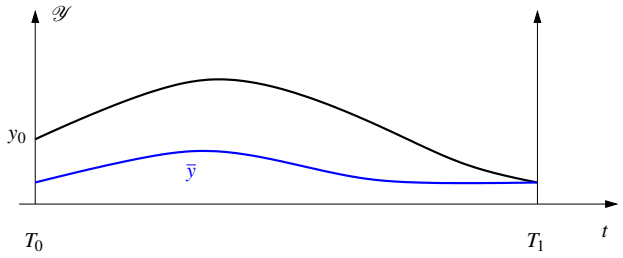

Fig. 4 Controllability to trajectories

Remark 2. The notions of zero-controllability and controllability to trajectories are particularly important for (irreversible) systems having a regularizing effect, since in that case, one cannot hope the exact controllability to hold. For instance, consider the internal control of the heat equation (or the Navier-Stokes equation with suitable assumptions), it can be proved that, whatever the choice of the control, the final state of the system is smooth when restricted to a part of Ω at a positive distance from ω.

Let us underline that many other types of controllability can be considered...

Now let us discuss another problem that one consider for a control system. To simplify the discussion, let us consider autonomous control systems:

$$\dot{y} = f(y,u).$$

We would like to ensure some *robustness* of the control. Indeed, the control which are considered in the above controllability problems are "open-loop", that is, depend on t, y_0 and y_1. But if the system deviates from its planned trajectory, the control may no longer be adapted to the situation. A way to find a control which is more robust to perturbations (which can come from noise, imprecisions of the model, etc.) is to look for a control in "closed-loop" form, that is, depending on the state $y(t)$ at time t, rather than on the memory of y_0. An important control problem connected to this discussion is the following.

6. Asymptotic stabilization. Given an equilibrium state (y_e, u_e) of the system (that is, a point such that $f(y_e, u_e) = 0$), can one find a **state feedback function** $u = u(y)$, such that $u_e = u(y_e)$ and that the so-called **closed-loop system**:

$$\dot{y} = f(y, u(y)), \tag{CLS}$$

is (globally) asymptotically stable at the point y_e, i.e.

- for all $\varepsilon > 0$, there exists $\eta > 0$ such that all solution starting from $y_0 \in B(y_e, \eta)$ are global and satisfy that for all $t \geq 0$, $y(t) \in B(y_e, \varepsilon)$,
- any maximal solution is global in time and satisfies $y(t) \to y_e$ as $t \to +\infty$?

Remark 3. The properties as described above, in both controllability and stabilization contexts, are called **global**. One can consider their **local** versions as follows. The local exact controllability near y_* allows to drive any y_0 to any y_1 in some neighborhood of y_*. The local zero controllability allows to drive any small y_0 to 0. The local controllability to trajectories allows to drive any initial y_0 sufficiently close to $\overline{y}(T_0)$ to $\overline{y}(T_1)$. The local asymptotic stabilization makes the closed-loop system merely locally asymptotically stable at y_e.

2 Controllability of the Euler equation

In this section, we consider the problem of exact boundary controllability of the Euler equation for incompressible inviscid fluids. We begin by describing more precisely the control system under view.

2.1 The control problem

We consider a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3. For a positive time $T > 0$, we consider the Euler equation for perfect incompressible fluids in $[0,T] \times \Omega$:

$$\begin{cases} \partial_t v + (v.\nabla)v + \nabla p = 0 \text{ in } [0,T] \times \Omega, \\ \text{div } v = 0 \text{ in } [0,T] \times \Omega. \end{cases} \tag{1}$$

Here, $v : [0,T] \times \Omega \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is the velocity field, $p : [0,T] \times \Omega \to \mathbb{R}$ is the pressure field. This equation describes the evolution of a homogeneous, incompressible and inviscid fluid. As is classical, the first equation stands for the conservation of momentum, and the second equation is the incompressibility constraint. Of course, the system needs boundary conditions to be determined. In general, to close the system, one adds the usual *impermeability condition* on the boundary:

$$v.n = 0 \text{ on } [0,T] \times \partial\Omega, \tag{2}$$

with n the unit outward normal on $\partial\Omega$. In other words, the fluid cannot cross the boundary (but it can slip on it).

As noticed before in the context of the Navier-Stokes equation, in incompressible fluid mechanics, the pressure is not a real unknown of the system, which can be reformulated in terms of v only. A way to look at the pressure is to consider it as a Lagrange multiplier associated to the incompressibility constraint.

This closed system (1)-(2) has been studied for a very long time. And it is known that the system in 2-D (respectively 3D), is well-posed globally (resp. locally) in time: see for instance the classical references: Lichtenstein [65], Wolibner [84], Yudovich [85], Kato [56], Ebin-Marsden [39], etc. The main condition is that the state space where $v(t,\cdot)$ is taken should be a Hölder or a Sobolev space which is continuously embedded in the Lipschitz space.

Now, we would like to understand the properties of this equation under the influence of a boundary control, see Figure 2. To make a precise statement, we consider a nonempty open part Σ of the boundary $\partial\Omega$. Instead of imposing the impermeability condition (2) everywhere on $\partial\Omega$, we consider the possibility of choosing non-homogeneous boundary conditions on the "control zone" Σ as follows:

- on $\partial\Omega\setminus\Sigma$, the fluid does not cross the boundary, so

$$v.n = 0 \text{ on } [0,T]\times(\partial\Omega\setminus\Sigma), \tag{3}$$

- on Σ, we suppose that one can choose the boundary conditions, that is, use them as a control.

The non-homogeneous boundary value problem for the Euler system is not completely standard; for instance it is not sufficient to prescribe the normal velocity on Σ to determine the system. There are several possibilities to make the system determined. The most usual notion of non-homogeneous boundary conditions for the 2-D Euler equation is due to Yudovich [86] and consists in prescribing:

- the normal velocity on Σ, that is,

$$v(t,x).n(x) \text{ on } [0,T]\times\Sigma, \tag{4}$$

- the *entering vorticity*, that is, the vorticity (i.e. the curl of the velocity field) at points of Σ where the velocity field points *inside* Ω. In other words, one prescribes

$$\text{curl } v(t,x) \text{ on } \Sigma_T^- := \{(t,x)\in[0,T]\times\Sigma \ / \ v(t,x).n(x)<0\}. \tag{5}$$

(Recall that n is the *outward* unit normal on $\partial\Omega$.)

Yudovich proves that under suitable assumptions on Σ and the boundary data, there exists a unique solution to the initial-boundary value problem. Let us underline that

this result concerns quite regular solutions, not the solution of the celebrated reference [85] concerning the homogeneous case. In particular, this regularity is useful to prove uniqueness.

Concerning the 3D equation, Kazhikov [58] proved that one can prescribe as a natural boundary condition for the Euler equation:

- the normal velocity on Σ,
- the *tangential part* of entering vorticity

$$\text{curl } v(t,x) \wedge n \text{ on } \Sigma_T^- := \{(t,x) \in [0,T] \times \Sigma \ / \ v(t,x).n(x) < 0\}.$$

This difference with regards to the bidimensional case can be explained as follows. In 3-D, the vorticity is a divergence-free vector field, while in 2-D it is merely a scalar field. Due to the divergence-free condition of the vorticity, it is enough to prescribe the tangential part of curl v in order to recover this vector field completely.

In both cases the form of the boundary data seems rather involved, but as we will see later, we will express the controllability problem in a way that circumvents this difficulty.

Now, for what concerns the state of the system, the natural space will consist in smooth enough vector fields, satisfying the incompressibility condition $\text{div}\,(v) = 0$ and the constraint (3) on the wall. Since the regularity is not a real issue here (as long as the state v belongs to a Hölder or a Sobolev space contained in the Lipschitz space), we will consider velocities in $C^\infty(\overline{\Omega};\mathbb{R}^n)$. The arguments could be adapted to less regular spaces.

The controllability problem becomes the following one: given a time $T > 0$, and two states v_0, v_1 in $C^\infty(\overline{\Omega};\mathbb{R}^n)$ satisfying the compatibility conditions

$$\text{div}\,(v_0) = \text{div}\,(v_1) = 0 \text{ in } \Omega, \tag{6}$$

$$v_0.n = v_1.n = 0 \text{ on } \partial\Omega, \tag{7}$$

can one find a boundary control such that the corresponding solution v starting from v_0 satisfies

$$v_{|t=T} = v_1 ? \tag{8}$$

But as we saw, the form of the boundary control is a difficulty in itself. To overcome this difficulty, we reformulate the controllability problem as follows: given a time $T > 0$, and two states v_0, v_1 in $C^\infty(\overline{\Omega};\mathbb{R}^n)$ satisfying the compatibility conditions (6)-(7), can one find a *solution* $v \in C^\infty([0,T] \times \overline{\Omega};\mathbb{R}^n)$ of (1) starting from v_0, satisfying the constraint (3) and such that (8) holds?

This formulation can be found in many other contexts of PDE controllability contexts. Let us underline that there is no real difference between the two problems. Should one be able to construct a solution v satisfying the constraint (3), it suffices to take the appropriate trace of v on the boundary to get the control.

Note that the same reformulation works for the approximate controllability problem as well. The standard way to formulate the approximate controllability for the norm $\|\cdot\|$ is the following question: given $T>0$, v_0, v_1 in $C^\infty(\overline{\Omega};\mathbb{R}^n)$ satisfying (6)-(7) and $\varepsilon>0$, does there exist a boundary control such that the corresponding solution v starting from v_0 satisfies

$$\|v_{|t=T}-v_1\|<\varepsilon? \tag{9}$$

And the way we will look to it is to ask if there exists a solution v starting from v_0, satisfying (3) and (9).

2.2 Controllability results

The first result concerning the controllability of the Euler equation is the following, see [25].

Theorem 1 (Coron). *The 2-dimensional Euler equation is exactly controllable in arbitrary time if and only if Σ meets all the connected components of the boundary. In other words, under this condition, for all $T>0$, for all $v_0,v_1\in C^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying* (6)-(7)*, there exists $v\in C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$, solution of* (1)*,* (3) *and satisfying*

$$v_{|t=0}=v_0 \ \text{ and } \ v_{|t=T}=v_1 \ \text{ in } \ \Omega.$$

Note that this result is global, and that when the controllability holds, it holds for all time T. The fact that the controllability holds for all time is far from being true for all PDE controllability problems. For instance, it is well-known that the controllability of the wave equation can hold only for a sufficiently large time, due to the finite speed of propagation.

The controllability of the Euler equation was afterwards established in the 3-D case, see [44].

Theorem 2 (G.). *The previous result also holds in 3-D.*

An interesting fact concerning the 3-D case is that it is not known whether the regular (uncontrolled) solutions of the 3-D Euler equation are global in time or not. As a matter of fact, a possible blow-up is suspected. But here, the result states that, if Σ meets all the connected components of $\partial\Omega$, then one can use the control to make the solution "live" during any time interval $[0,T]$, and even, should one choose $v_1=0$, make the solution global in time. Hence the boundary control is strong enough to *prevent a possible blow-up*.

That the condition on Σ is necessary to get the exact controllability is not difficult to prove. Indeed, two different conservations prove that if Σ does not meet all the connected components of the boundary, then the controllability does not hold. Let us first discuss them in the 2-D case:

- First, Kelvin's law states that the circulation of velocity around a Jordan curve is constant as the curve follows the flow. Now, suppose that a certain connected component of the boundary, let us say γ, does not meet Σ. It follows that, whatever the solution v, this connected component is left (globally) invariant by the flow of v. Hence the circulation of velocity along this component is a conserved quantity, no matter the choice of the control. Hence it suffices to choose v_0 and v_1 having different circulations along γ to prove that the exact controllability does not hold.
- In 2D, the vorticity $\omega := \text{curl } v$ is constant along the flow of v. And again an uncontrolled component of $\partial\Omega$ is preserved by the flow. So it suffices to choose v_0 and v_1 having vorticity distributions on γ that cannot be driven one to another by a smooth deformation to prove that the exact controllability does not hold. Note that this invariant is different from the previous one. One can easily construct velocity fields having the same first invariant and not the same second one.

These obstructions are in fact also valid in 3-D. For the first invariant, one considers a curve γ on an uncontrolled connected component of $\partial\Omega$, a velocity field v_0 with non trivial circulation along γ, and $v_1 = 0$. For what concerns the second obstruction, the vorticity does no longer follow the flow of the velocity in 3-D, but however the support of the vorticity does. Hence it suffices to choose v_0 such that $\text{Supp}(\text{curl } v_0)$ meets an uncontrolled connected component of $\partial\Omega$ and $v_1 = 0$ to see that the exact controllability does not hold.

Now, one could wonder what happens when Σ does not meet all the connected components of the boundary. The exact controllability does not hold, but can one at least hope to get some approximate controllability? Here is a positive answer, see [25].

Theorem 3 (Coron). *If Σ is non empty (but does not meet all the connected components of the boundary), the system is approximately controllable for the norm $L^p(\Omega)$, $p < \infty$. Not for $p = \infty$.*

The same conserved quantity as previously (the velocity circulation along uncontrolled connected components of $\partial\Omega$) shows that the result is false in general if $p = +\infty$. Hence this result cannot be improved. But one could wonder whether this is the only obstruction. The following answer is given in [45].

Theorem 4 (G.). *If v_0 and v_1 have the same velocity circulation on the uncontrolled components of the boundary, then the approximate controllability occurs in $W^{1,p}(\Omega)$, $p < \infty$. Not for $p = \infty$.*

Here, the second conserved quantity (the distribution of vorticity along uncontrolled connected components of $\partial\Omega$, up to regular deformations) shows that the result is false in general if $p = +\infty$. It is natural to ask again whether this is the only obstruction to a better approximate controllability. This is also proven in [45].

Theorem 5 (G.). *If v_0 and v_1 have the same velocity circulation on the uncontrolled components of the boundary, and moreover there exist smooth deformations on these*

components sending the vorticity distributions of v_0 on the ones of v_1, then the approximate controllability occurs in $W^{2,p}(\Omega)$, $p<\infty$. Not for $p=\infty$.

One can show that the case $p=\infty$ is not true by considerations on the derivatives of the vorticity. However, we are not able to describe the invariant properly. So the following open problem remains.

Open problem 1 *What happens next (e.g. for $W^{3,p}(\Omega)$) is open.*

For what concerns the 3-D case, we are not able to extend the above results. The main problem is a possibility of blow-up. It is not clear how to get rid of possible "germs of explosion" near the uncontrolled components of the boundary. This lefts us with the following.

Open problem 2 *The question of approximate controllability of the 3-D Euler equation in $L^p(\Omega)$ (when Σ does not meet all the connected components of the boundary), is still open.*

2.3 Proof of the exact controllability

In this paragraph, we try to explain the main ideas of the proof of Theorem 1. We will first consider the simpler case when Ω is simply connected, and then we will describe what is needed to extend the result to general bidimensional domains. We will also give a few ideas about Theorem 2.

2.3.1 Introduction

We are considering the controllability problem for a nonlinear PDE. Let us explain how this is often dealt with.

Standard approach to nonlinear PDE controllability problems. The most standard method to establish the controllability of a nonlinear PDE is the following.

1. Linearize the equation,
2. Prove a controllability result on the linearized equation,
3. Deduce a controllability result on the nonlinear system by a fixed point or an inverse mapping theorem.

Now to prove the controllability of the linearized equation, there is a standard approach by *duality* (D. Russell [76], J.-L. Lions [66]). This consists in proving an *observability inequality on the (homogeneous) dual system*. Roughly speaking, one has to prove the surjectivity of the control $\mapsto$ final state map, and the argument is somewhat close to the standard

$$A \text{ surjective } \iff \exists c>0,\ \forall u,\ \|A^*u\| \geq c\|u\|.$$

But this is not (by far) the end of the story. Indeed, in general these observability inequalities (which measure the solution everywhere in terms of this solution measured in the control zone only) are very difficult to prove. Also, this is not the only way to establish controllability, even in the linear case. But this gives a good start: we have to prove some inequality, so the problem seems more standard than to find a control driving the solution from one place to another.

However, this method can have at least two drawbacks:

- Frequently, this merely leads to *local* results, unless the nonlinearity is nice (see e.g. Zuazua [87]).
- In many physical cases, and in particular for what concerns the Euler equation, the linearized equation is **not** (unconditionally) controllable.

Let us indeed consider the *linearized Euler equation* around 0:

$$\begin{cases} \partial_t v(t,x) + \nabla p(t,x) = 0 \text{ in } [0,T] \times \Omega, \\ \text{div } v(t,x) = 0 \text{ in } [0,T] \times \Omega. \end{cases} \tag{10}$$

As noticed by J.-L. Lions [67], this equation is not controllable, because solutions of (10) satisfy

$$v_{|t=T} - v_{|t=0} \text{ is the gradient of a harmonic function.}$$

The return method. A method designed by J.-M. Coron to tackle this kind of situation is the *return method*. This method was introduced in the context of finite-dimensional control systems, see [23]. The idea is the following:

find a particular solution $\overline{y}$ of the (nonlinear) system (with control), such that $\overline{y}(0) = \overline{y}(T) = 0$ and such that the linearized system around $\overline{y}$ is controllable.

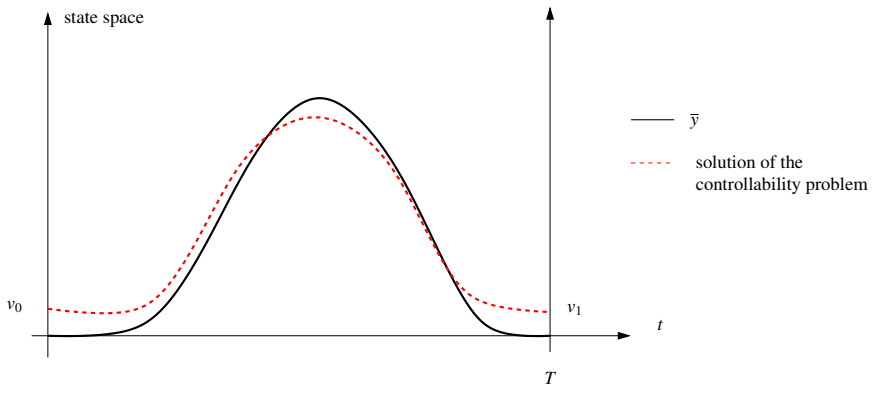

Fig. 5 The return method

One may then hope to find a solution of the nonlinear controllability problem close to $\bar{y}$.

In general, it is not easy to construct such a solution of the nonlinear system. But it turns out that in many different physical situations, this method have proved very useful. It can be seen as a way to exploit the nonlinearity of the system. We refer to [29] for examples and references on that subject.

2.3.2 The solution $\bar{y}$

In this part, we explain the construction of the function $\bar{y}$ used to prove Theorem 1. The vast majority of the arguments here come from [25]; a slight part of the construction that we show is a little bit different from the one of [25] and uses arguments from [45]. First, in an informal manner, we explain how we are led to look for particular properties for $\bar{y}$.

The choice of $\bar{y}$: what should it do? Let $\omega := \mathrm{curl}\,(v)$ the vorticity (either a scalar in 2D or a vector in 3D). It satisfies

$$\partial_t \omega + (v\cdot\nabla)\omega = 0 \ \ (2D \text{ case}), \tag{11}$$

or

$$\partial_t \omega + (v\cdot\nabla)\omega = (\omega.\nabla)v \ \ (3D \text{ case}). \tag{12}$$

Calling Φ the flow of v, that is, the solution of the ODE associated to v:

$$\partial_t \Phi(t,s,x) = v(t,\Phi(t,s,x)) \ \text{ and } \ \Phi(t,t,x) = x, \tag{13}$$

Hence, in the 2D case, the vorticity follows the flow of v, that is to say, it is constant along flow lines:

$$\omega(t,\Phi(t,0,x)) = \omega(0,x).$$

This is no longer true for what concerns the 3D case, but, at least, the support of the vorticity follows the flow in that case. A consequence of this is the following. If one wants to steer a state v_0 such that $\mathrm{curl}\,(v_0)$ which does not vanish anywhere on Ω, to $v_1 = 0$, then even if $\|v_0\| \ll 1$, one should expect the following property to hold:

$$\textit{the flow of } \bar{y} \textit{ makes every point of } \overline{\Omega} \textit{ leave the domain}. \tag{14}$$

Indeed, we will look for a solution v close to $\bar{y}$; but v must satisfy such a property, because if it does not, there remains inside Ω at time T, points where the vorticity is inherited directly from $\mathrm{curl}\,(v_0)$, which contradicts $v(T) = 0$.

The choice of $\bar{y}$: what can it do? But on another side, the question is: how can we construct a solution of the nonlinear system (with control)? In general, we have not so many particular solutions of a nonlinear system at our disposal. But for what

concerns the Euler equations, a very classical form of particular solutions is known for a very long time. These are the *potential flows*:

$$\overline{y}(t,x) = \nabla_x \theta,$$

where

$$\Delta_x \theta(t,x) = 0 \text{ for all } (t,x) \in [0,T] \times \Omega.$$

These are solutions of the Euler equation: taking

$$p = -\theta_t - \frac{|\nabla\theta|^2}{2},$$

it is elementary to check that $\overline{y}$ satisfies (1) (and it is remarkable that the dependence in time is in some sense arbitrary). The boundary condition (3) translates into

$$\partial_n \theta(t,x) = 0 \text{ for all } (t,x) \in [0,T] \times (\partial\Omega \setminus \Sigma).$$

Main proposition. As one may expect, the solution $\overline{y}$ that one constructs is at the intersection of the two above constraints. An important part of the proof is devoted to showing the following existence result.

Proposition 1 (Coron). *There exists $\overline{\theta} \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R})$, compactly supported in time in $(0,T)$, such that*

$$\Delta_x \overline{\theta}(t,x) = 0 \text{ in } [0,T] \times \Omega, \ \partial_n \overline{\theta}(t,x) = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Sigma), \tag{15}$$

and such that the flow of $\nabla\overline{\theta}$ makes all the points in $\overline{\Omega}$ leave the domain.

Remark 4. The flow of $\nabla\overline{\theta}$ is not very well-defined, because $\nabla\overline{\theta}$ is not everywhere tangent on the boundary, and hence the flow is not "internal" to Ω. It follows that the flow $\Phi(t,s,x)$ solution to:

$$\partial_t \Phi(t,s,x) = \overline{y}(t, \Phi(t,s,x)) \text{ and } \Phi(t,t,x) = x,$$

is not defined for all time (t,s). An elementary way to define this flow properly is to extend $\overline{\theta}$ into a function of $C^\infty([0,T]; C_c^\infty(\mathbb{R}^n))$ (which of course is no longer harmonic outside Ω). This allows to define a flow globally and make the statement mathematically accurate.

Idea of the construction of $\overline{y}$. For the rest of Paragraph 2.3.2, we explain the steps to prove Proposition 1. As we will see, it is the consequence of the following one:

Proposition 2. *Given a curve $\gamma \in C^k([0,1]; \Omega \cup \Sigma)$, there exists $\theta \in C^k([0,1] \times \overline{\Omega}; \mathbb{R})$ satisfying* (15) *such that the flow Φ of $\nabla\theta$ satisfies:*

$$\Phi(t,0,\gamma(0)) = \gamma(t).$$

The same holds for $\gamma \in C^k([0,1]; \partial\Omega)$.

Idea of the proof of Proposition 1 assuming Proposition 2. This is mainly a matter of compactness of $\overline{\Omega}$. For each x in $\overline{\Omega}$, one can find a curve γ in $\overline{\Omega}$ driving x outside of $\overline{\Omega}$. To make this statement rigorous, extend a little bit Ω across Σ, to obtain a new smooth domain $\tilde{\Omega}$. Hence one can find a corresponding harmonic flow $\nabla\theta$ (depending on x), defined on $[0,1]\times\overline{\tilde{\Omega}}$.

Now, by continuity of the flow, together with x, a small neighborhood of x in $\overline{\Omega}$ is sent outside $\overline{\Omega}$ by this flow, say $\mathscr{V}_x$. By compactness of $\overline{\Omega}$, we can find a finite number of points $x_1,\dots x_n$ such that $\overline{\Omega}=\mathscr{V}_{x_1}\cup\dots\cup\mathscr{V}_{x_n}$. Call $\nabla\theta_1,\dots\nabla\theta_n$ the corresponding flows.

Now, let us notice that the time T is not an issue here. If one is able to find a function $\theta\in C^\infty([0,\tilde{T}]\times\overline{\Omega};\mathbb{R})$, harmonic in x, satisfying the homogeneous Neumann boundary condition on $\partial\Omega\setminus\Sigma$ for all $t\in[0,\tilde{T}]$ and whose flow satisfies (14), then it is just a matter of time-rescaling to prove Proposition 1.

Now since we do not care about the size of the time interval, the function θ is obtained by gluing in time several flows of this type. Precisely, we construct θ as follows:

- $\theta(t,x):=\theta_1(t,x)$ during $[0,1]$,
- then $\theta(t,x):=-\theta_1(2-t,x)$ during $[1,2]$. In this way, we know that the corresponding Φ satisfies $\Phi(2,0,x)=x$ for all $x\in\overline{\Omega}$.
- And then we iterate: we set $\theta(t,x):=\theta_3(t+2,x)$ during $[2,3]$, and then $\theta(t,x):=-\theta_3(4-t,x)$ during $[3,4]$, etc.

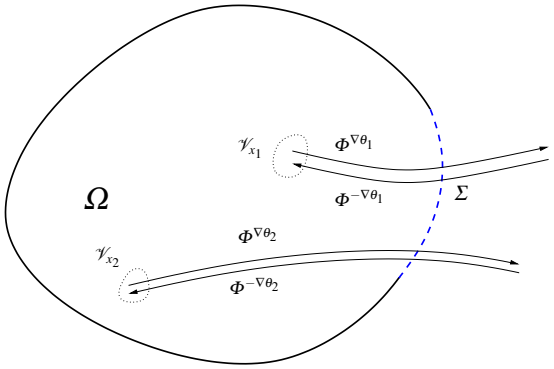

Fig. 6 The flow of θ

It is then not difficult to see that the θ that we have constructed is convenient. □

Following a curve. Now we have to establish Proposition 2. A somewhat close statement was noticed independently by T. Kato in another context. In [57], Kato

proves that without control, the trajectories of the flow of solutions of the Euler equation are C^∞ with respect to time; with non-homogeneous boundary conditions, he notices that this is no longer true.

A way to prove Proposition 2 is to establish the following lemma.

Lemma 1. *For $x \in \overline{\Omega}$, one has the following:*

$$\left\{\nabla\theta(x),\ \theta \in C^\infty(\overline{\Omega};\mathbb{R}) \text{ satisfying (15)}\right\} = \begin{cases} \mathbb{R}^n \text{ for } x \in \Omega \cup \Sigma, \\ T_x\partial\Omega \text{ for } x \in \partial\Omega \setminus \Sigma. \end{cases}$$

Proof of Proposition 2 assuming Lemma 1. We suppose that Lemma 1 is established. Then for each $t \in [0,1]$, one can find a finite number of functions $\theta_1, \dots, \theta_N$, satisfying (15), such that

$$\text{Span}\{\nabla\theta_1(\gamma(t)), \dots, \nabla\theta_N(\gamma(t))\} = \mathbb{R}^n \text{ (or } T_{\gamma(t)}\partial\Omega). \tag{16}$$

Using the continuity in time of the curve γ, the continuity in space of these functions $\nabla\theta_i$ and the openness of the condition (16), we see that we still have $\text{Span}\{\nabla\theta_1(\gamma(s)), \dots, \nabla\theta_N(\gamma(s))\} = \mathbb{R}^n$ or $T_{\gamma(s)}\partial\Omega$ for s in a small neighborhood $\mathscr{U}_t$ of t. In particular, for each t, in such a neighborhood $\mathscr{U}_t$ we are able to describe $\dot\gamma(s)$ as follows:

$$\begin{aligned} \forall s \in \mathscr{U}_t,\ \ \dot\gamma(s) &= \sum_{i=1}^N \lambda_i(s)\nabla\theta_i(\gamma(s)), \\ &=: \nabla\theta^t(s,\gamma(s)), \end{aligned}$$

for suitable functions $\lambda_i(s)$.

Now we use the compactness of $[0,1]$, and extract a finite subcover of $[0,1]$ by $\mathscr{U}_{t_1}, \dots, \mathscr{U}_{t_n}$. Then one can construct $\theta(t,x)$ with the form

$$\theta(t,x) = \sum_{i=1}^N \rho_i(t)\theta^{t_i}(t,x).$$

where $\rho_1, \dots, \rho_n$ is a partition of unity adapted to this covering of $[0,1]$. Then one can check easily that this θ is convenient. □

The possible directions of $\nabla\theta(x)$. Now it remains to prove Lemma 1. In 2D, this can be proved by using Runge's theorem (of approximation of holomorphic functions by rational functions). Indeed, as is very classical, complex analysis is very useful to construct such flows in dimension 2 because, setting $Vf := (\text{Re}\, f, -\text{Im}\, f)$, we have:

$$f \text{ satisfies the Cauchy-Riemann equations} \Longleftrightarrow \text{curl}\, Vf = \text{div}\, Vf = 0.$$

Of course, in a simply connected domain, there is no difference between a curl-free vector field and a gradient field.

1. Let us first consider the case $x \in \Omega \cup \Sigma$. The idea is the following: define the

holomorphic function φ as follows: in some neighborhood of $\partial\Omega \setminus \Sigma$ in the complex plane, $\varphi(z) = 0$, and in some neighborhood of x (disjoint from the latter), $\varphi(z) = c$ where c is an arbitrary complex constant, see Figure 7. Now approximate φ by a

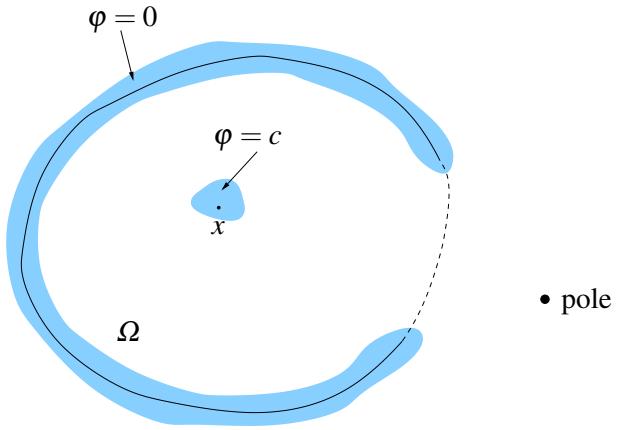

Fig. 7 Use of Runge's theorem

rational function f whose only pole belongs to the unbounded component of $\mathbb{C} \setminus \overline{\Omega}$. (Recall that Σ meets all the connected components of $\partial\Omega$; hence we do not need more than one pole.)

The resulting rational function is in particular holomorphic in a neighborhood of $\overline{\Omega}$, and even in a neighborhood of the complement of the unbounded component of $\mathbb{C} \setminus \overline{\Omega}$. And since the only pole is in the unbounded component of $\mathbb{C} \setminus \overline{\Omega}$, one can see that Vf is a gradient field. But it is not quite satisfactory yet, because this vector field $V(f) = \nabla\theta$ does not satisfy $\partial_n\theta = 0$ on $\partial\Omega \setminus \Sigma$ exactly, but merely $\partial_n\theta = \mathscr{O}(\varepsilon)$, where ε is the approximation parameter, in any C^k norm.

But it suffices to subtract a solution of a Neumann problem to get $\partial_n\theta = 0$ on $\partial\Omega \setminus \Sigma$ exactly. To that purpose, choose g on the boundary such that $g = \partial_n\theta$ on $\partial\Omega \setminus \Sigma$, $\int_{\partial\Omega} g = 0$ and $\|g\|_{C^k(\Sigma)} = \mathscr{O}(\|g\|_{C^k(\partial\Omega\setminus\Sigma)})$ on Σ. Then solve

$$\Delta\psi = 0 \text{ in } \Omega, \quad \partial_n\psi = g \text{ on } \partial\Omega.$$

Using standard elliptic estimates, we deduce that the size of ψ (in $C^{k,\alpha}(\Omega)$ norm for instance) is also of order ε, so $\theta - \psi$ is convenient.

2. If $x \in \partial\Omega \setminus \Sigma$, the situation is more difficult. Of course, we can no longer approximate 0 near $\partial\Omega \setminus \Sigma$ and c near x at the same time. Instead, we approximate the following function introduced for $a \in \Omega$:

$$R_a(z) := N(a)\left[\frac{1}{z-a} - \frac{1}{z-\hat{a}}\right], \quad \text{as } a \to x, \tag{17}$$

where a belongs to Ω and $\hat{a}$ is as in Figure 8, symmetrically disposed with respect to $T_x\partial\Omega$. Moreover, $N(a) = \mathcal{O}(d(x,a))$ is a normalization factor intended to get that $R_a(x)$ is of order 1, while on $\{z \in \partial\Omega \ / \ d(x,z) \geq c > 0\}$, one has $R_a(z) \to 0$ uniformly as $a \to x$. One can interpret $R_a(z)$ as a dipolar expansion.

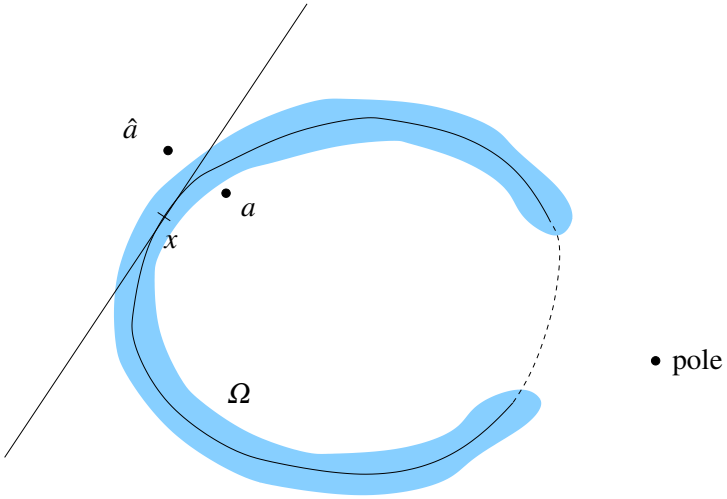

Fig. 8 The boundary case

Now we consider a sufficiently close to x (in a way that $x-a$ is not orthogonal to $T_x\partial\Omega$), and introduce a neighborhood $\mathcal{V}$ of $\partial\Omega$ which does not contain a nor $\hat{a}$, see again Figure 8. Again we apply Runge's theorem to approximate R_a on $\mathcal{V}$, with a pole in the unbounded component of $\mathbb{C} \setminus \overline{\Omega}$. This results in a rational function f, which itself yields a vector field Vf. Again, this vector field is a gradient in Ω, but it does not necessarily satisfy $\partial_n\theta = 0$. Therefore, as previously, we have to subtract a function ψ defined as above. Then the result follows from asymptotic developments as $a \to x$, which allow to prove that the error between $V(R_a)$ and $\nabla\psi$ is small as $a \to x$. □

2.3.3 Using the function $\bar{y}$

Now let us explain how we can use the function $\bar{y}$ constructed above to establish Theorem 1.

Local zero-controllability, rough idea. We first consider the case where v_0 is small enough (in some fixed, sufficiently strong norm), and $v_1 = 0$. The general case will be deduced from this particular one. Let us also suppose that Ω is simply connected;

we will come back to the additional difficulty of a non-trivial topology of the domain later.

In the following construction, we will put aside the regularity/compatibility conditions issues in a first time. Given a small v_0, we try to construct a solution of the following system:

$$\begin{cases} \mathrm{curl}\,(v) = \omega \text{ in } [0,T]\times\Omega, \\ \mathrm{div}\,(v) = 0 \text{ in } [0,T]\times\Omega, \\ v.n = \bar{y}.n \text{ on } [0,T]\times\partial\Omega, \\ \partial_t\omega + (v.\nabla)\omega = 0 \text{ in } [0,T]\times\Omega, \\ \omega_{|t=0} = \omega_0 = \mathrm{curl}\,(v_0) \text{ in } \Omega, \\ \omega = 0 \text{ on } \{(t,x)\in[0,T]\times\Sigma \ / \ \nabla\theta.n < 0\}. \end{cases} \tag{18}$$

Mainly, this is exactly equation (1) rewritten in vorticity form. This is very classical in fluid mechanics (and by the way this allows to get rid of the pressure): the vorticity follows a transport equation depending on the velocity field, and the velocity field can be recovered from the vorticity by using the div/curl elliptic system. The important fact here concerns the boundary conditions (that is, the control); in this form, Yudovich's boundary conditions become more natural. The normal velocity is directly inherited from $\bar{y}$, and the entering vorticity is 0; this clearly involves compatibility conditions issues on Σ and at $t = 0$, but this gives the main idea.

Hence we assume that by some procedure we managed to find a solution of (18). If there were no regularity issues due to the non-homogeneous boundary conditions, it would mainly be a matter of finding a fixed point of some operator defined as follows. First, one maps ω to v by the elliptic div-curl system. Then, to v a new vorticity, say $\tilde{\omega}$, by the transport system. Using the smallness of v_0 this would yield a fixed point of the operator.

Let us now explain why such a solution would drive the state of the system from v_0 at $t = 0$ to 0 at time T, provided that v_0 is sufficiently small.

- The main principle —this is the core of the application of the return method to the Euler equation— is the following. The vorticity is transported by the flow of the velocity; hence its value $\omega(t,x)$ comes either from the initial datum ω_0 (if in the flow of v the point x does not come from Σ), either from Σ and in that case it is 0 (since the entering vorticity is null). It follows that if $\|v_0\| \ll 1$ (for a norm stronger than Lipschitz), then the vorticity of the solution will also be small for all times. Hence the solution is close to the one obtained with no vorticity, that is to say with the solution associated to $v(0,\cdot) = 0$.
- But this solution corresponding to $v(0,\cdot) = 0$ is precisely $\bar{y}$.
- Consequently, for $\|v_0\|$ small enough, v stays close to $\bar{y}$ for all time. Therefore, using a Gronwall argument and (14), one can show that the flow of v makes all points in $\overline{\Omega}$ at $t = 0$ leave the domain.

- It follows that all points at time T in Ω come from Σ in the flow of v. Hence, since the vorticity follows the velocity flow and since the entering vorticity is null, we deduce that $\omega(T) = 0$.
- Now $v(T).n = \bar{y}(T).n = 0$. Since $\operatorname{curl}(v(T)) = \operatorname{div}(v(T)) = 0$ and since Ω is simply connected, we can affirm that $v(T) = 0$. (Not when Ω is multiply connected, as we will see later.)

How to make the construction smooth. Now, of course, we would like to do this in a smooth manner. Indeed putting the entering vorticity to 0 makes it unlikely to construct a regular solution. The main idea is the following. We have to elaborate a new fixed point strategy. Let us be given a velocity vector field v in $[0,T] \times \Omega$, starting from v_0 and close to $\bar{y}$ (remember that v_0 is small, so these two conditions are compatible). Then the construction is as follows.

- Extend the velocity field v to a velocity field $\tilde{v}$ defined on $\mathbb{R}^2$ and compactly supported in x.
- Transport the initial vorticity ω_0 (also extended on $\mathbb{R}^2$ with compact support) by $\tilde{v}$, and a finite number of times, transform the vorticity by

$$\omega(t^+,x) = \varphi(x)\omega(t^-,x), \tag{19}$$

where φ is a cutoff function such that $\varphi = 1$ on $\overline{\Omega}$, see Figure 9. Here we use the

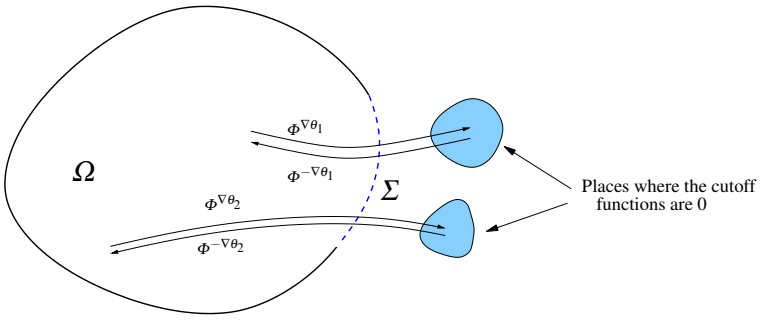

Fig. 9 The smooth process

particular form of $\bar{y}$: its flow brings points of Ω outside of the domain "one piece after another" (recall the covering of $\overline{\Omega}$ by $\mathcal{V}_{x_1}, \ldots, \mathcal{V}_{x_n}$ in the construction of $\bar{y}$). Hence the idea is to put the vorticity to zero on these "pieces" $\mathcal{V}_{x_i}$ of Ω, one after another, when the flow makes them go out of $\overline{\Omega}$.

- Then one associates a new velocity field in Ω by the div/curl elliptic system in Ω, where the normal velocity on the boundary has to be close to $\bar{y}$ and compatible with $v_0.n$ at $t = 0$.

Then one show that:

- This operator has a fixed point when v_0 is small enough.

- This fixed point is regular. Of course we introduce discontinuities in time by the above process when applying a cutoff function between t^+ and t^- in (19). However, these discontinuities are only with respect to the variable t, and take place outside of $\overline{\Omega}$. It follows that **inside** $\overline{\Omega}$, the solution is smooth...

Passage to the global controllability. The natural question now is: what if $v_1 \neq 0$ and v_0 is not small? The main point is to use the *time-scale invariance* of the equation: for $\lambda > 0$,

$$\begin{aligned} &v(t,x) \text{ is a solution of the equation defined in } [0,T]\times\Omega \\ &\qquad \Longleftrightarrow v^\lambda(t,x) := \lambda v(\lambda t,x) \text{ is a solution of the equation} \\ &\qquad\qquad \text{defined in } [0,T/\lambda]\times\Omega. \end{aligned} \tag{20}$$

The pressure associated to v_λ is

$$p_\lambda(t,x) = \lambda^2 p(t,x) \text{ in } [0,T/\lambda]\times\Omega.$$

It is remarkable that the Euler equation has this particular scale invariance, which concerns the time variable only and not the space one.

Now, the idea is the following.

- Using this scale invariance, we see that bringing v_0 to 0 in time T is equivalent to bringing λv_0 to 0 in time T/λ.
- We know how to bring any v_0 such that $\|v_0\| \leq \varepsilon$ to 0 in time T. Hence we know how to bring any v_0 with **larger** norm in **smaller** time (take λ large). In particular, we can bring a large initial condition very fast to 0... and then stay at 0 till the planned time of controllability.
- For what concerns $v_1 \neq 0$, use the **reversibility** of the equation, which corresponds to $\lambda = -1$ in (20). If a solution $v(t,x)$ goes from v_0 to 0 in time T, then $-v(T-t,x)$ is again a solution, going this time from 0 to $-v_0$. Hence to go from v_0 to v_1, apply the following recipe: bring v_0 to 0 in time $T/2$, and then 0 to v_1 in time $T/2$.

Multiply connected domains. Above we treated the case where Ω is simply connected. But in multiply connected domains,

$$\left.\begin{array}{l} \operatorname{curl}(v(T)) = 0 \text{ in } \Omega, \\ \operatorname{div}(v(T)) = 0 \text{ in } \Omega, \\ v.n = 0 \text{ on } \partial\Omega, \end{array}\right\} \not\Rightarrow v(T) = 0, \tag{21}$$

because there is a non-trivial finite-dimensional vector space of **harmonic tangent vector fields** (representing the first tangential de Rham cohomology space of the domain), that is, of solutions of the above system homogeneous div/curl elliptic

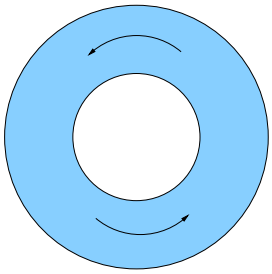

Fig. 10 Harmonic tangent field in the annulus

system. Precisely, if $\partial\Omega$ has $g+1$ connected components (recall that we are still in dimension 2), then the dimension of this space is g.

Hence applying the strategy above, we can bring the vorticity of the solution to 0, but not the velocity field. We only know that $v(T)$ is a harmonic tangent vector field. Hence we have in fact controllability up to a finite dimensional space; but it turns out that solving this "finite-dimensional" problem has the same level of difficulty than bringing "the other directions to zero". (Even, as we will see, in 3D, this part is by far the most difficult part.)

Now, the space of harmonic tangent field is characterized by the g velocity circulations around the g inner boundary components (for instance), which we call Γ_1, $\ldots$, Γ_g. Instead of (21), we have then

$$\left.\begin{aligned} \operatorname{curl}(v(T)) &= 0 \text{ in } \Omega, \\ \operatorname{div}(v(T)) &= 0 \text{ in } \Omega, \\ v.n &= 0 \ \text{ on } \partial\Omega, \\ \int_{\Gamma_i} v(T).\tau\, d\sigma &= 0 \ \text{ for } i = 1,\ldots,g, \end{aligned}\right\} \Rightarrow v(T) = 0. \tag{22}$$

Hence our goal is to bring these g velocity circulations to zero. To do so, the idea consists in making some vorticity pass across the domain (from one component of $\mathbb{R}^2 \setminus \overline{\Omega}$ to another, typically from an inner component to the outer one) as in Figure 11.

The rough idea is the following. We want to bring the velocity circulation around some Γ_i to 0. But by Kelvin's circulation theorem, this circulation is constant, when the curve follows the flow. If we make some vorticity cross the domain, by Stokes theorem the difference between the velocity circulation around Γ_i at time 0 (plain line in Figure 11) and the one at time T (dotted line in Figure 11), will be given by the total flux of the vorticity across $[0,T]\times\Gamma_i$. Hence by using this principle we can fix the circulation around Γ_i.

Of course, to do this properly, we have to construct another reference solution $\bar{y}$ making points going from an inner component to the outer one, and to construct

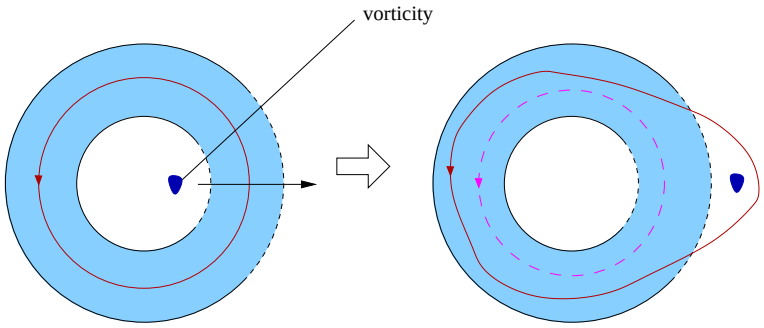

Fig. 11 Modifying the velocity circulation by making vorticity cross the domain

the solution close to this new $\bar{y}$. This uses the principles showed above, we omit the details.

2.3.4 What about 3D?

Here we only give a few ideas about the 3D case.

There are three main differences for what concerns dimension 3.

- **1. Construction of $\bar{y}$.** We have no longer access to the complex variable arguments to describe the potential flows. We will see however that there are similar tools that we can use in dimension 3.
- **2. Transport of the vorticity**. In dimension 3, the vorticity is no longer transported by the flow but is affected by a "stretching" term:

$$\partial_t \omega + (v.\nabla\omega) = (\omega.\nabla)v. \tag{23}$$

  However using (23), one can prove easily that the support of the vorticity is transported by the velocity flow. This property suffices to our purpose, when following the ideas above.
- **3. Blow up**. The solution could blow up. Indeed, it is still unknown whether regular solutions of the 3D Euler equation, which exist locally in time, are global in time or can blow up in finite time. But mainly, as we follow the lines of the proof described above, we see that the main part of the work is done with initial states v_0 satisfying $\|v_0\| \leq \varepsilon$. Even if it is not know that the solutions of the 3D Euler equation remain regular for all time, we know that they have a time of existence at least of $1/\|v_0\|$. Hence we can work with solution which will not be singular before the final time T.
  After the time-rescaling procedure, this means that we act **sufficiently fast** to avoid the blow up, and bring the solution to 0 exactly for some small time.

- **4. Topology.** This is by far the most difficult issue. Of course, the topology of regular open sets in 3D is different and more complex that in 2D.

Let us say a few words about the new ingredients needed for the 3D case.

Construction of $\bar{y}$. We cannot use the complex variable argument here, but in 3-D, there exist Runge-type theorems of approximations of **harmonic functions** by harmonic functions defined on a larger set. For instance, the following result of Walsh can be used (see e.g. [43]).

Theorem 6 (Walsh, 1929). *Let K a compact set in $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ is connected. Then for each function u harmonic in an open set containing K, for each $\varepsilon > 0$, there exists a harmonic polynomial v such that*

$$\|u - v\|_\infty \leq \varepsilon.$$

This can replace the use of Runge's theorem in the above steps. For what concerns (17), we can replace the functions $z \mapsto \frac{1}{z-a}$ by the fundamental solution of the Laplacian in $\mathbb{R}^3$, and one can make the same type of dipolar developments. The projection of the direction of $a - x$ on $T_{\gamma(t)}\partial\Omega$ will give the direction of the resulting $\nabla\theta$ up to small errors. See [44] for more details.

The difficulty coming from the topology. Of course, the topology of smooth bounded open of $\mathbb{R}^3$ is by far more complex that in the 2D case. Note in particular that in 3D, multiply connected domains can have a connected boundary, and simply connected domains can have several boundary components. The difficulty concerns as before multiply connected domains (whether $\partial\Omega$ is connected or not.)

In dimension 3, to get rid of tangential harmonic vector fields, one uses **vortex filaments** (or regularization of vortex filaments), that has to cross the domain, as described in Figure 12.

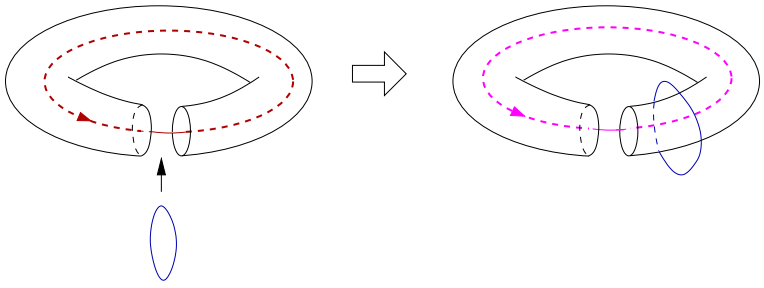

Fig. 12 Making vortex filaments cross the domain

We recall that given a Jordan curve J, a vortex filament located at J is the distribution of vorticity given by

$$\omega = \alpha \delta_J(x)\tau(x),$$

where δ_J is the linear measure on J, τ the unit tangent on J, and α is a real parameter. This distribution is naturally divergence-free and is an important object in three-dimensional fluid mechanics. Of course, to get a smooth solution, one has to mollify it at some stage.

One can check that the "change of velocity circulation" around the curve represented in dotted lines in Figure 12, is given by the intensity of the vortex filament. The main part consists in finding $\bar{y}$ whose flow makes the vortex filament cross the domain, using the previous tools. We refer again to [44] for more details.

2.4 References

An important majority of the arguments described above come from the seminal work of Coron [25] (see also [24]). Extensions of this work concerning the controllability of the Euler equation can be found in [44, 45].

For what concerns the connected problem of asymptotic stabilization by the boundary control, we refer to Coron [27, 26] and to [46].

A general reference concerning this problem and the use of the return method is Coron's book [29].

Let us finally give some other references for what concerns the Navier-Stokes equations, which is closely related to Euler equation (let us underline that this bibliography is far from being complete). For Navier-Stokes, due to the regularizing effect of the equation, one would like to show the controllability to trajectories. Several results on this direction:

- Fursikov-Imanuvilov [41], Imanuvilov [55], Fernandez-Cara-Guerrero-Imanuvilov-Puel [40], have obtained results of **local** controllability to trajectories,
- Coron [28], Coron-Fursikov [30], Chapouly [18]: obtained **global** approximate controllability, with Navier boundary conditions. (These results rely on the controllability of the Euler equation!)

However the global controllability to zero for the Navier-Stokes equation with Dirichlet boundary conditions leaves us the following problem.

Open problem 3 *The problem of global controllability to trajectories of the Navier-Stokes equations with $v = 0$ on $\partial\Omega \setminus \Sigma$ is still open.*

Let us finally give some references concerning the controllability of the Navier-Stokes equation by means of low modes: Agrachev-Sarychev [1, 2] and Shirikyan [80]. A related technique was used recently by Nersesian [72] for the compressible Euler equation.

3 Approximate Lagrangian controllability of the Euler equation

In this section, we describe how the techniques developed in Section 2, can be used to fulfill other purposes for the fluid, namely, to control the displacement of the fluid during the time interval $[0,T]$ rather than its velocity field at final time T.

3.1 The question of Lagrangian controllability

3.1.1 Controlling the displacement of a fluid

Again we consider a smooth bounded domain $\Omega \subset \mathbb{R}^2$ (we consider only $n=2$ here), and Σ a nonempty open set of $\partial\Omega$ and the control system

$$\begin{cases} \partial_t v + (v.\nabla)v + \nabla p = 0 \text{ in } [0,T]\times\Omega, \\ \text{div } v = 0 \text{ in } [0,T]\times\Omega, \\ v.n = 0 \text{ on } [0,T]\times[\partial\Omega\setminus\Sigma]. \end{cases} \tag{24}$$

As previously the control is the boundary data on Σ, e.g.

$$\begin{cases} v(t,x).n(x) \text{ on } [0,T]\times\Sigma, \\ \text{curl } v(t,x) \text{ on } \Sigma_T^- := \{(t,x)\in[0,T]\times\Sigma \ / \ v(t,x).n(x)<0\}. \end{cases} \tag{25}$$

But here, we will be interested in another type of controllability, which is natural for equations from fluid mechanics: is possible **to drive a zone of fluid** from a given place to another by using the control? This question is based on a suggestion by J.-P. Puel. The first study on the subject is due to Horsin [54] where the Burgers equation is considered. One can think for instance to a polluted zone in the fluid, which we would like to transfer to a zone where it can be treated.

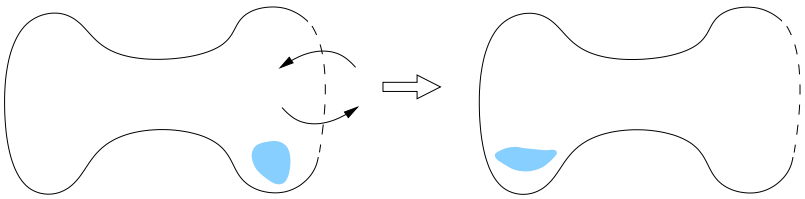

Fig. 13 Controlling the displacement of a fluid zone

First definition. Now before giving the precise definition of the problem under view, let us make a few remarks to motivate it.

- First, it is natural, in order to control the fluid zone during the whole displacement to ask that it remains inside the domain Ω during the whole time interval. This is not straightforward, since the condition $v.n = 0$ is not imposed on the whole boundary.
- In the sequel, we will consider only fluids zones given by the interior (inside Ω) of smooth (C^∞) Jordan curves. This seems a natural class of domains; of course generalizations could be considered.
- Due to the incompressibility of the fluid, the starting zone and the target zone must have the same area, if one wants to be able to drive one to another by the flow of the velocity field.
- We have also to require that there is no topological obstruction to move a zone to the other one. In other words, one should be able to deform continuously one curve one the other one. Hence we will suppose that the two curves are *homotopic* in Ω.

Definition 2. We will say that the system satisfies the exact *Lagrangian* controllability property, if given two smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area, a time $T > 0$ and an initial datum v_0, there exists a control such that the flow given by the velocity field drives γ_0 to γ_1, by staying inside the domain.

An objection. But one can see that the exact Lagrangian controllability does not hold in general. As way to see this is the following. Denote $\Phi^v(t,s,x)$ the flow associated to the velocity field v; see (13).

- Let us suppose that $\omega_0 := \text{curl } v_0 = 0$. In that case if the flow $\Phi^v(t,0,x)$ maintains γ_0 inside the domain, then for all t we have that in the neighborhood of $\Phi^v(t,0,\gamma_0)$,
$$\omega(t,\cdot) = \text{curl } v(t,\cdot) = 0,$$
since, due to (11), the vorticity satisfies $\omega(t,\Phi^v(t,0,x)) = \omega_0(x)$.
- Since curl $v = \text{div } v = 0$, locally around the points of γ_0, u is the gradient of a harmonic function; u is therefore real-analytic in a neighborhood $\Phi^v(t,0,\gamma_0)$.
- Hence if γ_0 is real-analytic, its real-analyticity is propagated over time.
- Now if γ_1 is smooth but non real-analytic, we see that we cannot drive γ_0 to γ_1 by keeping the curve inside Ω. Hence the exact Lagrangian controllability cannot hold.

Approximate Lagrangian controllability. Since the exact Lagrangian controllability does not hold, this leads us, as for the controllability in the usual sense, to soften the question and wonder if it is possible, at least, to drive the initial fluid zone *arbitrarily close* to the target.

This motivates the following definition.

Definition 3. We will say that the system satisfies the property of *approximate Lagrangian controllability* in C^k, if given two smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area, a time $T > 0$, an initial datum v_0 **and**

a real number $\varepsilon > 0$, we can find a control such that the flow of the velocity field maintains γ_0 inside Ω for all time $t \in [0,T]$ and satisfies, up to reparameterization of the curves:

$$\|\Phi^v(T,0,\gamma_0) - \gamma_1\|_{C^k} \leq \varepsilon.$$

Here, $(t,x) \mapsto \Phi^v(t,0,x)$ is again the flow of the vector field v. One parameterizes Jordan curves by the circle $\mathbb{S}^1$.

Main result. The main result that we describe in this section is the following. It can be found in [48].

Theorem 7 (G.-Horsin). *Provided that $\Sigma \neq \emptyset$, the approximate Lagrangian controllability holds in all C^k.*

In other words, consider two smooth smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area. Let $k \in \mathbb{N}$. We consider $v_0 \in C^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying

$$div\,(v_0) = 0 \text{ in } \Omega \text{ and } v_0.n = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Sigma).$$

For any $T > 0$, $\varepsilon > 0$, there exists a solution v of the Euler equation in $C^\infty([0,T] \times \overline{\Omega};\mathbb{R}^2)$ with

$$v.n = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Sigma) \text{ and } v_{|t=0} = v_0 \text{ in } \Omega,$$

and whose flow satisfies

$$\forall t \in [0,T],\ \Phi^v(t,0,\gamma_0) \subset \Omega,$$

and up to reparameterization

$$\|\gamma_1 - \Phi^v(T,0,\gamma_0)\|_{C^k} \leq \varepsilon.$$

3.1.2 A connected result: vortex patches

We now discuss a closely related problem. Indeed, it turns out that the techniques used to prove Theorem 7 can be used to answer the following question: is it possible to control the shape of vortex patches?

Let us first explain what vortex patches are. The starting point is the following, see [85].

Theorem 8 (Yudovich, 1961). *For any $v_0 \in C^0(\overline{\Omega};\mathbb{R}^2)$ such that $div\,(v_0) = 0$ in Ω, $v_0.n = 0$ on $\partial\Omega$ and* **curl** *$v_0 \in L^\infty$, there exists a unique (weak) global solution of the Euler equation starting from v_0 and satisfying $v.n = 0$ on the boundary.*

A particular case of initial data with vorticity in $L^\infty(\Omega;\mathbb{R})$ is the one of *vortex patches*.

Definition 4. A vortex patch is a solution of the Euler equation whose initial datum is the characteristic function of the interior of a smooth Jordan curve (at least $C^{1,\alpha}$).

An important result in the theory of vortex patches is the following, see [19, 20].

Theorem 9 (Chemin, 1993). *In $\mathbb{R}^2$, the regularity of the boundary of the vortex patch is propagated globally in time.*

There are many other references on the subject of vortex patches, see also for instance: Bertozzi-Constantin [10], Danchin [33], Depauw [34], Dutrifoy [38], Gamblin & Saint-Raymond [42], Serfati [78], Sueur [81],...

Hence one can wonder whether it is possible, in the framework of the control system (24)-(25), to control the shape of a vortex patch, that is, to control the evolution of its boundary. Let us underline that this problem is different from the one considered above, because in the context of vortex patches, the solutions are not regular, while in Theorem 7 the solutions are smooth.

A result that one can prove is the following.

Theorem 10 (G.-Horsin). *Consider two smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area. Suppose also that the control zone Σ is in the exterior of these curves. Let $v_0 \in \mathscr{L}ip(\overline{\Omega};\mathbb{R}^2)$ with $v_0.n \in C^\infty(\partial\Omega)$ a vortex patch initial condition corresponding to γ_0, i.e. such that*

$$curl\,(v_0) = \mathbf{1}_{Int(\gamma_0)} \text{ in } \Omega, \;\; div\,(v_0) = 0 \text{ in } \Omega, \;\; v_0.n = 0 \text{ on } \partial\Omega \setminus \Sigma.$$

Then for any $T > 0$, any $k \in \mathbb{N}$, any $\varepsilon > 0$, there exists $u \in L^\infty([0,T];\mathscr{L}ip(\overline{\Omega}))$ a solution of the Euler equation such that

$$curl\, v = 0 \text{ on } [0,T] \times \Sigma,$$
$$v.n = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Sigma) \text{ and } v_{|t=0} = v_0 \text{ in } \Omega,$$

that $\Phi^v(T,0,\gamma_0)$ does not leave the domain and and that, up to reparameterization, one has

$$\|\gamma_1 - \Phi^v(T,0,\gamma_0)\|_{C^k} \leq \varepsilon.$$

Note that in the above result, we impose the entering vorticity to be zero. The reason for this is that we want the vortex patch to stay a vortex patch; hence we do not want to add vorticity inside the domain.

Remark 5. Let us focus on the regularity of the velocity field:

- As long as the patch stays regular, one has $v(t,\cdot) \in \mathscr{L}ip(\Omega)$ (see for instance [20]).
- Without the regularity of the patch, the velocity field $v(t,\cdot)$ is merely log-Lipschitz:
$$|v(t,x) - v(t,y)| \lesssim |x-y| \max(1, -\log(|x-y|).$$

This estimate is a central argument in [84, 85].

Hence we obtain a result on the shape of the patch in C^k, despite the fact that the velocity field is Lipschitz only. This is connected to the fact that this velocity field is in fact more regular in several "good directions", see the references above.

3.2 Ideas of proof

Let us now give a few ideas of the proofs of Theorems 7 and 10.

3.2.1 The main proposition.

We exploit the same idea to use *potential flows* and complex analysis as in Section 2. If we follow the ideas of Section 2, we would like to find a potential flow which makes the fluid approximately go from one zone to another. In particular, this will answer to the problem in the particular case where $v_0 = 0$.

Precisely, the core of the proof is to show the following proposition.

Proposition 3. *Consider two smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area. For any $k \in \mathbb{N}$, $\varepsilon > 0$, there exists $\theta \in C_0^\infty([0,1]; C^\infty(\overline{\Omega}; \mathbb{R}))$ such that*

$$\Delta_x \theta(t,\cdot) = 0 \text{ in } \Omega, \text{ for all } t \in [0,1],$$
$$\frac{\partial \theta}{\partial n} = 0 \text{ on } [0,1] \times (\partial\Omega \setminus \Sigma),$$

whose flow satisfies

$$\forall t \in [0,1],\ \Phi^{\nabla\theta}(t,0,\gamma_0) \subset \Omega,$$

and, up to reparameterization of the curves,

$$\|\gamma_1 - \Phi^{\nabla\theta}(1,0,\gamma_0)\|_{C^k} \leq \varepsilon.$$

In other words, there exists a potential flow driving γ_0 to γ_1 (approximately in C^k) and fulfilling the boundary condition on $\partial\Omega \setminus \Sigma$. The time interval here is fixed to be $[0,1]$; one can change the parameterization in time to transform it into any $[0,T]$.

A large part of the proof consists in establishing this proposition. This is proven in two steps:

- **Part 1:** find a solenoidal (divergence-free) vector field driving γ_0 to γ_1.
- **Part 2:** approximate (at each time) the above vector field on the curve (or to be more precise, its normal part), by the gradient of a harmonic function defined on $\overline{\Omega}$ and satisfying the constraint on $\partial\Omega \setminus \Sigma$.

3.2.2 Part 1: finding a solenoidal vector field driving exactly γ_0 to γ_1.

In this paragraph, we consider the problem of driving γ_0 to γ_1 (exactly), by a divergence-free vector field. Of course, this constraint on the vector field is significantly weaker than the constraint to be a potential flow. In return, one can obtain an exact result. In more precise form, one can prove the following proposition. We denote by $\mathrm{Int}(\gamma)$ the interior of a Jordan curve γ in the sense of Jordan's theorem, that

is, the (unique) bounded component of $\mathbb{R}^2 \setminus \gamma$. We also denote by $|A|$ the Lebesgue measure of a measurable subset $A \subset \mathbb{R}^2$.

Proposition 4. *Consider γ_0 and γ_1 two smooth (C^∞) Jordan curves which are homotopic in Ω and satisfy*

$$|Int(\gamma_0)| = |Int(\gamma_1)|. \tag{26}$$

Then there exists $v \in C_0^\infty((0,1)\times\Omega;\mathbb{R}^2)$ such that

$$div\, v = 0 \text{ in } (0,1)\times\Omega,$$

$$\Phi^v(1,0,\gamma_0) = \gamma_1.$$

Note that, even without the divergence constraint, the result is not trivial (but it is known for a long time in that case). Indeed, the two Jordan curves being homotopic means that one can find a continuous function $\Gamma : [0,1]\times\mathbb{S}^1 \to \mathbb{S}^1$ such that $\Gamma(0,\cdot) = \gamma_0$ and $\Gamma(1,\cdot) = \gamma_1$. But it does not say that the deformation is regular, nor the fact that for $t \in (0,1)$, $\Gamma(t,\cdot)$ is still a Jordan curve...

Ideas of proof of Proposition 4. As a matter of fact, the case where $\mathrm{Int}(\gamma_0)$ and $\mathrm{Int}(\gamma_1)$ do not intersect can be treated rather simply. An idea in this case would be for instance to draw a "pipe" between the two domains, and to "blow" the first domain into the second one, through the pipe.

But this is less clear if the two domains intersect. In that case, one has in particular to be sure that the deformation of γ_0 does not self-intersect. And one should keep in mind that we have the constraint that the fluid zone under view should stay inside Ω: there could be very few room left inside Ω (in particular, it may be impossible to separate the two zones in order to apply the strategy described above.)

1. A way to treat the general case is to get in the opposite case where the two zones intersect:

$$\mathrm{Int}(\gamma_0) \cap \mathrm{Int}(\gamma_1) \neq \emptyset. \tag{27}$$

To prove that one can reduce the study to the case described by (27), it is enough to find a solenoidal vector field v driving some point of γ_0 inside $\mathrm{Int}(\gamma_1)$ (while letting some other point outside), and to consider $\Phi^v(1,0,\gamma_0)$ as a new initial curve. Note that one cannot have $\gamma_0 \subset \mathrm{Int}(\gamma_1)$ due to (26).

Constructing such a vector field is not difficult. Indeed, we have much flexibility to construct a solenoidal vector field: any vector field of the form

$$v(t,x) = \nabla^\perp \psi(t,x) = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \tag{28}$$

is automatically solenoidal. Hence a possible procedure is the following:

- draw a smooth curve $\mathscr{C}$ in Ω from some point of γ_0 to some point inside $\mathrm{Int}(\gamma_1)$,
- introduce the velocity vector $\dot{\mathscr{C}}$ on the graph $(t,\mathscr{C}(t))$ of the curve,
- extend this field on $[0,1]\times\Omega$ with the form (28); using a cutoff function (applied to ψ) if necessary to ensure that this field is compactly supported in Ω.

One can check that this procedure allows to construct v as claimed.

Let us add, that, using a small translation if necessary, we can moreover suppose that γ_0 and γ_1 intersect transversally. Of course, a translation is obtained by the flow of a solenoidal vector field, and again we can make the corresponding vector field compactly supported in Ω. And by a small translation of γ_0, one can make the curves transverse (by using the parametric form of Thom's transversality Theorem for instance).

2. We are now in the situation described by Figure 14. As a matter fact, things can be way messier, but let us give the idea of the proof when Ω is simply connected, so that $\partial\Omega$ is connected and no connected components of $\partial\Omega$ can be enclosed by γ_0 and γ_1. The goal is to deform γ_0 on γ_1 in an area-preserving way.

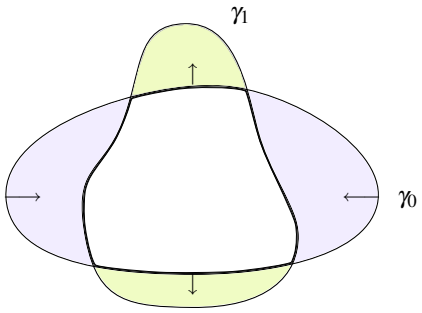

Fig. 14 Deforming one curve on another

Now to make the construction, as described above, we first define the vector field on the curve itself as it evolves through time, and then to extend it on the whole $[0,T]\times\Omega$. We work only inside the symmetric difference of the two interiors (colored on Figure 14) to deform one curve to another (see the arrows in Figure 14). The goal is, on each component of this symmetric difference, to find a vector field which drives the segment of γ_0 to the one of γ_1. This can be done inside $\mathrm{Int}(\gamma_0)$ (zones on the left and on the right in Figure 14) or inside $\mathrm{Int}(\gamma_1)$ (zones on the top and on the bottom in Figure 14). To do so, there are several steps.

- Denote S_k the component of the symmetric differences. Each S_k is circumscribed by a segment of γ_0 that we denote γ_0^k and a segment of γ_1 that we denote γ_1^k. We aim at constructing a vector field driving the interval γ_0^k of γ_0 on the interval γ_1^k of γ_1.
- A way to do this (inspired from [24]) is to consider $\nabla\phi$ where ϕ is the harmonic extension of a function g equal to 0 (respectively 1) on the interval γ_0^k (resp. on the interval γ_1^k) and "regularized near the intersections" $\gamma_0^k\cap\gamma_1^k$. The regularization near the intersections consists in introducing near the two intersection points a small curve joining γ_0^k and γ_1^k so that the resulting domain D is smooth (see Figure 15), and considering g going smoothly from 0 to 1 on these small curves.

We extend ϕ in a harmonic way in D, and then the extension of the vector field to the whole domain S_k can be done by using local coordinates near the two intersection points (which is made easy due to the transversality of γ_0 and γ_1).

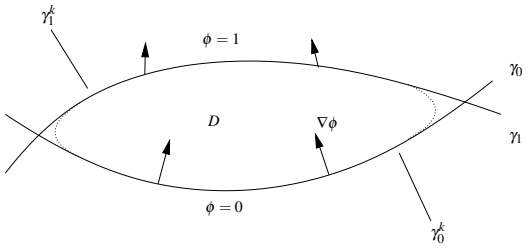

Fig. 15 Construction of a vector field on S^k

- Next we normalize the vector fields in order that the corresponding flow, satisfies in each S_k and for all $t \in [0,1]$:

$$\text{Area}(\gamma_0^k, \Phi(t,0,\gamma_0^k)) = t\,\text{Area}(\gamma_0^k, \gamma_1^k), \tag{29}$$

  where $\text{Area}(\gamma_0^k, \Phi(t,0,\gamma_0^k))$ denote the area enclosed between γ_0^k and $\Phi(t,0,\gamma_0^k)$.
- Then we have to glue these vector fields defined in each S_k together, in a way that it is smooth at the points of $\gamma_0 \cap \gamma_1$. Again, we can use local coordinates to make an explicit construction here.
- Then, thanks to (29), the vector field restricted to $\{(t, \Phi(t,0,\gamma_0))\}$ can be extended to a global solenoidal vector field.

3.2.3 Part 2: approximating the flow of the reference vector field by a potential flow.

Now that we have a reference vector field driving γ_0 to γ_1, we have to explain how we can approximate its flow on γ_0 by the action of a potential flow. This is given by the following proposition.

Proposition 5. *Let γ_0 a smooth (C^∞) Jordan curve; let $X \in C^0([0,1];C^\infty(\overline{\Omega}))$ a smooth solenoidal vector field, with $X.n = 0$ on $[0,1] \times \partial\Omega$. Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\theta \in C^\infty([0,1] \times \overline{\Omega};\mathbb{R})$ such that*

$$\Delta_x \theta(t,\cdot) = 0 \text{ in } \Omega, \text{ for all } t \in [0,1],$$
$$\frac{\partial \theta}{\partial n} = 0 \text{ on } [0,1] \times (\partial\Omega \setminus \Sigma),$$

and whose flow satisfies

$$\forall t \in [0,1],\ \Phi^{\nabla\theta}(t,0,\gamma_0) \subset \Omega,$$

and, up to reparameterization,

$$\|\Phi^X(t,0,\gamma_0) - \Phi^{\nabla\theta}(t,0,\gamma_0)\|_{C^k} \leq \varepsilon, \ \ \forall t \in [0,1].$$

Ideas of proof for Part 2. The proof follows three successive steps of growing generality, namely

- We first treat the case when all the data, that is, both γ_0 and X are real-analytic (in the x-variable for the latter),
- Then we relax the assumption by assuming only X to be real-analytic (while γ_0 is merely C^∞),
- And finally we relax the assumption by assuming only C^∞ smoothness of the data.

Again, for simplicity, we assume that Ω is simply connected (but this is not as crucial as for the controllability in the usual sense.)

First step: when the data are real-analytic: $\gamma_0 \in C^\omega(\mathbb{S}^1;\mathbb{R}^2)$ and $X \in C^0([0,1];C^\omega(\overline{\Omega}))$.

Let $\gamma(t) := \Phi^X(t,0,\gamma_0)$. For any t, this is a real-analytic curve. The main principle is the following. If we want the action of the potential flow $\nabla\theta$ on γ_0 to generate exactly $\gamma(t)$ (up to reparameterization), we only have to mimic the normal part of X on $\gamma(t)$ (the tangential part is "absorbed by the reparameterization".)

Hence it is natural to consider for each time the solution of the following elliptic problem:

$$\begin{cases} \Delta_x \psi(t,\cdot) = 0 \text{ in } \mathrm{Int}(\gamma(t)), \\ \dfrac{\partial \psi}{\partial n}(t,\cdot) = X(t,\cdot).n(\cdot) \text{ on } \gamma(t), \\ \displaystyle\int_{\gamma(t)} \psi(t,\cdot)\, d\sigma = 0. \end{cases}$$

This is the only harmonic function defined in $\mathrm{Int}(\gamma(t))$ which has exactly the "correct" normal part on $\gamma(t)$. Unfortunately, this function cannot be extended as a harmonic function on Ω in general, nor a fortiori in a way that satisfies $\partial_n \psi = 0$ on $[0,1] \times (\partial\Omega \setminus \Sigma)$.

But here is the place where the real-analyticity plays a crucial role: as $\gamma(t)$ and $X.n$ on $\gamma(t)$ are analytic, we can extend the solution ψ across the boundary $\gamma(t)$ (this is a classical Cauchy-Kowalewsky-style result, see for instance [71]).

Moreover, using the continuity in time of X and γ with values in C^ω (see e.g. [59] for more details on the topology of C^ω), we see that the size of the neighborhood of $\gamma(t)$ where this solution can be extended can be estimated from below.

Now this vector field is still not globally defined on Ω, but we can use Runge's theorem in a similar way as in Section 2. Proceeding in the same way, we can obtain approximations defined on $\overline{\Omega}$, and which satisfy

$$\nabla\tilde{\psi}(t,\cdot).n = 0 \text{ on } \partial\Omega \setminus \Sigma. \tag{30}$$

As previously, (30) is not obtained exactly in a first time, but one can remove the solution of a suitable Neumann problem to get this relation exactly.

Finally, we obtain the function θ as:

$$\theta(t,x) = \sum_{k=1}^{n} \rho_i(t)\tilde{\psi}(t_i,\cdot),$$

with ρ_i a certain partition of unity of $[0,1]$. (We use that Runge's approximation obtained at time t, is still an acceptable approximation in some neighborhood of t, so that by compactness of $[0,1]$ we can consider only a finite number of $\tilde{\psi}(t_i,\cdot)$.)

The rest of the proof consists in explaining why the cost of changing X by $\nabla\theta$ is small: this is mainly a Gronwall's lemma and the use of reparameterization to compensate the discrepancy of the tangential components of the vector fields. By this process, by choosing a sufficiently small parameter in Runge's theorem, and using the fact that the size of the neighborhood of $\gamma(t)$ on which ψ can be extended is uniform, one can also obtain estimates of $\|\nabla\theta\|_{C^k}$ on $\Phi^{\nabla\theta}(t,0,\gamma_0)$ in terms of $\|\nabla\psi(t)\|_{C^k}$ on $\gamma(t)$ only.

Second step: when only the vector field is real analytic: $X \in C^0([0,1];C^\omega(\overline{\Omega}))$ but $\gamma_0 \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$.

The idea is of course to use the previous step. We can approach γ_0 by real analytic curves, from the outside. This comes from a general result by H. Whitney [83], or in a simpler way in our case:

- We consider C_0 the complement of $\mathrm{Int}(\gamma_0)$ in the Riemann sphere. By Riemann's conformal mapping theorem, there exists φ a conformal transformation from C_0 to $\overline{B}_{\mathbb{C}}(0,1)$.
- Then, since such a conformal transformation is regular up to the boundary when γ_0 is regular (say, C^∞) (this is Kellogg-Warschawski's theorem, see e.g. [74]), the curve $\varphi(S(0,1-\nu))$ is an appropriate approximation as $\nu \to 0^+$.
- Next, we apply the process of Part 1 on the ν-approximation γ_0^ν of γ_0. We obtain a function θ^ν. Call $\gamma_\nu(t) := \Phi^{\nabla\theta^\nu}(t,0,\gamma_0^\nu)$.
- The central point is to show that, on $\gamma^\nu(t)$, we have uniform estimates on $\nabla\theta^\nu$ as $\nu \to 0^+$.
- Due to the construction in the Step 1, we have only to prove uniform estimates on the $\nabla\psi^\nu(t)$ constructed on $\gamma^\nu(t)$ as $\nu \to 0^+$.
- This is obtained by noting that the constants in elliptic estimates in $\mathrm{Int}(\gamma^\nu(t))$ are bounded independently from ν. Indeed, $\gamma^\nu(t)$ converges to $\gamma(t)$ for the C^∞ topology. It follows that we have uniform estimates on θ **inside Int**$(\gamma^\nu(t))$, in all the C^k-norms in terms of X, γ_0 and k only. In particular, these estimates do not blow up as $\nu \to 0^+$.
- We are then able to conclude by Gronwall's lemma, because $\Phi^{\nabla\theta^\nu}(t,0,\gamma_0)$ is precisely included in $\mathrm{Int}(\gamma^\nu(t))$ since γ_0 is in the inside of γ_0^ν.

Third step: when both data are merely C^∞: $\gamma_0 \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ and $X \in C^0([0,1];C^\infty(\overline{\Omega}))$.

Again this is a consequence of the previous step. We use Whitney's analytic approximation theorem [82]: X can be approached arbitrarily for the $C^0([0,1];C^\infty(\overline{\Omega}))$-topology by $X^n \in C^0([0,1];C^\omega(\overline{\Omega}))$.

Hence we construct by the step above potential flows corresponding to X^n. Then we can prove by using the previous step and Gronwall's lemma that for n sufficiently large, we have a good approximation of the flow on γ_0.

3.2.4 How to deduce the results from the main proposition

The idea uses here the same argument of time-scale invariance of the Euler equation.

1. As before, we first consider the case when $\|v_0\|_{C^{k+1,\alpha}} \ll 1$. In that case, proceeding as previously, one can construct a solution of the Euler equation, starting from v_0, such that the normal velocity on the boundary is mainly $\nabla\theta$ (if fact we have to take $v_0.n$ into account), and such that $\|v(t,\cdot)\|_{C^{k+1,\alpha}}$ is of the same order as $v_0\|_{C^{k+1,\alpha}}$. To make the construction, one can use an analogous fixed point scheme as in Section 2, using an extension operator (but here we do not need the cutoff functions). Then standard perturbation arguments show that one has

$$\begin{aligned}\|\Phi^v(T,0,\gamma_0)-\gamma_1\|_{C^k} &\leq \|\Phi^v(T,0,\gamma_0)-\Phi^{\nabla\theta}(T,0,\gamma_0)\|_{C^k}+\|\Phi^{\nabla\theta}(T,0,\gamma_0)-\gamma_1\|_{C^k}\\ &\lesssim \|v_0\|_{C^{k+1,\alpha}}+\varepsilon.\end{aligned}$$

2. Then one uses again the time scale invariance of the equation as follows. We cut the time interval in two parts: for $\nu > 0$ small, there are two phases, namely, during the time intervals $[0,T-\nu]$ and $[T-\nu,T]$, such as described in Figure 16. The control is performed as follows.

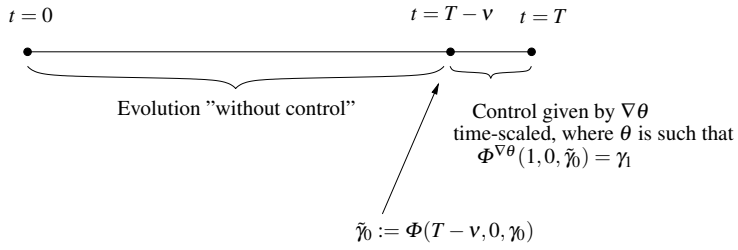

Fig. 16 The two phases of the control

- In a first time, during the time interval $[0,T-\nu]$, we "do nothing", that is we mainly wait. In fact, we have to take $v_0.n$ into account, and to preserve the reg-

ularity of the solution. But we have no other purpose during this time interval than to wait the second phase and to let the size of the solution v stay of the same order as v_0. This can be done by introducing the same type of fixed point scheme as previously. We can drive the normal part of the velocity on Σ to 0 during this phase.

- At the very end of the time interval, that is, during $[T-\nu,T]$, we act fast and violently to drive $\tilde{\gamma}_0 := \Phi(T-\nu,0,\gamma_0)$ to γ_1. The control is given by the normal part of $\frac{1}{\nu}\nabla\theta(t-T+\nu,\cdot)$ for what concerns the normal velocity. The part of the control concerning the vorticity is used just in order not to ruin the regularity and that the size of the vorticity stays of the same order as ω_0.

3. Now let v be the resulting solution in $[0,T]\times\Omega$. If we change back the time scale to get back the dynamics of the time interval $[T-\nu,T]$ to the time interval $[0,1]$, the evolution is driven by the Euler equation, with:

- as boundary condition (on the normal trace) the same as $\nabla\theta$,
- as initial condition $\nu u(T-\nu,\cdot)$, which is clearly small as $\nu\to 0^+$.

Hence as in Section 2, we are in the same situation as if the initial datum was small! And we can show that the solution that we constructed on $[0,T]$ satisfies:

$$\|\Phi^v(T,0,\gamma_0)-\gamma_1\|_{C^k}\lesssim \nu+\varepsilon.$$

This allows to prove Theorem 7.

The case of vortex patches. Let us now say a few words concerning the proof of Theorem 10. The construction is similar, but we can no longer use

$$\|\nu u(T-\nu,\cdot)\|_{C^{k+1,\alpha}}\lesssim \nu,$$

because v is Lipschitz only! But we use instead arguments due to:

- Depauw [34], which has studied vortex patches in a domain, and showed that the regularity propagates as in Chemin's theorem; hence the "first phase" $[0,T-\nu]$ can be done in the same way,
- Bertozzi-Constantin [10], who tackled the problem of the regularity of vortex patches by using the integro-differential equation satisfied by their boundary γ:

$$\frac{d}{dt}\gamma(t,s)=-\frac{1}{2\pi}\int_0^{2\pi}\log|x-\gamma(\sigma)|\,\tau(\sigma)\,d\sigma$$
$$+\text{here, terms due to the presence of } \partial\Omega \text{ and of the control.}$$

Using this approach, we can see that, despite the fact that the flow is merely Lipschitz, it propagates the regularity of the boundary of the patch. And including the terms due to the boundary and to the control is not a real issue, since these terms are regular.

3.3 Comments

The main reference concerning this section is [48], where the technical details are written. This lets several problems open, though.

What about 3D? Several problems appear when considering the dimension 3:

- How to deform (in a smooth, volume-preserving way) a domain to another one?
- How to prevent the solution from potentially blowing up?
- The others parts of the proof do not depend on the dimension...

Results in that direction are partial:

Proposition 6 (G.-Horsin, in progress). *If B_1 and B_2 are two smooth open sets in Ω, diffeomorphic to a ball, with same volume, at positive distance from $\partial\Omega$ and* **disjoint***, then one can smoothly deform B_1 to B_2 inside Ω in a volume-preserving manner.*

This yields to

Corollary 1 (G.-Horsin, in progress). *Let B_1 and B_2 as previously, and S_1, S_2 their boundary. Let $k \in \mathbb{N}$. We consider $v_0 \in C^\infty(\overline{\Omega};\mathbb{R}^3)$ satisfying*

$$div\,(v_0) = 0 \text{ in } \Omega \text{ and } v_0.n = 0 \text{ on } [0,T]\times(\partial\Omega\setminus\Sigma).$$

For any $\varepsilon > 0$, **there exists** *$T > 0$ and a solution v of the Euler equation in $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^3)$ with*

$$v.n = 0 \text{ on } [0,T]\times(\partial\Omega\setminus\Sigma) \text{ and } v_{|t=0} = v_0 \text{ in } \Omega,$$

and whose flow satisfies

$$\forall t \in [0,T],\ \Phi^v(t,0,B_1) \subset \Omega,$$

and up to reparameterization

$$\|S_2 - \Phi^v(T,0,S_1)\|_{C^k} \leq \varepsilon.$$

The fact that the result is valid *for short control times* comes as a way to avoid blow-up.

Open problems. Other open problems can be raised in this field.

- *More complex domains.* What can be said if the fluid zone to be displaced is no longer a Jordan domain, or about more general situations in 3D?
- *Numerics.* Can we find an efficient algorithm to compute the control?
- *Navier-Stokes equations.* Can we obtain a similar result for incompressible Navier-Stokes equations?

$$\begin{cases} \partial_t v + (v.\nabla)v - \Delta v + \nabla p = 0 \text{ in } [0,T]\times\Omega, \\ \text{div } v = 0 \text{ in } [0,T]\times\Omega. \end{cases}$$

This question can be raised both in the cases of Dirichlet's boundary conditions and with Navier's (for which one could try to use the techniques of Coron [28] and Chapouly [18]).

- *Stabilization.* Can we find a feedback control:

$$\text{control(t)} = f(\gamma(t), v(t)),$$

stabilizing a fluid zone at a fixed place?

4 Controllability of the 1D isentropic (compressible) Euler equation

In this last section, we consider a different model, namely the 1D isentropic compressible Euler equation. Despite the fact that this equation, like the incompressible Euler equation, models an inviscid fluid evolving under the influence of pressure, the mathematical properties of the two equations are rather different. However, for what concerns the controllability of this equation, the basic principle of using the return method is common, even if it takes different forms.

4.1 Introduction

The models that we consider here are the following. There are two versions of the one-dimensional isentropic Euler equations: in Eulerian coordinates or in Lagrangian coordinates (that is, when following the flow). These equations read:

- In Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \partial_x(m) = 0, \\ \partial_t(m) + \partial_x(\frac{m^2}{\rho} + \kappa\rho^\gamma) = 0. \end{cases} \tag{EI}$$

- In Lagrangian coordinates (the equation is also known as the p-system):

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x(\kappa\tau^{-\gamma}) = 0. \end{cases} \tag{P}$$

Above, the various notations are:

- $t \in \mathbb{R}^+$ is the time, $x \in \mathbb{R}$ is the position,
- $\rho = \rho(t,x) \geq 0$ is the density of the fluid,

- $m(t,x)$ is the momentum ($v(t,x) = \frac{m(t,x)}{\rho(t,x)}$ is the velocity of the fluid),
- $\tau := 1/\rho$ is the specific volume,
- the pressure law is $p(\rho) = \kappa\rho^\gamma = \kappa\tau^{-\gamma}$, $\gamma \in (1,3]$, $\kappa > 0$.

As is well-know, these two equations stand respectively for the conservation of mass and momentum in the fluid.

The controllability problem. What we consider in this chapter is the study of the above equations from the point of view of controllability. The equation will be posed on a bounded domain of the real line, say $[0,1]$; hence (t,x) belongs to $[0,T]\times[0,1]$.

The *state* of the system will be the couple of both unknowns, that is:

$$\text{Case (EI): } u = (\rho, m), \quad \text{Case (P): } u = (\tau, v). \tag{31}$$

The *control* will be the "boundary data", which is a very delicate matter for this type of equations (see for instance [37, 4, 5]). As before, in order to avoid dealing with this issue, we will not look for the control itself, but rather for the solution itself. Hence we will not focus on this aspect.

Finally the *controllability problem* is the following: given $u_0 = (\rho_0, m_0)$ (or $u_0 = (\tau_0, v_0)$) and $u_1 = (\rho_1, m_1)$ (or $u_1 = (\tau_1, v_1)$), can we find *a solution* of the system driving u_0 at initial time to u_1 at time T? For which T?

Class of solutions. Both equations (EI) and (P) are classical examples of *hyperbolic systems of conservation laws:*

$$u_t + f(u)_x = 0,$$

where $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ (see the next section for more details on this class of equations).

Hyperbolic systems of conservation laws are known to develop singularities in finite time. This is due to the mechanism of *formation of shocks*, which are easy to see for instance for what concerns the Burgers equation (for which $n = 1$):

$$u_t + (u^2)_x = 0.$$

One can use the method of characteristics to show that u is constant along the characteristics associated to u (that is, $u(t, \Phi^u(t,0,x)) = u(0,x)$). As a consequence, these characteristics are straight lines. For many u_0, the straight lines can cross. These leads to shock waves appearing in the solution, such as described in Figure 17.

When considering control problems associated to equations such as (EI) or (P), a possibility is to consider regular solutions (say C^1), whose existence for a relevant interval of time is ensured by the smallness of the data. See for instance the references given in Section 4.4 for such studies.

But from both mathematical and physical viewpoints, one should also consider the case of *discontinuous* weak solutions in which *shock waves* may appear, which

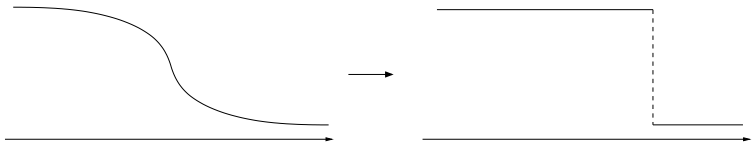

Fig. 17 Formation of shock

are to be understood in the sense of distributions (his makes sense for $u \in L^\infty$ for instance). But a classical issue for what concerns weak solutions containing discontinuities, is that in general in this context *uniqueness is lost*. Hence it is natural to consider weak solutions which satisfy additional requirements, aimed at selecting among all weak solutions, the physically relevant one. These will be called *entropy conditions*. Here we will consider a special class of *entropy solutions*. These solutions will be of bounded variation in the variable x uniformly in t, that is will belong to $L^\infty(0,T;BV(\mathbb{R}))$. Moreover these solutions will be of small total variation (mainly) and will avoid vacuum; as we will see they are constructed by a particular technique known as the *wave front tracking algorithm*. We discuss this more precisely in the next section.

Let us underline that it is very important to specify which class of solutions are considered (regular solutions or weak entropy solutions), because the properties of the equation in the two contexts are really different. For instance, the system is reversible in the case of C^1 solutions, not in the context of weak entropy solutions. As the reader knows (or guesses), this is not a detail when it comes to controllability questions.

4.2 Basic facts on systems of conservation laws

In this section, we recall some basic facts about (one-dimensional) systems of conservation laws and a particular way to construct solutions of these systems known as the *wave front tracking algorithm*. The reader familiar with this is encouraged to skip the section; the one who would like to know more precisely the theory (and to see the proofs) is referred to Bressan [13], Dafermos [32], Holden & Risebro [52] or LeFloch [61].

Systems of conservations laws. Equations (EI) and (P) are PDEs of a particular class, known as *systems of conservation laws*. Here, we consider only one-dimensional problems, and these are written as follows

$$u_t + f(u)_x = 0, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \tag{32}$$

where f is a smooth *flux* function (let us say, of class C^2 to fix the ideas) satisfying the following *strict hyperbolicity condition*:

for all $u \in \Omega$, $A(u) := df(u)$ has n real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$.

These scalar functions $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$ are the *characteristic speeds of the system*. Denote by $r_i(u)$, $i = 1 \ldots n$, some corresponding eigenvectors.

The theory concerning such systems is simplified when the characteristic fields (λ_i, r_i) satisfy a condition called the *genuine non-linearity* the sense of Lax [60]:

$$\nabla \lambda_i . r_i \neq 0 \quad \text{for all } u \text{ in } \Omega.$$

When this condition is satisfied —in particular this is the case for what concerns (EI) and (P)—, we normalize the vector fields r_i so that

$$\nabla \lambda_i . r_i = 1 \ \text{ in } \Omega. \tag{33}$$

For what concerns the two systems that we consider above, by standard computations one can show that (EI) and (P) satisfy the strict hyperbolicity condition (away from the vacuum $\rho = 0$) and that both characteristic fields are genuinely nonlinear. The characteristic speeds are as follows:

- Case (EI): $u = (\rho, m) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\lambda_1 = \frac{m}{\rho} - \sqrt{\kappa\gamma}\rho^{\frac{\gamma-1}{2}} \ \text{ and } \ \lambda_2 = \frac{m}{\rho} + \sqrt{\kappa\gamma}\rho^{\frac{\gamma-1}{2}},$$

- Case (P): $u = (\tau, v) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\lambda_1 = -\sqrt{\kappa\gamma\tau^{-\gamma-1}} \ \text{ and } \ \lambda_2 = \sqrt{\kappa\gamma\tau^{-\gamma-1}}.$$

One can see immediately an important difference between the two cases: for what concerns (P), the characteristic speeds have a constant sign, while this is not the case for (EI). This is very important for our problem, since the sign of the characteristic speed indicates the direction in which the solution propagates; and in particular the way the boundary control propagates inside the domain.

Entropy solutions. Now, as we indicated above, we will consider weak solutions which may contain discontinuities. Since in general uniqueness does not hold in this context, it is natural to introduce *entropy solutions*, which are weak solutions which fulfill additional admissibility conditions, aimed at selecting among the set of weak solutions, the physically acceptable one. A way to introduce the *entropy criterion* is the following.

One defines an *entropy/entropy flux couple* as a couple of functions $(\eta, q) : \Omega \to \mathbb{R}^2$ such that

$$\forall u \in \Omega, \ \ D\eta(u).Df(u) = Dq(u).$$

Then one defines an *entropy solution*: as a (weak) solution of (32) such that for any entropy couple (η, q) with η *convex*, one has:

$$\eta(u)_t + q(u)_x \leq 0, \tag{34}$$

in the sense of measures. Of course, if the solution u is regular, then (34) takes place as an equality, by the chain rule. This is no longer necessarily true for discontinuous solutions.

A way to justify the conditions (34) is the following. One can show that the solutions obtained by *vanishing viscosity*, i.e. as limits of solutions of the system where a small viscosity term has been added:

$$u_t^\varepsilon + (f(u^\varepsilon))_x - \varepsilon u_{xx}^\varepsilon = 0,$$

are entropy solutions. This explains the physical meaning of entropy solutions: in some sense, entropy solutions are solutions from which viscosity has disappeared, except for what concerns *the selection of admissible discontinuities*. We will see later another formulation of this selection at the level of a single discontinuity.

A celebrated result concerning hyperbolic systems of conservation laws with genuinely nonlinear fields is due to Glimm [50]. In this paper is shown the existence of global in time entropy solutions for such systems with the assumption that the initial data is of small total variation. The resulting entropy solution is then of small total variation uniformly in time.

There is now a huge literature on the subject, and it is virtually impossible to refer to all the works of the field in this course; see for instance the books [13, 32, 52, 61, 79] and references therein. Let us however underline that the situation is now well understood in the context of solutions with small total variation in the general case (not limited to the genuine nonlinearity assumption) for what concerns existence as well as uniqueness, stability issues, etc. See in particular Bianchini-Bressan [11].

Riemann problem. Now let us explain a way to construct solutions of (32). We will restrict ourselves to the case when $n = 2$ ("2×2 systems") and when both the fields are genuinely nonlinear. This is sufficient to treat (EI) and (P). The wave front tracking method uses as an elementary brick the solutions of the so-called *Riemann problem*, which consists in finding self-similar solutions $u = \overline{u}(x/t)$ to

$$\begin{cases} u_t + (f(u))_x = 0 \\ u_{|\mathbb{R}^-} = u_l \text{ and } u_{|\mathbb{R}^+} = u_r, \end{cases} \tag{35}$$

where u_l and u_r are constants of Ω. The fact that, given such initial data, one should look for self-similar solutions of (32) is due to the scale invariance of the equation under the change of variables $(t,x) \mapsto (\lambda t, \lambda x)$.

Of course, solutions of (35) are very particular cases of solutions of (32); however we will see that more general solutions can be constructed by "gluing together" pieces of solutions obtained as solutions to the Riemann problem.

Now in the particular case under view (the genuine nonlinearity is essential here), the Riemann problem can be solved by introducing *Lax's wave curves*. These are curves inside Ω which consist of all points $u_r \in \Omega$ that can be connected to a fixed u_l by particular solutions of (35), which are *shock waves* or *rarefaction waves*, which we now describe.

Elementary waves. Let us now describe these elementary waves.

• *Shock waves* are admissible discontinuous solutions joining u_l and u_r, as in Figure 18. More precisely, a shock is a simple discontinuity between the states u_l and u_r (on the left and the right, respectively), traveling at speed s satisfying *Rankine-Hugoniot relations*:

$$[f(u)] = s[u] \text{ (jump condition)}, \tag{36}$$

and *Lax's inequalities*:

$$\lambda_i(u_r) < s < \lambda_i(u_l) \text{ and } \lambda_{i-1}(u_l) < s < \lambda_{i+1}(u_r). \tag{37}$$

Lax's inequalities are associated to each characteristic family ($i = 1, \dots, n$), and each shock satisfies exactly one of them. As a consequence, there is a family of shocks associated to each characteristic family ($i = 1, \dots, n$).

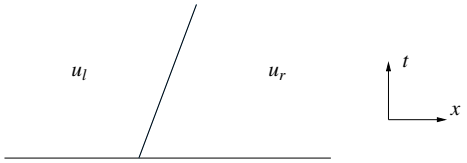

Fig. 18 A shock wave

In (36), the brackets denote the jump of the quantity across the discontinuity: $[f(u)] := f(u_r) - f(u_l)$ and $[u] := u_r - u_l$.

The Rankine-Hugoniot relation (36) ensures that this is a solution in the sense of distributions, Lax's inequalities (37) (associated to each characteristic family) give the entropy criterion.

One can show that, for each $i \in \{1, \dots, n\}$, there is a curve $S-i$, passing through u_l and tangent to $r_i(u_l)$ at u_l, corresponding to the points u_r that fulfill (36). Only **half** of this curve satisfies (37). It is elementary to see that, as u_r tends to u_l along the i-th shock curve, one has $s \to \lambda_i(u_l)$.

• *Rarefaction waves* are **regular** (self-similar) solutions joining u_l to u_r, as described in Figure 19. They are obtained with the help integral curves of r_i as follows. We introduce the orbits of the vector fields r_i

$$\begin{cases} \frac{d}{d\sigma} R_i(\sigma) = r_i(R_i(\sigma)), \\ R_i(0) = u_l. \end{cases} \tag{38}$$

Now, for $\sigma \geq 0$, if $u_r = R_i(\sigma, u_l)$, then one can construct the following function:

$$u(t,x) = \begin{cases} u_l & \text{if } \frac{x}{t} < \lambda_i(u_l), \\ R_i(\sigma)(u_l) & \text{if } \frac{x}{t} = \lambda_i(R_i(\sigma)(u_l)), \\ u_r & \text{if } \frac{x}{t} > \lambda_i(u_r). \end{cases} \tag{39}$$

Using (33) one sees that this gives a solution of (32).

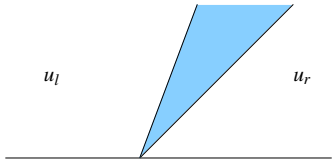

Fig. 19 A rarefaction wave

Again, for each $i \in \{1,\dots,n\}$, there is a curve, passing through u_l and tangent to $r_i(u_l)$ at u_l, corresponding to the orbit of the vector field r_i. But only **half** of this curve satisfies that the characteristic speed progresses across the wave (so that Figure 19 is valid). Due to (33), this corresponds indeed to $\sigma \geq 0$.

Lax's wave curves. Now *wave curves* are constructed as follows. Given u_l, we associate:

- the curves of i-shocks (or to be more precise, the half curves of i-shocks), given by all states u_r which can be connected by u_l through a shock of the i-th family,
- the curves of i-rarefactions (or again, the half curves of i-rarefactions), given by all states u_r which can be connected by u_l through a rarefaction waves of the i-th family,
- Lax's curves, which we will denote by Φ_i, obtained by gluing together these two half curves S_i and R_i.

One can show that Lax's curves are regular, because the i-shock curves and the i-rarefaction curves have a second-order tangency at u_l (with suitable parameterization). Figure 20 gives an example in the case of system (EI). The main point is that when u_r belongs to the i-th curve associated to u_l, that is to say, when $u_r = \Phi_i(\sigma, u_l)$, then there is an elementary wave joining u_l on the left to u_r on the right and giving an entropy solution to the Riemann problem.

In Figure 20, we have also represented the critical curves defined as the locus where one of the characteristic speeds vanishes.

Remark 6. The curves that we describe above are *right* shock, rarefaction or wave curves, because they describe the states that can be connected on the right to some fixed left state u_l. We could define in the same way *left* shock, rarefaction or wave curves describing the states that can be connected on the left to some fixed right state u_r.

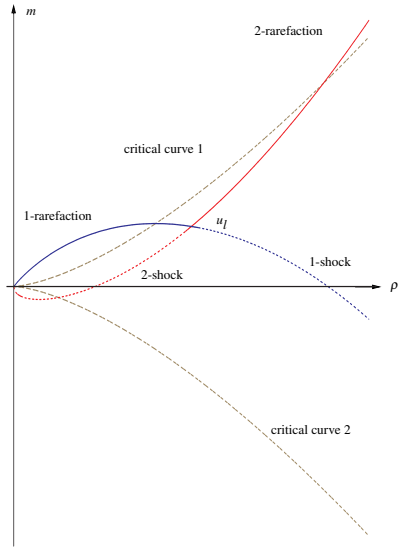

Fig. 20 Lax's curves for (EI) in (ρ, m) coordinates

These curves allow to solve the Riemann problem (at least, when u_l and u_r are sufficiently close one to another, which is sufficient to our purpose, since we will consider small *BV* solutions). Indeed, Lax [60] proved that one can solve (at least locally) the Riemann problem by first following the 1-curve then the 2-curve. In other words, one can connect any u_r sufficiently close to u_l by, first, a shock/rarefaction of the first family and, then, a shock/rarefaction of the second family, as in Figure 21. Another way to express this result is to see that $(\sigma_1, \varsigma_2) \mapsto \Phi_2(\sigma_2, \Phi_1(\sigma_1, u_l))$

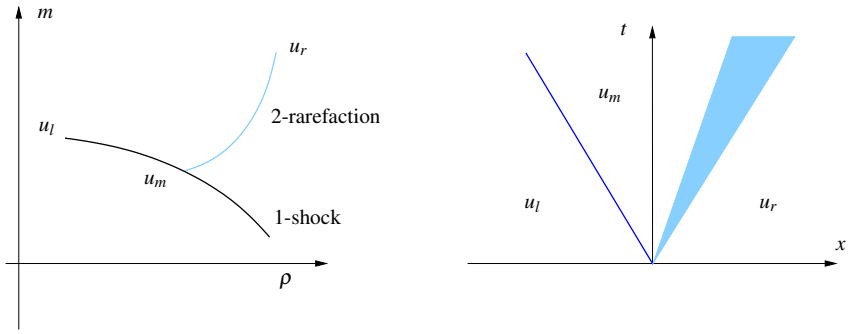

Fig. 21 Resolution of the Riemann problem

is locally onto near $(0,0)$; this is a consequence of the inverse mapping theorem. Moreover one can estimate (σ_1, σ_2) in terms of $\Phi_2(\sigma_2, \Phi_1(\sigma_1, u_l)) - u_l$ and vice

versa, with constants independent of u_l, that is :

$$c(|\sigma_1|+|\sigma_2|) \leq |\Phi_2(\sigma_2,\Phi_1(\sigma_1,u_l))-u_l| \leq (|\sigma_1|+|\sigma_2|). \tag{40}$$

Riemann invariants. Let us finally introduce the *Riemann invariants*. We will say that $w^i : \Omega \to \mathbb{R}$ is a *i-Riemann invariant* when we have

$$r_i.\nabla w^i = 0 \text{ in } \Omega. \tag{41}$$

It is elementary to determine for (EI) and (P) new coordinates given by a 1-Riemann invariant and a 2-Riemann invariant:

- Case (EI):

$$w^1(u) = \frac{m}{\rho} + \frac{2\sqrt{\kappa\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} \text{ and } w^2(u) = \frac{m}{\rho} - \frac{2\sqrt{\kappa\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}},$$

- Case (P):

$$w^1(u) = v + \frac{2\sqrt{\kappa\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}} \text{ and } w^2(u) = v - \frac{2\sqrt{\kappa\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}}.$$

It is particularly interesting to parameterize the wave curves by these Riemann invariants, because in these coordinates, naturally, rarefaction curves become half straight lines, see Figure 22.

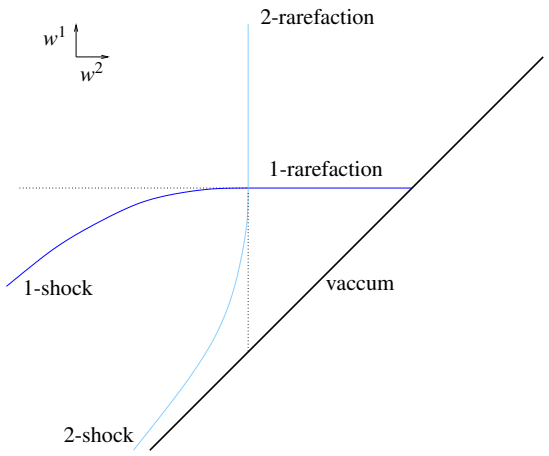

Fig. 22 Lax's curves in (w^1, w^2) coordinates

Front-tracking algorithm. Now let us discuss a particular way to construct entropy solutions to systems of conservation laws, known as the *wave-front tracking algorithm*. This algorithm was introduced by Dafermos [31] in the scalar case ($n = 1$), and extended by Di Perna [36] for 2×2 systems, and then extended by Bressan [12], Risebro [75], Ancona-Marson [9], G.-LeFloch [49], etc.

Let us underline that there are other ways to construct entropy solutions of systems of conservation laws, such as Glimm's random choice method [50], the vanishing viscosity approach [11], etc.

The basic principle is as follows:

- Construct a suitable sequence of *piecewise constant functions over a polygonal subdivision* of $\mathbb{R}^+ \times \mathbb{R}$. These approximations are called *front-tracking approximations*.
- Prove estimates in $L^\infty_t(BV_x)$ for the approximations,
- Extract by compactness a converging subsequence. Then prove that the limit is an entropy solution.

To fulfill this purpose, an algorithm is the following (we more or less follow Di Perna [36], who considers 2×2 genuinely nonlinear systems):

- Let $\nu > 0$ (which will go to 0^+).
- Approximate initial condition on $\mathbb{R}$ by piecewise constant functions: $u_0^\nu \to u_0$ in L^1_{loc} as $\nu \to 0^+$.
- At each discontinuity of u_0^ν, let us say x_0:
  - solve the corresponding Riemann problem (where the discontinuity is placed at x_0 rather than 0),
  - replace rarefaction waves by *rarefaction fans*. These are piecewise constant functions according to the variable $\frac{x-x_0}{t}$, approximating the solution given by (39) (recentered to (t_0,x_0) instead of $(0,0)$). To be more precise, let us consider as in Figure 23 a rarefaction wave at $x = 0$, separating u_m and u_r, let us say $u_r = R_i(\sigma,u_m)$, $\sigma > 0$. Then introduce states $u_1 := u_m$, $u_2, \dots, u_k = u_r$ in a way that $u_{j+1} = R_i(s_j,u_j)$ with $0 < s_j \leq \nu$ (and let us say, all s_j but s_{k-1} are equal to ν). Then the rarefaction fan is given by (for (t,x) close to $(0,x_0)$):

$$u^\nu(t,x) = \begin{cases} u_m & \text{if } \frac{x-x_0}{t} < \lambda_i(u_m), \\ u_{j+1} & \text{if } \lambda_i(u_j) \leq \frac{x-x_0}{t} < \lambda_i(u_{j+1}), \text{ for } j \leq k-1, \\ u_r & \text{if } \frac{x-x_0}{t} \geq \lambda_i(u_r). \end{cases} \tag{42}$$

At this stage, we can hence construct front-tracking approximations for small times, by extending the discontinuities along straight lines, see Figure 23. We have to explain how to extend them for all $t \geq 0$, precisely, to explain how we define the approximation after two such discontinuities meet. All discontinuities (representing a shock or approximating a rarefaction) are called *fronts*. We call an *interaction point* a point where to fronts meet.

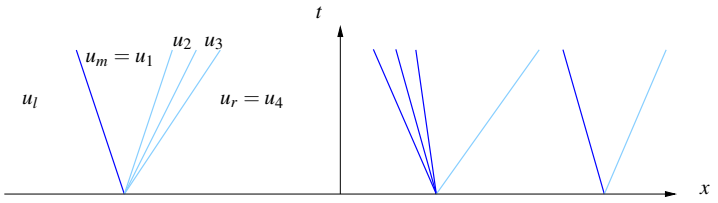

Fig. 23 A front-tracking algorithm, phase 1

- To extend the approximation across an interaction point, iterate the procedure without splitting again rarefactions (this is specific to 2×2 system). In other words, when two fronts meet, we solve the Riemann problem between the leftmost and the rightmost states, and for what concerns the rarefaction waves, we cut them into pieces as previously if there was no rarefaction front of the same family among the incoming fronts, or we approximate it by a single front otherwise. See Figure 24.

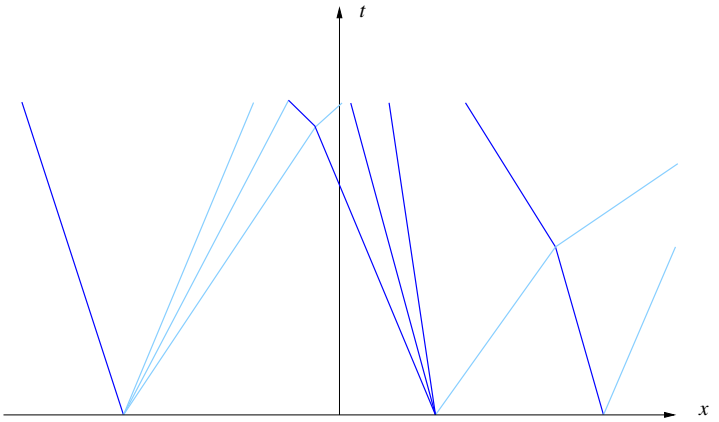

Fig. 24 A front-tracking algorithm, phase 2

One can show than this algorithm defines a piecewise constant function for all $t \geq 0$, with a finite number of fronts and discrete interaction points. (As matter of fact, to prove this, one uses estimates that are described below.)

Estimates for front tracking approximations. Now to complete the program, one has to prove estimates on front-tracking approximations, in order to get compactness and to be able to pass to the limit. (Actually, one already needs estimate to prove the above claim of well-defined approximations.)

A central argument is due to Glimm [50], and allows to obtain a bound on the total variation of the approximations, uniformly in time.

Consider the approximation u^ν obtained by the above process, defined on $\mathbb{R}^+ \times \mathbb{R}$. First, it is easy to see that the total variation of u^ν does not change except at interaction times. Hence one has only to analyze what happens at the interaction points. To that purpose, a first step is to decide a way to measure the size of a front in a front-tracking approximation. We will call σ_i the *strength* of a front, the real number such that $u_r = \Phi_i(\sigma_i, u_l)$ (so that $\sigma_i > 0$ for rarefactions, $\sigma_i < 0$ for shocks). The value $|\sigma_i|$ measures the size of the discontinuity (remember (40)); the sign of σ_i encodes the nature of the wave.

Now, at an interaction point where a i-wave meets a j-wave, one proves that, whether $i = j$ or $i \neq j$, one has the following relations between the strengths of the incoming waves, and the strengths of the outgoing ones:

$$\sigma_i'' = \sigma_i + \sigma_i' + \mathcal{O}(1)|\sigma_i \sigma_j'|. \tag{43}$$

Estimates (43) is known as *Glimm's interaction estimates*.

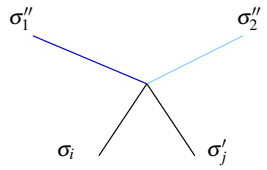

Fig. 25 Analysis of an interaction

In other words, what (43) proves is that:

- if $i = j$, the strength of the i- and j- outgoing waves are almost the same as the i- and j- incoming ones, up to a quadratic error;
- if $i \neq j$, the strength of the outgoing wave of family $i = j$ is almost the sum of the strengths of the incoming waves (up to a quadratic error), and the strength of the outgoing wave in the other family is of quadratic order.

Now consider the functionals

$$V(\tau) = \sum_{\alpha \text{ front at time } t} |\sigma_\alpha| \; ; \quad Q(\tau) = \sum_{\substack{\alpha,\beta \\ \text{approaching fronts}}} |\sigma_\alpha|.|\sigma_\beta|,$$

By *approaching fronts*, we mean fronts of different families where the leftmost front is of a faster family (that is, having a higher index), or fronts of the same family (provided that one of the two at least is a shock).

An important feature of the functional V is that, due to (40), there exists $C_1, C_2 > 0$ such that on a front-tracking approximation u^ν, one has

$$C_1 TV(u^\nu(\tau)) \leq V(\tau) \leq C_2 TV(u^\nu(\tau)).$$

Using the above interaction estimates, we see that, at an interaction time,

$$\sum_{\substack{\alpha \text{ outgoing fronts} \\ \text{after interaction at time } t}} |\sigma_\alpha| \;\leq \sum_{\substack{\alpha \text{ incoming fronts} \\ \text{interacting at time } t}} |\sigma_\alpha| \;+\mathscr{O}(1)[Q(t^-)-Q(t^+)].$$

It follows that for some $C>0$, if $TV(u_0)$ is small enough, then the functional,

$$V(t)+CQ(t),$$

known as *Glimm's functional*, is non-increasing over time.

From this we obtain a bound in $L^\infty(BV)$ of the sequence. Now one uses the *finite speed of propagation*: the slope of the fronts is bounded. This gives a $\mathscr{L}ip(L^1_{loc})$ bound.

Passage to the limit. Hence, with the help of these two bounds and of Helly's theorem, one obtains the *compactness* of the front-tracking sequence in L^1_{loc}.

Finally, one proves that a limit point of this sequence is indeed an entropy solution. To do so, given an entropy couple (η,q) with η convex, we have to estimate

$$\int_0^T \int_{\mathbb{R}} \varphi(t,x)(\eta(u^\nu)_t + q(u^\nu)_x), \tag{44}$$

for $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$ with $\varphi \geq 0$. We only need to see the contributions of the fronts in the integral (44). More precisely, by Stokes' theorem, one can transform this integral into:

$$\int_0^T \sum_{\alpha \text{ front at time } t} \varphi(t,\alpha(t))\Big\{\dot\alpha(t)\big[\eta(u^\nu(\alpha(t)^+)) - \eta(u^\nu(\alpha(t)^-))\big] \\ - \big[q(u^\nu(\alpha(t)^+)) - q(u^\nu(\alpha(t)^-))\big]\Big\}\,dt,$$

where $\alpha(t)$ denotes the position of the front α at time t, and $\dot\alpha(t)$ is speed. Then the analysis is as follows:

- Shock fronts give a negative contribution (this comes from the admissibility of shocks – they satisfy the entropy inequality),
- Rarefaction fronts are not exact entropy solutions. Hence each rarefaction front gives a contribution to (44). One can see that this contribution is of order $\mathscr{O}(\nu^2)$; this is due to the fact that they are of strength at most η, and travel to the correct velocity up to an error of size η. Since using the bound on the total variation, the total strength of rarefaction fronts is at most $\mathscr{O}(1)TV(u_0)$, the total contribution of rarefaction fronts in (44) is at most $\mathscr{O}(1)TV(u_0)\nu\ldots$

This completes our description of the existence theory by the front-tracking algorithm.

A remark. For the isentropic Euler system (in Eulerian or Lagrangian coordinates), existence theory of entropy solutions has been shown for much more general solutions [68, 69]:

Theorem 11 (Lions, Perthame, Souganidis, Tadmor). *Let* $(\rho_0, v_0) \in L^\infty(\mathbb{R})$, $\rho_0 \geq 0$. *Then for all* $\gamma > 1$, *there exists a global entropy solution of* (EI) *with initial data* (ρ_0, v_0).

4.3 The controllability problem

The problem. Let us now be more precise on the controllability problem that we consider. As explained above, we will not focus on finding the control on the boundary, but rather the solution itself; this allows to avoid the difficulties of the initial-boundary value problem. Next, in order to be able to use the front-tracking method, we will consider states with small total variation.

Hence the problem becomes the following: given u_0 an initial state (remember that the state is given by (31)) and u_1 a target, both supposed to have a small total variation, is it possible to find an entropy solution u, defined on $[0,T] \times [0,1]$ and driving u_0 to u_1, for *some* time $T > 0$? Note that one does not necessarily expect the controllability here to hold for any time $T > 0$. This is mainly a consequence of the finite speed of propagation of the equation.

But as we will see, the main problem here is the final state u_1. This is due to the fact that a nonlinear effect of genuinely nonlinear systems, known as the decay of positive waves (see [13]), probably prevents all u_1 to be reachable. On another side, describing exactly the set of u_1 that can be attained starting from u_0 seems out of reach for the moment. What we can prove is that, under *sufficient conditions*, u_1 can be reached (for some time T) starting from u_0.

Results. Precisely, here is what one can prove, see [47]. We begin with the Eulerian case.

Theorem 12 (G.). *There exists* $c > 0$ *depending on* γ *such that the following holds. Consider* $\overline{u}_0$ *and* $\overline{u}_1$ *two states in* $\mathbb{R}^{+*} \times \mathbb{R}$. *Set* $\overline{\lambda_1} := \lambda_1(\overline{u}_1)$ *and* $\overline{\lambda_2} := \lambda_2(\overline{u}_1)$. *There exist* $\varepsilon_1 = \varepsilon_1(\overline{u}_0) > 0$, $\varepsilon_2 = \varepsilon_2(\overline{u}_1) > 0$, *and* $T = T(\overline{u}_0, \overline{u}_1) > 0$, *such that, for any* $u_0, u_1 \in BV([0,1])$ *satisfying:*

$$\|u_0 - \overline{u}_0\| \leq \varepsilon_1 \text{ and } TV(u_0) \leq \varepsilon_1,$$
$$\|u_1 - \overline{u}_1\| \leq \varepsilon_2 \text{ and } TV(u_1) \leq \varepsilon_2,$$

and $\forall x, y \in [0,1]$ *such that* $x < y$,

$$\begin{cases} \dfrac{w^2(u_1(x))-w^2(u_1(y))}{x-y} \leq c\max\Big(\dfrac{\overline{\lambda_2}-\overline{\lambda_1}}{1-y}, \dfrac{\overline{\lambda_1}}{x}, \dfrac{-\overline{\lambda_1}}{1-y}\Big), \\ \dfrac{w^1(u_1(x))-w^1(u_1(y))}{x-y} \leq c\max\Big(\dfrac{\overline{\lambda_2}-\overline{\lambda_1}}{x}, \dfrac{-\overline{\lambda_2}}{1-y}, \dfrac{\overline{\lambda_2}}{x}\Big), \end{cases} \tag{45}$$

there is an entropy solution u of (EI) *in* $[0,T]\times[0,1]$ *such that*

$$u_{|t=0} = u_0 \ \text{ and } \ u_{|t=T} = u_1.$$

The statement concerning the Lagrangian system is the following.

Theorem 13 (G.). *Consider $\overline{u}_0$ and $\overline{u}_1$ two states in $\mathbb{R}^{+*}\times\mathbb{R}$. There exists $c = c(\gamma,\overline{u}_1) > 0$ such that the following holds. Set $\overline{\lambda_1} := \lambda_1(\overline{u}_1)$ and $\overline{\lambda_2} := \lambda_1(\overline{u}_2)$. There exist $\varepsilon_1 = \varepsilon_1(\overline{u}_0) > 0$, $\varepsilon_2 = \varepsilon_2(\overline{u}_1) > 0$, and $T = T(\overline{u}_0,\overline{u}_1) > 0$, such that, for any $u_0, u_1 \in BV([0,1])$ satisfying:*

$$\|u_0 - \overline{u}_0\| \leq \varepsilon_1 \text{ and } TV(u_0) \leq \varepsilon_1,$$
$$\|u_1 - \overline{u}_1\| \leq \varepsilon_2 \text{ and } TV(u_1) \leq \varepsilon_2,$$

and $\forall x,y \in [0,1]$ such that $x < y$,

$$\begin{cases} \dfrac{w^2(u_1(x))-w^2(u_1(y))}{x-y} \leq c\dfrac{\overline{\lambda_2}-\overline{\lambda_1}}{1-y}, \\ \dfrac{w^1(u_1(x))-w^1(u_1(y))}{x-y} \leq c\dfrac{\overline{\lambda_2}-\overline{\lambda_1}}{x}, \end{cases} \tag{46}$$

there is an entropy solution u of (P) *in* $[0,T]\times[0,1]$ *such that*

$$u_{|t=0} = u_0 \ \text{ and } \ u_{|t=T} = u_1.$$

In other words, for both systems, we consider u_0 and u_1 that have small total variation, more precisely which are close in the sense of the BV norm to two constant states $\overline{u}_0$ and $\overline{u}_1$. Provided that u_1 satisfy these "semi-Lipschitz" inequalities (45) or (46) (written in the coordinates given by the Riemann invariants), where the constant depends on $\overline{u}_1$ and can degenerate on the boundary, then one can drive u_0 to u_1.

The semi-Lipschitz inequalities. Let us comment a little bit these semi-Lipschitz inequalities that we require on the final state. These are close to *Oleinik's inequality*, which is valid for entropy solutions of uniformly convex scalar conservation laws. This inequality states that if $f:\mathbb{R}\to\mathbb{R}$ is such that $f'' \geq c > 0$, then the entropy solutions of

$$u_t + (f(u))_x = 0,$$

satisfy

$$\forall t > 0,\ \forall x < y,\ \ \frac{u(t,y)-u(t,x)}{y-x} \leq \frac{1}{ct}.$$

(See for instance [32]). The Oleinik inequality describes the spreading of rarefaction waves: shock waves yield a negative left hand side, while rarefaction waves given by formula (39) naturally spread and satisfy an inequality of this type.

Now, for what concerns the trajectories of systems (EI) or (P), the Oleinik-type conditions on the Riemann invariants are not satisfied in general. (See however Bressan-Colombo [15].)

In particular, it is not difficult to construct solutions of (EI) or (P) which violate this condition: the meeting of two shocks of the same family can generate a rarefaction wave in the other family, in contradiction with these inequalities, if the time T considered is very close after the interaction time, as in Figure 26. But as we

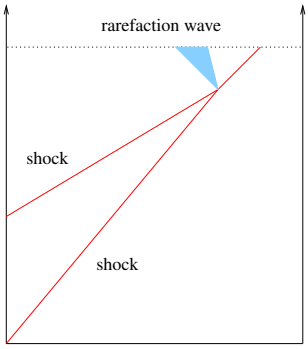

Fig. 26 A trajectory violating (45) or (46)

explained earlier, these are *sufficient conditions* for the final state to be reachable.

4.4 Some references

Before giving ideas of the proof, let us give several references concerning the control of systems of conservation laws.

Classical solutions. As we explained earlier, the theory for the control of systems of conservation laws highly depends on whether you consider classical solutions (let us say, of class C^1), or entropy solutions (with discontinuities). Concerning the former, a very important result is the following [63].

Theorem 14 (Li-Rao, 2002). *Consider*

$$\partial_t u + A(u)u_x = F(u),$$

such that $A(u)$ has n distinct real eigenvalues $\lambda_1(u) < \cdots < \lambda_k(u) \leq -c < 0$ and $0 \leq c < \lambda_{k+1}(u) < \cdots < \lambda_n(u)$ and $F(0) = 0$. Then there exists $\varepsilon > 0$ such that for all $\phi, \psi \in C^1([0,1])$ such that $\|\phi\|_{C^1} + \|\psi\|_{C^1} < \varepsilon$, there exists a solution $u \in C^1([0,T] \times [0,1])$ such that

$$u_{|t=0} = \phi, \text{ and } u_{|t=T} = \psi.$$

Note that in this context of classical solutions, Theorem 14 is an extremely general result. Not only this theorem considers general systems (not limited to $n = 2$ or to genuinely nonlinear fields), but it even considers the case where A is not the jacobian of some f (non-conservative systems) and a right-hand side (balance laws). Nothing so general is known in the context of entropy solutions. Note however that the condition of strict separation of the characteristic speeds from 0 is not required in Theorems 12 and 13.

Since the above result, **many** other developments and generalizations have appeared. For this, we refer in particular the recent book by Li Ta-Tsien [64] and references therein.

Entropy solutions. In this context, one does not expect to have a result with such a wide range as Theorem 14. In fact, new phenomena appear, proving that such a general result is not true in general. Let us list several results in the field.

- Ancona and Marson (1998) [7]: for the scalar equation $u_t + (f(u))_x = 0$ with $f'' \geq c > 0$, they give a complete description of the attainable set starting from 0.
- Horsin (1998) [53] has studied the controllability problem for the Burgers equation $u_t + (u^2/2)_x = 0$ with general $u_0 \in BV$ using Coron's return method.
- Bressan and Coclite (2002) [14]: for systems with genuinely nonlinear fields and satisfying $\lambda_1(\cdot) < \cdots < \lambda_k(\cdot) \leq -c < 0$ and $0 < c \leq \lambda_{k+1}(\cdot) < \cdots < \lambda_n(\cdot)$, for any constant state ω, one can find u such that

$$u(t,\cdot) \to \omega \text{ as } t \to +\infty.$$

- Ancona and Coclite (2002) [6]: Temple systems satisfying $\lambda_1(u) < \cdots < \lambda_k(u) \leq -c < 0$ and $0 < c \leq \lambda_{k+1}(u) < \cdots < \lambda_n(u)$, are controllable in L^∞ provided the final state satisfies the Oleinik-type condition.
- Bressan and Coclite (2002) [14]: for a class of systems containing Di Perna's system [35]:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t u + \partial_x \left(\frac{u^2}{2} + \frac{K^2}{\gamma - 1} \rho^{\gamma - 1} \right) = 0, \end{cases} \tag{47}$$

  there are initial conditions $\varphi \in BV([0,1])$ of arbitrary small total variation such that any entropy solution *u remaining of small total variation* satisfies:

$$\text{for any } t, u(t,\cdot) \text{ is not constant.}$$

This is particularly striking, when comparing to Theorem 14: in the C^1 framework, any small C^1 data can be driven to a constant in finite time.

- Ancona-Marson (2008) [8]: In this paper, they consider the asymptotic stabilization by controlling one side only.
- Perrollaz (2010) [73]: In this paper, the author considers the controllability of scalar conservation laws with an additional control on the left hand side:

$$u_t + f(u)_x = v(t),$$

and proves that this control can help in a very important way. This follows a study by Chapouly (2008) [17] in the C^1 framework.

4.5 Sketch of proof

The proof relies again on the **return method**: the idea is to connect u_0 and u_1 via a solution which which goes far away from u_0 and u_1. It is worth noticing that we will not use a *linearization* technique here; this is due to the low level of regularity.

The proof also uses a central difference between Euler system and DiPerna's one (47): for the Euler system, the interaction of two shocks of the same family generate **a rarefaction wave in the other family**. For DiPerna's system, it generates a shock. And this is central in Bressan and Coclite's negative result cited above.

The proof is split in three steps:

- Drive u_0 to a constant state,
- Drive the previous state to any constant state,
- Drive a constant state to u_1 or, in other words, find a solution from *some* constant state to u_1.

In the sequel, the argument is performed at the level of front-tracking approximations, which we almost consider as genuine solutions.

4.5.1 Driving u_0 to a constant state

A first idea. In the Eulerian case, an idea is the following: to make a (**very**) strong 2-shock enter the domain through the left side.

More precisely, one considers a state U_l such that the Riemann problem $(U_l, \overline{u}_0)$ is solved by a 2-shock. One computes easily that the set of $U_l = (\rho_l, m_l)$ that can be connected *from the left* to $\overline{u}_0$ by a 2-shock can be parameterized by ρ_l as follows:

$$[\overline{\rho_0}, +\infty) \ni \rho_l \mapsto (\rho_l, m_l) \text{ with } \frac{m_l}{\rho_l} = \frac{\overline{m_0}}{\overline{\rho_0}} + \sqrt{\kappa \frac{1}{\rho_l \overline{\rho_0}} \frac{\rho_l^\gamma - \overline{\rho_0}^\gamma}{\rho_l - \overline{\rho_0}}} (\rho_l - \overline{\rho_0}). \qquad (48)$$

The corresponding shock speed is given by:

$$s = \frac{\overline{m_0}}{\overline{\rho_0}} + \sqrt{\kappa \frac{\rho_l}{\overline{\rho_0}} \frac{\rho_l^\gamma - \overline{\rho_0}^\gamma}{\rho_l - \overline{\rho_0}}}, \tag{49}$$

and the 1-characteristic speed on the left (that is, at U_l) is:

$$\lambda_1(\rho_l, m_l) = \frac{\overline{m_0}}{\overline{\rho_0}} + \sqrt{\kappa \frac{\rho_l}{\overline{\rho_0}} \frac{\rho_l^\gamma - \overline{\rho_0}^\gamma}{\rho_l - \overline{\rho_0}}} \left(1 - \frac{\overline{\rho_0}}{\rho_l}\right) - \sqrt{\kappa\gamma} \rho_l^{\frac{\gamma-1}{2}}. \tag{50}$$

It follows that one can choose U_l so that:

$$s \geq 2 \text{ and } \lambda_1(U_l) \geq 2.$$

Now, one constructs a solution on the whole real line $\mathbb{R}$ with initial condition:

$$\begin{cases} U_l \text{ on } (-\infty, 0), \\ u_0 \text{ on } [0,1], \\ \overline{u_0} \text{ on } (1, +\infty). \end{cases} \tag{51}$$

Several authors (Alber [3], Schochet [77], Corli & Sablé-Tougeron [22], Chern [21], Lewicka-Trivisa [62], Bressan-Colombo [16],...) have studied the existence of *BV* solutions in the neighborhood of a strong shock, under Majda's stability condition [70]:

i. s is not an eigenvalue of $A(u^\pm)$,
ii. $\{r_j(u^+) \ / \ \lambda_j(u^+) > s\} \cup \{u^+ - u^-\} \cup \{r_j(u^-) \ / \ \lambda_j(u^-) < s\}$ is a basis of $\mathbb{R}^2$,

which is satisfied for any shock here. According to these studies, one can construct a global in time solution on $\mathbb{R}$ associated to the initial condition (51). As we will see, restricting this solution to $[0,T] \times [0,1]$ will give a solution steering u_0 to a constant state (in the Eulerian case).

Let us give more details about the way to construct a solution "near a strong shock". Schochet proved in this context that the Riemann problem is solvable in a neighborhood of the strong shock and gave interaction estimates on the interactions $\Gamma + \gamma \to \Gamma' + \gamma'$. That is, the interaction of the large shock with a small wave yields again a large shock (whose strength has been a little bit modified, but which stays strong) plus small waves. Moreover we have estimates on the strengths of the outgoing waves in terms of the incoming ones, replacing (43) which is valid for the interaction of small waves. Let us say the strong shock is of the family j and interacts with a small wave of the family k, then we have:

$$\sigma_i' = \mathscr{O}(1)|\sigma_k| \text{ for } i \neq j \text{ and } \sigma_j' = \sigma_j + \mathscr{O}(1)|\sigma_k|. \tag{52}$$

As opposed to Glimm's estimates (43), estimates (52) are linear with respect to the strength of incoming small waves; but this is compensated by the fact that there is only one strong shock in our solution. Using this tool, one can construct a global solution with the initial condition described above (because a standard wave crosses the strong shock at most once).

Now on the left of the 2-strong shock, all characteristic speeds (whether of the first or the second family) are positive and bounded away from 0, hence fronts leave the domain, see Figure 27.

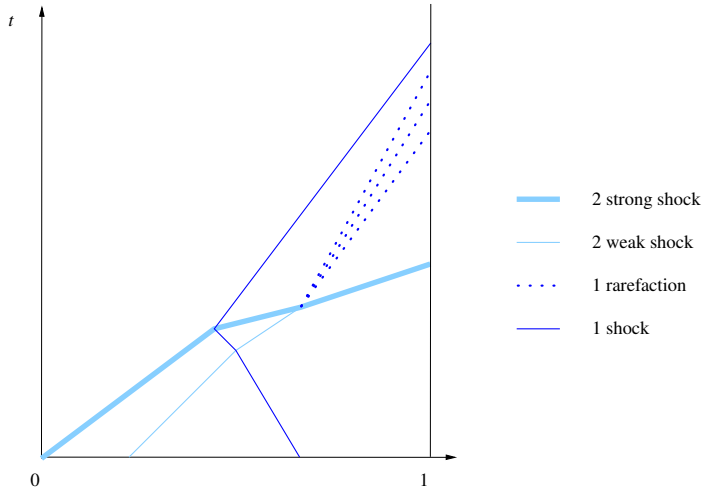

Fig. 27 The use of a (very) strong shock

Remark 7. In this context of the perturbation of a large shock, we call the shock with large amplitude a strong (or large) shock. By contrast, we call the other waves *weak*.

Drawbacks of the previous construction.

- A first problem is that even for a small perturbation of a constant, the solution constructed above is **huge**. One would like a control reasonably small when the perturbation is small.
- The previous strategy fails in the case of the p-system, for which λ_1 is always negative. One could have the impression that in a first time, the strong 2-shock "filters" the 2-waves from the initial datum, so that even in this case the above strategy allows to reach a constant state. But it should be noted that the interactions of 1-fronts do generate new 2-waves, see Figure 28...

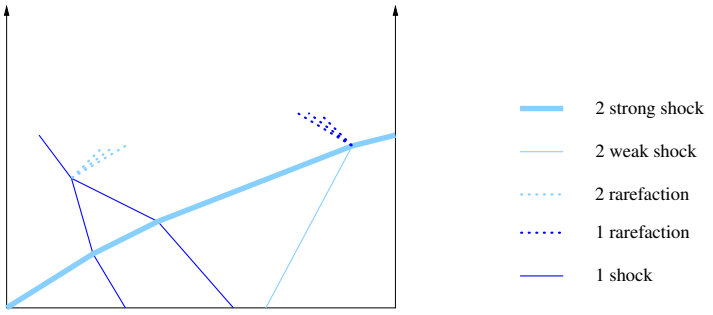

Fig. 28 A strong shock in the Lagrangian case

A better strategy. This leads us to invent another strategy. The starting point is the following. If above the 2-strong shock and within the first characteristic family, there were only 1-rarefaction waves, then the problem would be solved, because there would be no interaction above the strong shock. Let us explain why. The situa-

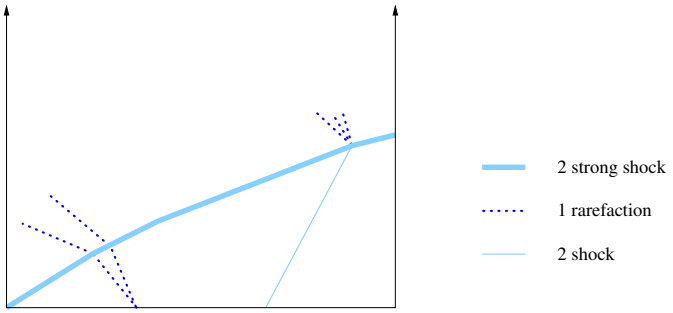

Fig. 29 Situation with only 1-rarefaction waves above the 2-strong shock

tion is described in Figure 29. Rarefaction fronts of the first family above the strong shock will not interact, since their behavior consists in going away one from another (this is due to genuine nonlinearity). But since the 2-waves have been absorbed by the strong 2-shock, there are no interaction at all above the 2-strong shock. Hence front travel without crossing above the 2-strong shock, and eventually leave the domain provided that the 2-strong shock has been chosen in a way that avoids null characteristic speeds on its left. This is made possible by the above formulas concerning the strong shock, for both the Eulerian and the Lagrangian system.

Consequently, one would like to understand how to prevent 1-shock waves to emerge from the 2-strong shock. There are two situations that can make a 1-shock enter the domain above the 2-strong shock:

- the meeting of the strong shock with a 1-shock,
- the meeting of the strong shock with a 2-rarefaction front,

see Figure 30.

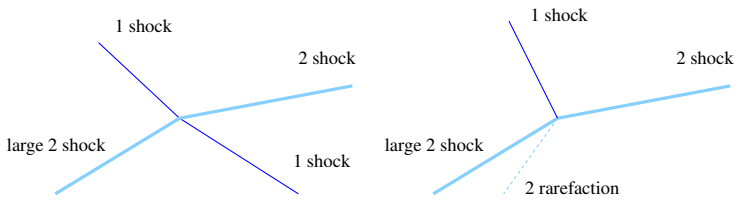

Fig. 30 The two situations generating a 1-shock above the strong shock

The main idea is the following. One can prove that it is possible to construct additional small 2-shocks that —provided that they arrive from the left at the right interaction time with the right intensity— kill the outgoing shock in the manner described in Figure 31. This is possible thanks to the fact that normally, the interaction of two shocks of the same family generate a **rarefaction** in the other family. Hence we use a **cancellation** effect. Indeed the interaction of the large shock with the incoming 1-shock or 2-rarefaction wave normally generates a 1-rarefaction wave, while the interaction of the large shock with the additional 2-shock normally generates a 1-shock. This is where we use the central difference with respect to DiPerna's model.

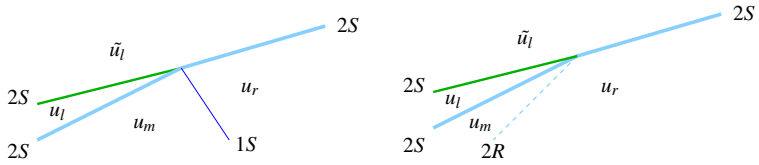

Fig. 31 Additional 2-shocks killing the emerging 1-shock wave

Together with this construction, we have an estimate on the size of these 2-shocks in terms of the incoming 1-shock or 2-rarefaction (as long as the strong shock is strong…). The proof of the existence of these two shocks and the corresponding estimates is obtained by the inverse mapping theorem and a precised version of (52).

An important problem remain: how to construct an approximation in which these 2-*shocks come at the right time and with the right strength?*

The construction. The idea in order to construct such an approximation is the following. First we construct the solution **under** the strong 2-shock, taking the additional 2-shocks described above in to account, as in Figure 32. In other words, we imagine that we have succeeded to send our additional 2-shocks exactly as we wish. Then taking this information into account (the additional 2-shocks influence the strong one), we can construct the front-tracking approximation under the strong 2-shock.

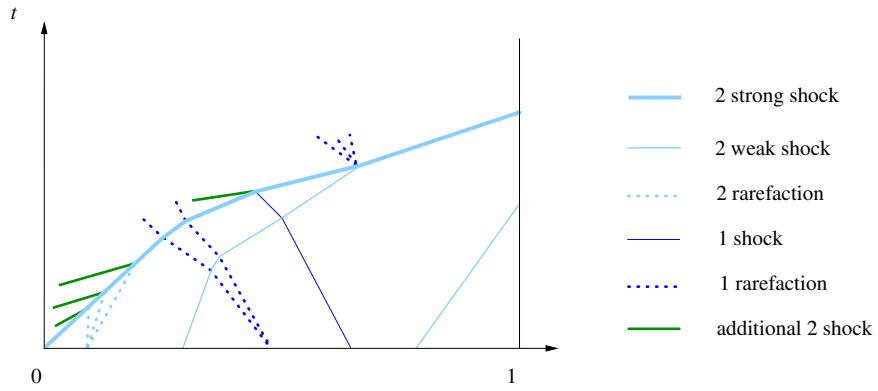

Fig. 32 First part of the construction

In a second time, we construct the approximations beyond the strong 2-shock. To that purpose, we have to extend:

- the 1-rarefaction waves **forward** in time,
- the 2-shocks **backward** in time.

We construct this approximation by using $1-x$ as the time variable. This is classical for what concerns C^1 solutions. But this raises more difficulties when it comes to entropy weak solutions. Indeed, the direction of time is very important in the selection of admissible discontinuities. Now, once we consider $1-x$ as the time (that is, after "rotating" the figure), we are led to an initial-boundary value problem, with a moving boundary (the strong 2-shock constructed in the previous step), see Figure 33. We have to describe how we complete the approximation.

The idea is to extend the front-tracking approximation by expanding the fronts emerging from the moving left boundary, that is, from the strong 2-shock; see Figure 33. These fronts are either 1-rarefaction fronts of 2-shocks. Then we have to solve the "interactions" that we meet in this situation. Let us see how we can "treat" these interactions. There are two possible types of interactions. Either the two incoming fronts are of the same family, or they are of opposite family.

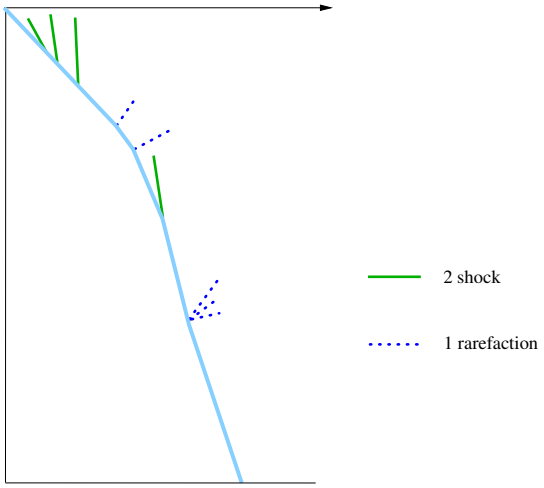

Fig. 33 Second part of the construction

But there are no interactions of fronts of the same family because:

- rarefaction fronts go forward in time, and hence do not meet because of the genuine nonlinearity (they spread),
- shocks go backward in time, and hence do not meet thanks to Lax's inequalities (in the usual direction of time they tend to run one into another).

And this is fortunate, because it is not clear how we could have solved these interactions in a way that would result in an entropy solution in the limit.

For what concerns interactions of front of opposite families, it is not difficult to see that this can be solved as in Lax's Theorem. In other words, one can extend the approximation above the interaction of a 2-shock going backward in time and a 1-rarefaction front go forward in time, as fronts of the same nature, see Figure 34. In the same way, we can obtained Glimm-type estimates (the strength of the waves is conserved across the interaction up to a quadratic error.)

Finally we get an approximation as described in Figure 35.

After this construction, it remains then to prove $L^\infty_t(BV_x)$ estimates by adapting Glimm's functionals to the situation, and then to use arguments comparable to the ones used for the Cauchy problem. The main idea for this is to consider functionals measuring the strength of the waves along curves which are not time slices. One can compare the total strength of the waves "under the strong shock" with their initial total strength, and then compare the total strength above the strong shock to the total strength under it. Using the above interaction estimates allows to bound the total variation of the solution uniformly in time.

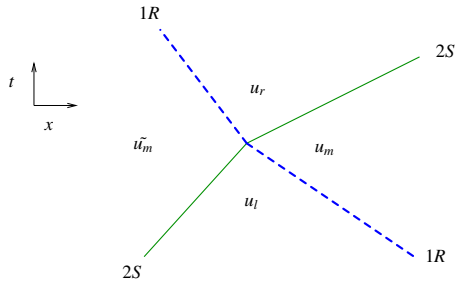

Fig. 34 "Side" interactions

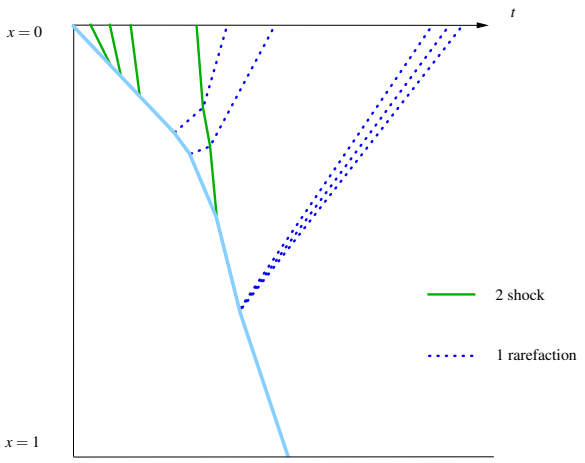

Fig. 35 The approximation

4.5.2 Travelling between constant states

This step is actually very elementary. There are three different zones in the case of (EI):

$$\mathscr{D}_1 := \{u\ ,\ 0 < \lambda_1 < \lambda_2\},$$
$$\mathscr{D}_2 := \{u\ ,\ \lambda_1 < 0 < \lambda_2\},$$
$$\mathscr{D}_3 := \{u\ ,\ \lambda_1 < \lambda_2 < 0\}.$$

In the case of equation (P), the situation is even simpler since there is only one zone, that is $\mathscr{D}_2$.

Now, inside each zone, one can move along (right or left) wave curves, using simple waves, such as described in Figure 36. This corresponds to simple waves (shock or rarefaction), that we make cross the domain one after another, from the

left or from the right, according to the sign of the speed of the wave. This works inside each $\mathscr{D}_i$, because inside each zone the zero characteristic speed is not met.

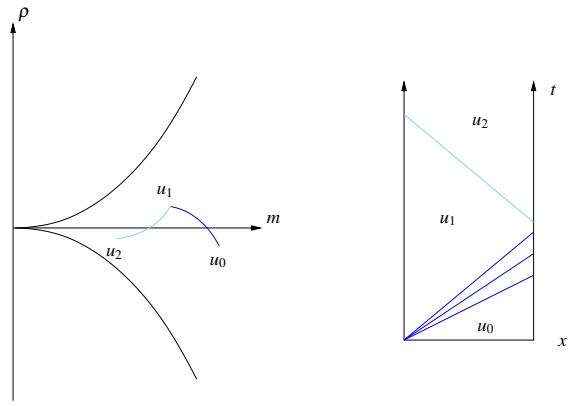

Fig. 36 Moving between constant states inside a zone

In the case of equation (EI), it remains to explain how to move from a zone to another. A way to do this is to use strong shocks as in Figure 37. Recall also formulas (48)-(50) in the first approach to treat the initial condition.

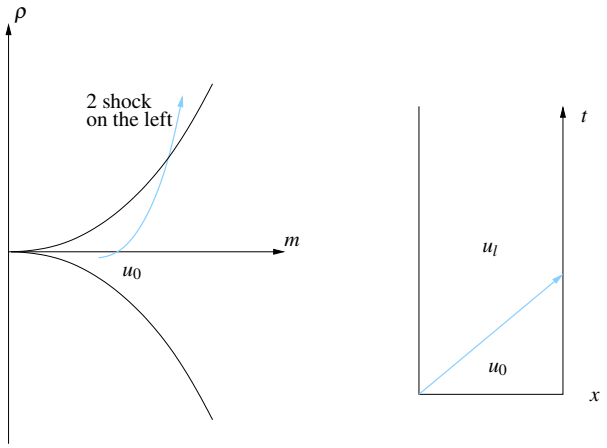

Fig. 37 Traveling between zones

4.5.3 Driving a constant state to u_1

Let us finally explain how one can reach u_1 from some constant state.

The construction consists in starting from u_1 and to build approximation of a solution by a *backward in time* front-tracking algorithm. For the usual front-tracking algorithm, the standard elementary brick is the Riemann problem. But the equivalent in the backward setting (in a way that will yield an entropy solution in the usual sense of time) is not clear.

A backward Riemann problem? The Riemann problem for negative times is ill-posed. There are two reasons for that. First, existence is not granted in general: typically, a rarefaction wave as in Figure 38 cannot be extended backward in a way that respects entropy criterions.

Fig. 38 A case with non existence for the backward Riemann problem

But even when one has existence, in general one does not have uniqueness. A simple example using wave curves is presented in Figures 39 and 40.

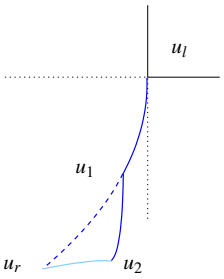

Fig. 39 Construction of two solutions of the backward Riemann problem

The difficulty with the well-posedness of the backward Riemann problem is the raison d'être of the semi-Lipschitz inequalities (45)-(46).

The construction. Let us now describe the construction. A first idea is again to construct a solution which includes strong shocks (backward in time), see Figure 41. The fact that we can use one or two strong shocks depends on the sign of the characteristic speeds in the case (EI) (and this explains the complex form of the semi-Lipschitz inequalities (45) in this case by the way).

Fig. 40 Two solutions of the backward Riemann problem

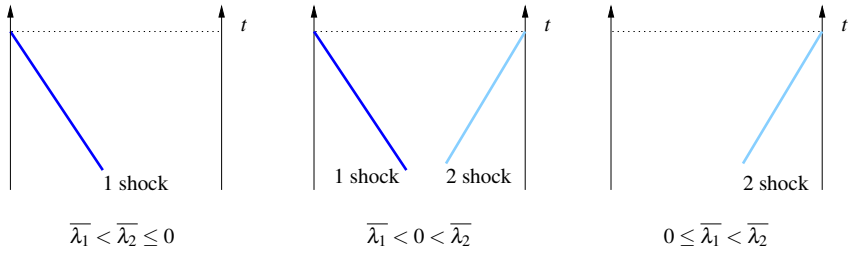

Fig. 41 Additional strong shocks

The presence of these strong shocks will help for both the question of non-existence for the backward Riemann problem (together of course with (45) and (46)) and for the following important issue. Another problem can indeed be posed by characteristic speeds which can be close to 0. Let us recall indeed that we have no assumption of separation of the characteristic speeds from 0 in Theorem 12. Of course, fronts having a velocity close to zero do not leave the domain; hence they make it impossible to reach a constant state. But we can manage to have non-zero characteristic speeds "under the strong shocks" constructed above, which excludes this difficulty.

Now, let us begin the construction of the backward front-tracking algorithm.

1. *Final state approximation.* Using the assumptions on u_1, we find *particular* piecewise constant approximations of u_1. These approximations are selected in order that at each discontinuity point, we can "approximately" solve the backward Riemann problem as in Figure 42 using:

- either "shock fans", that is a succession of small shocks focusing at the same point,
- either single rarefaction fronts with small amplitude, with this additional constraint that the distance between two successive rarefaction fronts of the same family is estimated from below,
- either the (strong) shocks from the boundary.

That this is actually possible is a consequence of (45) and (46). Let us describe this a little bit. A simple approximation of u_1 consists in constructing piecewise

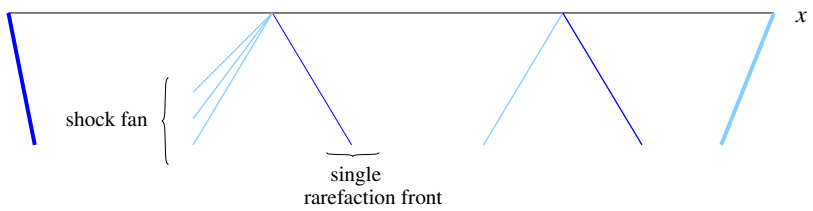

Fig. 42 Approximation of the final state

constant functions having only horizontal or vertical jumps in Riemann coordinates (that is, where w^2 or w^1 remains constant, respectively). But in fact, the negative jumps (for which w^i decreases) would not be yield a backward Riemann problem easily solvable in terms of simple waves (see Figure 22), because even in Riemann coordinates, shock curves are not straight lines.

Instead, one constructs an approximation of u_1 having the following features. There are two possible jumps:

- either a positive jump in w^i (with the other Riemann invariant w^j constant) of size at most ν (the approximation parameter), which gives naturally a rarefaction front, since rarefaction curves are straight lines in Riemann coordinates,
- either a negative jump in w^i, and in that case we approximate a horizontal/vertical segment by a succession of small shocks in the family $j \neq i$. Consequently in that case, w^j for $j \neq i$ is not constant across the discontinuity. However a succession of small shocks gives an accurate approximation of a horizontal/vertical negative jump in Riemann coordinates.

Using (45) and (46), we can moreover make sure that the successive positive jumps in w^i are distant one from another of at least $C\nu$.

2. *Extending the backward front-tracking approximations.* We extend the resulting fronts till two of these fronts meet at a *backward interaction point*. The backward interactions are treated as follows. Whether the two incoming fronts are of the same family or not is very important here.

Interactions inside a family. This depends on the nature of the fronts meeting.

- Shock/shock interactions: such interactions do not occur inside a characteristic family, as a consequence of Lax's inequalities.
- Rarefaction front/shock interactions: these do not occur either as a consequence of Lax's inequalities and estimates on the sizes of the rarefaction fronts, which stay small, hence with a speed close to the characteristic speed.
- Rarefaction/rarefaction interactions: these are likely to happen. These must be avoided, because if we allow many rarefaction fronts to merge, this will result in an non-entropic solution in the limit. As we will see later, the additional strong shock that we let enter the domain will be useful to kill the rarefaction fronts before this can happen.

Interactions of fronts of different families. There are two types of these interactions, depending on whether a strong shock is involved or not. We call weak waves the fronts that are not one of the strong shocks.

- Weak waves: if the two incoming fronts are weak, one can "solve" the interactions, just as in Lax's Theorem, see Figure 43.

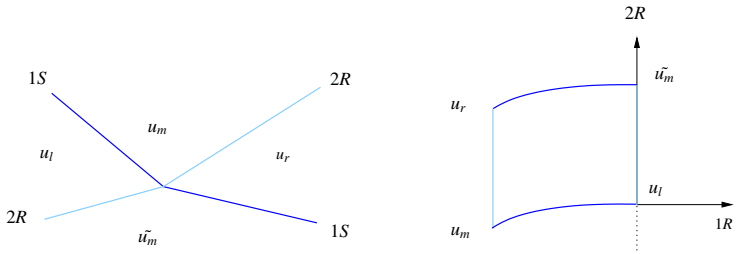

Fig. 43 A backward interaction between weak waves of opposite family

- Strong shock/weak shock interaction: again, one can extend the solution by a strong shock and a weak shock, satisfying Schochet's interaction estimates, see Figure 44.

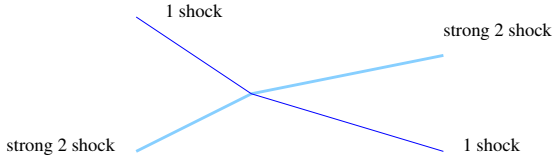

Fig. 44 A backward interaction between a strong shock and a weak shock of opposite family

- Strong shock/weak rarefaction interaction: we solve the backward interaction in terms of two incoming shocks of the same family (one strong, one weak), see Figure 45. In other terms, we use the opportunity of this meeting to kill the rarefaction fronts which are the main obstacle to get an entropy solution in the limit. Let us underline that we *choose* to do this, since there is no uniqueness in the backward Riemann problem.

Focusing of rarefaction waves. Now that we have given the main construction, we have to check that indeed this prevents rarefaction/rarefaction interactions to occur.

The main point is to estimate the distance between consecutive rarefaction fronts of the same family in order to prove that before possibly meeting, they must:

- either leave the domain,

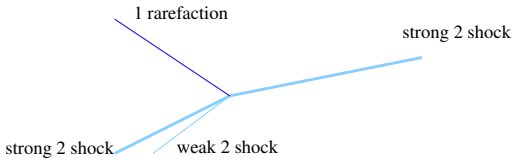

Fig. 45 A backward interaction between a strong shock and a weak rarefaction of opposite family

- either be killed by the meeting of a strong shock of the opposite family.

For that, one has to estimate the (backward-in time) focusing of rarefaction waves. This is done by using Glimm & Lax's estimates [51] on the spreading of rarefaction waves (forward-in-time).

Let us say a few words of this. We measure the distance between two successive 1-rarefaction fronts $\mathscr{C}_1$ and $\mathscr{C}_2$ "in the direction" of the second characteristic family, see Figure 46. Supposing that there has been no merging of rarefaction fronts yet, the strength of these rarefaction fronts are of order ν.

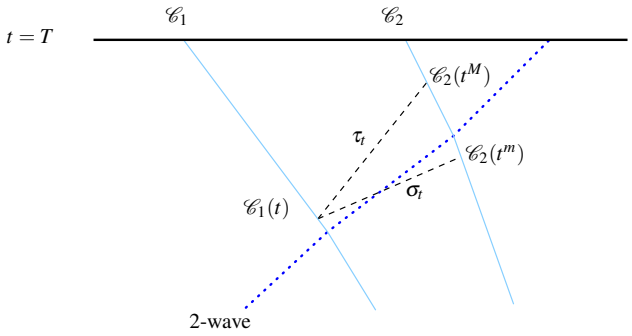

Fig. 46 Non crossing of characteristics

Now, roughly speaking: there are two "sources" in the difference of speed between $\mathscr{C}_1$ and $\mathscr{C}_2$:

- The strength of these fronts $\mathscr{C}_1$ and $\mathscr{C}_2$ themselves, of order $\mathscr{O}(\nu)$. More precisely if σ_1 and σ_2 are the strengths of the fronts (measured through w_1), this adds approximately $\frac{1}{2}\frac{\partial \lambda_1}{\partial w_1}[\sigma_1+\sigma_2]+\mathscr{O}(\nu^2)$ to $\dot{\mathscr{C}}_1-\dot{\mathscr{C}}_2$.
- The fronts of the second family that cross the two curves. When these fronts are between $\mathscr{C}_1$ and $\mathscr{C}_2$, they add an error between $\dot{\mathscr{C}}_1$ and $\dot{\mathscr{C}}_2$. The corresponding "additional deviation" is of order $\mathscr{O}(1)|\mathscr{C}_2(t)-\mathscr{C}_1(t)|$. Indeed, the fronts of the other family "do not stay long" between $\mathscr{C}_1$ and $\mathscr{C}_2$, see again Figure 46.

Using the construction of the approximations of u_1 and Oleinik-type semi-Lipschitz conditions on u_1, we can give a lower bound on $\mathscr{C}_1(T)-\mathscr{C}_2(T)$ in terms of ν. Then

one is able by using a Gronwall argument to estimate $\mathscr{C}_1(t) - \mathscr{C}_2(t)$ from below for $t \leq T$. If the constants in these semi-Lipschitz assumptions on u_1 are small enough, we are thus capable to affirm that the backward rarefaction fronts do not meet before either leaving the domain, or meeting a strong shock of the opposite family, in which case one kills rarefaction fronts.

Conclusion. Taking into account that backward rarefaction fronts do not merge, we get an approximation as described in Figure 47.

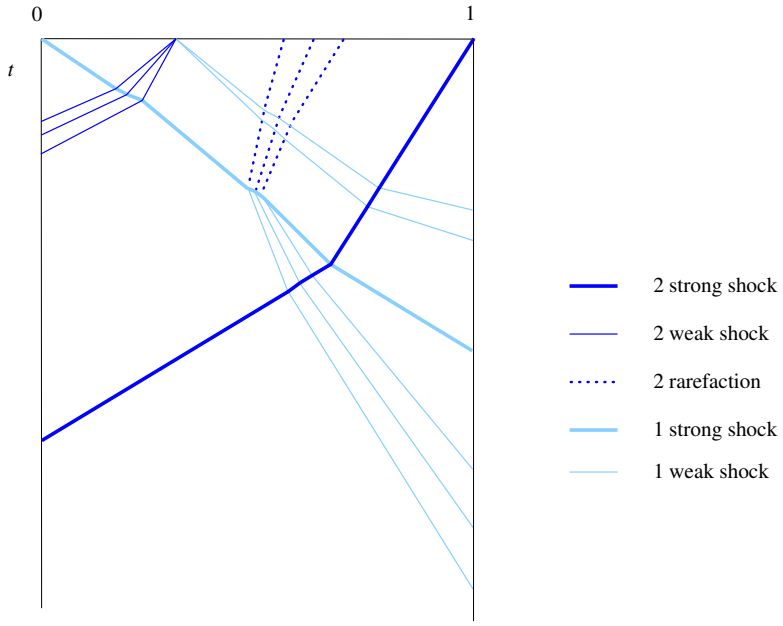

Fig. 47 The approximation of the backward problem

The rest of the proof consists in establishing estimates in $L^\infty_t(BV_x)$ for these approximations, as in the standard case. The main difference with the usual method is that one considers modified Glimm functionals V and Q to take into account the strong shocks and the construction, and that the estimates go backward in time (but are not of different nature).

Hence again, we can obtain the compactness of the sequence of approximations, and hence one can obtain a limit point of this sequence. It remains to prove that it is an entropy solution. The main point is that, since the rarefaction fronts never merge, they are always of order $\mathscr{O}(\nu)$. Hence we can be sure as for the usual front-tracking algorithm to obtain an entropy solution in the limit, and this ends the proof.

4.6 Comments

There is a **huge gap** between what is known in the framework of C^1 solutions and what is known for entropy solutions. For instance, the controllability of the *full compressible Euler equation*, with the equation of energy is a completely open problem.

Open problem 4 *What can be said on the controllability of the* 1*D full compressible Euler equation:*

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 2 + \rho\theta) = 0, \\ \partial_t(\frac{\rho u^2}{2} + \frac{\rho\theta}{\gamma-1}) + \partial_x(\frac{\rho u^3}{2} + \frac{\gamma\rho\theta u}{\gamma-1}) = 0, \end{cases}$$

where θ is the temperature, by means of boundary controls? More generally, can we widen the class of systems of conservation laws where one can prove the reachability of constant states?

As we underlined earlier, the situation is necessarily more complex in the context of entropy solutions than in the class of classical ones: in the case of C^1 solutions, both Euler and Di Perna's systems are locally controllable...

Acknowledgements. The author is very thankful to Prof. P. Cannarsa and J.-M. Coron, as well as to the CIME staff, for organizing this wonderful summer school in Cetraro.

References

1. Agrachev A. A., Sarychev A. V., Navier-Stokes equations: controllability by means of low modes forcing. J. Math. Fluid Mech. 7 (2005), no. 1, 108–152.
2. Agrachev A. A., Sarychev A. V., Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing. Comm. Math. Phys. 265 (2006), no. 3, 673–697.
3. Alber H. D., Local existence of weak solutions to the quasilinear wave equation for large initial values, Math. Z. 190 (1985), no. 2, 249–276.
4. D. Amadori, Initial-boundary value problems for nonlinear systems of conservation laws, NoDEA Nonlinear Differential Equations Appl. 4 (1997), no. 1, 1–42.
5. D. Amadori, R. M. Colombo, Continuous dependence for 2×2 conservation laws with boundary, J. Differential Equations 138 (1997), no. 2, 229–266.
6. Ancona F., Coclite G.M., On the attainable set for Temple class systems with boundary controls, SIAM J. Control Optim. 43 (2005), no. 6, 2166–2190.
7. Ancona F., Marson A., On the attainable set for scalar nonlinear conservation laws with boundary control, SIAM J. Control Optim. 36 (1998), no. 1, 290–312.
8. Ancona F., Marson A., Asymptotic stabilization of systems of conservation laws by controls acting at a single boundary point. *Control methods in PDE-dynamical systems*, Contemp. Math. 426 (2007), 1–43.
9. Ancona F., Marson A., Existence Theory by Front Tracking for General Nonlinear Hyperbolic Systems, Arch. Rational Mech . Anal. 185 (2007), 287–340.
10. Bertozzi A. L., Constantin P., Global regularity for vortex patches, Comm. Math. Phys. 152 (1993), no. 1, 19–28.

11. Bianchini S., Bressan A., Vanishing viscosity solutions of nonlinear hyperbolic systems, Ann. of Math. 161 (2005), no. 1, 223–342.
12. Bressan A., Global solutions of systems of conservation laws by wave front tracking, J. Math. Anal. Appl. 170 (1992), 414–432.
13. Bressan A., *Hyperbolic systems of conservation laws, the one-dimensional problem*, Oxford Lecture Series in Mathematics and its Applications 20, 2000.
14. Bressan A., Coclite G.M., On the boundary control of systems of conservation laws, SIAM J. Control Optim. 41 (2002), no. 2, 607–622.
15. Bressan A., Colombo R. M., Decay of positive waves in nonlinear systems of conservation laws, Ann. Sc. Norm. Sup. Pisa IV-26 (1998), 133-160.
16. Bressan A., Colombo R. M., Unique solutions of 2×2 conservation laws with large data. Indiana Univ. Math. J. 44 (1995), no. 3, 677–725.
17. Chapouly M., Global controllability of nonviscous and viscous Burgers-type equations. SIAM J. Control Optim. 48 (2009), no. 3, 1567–1599.
18. Chapouly M., On the global null controllability of a Navier-Stokes system with Navier slip boundary conditions. J. Differential Equations 247 (2009), no. 7, 2094–2123.
19. Chemin J.-Y., Persistance de structures géométriques dans les fluides incompressibles bidimensionnels, Ann. Sci. École Norm. Sup. 26 (1993), no. 4, 517–542.
20. Chemin J.-Y., *Fluides parfaits incompressibles*, Astérisque 230 (1995).
21. Chern I-L., Stability theorem and truncation error analysis for the Glimm scheme and for a front tracking method for flows with strong discontinuities, Comm. Pure Appl. Math. 42 (1989), 815-844.
22. Corli A., Sablé-Tougeron M., Perturbations of bounded variation of a strong shock wave. J. Differential Equations 138 (1997), no. 2, 195–228.
23. Coron J.-M., Global Asymptotic Stabilization for controllable systems without drift, Math. Control Signal Systems 5 (1992), 295–312.
24. Coron J.-M., Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), no. 3, 271–276.
25. Coron J.-M., On the controllability of 2-D incompressible perfect fluids, J. Math. Pures Appl. 75 (1996), 155–188.
26. Coron J.-M., Sur la stabilization des fluides parfaits incompressibles bidimensionnels. Séminaire : Équations aux Dérivées Partielles, 1998–1999, Exp. No. VII, École Polytechnique, Palaiseau, 1999.
27. Coron J.-M., On the null asymptotic stabilization of 2-D incompressible Euler equation in a simply connected domain, SIAM J. Control Optim. 37 (1999), no. 6, 1874–1896.
28. Coron J.-M., On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions, ESAIM Contrôle Optim. Calc. Var. 1 (1995/96), 35–75.
29. Coron J.-M., *Control and Nonlinearity*, Mathematical Surveys and Monographs 136, American Mathematical Society, Providence, RI, 2007.
30. Coron J.-M., Fursikov A. V., Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. Russian J. Math. Phys. 4 (1996), no. 4, 429–448.
31. Dafermos C. M., Polygonal approximations of solutions of the initial value problem for a conservation law. J. Math. Anal. Appl. 38 (1972), 33–41.
32. Dafermos C. M., *Hyperbolic conservation laws in continuum physics,* Grundlehren Math. Wissenschaften Series, Vol. 325, Springer Verlag, 2000.
33. Danchin R., Évolution d'une singularité de type cusp dans une poche de tourbillon, Rev. Mat. Iberoamericana 16 (2000), no. 2, 281–329.
34. Depauw N., Poche de tourbillon pour Euler 2D dans un ouvert à bord, J. Math. Pures Appl. 78 (1999), no. 3, 313–351.
35. DiPerna R. J., Global solutions to a class of nonlinear hyperbolic systems of equations. Comm. Pure Appl. Math. 26 (1973), 1–28.
36. DiPerna R. J., Global existence of solutions to nonlinear hyperbolic systems of conservation laws, J. Differential Equations 20 (1976), no. 1, 187–212.
37. Dubois F., LeFloch P.G., Boundary conditions for nonlinear hyperbolic systems of conservation laws. J. Differential Equations 71 (1988), no. 1, 93–122.

38. Dutrifoy A., On 3-D vortex patches in bounded domains, Comm. Partial Differential Equations 28 (2003), no. 7-8, 1237–1263.
39. Ebin D., Marsden J., Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math. 92 (1970), 102–163.
40. Fernàndez-Cara E., Guerrero S., Imanuvilov O. Yu., Puel J.-P., Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with $N-1$ scalar controls. SIAM J. Control Optim. 45 (2006), no. 1, 146–173.
41. Fursikov A. V.; Imanuvilov O. Yu., Exact local controllability of two-dimensional Navier-Stokes equations. (Russian) Mat. Sb. 187 (1996), no. 9, 103–138; translation in Sb. Math. 187 (1996), no. 9, 1355–1390
42. Gamblin P., Saint Raymond X., On three-dimensional vortex patches. Bull. Soc. Math. France 123 (1995), no. 3, 375–424.
43. Gardiner S. J., *Harmonic approximation*, London Mathematical Society Lecture Note Series, 221 (1995), Cambridge University Press, Cambridge.
44. Glass O., Exact boundary controllability of 3-D Euler equation, ESAIM Control Optim. Calc. Var. 5, 2000, 1–44.
45. Glass O., An addendum to a J.-M. Coron theorem concerning the controllability of the Euler system for 2D incompressible inviscid fluids, J. Math. Pures Appl. 80 (2001), no. 8, 845–877.
46. Glass O., Asymptotic stabilizability by stationary feedback of the two-dimensional Euler equation: the multiconnected case, SIAM J. Control Optim. 44 (2005), no. 3, 1105–1147.
47. Glass O., On the controllability of the 1-D isentropic Euler equation, J. European Math. Soc. 9, no. 3 (2007) 427–486.
48. Glass O., Horsin T., Approximate Lagrangian controllability for the 2-D Euler equation. Application to the control of the shape of vortex patches, J. Math. Pures Appl. 93 (2010), no. 1, 61–90.
49. Glass O., LeFloch P. G., Nonlinear Hyperbolic Systems: Nondegenerate Flux, Inner Speed Variation, and Graph Solutions, Arch. Rational Mech. Anal. 185 (2007), 409–480.
50. Glimm J., Solutions in the large for nonlinear hyperbolic systems of equations. Comm. Pure Appl. Math. 18 (1965), 697–715.
51. Glimm J., Lax, P. D., Decay of solutions of systems of nonlinear hyperbolic conservation laws. Memoirs of the A.M.S. 101 (1970).
52. Holden H., Risebro N. H., *Front tracking for hyperbolic conservation laws*, Applied Mathematical Sciences, 152, Springer-Verlag, New York, 2002.
53. Horsin T., On the controllability of the Burgers equation, ESAIM: Control Opt. Calc. Var. 3 (1998), 83-95.
54. Horsin T., Local exact Lagrangian controllability of the Burgers viscous equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), no. 2, 219–230.
55. Imanuvilov O. Yu., Remarks on exact controllability for the Navier-Stokes equations. ESAIM Control Optim. Calc. Var. 6 (2001), 39–72.
56. Kato T., Nonstationary flows of viscous and ideal fluids in $\mathbf{R}^3$, J. Functional Analysis 9 (1972), 296–305.
57. Kato T., On the smoothness of trajectories in incompressible perfect fluids, *Nonlinear wave equations (Providence, RI, 1998)*, Contemp. Math. 263 (2000), 109–130. Amer. Math. Soc., Providence, RI.
58. Kazhikov A. V., Note on the formulation of the problem of flow through a bounded region using equations of perfect fluid, PMM USSR, 44, 1981, 672–674.
59. S. G. Krantz, H. R. Parks, *A primer of real analytic functions*, Second edition, Basler Lehrbücher, Birkhäuser Verlag, Basel, 1992.
60. Lax P. D., Hyperbolic Systems of Conservation Laws II. Comm. Pure Appl. Math. 10 (1957), 537-566.
61. LeFloch P. G., *Hyperbolic systems of conservation laws: The theory of classical and nonclassical shock waves*, Lectures in Mathematics, ETH Zürich, Birkäuser, 2002.
62. Lewicka, M., Trivisa K., On the L^1 well posedness of systems of conservation laws near solutions containing two large shocks. J. Differential Equations 179 (2002), no. 1, 133–177.

63. Li T.-T., Rao B.-P., Exact boundary controllability for quasi-linear hyperbolic systems, SIAM J. Control Optim. 41 (2003), no. 6, 1748–1755.
64. Li T.-T., *Controllability and observability for quasilinear hyperbolic systems*. AIMS Series on Applied Mathematics, 3. American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.
65. Lichtenstein L., Über einige Existenzprobleme der Hydrodynamic, Math. Z. 28 (1928), no. 1, 387–415.
66. Lions J.-L., *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués*, Tomes 1 & 2. Masson, RMA 8 & 9, Paris 1988.
67. Lions J.-L., Are there connections between turbulence and controllability?, *Analysis and Optimization of Systems*, A. Bensoussan and J.-L. Lions, eds., Lecture Notes Control and Inform. Sci. 144, Springer-Verlag, Berlin, 1990.
68. Lions P.-L., Perthame B., Souganidis P. E., Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. Comm. Pure Appl. Math. 49 (1996), no. 6, 599–638.
69. Lions P.-L., Perthame B., Tadmor E., Existence and stability of entropy solutions to isentropic gas dynamics in Eulerian and Lagrangian variables. Comm. Math. Phys. 163 (1994), 415–431.
70. Majda A., *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New-York, 1984.
71. Morrey C. B., *Multiple integrals in the calculus of variations*, Die Grundlehren der mathematischen Wissenschaften, Band 130, Springer-Verlag, New York, 1966.
72. Nersisyan H., Controllability of the 3D compressible Euler system, preprint 2009.
73. Perrollaz V., Exact controllability of scalar conservation laws with an additional control in the context of entropy solutions, Preprint 2010.
74. Pommerenke Ch., *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften, 299. Springer-Verlag, Berlin, 1992.
75. Risebro, N. H., A front-tracking alternative to the random choice method. Proc. Amer. Math. Soc. 117 (1993), no. 4, 1125–1139.
76. Russell D. L., Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, SIAM Review 20 (1978), 639–739.
77. Schochet S., Sufficient conditions for local existence via Glimm's scheme for large BV data, J. Differential Equations 89 (1991), no. 2, 317–354.
78. Serfati P., Une preuve directe d'existence globale des vortex patches 2D, C. R. Acad. Sci. Paris Sér. I Math. 318 (1994), no. 6, 515–518.
79. Serre D., *Systèmes de lois de conservation. I. Hyperbolicité, entropies, ondes de choc,* & *II. Structures géométriques, oscillation et problèmes mixtes.* Fondations. Diderot Editeur, Paris, 1996.
80. Shirikyan A., Approximate controllability of three-dimensional Navier-Stokes equations. Comm. Math. Phys. 266 (2006), no. 1, 123–151.
81. Sueur F., Vorticity internal transition layers for the Navier-Stokes equations, preprint 2008, arXiv:0812.2145.
82. Whitney H., Analytic extensions of differentiable functions defined in closed sets. Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89.
83. Whitney H., The imbedding of manifolds in families of analytic manifolds. Ann. of Math. (2) 37 (1936), no. 4, 865–878.
84. Wolibner W., Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long, *Math. Z.* 37 (1933), 698–726.
85. Yudovich V. I., Non-stationary flows of an ideal incompressible fluid, Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 1032–1066 *(Russian). English translation in* USSR Comput. Math. & Math. Physics 3 (1963), 1407–1456.
86. Yudovich V. I., The flow of a perfect, incompressible liquid through a given region. Dokl. Akad. Nauk SSSR 146 (1962), 561–564 *(Russian). English translation in* Soviet Physics Dokl. 7 (1962), 789–791.
87. Zuazua E., Exact controllability for the semilinear wave equation. J. Math. Pures Appl. (9) 69 (1990), no. 1, 1–31.