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Number theory — Introduction

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If you toss a coin you certainly would suspect (even be sure) that in the long run the number of heads and tails will be the same. In same vain, if you pick numbers randomly in the long run, rather exactly, half the numbers you pick will be even and the other half odd. So there is a two-to-one chance that you pick an even number. Analogously, there will be three-to-one chance that your number will be divisible by 3. But what is the chance that you pick a prime number?

The prodigious youngster Gauss pondered over that question at the age about 15. He made long list of primes and counted the number of primes in intervals of the shape $[x, x+1000]$, and then observed that contained about $1/\log x$ primes. He thus answered your question: The answer depends on how big numbers you are allowed to pick, but if you confined your picks to be from an interval of shape $[x, x+y]$,the chance is about $y/\log x$. Of course the answer was conjectural, the computational capacity is limited even for Gauss.

It is common usage to let $\pi(x)$ denote the *prime counting function*; that is $\pi(x) = \#\{\, p \mid p \text{ a prime}, p \leq x \,\}$. It is step function that increase by one at each prime. For example do we have $\pi(7) = \pi(8) = \pi(9) = \pi(10) = 4$ but $\pi(11) = 5$. In our age of computers the values of $\pi(x)$ has been computed for very large x, for example is $\pi(10^{22}) = 201\,467\,286\,689\,315\,906\,290$. One of the main objectives of analytic number theory is to give approximations of $\pi(x)$ by functions easy to describe, like classical elementary functions; and of course one wants the approximations to be as good as possible.

The cumulative function of the prime distribution is thus $\pi(x)$, so if the distribution of primes goes like $1/\log x$, one suspects that

$$\pi(x) \sim \int_2^x \frac{dx}{\log x}. \tag{✗}$$

PNT

The integral on the right is often denoted by $\operatorname{li} x$ and is called the *logarithmic integral*, that is

$$\int_2^x \frac{dx}{\log x} = \operatorname{li} x.$$

The sign $\sim$ in ✗ means that the two sides are *asymptotical equal*: Two functions $f(x)$ and $g(x)$ are said to be *asymptotical equal*, symbolically written $f \sim g$, if $\lim_{x\to\infty} f(x)/g(x) = 1$. For example are $f(x) = \sqrt{x^2+a}$ and $g(x) = x$ asymptotical equal (to be precise, one should say "when $x \to \infty$"), and two polynomials are if they have the same dominating term. It does not mean that their difference tends to zero, for example it holds true that $x^3 + x^2 \sim x^3 + x$, the difference however tends to infinity (it is asymptotic to x^2!).

The statement in (✗) had for many years the status as Gauss' conjecture, but was finally proven by Hadamar and de la Vallée Poussin, and is now called the Prime Number Theorem, or PNT for short:

Theorem 1.1 (The Prime Number Theorem)

$$\pi(x) \sim \int_2^x \frac{dx}{\log x} = \operatorname{li}(x).$$

By performing a partial integration with $dv = dx$ and $u = 1/\log x$, one finds

$$\int_2^x \frac{dx}{\log x} = \frac{x}{\log x} + \int_2^x \frac{dx}{\log^2 x} + C$$

where C is the constant $-2/\log 2$. By splitting the interval of integration in the two intervals $[2, \sqrt{x}]$ and $[\sqrt{x}, x]$ (see the figure below) one arrives at the estimate

$$\left| \int_2^x \frac{dx}{\log^2 x} \right| < \frac{4x}{\log^2 x} + \frac{\sqrt{x}}{\log^2 2}$$

which shows that $\operatorname{li} x \sim x/\log x$. Hence, the relation $\sim$ being transitive, one has the following alternative formulation of the prime number theorem, which may be more speaking as the the function $x/\log x$ is easier accessible than $\operatorname{li} x$, (but it is less precise):

Theorem 1.2 (PNT, second version)

$$\pi(x) \sim \frac{x}{\log x}$$

Our principal goal of the course is to give a proof of the PNT, but as well to get some general understanding of the mathematical lore around the theorem.

PROBLEM 1.1. Show that for any n one has

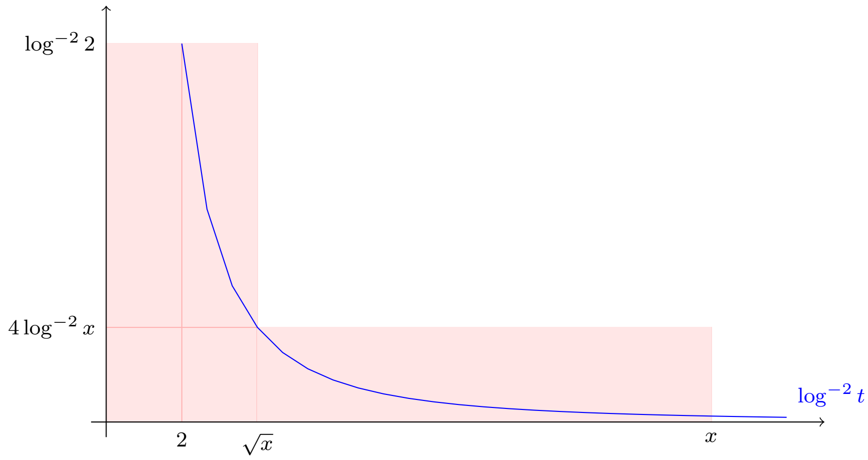
$$\operatorname{li} x = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{x}{\log^2 x} + \cdots + \frac{(n-1)!x}{\log^n x} + (n-1)! \int_2^x \frac{dt}{\log^{n+1} t} + C.$$

Give an explicit expression for the constant C. ✷

PROBLEM 1.2. Show that

$$\left| \int_2^x \frac{dt}{\log^n t} \right| \leq \frac{2^n x}{\log^n x} + \frac{\sqrt{x}}{\log^n 2} + C$$

✷



Estimating the logarithmic integral $\int_2^t \log^{-2} t \, dt$

BIG O AND SMALL o We use the opportunity to introduce some more notation constantly used in analytic number theory. If $f(x)$ and $g(x)$ are two functions, we say that f is '*'big O"* of g, in writing $f(x) = O(g(x))$ if there is a constant C such that $|f(x)| \leq C\,|g(x)|$. We say that f is *"small o"* of g as $x \to a$ of g—in writing $f(x) = o(g(x))$—if $f(x)/g(x) \to 0$ as $x \to a$.

The Riemann ζ-function

Taking the bull by the horns, we introduce immediately the may be most renown function in the whole of mathematics, the Riemann ζ-function. Encoded in its analytic properties lie many of the secrets of the prime numbers. The ζ-function was first studied by Euler in 1740, and some call it the Riemann-Euler function. Euler proved the property that is the bridge between the Riemann ζ and the primes, the so called Euler product. However most the fundamental properties was established by Riemann in his paper xxx where he also states the famous hypothesis about the zeros.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is a complex variable. It is the custom in this branch of mathematics to let $\sigma = \operatorname{Re} s$ an $\tau = \operatorname{Im} s$, so that $s = \sigma + \tau i$.

Proposition 1.1 *The series defining the ζ-function converges absolutely in the right half plane where $\sigma = \operatorname{Re} s > 1$. The convergence is uniform in the half planes $\sigma > \sigma_0 > 1$ and $\zeta(s)$ is an analytic function of s for $\sigma > 0$.*

PROOF: Indeed, by the elementary theory of real infinite series one learns in school (*e.g.,* the integral criterion) one knows that $\sum_{n\geq 1} n^{-\sigma}$ converges for any real $\sigma > 1$, and one has $|n^{-s}| = n^{-\sigma}$.

As $n^{-\sigma}$ is a decreasing function of σ, it follows that $\sum_{n=M}^{N} n^{\sigma} < \sum_{n=N}^{M} n^{-\sigma_0}$ whenever $\sigma > \sigma_0 > 1$, and this shows that the convergence is uniform in half planes $\sigma > \sigma_0 > 1$. The ζ-function therefore is an analytic function. In the same vain, it follows that $|\zeta(s)| < |\zeta(\sigma_0)|$ when $\operatorname{Re} s > \sigma_0$. ❑

The ζ-function may be extended to a meromorphic function in the whole complex plain with simple pole and residue 1 at $s = 1$ as the sole singularity. An argument using partial integration extends it to the right half plane $\sigma > 0$. Below we shall give this extension which also as a nice introduction to the commonly used technic of partial integration. The version we need is for Riemann-Stieltjes integrals, since step-functions are involved, and is more subtle than the usual calculus version. But first we introduce the Euler product:

The Euler product

As the name indicates, the Euler product was discovered by Euler. It connects the Riemann ζ-function with the prime numbers, and is in that respect really the hug of the whole theory. One of Euler's applications was to show there are infinitely many primes, which might be shooting sparrows with canons, but as we shall later on shall see, the idée has nice consequences.

EulerProduct

Proposition 1.2 (Euler product) *One has for such s that $\operatorname{Re} s = \sigma > 1$ the following equality*

$$\zeta(s) = \sum_n n^{-s} = \prod_p (1 - p^{-s})^{-1}.$$

PROOF: Summing the geometric series, one finds $(1 - p^{-s})^{-1} = \sum_{k\geq 0} p^{-ks}$, hence

$$\prod_{p\leq x} (1 - p^{-s})^{-1} = \prod_{p\leq x} \sum_{k\geq 0} p^{-ks} = \sum_{p|n \Rightarrow p\leq x} n^{-s}$$

where we in the second equality uses Mertens theorem saying that the product of finitely many absolutely convergent series equals their Cauchy-product. The Cauchy product is just the series whose terms are products of one term from each of the series involved. Clearly one has

$$\left|\sum_{n} n^{-s} - \sum_{p|n \Rightarrow p \leq x} n^{-s}\right| \leq \sum_{n \geq x} n^{-\sigma},$$

and the right side of the inequality can be made as small as we please the series $\sum_n n^{-\sigma}$ being absolute convergent. ❑

PROBLEM 1.3. Use the Euler product to show there are infinitely many primes. HINT: The harmonic series $\sum_n n^{-1}$ diverges. ✷

From the proof one obtains the following estimate that will be useful later on:

Useful

Lemma 1.1

$$\prod_{p \leq x}(1 - p^{-1}) < 1/\log x.$$

PROOF: Putting $s = 1$ we arrive at

$$\prod_{p \leq x}(1 - p^{-1})^{-1} = \sum_{p|n \Rightarrow p \leq x} n^{-1} \geq \sum_{n \leq x} n^{-1} \geq \log x.$$

❑

THE PRIMES ARE OF DENSITY ZERO As a teaser, we give the following result, certainly very weak compared to the PNT. Anyhow, it tells us that the density of the primes is zero; that is, the relative portion of primes less than x tends to zero when x grows.

Proposition 1.3

$$\lim_{x \to \infty} \pi(x)/x = 0.$$

PROOF: Let q be a natural number to be chosen later. We divide $[0, x]$ into intervals of length q; that is, intervals of shape $[(a-1)q, aq]$ where the a's are the natural numbers with $a \leq [x/q]$. In each of those subintervals there are $\phi(q)$ integers relatively prime to q, hence at most $\phi(q)$ primes not dividing q. If k is the number of different prime factors in q, this gives

$$\pi(x) \leq \phi(q)x/q + k,$$

and hence

$$\pi(x)/x \leq \phi(q)/q + k/x.$$

Now, if $\epsilon > 0$ is given, let y be such that $1/\log y < \epsilon$, then letting $q = \prod_{p \le y} p$, we have after lemma 1.1

$$\phi(q)/q = \prod_{p|q}(1 - p^{-1}) \le 1/\log y < \epsilon$$

and hence

$$\pi(x)/x < \epsilon + \epsilon$$

for $x > k\epsilon^{-1}$. ❏

The behavior at $s = 1$

The Riemann ζ-function has a simple pole at $s = 1$. For the moment the ζ-function is only defined for $\sigma > 1$, so this is not a meaningful statement at this stage. The following proposition is as close as we can come:

Proposition 1.4 *For $s > 1$ real one has*

$$\zeta(s) = \frac{1}{s-1} + g(s),$$

where $g(s)$ is a bounded function.

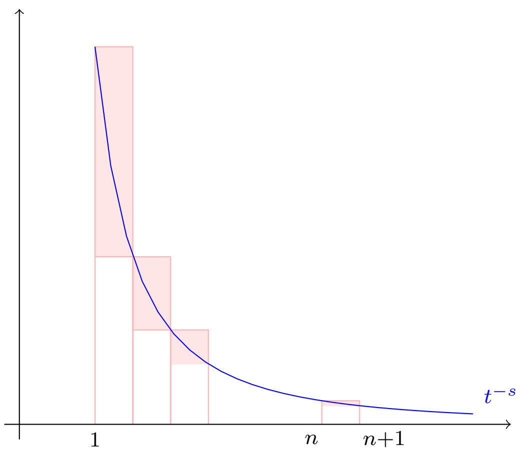
PROOF: The most elementary way to see this is to compare the series $\sum_{1 \le n \le x} n^{-s}$ with the integral $\int_1^x t^{-s}\, dt$. The usual Riemann sum approximation (see the figure below) gives:

$$\left| \sum_{1 \le n \le x} n^{-s} - \int_1^{x+1} t^{-s}\, dt \right| < 1.$$

Hence letting x tend towards ∞ and using that $\int_1^\infty t^{-s}\, dt = 1/(s-1)$, one arrives at

$$|\zeta(s) - 1/(s-1)| < 1.$$

❏



The Riemann sum for t^{-s} with $n = [x]$.

Extension to $s > 0$

There are several simple ways to extend $\zeta(s)$ to the right half plan $s > 0$ which go back to Riemann; we present two of then in this paragtaph. They use the partial integration formula for Riemann-Stieltjes integrals that reads

$$\int_a^b f(x)\,d\alpha(x) = f(x)\alpha(x)\big|_a^b - \int_a^b f'(x)\alpha(x)\,dx,$$

where α is a function of bounded variation. Typical for us α will be a step function, like any cumulative function $\alpha(x) = \sum_{n\leq x} a_n$.

THE FIRST WAY We take $\alpha(x) = [x]$. Recall that $x = [x] + \{x\}$ where $\{x\}$ denotes the *fractional part* of x. It lies in the interval $[0, 1)$. By partial integration we find

$$\begin{aligned}\zeta(s) &= \int_1^\infty x^{-s}\,d[x] = \int_1^\infty x^{-s}\,dx - \int_1^\infty x^{-s}\,d\{x\} = \\ &= \frac{1}{s-1} - \big(x^{-s}\{x\}\big|_1^\infty + s\int_1^\infty x^{-s-1}\{x\}\,dx\big) \\ &= \frac{1}{s-1} - s\int_1^\infty x^{-s-1}\{x\}\,dx.\end{aligned}$$

The partial integration appears at the second equality, and we use that $x^{-s} \to 0$ as $x \to \infty$ as long as $s > 0$, and that $\{1\} = 0$. The point is now that the integral

$$\int_1^\infty x^{-s-1}\{x\}\,dx$$

converges absolutely for $\sigma = \operatorname{Re} s > 0$ and uniformly for $\sigma \geq \sigma_0 > 0$ since the sawtooth function $\{x\}$ is bounded. From general theory it follows that the integral is an analytic function of the variable s. Hence we have

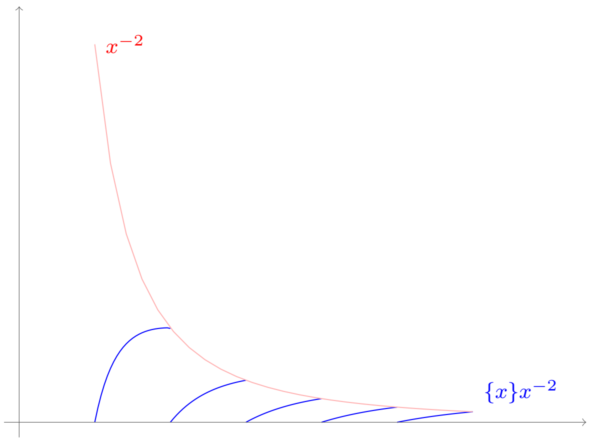
Proposition 1.5 *For* $\operatorname{Re} s = \sigma > 0$ *the integral*

$$\int_1^\infty x^{-s-1}\{x\}\,dx$$

converges absolutely and uniformly in the half planes $\sigma \geq \sigma_0 > 0$ *and therefore is an analytic function of* s*. Hence*

$$\frac{1}{s-1} - s\int_1^\infty x^{-s-1}\{x\}\,dx$$

is meromorphic with a simple pole and residue 1 *at* $s = 1$*. It coincides with* $\zeta(s)$ *when* $\sigma > 1$ *and hence is a meromorphic continuation of* $\zeta(s)$ *to the right half plane* $\sigma > 0$*.*



The function $x^{-s-1}\{x\}$ for $s = 2$ over the interval $[1, 6]$

THE SECOND WAY The trick is to subtract twice the "even part" of $\zeta(s)$ from $\zeta(s)$. On one hand we get:

$$\sum_n n^{-s} - 2\sum_{n \text{ even}} n^{-s} = \sum_{n \text{ odd}} n^{-s} - \sum_{n \text{ even}} n^{-s} = \sum_n (-1)^{n-1} n^{-s}$$

on the other

$$2\sum_{n \text{ even}} n^{-s} = 2\sum_n (2n)^{-s} = 2^{-s+1}\zeta(s)$$

Hence

$$(1 - 2^{-(s-1)})\zeta(s) = \sum_n (-1)^{n-1} n^{-s} = \Phi(s) \tag{1.1}$$

Zeta

What we have gained with this manipulation, is that the convergence of the series on the right is easier to handle. For real s it is an alternating series, and it is elementary that it converges for $s > 0$. We shall soon see that it converges in the right half plane $\sigma > 0$. The convergence is uniform in the half planes $\sigma \geq \sigma_0 > 0$, so the series defines an analytic function $\Phi(s)$. However, the convergence is *not* absolute in the vertical strip $0 < \sigma < 1$.

Before proceeding with extending the ζ-function, we prefer to give a general convergence result for a class of Dirichlet series in which our $\Phi(s)$ falls, namely the class of Dirichlet series whose coefficients have a bounded cumulative function. Later on in the course several other important Dirichlet series will belong to this class. The cumulative function $\alpha(x)$ for the coefficients of $\Phi(s)$ is certainly bounded; it alternates between the values 0 and 1. One has

Proposition 1.6 *Let $\{a_n\}$ be a sequence as above i.e., whose cumulative function $\alpha(x) = \sum_{n \leq x} a_n$ is bounded. Then the Dirichlet series*

$$L(\alpha, s) = \sum_n a_n n^{-s}$$

converges for $\sigma > 0$. The convergence is uniform in the half planes $\sigma > \sigma_0 > 0$, and hence the function $L(\alpha, s)$ is analytic for $\sigma > 0$.

PROOF: Pick a constant A such that $\left|\sum_{n\leq x} a_n\right| < A$. The proof is an easy and nice use of partial integration:

$$\sum_{N\leq n\leq M} a_n n^{-s} = \int_N^M t^{-s}\, d\alpha(t) = t^{-s}\alpha(t)\Big|_N^M + s\int_N^M t^{-s-1}\alpha(t)\, dt \tag{1.2}$$

$$= M^{-s}\alpha(M) - N^{-s}\alpha(N) + s\int_N^M t^{-s-1}\alpha(t)\, dt \tag{1.3}$$

Using that $\alpha(x)$ is a bounded function, we bound the integral

$$\left|\int_N^M t^{-s-1}\alpha(t)\, dt\right| \leq \int_N^M \left|t^{-s-1}\right| |\alpha(t)|\, dt \leq A\int_N^M t^{-\sigma-1}\, dt \leq AM^{-\sigma}.$$

All together we get

$$\left|\sum_{N\leq n\leq M} a_n n^{-s}\right| \leq 2M^{-\sigma}A + M^{-\sigma}A = 3AM^{-\sigma}$$

which tends to zero when M tends to infinity as long as $\sigma > 0$. This shows that the Dirichlet series converges in the right half plane where $\sigma > 0$, and in any domain where σ is bound away from zero, *i.e.*, where $\sigma > \sigma_0$, the convergence is uniform. ❑

We continue with the analytic continuation of the ζ-function. By the proposition we just established, the function $\Phi(s)$ is analytic for $\sigma > 0$. The function $1 - 2^{-(s-1)}$ is analytic everywhere and has $s = 1$ as is sole and simple zero. We have established the equality, valid for $\sigma > 1$,

$$\zeta(s) = (1 - 2^{-(s-1)})^{-1}\Phi(s)$$

where the right side is defined and analytic for $\sigma > 0$ except for a simple pole at $s = 1$.

PROBLEM 1.4. A bonus of the second way, is that it shows that $\zeta(s)$ has no real zeros between 0 and 1. Show that. HINT: $\Phi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} + \dots$ is positive ✷

The von Mangoldt function $\Lambda(n)$ and the logarithmic derivative

One of the arithmetic function that appears naturally in analytic number theory is the so called *von Mangoldt's function*. It was first used by the german mathematician Hans Carl Friedrich von Mangoldt who lived from 1854 to 1925. He was a student of Kummer and Weierstrass in Berlin, and became a professor at the polytechnic school in Aachen. His function is defined in the following way:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\nu \text{ is a power of a prime } p \\ 0 & \text{otherwise} \end{cases}$$

PROBLEM 1.5. Show that $\sum_{d|n} \Lambda(d) = \log n$ HINT: First show that $\Lambda(p^\nu) = \nu \log p$ for primes p. ✷

The von Mangoldt function appears in the logarithmic derivative of $\zeta(s)$; one has

Proposition 1.7

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_n \Lambda(n) n^{-s}$$

PROOF: Taking the logarithm of both sides in the Euler product, we find

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s})$$

Taking derivatives gives us

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \log p \frac{p^{-s}}{1 - p^{-s}} = \sum_p \log p \sum_{n \geq 1} p^{-ns} = \sum_n \Lambda(n) n^{-s}$$

❏

This indicates that there is a coupling between the zeros of ζ — that is, the poles of $\zeta'(s)/\zeta(s)$ — and von Mangoldt's function.

Chebychev's two cumulative functions

Pafnuty Lvovich Chebyshev was born in the little Russian town Okatov south eats of Moskow in 1821. He was a person with manifold interest, among other things he made a series of mechanical inventions useful for steam engines that was high-tech at that time, a mechanism called "Chebychev's linkage" transferring rotations into linear movements is still in use, but he is best known as a mathematician. He was a professor in St Petersburg and gave great contributions to analytic number theory in particular around the Prime Number Theorem. He died in 1894, two years before the Prime Number Theorem was proved by Hadamar and de la Vallée-Poussin.

Chebychev introduced two functions, the ψ-function and the θ-function. They are cumulative functions defined as

$$\psi(x) = \sum_{n \leq x} \Lambda(n) \qquad \theta(x) = \sum_{p \leq x} \log p$$

PROBLEM 1.6. Determine $\psi(8)$, $\psi(9)$ and $\psi(10)$. ✷

PROBLEM 1.7. Show that $\psi(n) = \log \operatorname{lcm}(\{\, d \mid d \leq n \,\})$. ✷

The introduction of the second Chebychev function $\psi(x)$ can be motivated as follows. The prime counting function $\pi(x)$ can be expresses as $\pi(x) = \sum_{p\leq x} 1$, so each prime contributes a unit to the sum. If we in stead give the prime p the weight $\log p$, we arrive at the function $\psi(x)$. According to the Prime Number Theorem, the primes should appear with the density $1/\log x$, so giving primes a logarithmic weight should compensate for their logarithmically declining probability. This to say that it is reasonable to belive that $\psi(x) \sim x$ is equivalent to the Prime Number Theorem. And indeed it is, as stated in the next proposition:

Proposition 1.8 *The following three statements are equivalent:* PsiAndThetaAndP

- $\psi(x) \sim x$
- $\theta(x) \sim x$
- $\pi(x) \sim x/\log x$

The proof of this will done two stages. First we establish the equaivlanecs of the two first statement, this is basically lemma 1.2 below. The equivalence between the last and the two first statements hinges on the so called Chebychev's estimates which we shall prove in the next paragraph.

The two Chebychev functions ψ and θ are intimately related. The following relation between them holds true:

$$\psi(x) = \sum_{p^\nu\leq x} \log p = \sum_{p\leq x^{1/\nu}} \log p = \sum_{\nu} \theta(x^{1/\nu}) =$$
$$= \theta(x) + \theta(\sqrt{x}) + \theta(\sqrt[3]{x}) + \dots$$

The first sum is understood to be over all natural numbers ν and all primes p satisfying $p^\nu \leq x$.

The last sum appears to be infinite, but is in fact finite since $\theta(x^{1/\nu}) = 0$ once $x^{1/\nu} < 2$, that is $\nu > \log x/\log 2 = \log_2 x$. As $\sqrt{x}$ is small compared to x when x is big— already for x around a million, $\sqrt{x}$ is just one per mille of $\sqrt{x}$—the terms $\theta(x^{1/\nu})$ are small compared to $\theta(x)$. This suggests that $\psi(x)$ and $\theta(x)$ have same behavior for large x; and indeed, this is true as expressed in the following lemma. This also proves the equivalence between the two first statements in proposition 1.8 since $\sqrt{x}\log^2 x = o(x)$:

PsiOgTheta

Lemma 1.2

$$\psi(x) = \theta(x) + O(\sqrt{x}\log^2 x)$$

PROOF: The proof consists of a few very crude estimates. The sum for $\theta(y)$ has at most y terms each being less than $\log y$, so $\theta(y) \leq y \log y$. This gives

$$0 < \psi(x) - \theta(x) = \sum_{2 \leq \nu \leq \log_2 x} \theta(x^{1/\nu}) < \sum_{2 \leq \nu \leq \log_2 x} x^{1/\nu} \log x^{1/\nu} <$$
$$< \log_2 x \sqrt{x} \log x = \sqrt{x} (\log x)^2 / \log 2,$$

❏

PROOF OF PROPOSITION 1.8: We already remarked that the two first statements are equivalent by lemma 1.2. To prove the remaining equivalence we appeal again to partial integration:

$$\theta(x) = \sum_{p \leq x} \log p = \int_2^x \log t \, d\pi(t) = \pi(x) \log x - \int_2^x \pi(t)/t \, dt.$$

Assuming the Chebychev's inequalities $c'x/\log x < \pi(x) < cx/\log x$ for appropriate constants c of c', valid for $x > A$, we obtain

$$c' \int_A^x 1/\log t \, dt < \int_A^x \pi(t)/t \, dt < c \int_A^x 1/\log t \, dt,$$

and since $\int_2^x 1/\log t \, dt = O(x/\log x)$, we arrive at the equality

$$\theta(x)/x = \pi(x) \log x/x + O(1/\log x),$$

and we are done. ❏

Chebychev's bounds

Around 1850 Chebyshev gave his upper and lower bounds for the prime counting function. These estimates were some kind of forerunners for the Prime Number Theorem, but certainly much weaker, and they are possible to prove with entirely elementary methods. Chebyshev result was

ChebychevBounds **Proposition 1.9 (Chebychev's bounds)** *There are constants c and c' and an A such that for $x \geq A$ one has*

$$c' \frac{x}{\log x} < \pi(x) < c \frac{x}{\log x}.$$

The depth and the difficulty of this statement depend on the constants. Chebychev originals were $c' = 0.92$ and $c = 1.055$. The PNT is equivalent to having Chebychev's estimates with $c' = 1 - \epsilon$ and $c = 1 + \epsilon$ for any $\epsilon > 0$, but admittedly, x must be larger and larger the smaller ϵ is. Given the depth of the Prime Number Theorem, is clear that the closer the c's are to the optimal (but unattainable) value one, the more difficult the proposition becomes. We shall prove it for $c' = 1/2$ and $c = 2$. The main idea of the proof is to use the divisibility properties of some binomial coefficients. For $m+1 < p \leq 2m+1$ the prime p divides the binomial coefficient $\binom{2m+1}{m}$ which is equal to $(2m+1)!/m!(m+1)!$. Now $\binom{2m+1}{m}$ appears twice in the binomial development $(1+1)^{2m+1}$ and therefore satisfies

$$\binom{2m+1}{m} < 2^{2m} = 4^m.$$

This gives

$$\prod_{m+1<p\leq 2m+1} p < \binom{2m+1}{m} < 4^m. \tag{1.4}$$

In a similar fashion one finds ChebFundUlikhet

$$\prod\nolimits_{m<p\leq 2m} p < \binom{2m}{n} < 4^m. \tag{1.5}$$

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PROBLEM 1.8. Show that $\prod_{p\leq x} p < 4^x$. HINT: Induction. ✷

Proposition 1.10 *For x large (in fact for $x \geq 65$) the following inequalities hold* ChebychevBound2

$$\frac{x}{2\log x} < \pi(x) < 2\frac{x}{\log x}.$$

PROOF: The prime counting function is insensitive to x being replaced by the integral value $[x]$, and the function on the right side is increasing. The function on the left is also increasing, but increases by an amount less than one when the argument is increased by one (Its derivative is $2^{-1}(1/\log x - 1/\log^2 x)$). Hence we may safely assume that $x = n$ is an integer.

We start by the leftmost inequality. Let $w_p(k)$ denote the number of times p appears as a factor in the binomial coefficient $\binom{n}{k}$. Then clearly

$$p^{w_p(k)} < n.$$

Hence $\binom{n}{k} = \prod_{p\leq n} p^{w_p(k)} < n^{\pi(n)}$, which further gives

$$2^n = \sum\nolimits_{0\leq k\leq n} \binom{n}{k} < (n+1)n^{\pi(n)}.$$

Taking logarithms we arrive at

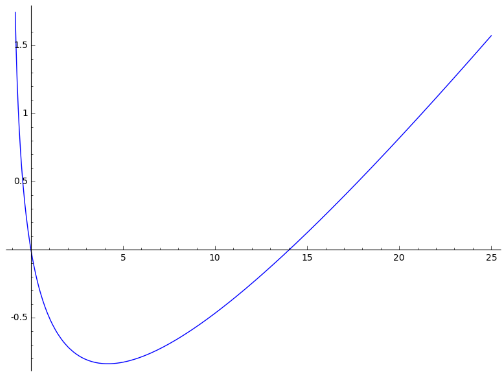
$$n\log 2 < \pi(n)\log n + \log(n+1)$$

or

$$\log 2\frac{n}{\log n} - \frac{\log(n+1)}{\log n} < \pi(n),$$

and we are done since $n\log 2 - \log(n+1) > \frac{1}{2}n$ for $n \geq 15$.

PROBLEM 1.9. Prove in detail that $n \log 2 - \log(n+1) > \frac{1}{2}n$ for $n \geq 15$. ✱



The graph of $x \log 2 - \log(x+1) - \frac{1}{2}x$

As to the rightmost inequality, we have from (1.5)

$$n^{\pi(2n)-\pi(n)} < 2^{2n},$$

and taking logarithms this leads to

$$\pi(2n) - \pi(n) < 2n \log 2/\log n.$$

The proof proceed by induction on n. If n is even, we obtain

$$\pi(2n) < 2n \log 2/\log n + \pi(n) < 2n \log 2/\log n + 2n/\log n = (\log 2 + 1)2n/\log n,$$

and we are saved by the inequality

$$(\log 2 + 1)2n/\log n < 4n/\log 2n, \tag{1.6}$$

LogUlikhet valid for ≥ 46.

PROBLEM 1.10. Prove the inequality (1.6). HINT: $\log x/\log 2x$ is increasing ✱

Now, if $x = 2n+1$ is an odd number, we find

$$\pi(2n+1) \leq 1 + \pi(2n) \leq 1 + (\log 2 + 1)2n/\log n.$$

To get ashore we need the inequality

$$1 + (\log 2 + 1)2n/\log n < 2(2n+1)/\log(2n+1),$$

which is equivalent with

$$\log 2 + 1 < \frac{\log n}{\log(2n+1)}\frac{2n+1}{n} - \frac{\log n}{2n}.$$

The right side approaches 2 as $n \to \infty$ (by l'Hôpital's rule) so for $n >> 0$ the inequality holds. A closer look shows that it in fact holds for $n \geq 65$. (The function on the right side is increasing, and checking the inequality for $n = 65$ is just a matter of computation). ❑

Korrektur lest hit