Chapter 8
Fluid Mechanics

8.1 Introduction

In continuum mechanics it is natural to define a fluid on the basis of what seems to be the most characteristic macromechanical aspects of liquids and gases as opposed to solid materials. Therefore we chose the following definition in Sect. 1.3:

A fluid is a material that deforms continuously when subjected to anisotropic states of stress.

An anisotropic state of stress in a particle implies surfaces through the particle subjected to shear stress. A fluid will therefore deform continuously when subjected to shear stresses. The fluid may be at rest without further deformation only when the state of stress is isotropic. This implies that the constitutive equations of any fluid at rest relative to any reference must reduce to:

$$\mathbf{T} = -p\,\mathbf{1}, \quad p = p(\rho, \theta) \tag{8.1.1}$$

p is the *thermodynamic pressure*, which is a function of the *density* ρ and the *temperature* θ. The relationship for p in (8.1.1) is called an *equation of state*.

An *ideal gas* is defined by the equation of state:

$$p = R\rho\,\theta \tag{8.1.2}$$

R is the *gas constant* for the gas, and θ is the absolute temperature, given in degrees Kelvin. The model ideal gas may be used with good results for many real gases, for example air.

Due to the large displacements and the chaotic motions of the fluid particles it is in general impossible to follow the motion of the individual particles. Therefore the physical properties of the particles or quantities related to particles are observed or described at fixed positions in space. In other words we employ *spatial description* and *Euler coordinates*. The primary kinematic quantity in Fluid Mechanics is the velocity vector $\mathbf{v}(\mathbf{r}, t)$.

The concept of *streamlines* is introduced to illustrate fluid flow. The streamlines are *vector lines to the velocity field*, i.e. lines that have the velocity vector as a tangent in every point in the space of the fluid. The stream line pattern of a *non-steady flow*: $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$, will in general change with time, see Problem 8.1. In a *steady flow*: $\mathbf{v} = \mathbf{v}(\mathbf{r})$, the streamlines coincide with the particle trajectories, called the *pathlines*. For a given velocity field $\mathbf{v}(\mathbf{r}, t)$ the streamlines are determined from the differential equations:

$$d\mathbf{r} \times \mathbf{v}(\mathbf{r}, t) = \mathbf{0} \quad \Leftrightarrow \quad \frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} \text{ at constant time } t \tag{8.1.3}$$

The pathlines are determined by the differential equation:

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}, t) \tag{8.1.4}$$

The streamlines through a closed curve in space form a *streamtube*.

The vector lines of the *vorticity field*:

$$\mathbf{c} = \operatorname{rot} \mathbf{v} \equiv \operatorname{curl} \mathbf{v} \equiv \nabla \times \mathbf{v} \tag{8.1.5}$$

are called *vortex lines*. The vortex lines through a closed curve in space form a *vortex tube*. If the velocity field is *irrotational*, which is the same as *vorticity free*, i.e. $\mathbf{c} = \mathbf{0}$, the velocity field may be developed from a velocity potential, see Theorem C.10:

$$\mathbf{v} = \nabla\phi, \quad \phi = \phi(\mathbf{r}, t) \tag{8.1.6}$$

This kind of flow is called *potential flow* and will be discussed in Sect. 8.5. The scalar field ϕ is called the *velocity potential*.

The fundamental field equations of fluid mechanics are the Cauchy equations in the form:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = \frac{1}{\rho} \operatorname{div} \mathbf{T} + \mathbf{b}, \quad \partial_t v_i + v_k v_{i,k} = \frac{1}{\rho} \mathrm{T}_{ik,k} + \mathrm{b}_i \tag{8.1.7}$$

the continuity equation, to be presented in Sect. 8.2.3, constitutive equations that relate the stress tensor to the velocity field, and finally an energy equation. The energy equation of a linearly viscous fluid is discussed in Sect. 8.4.4. The model *perfect fluid*, also called the *Eulerian fluid*, and which does not transfer shear stresses even when the fluid is deforming, is treated in Sect. 8.3. The most important fluid model: the *linearly viscous fluid*, also called the *Newtonian fluid*, is presented in Sect. 8.4. Non-Newtonian fluids are presented in Sect. 8.6.

The governing equations of Fluid Mechanics do not provide unique solutions, in contrast to the equations of the classical theory of elasticity. It is often necessary to check whether an obtained solution is stable. For instance, in pipe flow steady state conditions give steady state flow provided the velocities are small enough. Increasing the level of velocities the flow may change into a chaotic non-steady flow. Osborne Reynolds [1842–1912] performed in 1883 an experiment illustrating

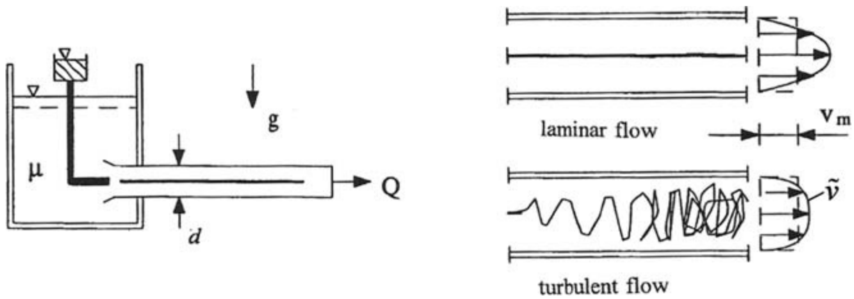

Fig. 8.1.1 The Reynolds experiment. Laminar and turbulent pipe flow

this phenomenon, see Fig. 8.1.1. A fluid of density ρ and viscosity μ flows through a pipe under steady state conditions. The quality of the flow is shown by injecting a thin colored fluid into the main flow. At low velocities the colored fluid is seen as a nearly straight line parallel to the axis of the pipe. This type of flow is called *laminar flow*: the fluid flows in cylindrical layers which move relative to each other. If the velocities are increased the colored fluid line becomes unstable and eventually disintegrates into a complex flow which gives the fluid in the pipe a general colored look. This type of flow is characterized as *turbulent flow*. The particle velocity at a specific place will vary strongly with time, but the time averaged velocity, or *time mean velocity* $\tilde{v}$, at the place over a certain time interval is constant. When a flow has become turbulent, it is customary to express the flow through the time mean velocity $\tilde{v}$. The equipments that record velocities may measure automatically the time average velocity at a place. Figure 8.1.1 shows the velocity distributions in the pipe for the two types of flow. Reynolds found in the experiment that the transition from laminar flow to turbulent flow is primarily dependent upon four factors: the *volumetric flow* Q, i.e. the fluid volume that per unit time flows through a cross-section of the pipe, the diameter of the pipe d, and the viscosity μ and the density ρ of the fluid. The result of the Reynolds' experiment may then be expressed thus: The *Reynolds number Re* defined by:

$$\mathrm{Re} = \rho \frac{v \cdot d}{\mu}, \; v \equiv v_m = \frac{4Q}{\pi d^2} \tag{8.1.8}$$

must be less than approx. 2000 for the flow to be laminar. In the expression for the Reynolds number v is the mean velocity over the cross-section of the pipe, $v = v_{\mathrm{m}} = Q/A$, where A is the cross-sectional area. For $Re > 2000$ the flow becomes turbulent. A Reynolds number may be defined for most flows, and we shall return to more general definition of the Reynolds number in Sect. 8.4 on linearly viscous fluids.

Figure 8.1.2 shows a rigid body in a *uniform flow*. Far away upstream from the body the velocity field is constant, independent of place and time. This situation occurs when a body moves with a constant velocity through a fluid at rest and when the reference for the motion is chosen fixed in the body. Apart from a thin region

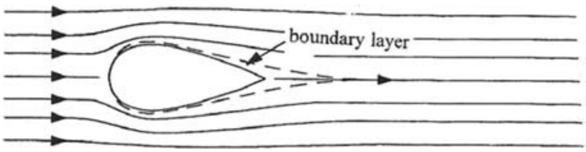

Fig. 8.1.2 Rigid Body in uniform flow. Free irrotational flow and boundary layer

near the surface of the body, we may neglect the viscosity of the fluid and assume irrotational flow. In the thin region near the surface of the body the viscosity has to be taken into account. This region is called the *boundary layer*.

Downstream of a rigid body in a flowing fluid a *wake* is created in which the flow is very chaotic and therefore is characterized as turbulent. The wake is due to the flow in the boundary layer. The fluid in the wake is rotational with a high content of vorticity. Bodies that create a narrow wake are often called *streamlined bodies* because the streamlines in a steady flow form a stable pattern surrounding the body, see Fig. 8.1.3. A rigid body creating a broad wake are called *blunt body*, see Fig. 8.1.3.

8.2 Control Volume. Reynolds' Transport Theorem

The fundamental laws of thermomechanics are: the principle of conservation of mass, the first and second axiom of Euler, the equation of mechanical energy balance, and the 1. law of thermomechanics. These laws are first expressed through equations for material bodies in motion, and the equations will be presented below in the form they are used when spacial or Eulerian description is chosen. This description is the natural one in Fluid Mechanics.

A fluid body of volume $V(t)$ and surface area $A(t)$ contains by definition the same mass m at any time t. The mass per unit volume at the place $\mathbf{r}$ and the present time t is represented by the *mass density*, or *density* for short, $\rho(\mathbf{r},t)$.

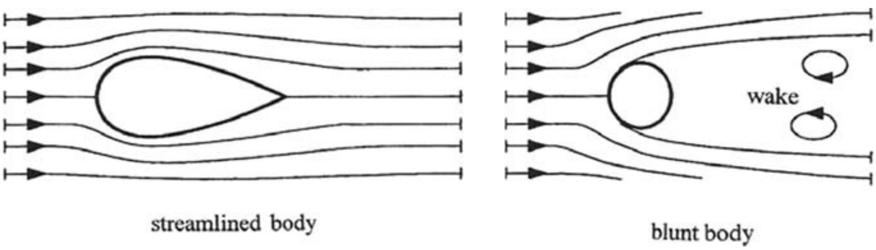

Fig. 8.1.3 Streamlined body with a negligible wake and blunt body with a broad wake

$$m = \int_{V(t)} \rho(\mathbf{r},t)\, dV = \text{constant} \tag{8.2.1}$$

Equation (8.2.1) expresses the *principle of conservation of mass*.

For a body of continuous matter we have in the Sect. 3.2.1, 6.1, and 6.3.2 defined a series of time-dependent *extensive quantities*, of which the most important are:

$$\mathbf{p}(t) = \int_{V(t)} \mathbf{v}\rho\, dV, \text{ linear momentum} \tag{8.2.2}$$

$$\mathbf{l}_O(t) = \int_{V(t)} \mathbf{r}\times\mathbf{v}\rho\, dV, \text{ angular momentum about a point } O \tag{8.2.3}$$

$$K(t) = \int_{V(t)} \frac{1}{2}v^2\rho\, dV, \text{ kinetic energy} \tag{8.2.4}$$

$$E(t) = \int_{V(t)} \varepsilon\,\rho\, dV, \text{ internal energy} \tag{8.2.5}$$

All these extensive quantities are expressed in the general form:

$$B(t) = \int_{V(t)} \beta\,\rho\, dV, \text{ extensive quantity} \tag{8.2.6}$$

$\beta = \beta(\mathbf{r},t)$ is a *specific intensive quantity*, representing the quantity per unit mass. The fundamental laws of thermomechanics contain material time derivatives of both intensive and extensive quantities. In Sect. 3.1.3 the following expression for the material derivative of an extensive quantity was developed. For the extensive quantity $B(t)$ we have:

$$\dot{B}(t) = \frac{d}{dt}\int_{V(t)} \beta\,\rho\, dV = \int_{V(t)} \dot{\beta}\,\rho\, dV \tag{8.2.7}$$

In Fluid Mechanics it is often more convenient to transform the equations of the fundamental laws to apply to a region fixed in space, or a region moving in a prescribed fashion. Such a region is called a *control volume* and is assumed to coincide with a fluid body with the volume $V(t)$ and the surface area $A(t)$ at the present time t, see Fig. 8.2.1. We shall first assume that the control volume V is fixed relative to the reference *Rf* chosen to describe the motion of the fluid. The surface A of the control volume V is called a *control surface*. To obtain the transformations of the equations of the laws of thermomechanics, which basically is meant to apply to a material body, such that they apply for a control volume, we shall derive an alternative expression for the material derivative of an extensive quantity B, which is related to the body.

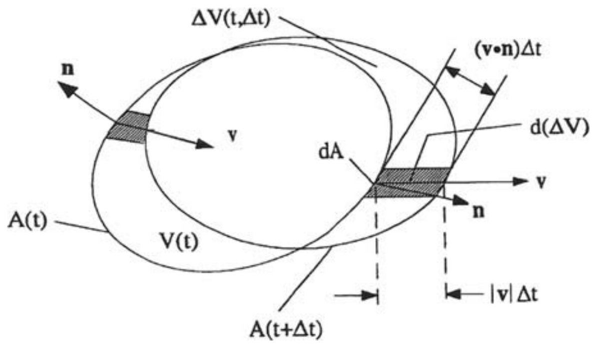

Fig. 8.2.1 Control volume V and control surface A

First let:

$$B(t) = \int_{V(t)} b\,dV,\; b = b(\mathbf{r},t) = \beta(\mathbf{r},t)\,\rho(\mathbf{r},t) \tag{8.2.8}$$

$b = \beta\rho$ is called *the density of the quantity* and expresses the quantity per unit volume. We may write:

$$\dot{B} = \lim_{\Delta t \to \infty} \frac{B(t+\Delta t) - B(t)}{\Delta t} \tag{8.2.9}$$

in which, see Fig. 8.2.1:

$$B(t+\Delta t) = \int_{V(t+\Delta t)} b(\mathbf{r},t+\Delta t)\,dV = \int_{V(t)} b(\mathbf{r},t+\Delta t)\,dV + \int_{\Delta V(t,\Delta t)} b(\mathbf{r},t+\Delta t)\,d(\Delta V)$$

$$\Delta V(t,\Delta t) = V(t+\Delta t) - V(t),\; d(\Delta V) = dA \cdot [(\mathbf{v}\cdot\mathbf{n})\,\Delta t] \tag{8.2.10}$$

The unit vector **n** is a normal to the control surface A directed out from the surface of the control volume. The third volume integral in (8.2.10) is transformed to a surface integral:

$$\int_{\Delta V(t,\Delta t)} b(\mathbf{r},t+\Delta t)\,d(\Delta V) = \int_{A(t)} b(\mathbf{r},t+\Delta t)\cdot[(\mathbf{v}\cdot\mathbf{n})\,\Delta t]\,dA \tag{8.2.11}$$

The contribution to the surface integral in (8.2.11) is positive whenever $\mathbf{v}\cdot\mathbf{n} > 0$ and mass is flowing out of the control volume V through the control surface A, and the contribution is negative whenever $\mathbf{v}\cdot\mathbf{n} < 0$ and mass is flowing into the control volume through the control surface A. We may now write:

$$\frac{B(t+\Delta t) - B(t)}{\Delta t} = \int_{V(t)} \frac{b(\mathbf{r},t+\Delta t) - b(\mathbf{r},t)}{\Delta t}\,dV + \int_{A(t)} b(\mathbf{r},t+\Delta t)\,(\mathbf{v}\cdot\mathbf{n})\,dA \tag{8.2.12}$$

Substitution of the result (8.2.12) into (8.2.9) yields:

$$\dot{B} = \int_V \frac{\partial b}{\partial t}\, dV + \int_A b\,(\mathbf{v}\cdot\mathbf{n})\, dA \tag{8.2.13}$$

This result expresses the *Reynolds' transport theorem*. The terms on the right side are:

term 1: The time rate of change of the quantity B inside the fixed control volume V.
term 2: The net flow out of the fixed control surface A.

Because the control volume V is fixed in space, the transport theorem may alternatively be expressed by:

$$\dot{B} = \frac{d}{dt}\int_V b\, dV + \int_A b\,(\mathbf{v}\cdot\mathbf{n})\, dA \tag{8.2.14}$$

In some application we choose a control volume V that moves and deforms. Let the velocity of points on the moving control surface be denoted $\bar{\mathbf{v}}(\mathbf{r},t)$. Equation (8.2.13) expressing the transport theorem is still valid, but (8.2.14) has to be replaced by:

$$\dot{B} = \frac{d}{dt}\int_{V(t)} b\, dV + \int_A b\,[(\mathbf{v}-\bar{\mathbf{v}})\cdot\mathbf{n}]\, dA \tag{8.2.15}$$

8.2.1 Alternative Derivation of the Reynolds' Transport Theorem

The result (8.2.13) may be derived directly using mathematics. The Theorem C.8 is first used to transform the integral in (8.2.8):

$$B(t) = \int_{V(t)} b\, dV = \int_{V_o} b\, J\, dV_o \tag{8.2.16}$$

V_o is the volume of the body in the reference configuration K_o, and J is the Jacobian to the deformation gradient:

$$J = \det\mathbf{F} = \det\, F_{ij} \equiv \det\left(\frac{\partial x_i}{\partial X_j}\right) \tag{8.2.17}$$

The transformation from the integral in (8.2.8) to the integral in (8.2.16) may also be obtained directly by the use of the results (5.5.64) and (5.5.65), from which:

$$dV = \frac{\rho_o}{\rho}\, dV_o = J\, dV_o \tag{8.2.18}$$

Because the volume V_o is independent of the time t, the material derivative of the integral in (8.2.8) may be obtained by performing the differentiation under the integral sign. Using formula (5.5.33):

$$\dot{J} = J \operatorname{div} \mathbf{v} \tag{8.2.19}$$

the following two formulas and Gauss' integral theorem C.3:

$$\dot{b} = \frac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b, \ \mathbf{v} \cdot \nabla b + b \operatorname{div} \mathbf{v} = \operatorname{div}(b\mathbf{v}), \ \int_{V(t)} \operatorname{div}(b\mathbf{v})\, dV = \int_A b\, (\mathbf{v} \cdot \mathbf{n})\, dA$$

we obtain:

$$\dot{B} = \int_{V_o} \left(\dot{b}J + b\dot{J}\right) dV_o = \ = \int_{V_o} \left(\dot{b} + b \operatorname{div} \mathbf{v}\right) J\, dV_o = \int_{V(t)} \left(\frac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b + b \operatorname{div} \mathbf{v}\right) dV$$

$$= \int_{V(t)} \left(\frac{\partial b}{\partial t} + \operatorname{div}(b\mathbf{v})\right) dV = \int_{V(t)} \frac{\partial b}{\partial t} dV + \int_A b\, (\mathbf{v} \cdot \mathbf{n})\, dA \quad \Rightarrow \tag{8.2.13}$$

8.2.2 Control Volume Equations

The Reynolds' transport theorem, (8.2.13), will now be used to transform the fundamental laws of thermomechanics for a body to a fixed control volume V with a control surface A.

The *principle of conservation of mass*, (8.2.1), implies:

$$\dot{m} = \frac{d}{dt} \int_{V(t)} \rho\, dV = 0 \quad \Rightarrow \quad \int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho\, (\mathbf{v} \cdot \mathbf{n})\, dA = 0 \tag{8.2.20}$$

The result (8.2.20) is called the *continuity equation for a control volume*.

The *law of balance of linear momentum*, Euler's 1. axiom (3.2.6):

$$\mathbf{f} = \dot{\mathbf{p}} = \frac{d}{dt} \int_{V(t)} \mathbf{v} \rho\, dV \quad \Rightarrow$$

$$\int_V \frac{\partial (\mathbf{v}\rho)}{\partial t} dV + \int_A \mathbf{v}\rho\, (\mathbf{v} \cdot \mathbf{n})\, dA = \int_A \mathbf{t}\, dA + \int_V \mathbf{b} \rho\, dV \tag{8.2.21}$$

The *law of balance of angular momentum*, Euler's 2. axiom (3.2.7):

$$\mathbf{m}_O = \dot{\mathbf{l}}_O \equiv \frac{d}{dt} \int_{V(t)} \mathbf{r} \times \mathbf{v} \rho\, dV \quad \Rightarrow$$

$$\int_V \frac{\partial (\mathbf{r} \times \mathbf{v}\rho)}{\partial t} dV + \int_A \mathbf{r} \times \mathbf{v}\rho\, (\mathbf{v} \cdot \mathbf{n})\, dA = \int_A \mathbf{r} \times \mathbf{t}\, dA + \int_V \mathbf{r} \times \mathbf{b} \rho\, dV \tag{8.2.22}$$

The *mechanical energy balance equation* (6.1.12):

$$\dot{K} = \frac{d}{dt}\int_{V(t)} \frac{v^2}{2}\,\rho\,dV = P - P^d \quad \Rightarrow$$

$$\int_V \frac{\partial}{\partial t}\left(\frac{v^2}{2}\rho\right) dV + \int_A \frac{v^2}{2}\rho\,(\mathbf{v}\cdot\mathbf{n})\,dA = \int_A \mathbf{t}\cdot\mathbf{v}\,dA + \int_V \mathbf{b}\cdot\mathbf{v}\,\rho\,dV - \int_V \mathbf{T}:\mathbf{D}\,dV \tag{8.2.23}$$

The *first law of thermomechanics*, (6.3.11):

$$\dot{E} = \frac{d}{dt}\int_{V(t)} \varepsilon\,\rho\,dV = \int_A q\,dA + P^d \quad \Rightarrow$$

$$\int_V \frac{\partial}{\partial t}(\varepsilon\,\rho)\,dV + \int_A \varepsilon\,\rho\,(\mathbf{v}\cdot\mathbf{n})\,dA = \int_A q\,dA + \int_V \mathbf{T}:\mathbf{D}\,dV \tag{8.2.24}$$

Example 8.1. Forces on a Turbine Vane

Figure 8.2.2a shows a vane hit by a water jet of cross-section A. The velocity of the water hitting the vane is $v_{in} = v$. The jet leaves the vane with the velocity $v_{out} = v$ and in a direction that makes an angle θ with respect to the direction of the incoming yet. Referring to Fig. 8.2.2b, we want to determine the forces K_x and K_y and the couple moment M at the point O where the vane is attached to a foundation, and which are due to the action of the water jet.

A control volume V is selected as shown by the dashed line in Fig. 8.2.2b. The law of balance of linear momentum (8.2.21) applied to this control volume gives:

$$\int_A \mathbf{v}\rho\,(\mathbf{v}\cdot\mathbf{n})\,dA = \int_A \mathbf{t}\,dA \quad \Rightarrow$$

$$v_{out}\cos\theta\cdot\rho\cdot(v_{out})A + v_{in}\cdot\rho\cdot(-v_{in})A = -K_x,\; v_{out}\sin\theta\cdot\rho\cdot(v_{out})A = K_y$$

The law of balance of angular momentum (8.2.22) with O as moment point and applied to the control volume V, reads:

$$\int_A \mathbf{r}\times\mathbf{v}\rho\,(\mathbf{v}\cdot\mathbf{n})\,dA = \int_A \mathbf{r}\times\mathbf{t}\,dA \quad \Rightarrow$$

$$-h\cdot(v_{out}\cos\theta)\cdot\rho\cdot(v_{out})A + c\cdot(v_{out}\sin\theta)\cdot\rho\cdot(v_{out})A - h\cdot v_{in}\cdot\rho\cdot(-v_{in})A = M$$

From these three equations we get the result:

$$K_x = \rho A v^2(1-\cos\theta),\; K_y = \rho A v^2\sin\theta,\; M = \rho A v^2\,[c\sin\theta + h(1-\cos\theta)]$$

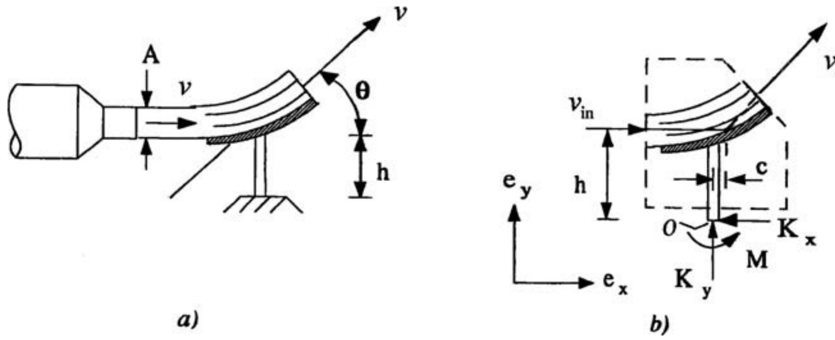

Fig. 8.2.2 Vane in a water jet

8.2.3 Continuity Equation

The *continuity equation at a place*, commonly just called the *continuity equation* in Fluid Mechanics, shall now be derived from the continuity (8.2.20) for a control volume. Using the Gauss' theorem C.3, (8.2.20) is rewritten to:

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho\, (\mathbf{v} \cdot \mathbf{n})\, dA = \int_V \left[\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\,\mathbf{v})\right] dV = 0$$

Because the equation must apply to an arbitrary control volume V, the integrand must be zero. Hence:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\,\mathbf{v}) = 0 \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} + (\rho\, v_i)_{,i} = 0 \tag{8.2.25}$$

Alternatively the result may be written as:

$$\dot{\rho} + \rho\; \mathrm{div}\,\mathbf{v} = 0 \quad \Leftrightarrow \quad \dot{\rho} + \rho\, v_{i,i} = 0 \tag{8.2.26}$$

Equations (8.2.25, 8.2.26) are alternative expressions for the *equation of continuity at a place*. For an incompressible fluid $\dot{\varepsilon}_v = \mathrm{div}\ \mathbf{v} = 0$, and (8.2.26) implies that $\dot{\rho} = 0$, i.e. the density in a fluid particle is constant. Note that $\partial_t \rho$ not necessarily has to be zero. The equality $\partial_t \rho = 0$ does apply only in the case the fluid is homogeneous, which means that $\rho(\mathbf{r},t) =$ constant. For incompressible fluids the equation of continuity, (8.2.25) or (8.2.26) may be replaced by the *condition of incompressibility*:

$$\mathrm{div}\,\mathbf{v} = 0 \quad \Leftrightarrow \quad v_{i,i} = 0 \tag{8.2.27}$$

The equation of continuity (8.2.25) or (8.2.26) can alternatively be derived as follows. Let the element of volume dV represent a small body of mass ρdV, where ρ is the mean value of the density in dV. The time rate of change of the volume dV may be expressed by the volumetric strain:

$$\dot{\varepsilon}_v = \operatorname{div} \mathbf{v} \quad \Rightarrow \quad d\dot{V} = (\operatorname{div} \mathbf{v})\, dV \tag{8.2.28}$$

Since the mass of a body is constant, we obtain:

$$\frac{d}{dt}(\rho\, dV) = \dot{\rho}\, dV + \rho\, d\dot{V} = (\dot{\rho} + \rho \operatorname{div} \mathbf{v})\, dV = 0 \quad \Rightarrow \quad \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0 \Rightarrow \text{(8.2.26)}$$

In a third alternative derivation of the equation of continuity at a place (8.2.26), we start with the *continuity equation in a particle* (5.5.65):

$$\rho\, J = \rho_o \tag{8.2.29}$$

Using (8.2.19), we obtain by material differentiation of (8.2.29):

$$\frac{d}{dt}(\rho\, J) = \dot{\rho}\, J + \rho\, \dot{J} = (\dot{\rho} + \rho \operatorname{div} \mathbf{v})\, J = 0 \quad \Rightarrow \quad \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0 \quad \Rightarrow \quad \text{(8.2.26)}$$

8.3 Perfect Fluid ≡ Eulerian Fluid

In many practical applications we may neglect shear stresses in fluid flows. For instance, when analyzing the flow of a liquid or a gas surrounding rigid bodies, it may often be sufficient to take into consideration the viscosity of the fluid only in a relatively thin layer, the *boundary layer*, near the solid surfaces. Outside of the boundary layer the shear stresses may be neglected, and the liquid or gas may be modelled as an *inviscid fluid*, i.e. a fluid without viscosity. The fluid model is called a *perfect fluid* or a *Eulerian fluid*. The constitutive equation of a perfect fluid is:

$$\mathbf{T} = -p\,\mathbf{1} \quad \Leftrightarrow \quad T_{ij} = -p\,\delta_{ij} \tag{8.3.1}$$

$$p = p(\rho, \theta) \tag{8.3.2}$$

$\rho(\mathbf{r},t)$ is the density of the fluid and $\theta(\mathbf{r},t)$ is the temperature in the fluid. The perfect fluid gives the simplest example of a *thermoelastic material*. The elasticity is expressed by the fact that the pressure $p(\rho, \theta)$ is a function of the density, which again is a function of the volumetric strain, as implied by (5.5.64) and (5.2.23).

The compressibility of a fluid may often be disregarded. Liquids are only rarely considered to be compressible. Gases, which are relatively easily compressible, may also in many practical cases be modelled as incompressible media. In elementary aerodynamics the compressibility of air may be neglected when the velocity of the flying body is less than approx. 1/3 of the speed of sound in air. For incompressible perfect fluids it is customary to replace formula (8.3.2) by:

$$p = p(\mathbf{r}, \theta) \tag{8.3.3}$$

because the pressure p no longer can be considered to be a state variable.

The motion of a perfect fluid is governed by the following four equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{b} \qquad \text{the Euler equations} \tag{8.3.4}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\mathbf{v}) = 0 \qquad \text{the continuity equation} \tag{8.3.5}$$

The *Euler equations* follow from the Cauchy equations (8.1.7) and the constitutive equations (8.3.1). The 4 equations (8.3.4) and (8.3.5) contain 5 unknown functions v_i, ρ, and p. In general the set of equations is supplemented by an energy equation and by an equation of state, for example (8.3.2), which then introduces the temperature θ as an additional 6. unknown field function.

It is often reasonable to assume as a boundary condition for the velocity field of a real fluid that the velocity relative to a rigid surface is zero, i.e. the fluid sticks to the rigid surface. For a perfect fluid only the relative velocity component normal to the boundary surface may be assumed to be zero.

An *ideal gas* is a perfect fluid having the equation of state:

$$p = R\rho\,\theta \tag{8.3.6}$$

R is the gas constant of the gas, and θ is the absolute temperature in degrees Kelvin [$^{\text{o}}$K]. This fluid model may be applied with success for many real gases, for instance air for which $R = 287\,\text{Nm/kg}^{\text{o}}\text{K} = 287\,\text{m}^2/\text{s}^{2\,\text{o}}\text{K}$.

We call a deformation process polytropic if the equation of state for the pressure may be presented as:

$$p = p_o \left(\frac{\rho}{\rho_o}\right)^{\alpha} \tag{8.3.7}$$

α is a constant, and p_o and ρ_o are reference values for pressure and density. The following known processes are represented by (8.3.7):

$$\begin{aligned} &a)\ \text{Isobaric process} && \Leftrightarrow && \text{constant pressure field: } \alpha = 0 \\ &b)\ \text{Isothermal process} && \Leftrightarrow && \text{constant temperature field: } \alpha = 1 \\ &c)\ \text{Isentropic process} && \Leftrightarrow && \text{constant entropy field: } \alpha = \kappa = c_p/c_v \end{aligned} \tag{8.3.8}$$

In the case c), where the specific entropy is constant, c_p and c_v are the *specific heats* at constant pressure and constant volume respectively.

The motion of a perfect fluid is called *barotropic* if a one-to one relation exists between pressure and density:

$$p = p(\rho) \quad \Leftrightarrow \quad \rho = \rho(p) \tag{8.3.9}$$

If we can assume that a barotropic relation exists in a particular problem, we may consider (8.3.9) as a property of the fluid. A fluid with (8.3.1) and (8.3.9) as constitutive equations is called an *elastic fluid*. The literature also applies the names *barotropic fluid*, autobarotropic fluid, and piezotropic fluid in this case. An incompressible fluid is also called a barotropic fluid, but this fluid can obviously not be considered to be elastic.

An elastic fluid, defined by the constitutive equations (8.3.1) and (8.3.9), is hyperelastic, as defined in the Sect. 7.6.1 and 7.10.2. To see this, we start by forming the stress power supplied to a fluid body per unit volume, and then using the continuity equation (8.2.26):

$$\omega = \mathbf{T} : \mathbf{D} = -p\delta_{ij}D_{ij} = -pD_{ii} = -p\,\mathrm{div}\,\mathbf{v} = p\,\frac{\dot{\rho}}{\rho} \quad \Rightarrow$$
$$\omega = \mathbf{T} : \mathbf{D} = p\,\frac{\dot{\rho}}{\rho} \tag{8.3.10}$$

We introduce the potential:

$$\psi = \int_{p_o}^{p} \frac{d\bar{p}}{\rho} - \frac{p}{\rho} = \int_{\rho_o}^{\rho} \frac{p}{\bar{\rho}^2}\, d\bar{\rho} + \text{ constant} \tag{8.3.11}$$

The task to show that the two expressions (8.3.11) for the potential ψ are equivalent, is left as an exercise in Problem 8.6. We now find that:

$$\omega = \rho\,\dot{\psi} \tag{8.3.12}$$

The stress power supplied to a body of volume V is:

$$P^d = \int_V \omega\, dV = \int_V \dot{\psi}\rho\, dV = \frac{d}{dt}\int_V \psi\rho\, dV = \dot{\Psi} \tag{8.3.13}$$

where:

$$\Psi = \int_V \psi\rho\, dV \tag{8.3.14}$$

The result (8.3.13) shows that the stress power P^d may be derived from a potential Ψ, and this proves that the elastic fluid is hyperelastic. The field $\psi = \psi(\mathbf{r},t)$ is the *elastic energy per unit mass*, i.e. the *specific elastic energy*, and Ψ is the elastic energy of the body.

8.3.1 Bernoulli's Equation

From the definition (8.3.11) we find:

$$\nabla\psi = \frac{d\psi}{d\rho}\nabla\rho = \frac{p}{\rho^2}\nabla\rho$$

Then the first term on the right-hand side of the Euler equation (8.3.4) may be transformed:

$$-\frac{1}{\rho}\nabla p = -\nabla\left(\frac{p}{\rho}\right) - \frac{p}{\rho^2}\nabla\rho = -\nabla\left(\frac{p}{\rho} + \psi\right) \qquad (8.3.15)$$

For barotropic fluids moving in a conservative body force field **b(r)**, such that:

$$\mathbf{b}(\mathbf{r}) = -\nabla\beta \qquad (8.3.16)$$

where $\beta = \beta(\mathbf{r})$ is the force potential, the Euler equations (8.3.4) may be presented as:

$$\dot{\mathbf{v}} \equiv \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla\left(\frac{p}{\rho} + \psi + \beta\right) \qquad (8.3.17)$$

$\beta = \beta(\mathbf{r})$ may also interpreted as the potential energy due to the conservative body force **b(r)**. The result (8.3.17) shows that the acceleration of a barotropic fluid in a conservative force field may be found from a scalar potential. From this result we may derive a series of important theorems.
Bernoulli's theorem for steady flow of a barotropic fluid: There exist surfaces, called *Bernoulli surfaces*, covered by stream lines and vortex lines, and defined by:

$$\frac{1}{2}v^2 + \pi + \beta = \text{ constant} \qquad (8.3.18)$$

The theorem is named after Daniel Bernoulli [1700–1782].

Proof. Using the identity c) in Problem 2.9 we can rewrite (8.3.17) to:

$$\partial_t\mathbf{v} + \mathbf{c}\times\mathbf{v} = -\nabla\left(\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right) \qquad (8.3.19)$$

c is the vorticity. In steady flows $\partial_t\mathbf{v} = 0$, and since the vector $\mathbf{c}\times\mathbf{v}$ at a place is normal to the stream line and the vortex line through the place, the left-hand side of (8.3.19) is zero. This fact proves the theorem.

From (8.3.18) follows the *Bernoulli equation*:

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta = \text{ constant along a stream line} \qquad (8.3.20)$$

For incompressible fluids the specific elastic energy ψ is constant and will for convenience be set equal to zero. It is now necessary to require a steady pressure because the pressure in an incompressible medium may be changed uniformly without influencing the motion. The reason for this is that velocity of pressure propagation, i.e. the velocity of sound, is infinite for incompressible media.

The Bernoulli equation may be interpreted as a statement of conservation of mechanical energy. In order to see that we apply (8.2.23) for mechanical energy balance on a control volume V of a stream tube between to surfaces A_1 and A_2 normal to the stream line, as shown in Fig. 8.3.1. The control surface A consists of the surfaces A_1, A_2, and A_3.

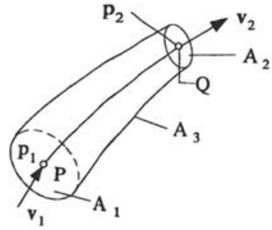

Fig. 8.3.1 Control volume V of a streamtube between the surfaces A_1 and A_2 normal to the stream lines. The control surface A consists of the surfaces A_1, A_2, and A_3

First we perform some initial manipulations. For steady flow the continuity (8.2.25) yields: $\nabla \cdot (\mathbf{v}\rho) = 0$. Equation (8.3.12) and formula (3.1.16) for the material derivative of the steady field $\psi(\mathbf{r})$ imply that:

$$\mathbf{T} : \mathbf{D} \equiv \omega = \rho\,\dot{\psi} = \rho\, v_i\, \psi_{,i} = \nabla \cdot (\psi \rho\, \mathbf{v})$$

Furthermore, with $\mathbf{t} = -p\mathbf{n}$ as the stress vector on any material surface and $\mathbf{b}$ as the body force defined by (8.3.16), we get:

$$\mathbf{t} \cdot \mathbf{v} = (-p\,\mathbf{n}) \cdot \mathbf{v} = -\frac{p}{\rho}(\rho\,\mathbf{v} \cdot \mathbf{n}),\ \mathbf{b} \cdot \mathbf{v}\rho = -\nabla\beta \cdot \mathbf{v}\rho = -\nabla \cdot (\beta\,\mathbf{v}\rho)$$

Using Gauss integration theorem C.3, we then obtain:

$$\int_V [\mathbf{b} \cdot \mathbf{v}\rho - \mathbf{T} : \mathbf{D}]\, dV = \int_V [-\nabla \cdot (\beta\,\mathbf{v}\rho + \psi\rho\,\mathbf{v})]\, dV = -\int_A [\beta + \psi\,]\, \rho\,\mathbf{v} \cdot \mathbf{n}\, dA$$

The energy equation (8.2.23) applied to the control volume V now becomes:

$$\int_A \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right] \rho\,\mathbf{v} \cdot \mathbf{n}\, dA = 0 \tag{8.3.21}$$

On the surface A_3 of the stream tube in Fig. 8.3.1 $\mathbf{v} \cdot \mathbf{n} = 0$. The mean value theorem C.6 then gives:

$$\left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_2 \int_{A_2} \rho\,\mathbf{v} \cdot \mathbf{n}\, dA + \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_1 \int_{A_1} \rho\,\mathbf{v} \cdot \mathbf{n}\, dA = 0$$

The terms $[\,]_1$ and $[\,]_2$ are calculated at places on A_1 and A_2, respectively. According to the continuity equation (8.2.20) the two integrals must, be equal but of opposite signs. If we let A_1 approach zero about P_1 and let A_2 approach zero about P_2, where P_1 and P_2 are two places on the same streamline, we obtain the result (8.3.20). All terms in (8.3.20) represent specific energies, i.e. energies per unit mass: $v^2/2$ is kinetic energy, $(p/\rho + \beta)$ is potential energy, and ψ is elastic energy.

A *Bernoulli equation for non-steady flow* is obtained by integration of (8.3.19) along a streamline between two points P_1 and P_2. The result is:

$$\int_{P_1}^{P_2} \frac{\partial \mathbf{v}}{\partial t} \cdot d\mathbf{r} + \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_{P_2} = \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_{P_1} \tag{8.3.22}$$

An important family of flows, presented in Sect. 8.5, are called *potential flows* or *irrotational flows*. These flows are characterized by zero vorticity: $\mathbf{c} = \nabla \times \mathbf{v} = \mathbf{0}$, which implies that the velocity may be expressed by a velocity potential ϕ, such that $\mathbf{v} = \nabla\phi$. From (8.3.19) we then get:

$$\nabla\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right] = 0$$

This result implies:
Bernoulli's theorem for irrotational flow of a barotropic fluid:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta = f(t) \quad \text{in the fluid} \tag{8.3.23}$$

where $f(t)$ is a function of time.

The result is also called the *Euler pressure equation*. For steady state flows (8.3.23) is reduced to:

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta = \text{constant in the fluid} \tag{8.3.24}$$

For incompressible fluids the specific elastic energy ψ is set equal to zero.

Example 8.2. The Torricelli Law

We want to determine the exit velocity v through the orifice in an open vessel containing a fluid. The vessel is assumed to have a large free surface as compared to the area of the orifice. The fluid is subjected to the constant specific gravitational body force g for which the specific potential energy is:

$$\beta = gz$$

We consider the streamline indicated in Fig. 8.3.2, from point A at the free surface to point B at the orifice. The fluid velocity at A may be neglected. The Bernoulli equation (8.3.24) then gives:

$$\frac{p_o}{\rho} + g(h_1 + h) = \frac{v^2}{2} + \frac{p_o}{\rho} + gh_1 \quad \Rightarrow \quad v = \sqrt{2gh}$$

The result is called *Torricelli's law*, named after Evangelista Torricelli [1608–1647].

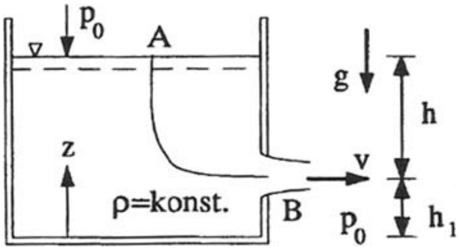

Fig. 8.3.2 Efflux from an orifice in an open vessel

8.3.2 Circulation and Vorticity

We shall discuss some properties of the vorticity field $\mathbf{c}(\mathbf{r},t)$ of barotropic fluids in conservative force fields. From (8.3.19) it follows that:

$$\nabla \times \partial_t \mathbf{v} + \nabla \times (\mathbf{c} \times \mathbf{v}) = 0 \tag{8.3.25}$$

The two terms will now in turn be transformed. The definition of the vorticity $\mathbf{c}$ yields:

$$\nabla \times \partial_t \mathbf{v} = \partial_t \mathbf{c} \tag{8.3.26}$$

The following identity is to be derived as Problem 8.7:

$$\nabla \times (\mathbf{c} \times \mathbf{v}) = \mathbf{c}\,\mathrm{div}\,\mathbf{v} + (\mathbf{v} \cdot \nabla)\ \mathbf{c} - (\mathbf{c} \cdot \nabla)\ \mathbf{v} \tag{8.3.27}$$

The formulas (8.3.26–8.3.27) are substituted into (8.3.25), and the result is:

$$\dot{\mathbf{c}} \equiv \partial_t \mathbf{c} + (\mathbf{v} \cdot \nabla)\ \mathbf{c} = (\mathbf{c} \cdot \nabla)\ \mathbf{v} - \mathbf{c}\,\mathrm{div}\,\mathbf{v} \tag{8.3.28}$$

Using the continuity (8.2.26), we can rewrite (8.3.28) to obtain *Nanson's formula* [E. Nanson 1874]:

$$\frac{d}{dt}\left(\frac{\mathbf{c}}{\rho}\right) = \mathbf{L} \cdot \frac{\mathbf{c}}{\rho} \tag{8.3.29}$$

$\mathbf{L} =$ grad $\mathbf{v}$ is the velocity gradient tensor, and the time derivative is the material derivative. The solution of this differential equation is:

$$\frac{\mathbf{c}}{\rho} = \frac{\mathbf{F} \cdot \mathbf{c}_o}{\rho_o} \tag{8.3.30}$$

$\mathbf{F}$ is the deformation gradient tensor, and $\mathbf{c}_o$ and ρ_o are respectively the vorticity and the density in the reference configuration K_o. Application of $(5.5.28)_1$ in the material derivative of (8.3.30) will show that (8.3.30) is the solution to (8.3.29).

From (8.3.30) we may draw two important conclusions:

a) *The vortex lines are material lines*: Let $d\mathbf{r}_o$ be tangent at the particle X to a material line that also is a vortex line in the reference configuration K_o, i.e $d\mathbf{r}_o$ is parallel to the vorticity $\mathbf{c}_o$. Then we may set $d\mathbf{r}_o = \mathbf{c}_o d\tau$, where $d\tau$ is a constant. In the present configuration K the material line has a tangent at the particle X that is parallel to $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}_o$. Thus from (8.3.30):

$$d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}_o = \mathbf{F} \cdot \mathbf{c}_o d\tau = \frac{\rho_o}{\rho} \mathbf{c} d\tau \tag{8.3.31}$$

The result shows that the tangent vector $d\mathbf{r}$ is parallel to the vorticity $\mathbf{c}$. This means that the material line is identical to the vortex line at all times. Using (8.3.31) and the relations $d\mathbf{r}_o = \mathbf{c}_o d\tau$ and $ds_\mathrm{o} = |d\mathbf{r}_\mathrm{o}|$, we get this formula for the line differential $ds = |d\mathbf{r}|$:

$$ds = \frac{\rho_o}{\rho} \frac{c}{c_o} ds_o \tag{8.3.32}$$

b) *The Lagrange-Cauchy theorem*: If a barotropic fluid subjected to a conservative body force field has irrotational motion, $\mathbf{c} = \mathbf{0}$, at a certain time, the motion will be irrotational at all times. This theorem may be used to identify potential flows.

Let C be a closed material line in a flowing fluid. The *circulation* around C is defined by the integral:

$$\Gamma(t) = \oint_C \mathbf{v} \cdot d\mathbf{r} \tag{8.3.33}$$

The *Kelvin circulation theorem*, named after William Thomson, Lord Kelvin [1824–1907] states that: For a barotropic fluid in a conservative force field the circulation around a closed material line C is time independent:

$$\Gamma(t) = \oint_C \mathbf{v} \cdot d\mathbf{r} = \text{constant} \tag{8.3.34}$$

Proof. Let the closed curve C be represented by $\mathbf{r} = \mathbf{r}(s_o, t)$, where s_o is the arc length parameter in the reference configuration K_o, and $0 \leq s_o \leq L_o$, where L_o is the length of the curve in K_o. Then the circulation around the curve C at time t is:

$$\Gamma(t) = \int_0^{L_o} \mathbf{v} \cdot \frac{\partial \mathbf{r}}{\partial s_o} ds_o$$

Because:

$$\mathbf{v} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}}{\partial s_o}\right) = \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial s_o} = \frac{\partial}{\partial s_o}\left(\frac{\mathbf{v} \cdot \mathbf{v}}{2}\right) = \frac{\partial}{\partial s_o}\left(\frac{v^2}{2}\right)$$

and the curve C is closed, we obtain:

$$\dot{\Gamma}(t) = \int_0^{L_o} \dot{\mathbf{v}} \cdot \frac{\partial \mathbf{r}}{\partial s_o} ds_o = \oint_C \dot{\mathbf{v}} \cdot d\mathbf{r} \tag{8.3.35}$$

For a barotropic fluid in a conservative force field (8.3.17) applies. Thus we get:

$$\dot{\Gamma}(t) = \oint_C \dot{\mathbf{v}} \cdot d\mathbf{r} = -\oint_C \nabla \left(\frac{p}{\rho} + \psi + \beta \right) \cdot d\mathbf{r} = -\oint_C d \left(\frac{p}{\rho} + \psi + \beta \right) = 0 \quad (8.3.36)$$

This result proves the theorem.

The circulation around a closed curve C may also be computed from the vorticity **c**. Let A be any surface bounded by the curve C. Then, using Stokes's theorem C.5, we get the result:

$$\Gamma(t) = \oint_C \mathbf{v} \cdot d\mathbf{r} = \int_A \mathbf{c} \cdot \mathbf{n} dA \quad (8.3.37)$$

n is the unit normal to the surface A. This result together with Kelvin's theorem gives an alternative proof of the Lagrange-Cauchy theorem.

Kelvin's theorem implies *the three vortex theorems of Helmholtz*, named after Herman Ludwig Ferdinand von Helmholtz [1821–1894]. Kelvin presented his theorem in 1869 to prove the three vortex theorems.

Theorem 1. The circulation is the same about any closed curve surrounding a vortex tube.

Theorem 2. The vortex lines are material lines.

Theorem 3. The *strength of a vortex tube*, defined by the surface integral in (8.3.37), is constant.

Proof. Figure 8.3.3 shows two curves C_1 and C_2 surrounding a vortex tube. Using Stokes' theorem C.5, we get:

$$\oint_{C_1} \mathbf{v} \cdot d\mathbf{r} - \oint_{C_2} \mathbf{v} \cdot d\mathbf{r} = \oint_C \mathbf{v} \cdot d\mathbf{r} = \int_A \mathbf{c} \cdot \mathbf{n}\, dA$$

A is a surface bounded by the curve C, marked in Fig. 8.3.3 by a broken line, and **n** is the unit normal to A. Since the vectors **n** and **c** are orthogonal, $\mathbf{c} \cdot \mathbf{n} = 0$, and we may conclude that:

$$\oint_{C_1} \mathbf{v} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{v} \cdot d\mathbf{r} \quad (8.3.38)$$

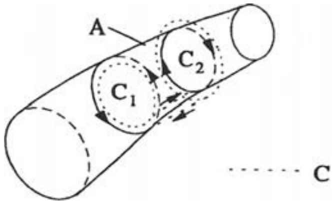

Fig. 8.3.3 Vortex tube

This result proves Theorem 1. Theorem 2 is already proved by (8.3.31). The strength of the vortex tube may through Theorem 1 be expressed by the circulation around any curve surrounding the vortex tube. Kelvin's theorem then completes the proof of Theorem 3.

8.3.3 Sound Waves

Sound propagates as elastic waves. In fluids the elastic waves represent small variations in the pressure. The loudest sound the human ear can receive without pain corresponds to an amplitude of the pressure variation of 28 Pa. The sound pressure $p_t = 28\,\text{Pa}$ is therefore called the *threshold of pain*. The weakest sound the human ear can hear is called the *threshold of hearing* and corresponds to a sound pressure of about $2 \cdot 10^{-5}\,\text{Pa}$. For comparison the normal atmospheric pressure is $p_o = 101.32\,\text{kPa}$.

Propagation of sound is an isentropic process governed by the Euler equations (8.3.4), the equation of continuity (8.3.5), and a constitutive equation on the form:

$$p = p(\rho) \tag{8.3.39}$$

A reference pressure p_o and the corresponding reference density ρ_o represent an equilibrium state governed by the equilibrium equation:

$$\mathbf{0} = -\frac{1}{\rho_o}\nabla p_o + \mathbf{b} \tag{8.3.40}$$

Sound waves are small variations $\tilde{p}$ in the pressure and $\tilde{\rho}$ in the density, such that:

$$\rho = \rho_o + \tilde{\rho}, \; p = p_o + \tilde{p} = p_o + \left.\frac{dp}{d\rho}\right|_{\rho=\rho_o} \tilde{\rho} \quad \Rightarrow \quad \frac{\tilde{p}}{\tilde{\rho}} = \left.\frac{dp}{d\rho}\right|_{\rho=\rho_o} \tag{8.3.41}$$

A linearization of the Euler equations and the continuity equation yields, after the equilibrium (8.3.40) has been applied, the set of equations:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_o}\nabla \tilde{p}, \; \frac{\partial \tilde{\rho}}{\partial t} + \rho_o \nabla \cdot \mathbf{v} = 0 \tag{8.3.42}$$

We introduce the constant parameter:

$$c = \sqrt{\frac{\tilde{p}}{\tilde{\rho}}} = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_o}} \tag{8.3.43}$$

which will be shown to be the *velocity of sound*. Formula (8.3.43) and the linearized basic equations (8.3.42) now yield:

$$\nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_o}\nabla^2 \tilde{p}, \; \frac{\partial^2 \tilde{p}}{\partial t^2} + \rho_o \nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = 0 \quad \Rightarrow$$

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = c^2 \nabla^2 \tilde{p} \tag{8.3.44}$$

This is a linear wave equation, as presented in Sect. 7.7, e.g. (7.7.5), and c may be identified as the velocity of sound in the fluid.

For gasses we may use the constitutive (8.3.7), and for air we choose:

$$p_o = 101.32\,\text{kPa} \quad , \quad \rho_o = 1.225\,\text{kg/m}^3, \; \alpha = 1.4 \quad \text{at } 15^{\text{o}}C$$

The velocity of sound in air is then:

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_o}} = \sqrt{p_o\,\alpha\,\frac{1}{\rho_o}} = 340 \text{ m/s at } 15^{\text{o}}C$$

For liquids we may use the constitutive equation:

$$p = C_1 \left(\frac{\rho}{\rho_o}\right)^{\gamma} - C_2 \tag{8.3.45}$$

C_1, C_2, and γ are constant parameters. For water we may set:

$$\rho_o = 1000\,\text{kg/m}^3, \; C_1 = 304.06\,\text{MPa} \quad , \quad C_2 = 303.96\,\text{MPa} \quad , \quad \gamma = 7$$

The velocity of sound in water then becomes:

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_o}} = \sqrt{C_1\,\gamma\frac{1}{\rho_o}} = 1460 \text{ m/s}$$

8.4 Linearly Viscous Fluid = Newtonian Fluid

8.4.1 Constitutive Equations

The presence of shear stresses in a flowing fluid is realized when we observe the velocity field near rigid boundary surfaces, see Fig. 8.4.1. The fluid particles are slowed down in the neighborhood of the rigid surface, and very close to the surface the relative velocity is practically zero. Shear stresses are present everywhere in the flow, but their influence on the velocity field is normally very slight, except in the boundary layer near the rigid boundary surface. In the analysis of fluid flow around rigid bodies it is customary to first model the fluid as perfect fluid, and then use the solution as an asymptotic external flow to a viscous boundary layer solution near the

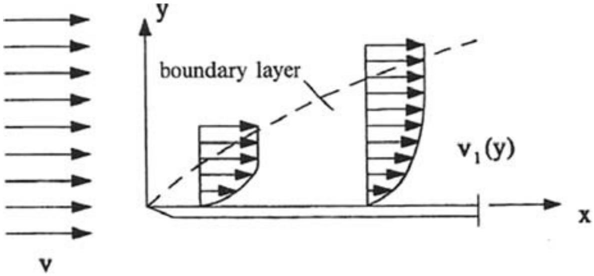

Fig. 8.4.1 Uniform flow and viscous boundary layer near a rigid surface

rigid surface. In the analysis of flows in pipes and through other narrow passages the viscous boundary layer fills the entire flow regime.

As a starting point for the development of a constitutive equation for a linearly viscous fluid we again look at the experiment with the viscometer in Fig. 1.3.2 in Sect. 1.3. The flow between the two cylindrical surfaces is considered to be a steady flow between two parallel planes, and called *simple shear flow*, see Fig. 8.4.2.

The velocity field of simple shear flow is:

$$v_1 = \frac{v}{h} x_2, \; v_2 = v_3 = 0 \tag{8.4.1}$$

The rate of deformation matrix becomes:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}}{2}, \; \dot{\gamma} = 2D_{12} = \frac{dv_1}{dx_2} = \frac{v}{h} \tag{8.4.2}$$

We assume that the result of the viscometer test is a shear stress proportional to the shear strain rate $\dot{\gamma}$:

$$\tau = \mu \dot{\gamma} \tag{8.4.3}$$

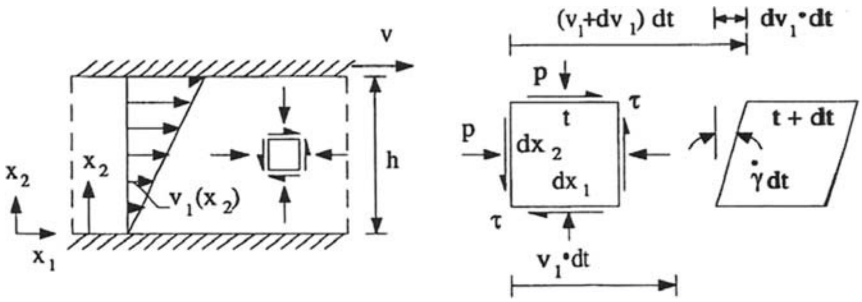

Fig. 8.4.2 Simple shear flow between two parallel planes

The *dynamic viscosity* μ is a temperature dependent material parameter. A first generalization of the constitutive (8.4.3) to apply to a general flow is called *Newton's law of fluid friction* and may be presented as:

$$T_{ij} = 2\mu D_{ij} = \mu\left(v_{i,j} + v_{j,i}\right) \text{ for } i \neq j \tag{8.4.4}$$

A further generalization of (8.4.4) leads to the *constitutive equation of a linearly viscous fluid*, or what is called a *Newtonian fluid*, and is presented below.

George Gabriel Stokes [1819–1903] presented the following four criteria for the relationships between the stresses and the velocity field in a viscous fluid:

1. The stress tensor $\mathbf{T}$ is a continuous function of the rate of deformation tensor $\mathbf{D}$.
2. The stress tensor $\mathbf{T}$ is explicitly independent of the particle coordinates, which implies that the fluid is homogeneous.
3. When the fluid is not deforming, i.e. $\mathbf{D} = \mathbf{0}$, the stress tensor is: $\mathbf{T} = -p(\rho, \theta)\mathbf{1}$.
4. Viscosity is an isotropic property.

The first three criteria imply the following general form of the constitutive equation of a viscous fluid:

$$\mathbf{T} = \mathbf{T}\left[\mathbf{D}, \rho, \theta\right], \; \mathbf{T}\left[\mathbf{0}, \rho, \theta\right] = -p(\rho, \theta)\,\mathbf{1} \tag{8.4.5}$$

The pressure $p(\rho, \theta)$ is the *thermodynamic pressure*. The requirement of isotropy is really superfluous because the (8.4.5) implies viscous isotropy. This fact will be demonstrated in Sect. 11.9.2 on *Stokesian fluids*. Equation (8.4.5) represents the constitutive equation of a Stokesian fluid.

We now restrict (8.4.5) to be linear with respect to $\mathbf{D}$. Using arguments along the lines used for isotropic, linearly elastic materials in Sect. 7.2, we may conclude that viscous isotropy implies that the tensors $\mathbf{T}$ and $\mathbf{D}$ are coaxial and furthermore that the linear version of (8.4.5) has the form:

$$\mathbf{T} = -p(\rho, \theta)\,\mathbf{1} + 2\mu\mathbf{D} + \left(\kappa - \frac{2\mu}{3}\right)(\operatorname{tr}\mathbf{D})\,\mathbf{1} \quad \Leftrightarrow$$
$$T_{ij} = -p(\rho, \theta)\delta_{ij} + 2\mu D_{ij} + \left(\kappa - \frac{2\mu}{3}\right) D_{kk}\,\delta_{ij} \tag{8.4.6}$$

The *viscosities* μ and κ are functions of the temperature and in some cases also of the pressure. The *dynamic viscosity* μ is relatively easy to determine experimentally, for instance using a viscometer of the type described in Sect. 1.3. The viscosity μ of water is $1.8 \cdot 10^{-3}\,\text{Ns/m}^2$ at $0\,^\circ\text{C}$ and $1.0 \cdot 10^{-3}\,\text{Ns/m}^2$ at $20\,^\circ\text{C}$. For air the viscosity μ is $1.7 \cdot 10^{-5}\,\text{Ns/m}^2$ at $0\,^\circ\text{C}$ and $1.8 \cdot 10^{-5}\,\text{Ns/m}^2$ at $20\,^\circ\text{C}$. The parameter κ is called the *bulk viscosity*, and is far more difficult to measure. Its physical implication will be discussed below. In the literature the (8.4.6) is often presented with λ in stead of $(\kappa - 2\mu/3)$, but the parameter λ has no direct physical interpretation, contrary to what is true for both μ and κ. With reference to (4.2.42) in Sect. 4.2.1 on isotropic tensors of 4. order, the parameters λ and μ may be identified as the Lamé-constants for viscous fluids.

In modern literature (8.4.6) is sometimes called the *Cauchy-Poisson law*. A material model defined by the constitutive (8.4.6) is called a *Newtonian fluid*. For an incompressible fluid, for which $\mathrm{tr}\mathbf{D} = 0$, (8.4.6) has to be replaced by:

$$\mathbf{T} = -p(\mathbf{r},t)\,\mathbf{1} + 2\mu\mathbf{D} \quad \Leftrightarrow \quad T_{ij} = -p(\mathbf{r},t)\delta_{ij} + 2\mu D_{ij} \tag{8.4.7}$$

The pressure $p(\mathbf{r},t)$ is a function of position $\mathbf{r}$ and time t, and can only be determined from the equations of motion and the boundary conditions. An equation of state, $p = p(\rho,\theta)$, loses its meaning when incompressibility is assumed.

If the symmetric tensors $\mathbf{T}$ and $\mathbf{D}$ are decomposed into isotrops and deviators, the constitutive equations (8.4.6) takes the alternative form:

$$\mathbf{T}^o = -p(\rho,\theta)\,\mathbf{1} + 3\kappa\mathbf{D}^o \quad \Leftrightarrow \quad T_{ij}^o = -p(\rho,\theta)\delta_{ij} + 3\kappa D_{ij}^o \tag{8.4.8}$$

$$\mathbf{T}' = 2\mu\mathbf{D}' \quad \Leftrightarrow \quad T_{ij}' = 2\mu D_{ij}' \tag{8.4.9}$$

For isotropic states of stress, we may replace (8.4.8) by:

$$\mathbf{T} = -\tilde{p}\mathbf{1}, \ \tilde{p} = p(\rho,\theta) - \kappa\,\dot{\varepsilon}_v, \ \dot{\varepsilon}_v = \mathrm{tr}\mathbf{D} = \mathrm{div}\,\mathbf{v} = -\frac{\dot{\rho}}{\rho} \tag{8.4.10}$$

Note that the total pressure $\tilde{p}$ is not the same as the thermodynamic pressure $p(\rho,\theta)$.

The bulk viscosity κ expresses the resistance of the fluid toward rapid volume changes. Due to the fact that it is difficult to measure κ, values are hard to find in the literature. Kinetic theory of gasses shows that $\kappa = 0$ for monatomic gasses. But as shown by Truesdell [49] this result is implied in the stress assumption that is the basis for the kinetic theory. Experiments show that for monatomic gasses it is reasonable to set $\kappa = 0$, while for other gasses and for all liquids the bulk viscosity κ, and values of $\lambda = \kappa - 2\mu/3$, are larger than, and often much larger than μ. The assumption $\kappa = 0$, which is sometimes taken for granted in older literature on Fluid Mechanics, is called the *Stokes relation*, since it was introduced by him. However, Stokes did not really believe the relation to be relevant. Usually the deviator $\mathbf{D}'$ dominates over $\mathbf{D}^o$ such that the effects of the bulk viscosity are small. The bulk viscosity κ has dominating importance for the dissipation and absorption of sound energy.

A Newtonian fluid provides an example of a *visco-thermoelastic material*. If the thermodynamic pressure p is a function only of the density, the Newtonian fluid represents a *viscoelastic fluid*.

Example 8.3. Flow Between Parallel Planes

In this example we shall use the *Saint-Venant's semi-inverse method*. By this method the unknown functions in a problem are partly assumed known. The governing equations and the boundary conditions in the problem are then used to determine these functions completely.

The flow of an incompressible Newtonian fluid between two rigid plates, as shown in Fig. 8.4.3, is driven by a constant pressure gradient $c = -\partial p/\partial x$ in the direction of the flow and by constant velocity v of one of the plates. Gravity

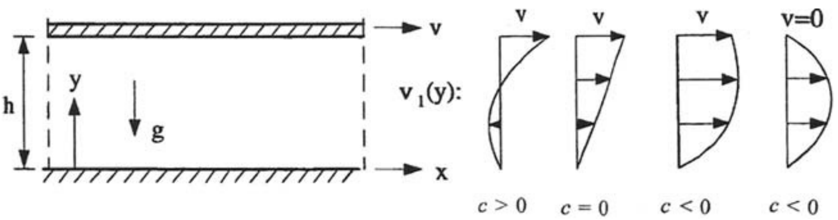

Fig. 8.4.3 Flow between parallel planes. Velocity profiles $v_x(y)$ for different combinations of the parameters c and v

represents a body force -g in the y-direction. We assume steady state motion and the velocity field:

$$v_x = v_x(y), \; v_y = v_z = 0 \tag{8.4.11}$$

which satisfies the incompressibility condition, div $\mathbf{v} = 0$. From the constitutive (8.4.7) we obtain the following expression for the stresses in the fluid:

$$\sigma_x = \sigma_y = \sigma_z = -p(x,y,z), \; \tau_{xy} = \mu \frac{dv_x}{dy}, \; \tau_{yz} = \tau_{zx} = 0$$

When these stresses are substituted into the Cauchy equations (8.1.7), we find:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 v_x}{dy^2}, \; 0 = -\frac{\partial p}{\partial y} - \rho g, \; 0 = -\frac{\partial p}{\partial z} \tag{8.4.12}$$

It follows from these equations that the pressure is independent of z and that the pressure gradient in the x-direction, $\partial p/\partial x = -c$, is constant as assumed. Integrations of the equations yield:

$$p(x,y) = -\rho g y - cx + A, \; v_x = -\frac{c}{2\mu} y^2 + By + C \tag{8.4.13}$$

A, B, and C are constants of integration. We assume that the fluid sticks to the rigid surfaces of the plates. The boundary conditions for the velocity and their implications are therefore:

$$v_x(0) = 0, \; v_x(h) = v \quad \Rightarrow \quad C = 0, \; B = \frac{v}{h} + \frac{ch}{2\mu}$$

The velocity field has then been determined.

$$v_x(y) = \frac{ch^2}{2\mu} \left[\frac{y}{h} - \left(\frac{y}{h} \right)^2 \right] + v \frac{y}{h}$$

Figure 8.4.3 illustrates the *velocity profile* $v_x(y)$ for different combinations of the constant parameters c and v.

A boundary condition for the pressure may be $p(0,0) = p_o$, which implies that $A = p_o$ and:

$$p(x,y) = p_o - \rho g y - cx$$

Example 8.4. Flow Around a Rotating Cylinder

Figure 8.4.4 shows a cylindrical container with inner radius b and a rigid cylinder with radius a. The container and the cylinder have a common vertical axis, and their length is L. The annular space between the two concentric cylindrical surfaces of the container and the cylinder contains a Newtonian fluid. The rigid cylinder rotates with a constant angular velocity ω due to a constant external couple moment M, which is counteracted by the shear stress from the fluid. The motion of the cylinder results in a steady flow of the fluid, and we assume the flow to be two-dimensional with the velocity field:

$$v_\theta = v_\theta(R), \; v_R = v_z = 0 \tag{8.4.14}$$

Using (5.4.19) for deformation rates in cylindrical coordinates, we find only one deformation rate different from zero:

$$\dot{\gamma} \equiv \dot{\gamma}_{\theta R} = \frac{dv_\theta}{dR} - \frac{v_\theta}{R} = R\frac{d}{dR}\left(\frac{v_\theta}{R}\right)$$

The state of stress in the fluid is thus given by a pressure p and a shear stress:

$$\tau \equiv \tau_{\theta R} = \mu\,\dot{\gamma}_{\theta R} = \mu\,R\frac{d}{dR}\left(\frac{v_\theta}{R}\right) \tag{8.4.15}$$

The law of balance of angular momentum, (8.2.22), for a cylindrical body of radius R containing the rigid cylinder and fluid, provides the following equilibrium equation:

$$0 = R\cdot\tau\cdot(2\pi RL) + M \quad \Rightarrow \quad \tau = -\frac{M}{2\pi L}\,\frac{1}{R^2} \tag{8.4.16}$$

A combination of (8.4.15) and (8.4.16) results in a differential equation for the unknown velocity field v_θ:

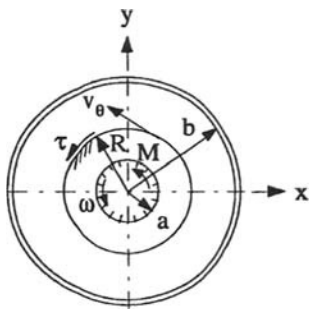

Fig. 8.4.4 Steady flow in a cylindrical container around a rotating cylinder

$$\frac{d}{dR}\left(\frac{v_\theta}{R}\right) = -\frac{M}{2\pi\mu L}\frac{1}{R^3} \tag{8.4.17}$$

We assume that the fluid sticks to the rigid cylindrical surfaces and obtain the boundary conditions:

$$v_\theta(a) = \omega a, \; v_\theta(b) = 0$$

The solution of the differential (8.4.17) is then:

$$v_\theta(R) = \frac{\omega a}{\left[1-(a/b)^2\right]}\left[\frac{a}{R} - \frac{aR}{b^2}\right], \; M = \frac{4\pi\mu L\omega a^2}{\left[1-(a/b)^2\right]}$$

An interesting special case, which will be referred to in Sect. 8.5, is obtained if we let $b \to \infty$. The result is the potential flow:

$$v_\theta(R) = \frac{\omega a^2}{R} = \nabla\phi, \; \phi = \omega a^2\theta \tag{8.4.18}$$

This flow is also discussed in Example 5.2 in Sect. 5.4, and called the *potential vortex*.

If the fluid is incompressible with constant density ρ, the pressure $p(R,\theta,z)$ in the fluid in a potential vortex may be determined as follows. In cylindrical coordinates the state of stress is expressed by:

$$\sigma_R = \sigma_\theta = \sigma_z = -p(R,\theta,z), \; \tau_{\theta R} \equiv \tau = -\frac{M}{2\pi L R^2}, \; \tau_{\theta z} = \tau_{zR} = 0$$

The body force is given by the gravitational force $-g\mathbf{e}_z$. The particle acceleration is:

$$\mathbf{a} = \dot{\mathbf{v}} = -\frac{v_\theta^2}{R}\mathbf{e}_R = -\frac{\omega^2 a^4}{R^3}\mathbf{e}_R$$

The Cauchy equations (3.2.39–41) yield:

$$-\frac{\partial p}{\partial R} = -\frac{\rho\omega^2 a^4}{R^3}, \; -\frac{\partial p}{\partial \theta} = 0, \; -\frac{\partial p}{\partial z} - \rho g = 0$$

The result of integrations of these equations is:

$$p(R,z) = -\frac{\rho\omega^2 a^4}{2R^2} - \rho g z + C$$

The constant of integration C may be determined from a pressure boundary condition. For comparison we may note that the pressure in a fluid in rigid-body rotation, see Problem 3.4, is:

$$p(R,z) = \frac{\rho\,\omega^2 R^2}{2} - \rho g z + C$$

8.4.2 The Navier-Stokes Equations

The general equations of motion of a linearly viscous fluid are called the *Navier-Stokes equations*. These equations are obtained by the substitution of the constitutive equations (8.4.6) into the Cauchy equations of motion (8.1.7). If it is assumed that the viscosities μ and κ may be considered to be constant parameters, the resulting equations are:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\kappa + \frac{\mu}{3}\right)\nabla(\nabla \cdot \mathbf{v}) + \mathbf{b} \tag{8.4.19}$$

In a Cartesian coordinate system the Navier-Stokes equations are:

$$\partial_t v_i + v_k\, v_{i,k} = -\frac{1}{\rho} p_i + \frac{\mu}{\rho} v_{i,kk} + \frac{1}{\rho}\left(\kappa + \frac{\mu}{3}\right) v_{k,ki} + b_i \tag{8.4.20}$$

For incompressible fluids $\nabla \cdot \mathbf{v} = 0$, and (8.4.19) is reduced to:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} + \mathbf{b} \tag{8.4.21}$$

The Navier-Stokes equations (8.4.19, 8.4.20, 8.4.21) are the most important equations in the study of viscous fluids. The complexity of the equations indicates that analytical solutions in most cases require major simplifications and approximations. Modern computer codes make it possible to use the Navier-Stokes equations in numerical solutions of very complex fluid flow problems.

For incompressible fluids it is often convenient to combine the pressure gradient ∇p and the body force term $\rho \mathbf{b}$ in the Navier-Stokes equations (8.4.21) by introducing the *modified pressure P*. First we compute the static pressure p_s that would exist in the fluid at rest only subjected to the body force $\mathbf{b}$. The static pressure p_s is thus determined from the equilibrium equation:

$$\mathbf{0} = -\frac{1}{\rho}\nabla p_s + \mathbf{b} \tag{8.4.22}$$

If the body force is conservative such that $\mathbf{b} = -\nabla\beta$, as presented in (8.3.16), we write for (8.4.22):

$$\mathbf{0} = -\nabla\left(\frac{p_s}{\rho} + \beta\right) \quad \Rightarrow \quad p_s = p_o - \rho\beta$$

p_o is constant reference pressure.

For example, let $\mathbf{b}$ be the constant gravitational force g and z the vertical height above a chosen reference level at which the pressure is p_o. Then we find $\beta = gz$ and:

$$p_s(z) = p_o - \rho g z$$

The modified pressure P is defined by:

$$P = p - p_s \quad \text{or} \quad P = p + \rho\beta - p_o \tag{8.4.23}$$

Then the Navier-Stokes equations for an incompressible fluid may be reduced to:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2 \mathbf{v} \tag{8.4.24}$$

In Example 8.4 we may set $p_s = C - \rho g z$ and the modified pressure becomes:

$$P(R) = -\frac{\rho\omega^2 a^4}{2R^2}$$

Example 8.5. Film Flow

Figure 8.4.5 illustrates the transportation of an incompressible fluid on a wide conveyer belt as a film with constant thickness h. The belt has the width w and is inclined an angle α with respect to the horizontal plane. The belt moves with a constant velocity v_o. The free fluid surface is only subjected to the atmospheric pressure p_a. The fluid sticks to the belt. The body force is given by the gravitational force g and is expressed by:

$$\mathbf{b} = -g\sin\alpha\,\mathbf{e}_x - g\cos\alpha\,\mathbf{e}_y$$

We assume steady two-dimensional flow with the velocity field and the pressure field:

$$v_x = v_x(y),\ v_y = 0,\ p = p(x,y)$$

The special situation at the edges of the belt needs not be considered if the width of the belt is sufficiently large. The rate of deformation field $\mathbf{D}$ will naturally be functions of the y-coordinate only. The stress field $\mathbf{T}$ will be dependent on both x and y. The boundary conditions for the flow are:

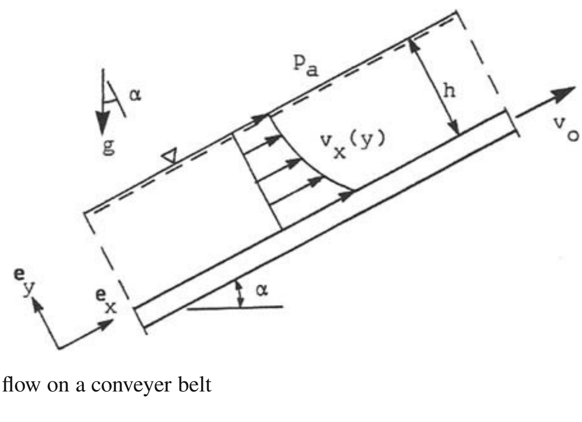

Fig. 8.4.5 Film flow on a conveyer belt

$$v_x(0) = v_o,\ \sigma_y(h,z) = -p(h,x) = -p_a,\ \tau_{xy}(h,z) = 0$$

The Navier-Stokes equations (8.4.21) are reduced to:

$$\mathbf{0} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 v_x \mathbf{e}_x + \mathbf{b} \Rightarrow 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{d^2 v_x}{dy^2} - g\cos\alpha,\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\sin\alpha$$

The solution to the two partial differential equations that also satisfies the boundary conditions is:

$$v_x(y) = v_o - \frac{\rho g \sin\alpha\, h^2}{2\mu}\left[1 - \left(1 - \frac{y}{h}\right)^2\right]$$
$$\sigma_x = \sigma_y = \sigma_z = -p = -p_a - \rho g \cos\alpha\,(h - y)$$
$$\tau_{zy} = -\rho g \sin\alpha\,(h - y),\ \tau_{xy} = \tau_{zx} = 0$$

The volumetric flow Q of fluid volume transported by the conveyer belt per unit time is calculated from:

$$Q = w\int_0^h v_x(y)\,dy = w\int_0^h \left\{v_o - \frac{\rho g \sin\alpha\, h^2}{2\mu}\left[1 - \left(1 - \frac{y}{h}\right)^2\right]\right\} dy \quad \Rightarrow$$
$$Q = v_o w h - \frac{\rho g \sin\alpha\, w h^3}{3\mu}$$

Example 8.6. Laminar Flow in Pipes

An incompressible Newtonian fluid flows through a circular cylindrical pipe of internal diameter d, see Fig. 8.4.6. The flow is driven by a constant modified pressure gradient in the axial direction z. The flow is assumed to be laminar and steady, with streamlines parallel to the axis of the pipe and with the velocity field:

$$v_z = v(R),\ v_R = v_\theta = 0 \tag{8.4.25}$$

In order to find the velocity function $v(R)$ we shall use the Navier-Stokes equations (8.4.24) in cylindrical coordinates, as given in Appendix B. First we note the

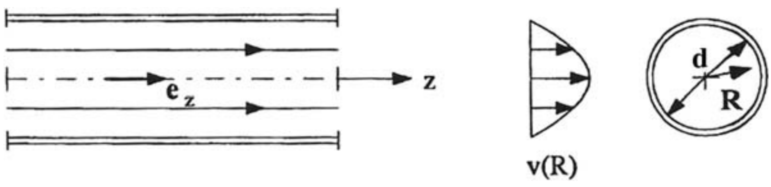

Fig. 8.4.6 Laminar pipe flow

particle acceleration is zero. Then the Navier-Stokes equations expressed in cylindrical coordinates are reduced to:

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial R},\ 0 = -\frac{1}{\rho R}\frac{\partial P}{\partial \theta},\ 0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu}{\rho R}\frac{\partial}{\partial R}\left(R\frac{\partial v}{\partial R}\right)$$

When these equations are integrated we end up with two integration constants. These are found by requiring a finite velocity at the center of the pipe and by assuming that the fluid sticks to the pipe wall:

$$v(0) \neq \infty,\ v(d/2) = 0$$

The solution of the partial differential equations is then:

$$P = P(z) = p_o - cz,\ v = \frac{d^2}{16\mu}c\left[1 - \left(\frac{2R}{d}\right)^2\right]$$

The parameter c is the constant negative modified pressure gradient in the direction of the flow. The velocity function may alternatively be expressed in terms of the maximum v_o:

$$v_z(R) \equiv v(R) = v_o\left[1 - \left(\frac{2R}{d}\right)^2\right],\ v_o = \frac{d^2}{16\mu}c \qquad (8.4.26)$$

The velocity profile is shown in Fig. 8.4.6.

According to the general expressions (5.4.19) for the rates of strain and rates of shear in cylindrical coordinates the assumed velocity field (8.4.25) provides only one non-zero value:

$$\dot{\gamma}_{zR} = \frac{dv}{dz}$$

The constitutive equations (8.4.7) then give the stresses:

$$\sigma_R = \sigma_\theta = \sigma_z = -p,\ \tau_{zR} = \mu\frac{dv_z}{dR} = -\frac{c}{2}R,\ \tau_{R\theta} = \tau_{\theta z} = 0 \qquad (8.4.27)$$

8.4.3 Dissipation

The viscosity results in dissipation of mechanical energy in a flowing fluid, i.e. mechanical energy is converted to heat and internal energy. We now compute the stress power per unit volume for a Newtonian fluid. Using the constitutive (8.4.6), we get:

$$\omega = \mathbf{T}:\mathbf{D} = -p\,\mathrm{div}\,\mathbf{v} + 2\mu\,\mathbf{D}:\mathbf{D} + \left(\kappa - \frac{2\mu}{3}\right)(\mathrm{tr}\,\mathbf{D})^2 \qquad (8.4.28)$$

A decomposition of the stress tensor **T** and the rate of deformation tensor **D** into isotrops and deviators, using the expressions (8.4.8) and (8.4.9) yields:

$$\omega = \mathbf{T} : \mathbf{D} = -p \operatorname{div} \mathbf{v} + 2\mu \mathbf{D}' : \mathbf{D}' + \kappa (\operatorname{div} \mathbf{v})^2 \tag{8.4.29}$$

For a fluid with barotropic pressure: $p = p(\rho)$, we introduce the specific elastic energy ψ defined by (8.3.11). Using the continuity (8.2.26), we obtain from (8.3.11):

$$\rho \, \dot{\psi} = p \, \frac{\dot{\rho}}{\rho} = -p \operatorname{div} \mathbf{v} \tag{8.4.30}$$

The expression (8.4.28) may then be rewritten to:

$$\omega = \mathbf{T} : \mathbf{D} = \rho \dot{\psi} + 2\mu \mathbf{D} : \mathbf{D} + \left(\kappa - \frac{2}{3}\mu \right) (\operatorname{tr} \mathbf{D})^2 \tag{8.4.31}$$

The first term on the right-hand side of this expression for the stress power per unit volume represents the recoverable part of the mechanical energy, while the two last terms, which are seen always to be positive, represent a loss or dissipation of mechanical energy.

The *viscous-dissipation function* is a positive semidefinite scalar-valued function and for any fluid defined by:

$$\delta = \omega - (-p \operatorname{div} \mathbf{v}) = \omega - p \frac{\dot{\rho}}{\rho} \tag{8.4.32}$$

For a fluid with barotropic pressure we get:

$$\delta = \omega - (-p \operatorname{div} \mathbf{v}) = \omega - p \frac{\dot{\rho}}{\rho} = \omega - \rho \, \dot{\psi} \tag{8.4.33}$$

For a Newtonian fluid the dissipation function becomes:

$$\delta = 2\mu \mathbf{D} : \mathbf{D} + \left(\kappa - \frac{2\mu}{3} \right) (\operatorname{tr} \mathbf{D})^2 = 2\mu \mathbf{D}' : \mathbf{D}' + \kappa (\operatorname{div} \mathbf{v})^2 \tag{8.4.34}$$

For a fluid with barotropic pressure the mechanical energy balance (6.1.12) for a fluid body with volume V may be presented as:

$$P = \dot{K} + \dot{\Psi} + \Delta \tag{8.4.35}$$

Ψ is the elastic energy of the body, confer (8.3.14), and Δ is the dissipation in the fluid body and given by:

$$\Delta = \int_V \delta \, dV \tag{8.4.36}$$

8.4.4 The Energy Equation

The general balance of thermal energy for a place or a particle is given (6.3.14):

$$\rho\dot{\varepsilon} = -\operatorname{div}\mathbf{h} + \mathbf{T} : \mathbf{D} \tag{8.4.37}$$

The equation will now be developed further for a Newtonian fluid. The specific internal energy may be replaced by the *specific enthalpy h* through thee relationship:

$$h = \varepsilon + \frac{p}{\rho} \tag{8.4.38}$$

p is the thermodynamic pressure. The heat flux vector $\mathbf{h}$ is expressed by the *Fourier heat conduction equation*, named after Jean Baptiste Joseph Fourier [1768–1830]:

$$\mathbf{h} = -k\nabla\theta \tag{8.4.39}$$

The parameter k is a temperature dependent *heat conduction coefficient*. In many cases k is taken to be a constant parameter. From the definition (8.4.38) we obtain:

$$\dot{\varepsilon} = \dot{h} - \frac{\dot{p}}{\rho} + p\frac{\dot{\rho}}{\rho^2} \tag{8.4.40}$$

For the last term in the energy (8.4.37) we use the definition (8.4.32) to write:

$$\mathbf{T} : \mathbf{D} = \omega = \delta + p\frac{\dot{\rho}}{\rho} \tag{8.4.41}$$

The results (8.4.39, 8.4.40, 8.4.41) are now substituted into the energy (8.4.37), and we obtain the alternative form of the thermal energy equation:

$$\rho\dot{h} = \dot{p} + \nabla\cdot(k\nabla\theta) + \delta \tag{8.4.42}$$

For an incompressible fluid we introduce the *specific heat at constant pressure*:

$$c_p = \left.\frac{\partial h}{\partial\theta}\right|_{p=\text{constant}} \tag{8.4.43}$$

It may be shown that for a gas the term $\dot{p}$ in (8.4.42) may be neglected if incompressibility is assumed. The energy equation for an incompressible gas then takes the form:

$$\rho\, c_p\,\dot{\theta} = \nabla\cdot(k\nabla\theta) + \delta \tag{8.4.44}$$

For a Newtonian fluid the dissipation function δ is given by (8.4.34).

If we had started with the energy equation in the form (8.4.37) and then assumed that the fluid was incompressible, we would have obtained the alternative energy (8.4.44) with c_p replaced by the *specific heat at constant volume, or constant density*:

$$c_v = \left.\frac{\partial \varepsilon}{\partial \theta}\right|_{\rho=\text{constant}} \tag{8.4.45}$$

For a liquid, which is nearly an incompressible material, $c_v = c_p$. It is customary for an incompressible liquid to replace both c_p and c_v by a common *specific heat c*.

A commentary to the definition of the specific heat may be of interest at this point. The specific heat c represents the heat that must be supplied per unit mass and per unit of temperature. From the thermal energy (8.4.37) we obtain:

$$c = \frac{\operatorname{div} \mathbf{h}}{\rho \dot{\theta}} = \frac{\dot{\varepsilon}}{\dot{\theta}} - \frac{\mathbf{T} : \mathbf{D}}{\rho \dot{\theta}} \tag{8.4.46}$$

For a perfect fluid we find, using (8.3.10) and (8.4.40), that:

$$c = \frac{\dot{\varepsilon}}{\dot{\theta}} - \frac{p \dot{\rho}}{\rho^2 \dot{\theta}} = \frac{\dot{h}}{\dot{\theta}} - \frac{\dot{p}}{\rho\, \dot{\theta}} \tag{8.4.47}$$

At constant volume, or constant density, i.e. $\dot{\rho} = 0$:

$$c = c_v = \left.\frac{\dot{\varepsilon}}{\dot{\theta}}\right|_{\rho=\text{constant}} = \left.\frac{\partial \varepsilon}{\partial \theta}\right|_{\rho=\text{constant}} \tag{8.4.48}$$

At constant pressure:

$$c = c_p = \left.\frac{\dot{h}}{\dot{\theta}}\right|_{p=\text{constant}} = \left.\frac{\partial h}{\partial \theta}\right|_{p=\text{constant}} \tag{8.4.49}$$

8.4.5 The Bernoulli Equation for Pipe Flow

An incompressible Newtonian fluid flows through a pipe. It is assumed that the flow is laminar and steady, and that the fluid sticks to the pipe wall. As shown in Fig. 8.4.7 the pipe is cylindrical at the two positions where the cross-sections are A_1 and A_2. It then follows from Example 8.6 that we may assume at these positions that the stream lines are parallel to the axis of the pipe. We shall derive a Bernoulli equation from the equation of balance of mechanical energy (8.2.23) for a control volume V

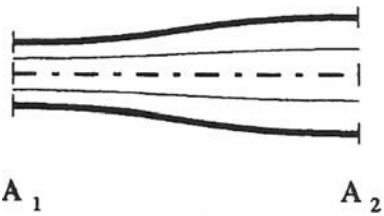

Fig. 8.4.7 Pipe flow

defined by the fluid body in the pipe and between the cross-sections A_1 and A_2. The control surface A consists then of the pipe wall A_w and the cross-sectionsA_1 and A_2.

For the body force we set $\mathbf{b} = -\nabla\beta$, and due to the condition of incompressibility: $\nabla \cdot \mathbf{v} = 0$, we can write: $\mathbf{b} \cdot \mathbf{v}\rho = -\nabla \cdot (\beta \mathbf{v}\rho)$. By the divergence theorem C.4 we get:

$$\int_V \nabla \cdot (\beta\,\mathbf{v}\rho)\,dV = \int_A \beta\,\rho\,\mathbf{v} \cdot \mathbf{n}\,dA \tag{8.4.50}$$

At the pipe wall A_w the fluid velocity $\mathbf{v}$ is zero, which implies that $\mathbf{t} \cdot \mathbf{v} = 0$ on A_w. Based on the result (8.4.27) in Example 8.6 we assume that on the cross-sections A_1 and A_2: $\mathbf{t} \cdot \mathbf{v} = -p\mathbf{v} \cdot \mathbf{n}$. Equation (8.4.32) yields: $\mathbf{T} : \mathbf{D} \equiv \omega = \delta$. It now follows that the energy (8.2.23) for the control volume V with the control surface $A = A_w + A_1 + A_2$ can be presented as:

$$\left[\int_A \left(\frac{v^2}{2} + \frac{p}{\rho} + \beta\right)\rho\,\mathbf{v} \cdot \mathbf{n}\,dA\right]_{A_2}^{A_1} = \Delta \equiv \int_V \delta\,dV = \int_V \mathbf{T} : \mathbf{D}\,dV = \int_V 2\mu\,\mathbf{D} : \mathbf{D}\,dV \tag{8.4.51}$$

Confer (8.3.21). The Navier-Stokes equation (8.4.20) may be presented as:

$$\dot{\mathbf{v}} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{v} - \nabla\beta = -\nabla\left(\frac{p}{\rho} + \beta\right) + \frac{\mu}{\rho}\nabla^2\mathbf{v}$$

At the cross-sections A_1 and A_2 we assume zero particle acceleration and zero velocity gradient. The Navier-Stokes equation then implies:

$$\nabla\left(\frac{p}{\rho} + \beta\right) = 0 \quad \Rightarrow \quad \frac{p}{\rho} + \beta = \text{constant over the cross-sections } A_1 \text{ and } A_2 \tag{8.4.52}$$

Hence:

$$\int_A \left(\frac{p}{\rho} + \beta\right)\rho\,\mathbf{v} \cdot \mathbf{n}\,dA = \left(\frac{p}{\rho} + \beta\right)\rho\,Q,\ Q = \int_A \mathbf{v} \cdot \mathbf{n}\,dA \text{ on } A = A_1 \text{ or } A_2$$

Q is the *volumetric flow*. We introduce the dimensionless parameter α such that:

$$\int_A \frac{v^2}{2}\rho\,\mathbf{v} \cdot \mathbf{n}\,dA = \left(\alpha\frac{v_m^2}{2}\right)\rho\,Q,\ v_m = \frac{Q}{A} = \text{ mean velocity over cross-section } A$$

Equation (8.4.51) may now be transformed into the *Bernoulli equation*:

$$\left[\alpha\frac{v_m^2}{2} + \frac{p}{\rho} + \beta\right]_{A_1} - \left[\alpha\frac{v_m^2}{2} + \frac{p}{\rho} + \beta\right]_{A_2} = \frac{\Delta}{\rho Q} \tag{8.4.53}$$

The right-hand side of this equation is called the loss term.

We shall compute the parameter α for a cylindrical pipe with circular cross section of diameter d. The velocity distribution over the cross section is given by (8.4.26). Thus:

$$\int_A \frac{v^2}{2}\rho\,\mathbf{v}\cdot\mathbf{n}\,dA = \int_0^{d/2} \frac{\rho\,v^3(R)}{2}\,2\pi R\,dR = \left(2\frac{v_m^2}{2}\right)\rho\,Q,\ v_m = \frac{v_o}{2}$$

Thus $\alpha = 2$ for circular pipes with laminar flow. For the same case we shall compute the loss term along a pipe of length L and with constant diameter d. The deformation rate matrix D in cylindrical coordinates is given by formulas (5.4.18–19). The velocity field (8.4.26) provides this rate of deformation matrix:

$$D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{1}{2}\frac{dv}{dR},\ \frac{dv}{dR} = -\frac{16\,v_m}{d^2}R$$

Then from the definition of Δ in (8.4.51) we get:

$$\Delta = \int_V 2\mu\,\mathbf{D}:\mathbf{D}\,dV = L\int_0^{d/2} 2\mu\left[\left(-\frac{1}{2}\frac{16v_m}{d^2}R\right)^2 2\right](2\pi R\,dR) = 32\frac{\mu\,v_m L}{d^2}Q$$

The loss term becomes:

$$\frac{\Delta}{\rho\,Q} = 32\frac{\mu\,v_m L}{d^2} \tag{8.4.54}$$

For steady turbulent flow in a pipe it is experimentally found that the time-averaged velocity is practically constant over the cross section, which means that $\alpha \approx 1$. The loss term for turbulent flow in a pipe of length L and with constant diameter d, is presented as:

$$\frac{\Delta}{\rho\,Q} = \lambda\,\frac{L}{d}\frac{v_m^2}{2} \tag{8.4.55}$$

The parameter λ has to be determined experimentally. It is found that the parameter λ depends on the *Reynolds number* $Re = v_m d/(\mu/\rho)$ and the roughness of the inside surface of the pipe. The last effect dominates, and it is customary to consider λ independent of Re in the case of turbulent flow.

For the special case of laminar flow discussed above, the loss term may also be computed from the Bernoulli (8.4.53). From (8.4.26) in Example 8.6 we obtained for the modified pressure gradient in the flow direction: $c = 16\mu v_o/d^2$. Using (8.4.23) we obtain:

$$cL = \frac{16\mu\,v_o L}{d^2} = 32\frac{\mu\,v_m L}{d^2} = P_{A_1} - P_{A_2} = [p+\rho\beta]_{A_1} - [p+\rho\beta]_{A_2} \quad\Rightarrow$$

$$[p+\rho\beta]_{A_1} - [p+\rho\beta]_{A_2} = 32\frac{\mu\,v_m L}{d^2} \tag{8.4.56}$$

Since the mean velocity over the cross-sections A_1 and A_2 are the same, the result (8.4.54) follows from (8.4.56) and the Bernoulli equation (8.4.53).

8.5 Potential Flow

It was shown in Sect. 8.3 that under the assumptions: 1) barotropic fluid and 2) conservative body forces, a fluid will remain in irrotational flow if the fluid has at any time been in irrotational flow. These conditions exist when for instance fluid flows from a reservoir as shown in Fig. 8.3.2. The same conditions are approximately satisfied in a flow created by a rigid body moving in fluid originally at rest. A similar case occurs in a fluid flow around a rigid body at rest if the fluid before it meets the body and far away from the body, has parallel stream lines and constant velocity, i.e. the fluid is in *uniform flow*. As illustrated in Fig. 8.5.1, the two last types of flows are essentially the same: The flow in the second case is obtained from the first if the motion is referred to the rigid body. Obviously the vorticity is zero in the uniform flow. The flow in the vicinity of rigid surfaces must be corrected by a boundary layer analysis, and the wake must be excluded from the irrotational flow analysis.

Irrotational flow is also called *potential flow* because the velocity field in the flow may be derived from a scalar valued function of position $\phi(\mathbf{r},t)$, called the *velocity potential*, such that:

$$\mathbf{v} = \nabla\phi \tag{8.5.1}$$

For potential flow the continuity (8.2.26) may be rewritten to:

$$\dot{\rho} + \rho\nabla^2\phi = 0 \tag{8.5.2}$$

In this section we shall assume that the fluid is incompressible. The continuity (8.5.2) is then reduced to the Laplace equation:

$$\nabla^2\phi = 0 \tag{8.5.3}$$

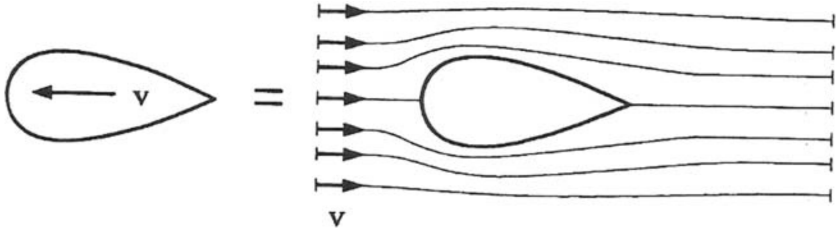

Fig. 8.5.1 Potential flow. Flow created by a rigid body moving in a fluid originally at rest is equivalent to the fluid flow around a rigid body at rest approached by a uniform flow

Solutions of this equation provide possible potential flows of incompressible fluids. The boundary conditions for ϕ are determined from the fact that the velocity component normal to rigid surfaces must be zero. Let $\mathbf{n}$ be the unit normal to a rigid surface. Then:

$$\mathbf{n}\cdot\nabla\phi = \frac{d\phi}{dn} = 0 \text{ on rigid surfaces} \tag{8.5.4}$$

The differential equation (8.5.3) and the boundary condition (8.5.4) are linear, which means that we may superimpose known solutions to obtain new solutions. This is demonstrated in Example 8.10 below. The theory of potential flows is highly developed and mathematically extensive. In two-dimensional flows conform mapping may be applied by which the real flow is transformed mathematically to a flow around a rigid cylinder. The reader is referred to the Fluid Mechanics literature for further presentation of this application.

Example 8.7. Uniform Flow

In a uniform flow: v_1 =constant and $v_2 = v_3 = 0$, referred to a Cartesian coordinate system Ox, the velocity potential is: $\phi = v_1 x_1$.

Example 8.8. The Potential Vortex

In Example 8.4 we found that the fluid flow created by a vertical cylinder of radius a rotating with a constant angular velocity ω in a cylindrical container of inner radius b, becomes a potential flow if the radius of the container is very large, i.e. $b\to\infty$, see (8.4.18). Now we introduce a constant parameter Γ such that the velocity potential and the corresponding velocity field are:

$$\phi = \frac{\Gamma\theta}{2\pi},\; v_\theta = \frac{\Gamma}{2\pi R},\; v_R = v_z = 0$$

The parameter Γ represents the circulation along any closed curve around the z-axis. To see this we compute the circulation along a circle C with center on the z-axis and of radius R:

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \int_0^{2\pi} \frac{\Gamma}{2\pi R} R\,d\theta = \Gamma = \text{constant} \tag{8.5.5}$$

This result seams to be in contradiction with what is implied by (8.3.37): Since the vorticity is zero in a potential flow, $\mathbf{c} = \nabla\times\mathbf{v} = \mathbf{0}$, (8.3.37) implies that the circulation Γ is zero. However, the result (8.3.37) is based on the application of Stokes' theorem C.5, which requires that the velocity $\mathbf{v}$ is regular on the integration surface A. This condition is not satisfied for the present flow because any surface bounded by C will intersect the z-axis, at which $\mathbf{v} = \infty$.

By proper choice of the integration path the result (8.5.5) may easily be generalized to apply to any closed curve around the z-axis.

Example 8.9. Rigid Cylinder in Uniform Flow

Potential flow around a rigid cylinder of radius a, as shown in Fig. 8.5.2, is represented by the velocity potential:

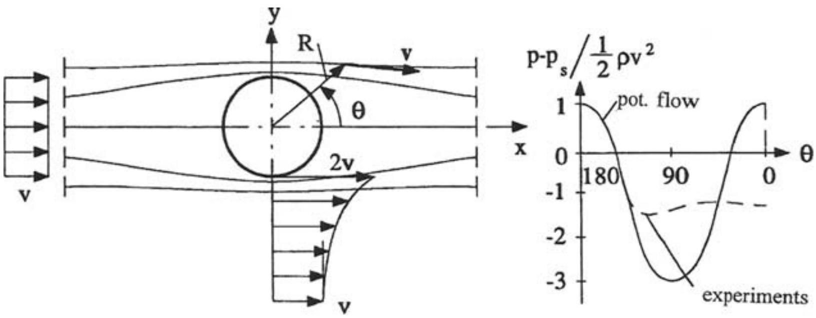

Fig. 8.5.2 Rigid cylinder in uniform flow

$$\phi = v\left[x + a^2\frac{\cos\theta}{R}\right] \equiv v\left[R + \frac{a^2}{R}\right]\cos\theta$$

v is the velocity of a uniform flow far away upstream from the cylinder.

The velocity field is found by applying (8.5.1) and (2.4.18) for the del-operator in cylindrical coordinates:

$$v_R = \frac{\partial\phi}{\partial R} = v\left[1 - \left(\frac{a}{R}\right)^2\right]\cos\theta,\ v_\theta = \frac{1}{R}\frac{\partial\phi}{\partial\theta} = -v\left[1 + \left(\frac{a}{R}\right)^2\right]\sin\theta$$

The Euler pressure equation (8.3.24) provides the pressure against the cylinder wall. First we introduce the modified pressure P from (8.4.23):

$$P = p - p_s = p + \rho\beta - p_o$$

p_s is the static pressure and p_o is a reference pressure. When the modified pressure is substituted into (8.3.24), we obtain an alternative form of the *Euler pressure equation*:

$$\frac{v^2}{2} + \frac{P}{\rho} = \text{ constant in the fluid} \tag{8.5.6}$$

In this general equation v is the fluid velocity at the chosen place in the fluid. In the present case the pressure p far away from the rigid cylinder is equal to the static pressure p_s and the modified pressure P is zero. Hence, when the pressure equation (8.5.6) is applied to the present flow example, we get for the modified pressure P on the rigid cylinder:

$$\frac{v^2}{2} + \frac{0}{\rho} = \frac{v_R^2 + v_\theta^2}{2} + \frac{P}{\rho} \quad \Rightarrow \quad P = p - p_s = \frac{\rho\, v^2}{2}\left(1\text{–}4\sin^2\theta\right)$$

Figure 8.5.2 shows how this theoretical pressure deviates from the pressure obtained experimentally. The theoretical velocity field and the corresponding pressure are only realistic near the front of the cylinder. The reason for this is the creation of the wake downstream, which is highly rotational. Confer Fig. 8.1.3.

Example 8.10. Rotating Cylinder in Uniform Flow

A long cylinder is lying on a horizontal table, as illustrated in Fig. 8.5.3. Around the cylinder is wound a tape. We pull the tape with a force T and thereby give the cylinder a horizontal velocity v and an angular velocity ω. Referred to the cylinder the air approaches the cylinder from a uniform flow with constant velocity v. The rotation of the cylinder creates a potential vortex as described in Example 5.2, Example 8.4, and Example 8.8. We will experience that the cylinder is subjected to a lifting force counteracting the gravitational force. In fact the cylinder may lift itself and rise higher then the level of the table it has left. The lift is obviously due to the rotation of the cylinder and thereby the circulation created by this rotation. The lifting effect is called the *Magnus effect*, named after Gustav Heinrich Magnus [1802–1870].

The Magnus effect is also present when balls in sports, like tennis, golf, and soccer, are thrown, kicked, or batted with a rotation. The ball may be given a higher or flatter vertical path, or a path curving to the left or to the right. The present example will give a theoretical explanation to the Magnus effect.

We shall compute the lifting force on a rigid cylinder in a flow described by the velocity potential:

$$\phi = v\left[R + \frac{a^2}{R}\right]\cos\theta - \frac{\Gamma\theta}{2\pi}, \; \Gamma = 2\pi\omega a^2$$

This potential is obtained by superposition of the velocity potentials in Example 8.9 and Example 8.8. Note however that the sense of ω is opposite in Example 8.8 and in the present example. The velocity field is obtained directly by addition of the velocities in the two examples:

$$v_R = v\left[1 - \left(\frac{a}{R}\right)^2\right]\cos\theta, \; v_\theta = -v\left[1 + \left(\frac{a}{R}\right)^2\right]\sin\theta - \frac{\Gamma}{2\pi R}$$

The modified pressure P on the rigid cylinder is obtained from the Euler pressure equation (8.5.6):

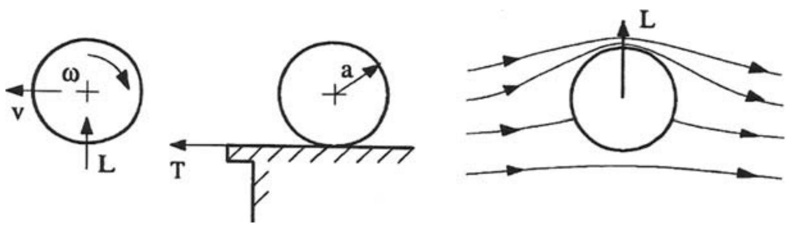

Fig. 8.5.3 The Magnus effect. Lifting force L due to the combined rotation and translation of the cylinder

$$\frac{v^2}{2}+\frac{0}{\rho}=\frac{v_R^2+v_\theta^2}{2}+\frac{P}{\rho} \quad \Rightarrow \quad P=p-p_s=\frac{\rho v^2}{2}\left[1-\left(2\sin\theta+\frac{\Gamma}{2\pi a v}\right)^2\right]$$

This pressure gives the resulting vertical lift force:

$$L=-\int_0^{2\pi} P\sin\theta\, a\, d\theta=\Gamma\rho v$$

8.5.1 The D'alembert Paradox

It may be shown theoretically that any body moving with constant velocity through a barotropic fluid, originally at rest, will not be subjected to a resulting force from the fluid. This phenomenon is called *the d'Alembert paradox* after Jean Le Rond d'Alembert [1717–1783]. Two-dimensional potential theory gives another result, as we just have seen in Example 8.10 above. In general we shall find that a rigid body in an originally uniform flow is subjected to a lift force L in the direction normal to the uniform flow which is given by the expression:

$$L=\Gamma\rho v_o \tag{8.5.7}$$

v_o is the velocity of the uniform flow and Γ is the circulation around any closed curve surrounding the two-dimensional body. This fascinating result is called *the Kutta-Joukowsky theorem* after Wilhelm Kutta [1867–1944] and Nikolai Egorovich Joukowsky [1847–1921].

8.6 Non-Newtonian Fluids

8.6.1 Introduction

Viscous fluids that do not follow Newton's law of fluid friction, (8.4.4) are called non-Newtonian fluids. These fluids are usually highly viscous fluids and their elastic properties are also of importance. The viscoelastic fluids discussed in Chap. 9 are also characterized as non-Newtonian. Typical real non-Newtonian fluids are polymer solutions, thermo plastics, drilling fluids, paints, fresh concrete and biological fluids. The theory of non-Newtonian fluids is called rheology.

The term rheology was invented in 1920 by Professor Eugene C. Bingham at Lafayette College in Indiana, USA. Bingham who was a professor of Chemistry, studied new materials with strange flow behavior, in particular paints. The syllable Rheo is from the Greek word "rhein", meaning flow, so the name rheology was taken to mean *the theory of deformation and flow of matter*. Rheology has also come to include the constitutive theory of highly viscous fluids and solids exhibiting viscoelastic and viscoplastic properties.

Materials in the solid state can behave fluid-like under special conditions. Plastic deformation of solids at yield and creep may be considered to be fluid-like behavior. At high temperatures ($> 400\,^\circ\mathrm{C}$) common structural steel shows creep and stress relaxation. In many simulations of forming processes with metals and polymers the material is modelled as a fluid although the temperature is below the melting temperature of the material.

Fluid models may be classified into three main groups:

A. Time independent fluids for which the fluid properties are explicitly independent of time.

  A1. Viscoplastic fluids. Two examples are presented in Sect. 8.6.2, and more examples are presented in Sect. 10.11.
  A2. Purely viscous fluids. Some examples are presented below, and some advanced models are discussed in the Sect. 11.9.

B. Time dependent fluids. The constitutive modeling of these fluids is very complex in will not be treated in this book.

  B1. Thixotropic fluids. For constant deformation rates the stresses in a thixotropic fluid decrease monotonically.
  B2. Rheopectic fluids or antithixotropic fluids. For constant deformation rates the stresses in a rheopectic fluid increase monotonically.

C. Viscoelastic fluids. Linear and non-linear models are discussed in Chap. 9.

8.6.2 Generalized Newtonian Fluids

The most commonly used models for incompressible non-Newtonian fluids are called *generalized Newtonian fluids*. The constitutive equation defining this fluid model is:

$$\mathbf{T} = -p\mathbf{1} + 2\eta\mathbf{D} \tag{8.6.1}$$

p is the pressure p ($\mathbf{r}$,t) and η, called the *viscosity function*, is a function of the *magnitude of shear rate* or the *shear rate measure* $\dot{\gamma}$:

$$\eta = \eta(\dot{\gamma}),\ \dot{\gamma} = \sqrt{2\mathbf{D}:\mathbf{D}} \equiv 2\sqrt{-II_D} \tag{8.6.2}$$

II_D is the 2. principal invariant of the rate of deformation tensor $\mathbf{D}$. This model is called a *purely viscous fluid* because the stress tensor depends solely on the rate of deformation tensor. The viscosity function, which is temperature dependent, is determined in experiments with simple shear flow, for instance as described in Sect. 1.3. In a simple shear flow the shear measure reduces to the absolute value of the rate of shear strain:

$$v_x = c(t)y,\ v_y = v_z = 0 \quad \Rightarrow \quad \dot{\gamma} = \left| \frac{dv_x}{dy} \right| = |c(t)| \tag{8.6.3}$$

Figure 8.6.1 shows a characteristic experimental curve of the viscosity function of real fluids. If η is constant and independent of the shear measure, (8.6.1) represents an incompressible Newtonian fluid, with $\eta \equiv \mu$ the shear viscosity.

Different analytical functions for the viscosity functions represent different fluid models, all of which are generalized Newtonian fluids. Some of the most common of these are presented in the following.

a) POWER-LAW FLUID (W. Ostwald 1925, A. de Waele 1923):

$$\eta = K\dot{\gamma}^{n-1} \tag{8.6.4}$$

K and n are temperature dependent material parameters. K is called the *consistency parameter* and n is the *power-law index*. Table 8.6.1 gives examples of values of K and n for some real fluids. It is often practical to set:

$$K = K_o \exp\left[-A\left(\theta - \theta_o\right)\right],\ n = \text{constant} \tag{8.6.5}$$

θ is the temperature and K_o, A, and the temperature θ_o are reference values. The power-law fluid has the weakness that it cannot fit the experimental curve of the viscosity function for very small and very large values of the shear measure. However, the model is relatively easy to work with in analytical solutions. For most real non-Newtonian fluids $n < 1$, see Fig. 8.6.1. The viscosity of the fluid decreases with increasing shear measure and the fluid is therefore called *shear-thinning*. For $n > 1$ the viscosity increases with increasing shear measure $\dot{\gamma}$, and the fluid is called *shear-thickening* or *dilatant*. The latter name is due to the fact that such a fluid normally also expands when deformed.

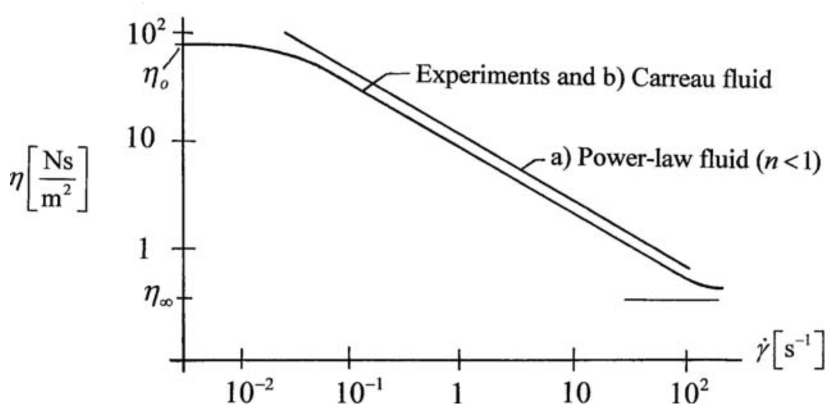

Fig. 8.6.1 The viscosity function

Table 8.6.1 Consistency K and power-law index n for some fluids

Fluid	Region of $\dot{\gamma}$ [s^{-1}]	$K[Ns^n/m^2]$	n
54.3% cement rock in water, 300 °K	10–200	2.51	0.153
23.3% Illinois clay in water, 300 °K	1800–6000	5.55	0.229
Polystyrene, 422 °K	0.03–3	$1.6 \cdot 10^5$	0.4
Tomato Concentrate, 90 °F 30% solids		18.7	0.4
Applesauce, 80 °F 11.6% solids		12.7	0.28
Banana puree, 68 °F		6.89	0.46

b) CARREAU FLUID (P.J. Carreau 1968):

$$\eta = \eta_\infty + (\eta_o - \eta_\infty)\left[1 + (\lambda\dot{\gamma})^2\right]^{(n-1)/2} \tag{8.6.6}$$

$\eta_\infty = \eta(\infty)$, called the *infinite-shear-rate viscosity*, $\eta_o = \eta(0)$, called the *zero-shear-rate- viscosity*, and λ is a time constant. This model adjusts to the experimental curve in Fig. 8.6.1 very well for all $\dot{\gamma}$-values.

c) ZENER-HOLLOMON FLUID (Zener, C. and Hollomon, J.H. 1944):

$$\eta(\dot{\gamma},\theta) = \frac{1}{\sqrt{3}\,\alpha\,\dot{\gamma}}\text{arcsinh}\left[\left(\frac{Z}{A}\right)^{1/n}\right],\; Z = \dot{\gamma}\exp\left[\frac{Q}{R\theta}\right] \tag{8.6.7}$$

α, A, and n are material parameters, and θ is the absolute temperature. The material parameter Q is called the activation energy, and R is the universal gas constant. The parameter Z, called the *Zener-Hollomon parameter*, is a temperature compensated shear measure. The model has been applied in simulations of forming processes, for instance in extrusion of light metals.

The next two models are not really purely viscous fluids but rather *viscoplastic fluids*. Section 10.11 presents a further discussion of viscoplastic fluids.

d) BINGHAM FLUID (E.C. Bingham 1922):

$$\eta = \infty \text{ when} |\tau_{\max}| < \tau_y,\; \eta = \mu + \frac{\tau_y}{\dot{\gamma}} \text{when} |\tau_{\max}| \geq \tau_y \tag{8.6.8}$$

$\tau_{\max}$ is the maximum shear stress in the fluid particle. τ_y is called the *yield shear stress*. μ is a constant viscosity parameter. This fluid model will be discussed further in Sect. 10.11.2.

e) CASSON FLUID (N. Casson 1959):

$$\eta = \infty \text{ when} |\tau_{\max}| < \tau_y,\; \eta = \mu + \frac{\tau_y}{\dot{\gamma}} + 2\sqrt{\frac{\mu\tau_y}{\dot{\gamma}}} \text{ when} |\tau_{\max}| \geq \tau_y \tag{8.6.9}$$

This model was originally introduced to describe flow of mixtures of pigments and oil. The model is now often used to describe flow of blood for low values of the shear measure. For high values of $\dot{\gamma}$ blood behaves as a Newtonian fluid.

8.6.3 Viscometric Flows. Kinematics

Steady flow between to parallel plates and without modified pressure gradient, Fig. 8.6.2, is called *steady simple shear flow*. As this type of flow has some characteristic features common for many more complex flows important in applications, we shall take a closer look at the characteristic aspects of steady simple shear flow. The velocity field is:

$$v_1 = \dot{\gamma} x_2, \; v_2 = v_3 = 0, \; \dot{\gamma} = \frac{v}{h} = \text{ constant} > 0 \tag{8.6.10}$$

resulting in the deformation rate matrix and the magnitude of shear rate:

$$D = \frac{1}{2}\dot{\gamma}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \sqrt{2\mathbf{D}:\mathbf{D}} = \sqrt{2tr D^2 = j} \tag{8.6.11}$$

The flow has the following characteristic features:

a) The flow is isochoric: $\nabla \cdot \mathbf{v} = \text{tr}D = 0$.
b) Material planes parallel to the x_1x_3-plane moves in the x_1-direction without in-plane strains. We say that these planes represent a one-parameter family of *isometric surfaces*. The coordinate x_2 is the parameter defining each plane in the family. The word "isometric" is used to indicate that the distances between particles in each surface, measured in the surface, do not change during the flow. The isometric surfaces are called *shearing surfaces*.
c) The deformation rate matrix D is given by (8.6.11).
d) The magnitude of shear rate $\dot{\gamma}$ in (8.6.11) is constant.

The traces of two shearing surfaces are shown in Fig. 8.6.2. The particles in the upper surface have the velocity $v_{1,2} \cdot dx_2$ relative to the lower surface. The stream-

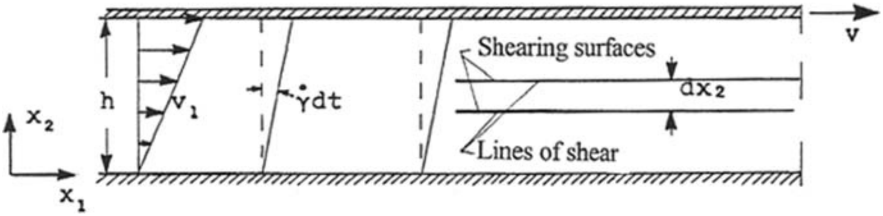

Fig. 8.6.2 Steady simple shear flow between a fixed plane and a moving parallel plate

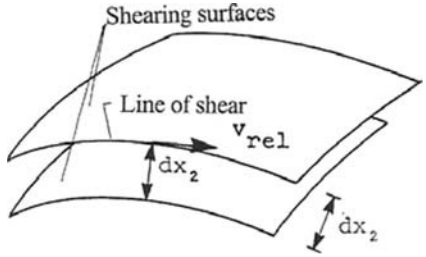

Fig. 8.6.3 Shear surfaces and line of shear

lines related to the velocity field $v_{1,2} \cdot dx_2$, when $dx_2 \to 0$, are called *lines of shear*. In the simple shear flow the shearing surfaces are planes and the lines of shear are straight lines parallel to the x_1-axis. Because the fluid particles are fixed to the same line of shear at all times, the lines of shear are material lines.

The general shear flow has features parallel to those of the simple steady shear flow. A flow is a *shear flow* if the following conditions are fulfilled:

a) The flow is isochoric: $\nabla \cdot \mathbf{v} = \mathrm{tr}D = 0$.

b) A one-parameter family of material surfaces exists that move isometrically, i.e. is without in-surface strains. These surfaces are called *shearing surfaces*, Fig. 8.6.3.

   The streamlines related to the velocity field v_{rel} of one shearing surface relative to a neighbor shearing surface are called *lines of shear*. The particles on one line of shear at the time t will not in general stay on the same line of shear at a later time. In other words, the lines of shear are not necessarily material lines. The condition a) implies zero strain rate normal to the shearing surfaces.

   A shear flow that in addition to the conditions a) and b) of a general shear flow, also satisfies the condition:

c) The lines of shear are material lines, is called a *unidirectional shear flow*. The material lines coinciding with the lines of shear at a particular time, will continue to be lines of shear as time passes. We may imaged that the lines of shear are "drawn" on the shearing surfaces and these material lines would than represent the lines of shear at later times. Unidirectional shear flow is the most common shear flow in applications and in particular in experiments designed to investigate the properties of non-Newtonian fluids.

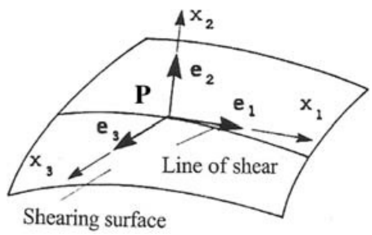

Fig. 8.6.4 Shear axes

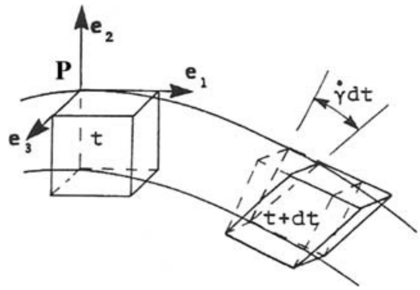

Fig. 8.6.5 Deformation of a fluid element

The analysis of the deformation kinematics of shear flows in the neighborhood of a particle P is simplified by introducing a local Cartesian coordinate system Px at the particle, as shown in Fig. 8.6.4. The coordinate axes are chosen such that the base vector $\mathbf{e}_1$ and $\mathbf{e}_3$ are tangents to the shearing surface, with $\mathbf{e}_1$ in the direction of the relative velocity $\mathrm{v}_{\mathrm{rel}}$ of the shearing surface relative to the neighbor shearing surface. The base vector $\mathbf{e}_1$ is thus tangent to the line of shear through the particle. The base vector $\mathbf{e}_2$ is normal to the shear surface. The three vectors $\mathbf{e}_i$ are called the *shear axes*, and the vector $\mathbf{e}_1$ is the *shear direction*.

A fluid element $dV = dx_1 dx_2 dx_3$ is during a short time interval dt deformed as indicated in Fig. 8.6.5. The deformation is govern by the shear strain rate $\dot{\gamma}_{12} = v_{1,2}$. The deformation rate matrix D in the Px-system is therefore equal to the deformation rate matrix (8.6.11) of a simple shear flow, and the magnitude of shear rate is $\dot{\gamma} = |\dot{\gamma}_{12}|$.

A unidirectional flow that also satisfies the condition:

d) For every particle the magnitude of shear rate $\dot{\gamma}$ is independent of time, is called a *viscometric flow*. Another name of this kind of flow is *rheological steady flow*. The flow is not necessarily a steady flow as defined in fluid mechanics. Rheological steady means that the magnitude of shear rate of the fluid is not changing with time. Viscometric flows play an important role in investigating the properties of non-Newtonian fluids. We shall now present a series of important viscometric flows and identify shearing surfaces, lines of shear, and shear axes for each flow.

Example 8.11. Steady Axial Annular Flow. Steady Pipe Flow

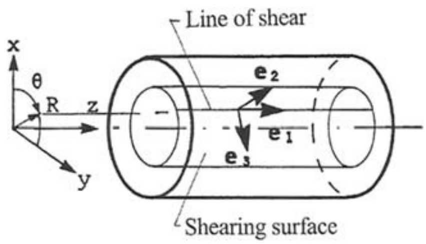

Fig. 8.6.6 Axial annular flow. Steady pipe flow

The fluid flows in the *annular space* between two solid, concentric cylindrical surfaces, or, as shown in Fig. 8.6.6, the fluid flows in a cylindrical pipe. The flow is steady and the velocity is parallel to the axis of the cylindrical surfaces:

$$v_z = v_z(R), \ v_R = v_\theta = 0 \tag{8.6.12}$$

The *shearing surfaces are concentric cylindrical surfaces*. The *lines of shear are straight lines* parallel to the axis of the cylindrical surfaces, and they coincide with the streamlines of the flow and with the pathlines of the fluid particles. The *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_z, \ \mathbf{e}_2 = \mathbf{e}_R, \ \mathbf{e}_3 = \mathbf{e}_\theta \tag{8.6.13}$$

The magnitude of shear rate is:

$$\dot{\gamma} = |\dot{\gamma}_{zR}| = \left|\frac{dv_z}{dR}\right| \tag{8.6.14}$$

Example 8.12. Steady Tangential Annular Flow

The fluid flows in the annular space between two concentric solid cylindrical surfaces. One of the solid surfaces rotates with a constant angular velocity ω. Figure 8.6.7 shows the case where the inner cylindrical surface rotates. The velocity field is:

$$v_\theta = v_\theta(R), \ v_z = v_R = 0 \tag{8.6.15}$$

The *shearing surfaces are concentric cylindrical surfaces*. The *lines of shear are circles* with constant R and z, and they coincide with streamlines of the flow and the pathlines of the particles. The *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_\theta, \ \mathbf{e}_2 = \mathbf{e}_R, \ \mathbf{e}_3 = -\mathbf{e}_z \tag{8.6.16}$$

The magnitude of shear rate is found from the formulas (5.4.19):

$$\dot{\gamma} = |\dot{\gamma}_{R\theta}| = \left|R\frac{d}{dR}\left(\frac{v_\theta}{R}\right)\right| \tag{8.6.17}$$

It is found that for a Newtonian fluid the velocity field presented by (8.6.15) is unstable when the angular velocity ω is increased above a certain limit. The instability introduces a secondary flow with velocities both in the z- and the R-directions and is described as *Taylor vortexes*. Instability and Taylor vortexes occur when:

$$T_a \equiv \left(\frac{\rho}{\mu}\omega\right)^2 r_i\,(r_o - r_i)^3 > 1700 \tag{8.6.18}$$

ρ is the density, μ is the viscosity, and r_i and r_o are the radii of the inner and outer solid boundary surfaces. T_a is called the Taylor number. At $T_a > 160 \cdot 10^3$ the flow becomes turbulent. Similar instabilities can occur for non-Newtonian fluids.

Example 8.13. Steady Torsion Flow

The fluid is set in motion between two plane concentric circular disks. One disk is at rest while the other disk rotates about its axis at a constant angular velocity ω. Figure 8.6.8 illustrates the situation. The dashed curved line indicates a free surface. In the case of a thick fluid this is really a free surface, while in the case of a thin fluid, the disks are submerged in a fluid bath. The rotating disk is touching the free surface of the bath and the dashed line marks an artificial free surface. Only the fluid between the disks is considered in the analysis.

The velocity field is assumed to be:

$$v_\theta = \frac{\omega}{h} R z, \; v_z = v_R = 0 \tag{8.6.19}$$

Based on the assumption that the fluid sticks to the solid disks, the velocity $v_\theta(R,z)$ satisfies the boundary conditions:

$$v_\theta\,(R,h) = \omega R, \; v_\theta\,(R,0) = 0 \tag{8.6.20}$$

The *shearing surfaces are planes normal to the axis of rotation*. The *lines of shear are concentric circles*, see Fig. 8.6.8b, and they coincide with the streamlines of the flow and the pathlines of the particles. Figure 8.6.8c shows an unfolded part of the cylinder surface $R \cdot dz$ between two shearing surfaces a distance dz apart. From the deformation of the fluid element shown in the figure we conclude that the *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_\theta, \; \mathbf{e}_2 = \mathbf{e}_z, \; \mathbf{e}_3 = \mathbf{e}_R \tag{8.6.21}$$

and that the magnitude of shear rate becomes, as seen from Fig. 8.6.8c:

$$\dot{\gamma} = |\dot{\gamma}_{\theta z}| = \left|\frac{\partial v_\theta}{\partial z}\right| = \frac{\omega}{h} R \tag{8.6.22}$$

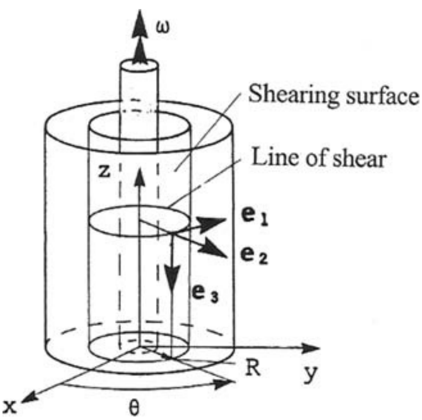

Fig. 8.6.7 Tangential annular flow

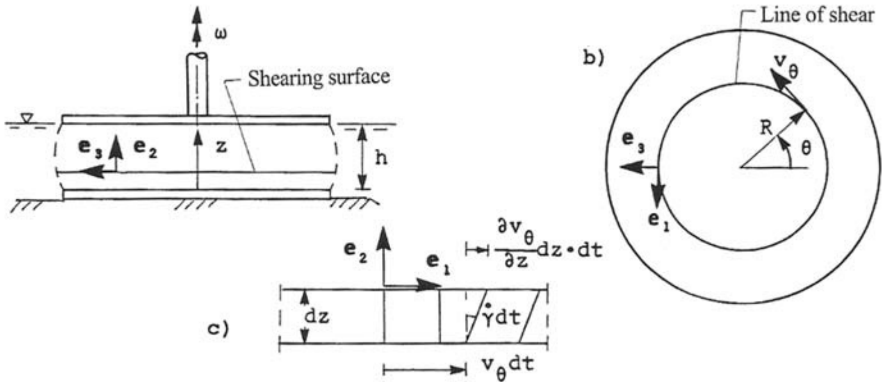

Fig. 8.6.8 Torsion flow

This result can is also be obtain from the formulas (5.4.19).

Example 8.14. Steady Helix Flow

The flow of the fluid in the annular space between to solid cylindrical surfaces is driven by the rotation and the axial translation of the inner cylindrical surface, see Fig. 8.6.9. The angular velocity ω and the axial velocity v are constants.

The velocity field is assumed as:

$$v_\theta = v_\theta(R),\ v_z = v_z(R),\ v_R = 0 \tag{8.6.23}$$

This kind of flow may also be obtained by a combination of a rotation of the inner cylinder and a constant modified pressure gradient $\partial P/dz$. The *shearing surfaces are concentric cylindrical surfaces*, which rotate and move in the axial direction. A fluid particle moves in a helix. Thus pathlines and streamlines are helices. A fluid particle on a shearing surface moves relative to a neighbor shearing surface also in a helix. Hence the *lines of shear are helices*, but they do not coincide with the

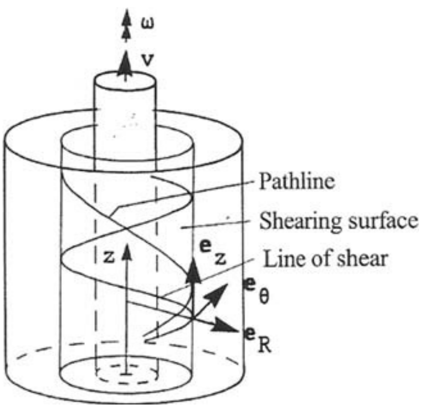

Fig. 8.6.9 Helix flow

streamlines or the pathlines. This is shown in Fig. 8.6.9. The rate of deformation matrix in cylindrical coordinates obtained from (5.4.18) and (5.4.19), contains only two independent elements for the flow (8.6.23):

$$\dot{\gamma}_{R\theta} = R\frac{d}{dR}\left(\frac{v_\theta}{R}\right),\ \dot{\gamma}_{zR} = \frac{dv_z}{dR} \tag{8.6.24}$$

The magnitude of shear rate becomes, see Problem 8.9:

$$\dot{\gamma} = \sqrt{2\mathbf{D}:\mathbf{D}} = \sqrt{2\left[\left(\frac{1}{2}\dot{\gamma}_{R\theta}\right)^2 2 + \left(\frac{1}{2}\dot{\gamma}_{zR}\right)^2 2\right]} \quad \Rightarrow$$

$$\dot{\gamma} = \sqrt{(\dot{\gamma}_{R\theta})^2 + (\dot{\gamma}_{zR})^2} \tag{8.6.25}$$

The shear axis normal to the shearing surface is $\mathbf{e}_2 = \mathbf{e}_R$. The *shear direction* and the third shear axis are found to be, Problem 8.9:

$$\mathbf{e}_1 = \frac{\dot{\gamma}_{R\theta}}{\dot{\gamma}}\mathbf{e}_\theta + \frac{\dot{\gamma}_{zR}}{\dot{\gamma}}\mathbf{e}_z,\ \mathbf{e}_3 = \frac{\dot{\gamma}_{zR}}{\dot{\gamma}}\mathbf{e}_\theta - \frac{\dot{\gamma}_{R\theta}}{\dot{\gamma}}\mathbf{e}_z \tag{8.6.26}$$

8.6.4 Material Functions for Viscometric Flows

Relations between stress components and deformation components, like strains and strain rates, in characteristic and simple flows are expressed by *material functions*. The *viscosity function* $\eta(\dot{\gamma})$ presented in Sect. 8.6.2 is an example of a material function for *unidirectional shear flows*. The characteristic flows for which the material functions are defined occur in standard experiments designed to investigate the properties of non-Newtonian fluids. In general the material functions may be functions of stresses, stress rates, strains, strain rates, temperature, time, and other parameters. The material functions are determined experimentally and are represented by data or mathematical functions representing these data.

In analyses of general flows fluid models are introduced. These models are defined by *constitutive equations*. A constitutive equation is a relationship between stresses and different measures of deformations, as strains, strain rates, and vorticities. A general constitutive equation is intended to represent a fluid in any flow, although it is experienced that most constitutive equations have limited application to only a few types of flows. The material functions may enter the constitutive equations or are used to determine material parameters in the constitutive equations. It might be a goal when constructing a fluid model that the constitutive equations of the model contain the material functions that is relevant for the special test flows that most resemblance the actual flow the fluid model is intended for.

We shall consider an *isotropic, incompressible, and purely viscous fluid* in a general viscometric flow, as described in Sect. 8.6.3. Figure 8.6.10a shows a particle

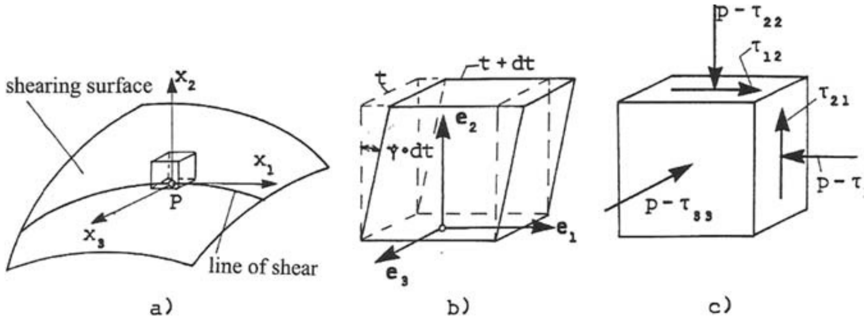

Fig. 8.6.10 Stresses in the viscometric flow

P and the shearing surface and the line of shear going through the particle. A local Cartesian coordinate system Px is introduced such that the base vectors $\mathbf{e}_i$ are the shear axes, see Fig. 8.6.10b. The rate of deformation matrix is given by (8.6.11), where $\dot{\gamma}$ is the magnitude of strain rate. The stress tensor $\mathbf{T}$ is decomposed into an isotropic part containing the pressure p, and a deviatoric extra stress $\mathbf{T}'$ due to the shear flow and assumed to be only a function of the magnitude of shear rate $\dot{\gamma}$ and the temperature θ. The temperature dependence will not be reflected implicitly in the following. Thus we set:

$$\mathbf{T} = -p\mathbf{1} + \mathbf{T}', \; T'_{ik} \equiv \tau_{ik}(\dot{\gamma}) \tag{8.6.27}$$

The condition of material isotropy implies that the state of stress must have the same symmetry as the state of deformation rate. With reference to Fig. 8.6.10 the x_1x_2-plane is a symmetry plane. This implies that the shear stresses $\tau_{13} = \tau_{31}$ and $\tau_{23} = \tau_{32}$ must be zero because these stresses act antisymmetrically with respect to the symmetry plane. The state of stress in the fluid is therefore given by the stress matrix:

$$T = (-p\delta_{ik} + \tau_{ik}) = \begin{pmatrix} -p+\tau_{11} & \tau_{12} & 0 \\ \tau_{12} & -p+\tau_{22} & 0 \\ 0 & 0 & -p+\tau_{33} \end{pmatrix} \tag{8.6.28}$$

Incompressibility implies that the pressure p cannot be given by a constitutive equation but has to be determined from the equations of motion and the boundary conditions for the flow. For an incompressible fluid the pressure level cannot influence the flow. Only pressure gradients are of importance.

In measuring directly or indirectly the normal stresses, it is not possible to distinguish between the pressure p and the contribution from the extra stresses due to the deformation of the fluid. The implication of this is that only normal stress differences may be expressed by material functions. In a viscometric flow we seek material functions for the following three stresses:

The shear stress: τ_{12}

The primary normal stress difference: $N_1 = T_{11} - T_{22} = \tau_{11} - \tau_{22}$

The secondary normal stress difference: $N_2 = T_{22} - T_{33} = \tau_{22} - \tau_{33}$ (8.6.29)

The third normal stress difference, $T_{11} - T_{33}$, is determined by the two others:

$$T_{11} - T_{33} = (T_{11} - T_{22}) + (T_{22} - T_{33}) = N_1 + N_2 \tag{8.6.30}$$

Three material functions, called *viscometric functions*, are introduced in a viscometric flow:

$$\eta(\dot{\gamma}) = \frac{|\tau_{12}(\dot{\gamma})|}{\dot{\gamma}} \text{ the viscosity function}$$

$$\psi_1(\dot{\gamma}) = \frac{N_1(\dot{\gamma})}{(\dot{\gamma})^2} \text{ the primary normal stress coefficient}$$

$$\psi_2(\dot{\gamma}) = \frac{N_2(\dot{\gamma})}{(\dot{\gamma})^2} \text{ the secondary normal stress coefficient} \tag{8.6.31}$$

$\dot{\gamma}$ is the magnitude of shear rate. The viscosity function is also called the *apparent viscosity*.

Figure 8.6.11 shows characteristic behavior of the viscometric functions for shear-thinning fluids. For low values of the magnitude of shear rate $\dot{\gamma}$ the viscosity function $\eta(\dot{\gamma})$ is nearly constant and equal to $\eta_o = \eta(0)$, called the *zero-shear-rate-viscosity*. For high values of the magnitude of shear rate $\dot{\gamma}$ the viscosity function $\eta(\dot{\gamma})$ may approach asymptotically a *infinite-shear-rate viscosity* η_∞.

For some fluids, for example highly concentrated polymer solutions and polymer melts, it may be impossible to measure η_∞. For the fluids mentioned the reason is that the polymer chains may be destroyed at very high shear rates.

The primary normal stress coefficient $\psi_1(\dot{\gamma})$ is positive, and is almost constant and equal to $\psi_{1,o} = \psi_1(0)$ for low magnitude of shear rate, and then decreases more rapidly with increasing magnitude of shear rate than the viscosity function $\eta(\dot{\gamma})$. A lower bound for ψ_1 when $\dot{\gamma} \to \infty$, is not registered.

The secondary normal stress coefficient $\psi_2(\dot{\gamma})$ is usually negative and is found for polymeric fluids to be approximately 10% of $\psi_1(\dot{\gamma})$ for the same fluid.

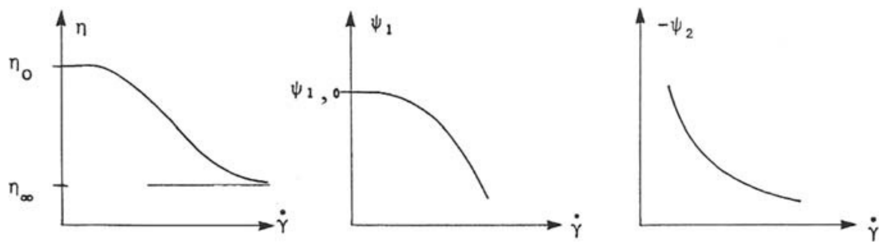

Fig. 8.6.11 Characteristic behavior of viscometric functions

8.6.5 Extensional Flows

As mentioned in Sect. 5.4, in any flow there exist through each particle at any time t three orthogonal material line elements that do not show shear strain rates: The lines remain orthogonal after a short time increment dt. Confer the elements 2 in the Examples 5.1 and 5.2. The three material line elements represent the principal directions of strain rates at the time t. We assume that the fluid is incompressible and introduce a local Cartesian coordinate system Px in the particle P, and with base vectors $\mathbf{e}_i$ coinciding with the *principal directions PD of strain rates* at the time t. Then the rate of deformation matrix takes the form:

$$D = \begin{pmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{pmatrix}, \quad \dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3 = 0 \tag{8.6.32}$$

A flow is called an *extensional flow if the same material line elements ML through each particle represent the principal directions PD of strain rates at all times*. This also implies the principal directions of strain rates are identical to the principal direction of strains. The literature also uses the names *elongational flow* and *shear free flow* for this type of flow. See Example 5.4.

A simple extensional flow is given by the velocity field:

$$v_x = \dot{\varepsilon}_x(t)\, x, \; v_y = \dot{\varepsilon}_y(t)\, y, \; v_z = \dot{\varepsilon}_z(t)\, z \tag{8.6.33}$$

The deformation of a volume element in this flow is illustrated in Fig. 8.6.12. Material lines parallel to the coordinate axes represent the principal directions *PD* of rates of strain at all times. The principal directions are fixed in space for this simple extensional flow.

It follows from Fig. 5.4.3 that in a simple shear flow the material line elements representing the principal directions of rates of strain at a time t do not represent the principal directions at a later time $t + dt$. The principal directions are fixed in space but the material lines coinciding with the principal directions at one time are not fixed in space. This difference between shear flows and extensional flows is very important in modelling of non-Newtonian fluids.

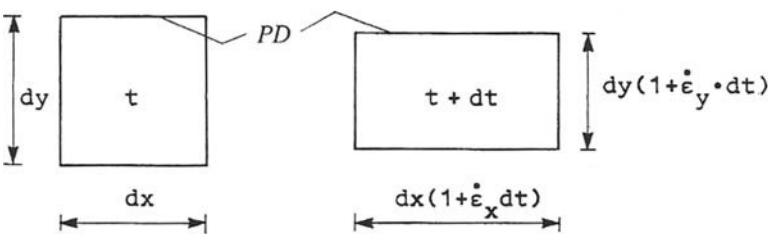

Fig. 8.6.12 Extensional flow. PD = principal directions of strain rates

Extensional flows are important in experimental investigations of the properties of non-Newtonian fluids. These flows are also relevant in connection with forming processes for plastics, as for example in vacuum forming, blow molding, foaming operations, and spinning. In metal forming extensional flows are important in milling and extrusion.

We continue to discuss incompressible and isotropic fluids for which the deviatoric stresses τ_{ik} only depend on the rate of deformation matrix (8.6.32). Isotropy implies that the principal axes of stress coincide with the principal directions of strain and strain rates. Based on the structure of the rate of deformation matrix (8.6.32) the stress matrix is presented as:

$$T = (-p\delta_{ik} + \tau_{ik}) = \begin{pmatrix} -p+\tau_{11} & 0 & 0 \\ 0 & -p+\tau_{22} & 0 \\ 0 & 0 & -p+\tau_{33} \end{pmatrix} \qquad (8.6.34)$$

Because the pressure is constitutively indeterminate, only normal stress differences may be modelled:

$$T_{11} - T_{22} = \tau_{11} - \tau_{33}, \; T_{22} - T_{33} = \tau_{22} - \tau_{33} \qquad (8.6.35)$$

Three special cases of steady extensional flow will now be presented.

UNIAXIAL EXTENSIONAL FLOW. See also Example 5.5. The rate of deformation matrix for this flow is given by (8.6.32) with:

$$\dot{\varepsilon}_1 = \dot{\varepsilon} = \text{ constant}, \; \dot{\varepsilon}_2 = \dot{\varepsilon}_3 = -\frac{\dot{\varepsilon}}{2} \qquad (8.6.36)$$

This type of flow is relevant when the fluid is stretched axisymmetrically in one direction. Material isotropy and the strain rates (8.6.36) imply that the normal stresses τ_{22} and τ_{33} are equal. Thus only one normal stress difference need be modelled, and the relevant material function is:

$$\eta_E(|\dot{\varepsilon}|) = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} \qquad (8.6.37)$$

called the *extensional viscosity* or the *Trouton viscosity*, F. T. Trouton(1906). For some fluids the extensional viscosity is decreasing with increasing strain rate. This is called *tension-thinning*. If the extensional viscosity is increasing with increasing strain rate the fluid is said to exhibit *tension-thickening*.

For Newtonian fluids with shear viscosity μ:

$$\tau_{11} = 2\mu\,\dot{\varepsilon}, \; \tau_{22} = \tau_{33} = -\mu\,\dot{\varepsilon} \quad \Rightarrow \quad \eta_E = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} = 3\mu \qquad (8.6.38)$$

The relationship between the extensional viscosity and the shear viscosity is in classical Newtonian fluid mechanics associated with the name Trouton.

The behavior of the extensional viscosity is often qualitatively different from that of the shear viscosity. It is found that highly elastic polymer solutions that show shear-thinning often exhibit a dramatic tension-thickening. Experiments and further analysis in continuum mechanics show that in the limit, as the strain rate approaches zero the extensional viscosity approaches a value three times the zero-shear rate viscosity:

$$\eta_E\left(|\dot{\varepsilon}|\right)|_{\dot{\varepsilon}\to 0} = 3\;\eta\left(\dot{\gamma}\right)|_{\dot{\gamma}\to 0} \quad \Rightarrow \quad \eta_E\left(0\right) = 3\eta\left(0\right) \quad \Leftrightarrow \quad \eta_{Eo} = 3\eta_o \quad (8.6.39)$$

BIAXIAL EXTENSIONAL FLOW. When a fluid is stretched or compressed equally in two orthogonal directions the flow is called a biaxial extension. The rate of deformation matrix for this flow is given by (8.6.32) with:

$$\dot{\varepsilon}_1 = \dot{\varepsilon}_2 = \dot{\varepsilon} = \text{ constant}, \quad \dot{\varepsilon}_3 = -2\dot{\varepsilon} \qquad (8.6.40)$$

The material function relevant for this type of flow is called *biaxial extensional viscosity*:

$$\eta_{EB}\left(|\dot{\varepsilon}|\right) = \frac{\tau_{11} - \tau_{33}}{\dot{\varepsilon}} \qquad (8.6.41)$$

A comparison of the biaxial extensional flow and uniaxial extensional flow shows that constitutive modelling of a fluid in either flow should be the same. In fact it follows that:

$$\eta_{EB}\left(|\dot{\varepsilon}|\right) = 2\eta_E\left(2\,|\dot{\varepsilon}|\right) \qquad (8.6.42)$$

PLANAR EXTENSIONAL FLOW. See also Example 5.6. The rate of deformation matrix for this flow is given by (8.6.32) with:

$$\dot{\varepsilon}_1 = -\dot{\varepsilon}_2 = \dot{\varepsilon} = \text{ constant}, \; \dot{\varepsilon}_3 = 0 \qquad (8.6.43)$$

The material function relevant for this type of flow is called *planar extensional viscosity*:

$$\eta_{EP}\left(|\dot{\varepsilon}|\right) = \frac{\tau_{11}}{\dot{\varepsilon}} \qquad (8.6.44)$$

Problems

Problem 8.1. A closed vessel filled with a fluid is given a translatoric motion defined by the velocity field:

$$v_1 = -v_o \sin\omega t, \; v_2 = v_o \cos\omega t$$

v_o and ω are constants. The fluid moves with the vessel as a rigid body. Show the streamlines at time t are straight lines, and that the path lines are circles.

Problem 8.2. Show that the streamlines and the path lines coincide for the following type of non-steady two-dimensional flow:

$$v_1 = f(t)\,g(x,y), \; v_2 = f(t)\,h(x,y), \; v_3 = 0$$

$f(t)$, $g(x,y)$, and $h(x,y)$ are arbitrary functions of the variables: time t and Cartesian coordinates x and y.

Problem 8.3. Let $\alpha(\mathbf{r}) = 0$ represent a fixed rigid boundary surface A in a flow of a perfect fluid. Show that the velocity field $\mathbf{v}(\mathbf{r},t)$ must satisfy the condition:

$$\mathbf{v} \cdot \nabla\alpha = 0 \text{ on } A$$

Problem 8.4. Let $\alpha(\mathbf{r},t) = 0$ represent a moving rigid boundary surface A in a flow of a perfect fluid. Show that the velocity field $\mathbf{v}(\mathbf{r},t)$ must satisfy the condition:

$$\partial_t\alpha + \mathbf{v} \cdot \nabla\alpha = 0 \text{ on } A$$

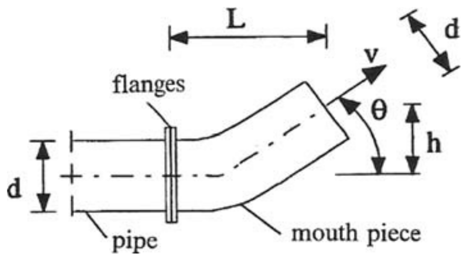

Fig. Problem 8.5

Problem 8.5. A mouth piece is attached to a pipe by flanges and bolts. The cross-sections of the piece and the pipe are the same with the area A. Water of constant velocity v flows through the pipe and out through the mouth piece. The pressure in the fluid is equal to the atmospheric pressure p_o. Determine the shear force, the axial force and the bending moment at the flanges.

Problem 8.6. Show that the two expressions for the elastic energy per unit mass in (8.3.11) are equivalent.

Problem 8.7. Use the identity (2.1.17) to prove the identity (8.3.27).

Problem 8.8. Use a differentiation test and (5.5.28) that (8.3.30) is the solution of the differential equation (8.3.29).

Problem 8.9. Derive the results presented as (8.6.25) and (8.6.26).