IDENTITIES DERIVED ON A FIBONACCI MULTIPLICATION TABLE

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A multiplication table constructed only with Fibonacci numbers assumes the appearance shown in Table 1. In any of its rows among three successive integers, the sum of the first two equals the third. This may be expressed as

$$F_mF_n + F_mF_{n+1} = F_mF_{n+2}. \tag{1}$$

Table 1

		F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	
		1	1	2	3	5	8	13	21	34	...
F_1	1	1	1	2	3	5	8	13	21	34	...
F_2	1	1	1	2	3	5	8	13	21	34	...
F_3	2	2	2	4	6	10	16	26	42	68	...
F_4	3	3	3	6	9	15	24	39	63	102	...
F_5	5	5	5	10	15	25	40	65	105	170	...
F_6	8	8	8	16	24	40	64	104	168	272	...
F_7	13	13	13	26	39	65	104	169	273	442	...
F_8	21	21	21	42	63	105	168	273	441	714	...
F_9	34	34	34	68	102	170	272	442	714	1156	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

While this result is rather trivial, it does suggest that the table should be scrutinized to uncover analogs. Doing this, an investigator perceives that along any descending diagonal the sum of two successive integers is a Fibonacci number. This is expressed as

$$F_mF_n + F_{m+1}F_{n+1} = F_{m+n+1}. \tag{2}$$

Combining formulas (1) and (2) "geometrically" leads to the following triangular representation in the table:

$$F_mF_n + F_{m+1}F_n + F_{m+1}F_{n-1} = F_{m+n+1}. \tag{3}$$

One of the identities known by practically every student of the Fibonacci numbers is

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_nF_{n+1}.$$

On the Fibonacci multiplication table, this assumes the following appearance:

$$\begin{array}{lllll}
1 & & & & \\
 & 1 & & & \\
 & & 4 & & \\
 & & & \ddots & \\
 & & & & F_n^2 = F_n F_{n+1}
\end{array}$$

When analogs are sought in the table, none appears. In view of the findings of identities (1), (2), and (3), this is surprising.

If, however, this well-known result is altered to assume the form

$$F_1F_2 + (F_2^2 + F_3^2 + \cdots + F_n^2) = F_n \cdot F_{n+1},$$

it remains numerically identical to $1^2 + 1^2 + 2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$.

As the revised form

$$\begin{array}{lllll}
F_1F_2 & & & & \\
F_2^2 & & & & \\
 & + & & & \\
 & & F_3^2 & & \\
 & & & + & \\
 & & & & \ddots \\
 & & & & + F_n^2 = F_n \cdot F_{n+1}
\end{array}$$

has analogs throughout the table, it is evident that $1 + 1 + 2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$ is just a special case of the more general identity

$$F_{m-1}F_n + F_mF_n + F_{m+1}F_{n+1} + F_{m+2}F_{n+2} + \cdots + F_{m+k}F_{n+k} = F_{m+k}F_{n+k+1},$$

that is,

$$\sum_{j=0}^{k} F_{m+j}F_{n+j} = F_{m+k}F_{n+k+1} - F_{m-1}F_n \text{ for } m \geqslant 2,\ n \geqslant 1. \tag{4}$$

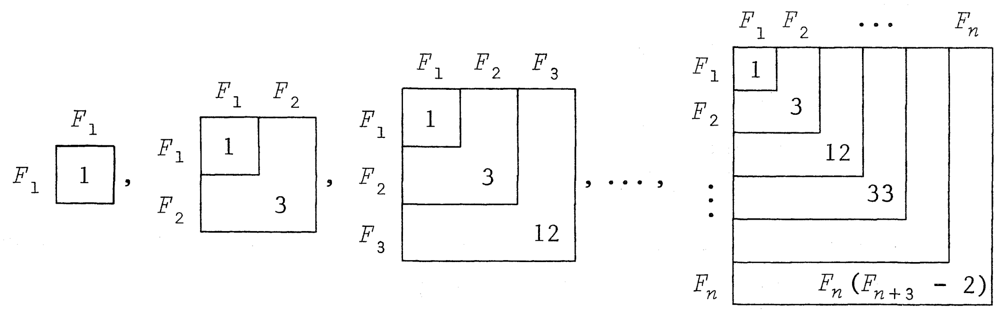
A sequence of squares beginning in the upper left-hand corner of the table may be built as follows:

$$\begin{array}{c|c|}
 & F_1 \\
\hline
F_1 & 1 \\
\hline
\end{array}, \quad
\begin{array}{c|cc|}
 & F_1 & F_2 \\
\hline
F_1 & 1 & 1 \\
F_2 & 1 & 1 \\
\hline
\end{array}, \quad
\begin{array}{c|ccc|}
 & F_1 & F_2 & F_3 \\
\hline
F_1 & 1 & 1 & 2 \\
F_2 & 1 & 1 & 2 \\
F_3 & 2 & 2 & 4 \\
\hline
\end{array}, \quad \ldots, \quad
\begin{array}{c|ccc|}
 & F_1 & F_2 \cdots & F_n \\
\hline
F_1 & 1 & 1 \ \cdots & F_n \\
F_2 & 1 & & \\
\vdots & \vdots & & \vdots \\
F_n & F_n & \cdots & F_n^2 \\
\hline
\end{array}$$

$$F_1^2 \qquad (F_1 + F_2)^2 \qquad (F_1 + F_2 + F_3)^2 \qquad (F_1 + F_2 + \cdots + F_n)^2$$

$$1^2 \qquad (1 + 1)^2 \qquad (1 + 1 + 2)^2 \qquad (1 + 1 + \cdots + F_n)^2$$

This same sequence could also be developed by summing rows and columns in the manner indicated below:



As these constructions cover identical squares, it becomes evident that the entries of any n by n square in the upper left-hand corner of the table may be summed in any two distinct ways both of which equal $(F_{n+2} - 1)^2$. This results in the following identities:

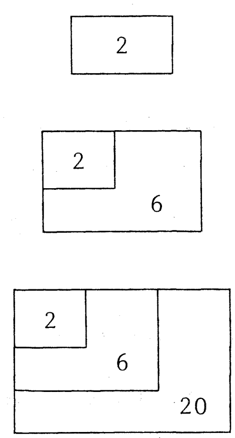
$$\left(\sum_{i=1}^{n} Fi\right)^2 = \sum_{i=1}^{n} Fi(F_{i+3} - 2) = (F_{n+2} - 1)^2. \tag{5}$$

An analog of the sequence of squares is the sequence of oblong rectangles of dimension n by $n + 1$.

1	1

1	1	2
1	1	2

1	1	2	3
1	1	2	3
2	2	4	6



By pursuing an analysis similar to that performed on the squares, the following oblong identities are obtained:

$$\left(\sum_{i=1}^{n} Fi\right)\left(\sum_{i=1}^{n+1} Fi\right) = \sum_{k=1}^{n} (F_{k+2}^2 - F_{k+2}) = (F_{n+2} - 1)(F_{n+3} - 1). \tag{6}$$

Other identities that may be gleaned from the table include

$$F_{2n+1}^2 = F_{2n} \cdot F_{2n+2} + 1 \tag{7}$$

and

$$F_{2n}^2 = F_{2n-1} \cdot F_{2n+1} - 1, \tag{8}$$

which can readily be combined into

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{9}$$

the basis for one of Charles Dodgson's favorite geometrical puzzles:

$$F_{n-1}F_{n+1} - F_{n+2}F_{n-2} = 2(-1)^n. \tag{10}$$

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actual ocean of yellows—were not only joyous, but also touched our mathematical souls. Do Fibonacci numbers not play an important role in deciphering nature's handiwork in sunflowers?

Volterra, situated about 550 metres above sea-level, immediately transplanted us into enigmatic Etruscan, as well as into problematic Medieval times. While we were fascinated both by the histroic memorabilia, as well as by the artifacts and master pieces, the magnificent panorama of the surrounding landscape enhanced our enjoyment still further.

As has become tradition in our conference, a banquet was held on the last night before the closing of our sessions. Lucca, the site of the meeting, provided a wonderful setting for a memorable evening, Ligurian in origin, it bespeaks of Etruscan culture, and exudes the charm of an ancient city.

The spirit at the banquet highlighted what had already become apparent during the week: that the Conference had not only been mind-streatching, but also heartwarming. Friendships which had been started, became knitted more closely. New friendships were formed. The magnetism of common interest and shared enthusiasm wove strong bonds amoung us. We had come from different cultural and ethnic backgrounds, and our native tongues differed. Yet, we truly understood each other. And we cared for each other.

I believe, I speak for all of us if I express my heartfelt thanks to all members of the International, as well as of the Local Committee whose dedication and industriousness gave us this unforgettable event. Our gratitude also goes to the University of Pisa whose generous hospitality we truly appreciated. I would also like to thank all participants, without whose work we could not have had this treat.

"Auf Wiedersehen" then, at Conference number Four in 1990.