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INTRODUCTION TO MAGNETO-FLUID MECHANICS

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By

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CONTENTS

INTRODUCTION TO MAGNETO-FLUID MECHANICS

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CHAPTER I – INTRODUCTION

Magneto-fluid mechanics, as the name implies, is a branch of fluid mechanics. The difference between magneto-fluid mechanics and ordinary fluid mechanics (in the restricted sense) lies in the forms of the external body forces. In the study of ordinary fluid mechanics, the body forces are either neglected or known in advance independent of the motion of the fluid medium.

In the study of magneto-fluid mechanics, the situation is much more complicated. Here we are working with a medium which is electrically conducting. When this medium moves in the presence of an externally applied magnetic field, a current is induced in the fluid medium. This induced current will interact with the magnetic field and cause a modification of the magnetic field. This current and the modified magnetic field will then interact and produce a body force (called the ponderomotive force) acting on the fluid medium and thereby influencing the subsequent motion of the fluid medium. This complicated interaction of the motion of the electrically conducting fluid medium with the applied magnetic field constitutes the central core of interest of the study of magneto-fluid mechanics.

Almost all of the observed phenomena in astrophysics are magento-fluid dynamic in nature. Current interests in hypersonic flow and containment of hot gases for the design of fusion reactors are also closely related to

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the study of the motion of partially or fully ionized gases in externally applied magnetic fields. Our own recent interest at the Reactor Division concerns the feasibility of using an applied magnetic field to stabilize the motion of a vortex heat-exchanger reactor and to reduce the influence of turbulence in the vortex flow. These and many other applications are the reasons why magneto-fluid mechanics plays so important a role in modern engineering sciences.

The purpose of this series of lectures is to develop the general theory of magneto-fluid mechanics by considering the fluid medium as a continuous, neutrally charged, electrically conducting fluid. Time will not allow us to develop the theory from a microscopic viewpoint using the concept of statistical mechanics of ionized gases.

CHAPTER II - METHOD OF ANALYSIS

2.1. Continuum Concept

As mentioned in the previous chapter, we are only going to be interested in the continuum concept of magneto-fluid mechanics. Let us now amplify this statement slightly. Consider a region of a fluid medium with a total volume V. If the total mass of the fluid medium contained in this region V is M, then the average density, $\bar{\rho}$, of the fluid medium of this region V is defined as the fraction of mass contained per unit volume in V if the mass is distributed uniformly within the region V, i.e.,

$$\bar{\rho} = \frac{M}{V} \quad . \tag{2.1.1}$$

The density at a point in a body of a fluid medium can be obtained by enclosing that point with a small volumetric element ΔV, and by taking the average density $\Delta M/\Delta V$ of this volumetric element, where ΔM is the fraction of the mass of the fluid medium in ΔV. The value of $\Delta M/\Delta V$ will become almost a constant as ΔV is taken smaller and smaller while always enclosing the point in consideration. In other words, the value of $\Delta M/\Delta V$ seems to possess a limit at that point. Actually, if we continue to reduce the value of ΔV, the value of $\Delta M/\Delta V$ will begin to fluctuate. This is because the volume ΔV will become too small to contain a sufficient number of molecules, or charged particles, to cancel out the effects introduced due to the random motions of the molecules or charged particles. In fact, when ΔV is made as small as the size of the particles, the value of $\Delta M/\Delta V$ will either be very large or nearly equal to zero depending upon whether at that instant of observation the volumetric element contains a particle or not. Therefore, in order to have a definitely defined value of density at a point in a fluid medium, the volumetric element ΔV cannot be made too small. In other words, the value of ΔV should be chosen such that it is small enough to give an apparent limit of the value of $\Delta M/\Delta V$ but not so small such that the value of $\Delta M/\Delta V$ fluctuates and becomes meaningless. The word "density" is meaningful only if the fluid medium can be observed this way. We shall now write the definition of the density, ρ, at a point P in a fluid medium as

$$\rho = \frac{dM}{dV} \, . \tag{2.1.2}$$

However, we should understand at the same time that dM/dV has the following physical meaning:

$$\frac{dM}{dV} = \lim_{\substack{\Delta V \to P \\ \Delta V > \Delta V^{\circ}}} \frac{\Delta M}{\Delta V} \, , \tag{2.1.3}$$

where $\substack{\Delta V \to P \\ \Delta V > \Delta V^{\circ}}$ mean that ΔV is a very small volumetric element enclosing P but is larger than a characteristic volume ΔV° which is the smallest bound of the size of the volumetric element to yield a meaningful limit of the ratio of $\Delta M/\Delta V$.

This type of restriction of the smallest size of observation should be considered in each and every discussion of the average properties of a continuum. For example, in our discussions, we shall treat volumetric elements of the size such that on the average they are neutrally charged. This is true also in the discussion of the forces acting on the fluid medium.

2.2. Eulerian Veiwpoint

Instead of considering the properties of the volumetric elements in a fluid in motion in terms of their initial positions and time (the Lagrangian viewpoint), it is usually more convenient to consider them as functions of their instantaneous positions and time. This approach is called the Eulerian method. It shall be the method used in the development of the basic theory of magneto-fluid mechanics in the subsequent lectures.

2.3. Cartesian Tensor Notation

The discussion of any physical theory of mechanics of continuous media can be treated and presented more precisely and efficiently if Cartesian tensor notation is used in place of the classical vector notation. Classical vector notation is a system of algebraic symbols which follow a special set of algebraic rules. Furthermore, the rules of vector calculus are many and usually complicated. The rules of Cartesian tensors, on the other hand, are very simple. The algebra and calculus of Cartesian tensors are the same as those for ordinary scalar quantities. One can learn these rules and the few formulas related to Cartesian tensors in a relatively short period. Therefore, we will no longer be burdened with the extra mathematical rules of the classical vector analysis while learning a new theory. In addition, physical quantities are usually tensorial quantities which cannot always be represented by vectors of the usual sense. In this sequence of lectures, we shall attempt to develop the theory of magneto-fluid mechanics using the Cartesian tensor notation.

2.4. Laws Governing the Motion of an Electrically Conducting Fluid in the Presence of an Externally Applied Magnetic Field

The laws governing the motion of a fluid medium are the laws of conservation of mass, the Newton's second law of motion, and the law of conservation of energy. Due to the interaction of the electrically conducting fluid with the externally applied magnetic field in magneto-fluid mechanics, additional laws pertaining to the electromagnetic interaction and the Ohm's law have to be considered in conjunction with the laws of ordinary fluid mechanics. It is the purpose of this sequence of lectures to introduce these laws of magneto-fluid mechanics mathematically in terms of a system of equations using the Cartesian tensor notation. These equations in general are very complicated and do not posses a general solution. Simple solved examples will be drawn to indicate the fundamental behavior of magneto-fluid flow. A discussion of the similarity parameters in magneto-fluid flow will also be briefly included.

CHAPTER III – TYPES OF FORCES AND THE STRESS TENSOR

3.1. Body Forces and Surface Forces

Forces acting on a body of a fluid medium may be divided into two parts; those which act across a surface due to direct contact with another body and those which act at a distance, not due to direct contact.

Body forces are forces which act on all the volumetric elements in the medium due to some external body or effect. An example of this is the gravitational force exerted on a medium due to another body at a distance. These types of forces can be conveniently discussed as force intensities, f_i (or simply forces) per unit mass. This definition is based on the apparent limit of the average value over a small volumetric element, ΔV,

$$f_i = \lim_{\substack{\Delta V \to P \\ \Delta V > \Delta V^{\circ}}} \frac{\Delta F_i}{\Delta M} = \frac{dF_i}{dM} = \frac{1}{\rho} \frac{dF_i}{dV} , \qquad (3.1.1)$$

where ΔF_i is the total force acting on the small volumetric element ΔV, and

ΔM is the total mass contained in ΔV.

Surface forces are contact forces which act across some surface of the fluid medium. This surface may be internal or external. These types of forces are conveniently discussed as force intensities (or stresses) per unit area. Let us consider a very small planar surface ΔS with unit normal n_i containing a point P in a continuum, Fig. 3.1.1. If the total force acting by the fluid medium on the positive n_i side across the surface element on the fluid medium on the negative n_i side is ΔF_i, then the stress vector, σ_i (or stress), acting across the surface element by the fluid medium on the n_i side at the point P is defined as

$$\sigma_i = \lim_{\substack{\Delta S \to P \\ \Delta S > \Delta S^\circ}} \frac{\Delta F_i}{\Delta S} = \frac{dF_i}{dS} , \qquad (3.1.2)$$

where ΔS° is the limit of the smallest size of ΔS for the fluid to be observed as a continuous medium.

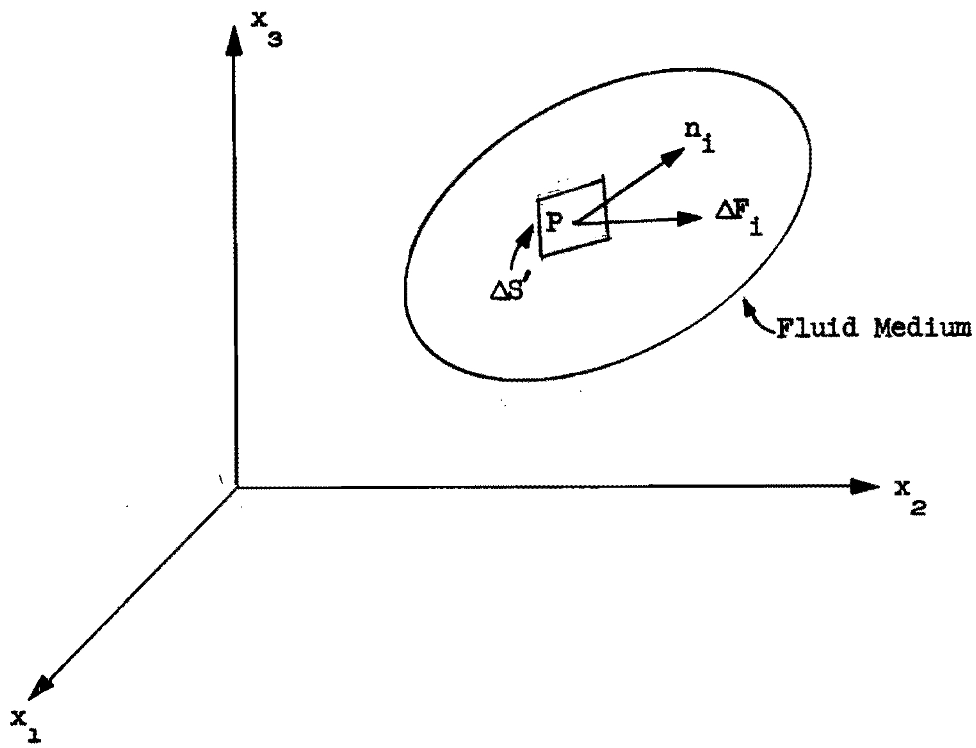

Fig. 3.1.1. Figure Depicting the Force ΔF_i Acting on a Small Surface Element ΔS Containing a Point P in a Fluid Medium.

We note that the stress or stress vector, σ_i, is a function of position, time, the orientation of the surface element, and the choice of the sense of direction of the unit normal, n_i. Every stress vector can be resolved into two components; one in the direction of n_i and one lying in the surface element. They are called the "normal" and "shearing" components of σ_i.

The usual assumption for both the body and surface forces is that the net moment due to the forces acting on the small volumetric or surface element vanishes.

3.2. Stress Tensor

Let us consider a point P in a fluid medium and a set of local Cartesian axes drawn from the point P. Visualize a small surface element containing P whose unit normal is in the positive x_1-direction, Fig. 3.2.1.

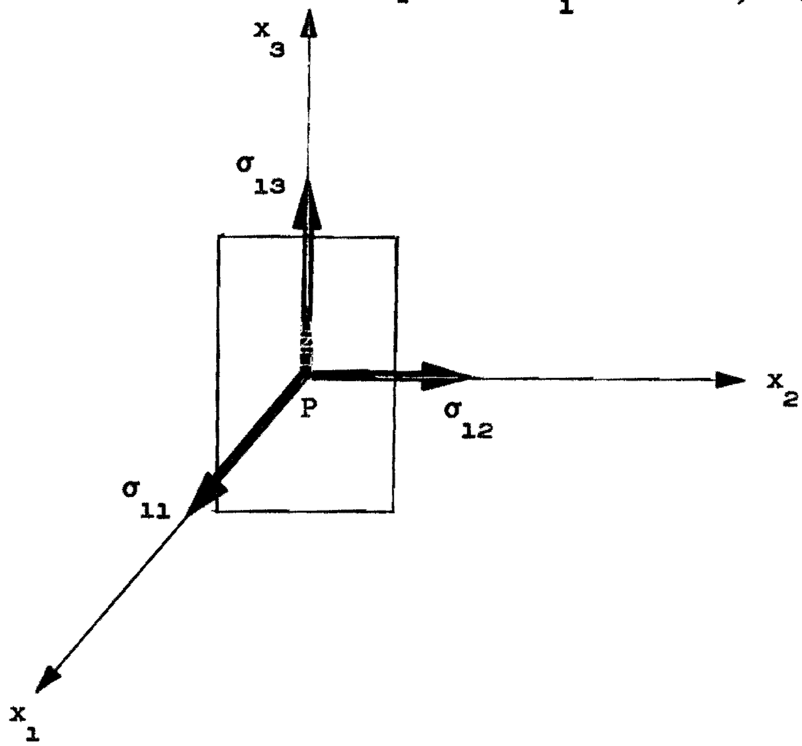

Fig. 3.2.1. Stress Vector σ_{1i} Acting on a Small Surface Element Whose Unit Normal is in the Positive x_1-Direction by the Portion of the Fluid Medium Containing the Unit Normal.

The stress vector σ_{1i} acting on this surface element by the medium containing the positive x_1-axis has three components; one normal component

σ_{11} acting in the positive x_1-direction and two shearing components σ_{12}, σ_{13} in the positive x_2- and x_3-directions, respectively. Similarly, we can visualize two other stress vectors σ_{2i}, σ_{3i} acting on surface elements whose unit normals are in the positive x_2- and x_3-directions. These three stress vectors σ_{1i}, σ_{2i}, σ_{3i} have a total of nine components. This set of nine components of stress is called a stress tensor. It can be denoted by a single symbol σ_{ji}. It is obvious that the reaction stress components acting by the fluid medium on the portion of the medium on the positive sides of the coordinate planes are equal and opposite to the nine components just defined.

It is possible to show that this stress tensor σ_{ji} completely defines the stresses acting at that point on an arbitrarily inclined plane with respect to the set of Cartesian coordinate axes x_i. To prove this, consider a very small tetrahedron as shown in Fig. 3.2.2.

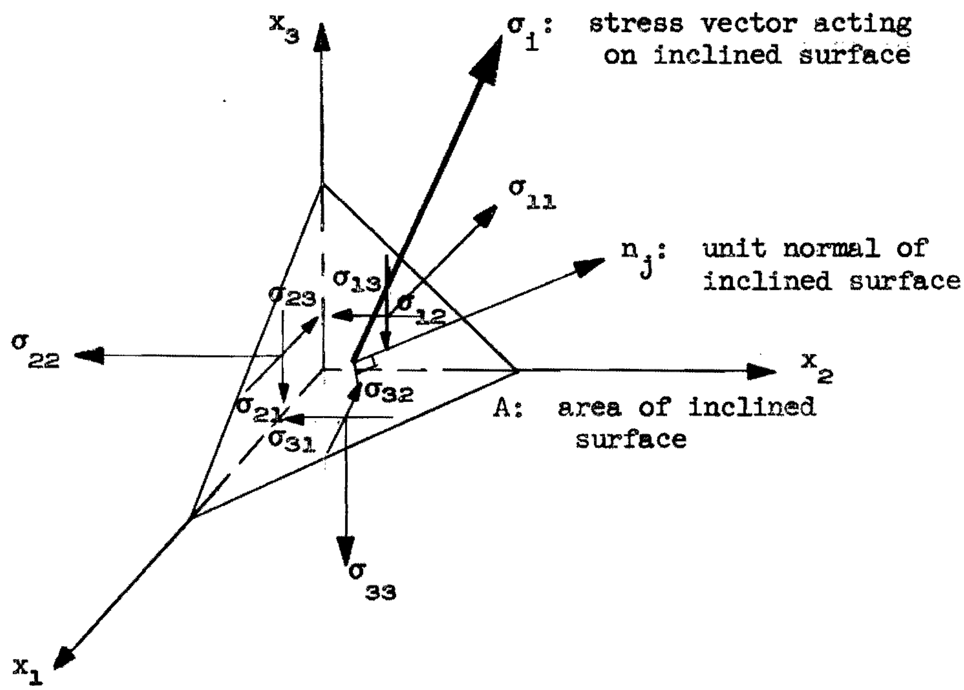

Fig. 3.2.2. Stresses Acting on a Differential Tetrahedron at a Point P in a Fluid.

Let us consider the forces acting on the free body of this tetrahedron. If we assume that the inertial and body forces are negligible compared to the magnitudes of the surface forces for a very small tetrahedron, then by balancing the forces on the tetrahedron, we obtain

$$\sigma_i A - \sigma_{ji} n_j A = 0 \quad , \tag{3.2.1}$$

where σ_i is stress vector acting on the inclined area A whose unit normal is n_j, and

σ_{ji} is the stress tensor at point P.

Equation (3.2.1) can be written as

$$\sigma_i = \sigma_{ji} n_j \quad . \tag{3.2.2}$$

This means that the stress σ_i at a point P acting on a plane whose unit normal is n_j is expressible in terms of the stress tensor σ_{ji} at P and the unit normal n_j.

It will not be hard to show, by using the equilibrium condition (with inertia and body forces neglected) of a small parallelepiped that the stress tensor is symmetrical, i.e.,

$$\sigma_{ji} = \sigma_{ij} \quad . \tag{3.2.3}$$

This means that there are only six independent components defining a stress tensor. They are

$$\begin{aligned} &\sigma_{11}, \sigma_{22}, \sigma_{33}, \\ &\sigma_{12} = \sigma_{21} \quad , \\ &\sigma_{23} = \sigma_{32} \quad , \\ &\sigma_{31} = \sigma_{13} \quad . \end{aligned} \tag{3.2.4}$$

CHAPTER IV – EQUATIONS GOVERNING THE MOTION OF A FLUID MEDIUM

4.1. Equation of Continuity

One of the most important equations governing the motion of a fluid is derived from the idea of conservation of mass. Consider a surface S enclosing a fixed region of space V in which fluid motion exists, Fig. 4.1.1. Let us call the outward normal of a surface element dS on S, n_j (x_1, x_2, x_3); the velocity components of a volumetric element of the fluid in the region V, q_j $(x_1, x_2, x_3; t)$; and the density of a volumetric element in the region V, ρ $(x_1, x_2, x_3; t)$. It is obvious from the concept

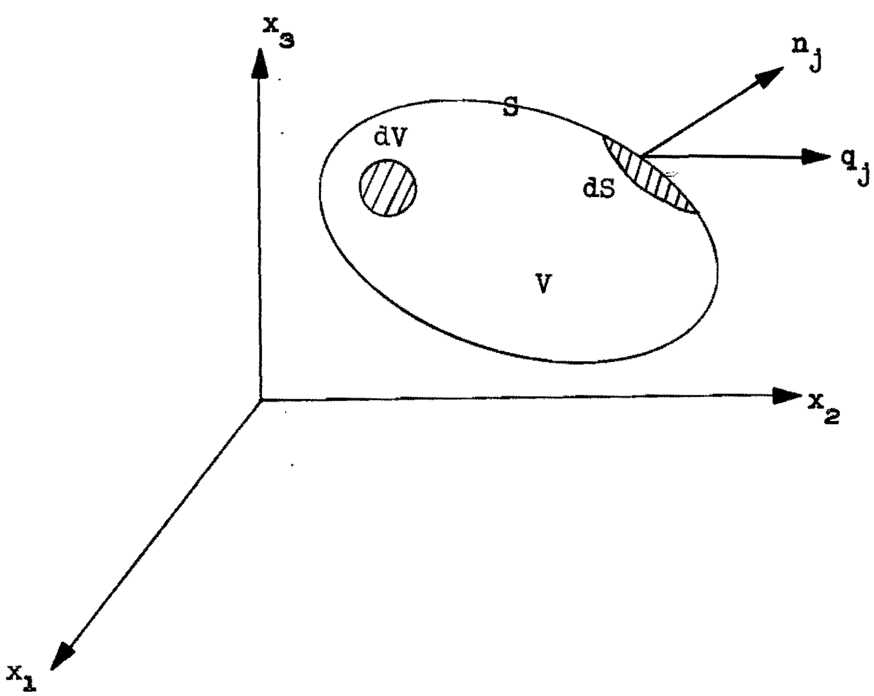

Fig. 4.1.1. Region of Space in Which Fluid Motion Exists.

of conservation of mass that the rate of increase of mass in the region V is exactly the amount of mass flowing into the region per unit time, or

$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_S \rho \, q_j \, n_j \, dS \quad , \qquad (4.1.1)$$

or

$$\int_V \frac{\partial \rho}{\partial t} \, dV + \int_S \rho \, q_j \, n_j \, dS = 0 \quad . \tag{4.1.2}$$

Equation (4.1.2) states that the total production of mass of the region V which includes both the net increase of mass within V and the amount of mass outflow is zero. Equation (4.1.1) or (4.1.2) is called the <u>integral continuity equation</u>.

Applying the Gauss Theorem, Eq. (4.1.2) becomes

$$\int_V \left[\frac{\partial \rho}{\partial t} + (\rho \, q_j),_j \right] dV = 0 \quad . \tag{4.1.3}$$

However, Eq. (4.1.3) should be satisfied for any fixed region of space. This means that the integrand of the left-hand side of Eq. (4.1.3) should be identically equal to zero, i.e.,

$$\frac{\partial \rho}{\partial t} + (\rho \, q_j),_j = 0 \quad , \tag{4.1.4}$$

or

$$\frac{\partial \rho}{\partial t} + q_j \, \rho,_j + \rho \, q_j,_j = 0 \quad . \tag{4.1.5}$$

The first two terms of the left-hand side of Eq. (4.1.5) can be considered as the total time rate of change of density of a fluid element if we follow the motion of this fluid element along. It is sometimes called the <u>co-moving derivative</u> of the density of the fluid element. Since in most of the equations in fluid mechanics, the total time derivatives are co-moving derivatives, we shall denote this type of differentiation by the symbol

$$\frac{d}{dt} ,$$

unless otherwise noted. Therefore, Eq. (4.1.5) can be written as

$$\frac{d\rho}{dt} + \rho \, q_{j,j} = 0 \quad . \tag{4.1.6}$$

Equations (4.1.4), (4.1.5), (4.1.6) are three alternative forms of the equation of continuity. The continuity equation relates the four field functions ρ $(x_1, x_2, x_3; t)$ and q_j $(x_1, x_2, x_3; t)$ in terms of a scalar partial differential equation. In order to solve a problem of fluid motion, it is generally necessary to find additional relationships for these field functions.

4.2. The Equation of Motion

The equations of motion which give three additional relationships between the field functions, ρ and q_j, can be obtained directly from the Newton's second law of motion.

Let us fix our attention to a fixed region of space V, in which fluid motion exists, Fig. 4.2.1.

The Newton's second law states that the total time rate of change of momentum of a body of fluid medium is equal to the external force acting on the fluid medium. Applying it to a subregion V_1 of V bounded by a surface S, we obtain

$$\int_S \sigma_i \, dS + \int_{V_1} \rho \, f_i \, dV = \int_{V_1} \frac{\partial (\rho \, q_i)}{\partial t} \, dV + \int_S \rho \, q_i \, q_j \, n_j \, dS \quad , \tag{4.2.1}$$

where σ_i $(x_1, x_2, x_3; t)$ is the stress vector acting on a surface element dS,

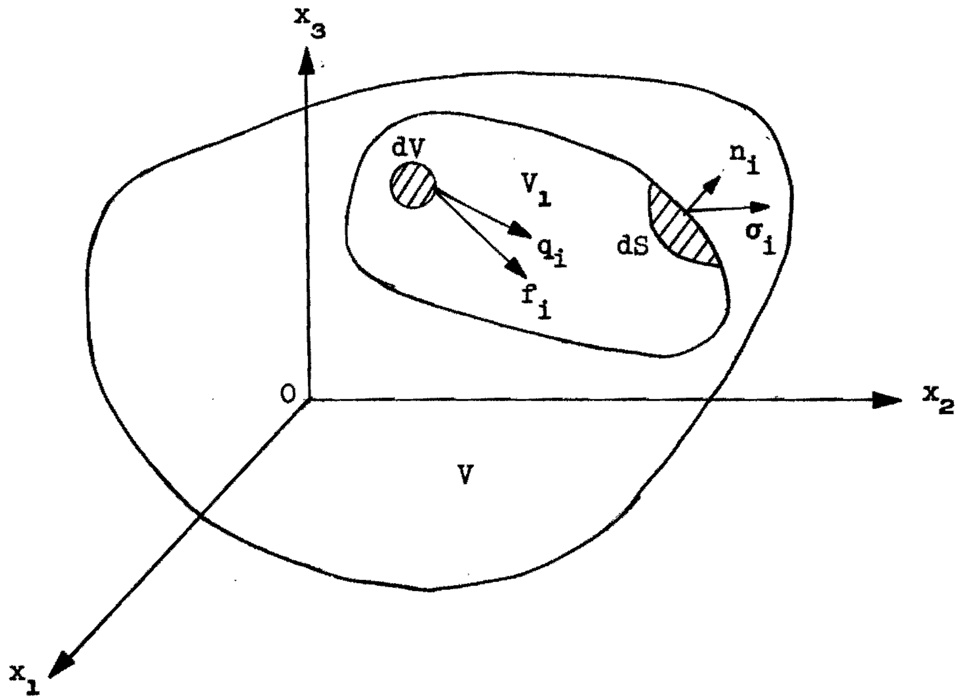

Fig. 4.2.1. Free Body Diagram of an Arbitrary Region of a Fluid Medium in Motion.

$\rho\ (x_1, x_2, x_3; t)$ is the density of the fluid medium of a volumetric element dV,

$f_i\ (x_1, x_2, x_3; t)$ is the body force per unit mass acting on a volumetric element dV, and

$q_i\ (x_1, x_2, x_3; t)$ is the velocity vector at a point in the fluid medium.

Expressing σ_i in terms of the stress tensor, σ_{ji}, we can rewrite Eq. (4.2.1) as

$$\int_S \sigma_{ji}\, n_j\, dS + \int_{V_1} \rho\, f_i\, dV = \int_{V_1} \frac{\partial\, (\rho\, q_i)}{\partial t}\, dV + \int_S \rho\, q_i\, q_j\, n_j\, dS \quad , \quad (4.2.2)$$

where $n_j\ (x_1, x_2, x_3)$ is the unit normal of a surface element dS on S.

Equation (4.2.2) is called the momentum integral equation governing the motion of a region of a fluid medium.

If we transform the surface integrals in Eq. (4.2.2) into volume integrals by means of the Gauss Theorem, we obtain

$$\int_{V_1} \left[\sigma_{ji,j} + \rho\, f_i - \frac{\partial\,(\rho\, q_i)}{\partial t} - (\rho\, q_i\, q_j)_{,j} \right] dV = 0 \quad . \qquad (4.2.3)$$

Equation (4.2.3) should hold true for any arbitrary region, V, of the fluid medium in motion. This means that the integrand of the left-hand side of Eq. (4.2.3) should be identically equal to zero, or

$$\frac{\partial\,(\rho\, q_i)}{\partial t} + (\rho\, q_i\, q_j)_{,j} = \rho\, f_i + \sigma_{ji,j} \quad . \qquad (4.2.4)$$

This is one form of the equation of motion. An alternative form of the equation of motion can be obtained by multiplying the continuity equation, Eq. (4.1.4), by q_i and subtracting it from Eq. (4.2.4).

$$\rho\, \frac{\partial\, q_i}{\partial t} + \rho\, q_j\, q_{i,j} = \rho\, f_i + \sigma_{ji,j} \quad . \qquad (4.2.5)$$

We note that the co-moving derivative of the velocity vector, q_i, is

$$\frac{dq_i}{dt} = \frac{\partial\, q_i}{\partial t} + q_j\, q_{i,j} \quad . \qquad (4.2.6)$$

This means that Eq. (4.2.5) can be written as

$$\frac{dq_i}{dt} = f_i + \frac{1}{\rho}\, \sigma_{ji,j} \quad . \qquad (4.2.7)$$

The three scalar partial differential equations of motion represented by Eq. (4.2.5) or Eq. (4.2.7) give the additional relationships among the functions ρ and q_i. However, they introduce at the same time nine independent components of the field functions of f_i $(x_1, x_2, x_3; t)$ and σ_{ji} $(x_1, x_2, x_3; t)$. It is therefore generally necessary to obtain additional equations to relate these unknown field functions.

4.3. The First Law of Thermodynamics

An additional relationship governing the unknown field functions, ρ, q_i, f_i, σ_{ji}, is given by the law of conservation of energy. This relationship or equation is called the energy equation or the first law of thermodynamics.

To derive this equation, let us refer to Fig. 4.2.1 again. If we call the internal energy of the fluid medium per unit mass, u $(x_1, x_2, x_3; t)$; and the heat transferred into the fluid medium per mass per unit time, c $(x_1, x_2, x_3; t)$; then the following energy balance equation is obtained for the arbitrary region V_1.

$$\underset{\text{(I)}}{\int_{V_1} \frac{\partial(\rho u)}{\partial t}\, dV} + \underset{\text{(II)}}{\int_{V_1} \frac{\partial(\frac{1}{2}\rho q^2)}{\partial t}\, dV} + \underset{\text{(III)}}{\int_S \rho u\, q_j\, n_j\, dS} + \underset{\text{(IV)}}{\int_S \tfrac{1}{2}\rho q^2\, q_j\, n_j\, dS} =$$

$$= \underset{\text{(V)}}{\int_S \sigma_{ji}\, n_j\, q_i\, dS} + \underset{\text{(VI)}}{\int_{V_1} \rho\, f_i\, q_i\, dV} + \underset{\text{(VII)}}{\int_{V_1} \rho\, c\, dV} \quad . \qquad (4.3.1)$$

The terms (I), (II), (III), and (IV) are the time rate of energy production due to the arbitrary region V_1 of the fluid medium; the terms (V) and (VI) are the time rate of work done on the fluid medium in region V_1 by the surface and body forces; and the term (VII) is the time rate of heat transfer into the region V_1.

Using the Gauss Theorem, Eq. (4.3.1) becomes

$$\int_{V_1} \left\{ \frac{\partial}{\partial t} \left[\rho(u + \tfrac{1}{2} q^2)\right] + \left[\rho(u + \tfrac{1}{2} q^2)\, q_j\right]_{,j} - (\sigma_{ji}\, q_i)_{,j} - \right.$$

$$\left. - \rho\, f_i\, q_i - \rho\, c \right\} dV = 0 \quad . \qquad (4.3.2)$$

Equation (4.3.2) should hold true for any arbitrary region V_1 of the fluid medium. This means that the integrand of the left-hand side of Eq. (4.3.2) should be identically equal to zero, or

$$\frac{\partial}{\partial t}\left[\rho(u + \tfrac{1}{2} q^2)\right] + \left[\rho(u + \tfrac{1}{2} q^2)\, q_j\right]_{,j} = (\sigma_{ji}\, q_i)_{,j} + \rho\, f_i\, q_i + \rho c \,. \qquad (4.3.3)$$

Equation (4.3.3) is the <u>energy equation</u> or <u>the first law of thermodynamics</u> of a fluid medium in motion.

An alternative form of the energy equation or the first law of thermodynamics can be obtained by multiplying the continuity equation, Eq. (4.1.4), by $(u + \frac{1}{2} q^2)$ and subtracting it from Eq. (4.3.3):

$$\rho\, \frac{\partial(u + \frac{1}{2} q^2)}{\partial t} + \rho\, q_j (u + \tfrac{1}{2} q^2)_{,j} = (\sigma_{ji}\, q_i)_{,j} + \rho\, f_i\, q_i + \rho c \quad , \qquad (4.3.4)$$

or

$$\frac{d(u + \frac{1}{2} q^2)}{dt} = \frac{1}{\rho}\, (\sigma_{ji}\, q_i)_{,j} + f_i\, q_i + c \quad . \qquad (4.3.5)$$

If we multiply the equation of motion, Eq. (4.2.7), by q_i and sum, we obtain

$$\frac{d(\frac{1}{2} q^2)}{dt} = \frac{1}{\rho}\, \sigma_{ji,j}\, q_i + f_i\, q_i \quad , \qquad (4.3.6)$$

which is called the work-kinetic energy equation. We note that the term on the left-hand side of Eq. (4.3.6) is the co-moving rate of change of the kinetic energy of a fluid element per unit time per unit mass, and that the terms on the right-hand side of Eq. (4.3.6) are the work done per unit time per unit mass on the element of the fluid medium.

Subtracting Eq. (4.3.6) from Eq. (4.3.5), we obtain still another form of the first law of thermodynamics:

$$\frac{du}{dt} = \frac{1}{\rho} \sigma_{ji} q_{i,j} + c \quad . \tag{4.3.7}$$

Equation (4.3.7) is one of the most useful forms of the first law of thermodynamics. It separates the first law of thermodynamics from the kinetic motion of the fluid medium. Therefore, many of the thermodynamic concepts pertaining to the equilibrium states of a fluid medium can now be carried over by the application of this equation.

In introducing the continuity equation, the equation of motion, and the first law of thermodynamics, we introduced the following unknown field functions: ρ, q_i, f_i, σ_{ji}, u, c. The total number of unknowns far exceeds the number of equations introduced. We therefore are forced to look for other independent relationships relating these unknowns. These relationships for magneto-fluid flow are introduced in the next two chapters.

CHAPTER V — FIELD THEORY OF ELECTROMAGNETISM*

5.1. Introduction

The usual approach in the discussion of classical electricity and magnetism is to deduce a set of field equations governing electromagnetic interaction with charged particles in vacuum from restricted experimental evidences. These laws are then carried over for electromagnetic interaction within a material medium by arbitrarily setting aside a portion of the charge density and electric currents as material properties. The remainder of the charge density and electric current are then treated as true charge density and current which interact with the modified electromagnetic field. Concepts such as polarization, magnetization, electric and magnetic permeabilities are introduced to discuss the material effects from a macroscopic point of view. When the medium is in motion, these laws are further modified to include the effects caused by the motion of the medium.

This concept of polarization and magnetization is very convenient in treating electromagnetic interactions within a solid continuum. This is not so in the case of conducting fluids. Permanent or slow-varying definitions of a polarized and magnetized material cannot be assumed for such a medium. Therefore, in the study of magneto-fluid flow, we shall treat the individual particles in the medium in direct interaction with the electromagnetic field and with each other. The concept of material electric and magnetic permeability becomes unnecessary in treating the motions of conducting fields. The currents produced in the medium will

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*Formulated for Rationatized MKS units.

be taken as they are in terms of their microscopic origin. The problem of co-moving variation with a medium becomes a consequence instead of a cause in electromagnetic interaction when treated this way.

In what follows, we shall attempt to derive the classical laws of electromagnetic interaction in vacuum through a set of postulates and the concept of retardation potentials without the consideration of the equivalent material effects. We shall then rely on the results of the physics of ionized gases to offer us an Ohm's law pertaining to the actual current in the moving fluid medium. This approach differs from the conventional method of deducing the general laws through a set of restricted equations.

5.2. Charge Density, Current Density, and Continuity Equation of the Law of Conservation of Charge

The charge density $\hat{\rho}$ at a point in a medium is defined as

$$\hat{\rho} = \lim_{\substack{\Delta V \to 0 \\ \Delta V > \Delta V^{\circ}}} \frac{\Delta Q}{\Delta V}. \tag{5.2.1}$$

Since both positive and negative charges may be present in a medium, we can define

$$\hat{\rho}_{\pm} = \lim_{\substack{\Delta V \to 0 \\ \Delta V \to \Delta V^{\circ}}} \frac{\Delta Q_{\pm}}{\Delta V} \tag{5.2.2}$$

where $\pm$ refer to the sign of the charges. Obviously, we have

$$\hat{\rho} = \hat{\rho}_{+} + \hat{\rho}_{-}.$$

The current density J_i at a point in a medium is defined as

$$J_i = \hat{\rho}_{+}(q_{+i}) + \hat{\rho}_{-}(q_{-i}) \tag{5.2.3}$$

where $q_{\pm i}$ are the velocities of the charges $\Delta Q_\pm$ at the point in consideration.

Continuity Equation of Conservation of Charge

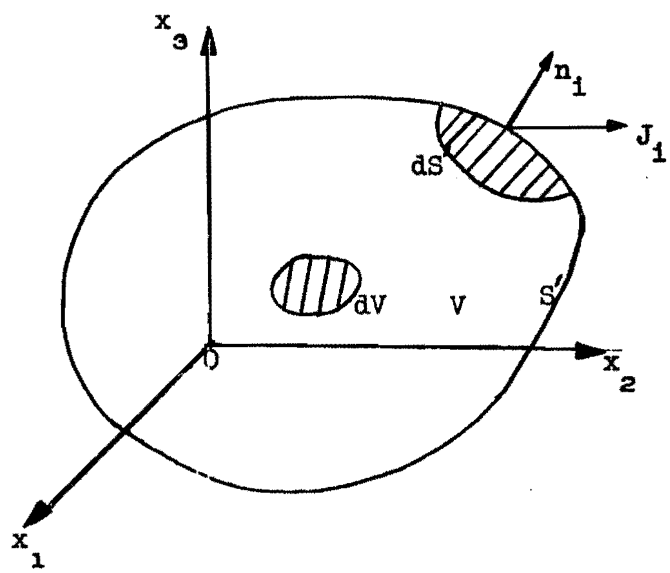

Fig. 5.2.1. Region in Which Charge Motion Exists

Consider a surface S' enclosing a fixed region of space V in which charge motion exists, Fig. 5.2.1. Let us call the unit outward normal of a surface element dS' on S', n_i (x_1, x_2, x_3); the current density. J_i $(x_1, x_2, x_3; t)$; and the charge density, $\hat{\rho}$ $(x_1, x_2, x_3; t)$. It is obvious from the concept of conservation of charge that

$$\int_V \frac{\partial\hat{\rho}}{\partial t}\, dV + \int_{S'} J_i\, n_i\, dS' = 0 . \qquad (5.2.4)$$

Applying the Gauss theorem, (5.2.4) becomes

$$\int_V \left(\frac{\partial\hat{\rho}}{\partial t} + J_{i,i} \right) dV = 0 . \qquad (5.2.5)$$

(5.2.5) should be satisfied for any fixed region of space. This means that the integrand of the left-hand side of (5.2.5) should be identically equal to zero, i.e.,

$$\frac{\partial \hat{\rho}}{\partial t} + J_{i,i} = 0. \tag{5.2.6}$$

(5.2.6) is called the Continuity Equation for the conservation charge.

5.3. Electric and Magnetic Fields

The electric field E_i, and the magnetic induction field B_i are defined as follows:

$$E_i = -\phi_{,i} - \frac{\partial A_i}{\partial t}, \tag{5.3.1}$$

$$B_i = \epsilon_{ijk} A_{k,j} . \tag{5.3.2}$$

where ϕ, A_i are the retarded scalar and vector potentials,

$$\phi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\tilde{\rho}\, dV'}{r},$$

$$A_i = \frac{\mu_0}{4\pi} \int_V \frac{\tilde{J}_i\, dV'}{r}, \tag{5.3.3}$$

with

$$\tilde{\rho} = \hat{\rho}\,(x_1', x_2', x_3'; t - \frac{r}{c_0})$$

$$\tilde{J}_i = J_i\,(x_1', x_2', x_3'; t - \frac{r}{c_0})$$

$$r_i = x_i - x_i'$$

$$r^2 = r_i\, r_i,$$

$$\epsilon_0\mu_0 = \frac{1}{c_0^2}. \tag{5.3.4}$$

The concept of retarded time

$$\tilde{t} = t - \frac{r}{c}, \tag{5.3.5}$$

is designed to take into account the finite speed of propagation c of electromagnetic interaction. The justification of these definitions is shown later when compared with the conventional results deduced from restricted experimental laws.

5.4. Properties of the Electric Field

From (5.3.1), we obtain

$$E_{i,i} = -\phi,_{ii} - \frac{\partial}{\partial t} A_{i,i}$$
$$= -(\phi,_{ii} - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2}) - \frac{\partial}{\partial t} (A_{i,i} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}) \tag{5.4.1}$$

By direct differentiation and using (5.3.3), (5.3.4), we can show that

$$\phi,_{ii} - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\hat{\rho}}{\epsilon_0}, \tag{5.4.2}$$

and

$$A_{i,i} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0. \tag{5.4.3}$$

Therefore, (5.4.1) becomes

$$E_{i,i} = \frac{\hat{\rho}}{\epsilon_0} \tag{5.4.4}$$

Also, from (5.3.1) and (5.3.2), we obtain

$$\epsilon_{ijk} E_{k,j} = -\frac{\partial}{\partial t} [\epsilon_{ijk} A_{k,j}]$$
$$= -\frac{\partial B_i}{\partial t} \tag{5.4.5}$$

5.5. Properties of the Magnetic Induction Field

From (5.3.2), we obtain

$$B_{i,i} = 0 \quad \text{(Solenoidal)} \tag{5.5.1}$$

By taking the curl of (5.3.2), we obtain

$$\begin{aligned} \epsilon_{ijk} B_{k,j} &= \epsilon_{ijk} \epsilon_{krs} A_{s,rj} \\ &= (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) A_{s,rj} \\ &= A_{j,ij} - A_{i,jj} \end{aligned} \tag{5.5.2}$$

Also, from (5.3.1)

$$\frac{\partial E_i}{\partial t} = -\left(\frac{\partial \phi}{\partial t}\right)_{,i} - \frac{\partial^2 A_i}{\partial t^2} \tag{5.5.3}$$

Therefore

$$\epsilon_{ijk} B_{k,j} = \frac{1}{c^2} \frac{\partial E_i}{\partial t} - \left(A_{i,jj} - \frac{1}{c^2} \frac{\partial^2 A_i}{\partial t^2}\right) + \left(A_{j,j} - \frac{1}{c_0^2} \frac{\partial \phi}{\partial t}\right)_{,i} \tag{5.5.4}$$

Now, by direct differentiation

$$A_{i,jj} - \frac{1}{c^2} \frac{\partial^2 A_i}{\partial t^2} = -\mu_0 J_i \tag{5.5.5}$$

Using (5.5.5) and (5.4.3), (5.5.4) becomes

$$\begin{aligned} \epsilon_{ijk} B_{k,j} &= \frac{1}{c_0^2} \frac{\partial E_i}{\partial t} + \mu_0 J_i \\ &= \mu_0 \left[\epsilon_0 \frac{\partial E_i}{\partial t} + J_i\right] \end{aligned} \tag{5.5.6}$$

5.6. Maxwell's Equations

(5.4.2), (5.4.5), (5.5.1), (5.5.6) form a set of interlocking equations relating the electric field E_i and the magnetic induction field B_i.

$$E_{i,i} = \frac{\hat{\rho}}{\epsilon_0}, \tag{5.4.2}$$

$$\epsilon_{ijk} E_{k,j} = -\frac{\partial B_i}{\partial t}, \tag{5.4.5}$$

$$B_{i,i} = 0, \tag{5.5.1}$$

$$\epsilon_{ijk} B_{k,j} = \mu_0 \left[\epsilon_0 \frac{\partial E_i}{\partial t} + J_i \right]. \tag{5.5.6}$$

They are called the Maxwells' equations in vacuum. The charge density $\hat{\rho}$ and current density J_i should be the total contributions of the medium when applied to magneto-fluid flow. "$\hat{\rho}$" is the total charge density at a point in the medium and J_i should include all types of currents other than the vacuum displacement current which is written out explicitly in (5.5.6).

It is possible to define the displacement vector D_i and magnetic field strength H_i as,

$$D_i = \epsilon_0 E_i,$$

$$H_i = \frac{B_i}{\mu_0}. \tag{5.6.1}$$

However, these do not introduce any additional advantage when polarization and magnetization concepts of material medium are not introduced.

The forms of the Maxwell's equation indicate that our initial postulates were correct and justified.

Another fact which is worth noting is the case for electromagnetic interaction in free space where both the charge and current densities are not present. From (5.4.2) and (5.5.5), we note that both the scalar and vector potentials satisfy the wave equation.

$$\phi_{,ii} - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (5.6.2)$$

$$A_{i,jj} - \frac{1}{c_0^2} \frac{\partial^2 A_i}{\partial t^2} = 0 \qquad (5.6.3)$$

The propagation velocity of these waves is $1/c_0^2 = \epsilon_0 \mu_0$. This is one of the assumptions made in introducing these retardation potentials.

5.7. Ohm's Law

Where a conducting medium is in motion, it is possible to separate the current J_i (excluding the displacement current) in terms of a part called the convection current J_i (convection) and a part called the conduction current J_i (conduction). The convection part is given as

$$J_i \text{ (convection)} = q_i \hat{\rho} \; . \qquad (5.7.1)$$

The conduction part should include all the currents not included in convection.

For a fully ionized gas, the conduction current can be shown to be given approximately by the following relationship if inertia effect of the electrons is neglected.

$$J_i + \frac{e\tau}{m(e)} \epsilon_{ijk} J_j B_k = \sigma \left[E_i + \epsilon_{ijk} q_j B_k + \frac{1}{ne} p(e)_{,i} \right] \qquad (5.7.2)$$

where τ is the mean electron collision period,

$m(e)$ is the mass of a single electron,

$p(e)$ is the mean electron pressure,

n is the number of electrons per unit volume, and

$\sigma = \dfrac{ne^2\tau}{m(e)}$ is called the conductivity.

In most of the applications, the terms

$$\frac{e\tau}{m(e)} \epsilon_{ijk} J_j B_k \text{ (Hall effect)} ,$$

$$\frac{\sigma}{ne} p(e)_{,i}$$

are small and can be neglected. The resulting expression when the electron pressure gradient and Hall effect are neglected for J_i becomes

$$J_i = \sigma \left[E_i + \epsilon_{ijk} q_j B_k \right] . \qquad (5.7.3)$$

(5.7.3) can be shown to be true for other types of conducting fluid media as well if secondary effects are neglected. It is called the <u>Ohm's law.</u>

(5.7.3) can be rewritten as

$$J_i = \sigma E_i' \qquad (5.7.4)$$

where

$$E_i' = E_i + \epsilon_{ijk} q_j B_k \qquad (5.7.5)$$

is the effective electric field seen by the moving medium. The term $\epsilon_{ijk} q_j B_k$ is the Lorentz contribution of an apparent electric field due to the motion of the medium.

The term

$$\frac{e\tau}{m(e)} \epsilon_{ijk} J_j B_k$$

becomes important when the spiraling of the electrons about the lines of magnetic field becomes important. It contributes a component of the current in a direction normal to E_i. This current is called the Hall current.

CHAPTER VI — FORMULATION OF MAGNETO-FLUID MECHANICS

6.1. Introduction

In discussing the continuity equation, the equations of motion, and the first law of thermodynamics governing the motion of a fluid medium in Chapter IV, we introduced a total of fifteen unknown field functions: ρ, q_i, f_i, σ_{ij}, u, c. In Chapter V, the classical nonrelativistic theory of electromagnetic interaction was formulated. This involves the introduction of fourteen additional unknowns; E_i, B_i, J_i, ϕ, A_i, ρ through the fourteen equations given by the fundamental postulates* of electromagnetism.

$$\text{(a)}\quad E_i = -\phi_{,i} - \frac{\partial A_i}{\partial t} ,$$

$$\text{(b)}\quad B_i = \epsilon_{ijk} A_{k,j} ,$$

$$\text{(c)}\quad \phi = \frac{1}{4\pi\epsilon_0} \int \frac{\tilde{\rho}}{r} \, dV' ,$$

$$\text{(d)}\quad A_i = \frac{\mu_0}{4\pi} \int \frac{\tilde{J}_i}{r} \, dV' ,$$

$$\text{(e)}\quad \frac{\partial\tilde{\rho}}{\partial t} + J_{i,i} = 0 ,$$

$$\text{(f)}\quad J_i = \sigma \left[E_i + \epsilon_{ijk} q_j B_k\right] ,$$

where

$$\text{(g)}\quad \tilde{\rho} = \hat{\rho}\left(x_1', x_2', x_3'; t - \frac{r}{c_0}\right) ,$$

$$\text{(h)}\quad \tilde{J}_i = J_i\left(x_1', x_2', x_3'; t - \frac{r}{c_0}\right) ,$$

$$\text{(i)}\quad r_i = x_i - x_i' ,$$

$$\text{(j)}\quad c_0 = 1/\sqrt{\epsilon_0 \mu_0} . \qquad (6.1.1)$$

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*Renumbered for ease of reference.

The concept of a Lorentz or ponderomotive force $\bar{f}_i$ was also introduced.

$$\rho \, \bar{f}_i = \epsilon_{ijk} \, J_j \, B_k \quad . \tag{6.1.2}$$

The purpose of this chapter is to formulate a continuum theory of magneto-fluid flow by combining the classical concepts of ordinary fluid mechanics and the concepts of electromagnetic interaction.

We shall assume that the electrically conducting fluid medium is neutrally charged. This means that the limit of observation of the volumetric elements of the fluid medium should be large enough such that the net charge density $\hat{\rho}$ vanishes everywhere in the medium.

$$\hat{\rho} = \hat{\rho}_+ + \hat{\rho}_- = 0 \quad . \tag{6.1.3}$$

This assumption in the flow of ionized gases and conducting liquids is usually realized and does not contribute a serious restriction on the theory to be formulated in this chapter.

Equation (6.1.2) implies that the convection current in the fluid medium vanishes; i.e.,

$$J_i \text{ (convection)} = \hat{\rho} \, q_i = 0 \quad . \tag{6.1.4}$$

From the continuity equation of charges,

$$\frac{\partial \hat{\rho}}{\partial t} + J_{i,i} = 0 \quad , \tag{6.1.1e}$$

and Eq. (6.1.2), we know that the conduction current or the <u>total current</u> J_i is always solenoidal.

$$J_{i,i} = 0 \quad . \tag{6.1.5}$$

This means that the Maxwell's vacuum displacement current $\epsilon_0 \frac{\partial E_i}{\partial t}$, for a neutrally charged fluid medium vanishes. This results in some simplification in the theory of magneto-fluid flow.

6.2. Ponderomotive or Lorentz Force

In a neutrally charged conducting fluid medium, the only body force of electromagnetic origin is the so-called ponderomotive or Lorentz force $\overline{f}_i$, given by one of the equations of the Biot-Savart law,

$$\rho \overline{f}_i = \epsilon_{ijk} J_j B_k \quad . \tag{6.1.2}$$

The magnetic induction field B_k in Eq. (6.1.2) is related to E_i, J_i, $\hat{\rho}$, ϕ, A_i by the fundamental postulates [Eq. (6.1.1)] and through the velocity components q_k.

The fundamental postulates [Eq. (6.1.1)] can be reduced to a set of interlocking equations as shown in Chapter V. For an electrically conducting, neutrally charged medium, these interlocking equations become:*

Maxwell's equations

(a) $E_{i,i} = 0$,

(b) $B_{i,i} = 0$,

(c) $\epsilon_{ijk} E_{k,j} = - \frac{\partial B_i}{\partial t}$,

(d) $\epsilon_{ijk} B_{k,j} = \mu_0 J_i$.

Ohm's law

(e) $J_i = \sigma [E_i + \epsilon_{ijk} q_j B_k]$

*Renumbered for ease of reference.

Continuity equation for charges

$$(f) \quad J_{i,i} = 0 \qquad (6.2.2)$$

These relations are not entirely independent of each other. However, they form a convenient set of equations in formulating problems of magneto-fluid flow.

6.3. Separation of the Stress Tensor, the Kinetic Equation of State, and the Newtonian Fluid

It is always possible to separate the stress tensor σ_{ij} in terms of a scalar function p and a new tensor τ_{ij} as follows:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \qquad (6.3.1)$$

where δ_{ij} is the "Kronecker delta".

We are at liberty to choose the magnitude of the scalar function p (the fluid pressure). For an incompressible isotropic fluid, it can be shown that the magnitude of p has to be equal to the negative of one-third of the algebraic sum of the normal components of the stress tensor, σ_{ij}. For a compressible fluid, one criterion to determine this separation of the stress tensor, σ_{ij} is to assume that the scalar function p will take on the same thermodynamic role, whether dynamic motion exists or not in the fluid, i.e., there exists a kinetic equation of state such that

$$F_1 (p, \rho, T) = 0 \ . \qquad (6.3.2)$$

Another criterion for the separation of the stress tensor σ_{ij} for a compressible fluid is to assume that the dissipation in the viscous fluid due to a dynamic process is contributed entirely by the new stress tensor

τ_{ij} (the viscous stress tensor). Using the kinetic theory of gases, it can be shown that for a monatomic gas, these two criteria imply that the scalar function p again has to be equal to the negative of one-third of the algebraic sum of the normal components of the stress tensor σ_{ij},

$$p = -\frac{1}{3}\sigma_{ii} . \tag{6.3.3}$$

Equation (6.3.3) determines the magnitude of the fluid pressure p for an incompressible isotropic fluid and a monatomic gas. It implies that

$$\tau_{ii} = 0 . \tag{6.3.4}$$

For other types of gases, however, this is not exactly true.

Inasmuch as we have assumed that the viscous stress tensor τ_{ij} contributed the dissipation during a dynamic process of the viscous fluid, it is logical for us to relate the components of the viscous stress tensor τ_{ij} to the velocity gradients $q_{i,j}$. We note that the velocity gradients $q_{i,j}$ can be separated into one symmetric tensor ϵ_{ij} and one anti-symmetric tensor ω_{ij} as follows:

$$q_{i,j} = \epsilon_{ij} - \omega_{ij} , \tag{6.3.5}$$

where

$$\epsilon_{ij} = \frac{1}{2}(q_{i,j} + q_{j,i}) \tag{6.3.6}$$

is called the <u>velocity strain tensor</u>, and

$$\omega_{ij} = \frac{1}{2}(q_{j,i} - q_{i,j}) \tag{6.3.7}$$

is called the vorticity tensor.

The vorticity tensor ω_{ij} can be shown to be a measure of the rate of rigid body rotation of the fluid elements and the velocity strain tensor ϵ_{ij} is a measure of the distortion of the fluid elements. Therefore, it is logical for us to relate the viscous stress tensor τ_{ij} in terms of the velocity strain tensor ϵ_{ij} only. If the relationship between τ_{ij} and ϵ_{ij} is linear then the fluid is called a Newtonian fluid.

$$\tau_{ij} = A_{ijk\ell}\,\epsilon_{k\ell} \quad , \tag{6.3.8}$$

where the components of the tensor $A_{ijk\ell}$ are constants. Equation (6.3.8) implies that when all the components of $\epsilon_{k\ell}$ are zero, the viscous stress tensor vanishes. Both the viscous stress tensor τ_{ij} and the velocity strain tensor ϵ_{ij} are symmetrical. This means that

$$A_{ijk\ell} = A_{jik\ell} = A_{ij\ell k} \quad . \tag{6.3.9}$$

For an isotropic medium, the tensor $A_{ijk\ell}$ should be invariant under rotations and reflections of the coordinate system. Combining this restriction with Eq. (6.3.9), we can show that $A_{ijk\ell}$ can be expressed in terms of two scalar constants λ and μ.

$$A_{ijk\ell} = \lambda\,\delta_{ij}\,\delta_{k\ell} + \mu\,(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) \quad , \tag{6.3.11}$$

for an isotropic Newtonian fluid.

Therefore, for an isotropic Newtonian fluid, the relationship between the viscous stress tensor and the velocity strain tensor is

$$\tau_{ij} = [\lambda\,\delta_{ij}\,\delta_{k\ell} + \mu\,(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk})]\,\epsilon_{k\ell} \tag{6.3.12}$$

or

$$\tau_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} , \qquad (6.3.13)$$

or

$$\tau_{ij} = \lambda q_{k,k} \delta_{ij} + \mu (q_{i,j} + q_{j,i}) . \qquad (6.3.14)$$

Experimental evidence has shown that the constant μ is always positive and real. It is called the "first coefficient of viscosity."

Contracting Eq. (6.3.14), we obtain,

$$\tau_{ii} = (3\lambda + 2\mu) q_{i,i} . \qquad (6.3.15)$$

The constant $(3\lambda + 2\mu)$ is called the "bulk (or second) coefficient of viscosity." For an incompressible fluid,

$$q_{i,i} = 0 . \qquad (6.3.16)$$

This means that

$$\tau_{ii} = 0 , \qquad (6.3.4)$$

and therefore

$$p = -\frac{1}{3} \sigma_{ii} , \qquad (6.3.3)$$

for an isotropic incompressible fluid. For a monatomic gas, τ_{ii} vanishes for another reason. Therefore, we deduce that

$$\lambda = -\frac{2}{3}\mu , \qquad (6.3.17)$$

for a monatomic gas.

In general, however, Eq. (6.3.17) does not hold exactly for an incompressible fluid or a polyatomic gas.

Summarizing, we can express the stress tensor σ_{ij} in terms of the fluid pressure p and the velocity gradients $q_{i,j}$ for an isotropic Newtonian fluid as follows:

$$\sigma_{ij} = (-p + \lambda q_{k,k}) \delta_{ij} + \mu (q_{i,j} + q_{j,i}) . \tag{6.3.18}$$

For a fluid which satisfies the condition,

$$\tau_{ii} = 0 , \tag{6.3.4}$$

Equation (6.3.18) becomes

$$\sigma_{ij} = -(p + \frac{2}{3} \mu q_{k,k}) \delta_{ij} + \mu (q_{i,j} + q_{j,i}) . \tag{6.3.19}$$

If the fluid is incompressible, ($\rho = C$), then the continuity equation for fluid motion states that

$$q_{k,k} = 0 . \tag{6.3.16}$$

Therefore, Eq. (6.3.18) or Eq. (6.3.19) becomes

$$\sigma_{ij} = -p \delta_{ij} + \mu (q_{i,j} + q_{j,i}) , \tag{6.3.20}$$

for an incompressible, isotropic, Newtonian fluid.

6.4. Fourier Law of Heat Conduction

The heat flux b_i due to a temperature gradient $T_{,i}$ is called heat conduction.

It is not hard to show that the heat transfer due to heat conduction, $\bar{c}$, into a fluid element per unit time per unit volume is

$$\bar{c} = - b_{i,i} \quad . \tag{6.4.1}$$

The heat flux b_i is usually related to the temperature gradient by the following expression:

$$b_i = - k\, T_{,i} \quad , \tag{6.4.2}$$

where k = constant is called the "heat-conductivity". Equation (6.4.2) is called the Fourier law of heat conduction.

6.5. Joule Heating

The work done $\frac{d\bar{w}}{dt}$ by the electromagnetic field on a neutrally charged, conducting fluid element per unit volume per unit time is obviously

$$\frac{d\bar{w}}{dt} = E_i\, J_i \quad , \tag{6.5.1}$$

where

$$J_i = \sigma\, [E_i + \epsilon_{ijk}\, q_j\, B_k] \quad . \tag{6.2.2e}$$

Therefore,

$$\frac{d\bar{w}}{dt} = \frac{J^2}{\sigma} + \epsilon_{ijk}\, q_i\, J_j\, B_k \quad . \tag{6.5.2}$$

The second term on the righthand side of Eq. (6.5.2) we notice is the work done by the ponderomotive force $\bar{f}_i$. The first term on the righthand side of Eq. (6.5.2) is a dissipative term (non-negative term) which can be considered as a heat transfer term due to electromagnetic interaction. This non-negative term $\frac{J^2}{\sigma}$ is usually called Joule heating.

$$\rho \, \bar{\bar{c}} \text{ (Joule heating)} = \frac{J^2}{\sigma} \tag{6.5.3}$$

The heat transfer term c in the first law of thermodynamics can therefore be written as

$$c = \bar{c} + \bar{\bar{c}} + c' \quad , \tag{6.5.4}$$

where

$\bar{c} = -\frac{1}{\rho} b_{i,i}$ is due to conduction,

$\bar{\bar{c}} = \frac{J^2}{\rho\sigma}$ is due to Joule heating, and

c' is the heat transfer due to radiation.

6.6. Caloric Equation of State

The internal energy per unit mass u as appeared in the first law of thermodynamics is usually assumed to be related to the fluid density ρ, and the absolute temperature T. through the Caloric equation of state,

$$F_2 (u, \rho, T) = 0 \tag{6.6.1}$$

The exact form of Eq. (6.6.1) depends on the kinetic equation of state, the second law of thermodynamics, and the specific heat at constant volume. This will be discussed in detail in Chapter VII.

6.7. Conservative Forces

Body forces $\bar{\bar{f}}_i$ of non-electromagnetic origin are usually conservative forces derivable from a scalar function of position

$$\bar{\bar{f}}_i = -\Omega_{,i} \quad , \tag{6.7.1}$$

where

$$\Omega = \Omega\,(x_1, x_2, x_3) \tag{6.7.2}$$

is called the force potential which is usually known in advance. The body force term in the equations of motion or the first law of thermodynamics can therefore be written as

$$f_i = \bar{f}_i + \bar{\bar{f}}_i + f'_i \quad , \tag{6.7.3}$$

where

$\bar{f}_i = \frac{1}{\rho}\,\epsilon_{ijk}\,J_j\,q_k$ is the Lorentz force,

$\bar{\bar{f}}_i = -\Omega_{,i}$ are known conservative forces, and

f'_i are other body forces not accounted for in $\bar{f}_i$ and $\bar{\bar{f}}_i$.

6.8. Formulation of Magneto-Fluid Flow

In the previous discussions, we have introduced no less than 48 equations governing the 48 unknown field functions: ρ, q_i, f_i, $\bar{f}_i$, $\bar{\bar{f}}_i$, σ_{ij}, p, τ_{ij}, u, T, E_i, B_i, J_i, ϕ, A_i, $\hat{\rho}$, c, $\bar{c}$, $\bar{\bar{c}}$, b_i. Combining these equations we obtain a set of equations governing the motion of an electrically conducting, neutrally charged, isotropic, Newtonian fluid medium within an externally applied magnetic field. The interlocking and the dependent characteristics of the electromagnetic equations cause an apparent excess of the number of equations over the unknowns. However, this set of equations is one of the most convenient sets of equations governing magneto-fluid flow.

Continuity equation for fluid motion.

$$\frac{\partial \rho}{\partial t} + (\rho\, q_j)_{,j} = 0 \qquad (4.1.4)$$

Equations of motion

$$\rho \frac{dq_i}{dt} = - p_{,i} + \tau_{ji,j} - \rho\, \Omega_{,i} + \epsilon_{ijk} J_j B_k \qquad (6.8.1)$$

First law of thermodynamics

$$\frac{du}{dt} + p \frac{d\left(\frac{1}{\rho}\right)}{dt} = \frac{1}{\rho} \tau_{ji}\, q_{i,j} - \frac{1}{\rho} b_{i,i} + \bar{\bar{c}} + c' \quad . \qquad (6.8.2)$$

Maxwell's Equations

(a) $E_{i,i} = 0$,

(b) $B_{i,i} = 0$,

(c) $\epsilon_{ijk} E_{k,j} = - \dfrac{\partial B_i}{\partial t}$,

(d) $\epsilon_{ijk} B_{k,j} = \mu_0 J_i$.

Ohm's law

(e) $J_i = \sigma\, [E_i + \epsilon_{ijk}\, q_j\, B_k]$.

Continuity equation of charges

$$(f) \quad J_{i,i} = 0 \quad . \qquad (6.2.2)$$

Fourier law of heat conduction

$$b_i = -k\, T_{,i} \quad . \qquad (6.4.2)$$

Newtonian viscous law for an isotropic fluid

$$\tau_{ij} = \lambda\, \delta_{ij}\, q_{k,k} + \mu\, (q_{i,j} + q_{j,i}) \quad . \qquad (6.3.14)$$

Kinetic equation of state

$$F_1\, (p, \rho, T) = 0 \quad . \qquad (6.3.2)$$

Caloric equation of state

$$F_2\, (u, \rho, T) = 0 \quad . \qquad (6.6.1)$$

Joule Heating

$$\bar{\bar{c}} = \frac{J^2}{\rho\sigma} \quad . \qquad (6.5.3)$$

In formulating the theory, we also introduced six constants μ_0, ϵ_0, σ, λ, μ, and k. The constants, "ϵ_0, μ_0", are given for a given set of electromagnetic units. "σ, λ, μ, and k" are either determined from experiments or based on the results of statistical mechanics. The fluid medium is assumed to be isotropic, Newtonian and follows the Fourier heat conduction law. The Ohm's law of the form of Eq. (6.2.2e) implies that the Hall current is neglected in the discussion. The unknown field functions are ρ, q_i, p, τ_{ij}, E_i, B_i, J_i, u, T, b_i, $\bar{\bar{c}}$.

CHAPTER VII – ALTERNATIVE FORMULATIONS AND SECOND LAW OF THERMODYNAMICS

7.1. Introduction

There are several alternative forms of the formulation of magneto-fluid flow. For example, the ponderomotive force can be easily expressed in terms of an equivalent tensor of the second rank called the Maxwell's stress tensor. The motion of the magnetic induction field can be described in terms of a vector equation called the induction equation. It is also possible to define a specific entropy per unit mass such that the first law and the second law of thermodynamics can be expressed in a single equation.

7.2. Maxwell's Stress Tensor

The ponderomotive or Lorentz force

$$\rho \bar{f}_i = \epsilon_{ijk} J_j B_k \quad , \tag{6.1.2}$$

can be combined with one of the equations of the Maxwell's laws

$$\epsilon_{ijk} B_{k,j} = \mu_0 J_i \quad , \tag{6.2.2d}$$

such that this force is expressed in terms of the magnetic induction field B_i alone.

$$\rho f_i = \epsilon_{ijk} \left(\frac{1}{\mu_0} \epsilon_{jrs} B_{s,r}\right) B_k$$

$$= \frac{1}{\mu_0} (\delta_{kr} \delta_{is} - \delta_{ks} \delta_{ir}) B_{s,r} B_k$$

$$= \frac{1}{\mu_0} \left(B_{i,k} B_k - B_{k,i} B_k \right)$$

$$= - \frac{1}{\mu_0} \left(\frac{B^2}{2} \right)_{,i} + \frac{1}{\mu_0} B_{i,k} B_k \; . \qquad (7.2.1)$$

But,

$$B_{k,k} = 0 \; . \qquad (6.2.2b)$$

Therefore, Eq. (7.2.1) becomes

$$\rho f_i = -\left(\frac{B^2}{2\mu_0} \right)_{,i} + \left(\frac{B_i B_k}{\mu_0} \right)_{,k} \; , \qquad (7.2.2)$$

or

$$\rho f_i = \sigma^*_{ij,j} \; , \qquad (7.2.3)$$

where

$$\sigma^*_{ij} = - p^* \delta_{ij} + \tau^*_{ij} \; , \qquad (7.2.4)$$

$$p^* = \frac{B^2}{2\mu_0} \; , \qquad (7.2.5)$$

$$\tau^*_{ij} = \frac{B_i B_k}{\mu_0} \; . \qquad (7.2.6)$$

"σ^*_{ij}" is called the Maxwell's stress tensor. It is composed of an equivalent magnetic pressure $p^* = \frac{B^2}{2\mu_0}$ and a tension $\frac{B^2}{\mu_0}$ along the lines of force.

Using Eq. (7.2.3), the equations of motion for the fluid medium can be written as

$$\rho \frac{dq_i}{dt} = \bar{\sigma}_{ji,j} + f'_i \; , \qquad (7.2.7)$$

where $$\bar{\sigma}_{ji} = \sigma_{ji} + \sigma^*_{ji} \text{ , and}$$

f'_i represents other body forces.

7.3. The "Induction Equation"

From the Ohm's law, we have

$$E_i = \frac{J_i}{\sigma} - \epsilon_{ijk}\, q_j\, B_k \quad . \tag{7.3.1}$$

Therefore,

$$\epsilon_{ijk}\, E_{k,j} = \frac{1}{\sigma}\, \epsilon_{ijk}\, J_{k,j} - \epsilon_{ijk}\, \epsilon_{krs}\, (q_r\, B_s)_{,j} \quad . \tag{7.3.2}$$

Substituting Eq. (7.3.2) into one of the Maxwell's equations

$$\epsilon_{ijk}\, E_{k,j} = - \frac{\partial B_i}{\partial t} \quad , \tag{6.2.2c}$$

we obtain

$$\frac{\partial B_i}{\partial t} - \epsilon_{ijk}\, \epsilon_{krs}\, (q_r\, B_s)_{,j} + \frac{1}{\sigma}\, \epsilon_{ijk}\, J_{k,j} = 0 \quad . \tag{7.3.3}$$

But from Eq. (6.2.2d)

$$J_k = \frac{1}{\mu_0}\, \epsilon_{krs}\, B_{s,r} \quad . \tag{7.3.4}$$

Therefore, Eq. (7.3.3) becomes

$$\frac{\partial B_i}{\partial t} - \epsilon_{ijk}\, \epsilon_{krs}\, (q_r\, B_s)_{,j} + \frac{1}{\mu_0\, \sigma}\, \epsilon_{ijk}\, \epsilon_{krs}\, B_{s,rj} = 0, \tag{7.3.4}$$

or,

$$\frac{\partial B_i}{\partial t} - (\delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr})\,(q_r\,B_s)_{,j} + \frac{1}{\mu_0\,\sigma}\,(\delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr})\,B_{s,rj} = 0\ , \tag{7.3.5}$$

or,

$$\frac{\partial B_i}{\partial t} - (q_i\,B_j)_{,j} + (q_j\,B_i)_j + \frac{1}{\mu_0\,\sigma}\,(B_{j,ij} - B_{i,jj}) = 0 . \tag{7.3.6}$$

But,

$$B_{i,i} = 0\ . \tag{6.2.2b}$$

Therefore, Eq. (7.3.6) becomes

$$\frac{\partial B_i}{\partial t} - (q_i\,B_j)_{,j} + (q_j\,B_i)_{,j} - \frac{1}{\mu_0\,\sigma}\,B_{i,jj} = 0\ . \tag{7.3.7}$$

Equation (7.3.7) is called the <u>induction equation</u>, which is sometimes quite useful in the study of magneto-fluid flow.

7.4. Second Law of Thermodynamics and Entropy Production

The first law of thermodynamics

$$\frac{du}{dt} = \frac{1}{\rho}\,\sigma_{ji}\,q_{i,j} + c \tag{4.3.7}$$

can be written as

$$\frac{du}{dt} + p\,\frac{d\left(\frac{1}{\rho}\right)}{dt} = \frac{1}{\rho}\,\tau_{ji}\,q_{i,j} + c\ , \tag{7.4.1}$$

where we have separated the stress tensor according to the separation equation

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij} \quad . \qquad (6.3.1)$$

The viscous stress tensor is assumed to contribute the material dissipation in the medium. For a reversible process the viscous stress tensor must not be present in the fluid medium. Therefore, for a reversible process

$$\frac{du}{dt} + p\,\frac{d\left(\frac{1}{\rho}\right)}{dt} = \frac{db_{(rev.)}}{dt} \quad , \qquad (7.4.2)$$

where we have called the time rate of heat transferred reversibly into the medium $\frac{db_{(rev.)}}{dt}$. Or in differential form (following the motion of the particles along)

$$du + p d\left(\frac{1}{\rho}\right) = đb_{(rev.)} \quad , \qquad (7.4.3)$$

where the dash on the symbol đb indicates that it is not an exact differential. Equation (7.4.3) is identical with the differential form of the first law of thermodynamics for a fluid element undergoing an equilibrium thermodynamic process. The second law of thermodynamics for reversible processes states that

$$ds = \frac{đb_{(rev.)}}{T} = \frac{du + p d\left(\frac{1}{\rho}\right)}{T} \quad , \qquad (7.4.4)$$

where

$$s = s\,(p, \rho) \quad , \qquad (7.4.5)$$

is a thermodynamic variable defined by Eq. (7.4.4) called the specific entropy per unit mass.

If we consider the specific internal energy u to be given by the caloric equation of state

$$F_2 \; (u, \rho, T) = 0 \; , \tag{6.6.1}$$

or

$$\begin{aligned} u &= u\,(\rho, T) \\ &= u\,(v, T) \; , \end{aligned} \tag{7.4.6}$$

where

$$v = \frac{1}{\rho} \; . \tag{7.4.7}$$

is the specific volume per mass; then from Eq. (7.4.4), we obtain

$$ds = \frac{\frac{\partial u}{\partial T} \; dT + (p + \frac{\partial u}{\partial v}) \; dv}{T} \; . \tag{7.4.8}$$

Since ds is a perfect differential, we know that

$$\frac{\partial}{\partial v}\left(\frac{1}{T} \; \frac{\partial u}{\partial T}\right) = \frac{\partial}{\partial T}\left[\frac{1}{T} \; (p + \frac{\partial u}{\partial v})\right] \; , \tag{7.4.9}$$

or

$$\frac{\partial u}{\partial v} = T \; \frac{\partial p}{\partial T} - p \; . \tag{7.4.10}$$

Therefore,

$$du = \left(\frac{\partial u}{\partial T}\right)_v dT + \left(\frac{\partial u}{\partial v}\right)_T dv$$

$$= \left(\frac{\partial u}{\partial T}\right)_v dv + \left(T\frac{\partial p}{\partial T} - p\right) dv \quad . \qquad (7.4.11)$$

This means that the specific energy per unit mass u is defined if the kinetic equation of state

$$F_1 (p, \rho, T) = 0 \qquad (6.3.2)$$

is given and if the <u>specific heat for constant volume</u>

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v \qquad (7.4.12)$$

is known.

Using the definition of specific entropy in Eq.(7.4.8), Eq. (7.4.2) becomes

$$T\frac{ds}{dt} = \frac{du}{dt} + p\frac{d\left(\frac{1}{\rho}\right)}{dt} \quad . \qquad (7.4.13)$$

This is the <u>combined first and second law of thermodynamics for a reversible process</u>.

For a fluid process with viscous dissipation and irreversible heat transfer in magneto-fluid mechanics, the first law of thermodynamics given in Eq. (6.8.2) should be used

$$\frac{du}{dt} + p\frac{d\left(\frac{1}{\rho}\right)}{dt} = \frac{1}{\rho}\tau_{ji}\, q_{i,j} - \frac{1}{\rho} b_{i,i} + \bar{\bar{c}} + c' \quad . \qquad (6.8.2)$$

Neglecting the radiation heat transfer c' and using Eqs. (6.5.3) and (7.4.8), we obtain

$$T \frac{ds}{dt} = \frac{1}{\rho} \tau_{ji} q_{i,j} + \frac{J^2}{\rho \sigma} - \frac{1}{\rho} b_{i,i} . \qquad (7.4.14)$$

If the net heat transfer due to heat conduction and Joule heating vanishes, then

$$T \frac{ds}{dt} = \frac{1}{\rho} \tau_{ji} q_{i,j} . \qquad (7.4.15)$$

But the <u>second law of thermodynamics for irreversible processes</u> states that

$$\frac{ds}{dt} > 0 \qquad (7.4.16)$$

for an adiabatic process ($c = 0$). Therefore, Eqs. (7.4.15) and (7.4.16) state that

$$\nu = \frac{1}{T \rho} \tau_{ji} q_{i,j} \qquad (7.4.17)$$

is always <u>non-negative</u>. "ν" is called the <u>viscous dissipation function</u>.

Combining Eqs. (7.4.14) and (7.4.17), we obtain

$$\frac{ds}{dt} = \nu + \frac{J^2}{T \rho \sigma} - \frac{b_{i,i}}{T \rho} \qquad (7.4.18)$$

or,

$$\frac{ds}{dt} + \frac{1}{\rho} \left(\frac{b_i}{T} \right)_{,i} = \nu + \frac{J^2}{T \sigma \rho} - \frac{b_i T_{,i}}{\rho T^2} . \qquad (7.4.19)$$

Equation (7.4.19) is the combined statement of the first and second laws of thermodynamics for irreversible processes. Since the heat flux b_i is always in the opposite direction of $T_{,i}$, the terms on the right of Eq. (7.4.19) are all non-negative. For a flow process where the co-moving rate of change of the specific entropy per unit time per unit mass of a fluid element vanishes, i.e.

$$\frac{ds}{dt} = 0 \quad ; \qquad (7.4.20)$$

there is a net outflux of entropy flow per unit time per unit mass from the fluid element

$$\frac{1}{\rho}\left(\frac{b_i}{T}\right)_{,i} = \nu + \frac{J^2}{T\,\sigma\,\rho} - \frac{b_i\,T_{,i}}{\rho\,T^2} \ , \qquad (7.4.21)$$

which is always non-negative. This outflux of entropy must somehow be produced within the fluid element. It is called the entropy production. The entropy production which characterizes irreversible magneto-fluid flow is due to viscous dissipation, Joule heating, and irreversible heat conduction.

CHAPTER VIII — SIMILARITY PARAMETERS OF MAGNETO-FLUID FLOW

8.1. Introduction

In this chapter, we shall compare the relative magnitudes of the terms appearing in the governing equations of magneto-fluid mechanics in terms of the so-called dimensionless similarity parameters.

When a particular term or set of terms in the governing equations appears to contribute negligible effects on a given problem in magneto-fluid flow, this term or set of terms can be deleted from the governing equations and thereby simplifying the analysis of the given problem. The same idea applies in experimental investigations of magneto-fluid flow. When an experiment is simulated in the laboratory, it will only be necessary to keep those similarity parameters which arose from the more important terms in the governing equations alike.

8.2. Nondimensionalized Equations and Similarity Parameters for Magneto-Fluid Flow with a Unique p-ρ-relationship

Let us first consider the simple case of magneto-fluid flow where the fluid pressure p and the fluid density ρ are uniquely related,

$$p = p(\rho) \quad . \tag{8.2.1}$$

In addition, let us also assume that

(a) $\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$,

(b) $\tau_{ij} = \lambda\,\delta_{ij}\,q_{k,k} + \mu\,(q_{i,j} + q_{j,i})$,

(c) $\lambda = -\frac{2}{3}\,\mu$,

(d) $f_i = \frac{1}{\rho}\,\epsilon_{ijk}\,J_j\,B_k$, and

(e) $J_i = \sigma\,[E_i + \epsilon_{ijk}\,q_j\,B_k]$,

(f) $$\frac{\partial}{\partial t} = 0 \quad . \tag{8.2.2}$$

The set of equations governing this type of magneto-fluid flow is given as follows:

(a) Continuity Equation

$$(\rho \, q_j)_{,j} = 0 \quad ,$$

(b) Equations of Motion

$$\rho \, q_j \, q_{i,j} = -p_{,i} + \mu \left(\frac{1}{3} q_{j,ji} + q_{i,jj} \right) - \frac{1}{\mu_0} \left[\left(\frac{B^2}{2} \right)_{,i} - B_j \, B_{i,j} \right] \quad ,$$

(c) Induction Equation

$$(q_j \, B_i - q_i \, B_j)_{,j} = \frac{1}{\mu_0 \, \sigma} \, B_{i,jj} \quad ,$$

(d) Solenoidal Property of B_i

$$B_{i,i} = 0 \quad , \tag{8.2.3}$$

and

$$p = p \, (\rho) \quad . \tag{8.2.1}$$

We note that due to the assumption of the existence of a unique p-ρ-relationship, the first law of thermodynamics and the equations of state are not included in the set of equations governing the fluid motion.

Let us now choose the following set of dimensionless variables:

$$\hat{p} = \frac{p}{\tilde{\tilde{p}}} \quad ,$$

$$\hat{\rho} = \frac{\rho}{\tilde{\tilde{\rho}}} \quad ,$$

$$\hat{q}_i = \frac{q_i}{\tilde{q}} ,$$

$$\hat{B}_i = \frac{B_i}{\tilde{B}} ,$$

$$\hat{x}_i = \frac{x_i}{\tilde{x}} , \qquad (8.2.4)$$

where $\tilde{p}$ is a certain constant characteristic pressure in the flow,

$\tilde{q}$ is a certain constant characteristic velocity in the flow,

$\tilde{B}$ is a certain constant characteristic magnetic induction field, and

$\tilde{x}$ is a certain constant characteristic length.

Transforming Eq. (8.2.3) by Eq. (8.2.4), we obtain

$$\text{(a)} \quad (\hat{\rho}\, \hat{q}_j)_{,\hat{j}} = 0 ,$$

$$\text{(b)} \quad \hat{q}_j\, \hat{q}_{i,\hat{j}} = -\frac{1}{2} C_{(P)}\, \hat{p}_{,\hat{i}} + \frac{1}{R}\left[\frac{1}{3}\, \hat{q}_{j,\hat{j}\hat{i}} + \hat{q}_{i,\hat{j}\hat{j}} \right] - \frac{1}{A^2}\left[\left(\frac{\hat{B}^2}{2}\right)_{,\hat{i}} - \hat{B}_j\, \hat{B}_{i,\hat{j}} \right] ,$$

$$\text{(c)} \quad (\hat{q}_j\, \hat{B}_i - \hat{q}_i\, \hat{B}_j)_{,\hat{j}} = \frac{1}{R_{(m)}}\, \hat{B}_{i,\hat{j}\hat{j}} ,$$

$$\text{(d)} \quad \hat{B}_{i,\hat{i}} = 0 ,$$

$$\text{(e)} \quad \hat{p} = \hat{p}\,(\hat{\rho}) . \qquad (8.2.5)$$

where $C_{(P)} = \dfrac{\tilde{p}}{\frac{1}{2}\tilde{\rho}\,\tilde{q}^2}$ is called the characteristic <u>pressure coefficient</u>,

$R = \dfrac{\tilde{\rho}\,\tilde{q}\,\tilde{x}}{\mu}$ is called the characteristic <u>Reynolds number</u>,

$A^2 = \dfrac{\tilde{\rho}\,\mu_0\,\tilde{q}^2}{\tilde{B}^2}$ is called the characteristic Alfvén number, and

$R_{(m)} = \mu_0 \sigma \tilde{q} \tilde{x}$ is called the characteristic magnetic Reynolds number.

From elementary gas dynamics, we know that the ratio

$$\frac{\tilde{p}}{\tilde{\rho}}$$

is a measure of certain characteristic sonic velocity in the flow. Therefore, the characteristic pressure coefficient $C_{(P)}$ can be visualized as the measure of the reciprocal of the square of certain characteristic Mach number M in the flow, i.e.,

$$C_{(P)} \approx \frac{1}{M^2} \quad . \tag{8.2.6}$$

As we shall see later in Chapter IX, the value

$$\frac{\tilde{B}^2}{\tilde{\rho}\, \mu_0}$$

is equal to the square of the speed of propagation of nondissipative magneto-fluid waves in a conducting medium. Therefore, the Alfvén number is the ratio of the magnitudes of the characteristic flow velocity to the characteristic Alfvén wave velocity.

For magneto-fluid flows satisfying the restrictions given by Eqs. (8.2.1) and (8.2.2) to be dynamically similar, it is necessary for them to have the same characteristic values of M, R, A, and $R_{(m)}$ in addition to the requirement of having identical dimensionless boundary conditions. These characteristic numbers are usually called the similarity parameters.

In practical problems of magneto-fluid flow, these characteristic numbers have different relative magnitudes. It is usually only necessary to retain those terms which are predominating in the governing equations.

Some of the possible types of magneto-fluid flow categorized according to the magnitudes of R and $R_{(m)}$ are listed as follows:

1. Inviscid, magnetic-predominating flow: $R \to \infty$, $R_{(m)} \ll 1$.
2. Inviscid, magnetic boundary layer flow: $R \to \infty$, $R_{(m)} \gg 1$.
3. Viscous and magnetic boundary layer flow: $R \gg 1$, $R_{(m)} \gg 1$.
4. Viscous and magnetic-predominating flow: $R \ll 1$, $R_{(m)} \ll 1$.

There is <u>another</u> dimensionless similarity parameter characterizing the relative magnitudes of the magnetic and viscous forces in magneto-fluid flow. It can be obtained from the equations of motion by expressing the ponderomotive force in terms of the conductivity σ using the Ohm's law and by comparing this term with the viscous term. It is called the <u>Hartmann number</u>, and defined as follows:

$$\hat{H} = \frac{R\, R_{(m)}}{A^2} = \sqrt{\frac{\sigma}{\mu}}\ \tilde{B}\,\tilde{x} \ . \qquad (8.2.7)$$

From Eq. (8.2.7), we know that $\hat{H}$ is not a new independent similarity parameter. However, it is a very convenient parameter to use when comparing the viscous and magnetic forces in magneto-fluid flow if the fluid is not perfectly conducting.

8.3. <u>Additional Similarity Parameters Arising from the First Law of Thermodynamics</u>

If there does not exist a unique p-ρ-relationship, then the magneto-fluid flow should also be governed by the first law of thermodynamics and the equations of state. For an ideal gas, it will not be hard to show that the additional similarity parameters introduced for such type of flow are the <u>Prandtl number</u>

$$P = \frac{c_{(p)} \mu}{k} , \tag{8.3.1}$$

where $c_{(p)}$ is the specific heat per unit mass for constant pressure, and the relative energy parameter

$$\xi = \frac{q^2}{c_{(p)} T} . \tag{8.3.2}$$

The Prandtl number P characterizes the relative magnitudes of viscous dissipation and heat conduction. The relative energy parameter ξ, as the name implies, characterizes the relative magnitudes of the kinetic energy per unit mass to the specific enthalpy h of the fluid defined by

$$h = u + \frac{p}{\rho} . \tag{8.3.3}$$

The characteristic parameter J indicating the relative magnitudes of Joule heating and heat conduction can be expressed in terms of the Afvén number and the magnetic Reynolds number as follows:

$$J = \frac{\tilde{B}^2}{\tilde{\rho} \mu_0 \tilde{q}^3 \tilde{x}} = \frac{1}{A^2 R_{(m)}} , \tag{8.3.4}$$

and is not an independent parameter.

Therefore, a complete set of similarity parameters for magneto-fluid flow can be chosen as follows: M, R, A, $R_{(m)}$, P, and ξ. Another convenient set is: M, R, A, H, P, and ξ.

CHAPTER IX — ALFVÉN WAVES

9.1. Introduction

It is possible to deduce a propagation velocity for small disturbances in an incompressible, inviscid, and perfectly conducting fluid in the presence of a uniform magnetic field in analogy with the discussion of sonic disturbances in an ordinary compressible inviscid fluid medium. This type of wave propagation is called an Alfvén wave.

9.2. Governing Equations for Nondissipative, Incompressible Magneto-Fluid Flow

For a perfectly conducting fluid, the Ohm's law becomes

$$E_i = -\epsilon_{ijk} q_j B_k \quad , \tag{9.2.1}$$

and the current is determined from Eq. (6.2.2e)

$$J_i = \frac{1}{\mu_0} \epsilon_{ijk} B_{k,j} \quad . \tag{9.2.2}$$

Therefore, the complete set of equations for nondissipative, incompressible magneto-fluid flow in the absence of other body forces is:

$$\text{(a)} \quad \rho \frac{\partial q_i}{\partial t} + \rho q_j q_{i,j} = -p_{,i} - \left(\frac{B^2}{2 \mu_0} \right)_{,i} + \left(\frac{B_i B_j}{\mu_0} \right)_{,j} \quad ,$$

$$\text{(b)} \quad \frac{\partial B_i}{\partial t} = B_j q_{i,j} - q_j B_{i,j} \quad ,$$

$$\text{(c)} \quad q_{i,i} = 0 \quad ,$$

$$\text{(d)} \quad B_{i,i} = 0 \quad . \tag{9.2.3}$$

We note that Eqs. (9.2.3c, d) are added restrictions on the field vectors q_i, B_i which are governed by Eqs. (9.2.3a, b).

9.3. Small Perturbation Equations from Equilibrium

Let us assume that the fluid is essentially in equilibrium with a uniform magnetic induction field, $B_i(0)$. Consider small disturbances in the fluid such that

$$
\begin{aligned}
&(a) \quad q_i = \epsilon\, \tilde{q}_i \;, \\
&(b) \quad B_i = B_i(0) + \epsilon\, \tilde{B}_i \;, \\
&(c) \quad p = p(0) + \epsilon\, \tilde{p} \;, \\
&(d) \quad \epsilon \ll 1 \;.
\end{aligned}
\qquad (9.3.1)
$$

where $B_i(0)$, $p(0)$ are constants.

Inserting Eq. (9.3.1) into Eq. (9.2.3) and combining, we obtain to the lowest order of ϵ,

$$
\begin{aligned}
&(a) \quad \rho \frac{\partial \tilde{q}_i}{\partial t} = -\left[\tilde{p} + \frac{1}{\mu_0} B_j(0)\, \tilde{B}_j\right]_{,i} + \frac{1}{\mu_0} B_j(0)\, \tilde{B}_{i,j} \;, \\
&(b) \quad \frac{\partial \tilde{B}_i}{\partial t} = B_j(0)\, \tilde{q}_{i,j} \;.
\end{aligned}
\qquad (9.3.2)
$$

These are the small perturbation equations governing wave propagations in a nondissipative medium which is initially at rest.

9.4. Reduction to the Wave Equation

From Eq. (9.3.2a), we know that

$$
\left[\tilde{p} + \frac{1}{\mu_0} B_j(0)\, \tilde{B}_j\right]_{,ii} = 0 \;. \qquad (9.4.1)
$$

However, we know that for the fluid region outside of the applied uniform magnetic induction field, $B_i(0)$,

$$
\begin{aligned}
\left[\tilde{p} + \frac{1}{\mu_0} B_j(0)\, \tilde{B}_j\right]_{(outside)} &= \tilde{p}_{(outside)} \\
&= 0 \;.
\end{aligned}
\qquad (9.4.2)
$$

Therefore, from the uniqueness theorem for the solution of a Laplace equation, we know that

$$\left[\tilde{p} + \frac{1}{\mu_0} B_j(0)\, \tilde{B}_j\right] = 0 \qquad (9.4.3)$$

everywhere in the fluid. Therefore, Eq. (9.3.2a) becomes,

$$\frac{\partial \tilde{q}_i}{\partial t} = \frac{1}{\rho\, \mu_0} B_j(0)\, \tilde{B}_{i,j} \quad , \qquad (9.4.4)$$

Equations (9.3.2b) and (9.4.4) can be combined, and we obtain two second order linear partial differential equations govering the small perturbations $\tilde{q}_i$ and $\tilde{B}_i$ as follows:

$$\text{(a)} \quad \frac{\partial^2 \tilde{q}_i}{\partial t^2} = \frac{1}{\rho\, \mu_0} B_j(0)\, B_k(0)\, \tilde{q}_{i,jk} \quad ,$$

$$\text{(b)} \quad \frac{\partial^2 \tilde{B}_i}{\partial t^2} = \frac{1}{\rho\, \mu_0} B_j(0)\, B_k(0)\, \tilde{B}_{i,jk} \quad . \qquad (9.4.5)$$

Calling the unit vector in the positive direction of the applied magnetic induction field b_i, Eq. (9.4.5) becomes

$$\text{(a)} \quad \frac{\partial^2 \tilde{q}_i}{\partial t^2} = \frac{B^2(0)}{\rho\, \mu_0} \frac{\partial^2 \tilde{q}_i}{\partial b^2} \quad ,$$

$$\text{(b)} \quad \frac{\partial^2 \tilde{B}_i}{\partial t^2} = \frac{B^2(0)}{\rho\, \mu_0} \frac{\partial^2 \tilde{B}_i}{\partial b^2} \quad . \qquad (9.4.6)$$

These are the standard one-dimensional wave equations. The solutions of these equation are

$$\text{(a)} \quad \tilde{q}_i = \alpha\,(b - At) + \beta\,(b + At) \quad ,$$

$$\text{(b)} \quad \tilde{B}_i = \gamma\,(b - At) + \delta\,(b + At) \quad , \qquad (9.4.7)$$

where α, β, γ, and δ are arbitrary functions and

$$\hat{A} = \sqrt{\frac{B^2(0)}{\rho \mu_0}} \tag{9.4.8}$$

is called the Alfvén velocity. The propagation is along the direction of the applied magnetic induction field $B_i(0)$.

CHAPTER X — STEADY PARALLEL INCOMPRESSIBLE MAGNETO-FLUID FLOW

10.1. Governing Equations

One of the simplest examples in magneto-fluid flow is the steady parallel flow of an incompressible, electrically conducting, Newtonian fluid within two parallel, infinite, insulating flat plates in the absence of other body forces. Let us choose a set of right-handed Cartesian coordinate axes x_i, such that the x_1-direction is in the direction of flow and the x_2-direction is in the positive direction of the applied uniform magnetic induction field. Due to the steady parallel flow assumption, all dependent variables are functions of x_2 only with the exception of the fluid pressure p, which can have a constant gradient in the x_1-direction.

The governing equations written explicitly for this case are:

Continuity Equation

$$\text{(a)} \quad \frac{d\, q_2}{d\, x_2} = 0 \ ,$$

Equations of Motion

$$\text{(b)} \quad p_{,1} = \mu \frac{d^2 q_1}{dx_2^{\ 2}} - J_3 B_2 + J_2 B_3 \ ,$$

$$\text{(c)} \quad p_{,2} = J_3 B_1 - J_1 B_3 \ ,$$

$$\text{(d)} \quad p_{,3} = J_1 B_2 - J_2 B_1 \ ,$$

Maxwell's Equations

$$\text{(e)} \quad \frac{dE_2}{dx_2} = 0 \ ,$$

$$(f) \quad \frac{dE_3}{dx_2} = 0 \,,$$

$$0 = 0 \,,$$

$$\frac{dE_1}{dx_2} = 0 \,,$$

$$(g) \quad \frac{dB_2}{dx_2} = 0 \,,$$

$$(h) \quad \frac{dB_3}{dx_2} = \mu_0 J_1 \,,$$

$$0 = \mu_0 J_2 \,,$$

$$-\frac{dB_1}{dx_2} = \mu_0 J_3 \,,$$

Ohm's Law

$$(i) \quad J_1 = \sigma E_1 \,,$$

$$J_2 = \sigma (E_2 - q_1 B_3) \,,$$

$$J_3 = \sigma (E_3 - q_1 B_2) \,,$$

Continuity Equation for Charges

$$(j) \quad \frac{dJ_2}{dx_2} = 0 \,. \qquad (10.1.1)$$

10.2. Reduction of Equations and Unknowns

We note that Eq. (10.1.1a) is automatically satisfied, since

$$q_1 = q_1 (x_2) \,,$$

$$q_2 = 0 \,,$$

$$q_3 = 0 \,. \qquad (10.1.2)$$

From Eq. (10.1.1j), we know that

$$J_2 = \text{constant} \quad . \tag{10.2.2}$$

But the flat plates are insulated. Therefore, we conclude that

$$J_2 = 0 \quad , \tag{10.2.3}$$

everywhere in the flow region. This result is consistent with one of the Maxwell equations (11.1.1h).

From Eq. (10.1.1g), we know that

$$B_2 = \text{constant} \quad , \tag{10.2.4}$$

the strength of the applied magnetic induction field.

Since

$$p_{,3} = 0 \quad , \tag{10.2.5}$$

we deduced in conjunction with Eqs. (10.1.1d) and (10.2.3), that

$$J_1 = 0 \quad . \tag{10.2.6}$$

Therefore, Eq. (10.1.1c) becomes

$$p_{,2} = J_3 B_1 \quad . \tag{10.2.7}$$

From Eq. (10.1.1h), we know that

$$B_3 = \text{constant} = 0 \quad . \tag{10.2.8}$$

Therefore, we deduce from Eqs. (10.1.1i), (10.2.6), and (10.2.3) that

$$E_1 = 0 \quad , \tag{10.2.9}$$

$$E_2 = 0 \quad . \tag{10.2.10}$$

From Eq. (10.1.1f), we also know that

$$E_3 = \text{constant} \quad . \tag{10.2.11}$$

Summarizing, we have the following set of equations* still to be satisfied. They are

$$\text{(a)} \quad p_{,1} = \mu \frac{d^2 q_1}{dx_2{}^2} - J_3 B_2 \ ,$$

$$\text{(b)} \quad p_{,2} = J_3 B_1 \ ,$$

$$\text{(c)} \quad - \frac{dB_1}{dx_2} = \mu_0 J_3 \ ,$$

$$\text{(d)} \quad \sigma \left[E_3 + q_1 B_2 \right] = J_3 \ . \qquad (10.2.12)$$

In these equations, $p_{,1}$, μ, μ_0, B_2 and E_3 are constants. Our task is to solve these equations simultaneously, together with the boundary conditions for the dependent variables: p, q_1, J_3, and B_1.

10.3. Solution of the Problem

Equations (10.2.12) are linear equations, and the solution of these equations is readily obtained with the aid of the given boundary conditions.

Let us substitute Eq. (10.2.12d) into Eq. (10.2.12a),

$$p_{,1} = \mu \frac{d^2 q_1}{dx_2{}^2} - \sigma B_2 (E_3 + q_1 B_2) \ . \qquad (10.3.1)$$

This is a second order ordinary differential equation for the single variable q_1. The boundary conditions for q_1 are

$$q = 0 \ \text{at} \ x_2 = \pm a \ , \qquad (10.3.2)$$

where 2a is the distance between the two plates and we have put the origin of the x_1-axis midway between the two plates. The solution of Eq. (10.3.1) subject to the boundary conditions [Eq. (10.3.2)] is

*Renumbered for convenience.

$$q = \frac{p_{,1} + \sigma B_2 E_3}{\sigma B_2^2} \left(\frac{\cosh \frac{\hat{H} x_2}{a}}{\cosh \hat{H}} - 1 \right) , \qquad (10.3.3)$$

where

$$\hat{H} = \sqrt{\frac{\sigma}{\mu}} \; B_2 \, a \; , \qquad (10.3.4)$$

is the Hartmann number defined in the previous chapter.

The current density J_3 can be easily obtained by inserting the result of Eq. (10.3.3) into Eq. (10.2.12d):

$$J_3 = \sigma E_3 + \frac{p_{,1} + \sigma B_2 E_3}{B_2} \left(\frac{\cosh \frac{\hat{H} x_2}{a}}{\cosh \hat{H}} - 1 \right) , \qquad (10.3.5)$$

But, from Eq. (10.2.12c),

$$B_1 = -\mu_0 \int J_3 \, dx_2 + \text{constant} \; . \qquad (10.3.6)$$

Therefore,

$$B_1 = -\mu_0 \left(\sigma E_3 - \frac{p_{,1} + \sigma B_2 E_3}{B_2} \right) x_2$$

$$- \frac{a \mu_0}{\hat{H}} \left(\frac{p_{,1} + \sigma B_2 E_3}{B_2} \right) \frac{\sinh \frac{\hat{H} x_2}{a}}{\cosh \hat{H}} + C \; . \qquad (10.3.7)$$

The constant of integration C and the value of E_3 are evaluated from the boundary conditions:

$$B_1 = 0, \text{ at } x_2 = \pm a \; . \qquad (10.3.8)$$

Therefore,

$$C = 0, \text{ and}$$

$$E_3 = \frac{p_{,1} (\hat{H} \coth \hat{H} - 1)}{\sigma B_2} \; . \qquad (10.3.9)$$

This means that

$$B_1 = \frac{\mu_0 \, p_{,1} \, a}{B_2} \left[\frac{x_2}{a} - \frac{\sinh \frac{\hat{H}}{a} x_2}{\sinh \hat{H}} \right] . \qquad (10.3.10)$$

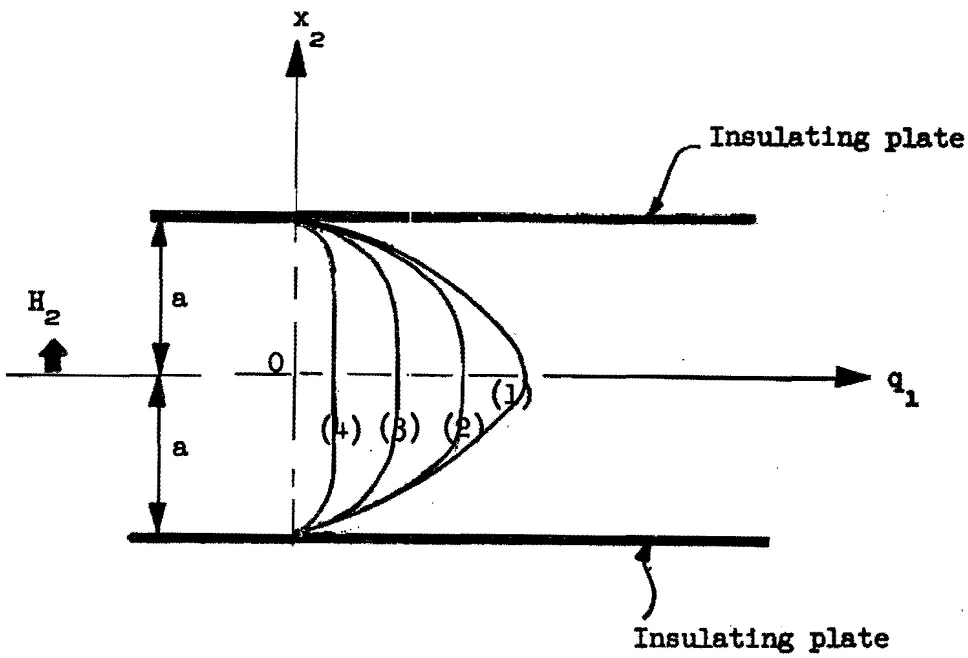
The pressure p can be evaluated as a function of H_1 by inserting Eq. (10.2.12c) into Eq. (10.2.12b) and integrating with respect to x_2. The result is

$$p + \frac{B_1^2}{2 \mu_0} = p(0) + p_{,1} \, x_1 \quad , \qquad (10.3.11)$$

where p(0) is the pressure at the origin if the induced magentic field B_1 is not present. "$B_1^2/2 \mu_0$" can be thought of as an <u>equivalent pressure</u> in the opposite direction of the fluid flow.

10.4. Velocity Profiles

The velocity profiles of this type of flow are predominately influenced by the Hartmann number $\hat{H}$, which is a measure of the relative magnitudes of the viscosity force and the induced drag force. The induced drag force tends to flatten the velocity profile, Fig. 10.4.1. For large values of $\hat{H}$, the velocity is nearly constant except at the boundary surfaces where the viscosity forces still predominate.



$\hat{H}(1) < \hat{H}(2) < \hat{H}(3) < \hat{H}(4)$

Fig. 10.4.1. Velocity Profiles of Steady, Parallel, Incompressible Magneto-Fluid Flow Between Two Parallel Insulating Plates under the Action of an Applied Magnetic Field Normal to the Surfaces of the Plates.

CHAPTER XI — MAGNETO-FLUID DYNAMIC SHOCK WAVES

11.1. Introduction

Analogous to ordinary compressible fluid flow, finite discontinuities of flow properties can occur across narrow transition zones in a nearly nondissipative magneto-fluid flow. Within these narrow transition zones, the gradients of the flow velocity, temperature, and magnetic induction field become extremely large. Therefore, the fluid medium should be considered as a viscous, heat-conducting, dissipative material within these narrow transition zones. Such a transition zone in magneto-fluid flow is called a magneto-fluid dynamic shock wave.

In this chapter, we shall derive the jump conditions (or shock conditions) for the flow properties and magnetic properties across such a shock wave in magneto-fluid flow. The derivation is restricted to steady flow conditions. These results can be easily generalized to include unsteady flow conditions.

11.2 Conservative Forms of the Governing Equations of Magneto-Fluid Flow

It will not be hard to show that the continuity equation, the equations of motion, the first law of thermodynamics, the induction equation, and the solenoidal condition of the magnetic induction field can be written in the following alternative forms for steady magneto-fluid flow in the absence of other body forces.

Continuity Equation

$$\text{(a)} \quad (\rho\, q_i)_{,i} = 0 \;,$$

Equations of Motion

$$\text{(b)} \quad \left(\rho\, q_j\, q_i + p\, \delta_{ji} - \tau_{ji} + \frac{B^2}{2\,\mu_0}\,\delta_{ij} - \frac{B_i\, B_j}{\mu_0}\right)_{,i} = 0 \;,$$

First Law of Thermodynamics

(c) $$\left[\rho\, q_i \left(h + \frac{1}{2} q^2\right) - \tau_{ji}\, q_j + b_i + \frac{1}{\mu_0} \epsilon_{ijk} E_j B_k\right]_{,i} = 0 \,,$$

Induction Equation

(d) $$(q_j B_i - q_i B_j)_{,j} = 0 \,,$$

Solenoidal Property of B_i

(e) $$B_{i,i} = 0 \,. \qquad (11.2.1)$$

Equations (11.2.1a, b, c, d, e) are called the conservative forms of the governing equations for steady magneto-fluid flow. As we shall see in the next section, conservative laws are directly derivable from these equations.

11.3. Shock Conditions

Without loss of generality, we can denote the direction normal to the shock-front, x_1, for a given streamline, Fig. 11.3.1.

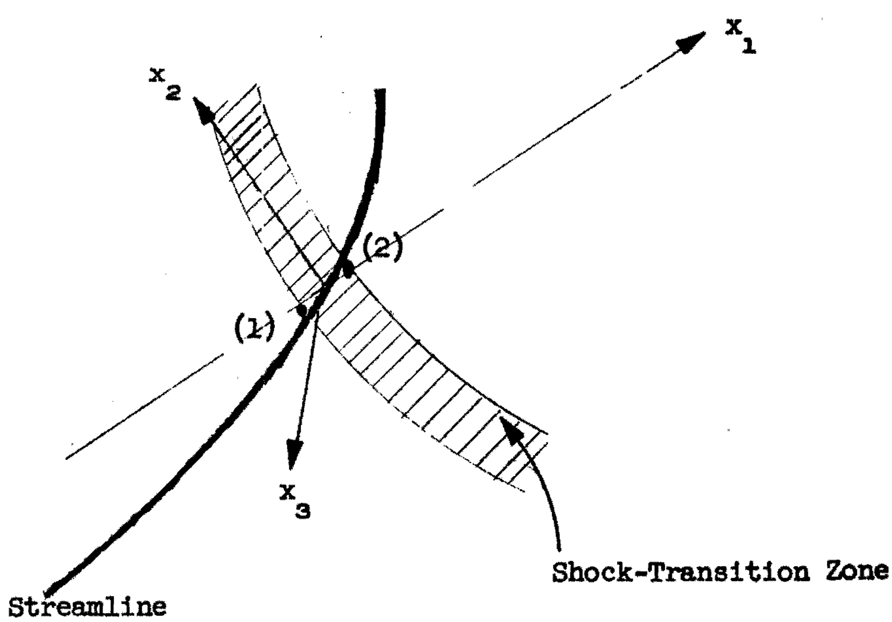


Fig. 11.3.1. Magneto-Fluid Dynamic Shock Wave

Equation (11.2.1a) can be written as

$$(\rho\, q_1)_{,1} + (\rho\, q_2)_{,2} + (\rho\, q_3)_{,3} = 0 \quad . \tag{11.3.1}$$

Let us integrate Eq. (11.3.1) across the shock transition zone along the x_1-axis.

$$\int_{x_1(1)}^{x_1(2)} (\rho\, q_1)_{,1}\, dx_1 + \int_{x_1(1)}^{x_1(2)} (\rho\, q_2)_{,2}\, dx_1 + \int_{x_1(1)}^{x_1(2)} (\rho\, q_3)_{,3}\, dx_1 = 0 \tag{11.3.2}$$

where $x_1(1)$ and $x_1(2)$ refer to the points just ahead of and behind the shock wave along the given streamline, respectively.

Since the shock-transition zone is very narrow and since the changes of flow properties in the x_2 and x_3 directions are small,

$$\lim_{x_1(2) - x_1(1) = 0} \int_{x_1(1)}^{x_1(2)} (\rho\, q_2)_{,2}\, dx_1 = \overline{(\rho\, q_2)}_{,2} \lim_{x_1(2) - x_1(1) = 0} \int_{x_1(1)}^{x_1(2)} dx_1 = 0 \quad \text{, and} \tag{11.3.3}$$

$$\lim_{x_1(2) - x_1(1) = 0} \int_{x_1(1)}^{x_1(2)} (\rho\, q_3)_{,3}\, dx_1 = 0 \quad . \tag{11.3.4}$$

Therefore, to the limit of

$$x_1(2) - x_1(1) = 0 \quad , \tag{11.3.5}$$

Equation (11.3.2) yields

$$\Big[\rho\, q_1\Big]_{(1)}^{(2)} = 0 \quad , \tag{11.3.6}$$

where the properties are to be evaluated at $x_1(1)$ and $x_1(2)$ as indicated by (1) and (2) behind the the square bracket in Eq. (11.3.6).

Similarly, we can show by integration of Eqs. (11.2.1b, c, d) that

$$\left[\rho\, q_1^2 + p - \tau_{11} + \frac{B^2}{2\,\mu_0} - \frac{B_1^{\,2}}{\mu_0}\right]_{(1)}^{(2)} = 0 \;, \tag{11.2.7}$$

$$\left[\rho\, q_1\, q_2 - \tau_{12} - \frac{B_1\, B_2}{\mu_0}\right]_{(1)}^{(2)} = 0 \;, \tag{11.3.8}$$

$$\left[\rho\, q_1\, q_3 - \tau_{13} - \frac{B_1\, B_3}{\mu_0}\right]_{(1)}^{(2)} = 0 \;, \tag{11.3.9}$$

$$\left[\rho\, q_1\, (h + \frac{1}{2} q^2) - \tau_{11}\, q_1 - \tau_{21}\, q_2 - \tau_{31}\, q_3 + b_1 + \frac{1}{\mu_0}\,(E_2\, B_3 - E_3\, B_2)\right]_{(1)}^{(2)} = 0 \;, \tag{11.3.10}$$

$$\left[q_1\, B_1 - q_1\, B_1\right]_{(1)}^{(2)} = 0 \;, \tag{11.3.11}$$

$$\left[q_1\, B_2 - q_2\, B_1\right]_{(1)}^{(2)} = 0 \;, \tag{11.3.12}$$

$$\left[q_1\, B_3 - q_3\, B_1\right]_{(1)}^{(2)} = 0 \;, \tag{11.3.13}$$

$$\left[B_1\right]_{(1)}^{(2)} = 0 \;. \tag{11.3.14}$$

Equation (11.3.11) is a trivial relationship. The term $\frac{1}{\mu_0}(E_2\, B_3 - E_3\, B_2)$ in Eq. (11.3.10) can be reduced by means of the Ohm's law.

$$E_2 = \frac{J_2}{\sigma} - (q_3 B_1 - q_1 B_3) \quad , \tag{11.3.15}$$

$$E_3 = \frac{J_3}{\sigma} - (q_1 B_2 - q_2 B_1) \quad . \tag{11.3.16}$$

Therefore,

$$\frac{1}{\mu_0} (E_2 B_3 - E_3 B_2) =$$

$$= \frac{1}{\mu_0 \sigma} (J_2 B_3 - J_3 B_2) + \frac{q_1}{\mu_0} (B_2^2 + B_3^2) -$$

$$- \frac{1}{\mu_0} (q_2 B_1 B_2 + q_3 B_1 B_3) \quad . \tag{11.3.17}$$

Therefore, Eq. (11.3.10) becomes

$$\Big[\rho\, q_1 (h + \frac{1}{2} q^2) - \tau_{11} q_1 - \tau_{21} q_2 - \tau_{31} q_3 + b_1 -$$

$$- \frac{1}{\mu_0 \sigma} (J_2 B_3 - J_3 B_2) + \frac{1}{\mu_0} (B_2^2 + B_3^2) -$$

$$- \frac{1}{\mu_0} (q_2 B_1 B_2 + q_3 B_1 B_3)\Big]_{(1)}^{(2)} = 0 \quad . \tag{11.3.18}$$

But the fluid is nearly nondissipative outside of the transition zone. Therefore, the dissipative terms in the integrated experssions when evaluated at the endpoints vanish. Thus we obtain the following set of magnetofluid dynamic shock conditions:

$$\text{(a)} \quad \Big[\rho\, q_1\Big]_{(1)}^{(2)} = 0 \quad ,$$

$$\text{(b)} \quad \Big[\rho\, q_1^2 + p + \frac{B_2^2}{2\,\mu_0} - \frac{B_1^2}{\mu_0}\Big]_{(1)}^{(2)} = 0 \quad ,$$

$$\text{(c)} \quad \left[\rho\, q_1\, q_2 - \frac{B_1\, B_2}{\mu_0}\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(d)} \quad \left[\rho\, q_1\, q_3 - \frac{B_1\, B_3}{\mu_0}\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(e)} \quad \left[\rho\, q_1\, (h + \frac{1}{2} q^2) + \frac{q_1}{\mu_0}\, (B_2^2 + B_3^2) - \right.$$

$$\left. - \frac{1}{\mu_0} (q_2 B_1 B_2 + q_3\, B_1\, B_3)\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(f)} \quad \left[q_1\, B_2 - q_2\, B_1\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(g)} \quad \left[q_1\, B_3 - q_3\, B_1\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(h)} \quad \left[B_1\right]_{(1)}^{(2)} = 0 \ . \qquad (11.3.19)$$

Equations (11.3.19) are the magneto-fluid dynamic shock conditons.

When the magneto-fluid dynamic shock wave and the direction of the magnetic field are both normal to the direction of the streamline, we obtain the following simple <u>normal shock</u> conditons:

$$\text{(a)} \quad \left[\rho\, q_1\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(b)} \quad \left[\rho\, q_1^2 + p + \frac{B_2^2}{2\, \mu_0}\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(c)} \quad \left[\rho\, q_1\, (h + \frac{1}{2} q_1^2) + \frac{q_1}{\mu_0}\, B_2^2\right]_{(1)}^{(2)} = 0 \ ,$$

$$\text{(d)} \quad \Big[q_1 B_2 \Big]_{(1)}^{(2)} = 0 \;, \qquad (11.3.21)$$

where we have arbitrarily set the x_1-direction as the direction of flow and the x_2-direction as the direction of the magnetic field. For such a normal shock, the first three equations can be obtained from the ordinary fluid dynamic normal shock equations, if we replace the fluid pressure p by

$$\left(p + \frac{B_2^2}{2\,\mu_0} \right) \;,$$

and the internal energy u by

$$\left(u + \frac{B_2^2}{\rho\,\mu_0} \right) \;.$$

It can be shown that such a transition can occur if

$$\left[\frac{q_1^2}{a^2 + \frac{B_2^2}{\rho\,\mu_0}} \right]^{(1)} > 1 \;, \qquad (11.3.22)$$

where

$$\left[\sqrt{a^2 + \frac{B_2^2}{\rho\,\mu_0}} \right]^{(1)}$$

can be shown to be the propagation velocity of small disturbances in a homogeneous, compressible, infinitely conducting medium in the direction normal to an applied uniform magnetic field equal to $B_2(1)$.

Appendix I

Vectors and Cartesian Tensors

A1.1 The Index Notation of a Vector

A vector in an ordinary three-dimensional space can be characterized by its magnitude and its direction with respect to a given reference frame. For example, the vector $\vec{r}$ as shown in Fig. A1.1.1 is represented by its algebraic magnitude and its directional cosines

$$\begin{aligned} \ell_1 &= \cos\theta_1, \\ \ell_2 &= \cos\theta_2, \\ \ell_3 &= \cos\theta_3, \end{aligned} \qquad \text{(A1.1.1)}$$

of the line containing $\vec{r}$ with respect to a set of right-handed Cartesian axes.

The magnitude of the rectangular components r_1, r_2, r_3 of this vector $\vec{r}$ in the three directions of the Cartesian axes are

$$\begin{aligned} r_1 &= r\ell_1, \\ r_2 &= r\ell_2, \\ r_3 &= r\ell_3, \end{aligned} \qquad \text{(A1.1.2)}$$

From the relations given in (A1.1.1) and (A1.1.2), we note that

$$r = \sqrt{r_1^2 + r_2^2 + r_3^2}, \qquad \text{(A1.1.3)}$$

and

$$\begin{aligned} \ell_1 &= r_1 \Big/ \sqrt{r_1^2 + r_2^2 + r_3^2}, \\ \ell_2 &= r_2 \Big/ \sqrt{r_1^2 + r_2^2 + r_3^2}, \\ \ell_3 &= r_3 \Big/ \sqrt{r_1^2 + r_2^2 + r_3^2}. \end{aligned} \qquad \text{(A1.1.4)}$$

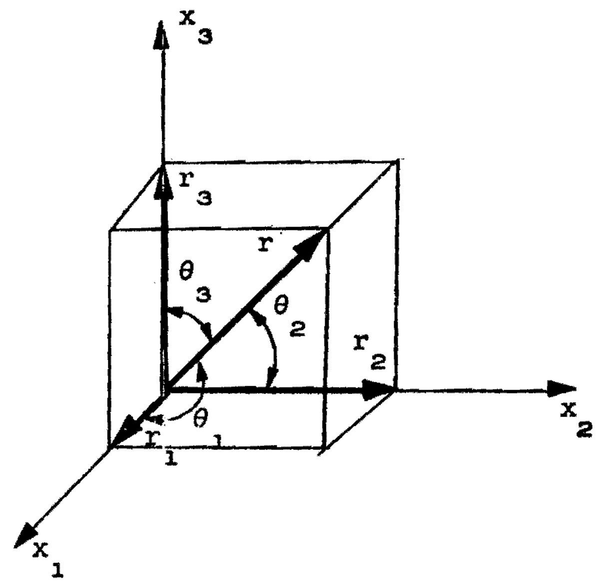


Fig. A1.1.1 Representation of a Vector $\vec{r}$

This means that a vector in an ordinary three-dimensional space can also be represented by the three magnitudes of the rectangular components of the vector in the directions of a given set of right-handed Cartesian axes. The three components of $\vec{r}$ can be written as

$$r_i,$$

where the subscript i is understood to take on the values of 1, 2, 3 in that order and therefore r_i in turn takes on the values of

$$r_1, r_2, r_3$$

in the same order. These represent the magnitudes of the three rectangular components of $\vec{r}$. "r_i" is the index or the Cartesian tensor notation of $\vec{r}$ in an ordinary three-dimensional space. It completely characterizes, and therefore represents, the vector $\vec{r}$.

A1.2 Transformation Law of a Vector

From the discussion of the previous section, it is obvious that we are at liberty to represent $\vec{r}$ using its index notation by referring it to any arbitrary set of right-handed Cartesian axes. In Fig. A1.2.1 a vector $\vec{r}$ is drawn from the common origin of a set of unprimed Cartesian axes and a set of primed axes. (The common origin is chosen for convenience without loss of generality). The index notation of the vector $\vec{r}$ in the unprimed axes is r_i and primed axes is r_i'. From analytic geometry, we know that r_i' can be expressed in terms of r_i.

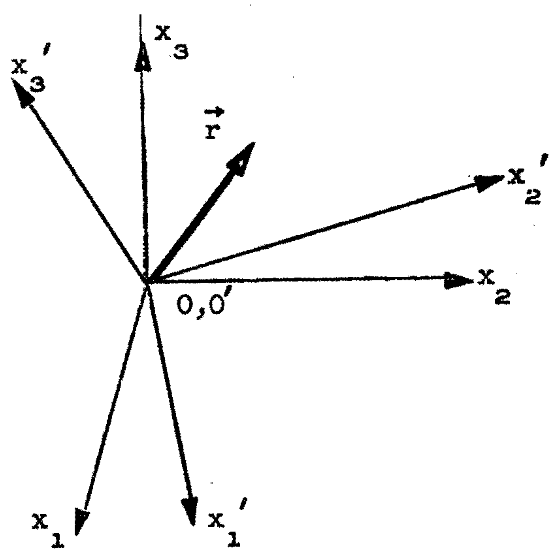


Fig. A1.2.1 Representation of a Vector in Two Sets of Right-Handed Cartesian Axes

$$r_1' = a_{11}r_1 + a_{12}r_2 + a_{13}r_3 ,$$

$$r_2' = a_{21}r_1 + a_{22}r_2 + a_{23}r_3 ,$$

$$r_3' = a_{31}r_1 + a_{32}r_2 + a_{33}r_3 . \quad \text{(A1.2.1)}$$

where

a_{11}, a_{12}, a_{13} are the directional cosines of the x_1'-axis with respect to the unprimed axes,

a_{21}, a_{22}, a_{23} are the directional cosines of the x_2'-axis with respect to the unprimed axes,

a_{31}, a_{32}, a_{33} are the directional cosines of the x_3'-axis with respect to the unprimed axes.

(A1.2.1) is called the transformation law of the vector $\vec{r}$ from one set of the right-handed Cartesian axes to another. It can be thought of as the definition of a vector in a three-dimensional space.

A1.3 The Index Notation and the Transformation Law of Cartesian Tensors

We are now in a position to state the two important rules used in "index notation".

(i) Range Convention: Whenever a small Latin suffix occurs unrepeated in a term, it is understood to take on the values of 1, 2, 3, (unless otherwise stated), the number of dimensions of the physical space. It represents a set of numbers or terms.

(ii) Summation Convention: Whenever a small Latin suffix occurs repeated in a term, it is understood to represent a summation over the range of 1, 2, 3 (unless otherwise stated).

Utilizing the above rules of the "index notation", the transformation law of a vector as given in (A1.2.1) can be written as

$$r_i' = a_{ij}\, r_j \quad , \tag{A1.3.1}$$

where a_{ij} represents a set of nine numbers which are the directional cosines between the primed and the unprimed axes. "a_{ij}" is called the transformation matrix. They satisfy the following <u>orthonormal relations</u>:

$$\begin{aligned} a_{ij}\, a_{kj} &= \delta_{ik}, \\ a_{ji}\, a_{jk} &= \delta_{ik}, \end{aligned} \tag{A1.3.2}$$

where δ_{ij} is known as the "Kronecker delta" defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i=j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{A1.3.3}$$

(A1.3.1) is known as the transformation law for "Cartesian tensors of the first rank". A vector can therefore be considered as a Cartesian tensor of the first rank when expressed in index notation.

The general transformation law for a "Cartesian tensor of the n'th rank" is

$$A'_{rst---} = a_{ri}\, a_{sj}\, a_{tk}\, A_{ijk---} \tag{A1.3.4}$$

where

A_{ijk---} is a Cartesian tensor of the n'th rank expressed in the unprimed system,

A'_{rst---} is the transformed tensor of A_{ijk---} in the primed system, and

a_{ij} is the transformation matrix.

We note that a scalar A can also be considered as a tensor. It is a Cartesian tensor of the zeroth rank.

A1.4 Addition and Subtraction of Vectors and Cartesian Tensors Using the Index Notation

The addition and subtraction of two Cartesian tensors of the same rank is defined as follows:

$$A_{ij---} \pm B_{ij---} = C_{ij---} \qquad \text{(A1.4.1)}$$

where

A_{ij---} and B_{ij---} are two Cartesian tensors of the same rank, and

C_{ij---} is the resulting Cartesian tensor due to addition or subtraction of A_{ij---} and B_{ij---}.

(A1.4.1) implies that the addition or subtraction is to be carried out for each pair of corresponding elements of A_{ij---} and B_{ij---}. We note that the indices of each term in (A1.4.1) are the same. The homogeneity of range indices is imperative in an indicial equation, for otherwise the equation becomes meaningless.

This law of addition and subtraction can be applied to vectors.

$$A_i \pm B_i = C_i, \qquad \text{(A1.4.2)}$$

where

A_i, B_i are two vectors, and

C_i is the resulting vector.

This is consistent with the usual definition and subtraction of vectors.

A1.5 Scaler (or Dot) Product of Two Vectors.

The scalar (or dot) product of two vectors $\vec{A}$, $\vec{B}$ is a scalar defined as

$$\vec{A} \cdot \vec{B} = AB \cos \theta, \tag{A1.5.1}$$

where θ is the angle between the two vectors $\vec{A}$, $\vec{B}$.

From elementary vector analysis, we know that in terms of the Cartesian components $\vec{A}$ and $\vec{B}$,

$$\vec{A} \cdot \vec{B} = A_1B_1 + A_2B_2 + A_3B_3. \tag{A1.5.2}$$

Therefore, the index notation of the dot product of two vectors is

$$\vec{A} \cdot \vec{B} = A_iB_i. \tag{A1.5.3}$$

A1.6 Vector (or Cross) Product of Two Vectors in Index Notation

The vector (or cross) product of two vectors $\vec{A}$, $\vec{B}$ is a vector $\vec{C}$ normal to $\vec{A}$, $\vec{B}$ such that $\vec{A}$, $\vec{B}$, $\vec{C}$ form a right-handed system. The magnitude of $\vec{C}$ is

$$C = AB \sin \theta, \tag{A1.6.1}$$

From elementary vector analysis, we know that the vector product $\vec{C}$ of $\vec{A}$, $\vec{B}$ can be represented by its three Cartesian components which are in turn related to the Cartesian components of $\vec{A}$ and $\vec{B}$.

$$\begin{aligned} C_1 &= A_2B_3 - A_3B_2, \\ C_2 &= A_3B_1 - A_1B_3, \\ C_3 &= A_1B_2 - A_2B_1. \end{aligned} \qquad (A1.6.2)$$

Let us define a "permutation symbol ϵ_{ijk}" such that

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if the values of } i, j, k \text{ do not form a permutation of } 1, 2, 3 \\ 1, & \text{if the values of } i, j, k \text{ form an even permutation of } 1, 2, 3 \\ -1, & \text{if the values of } i, j, k \text{ form an odd permutation of } 1, 2, 3 \end{cases}$$

Using this symbol, (A1.6.2) becomes

$$C_i = \epsilon_{ijk} A_j B_k. \qquad (A1.6.3)$$

This is the index notation of the cross product of two vectors.

A1.7 Index Notation of the Gradient of a Scalar Function of Position (Scalar Field)

From elementary vector analysis, we know that the gradient of a scalar function of position in a region R is

$$\text{grad } \phi = \vec{\nabla}\phi = \vec{i}_1 \frac{\partial\phi}{\partial x_1} + \vec{i}_2 \frac{\partial\phi}{\partial x_2} + \vec{i}_3 \frac{\partial\phi}{\partial x_3} \qquad (A1.7.1)$$

where

$\phi\ (x_1, x_2, x_3)$ is a scalar field which is single valued and with continuous derivatives in R,

$\vec{i}_1, \vec{i}_2, \vec{i}_3$ are the unit vectors in the x_1-, x_2-, x_3- directions,

$\vec{\nabla} = \vec{i}_1 \frac{\partial}{\partial x_1} + \vec{i}_2 \frac{\partial}{\partial x_2} + \vec{i}_3 \frac{\partial}{\partial x_3}$ is called the "del operator".

It is apparent that the gradient of ϕ can be expressed in index notation as

$$\frac{\partial \phi}{\partial x_i} = \phi,_i \qquad \text{(A1.7.2)}$$

where $,_i$ means partial differentiation with respect to x_i.

We note that for an infinitesural displacement dx_i within a surface of ϕ = constant in R,

$$d\phi = \phi,_i\ dx_i = 0. \qquad \text{(A1.7.3)}$$

This means that the vector $\phi,_i$, or the grad. ϕ, is everywhere normal to the surface of ϕ = constant in R.

A1.8 Index Notation of the Divergence of a Vector Function of Position (or Vector Field)

From elementary vector analysis, we know that the divergence of a vector field in a region R is a scalar field given by

$$\text{div. } \vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}, \qquad \text{(A1.8.1)}$$

where

$\vec{A}(x_1, x_2, x_3)$ is vector field which is single-valued with continuous derivatives in R,

A_1, A_2, A_3 are Cartesian components of $\vec{A}$.

The index notation of the divergence of a vector field in R is therefore

$$\vec{\nabla} \cdot \vec{A} = A_{i,i}. \qquad \text{(A1.8.2)}$$

A1.9 Index Notation of Curl of a Vector Function of Position (or Vector Field)

From elementary vector analysis, we know that the curl of a vector field is a vector field given as follows

$$\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A} = \vec{i}_1 (A_{3,2} - A_{2,3}) + \vec{i}_2 (A_{1,3} - A_{3,1}) + \vec{i}_3 (A_{2,1} - A_{1,2}), \qquad \text{(A1.9.1)}$$

where

$\vec{A}(x_1, x_2, x_3)$ is a vector field which is single-valued with continuous derivatives in R, and

A_1, A_2, A_3 are the Cartesian components of $\vec{A}$.

Therefore, the index notation of curl $\vec{A}$ is

$$\epsilon_{ijk} A_{k,j}.$$

A1.10 <u>The "ε-δ" Identity</u>

A very important identity involving the manipulation of the indicial expression is the "ε-δ identity" stated as follows:

$$\epsilon_{ijk}\,\epsilon_{irs} = \delta_{jr}\,\delta_{ks} - \delta_{js}\,\delta_{kr} \qquad \text{(A1.10.1)}$$

(A1.10.1) can be easily verified from the definitions of the permutation symbol and the Kronecker delta.

Using this identity, many of the vector identities become obvious. As an example, the vector identity

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{\nabla} \cdot \vec{B})\vec{A} - (\vec{\nabla} \cdot \vec{A})\vec{B}, \qquad \text{(A1.10.2)}$$

can be proven as follows:

$$\begin{aligned} &\epsilon_{ijk}\,\epsilon_{krs}\,(A_r\,B_s)_{,j} \\ &= (\delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr})(A_r\,B_s)_{,j} \\ &= (A_i\,B_j)_{,j} - (A_j\,B_i)_{,j} \qquad \text{(A1.10.3)} \\ &= B_j\,A_{i,j} - A_j\,B_{i,j} + B_{j,j}\,A_i - A_{j,j}\,B_i, \end{aligned}$$

where in the manipulation we have used the <u>ϵ-δ identity</u> and the obvious rule of

$$F_i\,\delta_{ij} = F_j. \qquad \text{(A1.10.4)}$$

A1.11 Scalar Line Integral and the Stokes Theorem

The scalar line integral of a vector field is defined as

$$\int_{x_i(1)}^{x_i(2)} A_i\,dx_i,$$

where

$A_i\,(x_1, x_2, x_3)$ is a vector field which is defined within a region R, and

$x_i(1)$, $x_i(2)$ are the end points of a continuous curve in R.

An important theorem in vector analysis which transforms a line integral into a surface integral is the "Stokes' Theorem". In index notation, it reads

$$\oint_C A_i \, dx_i = \int_{S'} n_i \, \epsilon_{ijk} \, A_{k,j} \, dS' \qquad \text{(A1.11.1)}$$

where

C is a closed continuous curve in R,

S' is a surface bounded by C,

n_i is the unit normal of a differential surface dS' on S', and

A_i (x_1, x_2, x_3) is single-valued and continuous in R.

The direction of n_i is depicted in Fig. A1.11.1.

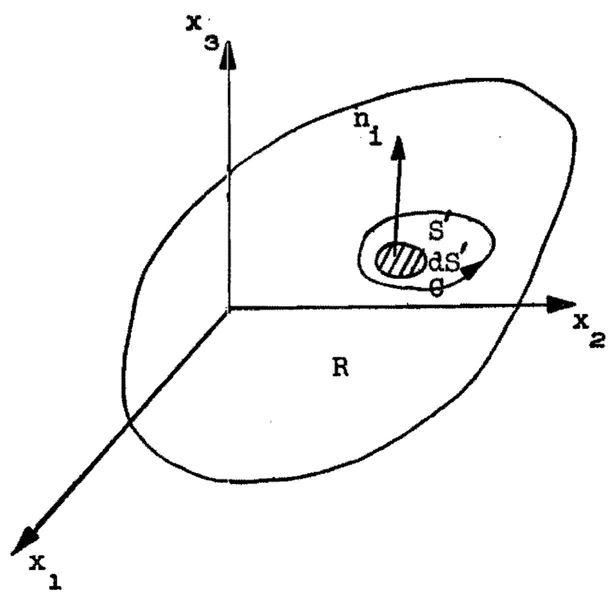


Fig. A1.11.1 Figure Depicting the Direction of the Normal Vector n_i used in the Stokes' Theorem

A1.12 Conservative Vector Field and the Concept of Scalar Potential

If a vector field A_i in R can be expressed as the gradient of a scalar field $\phi(x_1, x_2, x_3)$ in R,

$$A_i = \phi,_i, \tag{A1.12.1}$$

then the scalar line integral of A_i along a curve within R is

$$\int_{x_i(1)}^{x_i(2)} \phi,_i \, dx_i = \int_{\phi(1)}^{\phi(2)} d\phi = \phi(2) - \phi(1). \tag{A1.12.2}$$

This means that the line integral is dependent only on the end points. It is independent of the path of integration. If the line integral is evaluated along a closed path in R, then

$$\oint_{\substack{\text{Closed path} \\ \text{in R}}} A_i \, dx_i = \oint d\phi = 0. \tag{A1.12.3}$$

We note from elementary vector analysis

$$\vec{\nabla} \times \vec{\nabla}\phi = 0, \tag{A1.12.4}$$

which is an obvious statement in index notation,

$$\epsilon_{ijk} \, \phi,_{kj} = 0. \tag{A1.12.5}$$

Therefore, from the Stokes' Theorem, we can again show that

$$\begin{aligned} \oint A_i \, dx_i &= \int n_i \, (\epsilon_{ijk} \, A_{k,j}) dS' \\ &= \int n_i \, (\epsilon_{ijk} \, \phi_{,kj}) dS' \\ &= 0. \end{aligned} \tag{A1.12.6}$$

The converse of the above result is also true. It states that if the scalar line integral of any curve in R is independent of path, (or the scalar line integral around a closed path vanishes), then

$$\epsilon_{ijk} A_{k,j} = 0, \qquad \text{(A1.12.7)}$$

and

$$A_i = \phi_{,i}. \qquad \text{(A1.12.8)}$$

If a vector field can be expressed as the gradient of a scalar field, the vector field is called a Conservative vector field, and the scalar field is called the scalar potential of the conservative vector field.

A1.13 Generalized Gauss Theorem Stated in Indicial Form

The generalized Gauss theorem for a tensor field is

$$\int_{S'} A_{ijk---}\, n_i \, dS' = \int_{R_1} A_{ijk---,i} \, dV \qquad \text{(A1.13.1)}$$

where

$A_{ijk---}(x_1, x_2, x_3)$ is a tensor field which is single-valued and continuous in R,

S' is a surface enclosing a region R_1 and R, and

n_i is the unit outward normal of a surface element dS' on S'.

The Gauss theorem transforms a surface integral into a volume integral or vice versa.

A1.14 Green's Theorem in Indicial Form

The two common forms of the Green's theorem in indicial forms are given as follows:

$$\int_R [A,_i B,_i + AB,_{ii}]\, dV = \int_S AB,_i n_i \, dS' . \qquad (A1.14.1)$$

$$\int_R [AB,_{ii} - BA,_{ii}]\, dV = \int_{S'} (AB,_i n_i - BA,_i n_i) dS' . \qquad (A1.14.2)$$

where

A,B are scalar fields which are single-valued and continuous in R,

S' is a surface enclosing a region R_1 in R, and

n_i is the unit outward normal of a surface element dS' on S'.

The Green's theorem is extremely useful in developing the uniqueness theorems of boundary value problems.

A1.15 Vector Potential

If a vector function of position or vector field B_i in a region R is derivable from another vector field A_i as follows

$$B_i = \epsilon_{ijk} A_{k,j}, \qquad (A1.15.1)$$

where A_i is a single-valued and contionuous in R;

then A_i is called the vector potential of B_i.

An immediate consequence of (A1.15.1) is

$$B_{i,i} = 0. \qquad (A1.15.2)$$

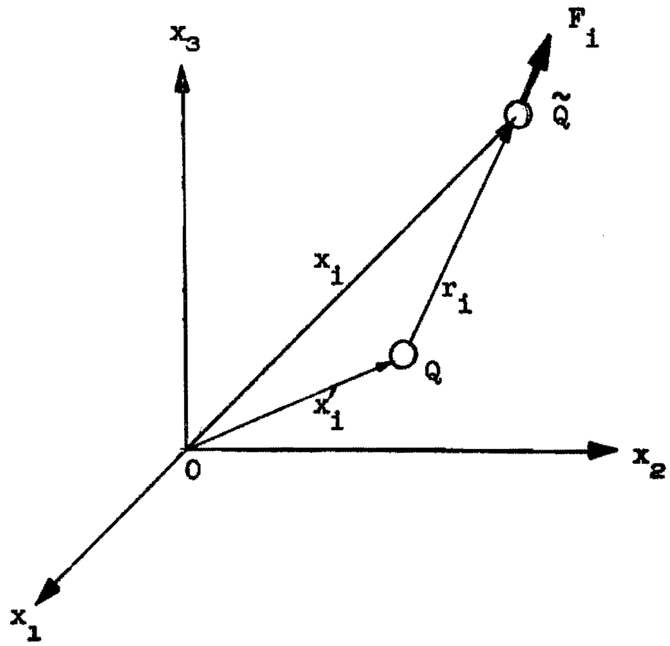
This means that the divergence of a vector field derivable from a vector potential always vanishes.

Appendix II

Outline of Elements of Electricity and Magnetism*

A2.1 Electrostatics in Vacuum

Coulomb's law:



$$F_i = \frac{1}{4\pi\epsilon_0}\frac{Q\tilde{Q}}{r^3} r_i \quad , \tag{A2.1.1}$$

$$r_i = x_i - x_i' \quad , \tag{A2.1.2}$$

$$r^2 = r_i r_i \quad . \tag{A2.1.3}$$

[ϵ_0: Vacuum electric permeability]

Definition of electric field for stationary charge:

$$E_i = \lim_{\substack{\Delta Q \to 0 \\ \Delta Q > \Delta Q^\circ}} \frac{\Delta F_i}{\Delta Q} = \frac{dF_i}{dQ} \quad . \tag{A2.1.4}$$

---

*Formulated for rationalized MKS units.

$\therefore E_i$ (due to Q at x_i')

$$= \frac{Q}{4\pi\ \epsilon_0\ r^3}\ r_i \qquad \text{(A2.1.5)}$$

$$= -\frac{Q}{4\pi\ \epsilon_0}\left(\frac{1}{r}\right)_{,i}$$

or, E_i (due to Q at x_i')

$$= -\phi_{,i}\ , \qquad \text{(A2.1.6)}$$

where $$\phi = \frac{Q}{4\pi\ \epsilon_0\ r}\ . \qquad \text{(A2.1.7)}$$

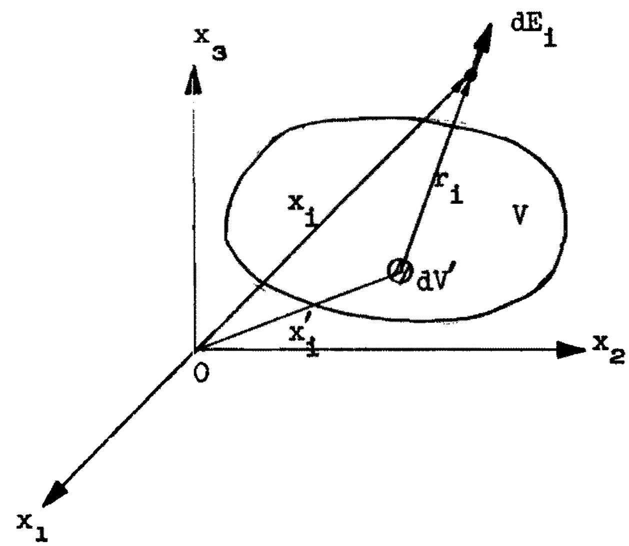
For a <u>distribution</u> of charges,

$$\phi = \frac{1}{4\pi\ \epsilon_0}\int_V \frac{\hat{\rho}\ dV'}{r}\ , \qquad \text{(A2.1.8)}$$

$$\hat{\rho} = \lim_{\substack{\Delta V \to 0 \\ \Delta V > \Delta V^{\circ}}} \frac{\Delta Q}{\Delta V} = \frac{dQ}{dV}\ , \qquad \text{(A2.1.9)}$$

$$E_i = -\phi_{,i} =$$

$$= -\frac{1}{4\pi\ \epsilon_0}\int_V \left(\frac{\hat{\rho}}{r}\right)_{,i} dV'\ . \qquad \text{(A2.1.10)}$$



$$\epsilon_{ijk} E_{k,j} = \epsilon_{ijk} \phi_{,kj} = 0 , \qquad (A2.1.11)$$

$$E_{i,i} = \frac{\hat{\rho}}{\epsilon_0} . \qquad (A2.1.12)$$

Electromotive Force, EMF

EMF = work done by E_i on a unit charge from $x_i(1)$ to $x_i(2)$

$$= \int_{x_i(1)}^{x_i(2)} E_i \, dx_i . \qquad (A2.1.13)$$

For an electrostatic field,

$$EMF = \int_{x_i(1)}^{x_i(2)} - \phi_{,i} \, dx_i$$

$$= -\int_{(1)}^{(2)} d\phi$$

$$= \phi(2) - \phi(1) , \qquad (A2.1.14)$$

and

$$\oint E_i \, dx_i = \oint d\phi = 0 \qquad (A2.1.15)$$

stationary current is impossible in electrostatic field.

A2.2 Polarization Vector and Displacement Vector

$$E_{i,i} = \frac{\hat{\rho}}{\epsilon_0} \qquad (A2.2.1)$$

$$\hat{\rho} = \hat{\rho}_{true} + \hat{\rho}_P \qquad (A2.2.2)$$

$$P_{i,i} = -\hat{\rho}_P \quad \begin{bmatrix} P_i: & \text{Polarization vector} \\ \rho_P: & \text{Polarization charge} \end{bmatrix} . \qquad (A2.2.3)$$

$$\left(E_i + \frac{P_i}{\epsilon_0}\right)_{,i} = \frac{\hat{\rho}_{true}}{\epsilon_0} . \qquad (A2.2.4)$$

Definition of displacement vector,

$$D_i = \epsilon_0 E_i + P_i \qquad (A2.2.5)$$

$$\therefore D_{i,i} = \hat{\rho}_{true} . \qquad (A2.2.6)$$

If

$$P_i = \epsilon_0 \chi E_i , \qquad (A2.2.7)$$

[χ: Electric susceptibility]

then

$$\begin{aligned} D_i &= \epsilon_0 (1 + \chi) E_i \\ &= \epsilon_0 \kappa E_i \\ &= \epsilon E_i \end{aligned} \qquad (A2.2.8)$$

$$\begin{bmatrix} \kappa: & \text{Dielectric constant} \\ \epsilon: & \text{Electric permeability} \end{bmatrix} .$$

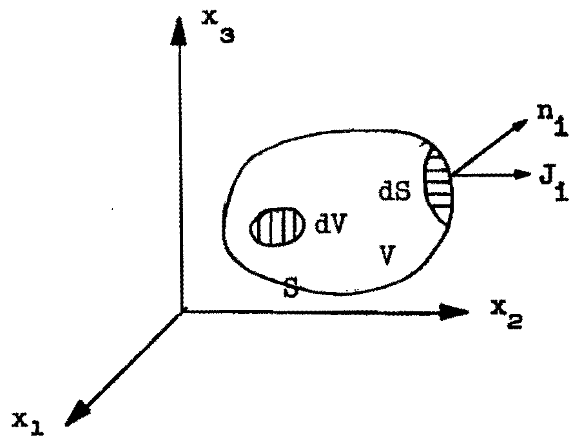
For a vacuum (i.e., all charges are separately considered),

$$\left[\begin{aligned} P_i &= 0 , \\ \chi &= 0 , \\ \kappa &= 1 , \\ \epsilon &= \epsilon_0 , \\ D_i &= \epsilon_0 E_i . \end{aligned}\right. \qquad (A2.2.9)$$

A2.3 Current Density and Continuity Equation for Charge

Current density

J_i = vector denoting time rate of flow of charge per unit area



$$\int_V \frac{\partial \hat{\rho}}{\partial t} \, dV + \int_S J_i \, n_i \, dS = 0 \; , \qquad \text{(A2.3.1)}$$

or

$$\int_V \left[\frac{\partial \hat{\rho}}{\partial t} + J_{i,i} \right] dV = 0 \; , \qquad \text{(A2.3.2)}$$

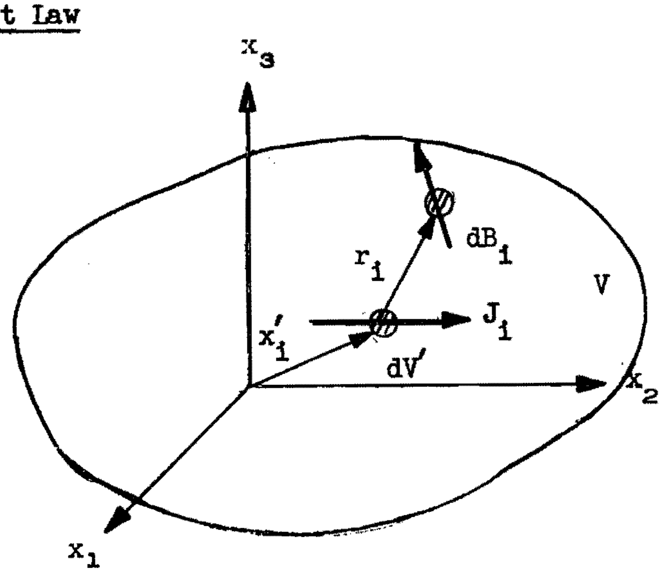
[V arbitrary]

$$\therefore \; \frac{\partial \hat{\rho}}{\partial t} + J_{i,i} = 0 \; . \qquad \text{(A2.3.3)}$$

(A2.3.3) is called the continuity equation for the conservation of charge.

A2.4 Magnetic Induction Field in Vacuum

Biot and Savart Law

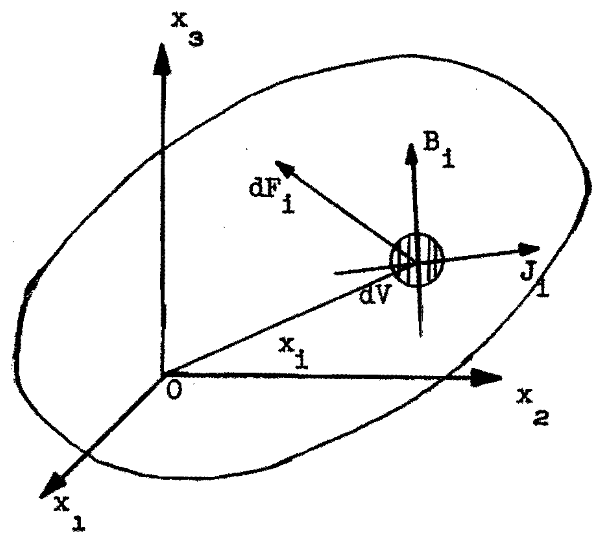


B_i (magnetic induction field)

$$= \int dB_i$$

$$= \frac{\mu_0}{4\pi} \int \frac{\epsilon_{ijk} J_j r_k}{r^3} dV' \qquad \text{(A2.4.1)}$$

[μ_0: Vacuum magnetic permeability]



F_i (Lorentz force)

$$= \int dF_i$$

$$= \int_V \epsilon_{ijk} J_j B_k \, dV . \qquad \text{(A2.4.2)}$$

<u>Therefore,</u>

$$B_{i,i} = 0 \quad (B_i \text{ is solenoidal}) \qquad \text{(A2.4.3)}$$

$$\epsilon_{ijk} B_{k,j} = \mu_0 J_i \qquad \text{(A2.4.4)}$$

$$B_i = \epsilon_{ijk} A_{k,j} \quad , \tag{A2.4.5}$$

$$A_i \text{ (vector potential)} = \frac{\mu_0}{4\pi} \int \frac{J_i}{r} \, dV \quad . \tag{A2.4.6}$$

A2.5 Magnetization and Magnetic Field Strength

$$M_i = \frac{1}{2} \epsilon_{ijk} r_j J_k(m) \quad . \tag{A2.5.1}$$

M_i: Magnetic moment per unit volume or magnetization,

r_j: Coordinate of the magnetization current density $J_k(m)$.

or,

$$J_i (m) = \epsilon_{ijk} M_{k,j} \quad . \tag{A2.5.2}$$

Therefore, if we assume that the current density can be separated as

$$J'_i + J_i(m) \quad ,$$

then

$$\epsilon_{ijk} B_{k,j} = \mu_0 \left[J'_i + J_i(m) \right] \tag{A2.5.3}$$

or,

$$\epsilon_{ijk} \left[B_k - \mu_0 M_k \right]_{,j} = \mu_0 J'_i \quad . \tag{A2.5.4}$$

Call H_i (magnetic field strength)

$$= \frac{B_i}{\mu_0} - M_i \quad . \tag{A2.5.5}$$

Therefore,

$$\epsilon_{ijk} H_{k,j} = J'_i \; . \tag{A2.5.6}$$

If $$M_i = \epsilon \, \chi(m) \, H_i , \tag{A2.5.7}$$

then,

$$\begin{aligned} B_i &= \mu_0 H_i + M_i \\ &= \mu_0 [1 + \chi(m)] H_i \\ &= \mu_0 \kappa(m) H_i \\ &= \mu H_i \ . \end{aligned} \qquad (A2.5.8)$$

$$\left[\begin{array}{ll} \chi(m): & \text{Magnetic susceptibility} \\ \kappa(m): & \text{Relative magnetic permeability} \\ \mu: & \text{Absolute magnetic permeability} \end{array} \right]$$

In vacuum (i.e., when current is individually considered),

$$\left[\begin{array}{lll} \chi(m) & = & 0 \ , \\ \kappa(m) & = & 1 \ , \\ \mu & = & \mu_0 \ , \\ B_i & = & \mu_0 H_i \ , \\ J_i' & = & J_i \ . \end{array} \right. \qquad (A2.5.9)$$

A2.6 Generalization of Magnetic Induction for Nonstationary Currents in Moving Media

$$\epsilon_{ijk} B_{k,j} = \mu_0 J_i \ , \qquad (A2.4.4)$$

where

$$\begin{aligned} J_i = \ & J_i \text{ (conduction)} \\ & + J_i \text{ (magnetization)} \\ & + J_i \text{ (polarization)} \\ & + J_i \text{ (convection)} \\ & + J_i \text{ (vacuum displacement current)} \end{aligned} \qquad (A2.6.1)$$

with

$$J_i \text{ (conduction)} = \sigma [E_i + \epsilon_{ijk} q_j B_k] \ ,$$

$$J_i \text{ (magnetization)} = \epsilon_{ijk} M_{k,j} \ ,$$

$$J_i \ \text{(polarization)} = \frac{dP_i}{dt}$$

$$= \frac{\partial P_i}{\partial t} + \epsilon_{ijk}\, \epsilon_{krs}\, (P_r\, q_s)_{,j} + (P_{j,j})\, q_i \quad ,$$

$$J_i \ \text{(convection)} = q_i\, [\hat{\rho}\ \text{true} + P_{j,j}] \quad ,$$

$$J_i \ \text{(vacuum displacement)} = \epsilon_0 \frac{\partial E_i}{\partial t} \,. \qquad \text{(A2.6.2)}$$

The <u>vacuum displacement current</u> $\epsilon_0 \frac{\partial E_i}{\partial t}$ is introduced by Maxwell to keep J_i solenoidal.

J_i (conduction) is related to the apparent E_i' for the moving medium through the <u>Ohm's law</u>.

$$J_i \ \text{(conduction)} = \sigma E_i'$$

$$= \sigma\, [E_i + \epsilon_{ijk}\, J_j\, B_k] \qquad \text{(A2.6.3)}$$

[σ: Conductivity].

A2.7 <u>Faraday's Law of Induction</u>

$$\text{EMF} = - \frac{d}{dt} \int_S B_i\, n_i\, dS \,. \qquad \text{(A2.7.1)}$$

This becomes (both for stationary and moving media),

$$\epsilon_{ijk}\, E_{k,j} = - \frac{\partial B_i}{\partial t} \quad , \qquad \text{(A2.7.2)}$$

by the Stokes' theorem.

A2.8 Maxwell's Equations for Moving Media

Since (A2.1.12),(A2.4.3) should also hold for a moving medium, they and (A2.4.4), (A2.7.2) form a set of interlocking equations defining E_i and B_i for a moving media. They are called the Maxwell's equations for a moving media.

$$\begin{cases} E_{i,i} = \dfrac{\hat{\rho}}{\epsilon_0} \ , \\ B_{i,i} = 0 \ , \\ \epsilon_{ijk} E_{k,j} = -\dfrac{\partial B_i}{\partial t} \ , \\ \epsilon_{ijk} B_{k,j} = \mu_0 \hat{J}_i + \mu_0 J_i \text{ (convection)} + \epsilon_0 \dfrac{\partial E_i}{\partial t} \ , \end{cases} \tag{A2.8.1}$$

where

$$\hat{\rho} = \hat{\rho}\text{ true} + \hat{\rho}_P \ ,$$

$$\hat{J}_i = J_i \text{ (conduction)} + J_i \text{ (magnetization)} + J_i \text{ (polarization)} \ ,$$

$$J_i \text{ (conduction)} = \sigma\,[E_i + \epsilon_{ijk}\, q_j\, B_k] \ ,$$

$$J_i \text{ (magnetization)} = \epsilon_{ijk} M_{k,j} \ ,$$

$$J_i \text{ (polarization)} = \frac{\partial P_i}{\partial t} + (P_{j,j})\, q_i + \epsilon_{ijk}\, \epsilon_{krs}\, (P_r\, q_s)_{,j}$$

$$J_i \text{ (convection)} = q_i\, [\hat{\rho}\text{ true} + P_{j,j}] \ . \tag{A2.8.2}$$

Appendix III

Selected Reference Books

The following is a list of selected reference books on the subject of magneto-fluid mechanics and its related topics. Most of these books contain within themselves lists of publications which can be referred to for further literature research on this subject.

For current publications on magneto-fluid mechanics, readers are suggested to consult the numerous technical journals published by the various societies of engineering and physical sciences.

1. W. P. Allis (ed.), Nuclear Fusion, D. Van Nostrand, New York, 1960.

2. D. Bershader (ed.), The Magnetodynamics of Conducting Fluids, Stanford University Press, Stanford, Calif., 1959.

3. A. B. Cambel, T. P. Anderson, and M. M. Slawsky (ed.), Magnetohydrodynamics, Northwestern University Press, Evanston, Ill., 1961.

4. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford University Press, Oxford, England, 1961.

5. T. S. Chang, Intermediate Fluid Mechanics, Edwards Brothers, Ann Arbor, Mich., 1962.

6. S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, Cambridge, England, 1939.

7. F. H. Clauser (ed.), Plasma Dynamics, Addison-Wesley, Reading, Mass., 1960.

8. T. G. Cowling, Magnetohydrodynamics, Interscience, New York, 1957.

9. V. C. A. Ferraro and C. Plumpton, Magneto-Fluid Mechanics, Oxford University Press, Oxford, England, 1961.

10. J. Fox (ed.), Electromagnetics and Fluid Dynamics of Gaseous Plasma, Polytechnic Press of BPI, Brooklyn, New York, 1962.

11. J. H. Jeans, The Dynamical Theory of Gases, 3d ed., Cambridge University Press, Cambridge, England, 1921.

12. S. DeGroot, Thermodynamics of Irreversible Processes, North-Holland Publishing Co., Amsterdam, 1958.

13. S. W. Kash, Plasma Acceleration, Stanford University Press, Stanford, Calif., 1960.

14. R. K. M. Landshoff, Magnetohydrodynamics, Stanford University Press, Stanford, Calif., 1957.

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