INTERSECTION MATH

And more!

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The following represents a sample activity based on the December 2006 newsletter of the St. Mark's Institute of Mathematics (www.stmarksschool.org/math). It also appears in chapter 2 of:

THINKING MATHEMATIS!
Volume 1: Arithmetic = Gateway to All

available at www.jamestanton.com.

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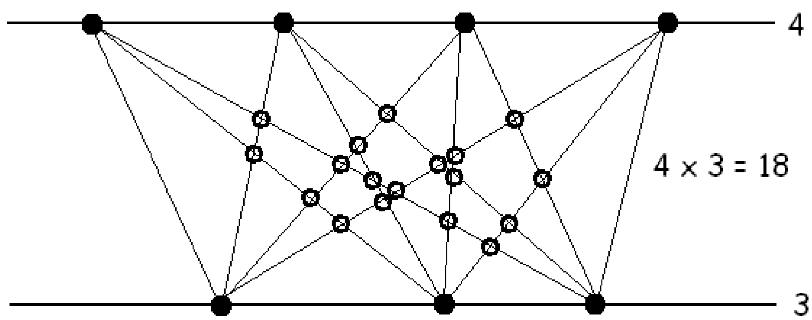
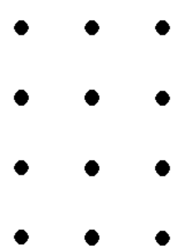
I: WEIRD MULITPLICATION!

MULTIPLICATION 1:
Here's a new type of math called intersection math. To compute the product of two numbers, say four times three, draw two horizontal lines, place four dots on the top line, three on the bottom, and then connect each dot on the top line to each and every dot on the bottom line. The number of intersections that occur between the two horizontal lines is the product. (One must make sure that the dots are sufficiently spaced so that no point of intersection is crossed multiple times.) In intersection math, $4 \times 3 = 18$.



a) Is 3×4 also 18 in intersection math? If so, why?
b) What is 1×107 in intersection math?
c) Draw a six-by-six multiplication table for intersection math. What patterns do you notice?
d) What's 963×4036 in intersection math?

MULTIPLICATION 2:
Here's another mathematical invention called rectangle math. To compute the product of two numbers, say, four times three, draw a four-by-three array of dots and count the number of (horizontal/vertical) rectangles one can draw with vertices on the grid. (Squares are also considered rectangles.)



Here there are six 1×1 rectangles, three 1×2 s, four 2×1 s, two 2×2 s, two 3×1 s, and one 3×2 , giving a total of eighteen rectangles:

$$4\times3=18.$$

a) Is it a coincidence that four times three in rectangle math is the same as four times three in intersection math?
b) Draw a six-by-six multiplication table for rectangle math. What do you notice?
c) What's 963×4036 in rectangle math?

MULTIPLICATION 3:
Here's yet another type of math, called dinner party math. To compute the product of two numbers, say, four times three(!), imagine you are the host of a dinner party. You have four male friends (Albert, Bilbert, Cuthbert, and Dilbert) and three female friends (Edwina, Fellina, and Gina) but can only invite two men and two women to your party. Now ask: How many different dinner parties could you host? Here's a list of all the options:

AB|EF AB|EG AB|FG AC|EF AC|EG AC|FG AD|EF AD|EG AD|FG
BC|EF BC|EG BC|FG BD|EF BD|EG BD|FG
CD|EF CD|EG CD|FG

That's eighteen possibilities. We thus say, in dinner party math, four times three is eighteen!

a) Coincidence?
b) Is 3×4 also 18 in dinner party math?
c) Draw a six-by-six multiplication table for dinner party math.
d) What's 963×4036 in dinner party math?

TOWARDS SOLUTIONS:

a) List all the ways to select two friends from the list: A B C D E F G H.
Do this in a systematic way so that you can see that the answer is the number $7+6+5+4+3+2+1$.

b) Explain why each "product" in dinner party math is the ordinary product of two numbers of this type.

c) Show how to convert a rectangle math problem into a dinner party math problem. Give each row of the rectangular array of dots a label, A, B, C, … and give each column a label E, F, G, …

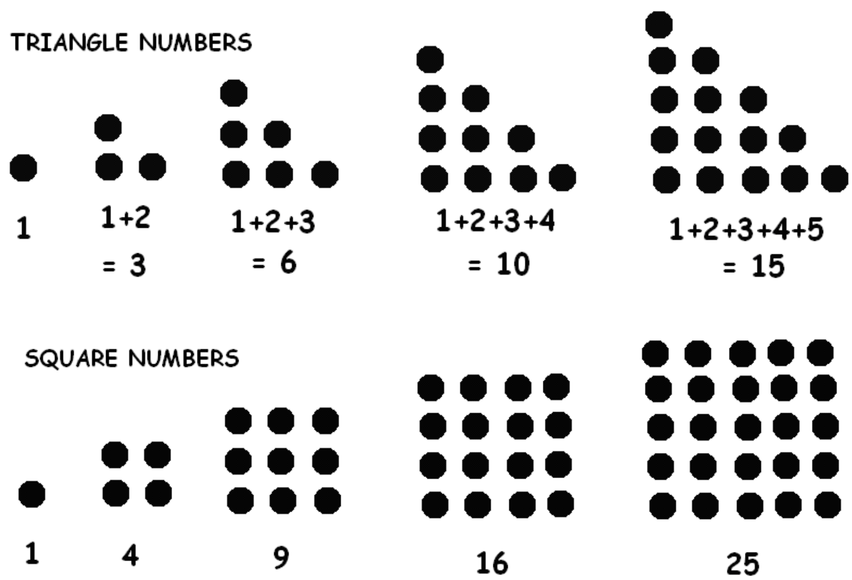
.

d) Show how to convert an intersection math problem into a dinner party problem. Give each dot on the top line a label A, B, C, … and each dot on the bottom line a label E, F, G… .

II: BRIEF ASIDE ON TRIANGLE NUMBERS

[A fuller version of these concepts appears in *THINKING MATHEMATICS! Volume 1.]*

The triangle numbers and square numbers are the numbers that arise from arranging dots into triangular and square configurations:



The N -th square number is given by $S_N = N^2$ and the N -th triangle number is the sum of the first N counting numbers:

$$T_N = 1 + 2 + 3 + \cdots + N$$

Write this sum forwards and backwards and sum column-wise:

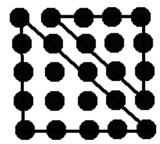
$$\begin{array}{ccccccccccc} 1 & + & 2 & + & \ldots & + & (N-1) & + & N & = & T_N \\ N & + & (N-1) & + & \ldots & + & 2 & + & 1 & = & T_N \\ \hline (N+1) & + & (N+1) & + & \ldots & + & (N+1) & + & (N+1) & = & 2T_N \end{array}$$

This shows that $N(N+1) = 2T_N$ yielding:

$$T_N = \frac{N(N+1)}{2}$$

There are some remarkable relationships twixt square and triangular numbers.

Two consecutive triangle numbers always sum to a square number.
A single picture reveals why.



Here we see that the fourth triangular number and the fifth triangular number fit together to make a 5×5 square.

Doubling a triangle number and adding its matching square number always yields another triangle number.

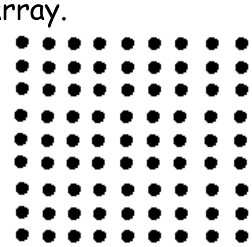
2x1 + 1 = 3 triangle
2x3 + 4 = 10 triangle
2x6 + 9 = 21 triangle
2x10+16= 36 triangle
2x15+25= 55 triangle
etc.

<u>Reason</u>: Two triangles and square make another triangle!

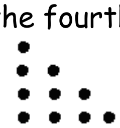
Multiply any triangle number by 8 and add1. The result is always an (odd) square number.

8x1 + 1 = 9 square
8x3 + 1 = 25 square
8x10+ 1= 81 square
8x15+1 = 121 square
8x21+1 = 169 square
etc.

EXERCISE: Here's a 9×9 square array.



Can you find in this grid eight copies of the fourth triangular number



leaving behind one single dot?

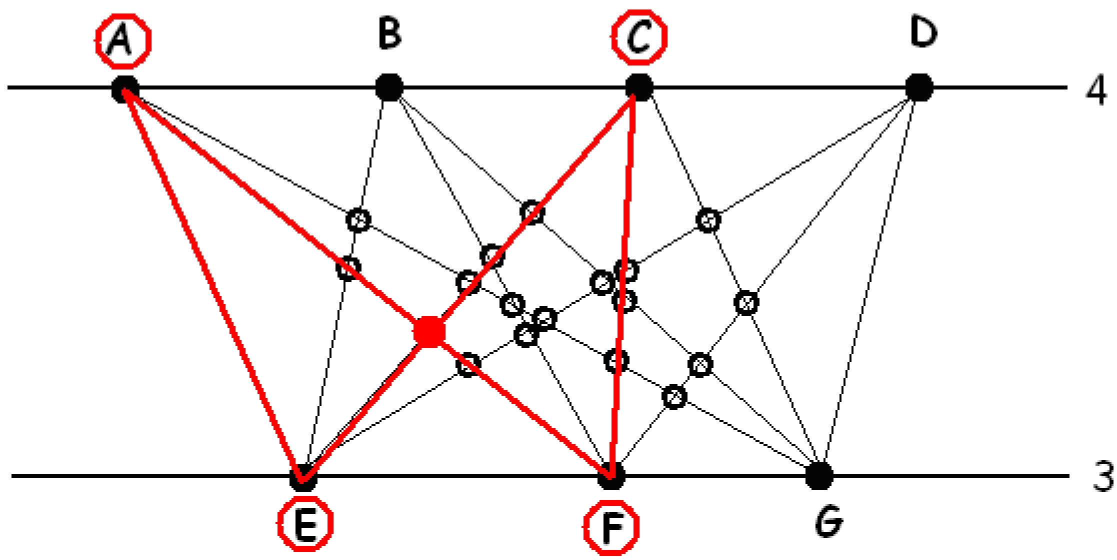
Try to come up with a design of eight triangles and a single dot that has a sense of symmetry to it and would clearly and easily generalize to 11×11, 13×13, and other square arrays of odd side-length. (HINT: Before you begin, what dot in the square array do you think might be the one singled out as special?)

TOUGH CHALLENGE: The numbers 1, 36 and 1225 are both square and triangular. Any more?

III: SOLUTIONS TO WEIRD MULTIPLICATION

The three different weird multiplications – intersection math, rectangle math, and dinner party math - are identical! To explain …

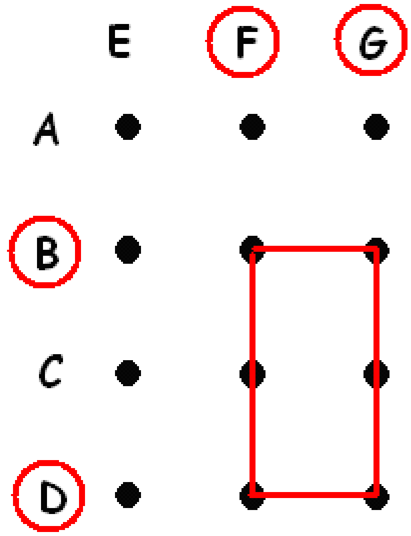
To compute 4x3 in intersection math imagine we labeled the four points on one line Albert, Bilbert, Cuthbert and Dilbert, and the three points on the second line, Edwina, Fellina and Gina:



Choosing a dinner party arrangement – two men and two women – identifies a unique intersection point and, conversely, each intersection point corresponds to a unique dinner party arrangement. (Check this! Choose a random intersection point and be clear that it does determine a specific dinner party arrangement.)

This shows that the number of intersection points matches the number of dinner parties. These two types of math are the same.

To compute 4x3 in rectangle math imagine we label the four rows of the diagram Albert, Bilbert, Cuthbert and Dilbert, and the three columns Edwina, Fellina and Gina.



Each dinner party configuration corresponds to a rectangle in the grid and each rectangle to a dinner party! Thus rectangle math is identical to dinner party math.

Here's the multiplication table for each type of math:

x	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	1	3	6	10	15
3	0	3	9	18	30	45
4	0	6	18	36	60	90
5	0	10	30	60	100	150
6	0	15	45	90	150	225

If you are observant you may have noticed an ordinary multiplication table sitting inside the design. It looks like intersection/rectangle/dinner-party math is the ordinary multiplication of triangle numbers!

To be clear about this, notice that the number of ways to select two people from a group of four really corresponds to the sum $1+2+3$, the third triangle number:

AB
AC BC
AD BD CD

In general, systematically listing pairs of men from a group of N options corresponds to the sum $1+2+3+\cdots+(N-1)$, the $(N-1)$ *th* triangle number.

In dinner party math with N men and M women to choose from there are T_{N-1} pairs of men to consider and T_{M-1} pairs of women. The number of possible dinner party options in all is:

$$T_{N-1} \times T_{M-1}$$

indeed the ordinary product of two triangle numbers!

In particular, in any of these maths:

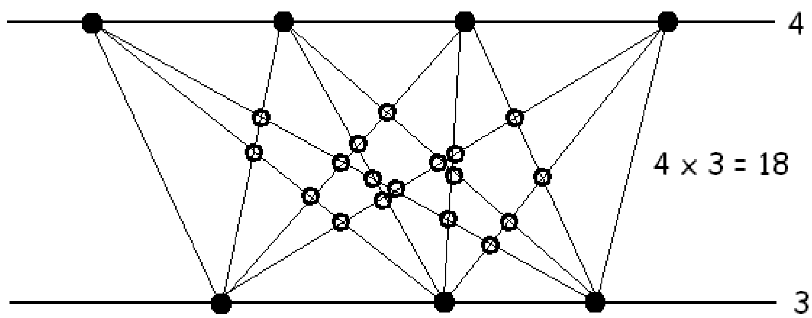
$$963 \times 4036 = T_{962} \times T_{4035} = \frac{962 \times 963}{2} \times \frac{4035 \times 4036}{2} = 3,771,690,643,890$$

EXTRA CHALLENGES:

These maths are commutative: $a \times b = b \times a$. Does the associative law also hold: $a \times (b \times c) = (a \times b) \times c$? Is there a multiplicative identity? That is, a special number *e* with the property that $a \times e = a$ for all *a*?

IV: WILD EXPLORATIONS

Let's go back to intersection math, but instead of counting points of intersection, let's count other things! Here again is the diagram for 4×3:



How many lines (apart from the original horizontal pair) did we draw in this diagram? Answer: 12.

Let's say: In line math we have $4 \times 3 = 12$.

CHALLENGE 1: Work out other products in "line math." Is there anything to observe?

How many spaces appear between the horizontal lines in the diagram for 4×3?

There is an issue here: Do we wish to include the infinitely long regions to the left and to the right, or just count the finite regions? Since I need to continue writing this essay I need to make a choice now(!), but do not feel you have to go with what I do here!

Here's my choice: I will only count the finite regions.

In this case I see 29 regions. I will say:

In region math we have $4 \times 3 = 29$.

CHALLENGE 2: Work out other products in "region math." Any patterns?

ALL THREE TOGETHER!

In the previous diagram for 4×3 we have:

Intersections $I=18$
Lines $L=12$
Regions $R=29$

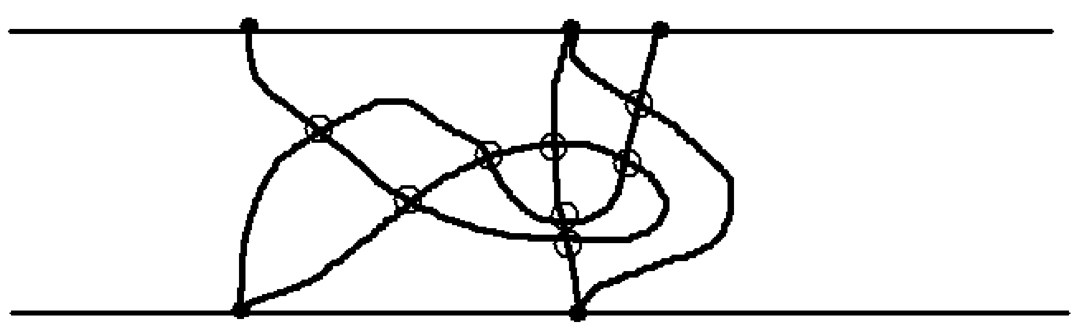
Collect data of this type for other product diagrams.

CHALLENGE 3: Is there a remarkable formula that seems to always hold true for the numbers I, L and R? Can you prove your formula must be true? (See challenge 4.)

OKAY … GOING WILD!

Who said lines have to be straight? Do we really have to connect each dot on the top line with each and every top on the bottom line?

Consider the following diagram:



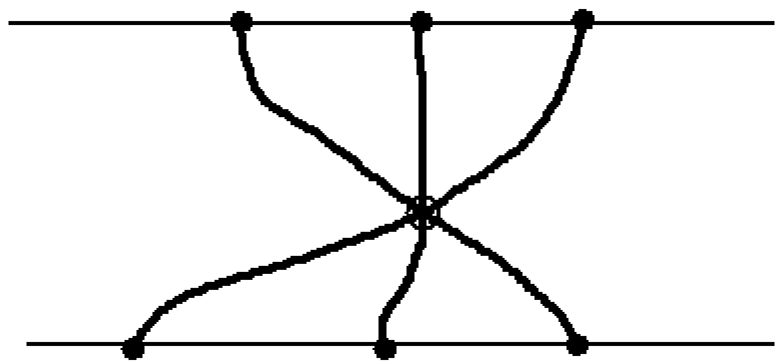
It has:

$I=8$
$L=4$
$R=11$

Does this fit your formula from challenge 3?

CHALLENGE 4: My personal instinct suggests it might be easier to prove the formula for challenge 3 in this context of allowing non-straight lines. Am I right? Try considering the results of adding one line at a time to the diagram.

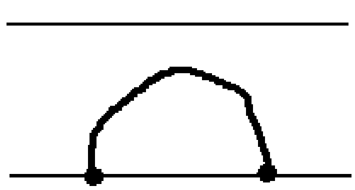
What about multiple intersections?

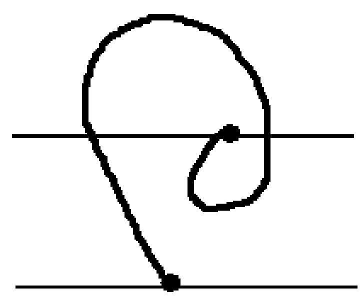


Should these intersections still count as "one intersection" or should they be worth more?

CHALLENGE 5: Is there a good way to count multiple intersections so that your formula from challenge 3 is still appropriate?

What about curvy lines that connect dots on the same horizontal line?
What about lines that wander outside the space between the horizontal lines?





CHALLENGE 7: Is there an ultimate general formula that makes good and appropriate sense for all possible scenarios?

And one final question …

CHALLENGE 8: What about those two infinite regions?
How does everything change if you do decide to include them in your counting?