The Big Dinner

Multiplication with the Ratio Table

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Contents

10 (10×10=100)
25 (25×4=100)
5 (5×20=100)
1 (1×100=100)
50 (50×2=100)
20

1, 2, 4, 5, 10.

4 (4×25=100)
20 (20×5=100)
100 (100×1=100)
2 (2×50=100)

Unit Overview
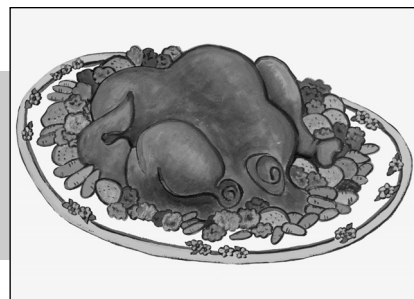
The focus of this unit is the development of multiplication, including automatizing the facts, using the ratio table, and developing the distributive property with large numbers. The unit begins with the context of the preparation for a big turkey dinner. This story context sets the stage for a series of investigations. First, students investigate the cost of a 24-pound turkey, selling for $1.25 per pound. As the unit progresses, students develop t-charts for the grocer that list the prices of turkeys of various sizes and of various amounts of apples, carrots, and potatoes. Students then use these charts to calculate the total cost of all the ingredients, working with the ratio table and the distributive property. The unit culminates in an investigation to determine how long the turkey should cook, at fifteen minutes per pound.

Several minilessons for multiplication are also included in the unit. These use strings of related problems as a way to explicitly guide learners toward computational fluency with whole-number multiplication and to build automaticity with multiplication facts by focusing on relationships.

The Landscape of Learning

BIG IDEAS

- ☀ Unitizing
- ☀ The distributive property of multiplication (over addition and over subtraction)
- ☀ The commutative property of multiplication
- ☀ The place value patterns that occur when multiplying by the base
- ☀ Proportional reasoning

STRATEGIES

- ☀ Using repeated addition
- ☀ Skip-counting
- ☀ Using partial products
- ☀ Using ten-times
- ☀ Using a t-chart or ratio table
- ☀ Doubling and halving

MODELS

- ☀ Open number line
- ☀ Ratio table

Note: This unit also incorporates aspects of the measurement strand as students calculate the cost and cooking time of the turkey (and other ingredients). If your students have not had prior experience with measuring weight and time, you should do several measurement activities first. Such activities might include weighing a variety of things in pound units and determining how many minutes there are in an hour (ensuring that students understand that 15 minutes is a quarter of an hour and 30 minutes is a half).

The Mathematical Landscape

Research has documented that different mathematical models have different effects on mental computation strategies: base-ten blocks may support the development of the standard algorithms, while the hundred chart supports the development of strategies such as counting by tens. The open number line is better aligned with students' invented strategies and it stimulates a mental representation of numbers and number operations that is more powerful for developing mental arithmetic strategies. Students using the open number line are cognitively involved in their actions (Beishuizen 1993; Gravemeijer 1991). In contrast, students who use base-ten blocks or the hundred chart tend to depend primarily on visualization, which results in a passive "reading off" behavior rather than cognitive involvement in the actions undertaken (Klein, Beishuizen, and Treffers 2002).

Traditionally, the array model has been used for multiplication. Research by Battista (1998), however, suggests that this model is often difficult for learners to understand because it requires a substantial cognitive reorganization to be able to coordinate rows and columns simultaneously—and thus an understanding of arrays goes through successive stages of development. Further, students' early strategies for multiplication are more representative of repeated addition and skip-counting—strategies that are often better represented on a number line and/or with a ratio table, such as a t-chart. For these reasons, this unit incorporates the open number line and the ratio table, rather than the array. Your students' understanding of the array model can be developed with the unit *Muffles' Truffles*—a nice unit to use right after this one.

This unit is designed to encourage the development of some of the big ideas related to multiplication, including:

- ❖ *unitizing*

- ❖ *the distributive property of multiplication (over addition and over subtraction)*

- ❖ *the commutative property of multiplication*

- ❖ *the place value patterns that occur when multiplying by the base*

- ❖ *proportional reasoning*

❖ *Unitizing*

Initially students think about multiplication additively. They use repeated addition or skip-count. To really think multiplicatively, students have to think about the group as a unit. Making the group a unit is called unitizing. When students are first introduced to multiplication, most have just reached solid ground with counting. They do know what six is; it is a group of six objects. And they know what three is; it is a group of three objects. But to think of 3 x 6, they have to cognitively reorganize everything they know about number. They have to think of the group of six as one unit because they need to make three sixes. To a young learner, this makes no sense. In fact, it is a contradiction. Something can't be six and one at the same time!

❖ *The distributive property of multiplication (over addition and over subtraction)*

The distributive property of multiplication over addition—knowing that factors can be broken up and distributed to make partial products, which can then be added together to produce the product of the original factors—is one of the most important ideas about multiplication. It is the reason that the standard pencil-and-paper algorithms for multiplication and division work, and its use for mental arithmetic is powerful as well.

Algebraically the distributive property is represented as $a(b + c) = ab + ac$. For example, 4×19 can be solved by decomposing the 19 into 10 and 9. Two partial products can now be made from 4×10 and 4×9. Adding those products, $40 + 36$, produces the answer to 4×19. The 19 can also be distributed in other ways, such as 12 and 7, or the 4 can be distributed into two groups to make $19 \times 2 + 19 \times 2$. In these examples we distributed addends and so the partial products had to be added, making use of the distributive property of multiplication over addition. Nineteen can also be thought of as $20 - 1$. Now, the partial products can be found by calculating 4×20 and 4×-1. This produces $80 - 4$, which also is equivalent to 4×19. In this case, we have used the distributive property of multiplication over subtraction.

For students, this property of multiplication can be quite difficult to construct. Understanding that 6×4 means six *groups* of four—not 6 *plus* 4—can itself be elusive until students construct unitizing. And now, with the distributive property, the challenge is even greater. Students have to think of making groups out of the groups! Students need much time to explore, arranging and rearranging the parts, in order to develop this big idea. This unit gives students many opportunities to explore these ideas.

❖ The commutative property of multiplication

Multiplication is also commutative: $a \times b = b \times a$. Picture four trains, each made with nineteen connecting cubes. Now imagine them right next to each other so they make a 4×19 array. If we turn this array ninety degrees, we have a 19×4 array—nineteen trains with four cubes in each. Using arrays like this is exactly what students do to convince each other of the commutative property. They often call it the turn-around rule.

❖ The place value patterns that occur when multiplying by the base

Precisely because multiplication is commutative, an interesting thing happens when we multiply by the base—the factor bumps over to the appropriate column. For example, $10 \times 4 = 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4$. The result of 40 seems like magic to students, who often say that they added a zero or refer to the zero trick. The reason this "trick" works is

that we can think of the ten groups of four as four groups of ten—so the 4 is placed (bumps over) into the tens column. It is important to support students in exploring why this pattern happens—to help them construct how place value is involved.

❖ Proportional reasoning

Proportional reasoning is very difficult for young learners, and yet it is the hallmark of genuine multiplicative thinking. If one car uses 4 tires and we want to know how many tires are needed for 6 cars, the thinking involved is about proportionality. If 4 tires are needed for 1 car, 8 are needed for 2, and 24 are needed for 6. Often multiplication has been taught as simple repeated addition, and the proportional reasoning—the essence of multiplicative thinking—has been sacrificed. Repeated addition is only one of several ways to think of multiplication. This unit is designed to develop multiplicative thinking not only by using repeated addition models, but by bridging them with proportionality with the use of the ratio table (in this unit the ratio table takes the form of a t-chart for the grocer).

STRATEGIES

As you work with the activities in this unit, you will notice that students will use many strategies to derive answers. Here are some strategies to notice:

- ❖ *using repeated addition*
- ❖ *skip-counting*
- ❖ *using partial products*
- ❖ *using ten-times*
- ❖ *using a t-chart or ratio table*
- ❖ *doubling and halving*

❖ Using repeated addition

Often the first strategy students use to solve multiplication problems is repeated addition. This is because they are viewing the situation additively, rather than multiplicatively. To solve 6×4 (how many tires needed for six cars), students will write $4 + 4 + 4 + 4 + 4 + 4$ and then add to find the total.

Repeated addition should be seen as a wonderful starting place in the journey, but not as the endpoint. As you confer with students you will need to help them keep track of the groups, and you can encourage more efficient grouping (such as turning the six fours into pairs, resulting in three eights).

❖ Skip-counting

The struggle to keep track of the groups usually pushes students to skip-count. Although they are still thinking about the situation additively rather than multiplicatively, they keep track of the groups mentally and skip-count. To figure out how many tires are needed for six cars, they would say 4, 8, 12, 16, 20, 24.

❖ Using partial products

The first shift to multiplicative thinking occurs when students begin to make partial products—they use a fact they know to make another. For example, they might reason that if 5 cars use 20 tires, and 2 cars use 8 tires, then 7 cars must need 28 tires. Here they are finally making the group a unit (unitizing) and adding it or subtracting it as needed. The big ideas underlying this strategy are unitizing and the distributive property.

❖ Using ten-times

Once students begin to make use of partial products, an important strategy to encourage is the use of the ten-times partial product. This can be very helpful for the nine-times table (just subtract one group from ten-times), for the five-times table (half of the product of ten-times), and for double-digit multiplication. Of course this strategy is helpful only if students have constructed an understanding of the place value patterns that occur when multiplying by the base.

❖ Using a t-chart or ratio table

The situations in this unit promote use of the ratio table as a tool to foster the efficient use of partial products. The first strategy you will probably see is a doubling strategy. If you double the number of cars, then you need to double the number of tires. This strategy is very helpful in some cases, such as when groups double, but it is not sufficient for all cases, such as when numbers are not as friendly. Students can also use the ratio table to make partial products (employing the distributive property) and add them.

For example, if 5 cars use 20 tires and 2 cars use 8 tires, 7 cars will use 28 tires.

❖ Doubling and halving

As students' multiplicative reasoning becomes stronger, they develop the ability to group more efficiently. They begin to realize that if you double the number of groups and want the total to be the same, you need to halve the amount in each group: $4 \times 6 = 8 \times 3$. This strategy can be generalized to tripling and thirding, or quadrupling and quartering, etc. The big idea underlying why these strategies work is the associative property of multiplication. In multiplication, the factors can be associated in a variety of ways. For example, $4 \times (2 \times 3) = (4 \times 2) \times 3$. These strategies are introduced and explored toward the end of this unit, but the associative property is not a primary focus of this unit. The unit in this series that develops the associative property deeply is *The Box Factory*. Doubling and halving can be revisited and generalized in that unit.

MATHEMATICAL MODELING

Two models are developed in this unit: the open number line and the ratio table. The open number line is used to illustrate repeated addition strategies and the relationship of equivalent expressions, such as 6 x 4 and 3 x 8. It is assumed that students have had experience with computation represented on number lines. If this is not the case, but they are familiar with measuring units such as a yard or meter stick, you might prefer to use those models.

The ratio table is also developed in the unit. No prior experience with it is needed as it will emerge from the situation of preparing t-charts for the grocer. With the ratio table model, students are supported in envisioning partial products and in using them to calculate other products. As the unit progresses, the ratio table model is used to support the development of the generalization of the distributive property.

Models go through three stages of development (Gravemeijer 1999; Fosnot and Dolk 2002):

❖ *model of the situation*

❖ *model of students' strategies*

❖ *model as a tool for thinking*

❖ Model of the situation

Initially models grow out of modeling the situation. For example, in this unit the ratio table emerges as charts for the grocer—charts that help customers calculate the cost of turkeys, carrots, potatoes, and apples, all of which are sold by the pound.

Chart for Turkey

Number of Pounds	Cost
1	$1.25
2	$2.50
4	$5.00

❖ Model of students' strategies

Students benefit from seeing the teacher model their strategies. Once a model has been introduced as a model of the situation, you can use it to model students' strategies as they explain their thinking. If a student says in solving 4×6, "I doubled and halved. I made it 2×12," you can use the number line and draw the following:
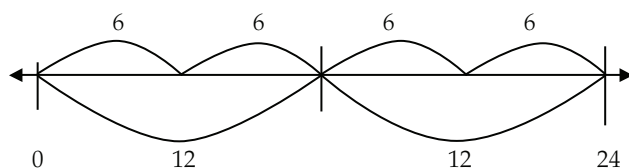
If you are working with a t-chart and a student calculating $4 \times \$1.25$ says, "I knew the cost of 2 pounds was $2.50. Four pounds is double, so I doubled the cost," you might represent the calculation as follows:

Chart for Turkey

Another student in calculating $6 \times \$1.25$ might say, "I knew 2 pounds costs $2.50 and 4 pounds costs $5.00

so I added them together to get the cost of 6 pounds." Here you can represent the partial products as follows:

Chart for Turkey

After you've represented the strategy on the t-chart, you might also want to write:

$$2 \times \$1.25 + \ 4 \times \$1.25 = \ 6 \times \$1.25$$
$$\$2.50 \ \ + \ \ \ \$5.00 \ \ \ = \ \ \ \$7.50$$

Representations like these give students a chance to reflect on and discuss each other's strategies.

❖ Model as a tool for thinking

Eventually students become able to use these models as tools to think with—to prove and explore their ideas about multiplicative and proportional reasoning.

Many opportunities to discuss these landmarks will arise as you work through this unit. Look for moments of puzzlement. Don't hesitate to let students discuss their ideas and check and recheck their strategies. Celebrate their accomplishments—they are young mathematicians at work!

A graphic of the full landscape of learning for multiplication and division is provided on page 11. The purpose of the graphic is to allow you to see the longer journey of students' mathematical development and to place your work with this unit within the scope of this long-term development. You may also find the graphic helpful as a way to record the progress of individual students for yourself. Each landmark can be shaded in as you find evidence in a student's work and in what the student says—evidence that a big idea, landmark strategy, or way of modeling has been constructed. In a sense you will be recording the individual pathways your students take as they develop as young mathematicians!

References and Resources

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MULTIPLICATION/DIVISION

relationship between branching, repeated addition, and arrays in multiplicative situations

the dimensions of length, width, and height can be used to produce a cubic unit measurement of volume

the surface area of rectangular solids increases as the measures of the 3 dimensions diverge

factoring and grouping flexibility

uses ratio table as a tool

uses open array as a tool

simplifying

the dimensions of length and width can be used to produce a square unit measurement of area

uses open array to represent strategies

models multiplicative situation as array

uses money as a model in calculating

the associative property of multiplication

using partial quotients

uses commutative property

relationship between rows and columns in arrays

use of automatized facts

the commutative property of multiplication

using ten-times

the place value patterns that occur when dividing by ten

tripling and thirding

the place value patterns that occur when multiplying by ten

uses familiar facts

proportional reasoning

using five-times

the generalized use of the distributive property of multiplication (over addition and subtraction) in solving division problems

using partial products

uses a t-chart or ratio table to represent a multiplicative situation

relationship between multiplication and division

doubling and halving

the relationship between partitive and quotative division

the distributive property of multiplication

doubling

unitizing

repeated additions can be regrouped

models multiplicative situation as repeated addition on an open number line

repeated addition or subtraction in division context

repeated addition

skip counting

dealing out or counting all, grouping, then counting the groups

represents groups and objects in the groups and counts by ones

tries to make equal-sized groups through trial and error

The landscape of learning: multiplication and division on the horizon showing landmark strategies (rectangles), big ideas (ovals), and models (triangles).