Multiplication to Ratio, Proportion, and Fractions within the Common Core

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CCSS Grade 6 Critical Area 1

Students use reasoning about multiplication and division to solve ratio and rate problems about quantities. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students connect their understanding of multiplication and division with ratios and rates. Thus students expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions. Students solve a wide variety of problems involving ratios and rates.

Definitions of rate and ratio

People do not agree about definitions of rate and ratio.

The CCSS learning path sought to support students to extend earlier understandings and avoid common errors and confusions.

See the R&P Progression for more explanations.
commoncoretools.wordpress.com

Notation Confusions

By Grade 6 what do students know about fractions and the notation $\frac{3}{5}$?

What does $\frac{3}{5}$ mean?

3.NF.1 $\frac{3}{5}$ is 3 parts of size $\frac{1}{5}$
($\frac{1}{5}$ is 1 part when a whole is partitioned into 5 equal parts)

5.NF.3 $3 \div 5 = \frac{3}{5}$ (a fraction)
The result of division can be expressed as a fraction.

Fractions versus ratios

Fractions and ratios are different in their basic meanings:

Fractions: are numbers telling how many parts of what size

Ratios: describe relationships between quantities
part A to part B or part B to part A or part A (or B) to total
or total to part A (or B)

It is too confusing to use the same notation for this new concept.

Levels in learning ratio

Level 1: Grade 6 early
Use 3 : 5 notation initially to build a new concept with whole number ratios.

Level 2: Grade 6 later
See the quotient meaning $\frac{3}{5}$ some people use for a ratio as a unit rate, the value of a ratio. Relate fractions and ratios and all notations.

Level 3: Grade 7
Ratios and proportions use fractions such as $\frac{3}{4} : \frac{2}{5}$. The constant of proportionality c in $y = cx$ is a unit rate.

The c in this equation is actually $\frac{B}{A}$, the unit rate for $B : A$, and is the *reciprocal* of the unit rate for $A : B$.

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Avoiding errors

Many proportion errors involve adding, not multiplying.
So get into multiplication-land first for ratio and proportion.

Research

Fuson, K. C. & Abrahamson, D. (2005). Understanding ratio and proportion as an example of the Apprehending Zone and Conceptual-Phase Problem-Solving Models. In J. Campbell (Ed.), Handbook of Mathematical Cognition (pp. 213-234). New York: Psychology Press.

And other articles you can get from Dor Abrahamson
dor “at” berkeley.edu

In our teaching experiments, Grade 5 students outperformed middle and high school students on proportion tasks.

▶ Discuss Patterns in the Multiplication Table

product
Factor Puzzle

Look for patterns in the multiplication tables.

Table 1

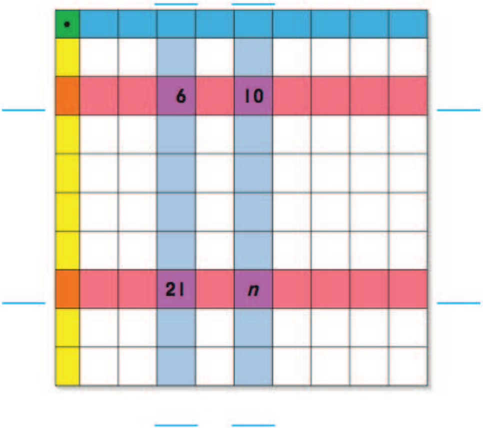
•	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Table 2

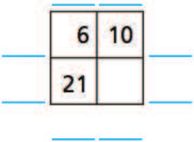
•	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

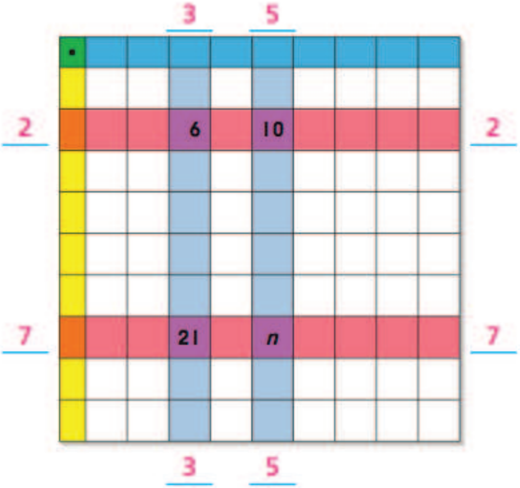
Write the missing **factors** and the missing **product**.

1. Table 3

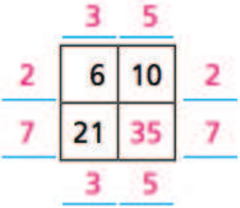


Factor Puzzle





Factor Puzzle



Extend a rate situation to be a class of rate situations with the same unit rate and show them in a table. The unit rate involves whole numbers.

Noreen started to save money. Every day she put three $1 coins into her duck bank.

Rate Table

Days	Dollars
1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24

+ 3
+ 3
+ 3
+ 3
+ 3
+ 3
+ 3

Rate

Discuss rate as an equal-groups situation.

The hiding 1: \$3 each day, \$3 per day, \$3 every day
\$3 each 1 day, \$3 per 1 day, \$3 every 1 day

The unit rate is the amount in 1 group but we do not say the 1.
This is how multiplication with 3 numbers becomes a proportion with 4 numbers: it uses the 1.
$2 \times 3 = 6$ becomes $1 : 3 = 2 : 6$

Rate tables

Start with the term “rate table” as showing many situations with the same rate.
First show multiples of the unit rate starting with 1 in the first column.
Notice that these are just two columns of the Multiplication Table.

After ratio tables are introduced, we will notice that rate tables and ratio tables really are quite similar and behave alike (rows are multiples of the unit rate or basic ratio), so we consider rate tables as a special case of ratio tables and can call them ratio tables.

What situations have a constant rate?

Students discuss what situations have a constant rate and which example tables are rate tables.

Arrays and areas can be considered as equal groups (one row or one column is the group), so rates can be used for such situations.
Each row is a multiple of the unit rate (later, of each other row, when multiplying by a fraction is included).

Finding unit rates

Find the unit rate given a product and the number of things:

$$P \div n = \text{unit rate}$$

Put this information in a scrambled rate table and fill in other scrambled rows of the table.

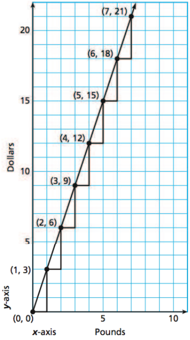
Relate table, equation, and graph

Number of Pounds	Cost in Dollars
1	3
2	6
3	9
4	12
5	15
6	18
7	21

Number of Pounds	• Unit Rate	=	Cost in Dollars
p	• *r*	=	*C*
1	• 3	=	3
2	• 3	=	6
3	• 3	=	9
4	• 3	=	12
5	• 3	=	15
6	• 3	=	18
7	• 3	=	21

Number of Pounds		Cost in Dollars
1	• 3	3
2	• 3	6
3	• 3	9
4	• 3	12
5	• 3	15
6	• 3	18
7	• 3	21

The unit rate is circled. Imagine the unit rate written on the vertical rule to be multiplied by the number in the left column to get the number in the right column.



Each point on the graph corresponds to an **ordered pair**. (0, 0) and (1, 3) are ordered pairs.

The first number is the ***x*-coordinate** and the second number is the ***y*-coordinate**.

In the ordered pair (1, 3), 1 is the *x*-coordinate and 3 is the *y*-coordinate.

From rate tables to ratio tables

Ratios as the product columns from two linked rate tables.

> *Noreen's brother Tim saves $5 a day. Noreen and Tim start saving on the same day.*

Constant Increases Ratio Table

	N	T	
	3	5	
	3	5	
+3	6	10	+5
+3	9	15	+5
+3	12	20	+5
+3	15	25	+5
+3	18	30	+5
+3	21	35	+5

Linking Multiplier Column Ratio Table

Days	N 3	T 5
1	3	5
2	6	10
3	9	15
4	12	20
5	15	25
6	18	30
7	21	35

Standard Ratio Table

N	T
3	5
3	5
6	10
9	15
12	20
15	25
18	30
21	35

Equivalent ratios

Equivalent ratios are two rows from a ratio table.
They can be written as

$$6 : 10 = 21 : 35$$

or

$$6 : 10 :: 21 : 35$$

a) A basic ratio (Confrey's littlest recipe) is the least possible whole number ratio (from the 1s row of the MT). Equivalent ratios are two multiples of the basic ratio.
b) Equivalent ratios are multiples of each other (where one multiple can be a fraction < 1).

Proportions

Two equivalent ratios make a proportion.

> *Grandma made applesauce using the same number of bags of red and yellow apples. Her red apples cost $6, and her yellow apples cost $14. I used her recipe but made more applesauce. I paid $35 for my yellow apples. How much did my red apples cost?*

Factor Puzzle

	Red	Yellow	
	3	7	
2	6	14	2
5	15	35	5
	3	7	

Ratio Table

	Bags	R 3	Y 7
	1	3	7
• 2	2	6	14
	3	9	21
	4	12	28
• 5	5	15	35
	6	18	42
	7	21	49
	8	24	56
	9	27	63

The Factor Puzzle and the Ratio Table as columns from a MT immediately makes a whole range of proportion problems solvable. Then it is important to explore the following three issues.

- Label the table.
- Practice with problems that have the information out of order: scrambled FP.
- State your assumption that makes the situation proportional.

Factor Puzzles, Ratio Tables, and Multiplication Tables

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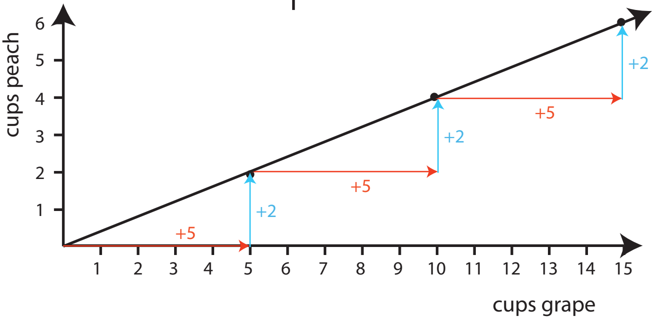
- Label the table.
- Practice with problems that have the information out of order: scrambled FP.
- State your assumption that makes the situation proportional.

Additive structure

cups grape	cups peach
5	2
10	4
15	6
20	8
25	10

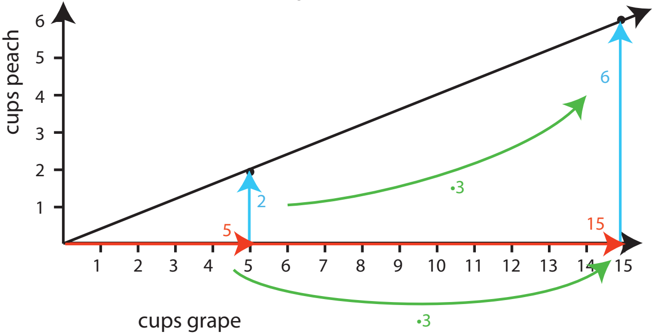
+5 +5 +5 +5 / +2 +2 +2 +2



Multiplicative structure

cups grape	cups peach
5	2
10	4
15	6
20	8
100	40

•2 •3 •20 •2 •3 •20



Fractional unit rates

By allowing entries in ratio and rate tables to be fractions (not just whole numbers), students can always find ratio or rate pairs where one of the entries is 1. This pair tells us a unit rate, namely the amount of one quantity per 1 unit of the other quantity. Students will see unit rates in vertical tables, in horizontal tables, or as factors in Factor Puzzles.

Vertical Ratio Table

Cherry : Orange

5	4
$\frac{5}{4}$	1

$\div 4$ (Cherry column) $\div 4$ (Orange column)

$\frac{5}{4}$ is the quotient of $5 \div 4$.

Sue has $\frac{5}{4}$ cups of cherry juice for every cup of orange juice.

The unit rate for the ratio 5:4 is $\frac{5}{4}$.

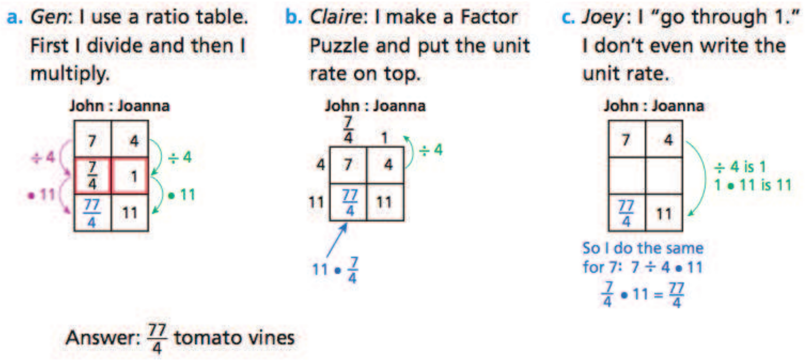
Fractional unit rates

For the reverse ratio $4 : 5$ orange to cherry, the value of the ratio is $\frac{4}{5}$.

- $\frac{4}{5}$ is the quotient of $4 \div 5$;
- $\frac{4}{5}$ is another unit rate:
  Sue has $\frac{4}{5}$ of a cup of orange for every 1 cup of cherry.

Variations in the unit rate strategy

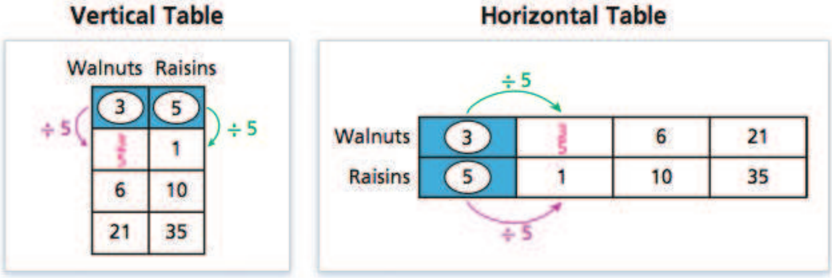
John can plant 7 tomato vines in the time it takes Joanna to plant 4 tomato vines. At that rate, when Joanna has planted 11 tomato vines, how many has John planted?



Vertical and horizontal ratio tables

The rows and columns of a multiplication table are symmetric and can be flipped into each other.
So ratio tables can be two rows of a multiplication table instead of two columns.

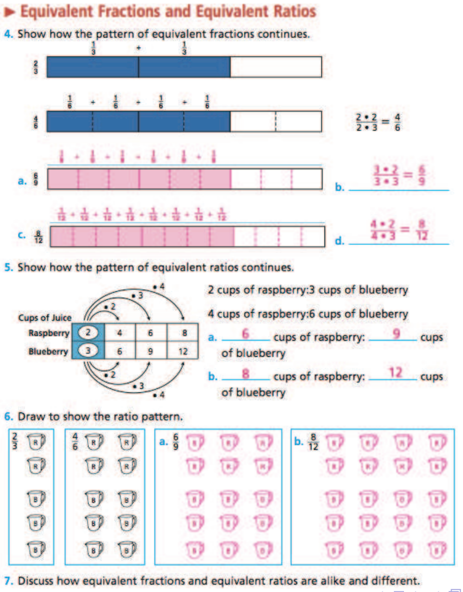
The ratio was horizontal and now is vertical, like a fraction.



Practice writing horizontal ratios in vertical fraction notation.

$$16 : 20 = 12 : a \quad \text{as} \quad \frac{16}{20} = \frac{12}{a}$$

Equivalent fractions and equivalent ratios are different



▶ Equivalent Fractions and Equivalent Ratios

4. Show how the pattern of equivalent fractions continues.

$\frac{2}{3}$: $\frac{1}{3} + \frac{1}{3}$

$\frac{4}{6}$: $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ $\qquad \frac{2 \cdot 2}{2 \cdot 3} = \frac{4}{6}$

a. $\frac{6}{9}$ $\qquad$ b. $\frac{3 \cdot 2}{3 \cdot 3} = \frac{6}{9}$

c. $\frac{8}{12}$ $\qquad$ d. $\frac{4 \cdot 2}{4 \cdot 3} = \frac{8}{12}$

5. Show how the pattern of equivalent ratios continues.

Cups of Juice

Raspberry	2	4	6	8
Blueberry	3	6	9	12

(• 2, • 3, • 4)

2 cups of raspberry:3 cups of blueberry

4 cups of raspberry:6 cups of blueberry

a. 6 cups of raspberry: 9 cups of blueberry

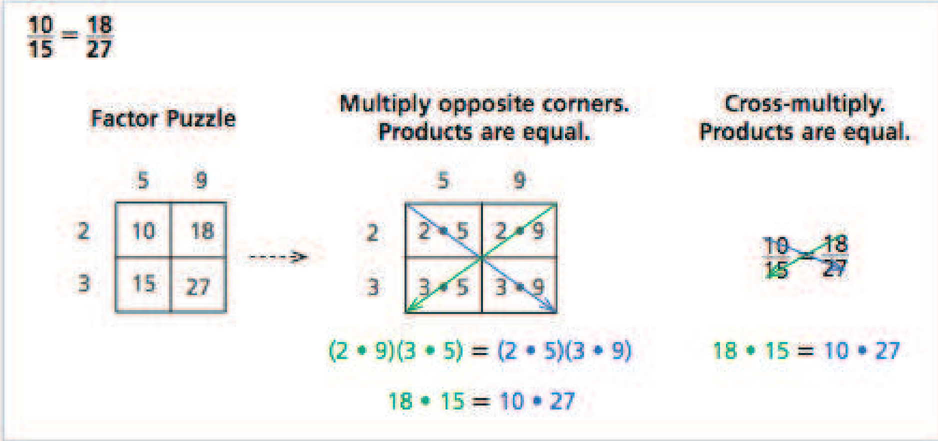
b. 8 cups of raspberry: 12 cups of blueberry

6. Draw to show the ratio pattern.

$\frac{2}{3}$ $\quad$ $\frac{4}{6}$ $\quad$ a. $\frac{6}{9}$ $\quad$ b. $\frac{8}{12}$

7. Discuss how equivalent fractions and equivalent ratios are alike and different.

Cross-multiplication



Cross-multiplication

9. Zander paid \$7 for 5 avocados. How much would 9 avocados cost?

Discuss how these solution strategies relate to each other.

Ratio Table

÷ 5 • 9

Dollars	7	$\frac{7}{5}$	$\frac{63}{5}$
Avocados	5	1	9

÷ 5 • 9

Factor Puzzle

		5	9
Dollars	$\frac{7}{5}$	7	$\frac{63}{5}$
Avocados	1	5	9

Cross-Multiplication

$$\frac{7}{5} = \frac{c}{9}$$

$$c \cdot 5 = 7 \cdot 9$$

$$5c = 63$$

$$c = \frac{63}{5}$$

The price for 9 avocados is $\frac{63}{5}$ dollars, or \$12.60.

Comparing ratios

Same amount of red.
Abby's has more yellow,
so Abby's is yellower,
Zack's is redder.

Abby's

cups red	cups yellow
1	3
2	6
3	9
4	12
5	15

Zack's

cups red	cups yellow
3	5
6	10
9	15
12	20
15	25

Comparing ratios

Abby's cups red	Abby's cups yellow
1	3
2	6
3	9
4	12
5	15

Zack's cups red	Zack's cups yellow
3	5
6	10
9	15
12	20
15	25

Same amount of yellow.
Zack's has more red.
So Zack's is redder,
Abby's is yellower.

Comparing ratios

Abby's

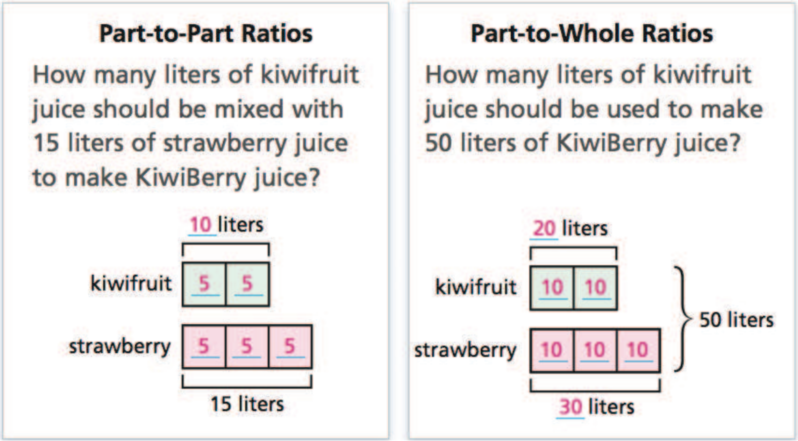
cups red	cups yellow	total cups
1	3	4
2	6	8
3	9	12

Zack's

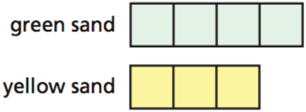
cups red	cups yellow	total cups
3	5	8
6	10	16
9	15	24

Same total.
Abby's has more yellow.
Zack's has more red.
So Abby's is yellower and
Zack's is redder.

A juice company's KiwiBerry juice is made by mixing 2 parts kiwifruit juice with 3 parts strawberry juice.



Multiplicative comparisons



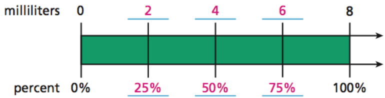
The amount of yellow sand is $\underline{\quad \frac{3}{4} \quad}$ times the amount of green sand.

The total amount of mixture is $\underline{\quad \frac{7}{3} \quad}$ times the amount of yellow sand.

Strategies for Percent Problems

The adult dose of a medicine is 8 milliliters. The child dose is 75% of the adult dose. How many milliliters is the child dose?

Double Number Line



Trey's Reasoning About Parts

100% is 4 parts, which is 8 mL.

25% is 1 part, which is 8 mL ÷ 4 = 2 mL.

75% is 3 parts and is 3 • 2 mL = 6 mL.

Quowanna's Factor Puzzle

		percent	milliliters
		25	2
portion	3	75	6
whole	4	100	8

Tomaslav's Equation

m is 75% of 8.

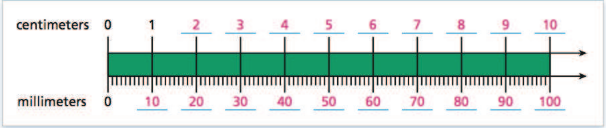
$$m = \frac{75}{100} \bullet 8 = \frac{3}{4} \bullet 8 = 3 \bullet 2 = 6$$

Jessica's Proportion

	percent		milliliters
portion / whole	$\frac{75}{100}$	=	$\frac{m}{8}$
	$\frac{3}{4}$	=	$\frac{m}{8}$
	m	=	6

Measurement Conversions

Double Number Line This model helps students understand how different units of measurement are related.



Proportional Reasoning Students also make measurement conversions using proportions or unit rates.

Write and Solve a Proportion

$$\frac{1\text{ cm}}{10\text{ mm}} = \frac{52\text{ cm}}{x\text{ mm}}$$

$$52 \bullet 10 = 1 \bullet x$$

$$520 = x$$

So, 52 cm = 520 mm.

Use a Unit Rate

$$52\ \cancel{\text{cm}} \bullet 10\ \frac{\text{mm}}{\cancel{\text{cm}}} = 520\text{ mm}$$

There are 52 cm and there are 10 mm in each cm.

The unit cm cancels, leaving the unit mm.

Level 3: Grade 7

Ratios and proportions use fractions such as

$$\frac{3}{4} : \frac{2}{5}$$

A unit rate for a ratio becomes a constant of proportionality c in $y = cx$.
For the ratio $A : B$, c is $\frac{B}{A}$, not $\frac{A}{B}$
This is because

$$\frac{y}{x} = \frac{B}{A}$$

so, multiplying both sides by x, we have

$$y = \frac{B}{A} \cdot x$$