Fluid Mechanics

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0. Introduction

The study of fluid mechanics encompasses many varied applications in both natural and engineering processes. Some examples are

Aerodynamics – flow past wings; computation of drag and lift; D'Alembert's paradox
Biological fluid dynamics – fish swimming; flow in arteries and veins; blood flow in vascular tumours; Korotkoff sounds
Geophysical fluid dynamics (GFD) – oceanography and meteorology; climate and weather prediction
Industrial fluid dynamics – flow past structures and buildings; coating flows (e.g. deposition of photographic films); spraying processes
Magnetohydrodynamics (MHD) – flow of conducting fluids; Maxwell's equations
Physiochemical fluid dynamics – flow under surface tension in the presence of chemical contaminants
Turbulence – transition to turbulence from laminar flow; boundary layer receptivity

All of these are active areas of research

We will show that the Navier-Stokes equations for an incompressible fluid are:

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -\nabla p/\rho + (\mu/\rho)\nabla^2 \mathbf{u},$$

where
  $\mathbf{u}$ is the velocity in the fluid
  p is the pressure
  ρ is the density
  μ is the viscosity.

If U and L are typical velocity and length scales for the problem in hand, we will also see that a given flow may be characterised by the value of the dimensionless *Reynolds number*

$$R = \frac{UL}{\nu}.$$

Simplifications can be made if $R \ll 1$ (see Section 6) or $R \gg 1$ (see Section 8).

1. Preliminary ideas

What is a fluid?

A simple fluid cannot withstand an externally applied stress without deforming in some manner. This definition of a fluid encompasses both liquids and gases. In order to develop a mathematical basis for the study of fluid motion, we adopt the *continuum hypothesis*, which states that the material under scrutiny is assumed to behave as if it is perfectly continuous. In real materials, fluctuations at the molecular level mean that this hypothesis is inaccurate at very small scales. However, in most situations we are concerned with macroscopic observations of a given fluid motion and are not troubled by the irregular variations apparent on molecular lengthscales. Focussing on a tiny volume of a given fluid, we make the approximation that physical quantities, such as mass and momentum, are spread uniformly over that volume rather than unevenly distributed within it. Under this assumption we may define, for example, the fluid velocity $\mathbf{u}(\mathbf{x}, t)$, density $\rho(\mathbf{x}, t)$, or temperature $T(\mathbf{x}, t)$, valid at a point $\mathbf{x}$ and time t in the flow field and treat them as continuous, differentiable quantities over the domain of interest.

A *viscous* fluid is distinguished by its ability to resist shearing motions. A *perfect* or *inviscid* fluid offers no resistance to shear. The *viscosity*, μ, of a fluid is a measure of its ability to withstand such shearing action. The *kinematic viscosity* ν of a fluid is defined to be the ratio between the viscosity and the density, so $\nu = \mu/\rho$.

Describing a fluid flow

There are two different frameworks avaliable within which to describe a fluid's motion. These are the:

Eulerian framework: In this approach, the fluid is considered as a whole. Global vector and scalar variables such as velocity and temperature are defined and solved for over the entire flow-field simultaneously. This is by far the most widely used.

Lagrangian framework: In this formulation, individual fluid particles are tracked. Each particle is labelled according to its known location at a given time. For example, a particular particle is labelled by its position $\mathbf{x}_0$ at time $t = 0$. Then, at a later time $t > 0$, it has position $\mathbf{x}(\mathbf{x}_0, t)$, velocity $\mathbf{u} = (\partial \mathbf{x}/\partial t)_{\mathbf{x}_0}$, and acceleration $(\partial \mathbf{u}/\partial t)_{\mathbf{x}_0}$, where the labels on the brackets indicate that the differentiations are performed for constant $\mathbf{x}_0$, i.e. for this one particular particle. This description of the flow tends to be rather cumbersome and is little used.

Compressibility

What role do compressibility effects play in determining the motion of a fluid? Air is clearly much more easily compressible than water, say, but to what extent is this important in a particular flow?

Bernoulli's theorem states that in a steady, inviscid flow with no body forces acting, the pressure is related to the velocity via

$$p + \frac{1}{2}\rho \mathbf{u}^2 = \text{constant along a streamline.}$$

It follows that a change in speed of a fluid element from zero to U requires a pressure change of $O(\rho U^2)$.

By definition,

$$c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho},$$

where c is the speed of sound. Thus a small change in density is related to a small change in pressure via

$$\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}p}{\rho c^2},$$

and so

$$\frac{\mathrm{d}\rho}{\rho} = O\Big(\frac{\rho U^2}{\rho c^2}\Big) = O\Big(M^2\Big),$$

on defining the *Mach number*

$$M = \frac{U}{c}.$$

We can deduce that if $M \ll 1$, the effects of compressibility can be safely ignored.

Examples:

1. In air, $c \approx 340ms^{-1}$, so that for $U = O(10^2 ms^{-1})$, $M = 0.29$ and compressibility will not be important.

2. For water, $c \approx 1.5 \times 10^3 ms^{-1}$, so would need $U \gg O(1500ms^{-1})$ for significant compressibility effects.

3. For an incompressible fluid, changes in pressure occur with *no* corresponding change in density. In this case $\mathrm{d}p/\mathrm{d}\rho$ is infinite, so the speed of sound in the incompressible medium is infinite. A consequence of this is that the fluid feels the effect of a local change everywhere in the fluid *instantaneously*.

2(a). Stress in fluids

The forces acting on a fluid in motion are divided into two categories:

1. *Long-range forces: Externally applied forces such as gravity, an imposed pressure gradient, or electromagnetic forces in the case of conducting fluids.*
2. *Short-range forces: These refer to local dynamic stresses developing within the fluid itself as it moves; specifically, the forces acting on a given element of fluid by the surrounding fluid.*

To describe those in the second category, we introduce the stress tensor σ_{ij}, defined as

> *the i-component of stress acting on an element of surface with unit normal* $\mathbf{n}$
> *in the j-direction.*

where i, j both run over 1,2,3, corresponding to x, y, z in a Cartesian frame.

This makes the stress on a surface easy to find if the surface in question has normal pointing in one of the coordinate directions. We just pick out the relevant entry of σ_{ij}. For example, the stress on a plane wall at $y = 0$ is found by noting that the normal points in the y direction. So, for the component of stress pointing in the x direction, for example, we need

$$\sigma_{12}.$$

Similarly, the component of stress in the z direction on a wall at $x = 0$ is σ_{31}.

In general, of course, the unit normal will not point in a coordinate direction. Suppose instead we have a solid boundary over which flows a fluid. We show below that the force exerted at a point on the boundary by the passing fluid is

$$\sum_{j=1}^{3} \sigma_{ij}\, n_j$$

where the unit vector $\mathbf{n}$ is normal to the boundary and points *into* the fluid.

Einstein's summation convention

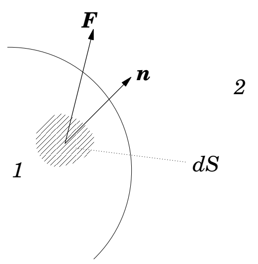
Einstein introduced the nifty convention of summing over a repeated index to save repeatedly writing the summation sign Σ (this shows up a lot in relativity!). We will adopt the same convention throughout this course, and thereby write the previous formula as just

$$\sigma_{ij}\, n_j$$

where the summation over j is now understood.

Stress within a fluid

As a fluid moves, it may be acted upon by long-range forces such as gravity. But there are also internal stresses active inside the fluid.



Consider an imaginary surface drawn in the fluid. Shade on this surface a small patch of area dS, as shown in the diagram. Label the fluid on one side of the dividing surface 1 and on the other side 2. The unit normal to the patch is $\mathbf{n}$ and is chosen to point into fluid 2. We define the stress (force per unit area) acting on the patch on side 1 due to the fluid on side 2 to be $\mathbf{H}$ (so the force is $\mathbf{H}\,\mathrm{d}S$), when $\mathbf{n}$ points *into the fluid on side 2*.

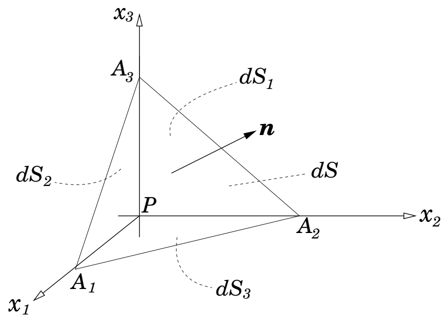
In general $\mathbf{H}$ will depend on $\mathbf{n}$, the position in the fluid $\mathbf{x}$ and time t, i.e. $\mathbf{H} = \mathbf{H}(\mathbf{x},\ \mathbf{n},\ t)$. From the definition of σ_{ij} above, we have

$$\sigma_{ij} = H_i(\mathbf{x},\ \mathbf{N}_j) \tag{2.1}$$

where $\mathbf{N}_1$=$\mathbf{i}$, $\mathbf{N}_2 = \mathbf{j}$, $\mathbf{N}_3 = \mathbf{k}$. The dependence on t in (2.1) is taken to be implicit.

How can we evaluate $\mathbf{H}$ for an arbitrary normal direction $\mathbf{n} = (n_1, n_2, n_3)$?

To do this, consider a small tetrahedron of fluid of volume $\mathrm{d}V$ as shown below.



We let

$$\text{the area of} \quad \left\{ \begin{array}{c} A_1A_2A_3 \\ PA_2A_3 \\ PA_1A_3 \\ PA_1A_2 \end{array} \right\} \quad \text{be} \quad \left\{ \begin{array}{c} \mathrm{d}S \\ \mathrm{d}S_1 \\ \mathrm{d}S_2 \\ \mathrm{d}S_3 \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{d}S \\ n_1\mathrm{d}S \\ n_2\mathrm{d}S \\ n_3\mathrm{d}S \end{array} \right\}. \tag{2.2}$$

(Note: To find the areas of each surface of the tetrahedron, use the simple half-base times height rule.)

Now,

$$\text{the force exerted by the fluid outside } \mathrm{d}V \text{ on} \quad \left\{ \begin{array}{c} \mathrm{d}S \\ \mathrm{d}S_1 \\ \mathrm{d}S_2 \\ \mathrm{d}S_3 \end{array} \right\} \quad \text{is} \quad \left\{ \begin{array}{c} H_i(\mathbf{x},\ \mathbf{n})\,\mathrm{d}S \\ -H_i(\mathbf{x},\ \mathbf{N}_1)\,\mathrm{d}S_1 \\ -H_i(\mathbf{x},\ \mathbf{N}_2)\,\mathrm{d}S_2 \\ -H_i(\mathbf{x},\ \mathbf{N}_3)\,\mathrm{d}S_3 \end{array} \right\}. \tag{2.3}$$

So the total force exerted on the tetrahedron is

$$\mathrm{d}S\,[H_i(\mathbf{x},\ \mathbf{n}) - \sigma_{ij}n_j].$$

Note: Here we are using the *Einstein summation convention* of summing over a repeated index. So $\sigma_{ij}n_j$ means $\sigma_{i1}n_1 + \sigma_{i2}n_2 + \sigma_{i3}n_3$.

Newton's second law states that

$$\text{Force} = \text{Mass} \times \text{Acceleration}. \tag{2.4}$$

If the dimensions of the tetrahedron are of size $O(\epsilon)$, the left hand side of (2.4) is $O(\mathrm{d}S) = O(\epsilon^2)$, while the right hand side is $O(\mathrm{d}V) = O(\epsilon^3)$. Therefore, if we let ϵ tend to zero, (2.4) produces the result that Force $= 0$, and we conclude that and so

$$H_i(\mathbf{x},\ \mathbf{n}) = \sigma_{ij}n_j. \tag{2.5}$$

Example 1: Find the stress in the x direction on a wall placed at $y = 0$. In this case $n_1 = n_2 = 0, n_3 = 1$. So $F_i = \sigma_{ij}n_j = \sigma_{i2}$. The component in the x direction is $F_1 = \sigma_{12}$, as we had above.

Example 2: Find the stress in the z direction on the upper side of the plane $x + y + z = 1$ due to the fluid above it. In this case

$$\mathbf{n} = +\frac{\nabla(x + y + z - 1)}{|\nabla(x + y + z - 1)|} = \frac{1}{\sqrt{3}}(1, 1, 1).$$

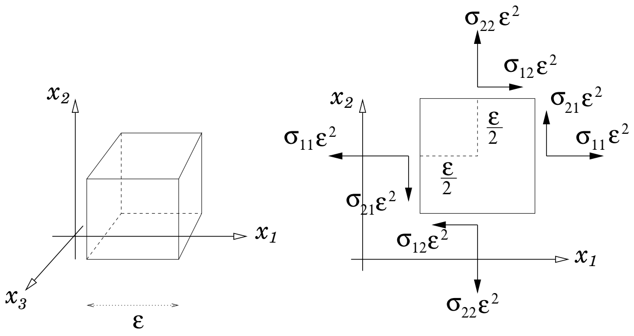
Then the stress in the z direction is F_3, with

$$F_3 = \sigma_{3j} n_j = \sigma_{31}\frac{1}{\sqrt{3}} + \sigma_{32}\frac{1}{\sqrt{3}} + \sigma_{33}\frac{1}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}(\sigma_{31} + \sigma_{32} + \sigma_{33}).$$

Symmetry of the stress tensor

In the absence of external couples applied to the fluid (so-called "body-couples"), the stress tensor σ_{ij} is symmetric.

To demonstrate this, consider a small cube of fluid of dimension ϵ as shown.



From Euler's postulate of the principle of the moment of momentum[1] We know that

$$\sum \text{Moments} = \frac{\mathrm{d}}{\mathrm{d}t}\{\text{Angular momentum}\}.$$

The left hand side is made up of moments due to

$$\underset{O(\epsilon^4)}{\text{body forces (e.g. gravity)}} \quad + \quad \underset{O(\epsilon^3)}{\text{surface forces due to surrounding fluid}}$$

while the right hand side is of size (cube volume) $\times$ (cube length) $= O(\epsilon^4)$. Therefore, on the face shown in the figure,

$$2\frac{\epsilon}{2}\epsilon^2[\sigma_{12} - \sigma_{21}] = O(\epsilon^4) \implies \sigma_{12} = \sigma_{21}.$$

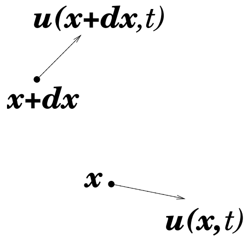
Following similar arguments, we deduce that

$$\sigma_{ij} = \sigma_{ji}.$$

2(b). Strain in fluids

Consider the relative positions of two individial fluid particles under any particular fluid motion. At time t, the particles are close together, with the first at position $\mathbf{x}$ and moving with velocity $\mathbf{u}(\mathbf{x}, t)$, and the second at $\mathbf{x} + \mathrm{d}\mathbf{x}$, moving with velocity $\mathbf{u}(\mathbf{x} + \mathrm{d}\mathbf{x}, t)$.

[1] See, for example, Acheson *Elementary Fluid Dynamics*, p. 202. It's worth emphasising here that Newton's original ideas for motion were solidified by Euler in 1752 as the *principle of linear momentum* and then later in 1775 as the *principle of moment of momentum*, these principles applying to any continuous medium such as a solid body (rigid or elastic) and a fluid. We have will effectively apply the principle of linear momentum later when we derive the equations of motion for a viscous fluid, but it is worth emphasising here that it is the moment of momentum principle that gives us the symmetry of the stress tensor (see also the argument on page 220 (Exercise 6.14) of Acheson.



Expanding the second particle's velocity in a Taylor series, we have the following expression for the relative velocity of the two particles:

$$u_i(\mathbf{x} + \mathrm{d}\mathbf{x}, t) - u_i(\mathbf{x}, t) = \frac{\partial u_i}{\partial x_j}\bigg|_{\mathbf{x}} \mathrm{d}x_j + O(\mathrm{d}x_j^2). \tag{2.6}$$

We now re-write the derivative on the right hand side like this

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \xi_{ij}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \xi_{ij} = -\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right). \tag{2.7}$$

Note that e_{ij} is symmetric, while ξ_{ij} is antisymmetric (So $e_{ij} = e_{ji}$, $\xi_{ij} = -\xi_{ji}$).

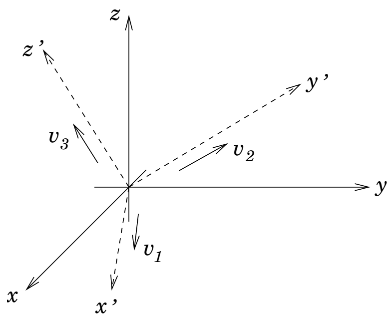
e_{ij} is called the *rate of deformation* or *strain* tensor.

ξ_{ij} is sometimes called the *vorticity* tensor.

To see why these two tensors are given their names, we examine their meanings in terms of the kinematics of the fluid particles.

The strain tensor e_{ij}

Since e_{ij} is a symmetric, second order tensor, we may rotate to a new set of axes, called *principal* axes, in which e_{ij} is diagonal (see Appendix A).



A set of Cartesian axes (x, y, z) and a rotated set of principal axes (x', y', z') along which are aligned the three orthogonal eigenvectors of e'_{ij}.

We denote it as e'_{ij} in the new frame. Writing both out as a matrix we have

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \qquad \text{and} \qquad e'_{ij} = \begin{pmatrix} e'_{11} & 0 & 0 \\ 0 & e'_{22} & 0 \\ 0 & 0 & e'_{33} \end{pmatrix}$$

The three eigenvalues of the new matrix are $\lambda_1 = e'_{11}$, $\lambda_2 = e'_{22}$ and $\lambda_3 = e'_{33}$ and, since e'_{ij} is symmetric, they are all real and have mutually orthogonal eigenvectors. These eigenvectors, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are aligned along the principal axes and form a basis.

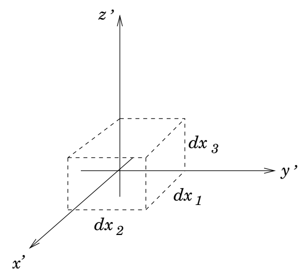
The action of matrix e'_{ij} on a general vector, $\mathbf{q} = \sum_{i=1}^{3} \alpha_i \mathbf{v}_i$, for constants α_i, is

$$\mathbf{e}' \cdot \mathbf{q} = \sum_{i=1}^{3} \alpha_i \, \mathbf{e}' \cdot \mathbf{v}_i = \sum_{i=1}^{3} \lambda_i \alpha_i \mathbf{v}_i.$$

Therefore the vector $\mathbf{q}$ gets *stretched* or *contracted* in directions $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ according to whether each eigenvalue λ_i is positive or negative.

Local expansion of the fluid

Suppose we try the above ideas out on a small cube of fluid aligned with the principal axes and with sides of length $\mathrm{d}x'_1$, $\mathrm{d}x'_2$, $\mathrm{d}x'_3$ and volume $V = \mathrm{d}x'_1 \mathrm{d}x'_2 \mathrm{d}x'_3$.



After a short time $\mathrm{d}t$, the side $\mathrm{d}x'_1$ will have become deformed so that its new length is

old length + (rate of stretching in the x'_1 direction)$\times \mathrm{d}t$

So its new length will be $(1 + e'_{11}\mathrm{d}t)\, \mathrm{d}x'_1$. Similar arguments apply for the other sides so the new edges are of lengths

$$\begin{gathered}(1 + e'_{11}\, \mathrm{d}t)\, \mathrm{d}x'_1 \\ (1 + e'_{22}\, \mathrm{d}t)\, \mathrm{d}x'_2 \\ (1 + e'_{33}\, \mathrm{d}t)\, \mathrm{d}x'_3\end{gathered}$$

Therefore, the change in volume of the cube after time $\mathrm{d}t$ is equal to

$$\{(1 + e'_{11}\mathrm{d}t)(1 + e'_{22}\mathrm{d}t)(1 + e'_{33}\mathrm{d}t) - 1\}V \;=\; (e'_{11} + e'_{22} + e'_{33})V\mathrm{d}t \;=\; \mathrm{Trace}(e'_{ij})V\mathrm{d}t,$$

neglecting terms of size $O(\mathrm{d}t^2)$. Now, the trace of a matrix is invariant under a rotation to a new coordinate frame (see Appendix A). So, the change in volume (divided by $V\mathrm{d}t$) is equal to

$$\mathrm{Trace}(e'_{ij}) = \mathrm{Trace}(e_{ij}) = e_{ii} = \frac{\partial u_i}{\partial x_i} = \nabla \cdot \mathbf{u}.$$

A fluid is defined to be *incompressible* if the local change in volume of a fluid element such as the box just considered is zero. In other words

$$\nabla \cdot \mathbf{u} = 0 \quad \text{for an incompressible fluid.}$$

The vorticity tensor ξ_{ij}

As a preliminary, we introduce the *alternating* tensor defined as

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if} \quad \text{any of} \;\; i, j, k \;\; \text{are equal} \\ 1 & \text{if} \quad i, j, k \;\; \text{are in cyclic order} \\ -1 & \qquad\qquad \text{otherwise,} \end{cases}$$

so, for example, $\epsilon_{123} = \epsilon_{231} = 1$, $\epsilon_{132} = -1$, $\epsilon_{133} = 0$. The alternating tensor provides a convenient way of expressing a vector (cross) product in index notation.

Exercise: Show that $\epsilon_{ijk}a_j b_k$ is equivalent to $\mathbf{a} \times \mathbf{b}$ by writing out the components.

To understand the kinematics associated with ξ_{ij}, we reason as follows. Since ξ_{ij} is antisymmetric, it has only 3 independent components and may therefore be written quite generally as

$$\xi_{ij} = -\tfrac{1}{2}\epsilon_{ink}\omega_k, \quad \text{for some vector } \boldsymbol{\omega} \text{ with components } w_k,\ k = 1, 2, 3. \tag{2.8}$$

The factor $-1/2$ is included for convenience. Check for yourself that the right hand side of (2.8) has only 3 independent components, ω_k.

Referring back to the Taylor expansion (2.6), the antisymmetric contribution to the relative velocity of the two particles is

$$\xi_{ij}\mathrm{d}x_j \;=\; -\tfrac{1}{2}\epsilon_{ijk}\mathrm{d}x_j\omega_k \;=\; \tfrac{1}{2}\epsilon_{ikj}\omega_k\mathrm{d}x_j \;=\; (\tfrac{1}{2}\boldsymbol{\omega} \times \mathrm{d}\mathbf{x})_i.$$

This is of the form

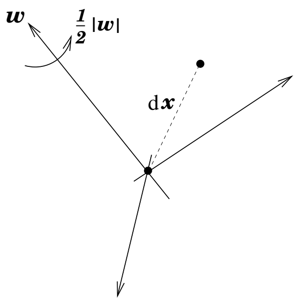
$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r},$$

where $\mathbf{r}$ is the position vector of a point relative to a suitably defined origin, which describes rotation of a particle about at axis with angular velocity $|\boldsymbol{\omega}|$.

So,

$$(\tfrac{1}{2}\boldsymbol{\omega} \times \mathrm{d}\mathbf{x})_i$$

represents a rotation of one fluid particle about the other with angular velocity $\frac{1}{2}\boldsymbol{\omega}$.



Local rotation of two fluid particles at angular rate $\frac{1}{2}|\omega|$ about an axis pointing in the direction $\boldsymbol{\omega}$.

What are the components of $\boldsymbol{\omega}$ in terms of the fluid velocity? First we note the useful identity involving the alternating tensor,

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl},$$

where the Kronecker delta δ_{ij} is 1 if i, j are equal and 0 otherwise. Hence

$$\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}.$$

Now, operating with ϵ_{ijl} on both sides of (2.8) and using the definition (2.7), we find

$$\omega_l \;=\; -\epsilon_{lij}\xi_{ij} = \tfrac{1}{2}\epsilon_{lij}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) = \epsilon_{lij}\frac{\partial u_j}{\partial x_i} \qquad \text{i.e.} \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

The vector $\boldsymbol{\omega}$ is called the *vorticity*. Any flow for which $\boldsymbol{\omega} = \mathbf{0}$ is called *irrotational*.

2(c). Stress-strain relationship for a Newtonian fluid

To proceed with the modelling of fluid flow, we need to understand the relationship between the stress in a fluid and the local rate of strain. In other words, how much is the fluid being deformed by the stresses exerted by the surrounding fluid?

For a fluid at *rest*, there are no viscous forces, and the stress is simply related to the static pressure p by

$$\sigma_{ij} = -p\,\delta_{ij}. \tag{2.9}$$

So $\sigma_{ii} = -p\,\delta_{ii} = -3p$ and therefore $p = -\sigma_{ii}/3$. Hence the stress exerted on an element with unit normal $\mathbf{n}$ is $\sigma_{ij}n_j = -pn_i$ and has magnitude p in the $-\mathbf{n}$ direction. So the pressure force exterted on a wall by a fluid at rest acts in a direction perpendicular to that wall.

For a fluid in *motion*, we write

$$\sigma_{ij} = -p\,\delta_{ij} + d_{ij}, \tag{2.10}$$

where p is the *dynamic pressure* and d_{ij} is the *deviatoric stress tensor*.

We define $d_{ii} = 0$ so that once again $\sigma_{ii} = -3p$ and thus $p = -\sigma_{ii}/3$. However, note that since the fluid is now moving, p in equation (2.10) is **not** in general the same as p in equation (2.9).

The deviatoric stress d_{ij} is due to the motion of the fluid, which consists of translation, rigid rotation and stretching or contracting.

Recall that ξ_{ij} corresponds to rigid rotation.
e_{ij} corresponds to stretching/straining.

Newtonian fluid

For a Newtonian fluid, the deviatoric stress depends *only* on e_{ij} and the dependence is linear.

Notes

1. For a Newtonian fluid d_{ij} is therefore symmetric.

2. The instantaneous stress at any point in the fluid does not depend on the past history of the fluid motion. Non-Newtonian, visco-elastic fluids can have a "memory" whereby the stress depends on the state of the fluid at previous times.

The stress-strain relationship

Since the deviatoric stress is a linear function of the strain rate for a Newtonian fluid, we may write

$$d_{ij} = A_{ijkl}e_{kl} \tag{2.11}$$

where A_{ijkl} is a constant fourth-order tensor.

We assume that A_{ijkl} is isotropic, which means that the stress d_{ij} generated on a fluid element by a given straining motion is independent of the orientation of that element. In other words, the fluid has the same properties in whichever direction you look. It is known that all isotropic tensors of even order can be written as sums of products of Kronecker deltas. Thus

$$A_{ijkl} = \mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk} + \lambda\delta_{ij}\delta_{kl},$$

where μ, μ' and λ are constants.

Now, since $d_{ij} = d_{ji}$,

$$A_{ijkl} = A_{jikl} \quad \text{and so} \quad \mu' = \mu.$$

Therefore,

$$\begin{aligned} d_{ij} &= \Big\{\mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + \lambda\delta_{ij}\delta_{kl}\Big\}e_{kl} \\ &= 2\mu e_{ij} + \lambda\delta_{ij}e_{kk}. \end{aligned} \tag{2.12}$$

But $d_{ii} = 0$ by definition and so, by (2.12),

$$d_{ii} = 0 = [2\mu + 3\lambda]e_{ii} \implies \lambda = -\frac{2}{3}\mu \tag{2.13}$$

Now, substituting d_{ij} into relation (2.10), we have

$$\sigma_{ij} = -p\delta_{ij} + 2\mu[e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}] \tag{2.14}$$

Remember that $e_{kk} = \nabla \cdot \mathbf{u}$ represents the amount of dilatation or local expansion of the fluid. Therefore, for an incompressible fluid, $e_{kk} = 0$ and so

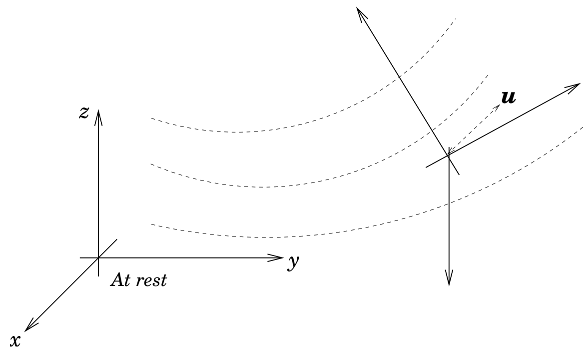
$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \tag{2.15}$$

This provides a constitutive relation between the stress and strain for an incompressible, Newtonian fluid. The constant μ is called the *viscosity* of the fluid. We assume equation (2.15) throughout the rest of the course.

3. The Navier-Stokes equations

3.1 The convective derivative

Our ultimate goal is to derive an equation describing the motion of a fluid. To move towards this goal, we introduce the important concept of a derivative which tells us how things change *as we move with the fluid*. The full usefullness of this idea will become apparent later on when we apply Newton's second law to derive the equations of fluid motion.



Consider a set of axes moving with the local fluid velocity $\mathbf{u}$ and another set fixed in space as shown in the diagram. If the origin of the moving set of axes is at $\mathbf{x} = (x(t),\ y(t),\ z(t))$ at time t, we can write its velocity as

$$\mathbf{u}(\mathbf{x}, t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

Suppose now we wish to calculate the acceleration of this set of axes. By definition, the acceleration is given by

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \lim_{\mathrm{d}t\to 0}\Big\{\frac{\mathbf{u}(\mathbf{x}(t+\mathrm{d}t),\ t+\mathrm{d}t) - \mathbf{u}(\mathbf{x},t)}{\mathrm{d}t}\Big\}. \tag{3.1}$$

Using Taylor's theorem,

$$\begin{aligned} \mathbf{x}(t+\mathrm{d}t) &= \mathbf{x}(t) + \mathrm{d}t\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \cdots \\ &= \mathbf{x}(t) + \mathrm{d}\mathbf{x} + \cdots, \end{aligned} \tag{3.2}$$

on writing $\mathrm{d}\mathbf{x} = \mathrm{d}t.(\mathrm{d}\mathbf{x}/\mathrm{d}t)$. Also by Taylor's theorem,

$$\mathbf{u}(x + \mathrm{d}\mathbf{x}, t + dt) = \mathbf{u}(\mathbf{x}, t) + \mathrm{d}\mathbf{x} \cdot \nabla\mathbf{u} + \mathrm{d}t\,\frac{\partial \mathbf{u}}{\partial t} + \cdots . \tag{3.3}$$

Therefore, (3.1) becomes

$$\begin{aligned} \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} &= \lim_{\mathrm{d}t\to 0} \frac{1}{\mathrm{d}t}\left\{\mathrm{d}t\frac{\partial \mathbf{u}}{\partial t} + \mathrm{d}\mathbf{x}\cdot\nabla\mathbf{u} + \cdots\right\} \\ &= \frac{\partial \mathbf{u}}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\nabla\mathbf{u} \\ \text{Thus}\quad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} &= \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}. \end{aligned} \tag{3.4}$$

The derivative on the left hand side of (3.4) is often written with a large D and referred to as the *convective* derivative (sometimes the *material* or *substantial* derivative).

We will follow this convention and henceforth write

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} \tag{3.5}$$

The convective derivative may be applied to any scalar or vector function. For example, consider the function $f(x(t),\ y(t),\ z(t))$. Proceeding as above, we find

$$\begin{aligned} \frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f \\ &= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}. \end{aligned}$$

This is really just the statement of the chain rule:

$$\frac{\mathrm{D}f}{\mathrm{D}t} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}. \tag{3.6}$$

$\mathrm{D}f/\mathrm{D}t$ tells us how the function f changes as we move with the flow.

For example, $\mathrm{D}\rho/\mathrm{D}t$ expresses the amount by which the density of a fluid particle changes as it is convected along with the flow. For an incompressible fluid, we therefore have

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0.$$

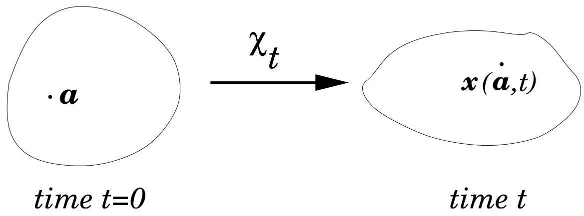
3.2 Lagrangian mappings

Working in the *Lagrangian* framework, we consider the motion of individual point particles; that is, given a fluid particle at a particular point in space at a particular time, we monitor how that particle travels through space as time progresses. To this end, we label all particles over the entire fluid domain according to their position at time $t = 0$. Then, at a later time, an individual fluid particle is identified by its label, as if it was carrying a flag along with it as it moves in the flow.

We therefore ascribe to each particle a label $\mathbf{a}$, where $\mathbf{a}$ is the position vector of that particle at $t = 0$.

Any position in the flow-field $\mathbf{x}$ may be regarded as a function of its label and time. Moreover we can write

$$\mathbf{x}(\mathbf{a}, t) = \boldsymbol{\chi}_t(\mathbf{a}) \tag{3.7}$$



where $\boldsymbol{\chi}_t(\mathbf{a})$ is a mapping from the initial particle positions at $t = 0$, to their current positions at time t. The subscript on the mapping function indicates that the particles are being mapped from where they were at $t = 0$ to their location at time t.
The mapping itself therefore changes in time. Furthermore, $\boldsymbol{\chi}_0(\mathbf{a}) = \mathbf{a}$.

We will refer to the initial configuration of fluid particles at $t = 0$ as *labelling space* as that is where all the labels sit. In addition, we will refer to the state of the fluid at time t as *convected space* as this is where all the particles have been carried by the flow.

Consider now the following important question: How is a small volume element in labelling space related to its counterpart at the later time t?

Suppose we define axes a_1, a_2, a_3 in labelling space, so $\mathbf{a} = (a_1, a_2, a_3)$. Then a volume element in this space is given by

$$\mathrm{d}V_l = \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3.$$

Consider now the differential vector ($\mathrm{d}a_1$,0,0) in labelling space and denote its counterpart in convected space as $\mathrm{d}\mathbf{x}^{(1)}$. Using (3.7) and applying the chain rule we can write:

$$\mathrm{d}\mathbf{x}^{(1)} = \frac{\partial \boldsymbol{\chi}_t}{\partial a_1}\,\mathrm{d}a_1. \tag{3.8}$$

Similar expressions follow for $\mathrm{d}\mathbf{x}^{(2)}$ and $\mathrm{d}\mathbf{x}^{(3)}$.

At this point, we note that the expression

$$\mathrm{d}V_c = (\mathrm{d}\mathbf{x}^{(1)} \times \mathrm{d}\mathbf{x}^{(2)}) \cdot \mathrm{d}\mathbf{x}^{(3)}$$

represents a volume element in the convected space. Convince yourself that this is true.

Therefore, using (3.8) we can write

$$(\mathrm{d}\mathbf{x}^{(1)} \times \mathrm{d}\mathbf{x}^{(2)}) \cdot \mathrm{d}\mathbf{x}^{(3)} = \Big(\frac{\partial \boldsymbol{\chi}_t}{\partial a_1} \times \frac{\partial \boldsymbol{\chi}_t}{\partial a_2}\Big) \cdot \frac{\partial \boldsymbol{\chi}_t}{\partial a_3}\,\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3.$$

Using the notation χ_t^i to denote the ith component of $\boldsymbol{\chi}_t$, we note that

$$\Big(\frac{\partial \boldsymbol{\chi}_t}{\partial a_1} \times \frac{\partial \boldsymbol{\chi}_t}{\partial a_2}\Big) \cdot \frac{\partial \boldsymbol{\chi}_t}{\partial a_3} = \det\Big(\frac{\partial \chi_t^i}{\partial a_j}\Big),$$

[see Exercise Sheet 1].

This allows us to relate the two volume elements as follows,

$$\mathrm{d}V_c = J\,\mathrm{d}V_l, \tag{3.9}$$

where

$$J = \det\Big(\frac{\partial \chi_t^i}{\partial a_j}\Big) = \begin{vmatrix} \dfrac{\partial \chi_t^1}{\partial a_1} & \dfrac{\partial \chi_t^1}{\partial a_2} & \dfrac{\partial \chi_t^1}{\partial a_3} \\ \dfrac{\partial \chi_t^2}{\partial a_1} & \dfrac{\partial \chi_t^2}{\partial a_2} & \dfrac{\partial \chi_t^2}{\partial a_3} \\ \dfrac{\partial \chi_t^3}{\partial a_1} & \dfrac{\partial \chi_t^3}{\partial a_2} & \dfrac{\partial \chi_t^3}{\partial a_3} \end{vmatrix} \tag{3.10}$$

is the Jacobian of the mapping function χ_t.

Equation (3.9) therefore shows how a volume element in labelling space is related to its counterpart in convected space.

The transformation between the two spaces is very useful as it allows us to relate objects which are varying in time (those in convected space) to objects which are static (those in labelling space).

Its usefulness is demonstrated by trying to work out the rate of change of volume of a parcel of fluid as it is convected along with the flow. Thus, defining a parcel of fluid with volume

$$V_c = \int_{V_c} \mathrm{d}V_c = \int_{V_l} J \, \mathrm{d}V_l, \tag{3.11}$$

we have immediately managed to change an integral defined over a volume which is changing in time to one over a volume which is fixed. This will make differentiation a lot easier.

To find the rate of change of the parcel's volume, we form

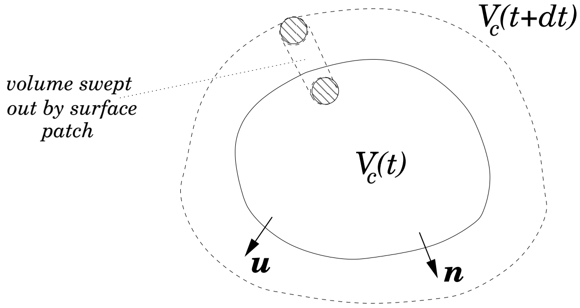
$$\frac{\mathrm{d}V_c}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V_l} J \, \mathrm{d}V_l = \int_{V_l} \frac{\mathrm{d}J}{\mathrm{d}t} \, \mathrm{d}V_l = \int_{V_l} \frac{\mathrm{D}J}{\mathrm{D}t} \, \mathrm{d}V_l, \tag{3.12}$$

since J is a function of $\mathbf{x}$, the position in convected space, and so it follows by the definition of the convective derivative that

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\partial J}{\partial t} + \mathbf{u} \cdot \nabla J \equiv \frac{\mathrm{D}J}{\mathrm{D}t}$$

Note that we were able to take the $\mathrm{d}/\mathrm{d}t$ inside the integral in (3.12) since V_l is independent of time.

Equally, we may argue as follows: Since the particles making up the parcel move with the fluid velocity $\mathbf{u}$,



the volume swept out by a small element $\mathrm{d}S_c$ of its surface S_c in time $\mathrm{d}t$ (see figure) must equal $\mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S_c$, where $\mathbf{n}$ is the surface unit normal, and thus

$$\frac{\mathrm{d}V_c}{\mathrm{d}t} = \int_{S_c} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S_c.$$

Applying the divergence theorem and using (3.9), we find

$$\frac{\mathrm{d}V_c}{\mathrm{d}t} = \int_{V_c} \nabla \cdot \mathbf{u} \, \mathrm{d}V_c = \int_{V_l} \nabla \cdot \mathbf{u} \, J \, \mathrm{d}V_l. \tag{3.13}$$

Now, comparing (3.12) and (3.13) we deduce that

$$\frac{\mathrm{D}J}{\mathrm{D}t} = J \, \nabla \cdot \mathbf{u}. \tag{3.14}$$

We will find this useful in the proof of the Reynolds transport theorem to be discussed below.

3.2.1 Lagrangian/Eulerian coordinates

As was discussed above, there are two main frameworks for describing a fluid flow: the Lagrangian framework and the Eulerian framework.

Eulerian coordinates

These are by far the most commonly used system of describing a fluid motion. The main alternative is Lagrangian coordinates, to be discussed below. In Eulerian coordinates $(\mathbf{x}, t)$ the state of a fluid (velocity, pressure, stress and so on) and described in terms of their values at a fixed location in space, $\mathbf{x}$, at a fixed time t. The Navier-Stokes equations are usually written in these coordinates, and it is these which we shall adopt in these lecture notes.

Lagrangian coordinates

Alternatively, one may describe the state of a fluid by making reference to the motion of individual fluid elements. In this system, reference is made to the state of the system at some appropriate point in the history of the fluid motion (at, say, time $t = 0$). Fluid elements at later times are described in terms of their original positions in this reference state. So, for example, a fluid particle at a general time t is located at the point

$$\mathbf{x}(\mathbf{a}, t),$$

where t is the present time and $\mathbf{a}$ is the position vector of the particle in the reference state (at time $t = 0$, say). The velocity of this fluid particle is

$$\mathbf{u} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{a}}.$$

We refer to $(\mathbf{a}, t)$ as Lagrangian coordinates for the flow. For the most part it is extremely cumbersome to express the equations in terms of Lagrangian coordinates. However, this is not always the case: a landmark achievement in fluid mechanics concerns the analysis of the singularity which occurs in the developing boundary layer on an impulsively started cylinder. This problem was successfully taked by van Dommelen & Shen (1980, J. Comp. Phys, **38**) using Lagrangian coordinates. The notion of a boundary layer is discussed later in these notes. In Appendix C we derive the boundary layer equations in Lagrangian coordinates.

3.3 The Reynolds Transport Theorem

This result tells us how an integral of any scalar quantity defined over a time-dependent volume of fluid $V(t)$ changes in time. In fact, it states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G\,\mathrm{d}V = \int_{V(t)} \left(\frac{\mathrm{D}G}{\mathrm{D}t} + G\,\nabla\cdot\mathbf{u}\right)\mathrm{d}V. \tag{3.15}$$

where $G(\mathbf{x}, t)$ is any scalar.

Proof

Using (3.9), we may convert the volume integral into one defined over the equivalent volume in labelling space, V_l. So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_l} G\,J\,\mathrm{d}V_l$$

Then, since V_l is time-independent, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G\,\mathrm{d}V = \int_{V_l} \frac{\mathrm{d}}{\mathrm{d}t}(GJ)\,\mathrm{d}V_l = \int_{V_l} \frac{\mathrm{D}}{\mathrm{D}t}(GJ)\,\mathrm{d}V_l = \int_{V_l} \left(J\frac{\mathrm{D}G}{\mathrm{D}t} + G\frac{\mathrm{D}J}{\mathrm{D}t}\right)\mathrm{d}V_l$$

by the product rule of differentiation.

Using the result (3.14), this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G\,\mathrm{d}V = \int_{V_l}\Big(J\frac{\mathrm{D}G}{\mathrm{D}t} + GJ\,\nabla\cdot\mathbf{u}\Big)\,\mathrm{d}V_l \tag{3.16}$$

$$= \int_{V_l}\Big(\frac{\mathrm{D}G}{\mathrm{D}t} + G\,\nabla\cdot\mathbf{u}\Big)J\,\mathrm{d}V_l \tag{3.17}$$

$$= \int_{V}\Big(\frac{\mathrm{D}G}{\mathrm{D}t} + G\,\nabla\cdot\mathbf{u}\Big)\,\mathrm{d}V, \tag{3.18}$$

using (3.9). $\diamond$.

EXAMPLE

If we identify the scalar $G(\mathbf{x},t)$ with the fluid density $\rho(\mathbf{x},t)$, we can use (3.15) to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\,\mathrm{d}V = \int_{V(t)}\Big(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\nabla\cdot\mathbf{u}\Big)\,\mathrm{d}V.$$

Since mass is conserved, it follows immediately that the integrand on the right hand side is zero, so

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\nabla\cdot\mathbf{u} = 0.$$

We noted above that if the fluid is incompressible, then

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0.$$

It therefore follows that, for an incompressible fluid,

$$\nabla\cdot\mathbf{u} = 0,$$

which is a restatement of the result $e_{kk} = 0$ found before using the small cube of fluid argument.

NOTE: It is important to realise that incompressible does **not** mean constant density. This is a common mistake. For example, consider the two-dimensional flow with velocity and density fields

$$\mathbf{u} = u(x,y)\mathbf{i} + v(x,y)\mathbf{j}, \qquad \rho = \rho(z).$$

Then

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} = 0$$

so the condition for incompressibility is satisfied, but the density field varies with z.

We will make use of the Reynolds transport theorem when we derive the Navier-Stokes equations.

3.4 EQUATIONS OF FLUID MOTION

Up to now, we have been concerned with the kinematics of a flow due to a predetermined velocity field $\mathbf{u}(\mathbf{x},t)$. But how do we determine the underlying velocity field in the first place?

To formulate the necessary equations governing the motion of a viscous fluid we will apply the following fundamental principles:

1. Conservation of mass.
2. Newton's second law of motion.

We will also discuss how energy is dissipated by viscous forces in a moving fluid and derive an equation for the conservation of energy in a parcel of fluid.

3.4.1. Conservation of mass

Consider a closed volume V which is *fixed* in space inside a moving fluid. As the fluid moves past, the rate of change of the mass of fluid within V must equal the net amount of fluid coming into (or out of) it, assuming there are no sources of fluid within V. Mathematically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V + \int_S \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = 0,$$

where ρ is the fluid density, S is the volume surface, and $\mathbf{n}$ is the normal to V pointing *outwards*. Using the fact that V is fixed in space and applying the divergence theorem, this becomes

$$\int_V \left\{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right\}\mathrm{d}V = 0.$$

Since our choice of volume V was arbitrary, it must be true that

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{3.19}$$

Equation (3.19) therefore expresses the principle of mass conservation. It is usually referred to as the *continuity equation*, which will frequently be abbreviated to *cty* equation.

Using the definition of the convective derivative, we can re-express the continuity equation as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0. \tag{3.20}$$

This is just the equation we found above using the Reynolds transport theorem.

3.4.2. Newton's second law

In order to apply Newton's second law of motion to a parcel of fluid, we will need to recall the Reynolds transport theorem introduced earlier:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G\,\mathrm{d}V \;=\; \int_{V(t)}\left(\frac{\mathrm{D}G}{\mathrm{D}t} + G\nabla\cdot\mathbf{u}\right)\mathrm{d}V.$$

for any scalar quantity G.

We are now in a position to derive the equation of motion of a fluid. Let $\mathbf{F}$ represent some body force per unit mass acting on the fluid (i.e. a long-range force such as gravity). We apply Newton's second law of motion to a volume of fluid $V(t)$, with surface area $S(t)$ and unit outward normal $\mathbf{n}$, moving with the flow. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_i\,\mathrm{d}V \;=\; \int_{V(t)} \rho F_i\,\mathrm{d}V + \int_{S(t)} \sigma_{ij}n_j\,\mathrm{d}S.$$

The left hand side expresses the rate of change of momentum of the fluid; the right hand side the forces acting on the fluid. The last term represents the stresses acting over the surface of the chosen volume by the surrounding fluid.

Applying the Reynolds transport theorem to the first term and the divergence theorem to the last, and using (3.20) we find

$$\int_{V(t)} \rho\frac{\mathrm{D}u_i}{\mathrm{D}t}\,\mathrm{d}V \;=\; \int_{V(t)} \rho F_i\,\mathrm{d}V + \int_{V(t)} \frac{\partial\sigma_{ij}}{\partial x_j}\,\mathrm{d}V. \tag{3.21}$$

Note that we have used the fact that

$$\frac{\mathrm{D}(\rho u_i)}{\mathrm{D}t} = \rho \frac{\mathrm{D}u_i}{\mathrm{D}t} + u_i \frac{\mathrm{D}\rho}{\mathrm{D}t}$$

and that $\mathrm{D}\rho/\mathrm{D}t = 0$ since the fluid is incompressible.

If the fluid is Newtonian, we may use the constitutive relation introduced earlier,

$$\sigma_{ij} = -p\,\delta_{ij} - \frac{2}{3}\mu\Delta\delta_{ij} + 2\mu \mathrm{e}_{ij},$$

where

$$\Delta = \nabla \cdot \mathbf{u} = e_{kk}$$

represents the local expansion of the fluid.

Accordingly,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \frac{2}{3}\frac{\partial}{\partial x_i}(\mu\Delta) + 2\frac{\partial}{\partial x_j}(\mu e_{ij}).$$

Substituting this relation into the above, we obtain

$$\int_{V(t)} \left(\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} - \rho\,F_i + \frac{\partial p}{\partial x_i} + \frac{2}{3}\frac{\partial}{\partial x_i}(\mu\Delta) - 2\frac{\partial}{\partial x_j}(\mu e_{ij}) \right) \mathrm{d}V \;=\; 0. \tag{3.22}$$

Since we are restricting our attention to *incompressible fluids*, we have that

$$\Delta = 0,$$

(see the earlier discussion of the expansion of a small cube of fluid).

Assuming μ is constant, (3.22) therefore reduces to

$$\int_{V(t)} \left(\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} - \rho\,F_i + \frac{\partial p}{\partial x_i} - \mu \left\{ \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right\} \right) \mathrm{d}V \;=\; 0. \tag{3.23}$$

But $\partial u_j/\partial x_j \equiv \nabla \cdot \mathbf{u} = 0$. Also, since our original volume was chosen arbitrarily, the integrand in (3.23) must vanish, as the statement must be true for all possible volume choices. Finally, we have

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} \;=\; \rho\,F_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}.$$

Alternatively, we can write this as

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} \;=\; \rho\,F_i + \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{3.24}$$

direct from (3.21).

In vector form, together with the continuity equation,

$$\nabla \cdot \mathbf{u} = 0, \qquad \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \;=\; \rho\,\mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{u}. \tag{3.25}$$

These are the *Navier-Stokes equations* for an incompressible Newtonian fluid.

In *Cartesian coordinates*, they are written in component form as

$$\begin{aligned}
\rho(\,u_t + u u_x + v u_y + w u_z\,) &= -p_x + F_x + \mu\nabla^2 u,\\
\rho(\,v_t + u v_x + v v_y + w v_z\,) &= -p_y + F_y + \mu\nabla^2 v,\\
\rho(\,w_t + u w_x + v w_y + w w_z\,) &= -p_z + F_z + \mu\nabla^2 w,\\
u_x + v_y + w_z &= 0.
\end{aligned}$$

where the velocity field $\mathbf{u} = u\,\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In *cylindrical polar coordinates*, they are written in component form as

$$\begin{aligned}
\rho\left(u_t + uu_r + \frac{1}{r}vu_\theta + wu_z - \frac{v^2}{r} \right) &= -p_r + F_r + \mu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}v_\theta\right), \\
\rho\left(v_t + uv_r + \frac{1}{r}vv_\theta + wv_z + \frac{uv}{r} \right) &= -\frac{1}{r}p_\theta + F_\theta + \mu\left(\nabla^2 v + \frac{2}{r^2}u_\theta - \frac{v}{r^2}\right), \\
\rho(w_t + uw_r + \frac{1}{r}vw_\theta + ww_z) &= -p_z + F_z + \mu\nabla^2 w, \\
u_r + \frac{u}{r} + \frac{1}{r}v_\theta + w_z &= 0.
\end{aligned}$$

where the velocity field $\mathbf{u} = u\,\hat{\mathbf{r}} + v\,\hat{\boldsymbol{\theta}} + w\mathbf{k}$ and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

If the body force, **F**, is conservative, i.e. we can write it as the gradient of a scalar potential,

$$\mathbf{F} = -\nabla\Omega,$$

then it can be incorporated into the pressure term by writing

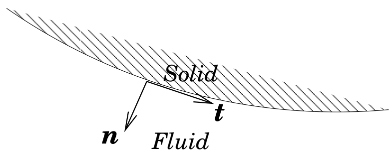
$$\rho\,\mathbf{F} - \nabla p = -\nabla(\rho\,\Omega + p) = -\nabla P,$$

where we have defined the *modified pressure* P to be

$$P = p + \rho\,\Omega.$$

3.4.3 Boundary conditions

Suitable boundary conditions must be appended to the system (3.25) in order to fully define the flow problem in hand. We divide them into the following categories:



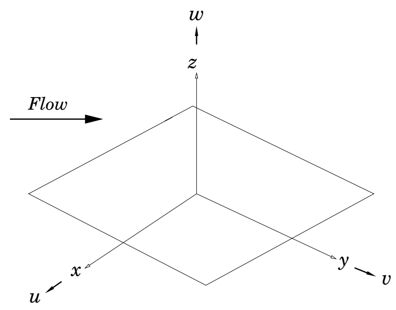
Fluid-Solid

The usual inviscid no-penetration condition applies at a solid impermeable surface. This states that $\mathbf{u}\cdot\mathbf{n} = 0$, i.e. fluid cannot flow into the wall.

For a viscous fluid a further boundary condition is required to complete the flow description due to the extra derivatives imported by the last term in (3.25). Experiments reveal that this supplementary condition should stipulate that the fluid does not slip over the solid boundary; that is to say the local fluid velocity at the wall is zero. Mathematically this is stated as $\mathbf{u}\cdot\mathbf{t} = 0$, where $\mathbf{t}$ is any vector tangential to the wall.

In summary, we require

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{and} \qquad \mathbf{u}\cdot\mathbf{t} = 0 \qquad \text{at the solid surface.}$$



The latter is usually referred to as the *no slip condition*.

Example: Fluid flowing over a plane wall at $z = 0$.

The no normal flow condition states that $w = 0$.

The no-slip condition states that $u = v = 0$.

NOTE: We can use the continuity equation to deduce a further piece of information about the flow at $z = 0$. We know that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

and, on $z = 0$,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

since u and v are both zero *everywhere* on $z = 0$. Therefore,

$$\frac{\partial w}{\partial z} = 0 \quad \text{on} \quad z = 0.$$

3.4.4 The kinematic condition

This applies at any surface which is *moving*. If the shape of the surface is described by the equation

$$f(x, y, z, t) = 0,$$

it states that a particle sitting on the surface $f = 0$ must move along with the fluid.

We know from the definition of the convective derivative that the rate of change of f following the fluid is $\mathrm{D}f/\mathrm{D}t$. Therefore, if the surface is convected along with the fluid, we have

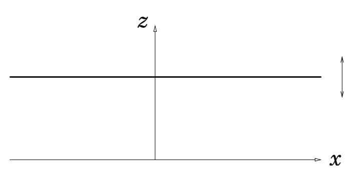
$$\frac{\mathrm{D}f}{\mathrm{D}t} = 0 \quad \text{at the surface.}$$

Thus,

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{surface} = \frac{\partial f}{\partial t} + u_s\frac{\partial f}{\partial x} + v_s\frac{\partial f}{\partial y} + w_s\frac{\partial f}{\partial z} = 0,$$

where u_s, v_s, w_s are the velocity components of the fluid at the surface.

Example: A flat plate moves up and down in a surrounding fluid so that at time t, its position is given by $z = \sin t$. What is the kinematic condition on the moving wall?



The equation of the wall is $z = \sin t$. Set $f = z - \sin t$. Then,

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{wall} = \frac{\partial f}{\partial t} + u_s\frac{\partial f}{\partial x} + v_s\frac{\partial f}{\partial y} + w_s\frac{\partial f}{\partial z}$$

$$= -\cos t + w_s = 0,$$

$$\Longrightarrow\ w_s = \cos t.$$

Therefore, the fluid velocity at the wall is $\cos t$ in the z direction.

3.4.5 Boundary condition at the interface between two fluids

This situation is more complicated as the position and shape of the interface are themselves unknown and must be found as part of the solution to the problem.

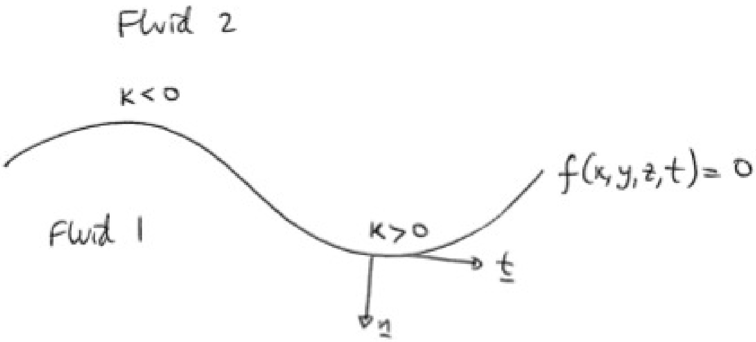


Figure 1: An interface between two fluids, with local unit normal and tangent vectors, $\mathbf{n}(s)$ and $\mathbf{t}(s)$ respectively, where acrlength s increases in the direction of the unit tangent $\mathbf{t}$.

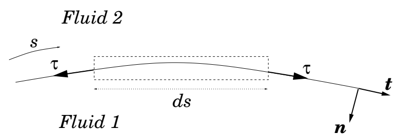
Here the function $f(x, y, z, t)$ must be determined.

Kinematic conditions: These state that the fluid velocity must be continuous across the interface. Using superscripts to denote variables in each of the fluids we demand that

$$\mathbf{u}^{(1)} \cdot \mathbf{n} = \mathbf{u}^{(2)} \cdot \mathbf{n} \qquad \text{and} \qquad \mathbf{u}^{(1)} \cdot \mathbf{t} = \mathbf{u}^{(2)} \cdot \mathbf{t} \qquad \text{on the interface.}$$

These state that the normal and tangential components of the velocities are continuous across the interface.

Dynamic condition: We also need a condition expressing the balance of forces prevailing at the interface between the fluids. To obtain this condition, we consider a small rectangular volume of fluid of length $\mathrm{d}s$ as shown in the figure below.



The height of the rectangular volume is assumed negligible in comparison with its length $\mathrm{d}s$. If s measures arc length along the interface, the local unit normal and tangent vectors to the interface are denoted as $\mathbf{n}(s)$

and $\mathbf{t}$ respectively at position s. Surface tension $\gamma(s)$ tugs on the volume at either end in the tangential direction. The force exerted on the lower portion of the rectangular volume by fluid 1 is given by

$$(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{n})\, \mathrm{d}s,$$

where $\boldsymbol{\sigma}^{(1)}$ is the stress tensor in fluid 1. This follows from the definition of the stress tensor given above. Similarly, the force exerted on the top by fluid 2 is given by $-(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{n})\, \mathrm{d}s$. The minus sign is needed since the unit normal must point *into* the fluid exerting the force.

Applying Newton's second law to the rectangular control volume (and disregarding its negligible inertia), we obtain the force balance

$$\gamma(s + \mathrm{d}s)\mathbf{t}(s + \mathrm{d}s) + \gamma(s)\mathbf{t}(s) + \mathrm{d}s \left[\boldsymbol{\sigma}^{(1)} \cdot \mathbf{n} - \boldsymbol{\sigma}^{(2)} \cdot \mathbf{n}\right] = \mathbf{0}.$$

The first two terms represent surface tension at either ends of the control volume. Since $\mathrm{d}s$ is small, we may expand in Taylor series, so,

$$\left[\gamma(s) + \mathrm{d}s\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right]\left[\mathbf{t}(s) + \mathrm{d}s\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\right] + \gamma(s)\mathbf{t}(s) + \mathrm{d}s \left[\boldsymbol{\sigma}^{(1)} \cdot \mathbf{n} - \boldsymbol{\sigma}^{(2)} \cdot \mathbf{n}\right] + O(\mathrm{d}s^2) = \mathbf{0}.$$

Expanding the first pair of square brackets and cancelling relevant terms,

$$\mathrm{d}s\, \gamma\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} + \mathrm{d}s\, \frac{\mathrm{d}\gamma}{\mathrm{d}s}\mathbf{t} + \mathrm{d}s \left[\boldsymbol{\sigma}^{(1)} \cdot \mathbf{n} - \boldsymbol{\sigma}^{(2)} \cdot \mathbf{n}\right] + O(\mathrm{d}s^2) = \mathbf{0}.$$

Dividing by $\mathrm{d}s$, taking the limit as $\mathrm{d}s \to 0$, and defining the local surface curvature κ so that

$$\kappa\mathbf{n} = -\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s},$$

we obtain,

$$-\kappa\gamma\mathbf{n} + \frac{\mathrm{d}\gamma}{\mathrm{d}s}\mathbf{t} + \left[\boldsymbol{\sigma}^{(1)} \cdot \mathbf{n} - \boldsymbol{\sigma}^{(2)} \cdot \mathbf{n}\right] = \mathbf{0}.$$

Rearranging, this yields

$$\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) \cdot \mathbf{n} = \kappa\gamma\mathbf{n} - \frac{\mathrm{d}\gamma}{\mathrm{d}s}\mathbf{t} \tag{3.26}$$

The sign of the curvature is indicated in figure 1.

If the surface tension is constant, as it is generally speaking, the last term in (3.26) disappears. In index notation, the jump in stress is then written as

$$\Delta F_i \equiv [\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}]\, n_j \;=\; 2\gamma\kappa\, n_i.$$

This states that the jump in stress ΔF_i at the interface is balanced by surface tension. A more general derivation valid for a two-dimensional curved surface can be found in Batchelor. Refer to pages 64 and 69.

N.B. Surface tension is a very weak force. For example, at $20°C$ the surface tension between air and water is

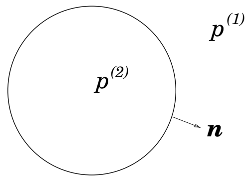
$$\gamma = 72.8 \text{ dyne/cm}.$$

Recall that 1 dyne $= 10^{-5}$ N.

Example: Consider a spherical bubble of radius a suspended motionless in air.

The mean curvature of the spherical bubble is $1/a$. Since neither fluid 1 nor 2 is moving, the jump in stress,

$$\Delta F_i = [\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}]\, n_j = [-p^{(1)}\, \delta_{ij} + p^{(2)}\, \delta_{ij}]\, n_j = -[p^{(1)} - p^{(2)}]\, n_i = 2\gamma\kappa\, n_i,$$



So, the pressure jump,

$$\Delta p = p^{(1)} - p^{(2)} = -2\gamma/a \implies \gamma = -\frac{a\Delta p}{2}$$

which is known as the Laplace-Young equation.

In the absence of surface tension, $\gamma = 0$ and the right hand side vanishes. In this case the fluid stresses are continuous across the interface. If $\gamma \neq 0$, since the vector on the right hand side is in the direction $\mathbf{n}$, there is only a jump in the normal stress. The tangential stress is still continuous.

Other examples: The interface between the atmosphere and the ocean. The interface between the tear film on your eye and the surrounding air.

3.5 The Energy Equation

How much energy does a given fluid flow carry with it as it flows? Suppose we track a chosen blob of fluid - how does the energy within the blob change with time?

We will derive an energy equation to describe how energy changes within the flow. The energy equation follows by mathematical manipulation of the equation of motion. We'll start with the easier case of purely inviscid flow, and then see how viscous effects modify our observations later.

3.5.1 Energy equation for inviscid flow

For an incompressible, inviscid fluid flow, Euler's equation holds:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathbf{F}$$

where $\rho\mathbf{F}$ is the body force per unit mass. To proceed, we take the dot product of this equation with the velocity $\mathbf{u}$:

$$\rho \mathbf{u} \cdot \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\mathbf{u} \cdot \nabla p + \rho \mathbf{F} \cdot \mathbf{u}.$$

Integrating over a volume V moving with the fluid, we find

$$\int_V \rho \mathbf{u} \cdot \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \,\mathrm{d}V = -\int_V \mathbf{u} \cdot \nabla p \,\mathrm{d}V + \int_V \rho \mathbf{F} \cdot \mathbf{u} \,\mathrm{d}V.$$

So,

$$\int_V \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{1}{2} \rho \mathbf{u}^2 \right) \,\mathrm{d}V = -\int_V \mathbf{u} \cdot \nabla p \,\mathrm{d}V + \int_V \rho \mathbf{F} \cdot \mathbf{u} \,\mathrm{d}V.$$

Note that

$$\nabla \cdot (p\mathbf{u}) = p \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p = \mathbf{u} \cdot \nabla p$$

since $\nabla \cdot \mathbf{u} = 0$, and so,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2} \rho \mathbf{u}^2 \,\mathrm{d}V = -\int_V \nabla \cdot (p\mathbf{u}) \,\mathrm{d}V + \int_V \rho \mathbf{F} \cdot \mathbf{u} \,\mathrm{d}V.$$

on using the Reynolds Transport Theorem on the LHS. Using the divergence theorem, we have finally

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2} \rho \mathbf{u}^2 \,\mathrm{d}V = \int_S (-p\mathbf{n}) \cdot \mathbf{u} \,\mathrm{d}S + \int_V \rho \mathbf{F} \cdot \mathbf{u} \,\mathrm{d}V. \tag{3.27}$$

Now, since $-p\mathbf{n}$ is the pressure force acting on the blob due to the surrounding fluid, the first term on the RHS represents the rate of working of the external pressure force on the blob. Similarly, the second term on the RHS represents the rate of working of the body force on the blob. Recognising

$$\int_V \frac{1}{2}\rho\mathbf{u}^2 \,\mathrm{d}V = K$$

as the total kinetic energy in the blob, we can read the equation as meaning that in each unit of time, the

Increase of K.E. of blob = work done by pressure force + work done by body force

In other words, *no energy at all is lost*. All work put in by the pressure force and the body force goes directly into increasing the kinetic energy of the blob. This remark does *not* hold true if viscosity is included, as we shall see in a moment.

Firstly, we recall that equation (3.27) applies for a volume $V(t)$ moving with the fluid. If instead we consider a volume V which is *fixed* in the fluid, we proceed as follows.

As before, we take the dot product of the Euler equation with the velocity $\mathbf{u}$:

$$\rho\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\mathbf{u}\cdot\nabla p + \rho\mathbf{F}\cdot\mathbf{u}.$$

Integrating over the fixed volume V, we find

$$\int_V \rho\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V = -\int_V \mathbf{u}\cdot\nabla p\,\mathrm{d}V + \int_V \rho\mathbf{F}\cdot\mathbf{u}\,\mathrm{d}V.$$

For a *steady flow* with a conservative body force such that $\mathbf{F} = -\nabla V$, this becomes

$$\int_V \rho\mathbf{u}\cdot\nabla\left(\frac{1}{2}|\mathbf{u}|^2\right)\,\mathrm{d}V = -\int_V \mathbf{u}\cdot\nabla p\,\mathrm{d}V + \int_V \rho\mathbf{u}\cdot\nabla V\,\mathrm{d}V,$$

where we have used the fact that $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla(\mathbf{u}^2/2)$ for a divergence free velocity field. Next we note that

$$\rho\mathbf{u}\cdot\nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) = \nabla\cdot\left(\frac{1}{2}\rho|\mathbf{u}^2|\mathbf{u}\right), \qquad \mathbf{u}\cdot\nabla V = \nabla\cdot(V\mathbf{u}),$$

since $\nabla\cdot\mathbf{u} = 0$. So, using the divergence theorem, we have

$$\int_S \frac{1}{2}\rho|\mathbf{u}^2|\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = -\int_S p\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S + \int_S \rho V\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S.$$

Therefore,

$$\int_S \left[\frac{1}{2}\rho|\mathbf{u}^2| + p + \rho g y\right]\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = 0$$

is the statement of conservation of energy over a closed volume fixed in space[2].

3.5.2 Energy equation for viscous flow

For an incompressible, viscous fluid flow, the Navier-Stokes equation holds:

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \rho F_i + \frac{\partial\sigma_{ij}}{\partial x_j}$$

where ρF_i is the body force per unit mass (see equation 3.24).

As above, we take the dot product with $\mathbf{u}$. Skipping steps identical to before (albeit this time in index notation) we find

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \int_V u_i\Big(\rho F_i + \frac{\partial\sigma_{ij}}{\partial x_j}\Big)\,\mathrm{d}V = \int_V \rho\,\mathbf{u}\cdot\mathbf{F}\,\mathrm{d}V + \int_V u_i\frac{\partial\sigma_{ij}}{\partial x_j}\,\mathrm{d}V.$$

[2]Compare *Longuet-Higgins* (1975), Proceedings of the Royal Society of London, **342**, equation 1.7).

Note that the effect of the pressure is here contained within the σ_{ij} of the final term since

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

This term therefore also incorporates the new effect due to viscosity.

Now,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial (u_i \sigma_{ij})}{\partial x_j} - \sigma_{ij}\frac{\partial u_i}{\partial x_j},$$

and so

$$\begin{aligned} \int_V u_i \frac{\partial \sigma_{ij}}{\partial x_j}\,\mathrm{d}V &= \int_V \frac{\partial (u_i\sigma_{ij})}{\partial x_j}\,\mathrm{d}V - \int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}V \\ \text{(using divergence theorem)} \qquad &= \int_S u_i\sigma_{ij}n_j\,\mathrm{d}S - \int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}V. \end{aligned}$$

But, writing

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\Big),$$

we have

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} \;=\; \frac{1}{2}\sigma_{ij}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\Big) \;=\; \frac{1}{2}\sigma_{ij}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\sigma_{ij}\frac{\partial u_i}{\partial x_j} \;=\; \frac{1}{2}\sigma_{ij}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\sigma_{ij}\frac{\partial u_j}{\partial x_i}$$

on swapping i and j in the second term and using the fact that σ_{ij} is symmetric. Thus,

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} \;=\; \frac{1}{2}\sigma_{ij}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \;=\; \sigma_{ij}e_{ij}.$$

To summarize, we have

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \int_V \rho\,\mathbf{u}\cdot\mathbf{F}\,\mathrm{d}V + \int_S u_i\sigma_{ij}n_j\,\mathrm{d}S - \int_V \sigma_{ij}e_{ij}\,\mathrm{d}V. \tag{3.28}$$

For an *incompressible* fluid, $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$, and so

$$\sigma_{ij}n_j = -pn_i + 2\mu e_{ij}n_j.$$

Also,

$$\sigma_{ij}e_{ij} = -pe_{ii} + 2\mu e_{ij}e_{ij} = 2\mu e_{ij}^2,$$

since $e_{ii} = \nabla\cdot\mathbf{u} = 0$.

Therefore, finally, the rate of change of the volume's kinetic energy

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \int_V \rho\,\mathbf{u}\cdot\mathbf{F}\,\mathrm{d}V + \int_S (-p\mathbf{n})\cdot\mathbf{u}\,\mathrm{d}S + 2\mu\int_S u_ie_{ij}n_j\,\mathrm{d}S - 2\mu\int_V e_{ij}^2\,\mathrm{d}V. \tag{3.29}$$

Compare this with equation (3.27) for inviscid flow. If we set $\mu = 0$, equation (3.29) reduces to (3.27).

The first new term in (3.29),

$$2\mu\int_V u_ie_{ij}n_j\,\mathrm{d}V$$

is the rate of working of the deviatoric viscous stresses on the surface of the fluid blob V. Thus the viscous stress makes a contribution to the increase of kinetic energy in the blob, as we would have expected.

So far so good – all of the first three terms on the RHS of (3.29) represent work done on the blob to increase its kinetic energy. But...

...the second new term in (3.29),

$$-2\mu \int_V e_{ij}^2 \,\mathrm{d}V$$

is definitely negative and marks a *reduction* in kinetic energy. So some energy is being *lost*. We say is it lost due to *viscous dissipation* and define the dissipation function Φ, where

$$\Phi = 2\mu e_{ij} e_{ij}.$$

This represents the rate at which energy is dissipated per unit volume by viscous action.

For a *compressible* fluid we find

$$\begin{aligned} \Phi &= 2\mu\Big(e_{ij}e_{ij} - \frac{\Delta^2}{3}\Big), \qquad (\Delta = e_{ii}) \\ \text{(non-trivial step)} \qquad &= 2\mu\Big(e_{ij} - \frac{\Delta\delta_{ij}}{3}\Big)^2. \end{aligned}$$

Summary

No energy is lost in an inviscid flow. This does not include such phenomena as hydraulic jumps and breaking waves - in these examples discontinuities are introduced into the flow, the preceding arguments do not work, and energy is in general lost.

Energy is lost from a viscous flow. The rate of loss of energy is represented by the dissipation function, Φ.

3.6 Conservation of energy

We saw above that shear forces dissipate energy in a viscous flow. But where does this energy go to?

Consider a fluid moving with no body forces acting. The total energy, E, inside a moving volume of fluid, V, is given by

$$E = K + I,$$

where K is the kinetic energy, and I is the *internal energy* associated with the movement of the constituent molecules within the fluid. (Note there is no potential energy because we've left out body forces.) When work is done on the volume by the surrounding fluid, some of the work goes into increasing the kinetic energy (K) and some into increasing the internal energy (I).

From equation (3.28), we know that

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \int_S u_i \sigma_{ij} n_j \,\mathrm{d}S - \int_V 2\mu e_{ij}^2 \,\mathrm{d}V = \int_S u_i \sigma_{ij} n_j \,\mathrm{d}S - \int_V \Phi \,\mathrm{d}V,$$

(with the body force taken as zero).

So,

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t} \\ &= \int_S u_i \sigma_{ij} n_j \,\mathrm{d}S + \left(\frac{\mathrm{d}I}{\mathrm{d}t} - \int_V \Phi \,\mathrm{d}V\right). \end{aligned}$$

The rate of change of the total energy E in the moving volume must be balanced by the rate of working of the local stresses on the volume surface, so that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_S u_i \sigma_{ij} n_j \,\mathrm{d}S.$$

This means that

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_V \Phi \,\mathrm{d}V,$$

and so viscous dissipation (in the form of heat) goes toward increasing the internal energy in the volume.

It seems natural, then, to ask the following question: How does viscous dissipation affect the temperature of a fluid? Since we wish to focus on temperature, we'll generalise the argument a little to allow for the effect of heat conduction in the fluid.

Heat is conducted through the surface *into* the moving volume at a rate q given by *Fick's law*,

$$q = \kappa \mathbf{n} \cdot \nabla T$$

per unit area of the volume's surface, where T is the local temperature. The constant κ is called the *thermal conductivity* of the fluid.

The First Law of Thermodynamics dictates that, in a unit of time, the internal energy of the volume is increased by (*a*) any work done, W, on the volume, and (*b*) any heat, Q, entering the volume[3]. In symbols,

$$\Delta E = Q + W, \tag{3.30}$$

where ΔE means the change in internal energy per unit time. From above, we have over the time interval $\mathrm{d}t$,

$$Q = \mathrm{d}t \int_S q \,\mathrm{d}S = \mathrm{d}t\,\kappa \int_S \mathbf{n} \cdot \nabla T \,\mathrm{d}S = \mathrm{d}t\,\kappa \int_V \nabla^2 T \,\mathrm{d}V,$$

$$W = \mathrm{d}t \int_V 2\mu e_{ij}^2 \,\mathrm{d}V = \mathrm{d}t \int_V \Phi \,\mathrm{d}V.$$

The internal energy is related to the fluid temperature by

$$I = \int_V \rho\, c\, T \,\mathrm{d}V,$$

where c is the specific heat capacity of the fluid and T is the temperature of the fluid. So, according to the First Law of Thermodynamics (3.30),

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho\, c\, T \,\mathrm{d}V = \int_V \left[\kappa \nabla^2 T + \Phi\right] \,\mathrm{d}V.$$

Using the Reynolds Transport Theorem,

$$\int_V \rho\, c \frac{\mathrm{D}T}{\mathrm{D}t} \,\mathrm{d}V = \int_V \left[\kappa \nabla^2 T + \Phi\right] \,\mathrm{d}V.$$

Hence, since V was chosen arbitrarily,

$$\rho\, c \frac{\mathrm{D}T}{\mathrm{D}t} = \Phi + \kappa \nabla^2 T.$$

Conclusion: The rate of increase of temperature as we move with the fluid is proportional to the rate of loss of energy due to viscous dissipation.

NOTE: In general the viscosity μ depends on temperature, so $\mu = \mu(T)$. However, in this course we will assume that Φ is small and that changes in the temperature of the fluid are insignificant. We will therefore take μ to be constant.

[3]Note that (*a*) does not include work done to increase the kinetic energy of the fluid in the volume. With reference to the energy equation (3.29), the first three terms on the RHS represent work done to increase the kinetic energy, and the fourth term represents work done to increase the internal energy.

3.7 The momentum integral equation

It is sometimes convenient to consider the balance of forces acting on a given parcel of fluid. To this end, it is useful to consider a volume of fluid which is *fixed* in space, as we did for the conservation of mass argument above. However, instead of considering the mass flux into and out of the fixed volume, we now examine the flux of momentum.

Newton's second law states that the force acting on the fixed volume V is equal to rate of change of its momentum. Hence[4],

$$\int_V \frac{\partial}{\partial t}(\rho\,\mathbf{u})\,\mathrm{d}V = -\int_S \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S + \int_V \rho\mathbf{F}\,\mathrm{d}V + \int_S \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S. \tag{3.31}$$

In turn:

The *term* on the left hand side is the rate of change of momentum inside the fixed volume.

The *first term* on the right hand side expresses the flux of momentum through the surface S. Since the volume is fixed, some fluid particles are entering and some are leaving the volume, carrying momentum with them.

The *second term* on the right hand side expresses the total body force acting on the fixed volume.

The *third term* on the right hand side expresses the total of the surface forces acting on S due to the surrounding fluid.

For an *inviscid fluid*, $\boldsymbol{\sigma} = -p\mathbf{I}$, where $\mathbf{I}$ is the identity matrix[5]. For a steady flow in the absence of a body force, we then have

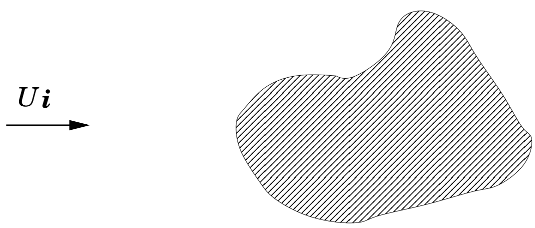
$$\int_S \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n}\,\mathrm{d}S = \mathbf{0}.$$

This relation may also be obtained by integrating the Euler equations directly over a volume V and using appropriate vector identities.

3.8 D'Alembert's Paradox

Use may be made of the momentum integral equation to determine the drag on an object placed in an inviscid flow.

Consider uniform inviscid flow past an object as shown



Suppose we now ask: *what is the drag on the object due to the flow?*

We assume that a long way from the object the flow is the uniform stream $U\mathbf{i}$ in the x direction at constant pressure P. Let $\mathbf{D}$ be the total force on the object. Then

$$\mathbf{D} = -\int_P p\,\mathbf{n}\,\mathrm{d}S,$$

where P is the surface of the object.

[4]This is obtained by integrating Newton's second law. See Appendix B

[5]This is the same as saying $\sigma_{ij} = -p\delta_{ij}$

To determine $\mathbf{D}$, we use the inviscid steady form of the momentum integral equation (3.31),

$$0 = -\int_S \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S - \int_S p\,\mathbf{n}\,\mathrm{d}S + \int_V \rho\mathbf{F}\,\mathrm{d}V. \tag{3.32}$$

Assuming that the only body force acting is gravity, $\mathbf{F} = -g\mathbf{k}$, and writing

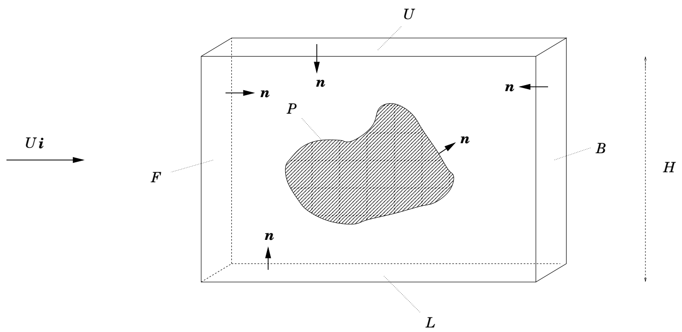
$$\mathbf{F} = \nabla\varphi, \qquad \varphi = -gz,$$

since gravity is a conservative force, the volume integral becomes a surface integral courtesy of the divergence theorem,

$$\int_V \rho\mathbf{F}\,\mathrm{d}V = \int_V \rho\nabla\varphi\,\mathrm{d}V = \int_S \rho\varphi\,\mathbf{n}\,\mathrm{d}S = -\int_S \rho g z\,\mathbf{n}\,\mathrm{d}S,$$

where $\mathbf{n}$ is the unit outward normal to the bounding surface S.

To determine the drag, we apply the momentum integral equation (3.31) over the volume bounded by the rectangular box of height H as shown with faces Front (F), Back (B), Upper (U) and Lower (L), and the object surface P.



Let Ω denote the union of F, B, U, L and P. Equation (3.32) gives

$$0 = -\int_\Omega \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S - \int_S p\,\mathbf{n}\,\mathrm{d}S + \int_\Omega \rho\varphi\,\mathbf{n}\,\mathrm{d}S. \tag{3.33}$$

Assuming that the bounding box is sufficiently large that the disturbance caused by the object has decayed and the flow has settled to a uniform stream, then $\mathbf{u}\cdot\mathbf{n} = 0$ on U and L, and $\mathbf{u}\cdot\mathbf{n} = U$ on F and $\mathbf{u}\cdot\mathbf{n} = -U$ on B. Also, since the fluid can't penetrate the object, $\mathbf{u}\cdot\mathbf{n} = 0$ on P. So the first integral in (3.33) gives

$$-\int_\Omega \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S = -\int_F \rho\,U^2\,\mathbf{i}\,\mathrm{d}S + \int_F \rho\,U^2\,\mathbf{i}\,\mathrm{d}S = \mathbf{0}.$$

To determine the second integral, we note that Bernoulli's equation in the fluid states that, along a streamline,

$$p + \frac{1}{2}\rho\mathbf{u}^2 + \rho g y = C,$$

for constant C. Letting the size of the box approach infinity we see that

$$p_U - p_L = -\rho g H$$

where p_U and p_L denote the pressures on U and L respectively. Also,

$$p_B = p_F,$$

with a similar notation. So the second integral gives

$$-\int_S p\,\mathbf{n}\,\mathrm{d}S = -\int_P p\,\mathbf{n}\,\mathrm{d}S - \int_\Omega p\,\mathbf{n}\,\mathrm{d}S = \mathbf{D} - \int_L p_L\,\mathbf{k}\,\mathrm{d}S + \int_U p_U\,\mathbf{k}\,\mathrm{d}S$$

$$= \mathbf{D} - \int_U p_U + \rho g H\,\mathbf{k}\,\mathrm{d}S + \int_U p_U\,\mathbf{k}\,\mathrm{d}S = \mathbf{D} - \rho g H A\,\mathbf{k}$$

where A is the area of U. Recalling that $\phi = -gz$, the third integral gives (assuming constant density)

$$\int_\Omega \rho\varphi\,\mathbf{n}\,\mathrm{d}S = \left[\int_B \rho g z\,\mathbf{i}\,\mathrm{d}S - \int_F \rho g z\,\mathbf{i}\,\mathrm{d}S\right] + \left[\int_U \rho g z\,\mathbf{k}\,\mathrm{d}S - \int_L \rho g z\,\mathbf{k}\,\mathrm{d}S\right] + \int_P \rho\nabla\phi\,\mathrm{d}V$$

$$= \left[\int_U \rho g z\,\mathbf{k}\,\mathrm{d}S - \int_U \rho g(z-H)\,\mathbf{k}\,\mathrm{d}S\right] + \int_P \rho\nabla\phi\,\mathrm{d}V$$

$$= \rho g H A\,\mathbf{k} - \rho g V_P\,\mathbf{k} = \rho g[HA - V_P]\,\mathbf{k}.$$

where A is the area of the upper (or lower) face and V_P is the volume occupied by the object.

So (3.33) gives

$$0 = \mathbf{D} - \rho g H\,\mathbf{k} + \rho g[HA - V_P]\,\mathbf{k}.$$

and thus

$$\mathbf{D} = \rho g V_P\,\mathbf{k}.$$

So the total force just involves the Archimedes upthrust of the solid object in the vertical direction. Taking the horizontal component in the direction of the flow,

$$\mathbf{D}\cdot\mathbf{i} = \mathbf{0},$$

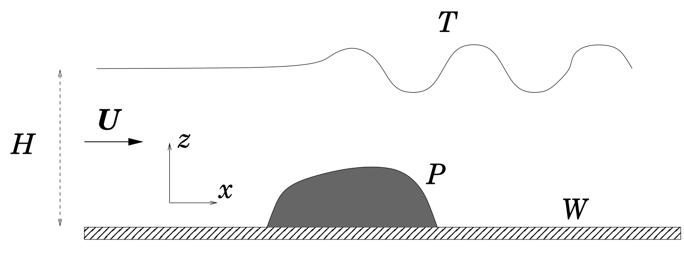
and we predict zero drag on the object!

This is *D'Alembert's Paradox*. It states that an object of an shape placed in a uniform stream in an inviscid fluid experiences zero drag.

It is important to realise that D'Alembert's paradox does **not** mean than any object placed in any inviscid flow experiences zero drag. The previous argument applies for an object placed in a uniform stream. If there is a free surface, for example, an object such as a ship may experience drag to wave resistance, as is explored in the next section.

3.9 Wave resistance

Consider the inviscid two-dimensional flow of a stream of water over an object P as shown in the figure. Assume the density of the water, ρ, is constant.



The wall is at $z = 0$ and the undisturbed height of the water is H. If the Froude number, F, defined as

$$F = \frac{U}{(gH)^{1/2}},$$

is less than 1 then the free surface of the stream, T, may develop waves as shown in the figure.

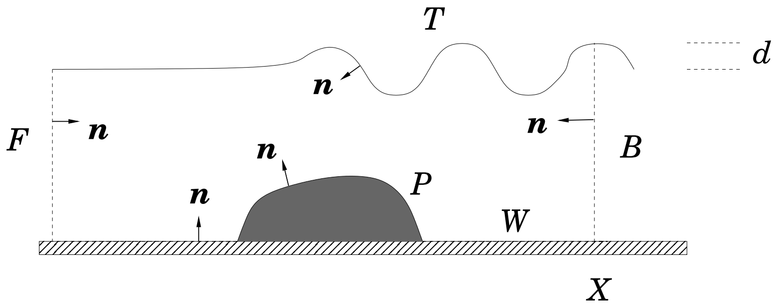
Let us calculate the drag on the object P for this flow. The inviscid steady form of the momentum integral equation (3.31) in two-dimensions is

$$0 = -\int_C \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}C - \int_C p\,\mathbf{n}\,\mathrm{d}l + \int_A \rho\mathbf{F}\,\mathrm{d}A,$$

over an area A enclosed by the closed contour C of incremental arc length $\mathrm{d}l$. Assuming that the only body force acting is gravity, $\mathbf{F} = -g\mathbf{k}$, and writing $\mathbf{F} = -\nabla(gz)$ as in the previous section, the equation becomes

$$0 = -\int_C \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}l - \int_C p\,\mathbf{n}\,\mathrm{d}l - \int_C \rho g z\,\mathbf{n}\,\mathrm{d}l. \tag{3.34}$$

Take S to be the dotted contour shown in the figure below. Note that there is no direction associated with the contour as we are integrating with respect to arc length.



The contour comprises the ends F and B, the wall W, the object P, and the free surface up to one of the crests T. Without loss of generality, we take $p = 0$ on the free surface, T. Noting that $\mathbf{u}\cdot\mathbf{n} = 0$ on the object, on the free surface T, and on the wall W, and that n_x is zero on the wall, where $\mathbf{n} = (n_x, n_y)$, the horizontal component of (3.34) gives

$$-D_x = -\mathbf{i}\cdot\mathbf{D} = -\rho\int_{F,B} u(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}l - \int_{F,B} p\,n_x\,\mathrm{d}l - \rho g\int_{P,F,B,T} z\,n_x\,\mathrm{d}l,$$

where

$$\mathbf{D} = -\int_P p\,\mathbf{n}\,\mathrm{d}l$$

is the hydrodynamic force on the object. Now,

$$\int_P z\,n_x\,\mathrm{d}l = \int_P \eta\eta'\mathrm{d}x = \frac{1}{2}[\eta^2]_{-L}^{L} = 0$$

if $z = \eta(x)$ describes the bubble shape. We also note that

$$-\int_{F,B} z\,n_x\,\mathrm{d}l \quad = \quad \int_0^{H+d} z\,\mathrm{d}z - \int_0^H z\,\mathrm{d}z = \frac{1}{2}[(H+d)^2 - H^2],$$

where d is the height of a wave crest over and above the undisturbed stream height (see figure). Moreover,

$$\int_T z\,n_x\,\mathrm{d}l = \int_T f f'\,\mathrm{d}x = \frac{1}{2}[f^2]_{-\infty}^{X} = \frac{1}{2}[(H+d)^2 - H^2]$$

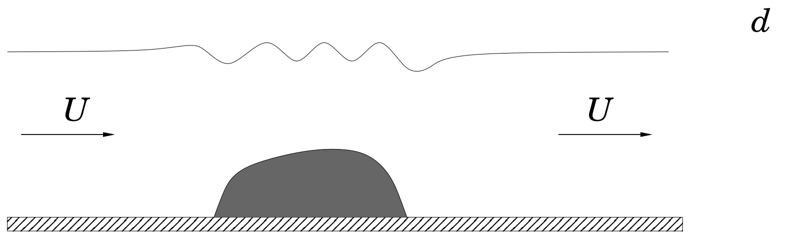
if $z = f(x)$ is the shape of the free surface. Also,

$$\int_{F,B} u(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}l = \int_F u^2\,\mathrm{d}l - \int_B u^2\,\mathrm{d}l.$$

So,

$$\begin{aligned}
-D_x \quad &= \quad \rho\int_B u^2\,\mathrm{d}l - \rho\int_F u^2\,\mathrm{d}l - \int_{F,B} p\,n_x\,\mathrm{d}l + \frac{1}{2}\rho g[(H+d)^2 - H^2] - \frac{1}{2}\rho g[(H+d)^2 - H^2] \\
&= \quad \int_B (p + \rho u^2)\,\mathrm{d}l - \int_F (p + \rho u^2)\,\mathrm{d}l
\end{aligned}$$

which in general is non-zero. Longuet Higgins shows how this expression may be written in terms of the potential energy of the wave train at infinity downstream (see Vanden-Broeck, JFM 1980, V.96, p.610).



Hence the drag on an object is non-zero if on one side there is a train of waves extending to infinity. Under some circumstances, one can obtain waves trapped over the obstacle with the free surface becoming flat at the same height on either side like this:

In this case, since the conditions are the same a long way upstream or downstream, we get

$$D_x = \int_B (p + \rho u^2)\, \mathrm{d}l - \int_F (p + \rho u^2)\, \mathrm{d}l = 0$$

and so there is no drag on the object (see Forbes, 1982, J.Eng.Math V 16, p.171-180).

3.10 D'Alembert's Paradox for a bubble

It is interesting to note that a two-dimensional bubble at equilibrium within an inviscid flow field experiences zero net force regardless of the nature of the velocity field around it. To see this, we note that at the surface of the bubble, we have the Laplace-Young equation

$$p = p_B + \gamma\kappa.$$

The force on the bubble is given by

$$-\int_C p\mathbf{n}\, \mathrm{d}s,$$

where C is the bubble contour and s is arc length around the bubble contour. Using the Laplace-Young equation, this becomes

$$-\int_C p_B\mathbf{n}\, \mathrm{d}s - \gamma\int_C \kappa\mathbf{n}\, \mathrm{d}s.$$

The first integral is easily shown to be zero by the divergence theorem. For the second integral, we note that, by the definition of curvature,

$$\kappa\mathbf{n} = -\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s},$$

where $\mathbf{t}$ is the unit tangent to the bubble contour pointing in the direction of increasing arc length. The force becomes

$$-\gamma\int_C \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\, \mathrm{d}s = -\gamma\int_C \mathrm{d}\mathbf{t} = [\mathbf{t}]_C = 0$$

since the bubble is closed. Hence the bubble experiences no net force in any direction.

A spherical bubble placed within an inviscid flow also experiences zero no force whatever the far-field flow (i.e. we do not need a free stream at infinity as in section 3.8). The force on the bubble is

$$D = -\int_B p\mathbf{n}\, \mathrm{d}S,$$

where B is the closed surface of the bubble. By the Laplace-Young equation given above, we have

$$p = p_B + \gamma\kappa,$$

where p_B is the constant pressure inside the bubble, and κ is the bubble curvature, and γ is the surface tension. So,

$$\int_B p\mathbf{n}\, \mathrm{d}S = p_B\int_B \mathbf{n}\, \mathrm{d}S + \gamma\int_B \kappa\mathbf{n}\, \mathrm{d}S$$

But, using the divergence theorem, it can be shown that

$$\int_B \mathbf{n}\,\mathrm{d}S = 0.$$

Since the bubble is spherical, then κ is a constant and the second integral also vanishes, with the result that the bubble experiences no drag regardless of the nature of the surrounding flow.

In fact, a closed three-dimensional bubble of *any* shape experiences zero net force in an inviscid fluid since it can be shown that

$$\int_S \kappa\mathbf{n}\,\mathrm{d}S = 0$$

where κ is the mean curvature at a point on the surface S, for any choice of S (see Blackmore & Ting, *SIAM Review*, **27**(4), p.569-572). This is consistent with physical thinking: since the bubble interface has zero inertia, the resultant force acting upon it must be zero (see also the force balance on an interfacial segment in section 3.4.5).

So the total force on the bubble is zero. This means that we cannot have a bubble of fixed shape in a fluid with gravity, since then the total force on the bubble is

$$-\int_S p^+\mathbf{n}\,\mathrm{d}S + \int_S p^-\mathbf{n}\,\mathrm{d}S$$

where p^+ and p^- are the pressures on the inside of the bubble respectively. In a static field, $p = -\rho g y + B$ for constant B. So, on using the divergence theorem, the total force is

$$-\int_V \nabla p^+\,\mathrm{d}V + \int_V \nabla p^-\,\mathrm{d}V$$

$$= \int_V \rho^+ g\,\mathrm{d}V + \int_V \rho^- g\,\mathrm{d}V = (\rho^+ - \rho^-)gV.$$

But this must be zero by the previous argument. So there must be another force to balance this buoyancy force. This comes, for example, from the visous drag on the bubble, in which case the total force acting is a combination of the pressure force and the normal viscous stress, which must sum to zero to give a total overall force of zero on the bubble.

Summary

We have derived equations governing

1. Conservation of mass in a fluid.
2. The dynamic motion of a fluid.
3. Conservation of energy in a fluid.

Vector identities

The following vector identities are useful in fluid mechanics:

$$\begin{aligned}
\nabla^2\mathbf{u} &= \nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u}),\\
\mathbf{u}\cdot\nabla\mathbf{u} &= \nabla(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\omega}\times\mathbf{u}, \quad \text{where} \quad \boldsymbol{\omega} = \nabla\times\mathbf{u},\\
\nabla\cdot(\phi\mathbf{u}) &= \phi\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\phi,\\
\nabla\times(\phi\mathbf{u}) &= \phi\nabla\times\mathbf{u} + (\nabla\phi)\times\mathbf{u}.
\end{aligned}$$

Other vector identities are also available. See Acheson p. 348..

4. Vorticity dynamics and the stream function

4.1 Vorticity dynamics

We have defined the fluid vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

to be the local rotation of fluid elements. This leads naturally to the definition of vortex lines (compare streamlines):

Vortex lines

A vortex line in a flow has an analogous definition to a streamline: a vortex line is everywhere tangent to the instantaneous vorticity vector. This is to say that for a fixed time t, a vortex line is described by $\mathbf{x}(s)$ for parameter s, where

$$\frac{d\mathbf{x}}{ds} = \boldsymbol{\omega}.$$

Therefore, similar to the calculation for streamlines, we may compute vortex lines by solving

$$\frac{dx}{\omega_1} = \frac{dx}{\omega_2} = \frac{dz}{\omega_3} = ds,$$

where $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$.

Example: Find the streamlines and vortex lines for the shear flow with velocity field $\mathbf{u} = \lambda y \mathbf{i}$.

Since the flow is only in the x direction, the streamlines must be lines parallel to the x axis. The vorticity is

$$\boldsymbol{\omega} = -\lambda \mathbf{k}.$$

So $\omega_1 = \omega_2 = 0$ and $\omega_3 = -\lambda$. The vortex lines therefore satisfy

$$\frac{dz}{(-\lambda)} = ds, \qquad \text{i.e.} \qquad \frac{dz}{ds} = -\lambda$$

and so $z = -\lambda s$, and the vortex lines are straight lines in the z direction. This is as expected, since the vorticity vector points only in the z direction for this flow.

Vortex tubes

A vortex tube is a tube generated entirely by vortex lines. For example in the flow with velocity field $\mathbf{u} = u(y,t)\mathbf{i}$, the vorticity is $\boldsymbol{\omega} = \omega \mathbf{k}$. We could imagine a vortex tube composed of vortex lines parallel to the z axis and lying on the surface of a circular cylinder of some chosen radius.

If the flow is incompressible then

$$\nabla \cdot \boldsymbol{\omega} = 0,$$

and so if we apply the divergence theorem to a section of a vortex tube with flat, parallel ends A_1 and A_2 we obtain

$$\iiint_V \nabla \cdot \boldsymbol{\omega} \, dV = \iint_S \boldsymbol{\omega} \cdot \mathbf{n} \, dS$$

where V is the volume of the tube and S is its surface. Note that the surface integral

$$\iint_S \boldsymbol{\omega} \cdot \mathbf{n} \, dS$$

vanishes on the tube except at the ends, since $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on a vortex line by definition. Thus,

$$\iiint_V \nabla \cdot \boldsymbol{\omega}\, dV = \iint_{A_1} \boldsymbol{\omega} \cdot \mathbf{n}\, dS - \iint_{A_2} \boldsymbol{\omega} \cdot \mathbf{n}\, dS.$$

If we consider a very thin vortex tube then $\boldsymbol{\omega}$ is approximately constant inside, and we have

$$\iiint_V \nabla \cdot \boldsymbol{\omega}\, dV = \left(\omega_{A_1} \iint_{A_1} dS - \omega_{A_2} \iint_{A_2} dS\right) \mathbf{n} = (A_1\omega_{A_1} - A_2\omega_{A_2})\,\mathbf{n}$$

But we know that $\nabla \cdot \boldsymbol{\omega} = 0$ and so we have

$$A_1\omega_1 = A_2\omega_2$$

and so the product $A\omega$ is conserved inside a vortex tube. This means that if the cross-sectional area of a tube goes down, the vorticity goes up and the local rotation of fluid particles becomes more intense.

Thus if a vortex tube becomes stretched out by a straining flow, the vorticity inside it intensifies.

Vorticity transport equation

It is often useful to understand the dynamics of a flow by working with the vorticity rather than the velocity field. To this end, we would like to have a set of governing equations for $\boldsymbol{\omega}$ rather than $\mathbf{u}$. We can derive these by taking the curl of the Navier-Stokes momentum equation,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -\nabla p/\rho + \nu \nabla^2 \mathbf{u}. \tag{4.1}$$

It is convenient to use two vector identities to rewrite (4.1) as

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) + \boldsymbol{\omega} \times \mathbf{u} &= -\nabla p/\rho - \nu \nabla \times (\nabla \times \mathbf{u}) \\ &= -\nabla p/\rho - \nu \nabla \times \boldsymbol{\omega}. \end{aligned} \tag{4.2}$$

Taking the curl of equation (4.2) gives

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) &= -\nu \nabla \times (\nabla \times \boldsymbol{\omega}) \\ &= \nu \nabla^2 \boldsymbol{\omega}. \end{aligned} \tag{4.3}$$

Making use of the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) \;=\; (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a})$$

we obtain the *vorticity transport equation*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} \;=\; \nu \nabla^2 \boldsymbol{\omega}. \tag{4.4}$$

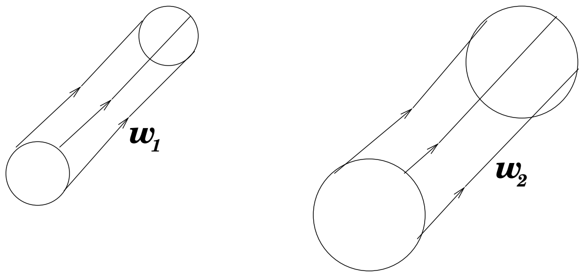
which may be written more compactly as

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} \;=\; \nu \nabla^2 \boldsymbol{\omega}. \tag{4.5}$$

Finally, we note that, since $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, we have

$$\nabla \cdot \boldsymbol{\omega} = 0. \tag{4.6}$$

Physical interpretation



Consider a thin vortex tube made up of vortex lines pointing in the direction of the vorticity vector $\boldsymbol{\omega}_1$ as shown in the left of the figure above. Since the tube is thin, the vorticity $|\boldsymbol{\omega}_1|$ is approximately constant inside the tube. Suppose that a while later the tube has expanded a little and the vorticity inside is now $|\boldsymbol{\omega}_2|$. Then the fact that $\nabla \cdot \boldsymbol{\omega} = 0$ means that $|\boldsymbol{\omega}_2| < |\boldsymbol{\omega}_1|$, since the "flux of vorticity" through the tube is conserved.

The term $\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t$ in (4.5) represents the convection of vorticity with the flow.

The $\nu\nabla^2\boldsymbol{\omega}$ in (4.5) represents viscous diffusion of vorticity.

Vortex stretching

The term $\boldsymbol{\omega} \cdot \nabla\mathbf{u}$ in (4.5) represents stretching of the vortex lines. To see why, we rewrite this term as

$$\boldsymbol{\omega} \cdot \nabla\mathbf{u} = \frac{1}{2}\boldsymbol{\omega} \cdot \left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) + \frac{1}{2}\boldsymbol{\omega} \cdot \left(\nabla\mathbf{u} - \nabla\mathbf{u}^T\right)$$

We recognise the terms within the brackets as the rate of strain tensor and the vorticity tensor (see section 2B). Therefore

$$\boldsymbol{\omega} \cdot \nabla\mathbf{u} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{e} + \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\xi}$$

In index notation

$$(\boldsymbol{\omega} \cdot \nabla\mathbf{u})_i = \frac{1}{2}\omega_j e_{ji} + \frac{1}{2}\omega_j \xi_{ji}.$$

Now, from (2.8), we have that

$$\xi_{ij} = -\tfrac{1}{2}\epsilon_{ijk}\omega_k$$

and so

$$\omega_j \xi_{ji} = -\tfrac{1}{2}\epsilon_{ijk}\omega_j\omega_k = 0$$

using the definition of the alternating tensor. So we conclude that

$$\boldsymbol{\omega} \cdot \nabla\mathbf{u} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{e}$$

The right hand side is the projection of the rate of strain tensor onto the vorticity vector and therefore represents the rate of straining along a vortex line, otherwise known as vortex stretching.

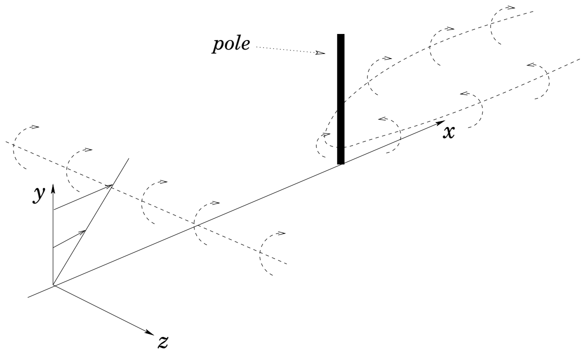
Note that there is no vortex stretching in a two-dimensional flow since in this case $\mathbf{u} = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$, and $\boldsymbol{\omega} = (v_x - u_y)\mathbf{k} = \omega\mathbf{k}$ and so

$$\boldsymbol{\omega} \cdot \nabla\mathbf{u} = \omega\frac{\partial\mathbf{u}}{\partial z} = \mathbf{0}.$$

So the vortex stretching term vanishes identically.

Example of vortex stretching: Snow clearing at telegraph poles

Wind blows in the x direction over flat ground at $y = 0$ with velocity $\mathbf{u} = y\mathbf{i}$. The vorticity is $\boldsymbol{\omega} = \nabla \times \mathbf{u} = -\mathbf{k}$, so the vortex lines point along the z axis as shown in the diagram. The flow encounters a telegraph pole at some x. The vortex lines wrap themselves around this obstacle, becoming very stretched close to the pole. Therefore, since $\nabla \cdot \boldsymbol{\omega} = 0$, thin vortex tubes experience greatly intensified vorticity at the pole and the rate of local spinning of fluid particles increases. This mechanism is responsible for the cleared areas you sometimes see at the foot of telegraph poles when it snows in the winter.



Burger's vortex

A famous example of a fluid flow which exhibits all of the features described above is Burger's vortex. In terms of cylindrical polar coordinates, this flow is given by

$$w = kz, \qquad u = -\frac{1}{2}kr, \qquad v = v(r) \tag{4.7}$$

$$\boldsymbol{\omega} = \omega(r)\mathbf{k}, \qquad \omega(r) = \frac{1}{r}\frac{\mathrm{d}(rv)}{\mathrm{d}r}, \tag{4.8}$$

for some chosen constant k. Note that none of the components depend on θ. This is an example of an *axisymmetric* flow.

Substituting (4.7, 4.8) into the vorticity transport equation (4.4), we obtain

$$k\frac{\mathrm{d}(\omega r^2)}{\mathrm{d}r} + 2\nu\left(r\frac{\mathrm{d}^2\omega}{\mathrm{d}r^2} + \frac{\mathrm{d}\omega}{\mathrm{d}r}\right) = 0. \tag{4.9}$$

Solving this for ω, and assuming $\omega \to 0$ as $r \to \infty$, the vorticity is given by

$$\boldsymbol{\omega} = \frac{k\Gamma}{4\pi\nu}\mathrm{e}^{-kr^2/4\nu}\,\mathbf{k}, \tag{4.10}$$

for some constant Γ. So most of the vorticity is concentrated in a 'core' region of radius of order $O(\sqrt{\nu/k})$.

It is now possible to find $v(r)$ by integrating the last of equations (4.8). We find

$$v = \frac{\Gamma}{2\pi r}(1 - \mathrm{e}^{-kr^2/4\nu}).$$

The Burger's vortex described above is a *steady* and, remarkably, *exact* solution of the Navier-Stokes equations. It involves a perfect balance between the three mechanisms of convection of vorticity, vortex stretching, and viscous diffusion of vorticity.

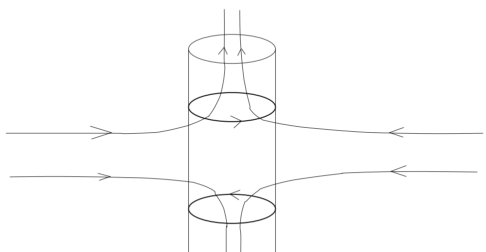


Illustration of Burger's vortex.

4.2 The Prandtl-Batchelor theorem

This pertains to the vorticity in regions of two-dimensional flow with *closed* streamlines when the viscosity is very weak. It states that within such regions, the vorticity is constant.

Consider the steady version of the vorticity transport equation for two-dimensional flow, so that $\boldsymbol{\omega} = \omega\mathbf{k}$. Recall that in 2-D the term $\boldsymbol{\omega} \cdot \nabla\mathbf{u}$ is identically zero. If the kinematic viscosity ν is very small, we may write, to a good approximation,

$$(\mathbf{u} \cdot \nabla)\omega = 0. \tag{4.11}$$

This means that the vorticity, ω, is constant along streamlines and hence just depends on the streamfunction ψ, i.e. $\omega = \omega(\psi)$. Now,

$$\nabla \times \boldsymbol{\omega} = \nabla \times (\omega\mathbf{k}) = \frac{\partial \omega}{\partial y}\mathbf{i} - \frac{\partial \omega}{\partial x}\mathbf{j} = \omega'(\psi)\left[\frac{\partial \psi}{\partial y}\mathbf{i} - \frac{\partial \psi}{\partial x}\mathbf{j}\right] = \omega'(\psi)\,\mathbf{u}. \tag{4.12}$$

We can show (see Problem Sheet 2) that, if there exists a closed streamline C in steady viscous flow with vorticity $\boldsymbol{\omega}$, then the line integral

$$\int_C (\nabla \times \boldsymbol{\omega}) \cdot \mathbf{dx} = 0.$$

Inserting (4.12) into this result yields (since ω is constant along the streamline C)

$$\int_C \omega'(\psi)\,\mathbf{u} \cdot \mathbf{dx} = \omega'(\psi)\int_C \mathbf{u} \cdot \mathbf{dx} = 0.$$

So unless by fluke this line integral happens to be zero, we must have $\omega'(\psi) = 0$. Therefore in a region of closed streamlines the vorticity is constant.

4.3 The stream function

It is sometimes convenient to describe a fluid in terms of a *stream function*, usually represented by the symbol ψ, which assumes constant values on streamlines. We can see how and when this is possible as follows.

The equation of conservation of mass (continuity) states that

$$\nabla \cdot \mathbf{u} = 0.$$

The general solution of this equation is

$$\mathbf{u} = \nabla \times \mathbf{A},$$

for some vector $\mathbf{A}$.

However, the choice of $\mathbf{A}$ is not unique and we can add onto it any so-called gauge function ∇q. For example, if we take $\mathbf{A}' = \mathbf{A} + \nabla q$, then

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla q) = \nabla \times \mathbf{A} + \nabla \times \nabla q = \nabla \times \mathbf{A}.$$

Now, the vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}),$$

using one of the vector identities.

Due to the aforementioned non-uniqueness, we are at liberty to insist that $\mathbf{A}$ is solenoidal, i.e. $\nabla \cdot \mathbf{A} = 0$.

Thus

$$\boldsymbol{\omega} = -\nabla^2 \mathbf{A}. \tag{4.13}$$

Examples

i) **Two-dimensional flow**

In this case, $\mathbf{u} = u(x,y)\,\mathbf{i} + v(x,y)\,\mathbf{j}$.

If we pick $\mathbf{A} = \psi(x,y)\mathbf{k}$, then

$$\nabla \cdot (\psi \mathbf{k}) = \frac{\partial \psi}{\partial z} = 0.$$

Thus we can represent the flow velocity as

$$\mathbf{u} \;=\; \nabla \times (\psi \mathbf{k}) \;=\; \frac{\partial \psi}{\partial y}\,\mathbf{i} - \frac{\partial \psi}{\partial x}\,\mathbf{j} \;=\; u\,\mathbf{i} + v\,\mathbf{j}.$$

So

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x},$$

and the vorticity is given by (4.13).

Constant on streamlines? A useful property of the streamfunction is that it is constant on a streamline. We note that

$$\mathbf{u} \cdot \nabla \psi = u\psi_x + v\psi_y = \psi_y \psi_x - \psi_x \psi_y = 0,$$

and so ψ is constant on a streamline.

ii) **Axisymmetric flow**

a) In cylindrical polar coordinates, the velocity is

$$\mathbf{u} = u(r,z)\,\hat{\mathbf{r}} + w(r,z)\,\mathbf{k}$$

and so does not depend on the azimuthal coordinate, θ.

If we choose $\mathbf{A} = (\psi/r)\,\hat{\boldsymbol{\theta}}$, where $\psi = \psi(r,z)$ then

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\psi/r)\,\hat{\boldsymbol{\theta}} = \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} = 0.$$

Thus we can represent the flow velocity as

$$\mathbf{u} = \nabla \times \left(\frac{\psi}{r}\hat{\boldsymbol{\theta}}\right) = \frac{1}{r}\left(\frac{\partial \psi}{\partial r}\,\mathbf{k} - \frac{\partial \psi}{\partial z}\,\hat{\mathbf{r}}\right) = w\,\mathbf{k} + u\,\hat{\mathbf{r}}.$$

So

$$w = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad u = -\frac{1}{r}\frac{\partial \psi}{\partial z},$$

with the vorticity given by (4.13). This is the standard definition of the axisymmetric streamfunction.

Constant on streamlines? We note that

$$\mathbf{u} \cdot \nabla \psi = u\psi_r + w\psi_z = -r^{-1}\psi_z\psi_r + r^{-1}\psi_r\psi_z = 0,$$

and so ψ is constant on a streamline.

Alternatively, we could have taken $\mathbf{A} = \psi\,\hat{\boldsymbol{\theta}}$, where $\psi = \psi(r, z)$. Then

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\psi\,\hat{\boldsymbol{\theta}}) = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0,$$

and

$$\mathbf{u} = \nabla \times (\psi\,\hat{\boldsymbol{\theta}}) = \Big(\frac{1}{r}\frac{\partial}{\partial r}(r\psi)\,\mathbf{k} - \frac{\partial \psi}{\partial z}\,\hat{\mathbf{r}}\Big) = w\,\mathbf{k} + u\,\hat{\mathbf{r}}.$$

So

$$w = \frac{1}{r}\frac{\partial}{\partial r}(r\psi), \qquad u = -\frac{\partial \psi}{\partial z}.$$

Constant on streamlines? We note that

$$\mathbf{u} \cdot \nabla\psi = u\psi_r + w\psi_z = -\psi_z\psi_r + r^{-1}(r\psi_r + \psi)\psi_z \neq 0,$$

and so ψ is **not** constant on a streamline. However, notice that

$$\mathbf{u} \cdot \nabla(r\psi) = u(r\psi_r + \psi) + rw\psi_z = -\psi_z(r\psi_r + \psi) + (r\psi_r + \psi)\psi_z = 0,$$

and so for this definition we have that the quantity $r\psi$ is constant on a streamline.

So the representation of a flow in terms of a stream function can therefore be done in different, but equivalent, ways.

b) In spherical polar coordinates, the velocity is

$$u(r, \theta)\,\hat{\mathbf{r}} + v(r, \theta)\,\hat{\boldsymbol{\theta}},$$

so there is no dependence on the coordinate ϕ.

If we choose

$$\mathbf{A} = \frac{\psi(r, \theta)}{r \sin\theta}\,\hat{\boldsymbol{\phi}},$$

then

$$\nabla \cdot \mathbf{A} = \nabla \cdot \Big(\frac{\psi(r, \theta)}{r \sin\theta}\,\hat{\boldsymbol{\phi}}\Big) = \frac{1}{r^2 \sin^2\theta}\frac{\partial \psi}{\partial \phi} = 0.$$

Thus we can represent the flow velocity as

$$\mathbf{u} = \nabla \times \Big(\frac{\psi(r, \theta)}{r \sin\theta}\,\hat{\boldsymbol{\phi}}\Big) = \frac{1}{r^2 \sin\theta}\Big(\frac{\partial \psi}{\partial \theta}\,\hat{\mathbf{r}} - r\frac{\partial \psi}{\partial r}\,\hat{\boldsymbol{\theta}}\Big) = u\,\hat{\mathbf{r}} + v\,\hat{\boldsymbol{\theta}}.$$

So

$$u = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \qquad v = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r},$$

with the vorticity given by (4.13).

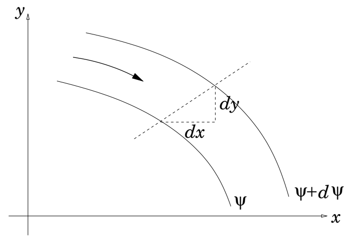
Physical interpretation

For simplicity, we concentrate on 2-D flow. So $\mathbf{u} = \nabla \times \psi\mathbf{k}$. Now, using the identity

$$\nabla \times (\phi\mathbf{A}) = \phi\nabla \times \mathbf{A} + \nabla\phi \times \mathbf{A},$$

we can write

$$\mathbf{u} \cdot \nabla\psi = (\nabla \times \psi\mathbf{k}) \cdot \nabla\psi = (\nabla\psi \times \mathbf{k}) \cdot \nabla\psi = \mathbf{k} \cdot (\nabla\psi \times \nabla\psi) = 0.$$



So the gradient of ψ in direction $\mathbf{u}$ is zero. In other words, ψ is constant along a streamline.

Let ψ and $\psi + \mathrm{d}\psi$ be two neighbouring streamlines:

The flux of fluid flowing between ψ and $\psi + \mathrm{d}\psi$ in unit time is

$$u\,\mathrm{d}y - v\,\mathrm{d}x = \frac{\partial\psi}{\partial y}\,\mathrm{d}y + \frac{\partial\psi}{\partial x}\,\mathrm{d}x = \mathrm{d}\psi.$$

Therefore, the difference in ψ measures the flux per unit time.

2-D equation of motion in terms of a stream function

Start with the vorticity transport equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} \;=\; \nu\nabla^2\boldsymbol{\omega}. \tag{4.14}$$

For two-dimensional flow,

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}, \qquad w = 0$$

and

$$\boldsymbol{\omega} = \omega\mathbf{k}, \qquad \omega = -\nabla^2\psi,$$

using (4.13).

So,

$$\nabla^2\boldsymbol{\omega} = -\nabla^4\psi\mathbf{k}.$$

Recall that in 2-D, there is no vortex stretching, and so $\boldsymbol{\omega}\cdot\nabla\mathbf{u} = 0$. Therefore the left hand side of (4.14) is

$$\frac{\partial}{\partial t}(-\nabla^2\psi\mathbf{k}) + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(-\nabla^2\psi\mathbf{k}) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(-\nabla^2\psi\mathbf{k}) = -\left\{\frac{\partial\nabla^2\psi}{\partial t} - \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}\right\}\mathbf{k},$$

where the Jacobian

$$\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}.$$

The equation of motion is therefore,

$$\frac{\partial\nabla^2\psi}{\partial t} - \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = \nu\nabla^4\psi.$$

4.4 Deducing the velocity field from a known vorticity distribution.

Suppose we know the vorticity field $\boldsymbol{\omega} = \nabla\times\mathbf{u}$. Can we deduce $\mathbf{u}$ from $\boldsymbol{\omega}$?

We start with a bit of potential theory. We study the function

$$\phi(\mathbf{x}_0) = \int_V \frac{\rho(\mathbf{x})}{r}\,\mathrm{d}V(\mathbf{x}), \tag{4.15}$$

where V is volume of fluid surrounding the field point $\mathbf{x}_0$, and ρ is some scalar field. Consider now $\nabla^2_{\mathbf{x}_0}\phi(\mathbf{x}_0)$, where

$$\nabla^2_{\mathbf{x}_0} \equiv \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + \frac{\partial^2}{\partial z_0^2}.$$

Then, we have

$$\nabla^2_{\mathbf{x}_0}\phi(\mathbf{x}_0) = \nabla^2_{\mathbf{x}_0} \int_V \frac{\rho(\mathbf{x})}{r}\,\mathrm{d}V(\mathbf{x}) = \int_V \nabla^2_{\mathbf{x}_0}\Big(\frac{1}{r}\Big)\,\rho(\mathbf{x})\,\mathrm{d}V(\mathbf{x}).$$

But recall that we can express the delta function $\delta(\hat{x})$, where $\hat{x} = \mathbf{x} - \mathbf{x}_0$, in the form

$$\delta(\hat{x}) = -\frac{1}{4\pi}\,\nabla^2\Big(\frac{1}{r}\Big)$$

where $r = |\hat{x}| = |\mathbf{x} - \mathbf{x}_0|$. We can do this since $G = 1/r$ is the Green's function for the singularly forced Laplace's equation,

$$\nabla^2 G = -4\pi\,\delta(\hat{x}).$$

In other words, $G = 1/r$ is a solution to this equation.

So,

$$\begin{aligned}\nabla^2_{\mathbf{x}_0}\phi(\mathbf{x}_0) &= -4\pi \int_V \delta(\hat{x})\,\rho(\mathbf{x})\,\mathrm{d}V(\mathbf{x}),\\ &= -4\pi\rho(\mathbf{x}_0),\end{aligned}$$

by the properties of the delta function.

Hence $\phi(\mathbf{x_0})$ satisfies the *Poisson equation*

$$\nabla^2\phi(\mathbf{x_0}) = -4\pi\rho(\mathbf{x_0}) \tag{4.16}$$

Stated another way, (4.15) is a solution to the Poisson equation (4.16).

Now consider the vorticity distribution $\boldsymbol{\omega}$ where

$$\boldsymbol{\omega} = \nabla\times\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0$$

for as yet unknown velocity field $\mathbf{u}$. Let's express $\mathbf{u}$ as the curl of its vector potential as follows,

$$\mathbf{u} = \nabla\times\mathbf{B}$$

where we have chosen $\nabla\cdot\mathbf{B} = 0$. Then,

$$\nabla\times\mathbf{u} = \nabla\times(\nabla\times\mathbf{B}) = -\nabla^2\mathbf{B} = \boldsymbol{\omega}.$$

So,

$$\nabla^2\mathbf{B} = -\boldsymbol{\omega}.$$

From the above, a solution to this Poisson equation is

$$\mathbf{B}(\mathbf{x}_0) = \frac{1}{4\pi}\int_V \frac{\boldsymbol{\omega}(\mathbf{x})}{r}\,\mathrm{d}V(\mathbf{x}).$$

We need to check $\nabla\cdot\mathbf{B} = 0$. So,

$$\nabla_{\mathbf{x_0}}\cdot\mathbf{B}(\mathbf{x_0}) = \frac{1}{4\pi}\nabla_{\mathbf{x_0}}\cdot\int_V \frac{\boldsymbol{\omega}(\mathbf{x})}{r}\,\mathrm{d}V(\mathbf{x}) = \frac{1}{4\pi}\int_V \nabla_{\mathbf{x_0}}\cdot\Big(\frac{\boldsymbol{\omega}(\mathbf{x})}{r}\Big)\,\mathrm{d}V(\mathbf{x})$$

$$= \frac{1}{4\pi}\int_V \boldsymbol{\omega}(\mathbf{x})\cdot\nabla_{\mathbf{x_0}}\Big(\frac{1}{r}\Big)\,\mathrm{d}V(\mathbf{x})$$

$$= \frac{1}{4\pi}\int_V \nabla_{\mathbf{x}} \cdot \left(\frac{\boldsymbol{\omega}(\mathbf{x})}{r}\right) \mathrm{d}V(\mathbf{x}).$$

Using the divergence theorem, we have

$$\nabla_{\mathbf{x_0}} \cdot \mathbf{B}(\mathbf{x_0}) = \frac{1}{4\pi}\int_S \frac{1}{r}\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \mathrm{d}S(\mathbf{x}),$$

where S is the surface containing the volume V. So we need $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on the boundary S (see Batchelor p. 86 for a discussion).

Then the velocity distribution is given by

$$\mathbf{u} = \frac{1}{4\pi}\int_V \nabla_{\mathbf{x_0}} \times \frac{\boldsymbol{\omega}(\mathbf{x})}{r}\, \mathrm{d}V(\mathbf{x}).$$

5. Exact solutions of the Navier-Stokes equations

Exact solutions to the Navier-Stokes equations which may be written down in closed form are very rare. Most problems are far too complicated to expect to be able to write down the solution in a simple way. Instead approximate solutions to the equations have to be found numerically on a computer.

However, there are a number of very well-known examples of exact solutions which can be found fairly easily. Often these are obtained by reducing the full equations using some symmetry or other special property of the problem. We begin with a discussion of flows which only move in one direction.

Keep in mind that we are always trying to find solutions to the Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -\nabla p/\rho + \nu \nabla^2 \mathbf{u}, \tag{5.1}$$

with boundary conditions appropriate to the particular problem in hand.

Unidirectional flows

These are very special types of flow for which there is only one component of velocity. We can write

$$\mathbf{u} = w\mathbf{k}$$

so that the fluid is moving with speed w in the z direction *only*.

The continuity equation gives us some useful information about w straightaway. In Cartesians,

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

but for this unidirectional flow, $u = v = 0$. So we have

$$\frac{\partial w}{\partial z} = 0$$

which states that w does not change as we move along in the z direction. Hence w can only be a function of x and y, i.e. $w(x, y)$.

This observation leads to the following remarkable property of unidirectional flows. Consider the non-linear term in the Navier-Stokes equations (5.1),

$$\begin{aligned}
\mathbf{u} \cdot \nabla \mathbf{u} &= (w\mathbf{k} \cdot \nabla)(w\mathbf{k}) \\
&= w\frac{\partial}{\partial z}(w\mathbf{k}) \\
&= 0
\end{aligned}$$

since neither w nor $\mathbf{k}$ depend on z. So the nonlinear term vanishes! This leaves us with a much simpler, linear problem to solve.

To summarize, for a unidirectional flow, we have to solve

$$\mathbf{u} = w\mathbf{k}, \qquad w = w(x, y), \qquad \frac{\partial \mathbf{u}}{\partial t} \; = \; -\nabla p/\rho + \nu \nabla^2 \mathbf{u}, \tag{5.2}$$

together with appropriate boundary conditions.

Steady examples of unidirectional flows

We start with the simpler case of steady flow. Now we need to solve

$$\mathbf{u} = w\mathbf{k}, \qquad w = w(x, y), \qquad \mathbf{0} \; = \; -\nabla p + \mu \nabla^2 \mathbf{u}. \tag{5.3}$$

In component form,

$$\begin{aligned} 0 &= -p_x + \mu(u_{xx} + u_{yy}), \\ 0 &= -p_y + \mu(v_{xx} + v_{yy}), \\ 0 &= -p_z + \mu(w_{xx} + w_{yy}). \end{aligned}$$

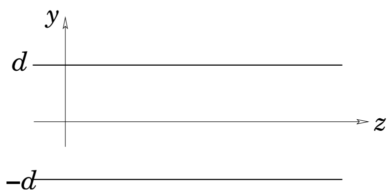
Since $u = v = 0$, the first two of these equations tells us that

$$p_x = p_y = 0.$$

So there are no pressure gradients in the x or y directions.

i) Plane Poiseuille Flow (PPF)

Consider two-dimensional (y, z) flow in a channel with centre line z and with walls placed a distance $2d$ apart at $y = \pm d$. In this case $u = 0$ and there is no dependence on x.



Suppose that fluid is driven along the channel by a pressure gradient of size $-G$, where $G > 0$, i.e.

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -G.$$

We try looking for a unidirectional solution with $w = w(y)$ in the z direction and $v = 0$ in the y direction. Then, from (5.3), we need to solve

$$0 = G + \mu \frac{\mathrm{d}^2 w}{\mathrm{d}y^2}, \qquad w(d) = w(-d) = 0. \tag{5.4}$$

The boundary conditions are those of no-slip at the walls $y = d$ and $y = -d$. Note that the normal flow condition $v = 0$ at $y = \pm d$ is satisfied automatically since $v \equiv 0$ everywhere.

Integrating (5.4) twice with respect to y, we find

$$Ay + B = \frac{1}{2} G y^2 + \mu w$$

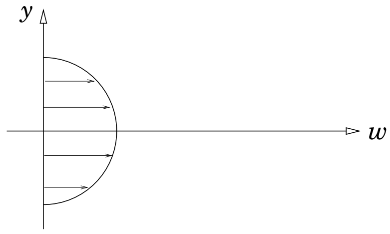
for arbitrary constants A, B. These are set by applying the boundary conditions:

$$\begin{aligned} w(d) = 0 \implies Ad + B &= \frac{1}{2}Gd^2, \\ w(-d) = 0 \implies -Ad + B &= \frac{1}{2}Gd^2 \end{aligned}$$

Solving for A and B we obtain the solution.

$$w = \frac{G}{2\mu}(d^2 - y^2). \tag{5.5}$$

It is customary to plot the *velocity profile* with the velocity on the horizontal axis like this:



The maximum velocity $w_{max} = Gd^2/2\mu$ occurs on the centre line $y = 0$.

Flux

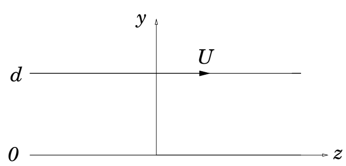
One feature of the flow we might be interested in is the volume flow rate or *flux* through the channel. This follows from integrating the velocity profile across the channel width. So the flux

$$\begin{aligned} Q = \int_{-d}^{d} w \, \mathrm{d}y &= \frac{G}{2\mu}\int_{-d}^{d}(d^2 - y^2)\,\mathrm{d}y \\ &= \frac{2Gd^3}{3\mu}. \end{aligned}$$

This represents the volume of fluid passing through a vertical line across the channel in unit time.

ii) *Plane Couette Flow*

Consider flow in a channel whose upper wall is moving at speed U in the z-direction.



In this case the fluid flow is driven purely by the upper wall motion. We expect a unidirectional solution of the form $w = w(y)$. In that case, equations (5.3) imply

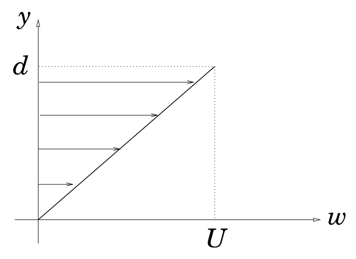
$$0 = \frac{\mathrm{d}^2 w}{\mathrm{d}y^2}.$$

Integrating twice, we have

$$w = Ay + B,$$

for constants A, B. The boundary conditions are $w(0) = 0$ and $w(d) = U$. Thus

$$A = \frac{U}{d}, \qquad B = 0.$$



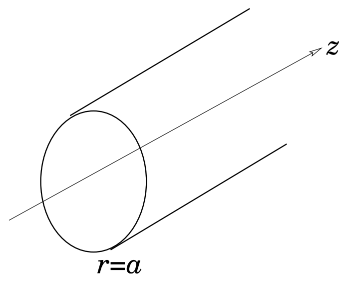
So

$$w = Uy/d,$$

and the velocity profile looks like this:

iii) *Circular Poiseuille Flow (CPF)*

This is also known as Hagen-Poiseuille flow after the work of Hagen (1839) and Poiseuille (1840). In this case we consider pressure driven flow through an infinite tube of circular cross-section of radius a. For



convenience we use cylindrical polar coordinates, taking

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -G, \qquad G > 0$$

as the pressure gradient driving the fluid flow.

The continuity equation states that

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0.$$

But, since $w = w\mathbf{k}$ and $u = v = 0$,

$$\frac{\partial w}{\partial z} = 0.$$

Therefore we just have $w = w(r, \theta)$. Assuming no variation in θ we take $w = w(r)$.

So now (5.3) requires us to solve

$$0 = G + \mu\Big(\frac{\mathrm{d}^2 w}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\Big), \qquad w(a) = 0. \tag{5.6}$$

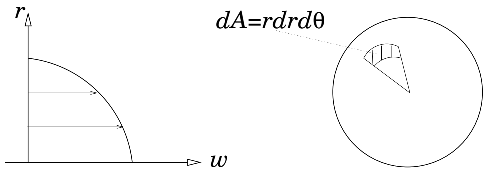
The boundary condition at the wall is that of no-slip. We will need to apply another condition at $r = 0$ later on.

We can re-write (5.6) as

$$0 = G + \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}w}{\mathrm{d}r}\Big). \tag{5.7}$$

Integrating once with respect to r,

$$A = \frac{1}{2}Gr^2 + \mu\, r\frac{\mathrm{d}w}{\mathrm{d}r},$$



for constant A. And again,

$$A \log r + B = \frac{1}{4}Gr^2 + \mu w.$$

To prevent w from diverging at $r = 0$, we need to set $A = 0$. This is the second boundary condition alluded to above. The constant B is fixed by requiring $w(a) = 0$, so

$$B = \frac{1}{4}Ga^2.$$

So the solution for CPF is

$$w = \frac{G}{4\mu}(a^2 - r^2). \tag{5.8}$$

The maximum velocity $w_{max} = Ga^2/4\mu$ occurs on the centre line $r = 0$.

Flux

To compute the flux through the circular tube, we once more integrate the velocity over the cross-section. Recall that an element of area in the cross-section is $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$. So,

$$\begin{aligned} Q = \int_0^{2\pi}\int_0^a w\, r\mathrm{d}r\mathrm{d}\theta &= 2\pi\frac{G}{4\mu}\int_0^a r(a^2 - r^2)\,\mathrm{d}r \\ &= \frac{\pi G a^4}{8\mu}. \end{aligned}$$

Furthermore, we define the mean velocity,

$$u_{mean} = \frac{\text{Flux}}{\text{x-sec area}} = \frac{Q}{\pi a^2} = \frac{Ga^2}{8\mu}.$$

Shear stress

To calculate the axial shear stress at the wall, recall that $F_i = \sigma_{ij}n_j$. In this case the unit normal at the wall pointing *into the fluid* is $\mathbf{n} = -\hat{\mathbf{r}}$. So the stress in the z direction at the wall is

$$F_z = -\sigma_{zr} = -2\mu e_{zr}.$$

We know that $e_{rz} = e_{zr}$ and we are given that

$$\begin{aligned} \mathrm{e}_{rz} &= \frac{1}{2}\Big(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\Big) \\ \text{(in this case)} &= -\frac{Gr}{4\mu}. \end{aligned}$$

Therefore, at the wall,

$$F_z\Big|_{r=a} = -2\mu e_{rz}|_{r=a} = \frac{Ga}{2}, \tag{5.9}$$

which is the shear stress exerted by the fluid on the wall in the downstream (i.e. z) direction.

iv) Indented circular Poiseuille Flow

Berker (1963)[6] discusses an exact solution of the Navier-Stokes equations corresponding to pressure-driven unidirectional flow along a circular pipe whose cross-section is a circle indented with part of another circle. Specifically, the domain of flow in the cross section lies in

$$b < r < 2a\cos\theta,$$

for constants a, b, where (r, θ) are polar coordinates. Note that $r = 2a\cos\theta$ describes a circle of radius a with centre at $x = a$. In Cartesians the domain of flow lies inside the circle

$$(x - a)^2 + y^2 = a^2,$$

and outside the circle

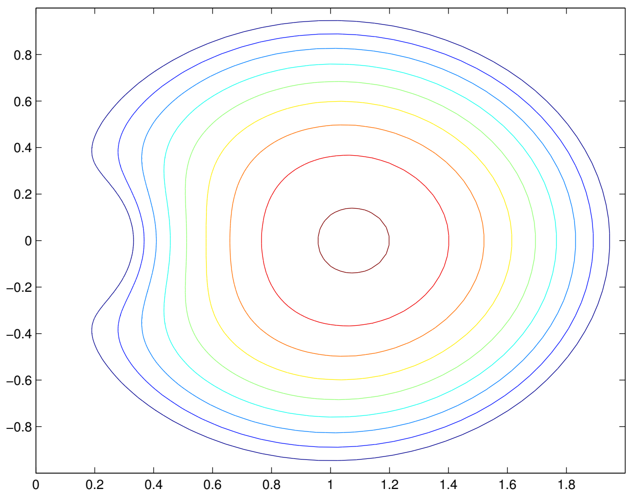
$$x^2 + y^2 = b^2.$$

The solution which satisfies the Poisson equation $\nabla^2 u = -1$ is

$$u = \frac{1}{2}(r^2 - b^2)\left(\frac{2a\cos\theta}{r} - 1\right),$$

where the first factor in brackets may be identified as the solution for Poiseuille flow in a circular pipe. Note that the first factor is identically zero on the smaller circle of radius b and the second factor is identically zero on the larger circle of radius a. In this sense the solution resembles that for undirectional flow in an equilateral triangular pipe where each factor is zero of one the three walls.

A contour plot showing lines of constant velocity u is shown in the figure below for the case $a = 1$ and $b = 0.3$.



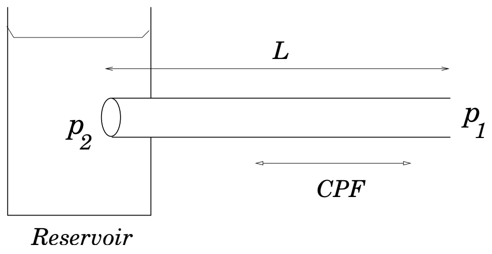
Notes

1. CPF is often used by medical workers as a model of blood flow in arteries. The shear stress is important because of its link with the onset of atherosclerosis, the disease responsible for about a third of deaths in Western society.

2. The result (5.9) shows that, for a fixed flux through the tube, the driving pressure gradient $G \propto a^{-4}$. Applying CPF to blood flow, this states that build-up of fatty deposits on the arterial walls due to an unhealthy lifestyle rapidly increases the amount of effort required of the heart to drive the same amount of blood through the body. If the artery's diameter is halved, for example, the pressure gradient needs to be 16 times larger to produce the same flux!

3. Experimentally, one might set up CPF as shown: A circular tube of length L has one end in a reservoir

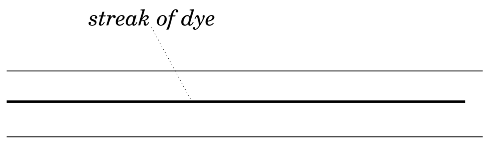
[6] Encyclopaedia of Physics, ed. S. Flugge, vol. VIII/2, Springer



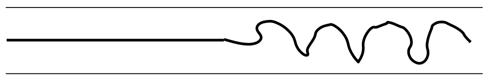
of water so that the entry pressure p_2 is held fixed. The pressure at the exit, p_1, is also constant. The pressure gradient is then

$$G = -\left(\frac{p_2 - p_1}{L}\right) = -\frac{\Delta p}{L}, \quad \text{say.}$$

This experiment was first performed by Osborne Reynolds in 1883. His original apparatus is kept in the Engineering department at Manchester University. Reynolds found that for sufficiently low flow rates Q, the CPF solution (5.8) was indeed observed. He injected a blob of dye at the entrance to the pipe and it stretched out in a perfectly straight line, as this solution predicts.



However, as Q was increased, Reynolds found that at some distance along the pipe the laminar CPF flow eventually broke up and thereafter became turbulent.

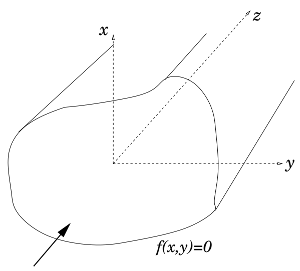


As Q was increased, the start of the turbulent region moved further and further upstream.

The reason for the break-up of the fluid flow in a circular pipe into turbulence is still not fully understood!

Steady unidirectional flow in pipes of arbitrary cross-section

Unidirectional solutions exist in pipes of general cross-section.



With a pressure gradient $-G$ in the z direction driving the flow, the problem to be solved is

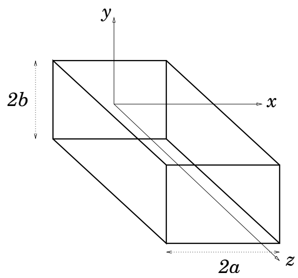
$$\nabla^2 u = -\frac{G}{\mu}, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

with

$$u = 0 \quad \text{on} \quad f(x, y) = 0.$$

For very special cross-sectional geometries (a circle, an ellipse, a triangle – see Sheet 3) the solution can be found in closed form. Otherwise a solution can be sought using a complex variable - conformal mapping technique. Alternatively, we can seek a series solution, as in the next example.

Flow in a rectangular channel



Consider steady undirectional flow in a channel of rectangular cross-section

$$-a \leq x \leq a, \qquad -b \leq y \leq b,$$

driven in the z direction by a pressure gradient $\mathrm{d}p/\mathrm{d}z = -G$.

We need to solve

$$u_{xx} + u_{yy} = -\frac{G}{\mu}, \quad \text{with} \quad u = 0 \quad \text{on} \quad x = \pm a, \quad y = \pm b. \tag{5.10}$$

If we look for a separable solution of type $u(x, y) = X(x)Y(y)$, we find that the velocity can be expressed in the series form

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos\left[(2m+1)\frac{\pi x}{2a}\right] \cos\left[(2n+1)\frac{\pi y}{2b}\right], \tag{5.11}$$

where the A_{mn} are constants to be found.

Exercise 1: Determine the A_{mn} as follows. Substitute (5.11) into (5.10). Then, multiplying both sides by

$$\cos\left[(2m+1)\frac{\pi x}{2a}\right] \cos\left[(2n+1)\frac{\pi y}{2b}\right],$$

integrate over the rectangular cross-section.

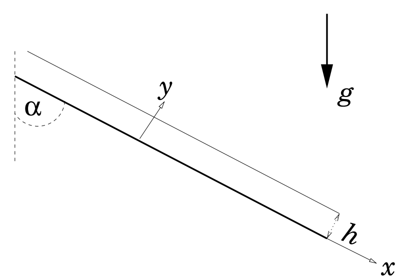
The result is:

$$A_{mn} = \frac{64G(-1)^{m+n}}{(2m+1)(2n+1)} \left\{ \frac{(2m+1)^2}{a^2} + \frac{(2n+1)^2}{b^2} \right\}^{-1}.$$

Exercise 2: Show that the channel solution above reduces to plane Poiseuille flow (PPF) in the limit $b/a \to 0$.

Film flow down an inclined plane

Consider the two-dimensional flow of a film of fluid of height h under gravity down an inclined plane:



Fix Cartesian axes as shown. The body force, $\mathbf{F}$, due to gravity is given by

$$\mathbf{F} = (F_x,\ F_y) = g(\cos\alpha,\ -\sin\alpha).$$

The pressure at $y = h$ is equal to atmospheric, p_a. We seek a steady unidirectional solution with $u = u(y)$, $v = 0$.

First, check that continuity is satisfied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0.$$

From before, we know that for unidirectional flow $\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{0}$. The momentum equations are therefore,

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \rho g \cos\alpha + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}, \\ 0 &= -\frac{\partial p}{\partial y} - \rho g \sin\alpha. \end{aligned}$$

Integrating the second,

$$p = -\rho g y \sin\alpha + f(x).$$

But $p = p_a$ at $y = h$ and so

$$p = p_a + (h - y)\rho g \sin\alpha.$$

From this we can see that

$$\frac{\partial p}{\partial x} = 0.$$

Therefore, the first momentum equation becomes

$$0 = \rho g \cos\alpha + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}.$$

Integrating twice,

$$u = -\frac{\rho g y^2}{2\mu} \cos\alpha + \beta y + \gamma,$$

for constants β, γ.

No-slip at $y = 0 \implies \gamma = 0$.

At $y = h$ there is a fluid-fluid interface between the water (say) in the film and the air above it. The stress condition at $y = h$ is given by (assuming negligible surface tension)

$$[\sigma_{ij}^{(air)} - \sigma_{ij}^{(water)}]\, n_j \; = \; 0,$$

where

$$\mathbf{n} = (0, 1, 0) = \mathbf{j}.$$

So,

$$-p_a\, n_i + 2\mu^{(air)} e_{ij}^{(air)}|_{y=h}\, n_j = -p|_{y=h}\, n_i + 2\mu\, e_{ij}|_{y=h}\, n_j,$$

i.e.

$$p|_{y=h} = p_a,$$

since $e_{22} = \partial v / \partial y = 0$, and

$$2\mu^{(air)} e_{12}^{(air)}|_{y=h} = 2\mu\, e_{12}|_{y=h}.$$

But, since air is much less viscous than water,

$$\mu^{(air)} \ll \mu,$$

the last equation becomes just

$$e_{12}|_{y=h} = 0,$$

i.e.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}y}\right|_{y=h} = 0.$$

Exercise: Follow a similar argument to above to derive the shear stress condition at the surface of a liquid film flowing down the outside of a cylindrical rod of radius a.

Applying the condition of zero surface stress, we find

$$0 = -\frac{\rho g h \cos\alpha}{\mu} + \beta,$$

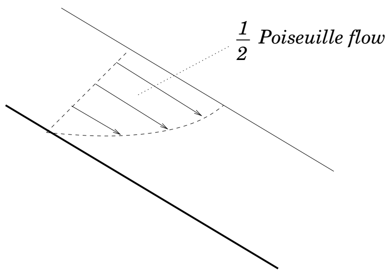
and so

$$u = \frac{\rho g y}{2\mu}(2h - y)\cos\alpha,$$

which can be re-written

$$u = \frac{\rho g}{2\mu}\left[h^2 - (h-y)^2\right]\cos\alpha.$$

This is the Nusselt solution. It represents 'half' of plane Poiseuille flow.



The maximum speed,

$$u_{max} = \frac{\rho g h^2}{2\mu}\cos\alpha.$$

The flux,

$$\begin{aligned} F &= \int_0^h u \, \mathrm{d}y = \frac{\rho g \cos\alpha}{2\mu}\int_0^h (2hy - y^2)\,\mathrm{d}y \\ &= \frac{\rho g h^3 \cos\alpha}{3\mu}. \end{aligned}$$

In other words, given a flux of fluid F and slope angle α, a steady unidirectional motion is possible for a film of height

$$h = \left(\frac{3\mu F}{\rho g \cos\alpha}\right)^{\frac{1}{3}}.$$

What happens if F is suddenly increased by a small amount $\mathrm{d}F$?

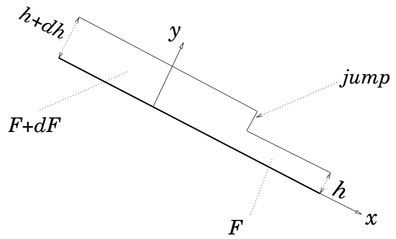
Since $\mathrm{d}F$ is small, we expect a corresponding increase in the layer thickness,

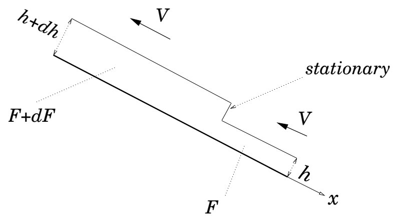
$$\mathrm{d}h = \frac{\mathrm{d}h}{\mathrm{d}F}\,\mathrm{d}F.$$

The jump travels downstream at a finite speed, V, so that, with axes fixed in the jump, we have the picture:

Continuity requires that

$$\text{net flux before jump} = \text{net flux beyond jump},$$





i.e.

$$(F + \mathrm{d}F) - (h + \mathrm{d}h)V = F - hV.$$

Rearranging, and taking the limit,

$$V = \frac{\mathrm{d}F}{\mathrm{d}h}.$$

From above,

$$\frac{\mathrm{d}F}{\mathrm{d}h} = \frac{\rho g h^2}{\mu} \cos\alpha = \Big(\frac{9gF^2 \cos\alpha}{\nu}\Big)^{\frac{1}{3}}.$$

But, we know that

$$u_{max} = \frac{\rho g h^2}{2\mu} \cos\alpha,$$

and so

$$V = 2u_{max}.$$

Thus, the speed of the jump is twice the maximum speed of the fluid.

Unsteady unidirectional flows

For an unsteady, unidirectional flow, we have to solve (5.2),

$$\mathbf{u} = w\mathbf{k}, \qquad w = w(x, y, t), \qquad \frac{\partial \mathbf{u}}{\partial t} \; = \; -\nabla p/\rho + \nu \nabla^2 \mathbf{u}.$$

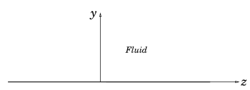
As in the steady case, the first and second momentum equations imply that

$$p_x = p_y = 0 \implies p = p(z, t).$$

1. Impulsively started flows - Rayleigh layers

a) Unbounded flow

Consider a semi-infinite mass of fluid at rest above a plane, rigid wall.



For $t < 0$, the wall is at rest. At $t = 0$ it instantaneously acquires a speed U in the z direction. There is no variation in z so all z derivatives are zero, $\partial/\partial z \equiv 0$. So $w = w(y,t)$ and, accordingly, continuity is satisfied. Since there is no imposed pressure gradient, $p = 0$ for all z and t.

The problem for w is thus:

$$\nu w_{yy} - w_t = 0,$$

$$\begin{aligned} w(0,t) &= U \quad \text{for} \quad t > 0, \\ w(y,t) &\to 0 \quad \text{as} \quad y \to \infty, \\ w(y,0) &= 0 \quad \text{for} \quad y > 0. \end{aligned}$$

If we write $w = UW(y,t)$, this becomes

$$\nu W_{yy} - W_t = 0,$$

$$\begin{aligned} W(0,t) &= 1 \quad \text{for} \quad t > 0, \\ W(y,t) &\to 0 \quad \text{as} \quad y \to \infty, \\ W(y,0) &= 0 \quad \text{for} \quad y > 0. \end{aligned}$$

We now make the observation that, for constant α, the transformation

$$y = \alpha\hat{y}, \qquad t = \alpha^2\hat{t}$$

leaves the equations and boundary conditions unchanged. This motivates us to seek a solution with the same property. We note that the grouping

$$\frac{y}{t^{1/2}}$$

is unchanged by the α-transformation. Accordingly, we try looking for a solution in the form

$$W = f(\eta), \quad \text{where} \quad \eta = \frac{y}{2(\nu t)^{1/2}}.$$

This is called a *similarity solution*, for reasons which will become clear below. It has the advantage of reducing the partial differential equation to an ordinary one. Applying the chain rule,

$$\frac{\partial}{\partial t} = -\frac{1}{2}\frac{\eta}{t}\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial y} = \frac{1}{2(\nu t)^{1/2}}\frac{\partial}{\partial \eta}, \quad \frac{\partial^2}{\partial y^2} = \frac{1}{4\nu t}\frac{\partial^2}{\partial \eta^2}.$$

Thus the equation becomes

$$f'' + 2\eta f = 0.$$

So

$$f' = A\mathrm{e}^{-\eta^2}, \quad \text{for} \quad \text{constant } A.$$

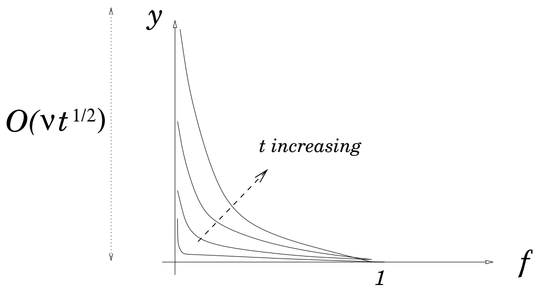
Then,

$$f = B + A\int_0^\eta \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi,$$

for constant B.

But, since $W = 1$ at $y = 0$, it follows that $f = 1$ at $\eta = 0$ and so $B = 1$. Also, since $W \to 0$ as $y \to \infty$, we have $f \to 0$ as $\eta \to \infty$, which means

$$A\int_0^\infty \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi = -1.$$



It is known that

$$\int_0^\infty \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi = \frac{\sqrt{\pi}}{2}.$$

Thus

$$f = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta),$$

where erf is the Error function. Plotting y against f:

Thus there is a layer of thickness $O((\nu t)^{1/2})$, in which the fluid is most affected by the wall motion. This is called a *Rayleigh layer*. The (y, f) profiles shown in the figure above are self-similar. This means that they are just re-scaled versions of each other. For example, we could multiply the first profile by a suitably-chosen and get it to lie directly on top of any of the other profiles.

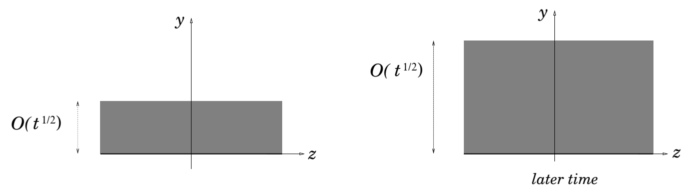
Considering the vorticity of this impulsively-driven flow,

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla \times (w\mathbf{k}) = \frac{\partial w}{\partial y}\mathbf{i} = \frac{U}{2(\nu t)^{1/2}}\frac{\mathrm{d}W}{\mathrm{d}\eta}\mathbf{i} = -\frac{U}{(\nu t)^{1/2}}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\eta^2}\mathbf{i}.$$

So,

$$\boldsymbol{\omega} = -\frac{U}{(\nu t)^{1/2}}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-y^2/(4\nu t)}\,\mathbf{i}.$$

Now, if $y^2 \gg 4\nu t$, the exponential is very small indeed and $\boldsymbol{\omega}$ is approximately zero. However, if $y^2 \sim 4\nu t$, the exponential is not small and $\boldsymbol{\omega}$ will have an appreciable value. So, as time progresses, measurable values of vorticity lie in the approximate y range $[0, 2(\nu t)^{1/2}]$. This range extends in time like the square root of t. Thus the vorticity (shown as a shaded area) diffuses away from the wall in time.



Thus, the vorticity behaves like heat diffusing away from an impulsively heated plate at $y = 0$.

Similarity solutions

We can now see why we called the previous solution a *similarity solution*. As time changes, the velocity profile is simply rescaled, but retains its original shape. Any profile at any given time may be simply rescaled to lie on top of any other profile.

A similarity solution is possible in a problem like this when there is *no natural lengthscale*. If the plate were finite we could choose its length, L, say, as a basis for non-dimensionalization. In this case, the plate is infinite and so we cannot do this. Consider the governing equation:

$$\nu W_{yy} = W_t.$$

We know that ν has dimensions $[\nu] = L^2T^{-1}$. Thus the dimensions of the LHS are

$$\frac{[\nu]}{[y^2]} = \frac{L^2}{T}\frac{1}{L^2} = \frac{1}{T}$$

and those of the RHS are

$$\frac{1}{[t]} = \frac{1}{T}.$$

A dimensional balance therefore gives

$$\frac{[\nu][t]}{[y^2]} = 1.$$

So the only possible dimensionless grouping of the variables is

$$\frac{y}{(\nu t)^{1/2}}$$

or any power thereof. So the solution must feature y, t and ν grouped *only* in this way.

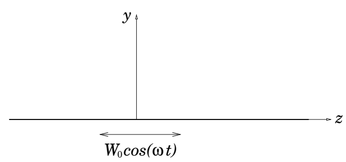
We can conclude that the fluid velocity must be expressed in the form

$$w = Wf(\eta), \qquad \eta = \frac{y}{(\nu t)^{1/2}},$$

a similarity solution with similarity variable η.

2. Oscillating flows - Stokes layers

Consider a similar flow to the above, but with the wall at $y = 0$ now oscillating in its own plane at a fixed frequency.



Again, it is reasonable to assume that the pressure gradient $G = 0$ and that the flow is unidirectional, with $w = w(y,t)$. So continuity is satisfied and the problem is

$$\nu w_{yy} = w_t,$$

with

$$w \to 0 \quad \text{as} \quad y \to \infty, \quad w = W_0 \cos(\omega t) \quad \text{on} \quad y = 0.$$

Since the frequency of the wall motion is ω and the problem is linear, we expect the frequency of the fluid velocity w also to be ω.

Thus we seek a solution by writing

$$w = W_0\, \Re\Big\{W(y)\mathrm{e}^{i\omega t}\Big\},$$

where $\Re$ means 'take the real part'.

Then $W(y)$ satisfies

$$\nu W'' - i\omega W = 0,$$

$$W = 1 \quad \text{on} \quad y = 0, \quad W \to 0 \quad \text{as} \quad y \to \infty.$$

So

$$W = A\mathrm{e}^{\sigma y} + B\mathrm{e}^{-\sigma y},$$

for constants A, B and where

$$\sigma = \sqrt{\frac{i\omega}{\nu}}.$$

Now, $\sqrt{i} = (1+i)/\sqrt{2}$ and so, to satisfy the infinity condition, we need to set $A = 0$. Thus,

$$W = B \exp\Big[-(1+i)y/(2\nu/\omega)^{1/2} \Big],$$

and so

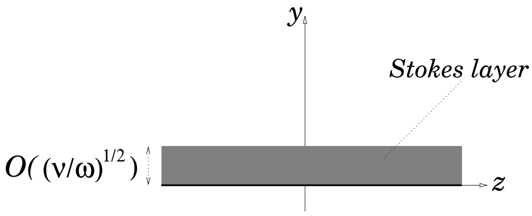
$$\frac{w}{W_0} = \mathrm{e}^{-\sqrt{\frac{\omega}{2\nu}}y} \cos\Big(\omega t - \sqrt{\frac{\omega}{2\nu}}y\Big).$$

This is called a *Stokes shear wave*.

The vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \;=\; \frac{\partial w}{\partial y}\mathbf{i} \;=\; -W_0\sqrt{\frac{\omega}{\nu}}\,\mathrm{e}^{-\sqrt{\frac{\omega}{2\nu}}y} \cos\Big(\omega t - \sqrt{\frac{\omega}{2\nu}}y + \frac{\pi}{4}\Big). \tag{5.12}$$

According to the exponentially decaying term in (5.12), most of the vorticity is confined in a region of thickness $O(\sqrt{\nu/\omega})$. This region is called a *Stokes layer*. The higher the frequency, the thinner the Stokes layer. Note that the Stokes layer thickness does not grow in time. This contrasts with the growing Rayleigh layer studied above.



NOTE: The wall shear stress is proportional to

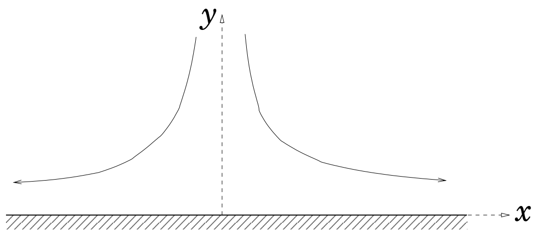
$$\left.\frac{\partial w}{\partial y}\right|_{y=0} \quad\propto\quad \cos\omega t - \sin\omega t \;=\; \sqrt{2}\cos(\omega t + \pi/4).$$

Therefore there is a phase difference of $\pi/4$ between the velocity w and the wall shear.

Other exact solutions

1. Stagnation-point (Hiemenz) flow.

Consider the flow of a viscous fluid towards a plane wall as shown in the diagram.



First consider the flow as if it were perfectly inviscid and irrotational. Introducing a two-dimensional stream function ψ, defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x},$$

the inviscid, irrotational flow is governed by Laplace's equation

$$\nabla^2\psi = 0.$$

The boundary condition is that of no normal flow at $y = 0$, namely

$$v = 0 \quad \text{on} \quad y = 0 \quad \text{or} \quad \psi = 0 \quad \text{on} \quad y = 0.$$

By inspection, we see that the solution is

$$\psi = kxy,$$

for some constant k.

Thus,

$$u = kx, \qquad v = -ky.$$

Clearly this solution does not satisfy the no slip condition at $y = 0$. To manage to do this, we need to incorporate viscous effects.

By analogy with the inviscid solution, we seek a solution to the viscous flow problem of a similar form. Specifically, we write

$$u = xf'(y), \qquad v = -f(y),$$

where f is a function to be determined. With this choice, we note that the continuity equation is satisfied straightaway, since

$$u_x + v_y = f' - f' = 0.$$

To determine f, we need to solve the x-momentum equation,

$$uu_x + vu_y = -\frac{1}{\rho}p_x + \nu[u_{xx} + u_{yy}].$$

However, we don't yet know the form of the pressure. To resolve this issue, consider the y-momentum equation,

$$uv_x + vv_y = -\frac{1}{\rho}p_y + \nu[v_{xx} + v_{yy}],$$

which becomes,

$$p_y = -\rho[ff' + f''] \equiv Q(y), \text{ say.}$$

$$\implies p = \overline{p}(x) + \rho\int_0^y Q(\xi)\,\mathrm{d}\xi,$$

for function of integration $\overline{p}(x)$.

Rearranging the x momentum equation, we find

$$p_x = \rho x[\nu f''' + ff'' - f'^2].$$

Combining these results, we deduce that

$$\overline{p} = \frac{1}{2}\rho x^2[\nu f''' + ff'' - f'^2] + p_0,$$

for constant p_0. Since $\overline{p} = \overline{p}(x)$, we have that

$$\nu f''' + ff'' - f'^2 = \beta, \tag{5.13}$$

for some constant β.

Now, as $y \to \infty$, we expect this solution to approach the inviscid flow calculated above. Thus we require

$$f' \to k \quad \text{as} \quad y \to \infty,$$

in which case $u \to ky$ as $y \to \infty$. Letting $t \to \infty$ in (5.13), we find that

$$\beta = -k^2.$$

Therefore, f must satisfy the system

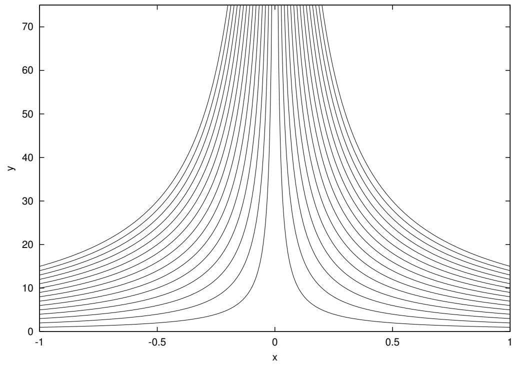
$$\nu f''' + ff'' + k^2 - f'^2 = 0,$$

$$f = f'(0) = 0 \quad \text{on} \quad y = 0; \quad f' \to k \quad \text{as} \quad y \to \infty.$$

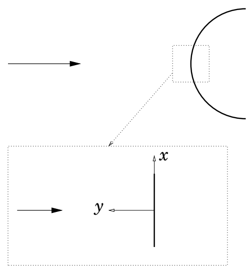
The first two boundary conditions are those of no normal flow and no slip at the wall.

NOTES

1. The solution for f must be found numerically.
2. The streamlines for this flow look like this:



3. The stagnation point flow on a plane wall is also relevant to that occuring on a curved, bluff body.



In the neighbourhood of the stagnation point, the body may be considered locally flat. For small x, the external inviscid flow may be expanded in a power series

$$U(x) = u_0 x + u_1 x^2 + u_2 x^3 + \cdots ,$$

and so locally $U \propto x$.

Exact Navier-Stokes solutions in cylindrical polar coordinates

If we express the Navier-Stokes equations in cylindrical polar coordinates (see sheet), we can make considerable simplifications by assuming that the flow in question is *axisymmetric*. This means that nothing changes with θ, i.e.

$$\frac{\partial}{\partial \theta} \equiv 0.$$

So, for an axisymmetric flow, the velocity field is given by

$$\mathbf{u} = u(r, z, t)\,\hat{\mathbf{r}} + w(r, z, t)\,\mathbf{k}.$$

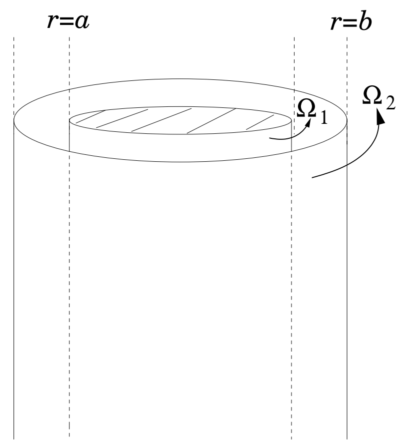
For an axisymmetric flow with *swirl*,

$$\mathbf{u} = u(r, z, t)\,\hat{\mathbf{r}} + v(r, z, t)\,\hat{\boldsymbol{\theta}} + w(r, z, t)\,\mathbf{k}.$$

We have already seen an example of an exact axisymmetric solution to the Navier-Stokes equations. This was circular Poiseuille flow (CPF). Next we look at an axisymmetric swirling flow.

Steady axisymmetric swirling flow between coaxial cylinders

Fluid fills the annular gap between two infinite coaxial circular cylinders. The inner cylinder rotates at angular speed Ω_1, and the outer at Ω_2.



Since the flow is steady and axisymmetric,

$$\frac{\partial}{\partial t} \equiv 0, \qquad \frac{\partial}{\partial \theta} \equiv 0,$$

and since it is just swirling around, with no dependence on the z coordinate,

$$\mathbf{u} = v(r)\,\hat{\boldsymbol{\theta}}.$$

Thus the streamlines for this flow are concentric circles.

The equations of motion are

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \qquad 0 = \frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r^2}.$$

Solving the second of these,

$$v = Ar + \frac{B}{r},$$

for constants A, B.

Thus the solution is a combination of a rigid-body rotation

$$\mathbf{u} = r\,\hat{\boldsymbol{\theta}},$$

and a line vortex

$$\mathbf{u} = \frac{1}{r}\,\hat{\boldsymbol{\theta}}.$$

The boundary conditions are

$$\begin{aligned} v &= a\Omega_1 \quad \text{on} \quad r = a \implies a\Omega_1 = Aa + \frac{B}{a}, \\ v &= b\Omega_2 \quad \text{on} \quad r = b \implies b\Omega_2 = Ab + \frac{B}{b}. \end{aligned}$$

Thus,

$$A = \frac{b^2\Omega_2 - a^2\Omega_1}{b^2 - a^2}, \quad B = a^2b^2\,\frac{\Omega_2 - \Omega_1}{a^2 - b^2}.$$

The vorticity

$$\boldsymbol{\omega} \;=\; \nabla \times \mathbf{u} \;=\; 2\,\frac{b^2\Omega_2 - a^2\Omega_1}{b^2 - a^2}\,\mathbf{k},$$

and is therefore constant.

What is the couple on the inner cylinder? [Recall that couple (or torque or moment) means the turning effect of a force acting at a distance from a given point.]

To determine this, we will need $\sigma_{\theta r}$ at $r = a$. We know that (see sheet),

$$e_{\theta r} = \frac{1}{2} r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v}{r}\right) = -\frac{B}{r^2}.$$

Thus,

$$\sigma_{\theta r} = -\frac{2\mu B}{r^2}.$$

Hence, the couple or torque per unit length in the z direction is given by

$$\tau_a = \int_0^{2\pi} (a\sigma_{\theta r}|_{r=a})\, a\, \mathrm{d}\theta = -4\pi\mu B.$$

So,

$$\tau_a = 4\pi\mu a^2 b^2\,\frac{\Omega_2 - \Omega_1}{b^2 - a^2}.$$

Similarly, the couple on the outer cylinder is found to be

$$\tau_b = -4\pi\mu a^2 b^2\,\frac{\Omega_2 - \Omega_1}{b^2 - a^2}.$$

Special cases:

1. *Equal rotation rates*

In this case $\Omega_1 = \Omega_2$ and so $B = 0$, $A = \Omega_1$. This means the fluid swirls round as if it were just a rotating solid body,

$$v = r\Omega_1.$$

Note that in this case, the couples on the inner and outer cylinder both vanish.

2. *No inner cylinder*

Now $a = 0$ and so we must choose $B = 0$ to avoid a singularity at $r = 0$. Again, we just have solid body rotation,

$$v = r\Omega_2.$$

3. *Stationary outer cylinder: Circular Couette flow*

If $\Omega_2 = 0$, we get

$$\frac{v}{r} = \frac{a^2\Omega_1}{b^2 - a^2}\left(\frac{b^2}{r^2} - 1\right).$$

Letting $b \to \infty$,

$$v = \frac{a^2\Omega_1}{r}.$$

This flow is irrotational since

$$\boldsymbol{\omega} = \frac{1}{r}\frac{\partial(rv)}{\partial r}\,\mathbf{k} = \mathbf{0}.$$

Exercise: Show that taking the double limit $a \to \infty$ and $b \to \infty$ such that the difference $(b - a)$ is held constant, we recover plane Couette flow (PCF).

Steady radial axisymmetric flow with swirl between coaxial cylinders

We can generalize the previous problem to include a radial component of flow, so

$$\mathbf{u} = u(r)\,\hat{\mathbf{r}} + v(r)\,\hat{\boldsymbol{\theta}},$$

but still everything is independent of θ. We continue to assume the flow is also independent of z.

The continuity equation implies,

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru) = 0,$$

and so

$$ru = \text{constant}.$$

Suppose that $u = -U$ at $r = a$, with $U > 0$, so there is suction of fluid at the inner cylinder. Then

$$u = -\frac{aU}{r},$$

and, at $r = b$,

$$u = -\frac{aU}{b},$$

so there is injection of fluid at the outer cylinder.

The r-momentum equation states that

$$u\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} + \nu\Big(\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2}\Big),$$

while the θ-momentum equation has

$$u\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{uv}{r} = \nu\Big(\frac{\mathrm{d}^2v}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r^2}\Big).$$

Substituting for u in the second equation,

$$\frac{\mathrm{d}^2v}{\mathrm{d}r^2} + \Big(1 + \frac{Ua}{\nu}\Big)\frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \Big(1 - \frac{Ua}{\nu}\Big)\frac{v}{r^2} = 0.$$

This is an equation for $v(r)$. Since u is already known, once v is determined, we can in principle solve the r-momentum equation to obtain the pressure p.

Defining a Reynolds number based on the suction velocity, $R = Ua/\nu$, we can re-write the v equation as

$$\frac{\mathrm{d}^2v}{\mathrm{d}r^2} + (1+R)\frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - (1-R)\frac{v}{r^2} = 0.$$

Seeking a solution of the form $v = kr^n$, we get

$$n(n-1) + n(1+R) - (1-R) = n^2 + Rn + (R-1) = (n+1)(n + [R-1]) = 0,$$

and so

$$n = -1,\ (1-R).$$

In the special case of $R = 2$, n has a repeated root of -1.

So there are two possibilities:

$$
\begin{aligned}
v &= \frac{C}{r} + Dr^{(1-R)} \quad \text{if} \quad R \neq 2, \\
v &= \frac{C}{r} + D\frac{\ln r}{r} \quad \text{if} \quad R = 2.
\end{aligned}
$$

The constants C and D come from the boundary conditions,

$$
\begin{aligned}
v &= a\Omega_1 \quad \text{on} \quad r = a, \\
v &= b\Omega_2 \quad \text{on} \quad r = b.
\end{aligned}
$$

We consider the special case $\Omega_2 = 0$ and $R \neq 2$. Then

$$
a\Omega_1 = \frac{C}{a} + Da^{(1-R)},
$$

$$
0 = \frac{C}{b} + Db^{(1-R)}.
$$

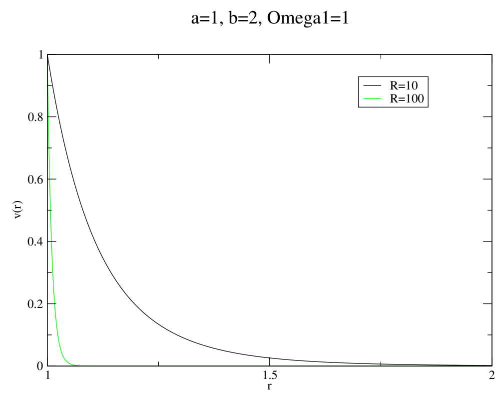
Then,

$$
\begin{aligned}
v &= \frac{a^2\Omega_1}{b^{2-R} - a^{2-R}}\left[\frac{b^{2-R}}{r} - r^{1-R}\right] \\
&= \frac{a^2\Omega_1}{r}\left[\frac{1-(r/b)^{2-R}}{1-(a/b)^{2-R}}\right] \\
&= \frac{a^2\Omega_1}{r}\left[\frac{1-(b/r)^{R-2}}{1-(b/a)^{R-2}}\right].
\end{aligned}
$$

Consider what happens when the Reynolds number becomes large, i.e. $R \gg 1$. Then,

$$
v \approx \frac{a^2\Omega_1}{r}\left(\frac{a}{r}\right)^{R-2} = a\Omega_1\left(\frac{a}{r}\right)^{R-1}
$$

Since $r > a$ and R is very large, this is essentially zero unless r is very close to a. In other words, the only significant swirl occurs very near to the inner cylinder.



The figure shows that the solution goes from near zero to one over a very short distance. This becomes more severe as R gets larger.

This is our first example of a *boundary layer*, a very narrow region over which there is a very rapid change in the solution. We will deal with these in more detail later in the course.

Spin down in a circular cylinder

Make a cup of coffee and stir it round a few times. How long does it take for the coffee to come to rest?

We consider a simplified version of this problem, that of how long it takes swirling fluid in an infinite cylinder to stop once the forcing is removed. Suppose viscous fluid occupies the cylinder $r = a$ and suppose that both the cylinder and the fluid are initially rotating with angular velocity Ω, so

$$\mathbf{u} = v\,\hat{\boldsymbol{\theta}}, \qquad v = \Omega\, r \quad \text{for} \quad t < 0.$$

This is the special case discussed above of rigid body rotation with no inner cylinder.

At $t = 0$ the cylinder is suddenly stopped. The ensuing *unsteady* fluid motion is governed by the θ-momentum equation

$$\frac{\partial v}{\partial t} = \nu\Big(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\Big),$$

with

$$v = 0 \quad \text{on} \quad r = a \quad \text{for} \quad t > 0.$$

We can seek a separable solution to this problem, writing

$$v = G(t)H(r).$$

Thus,

$$\dot{G}H = \nu\Big(GH'' + \frac{1}{r}GH' - \frac{1}{r^2}GH\Big),$$

so

$$\frac{H''}{H} + \frac{1}{r}\frac{H'}{H} - \frac{1}{r^2} = \frac{1}{\nu}\frac{\dot{G}}{G}.$$

The left hand side is purely a function of r, the right hand side of t. So, we can set

$$\frac{H''}{H} + \frac{1}{r}\frac{H'}{H} - \frac{1}{r^2} = -\lambda^2,$$

for separation constant λ. Thus,

$$\dot{G} = -\nu\lambda^2 G,$$

which has solution

$$G = \mathrm{e}^{-\nu\lambda^2 t}.$$

This leaves

$$H'' + \frac{1}{r}H' + \Big(\lambda^2 - \frac{1}{r^2}\Big)H = 0,$$

or

$$r^2H'' + rH' + (\lambda^2 r^2 - 1)H = 0.$$

This is Bessel's equation of order 1 (see MTH-2C23). It has the general solution

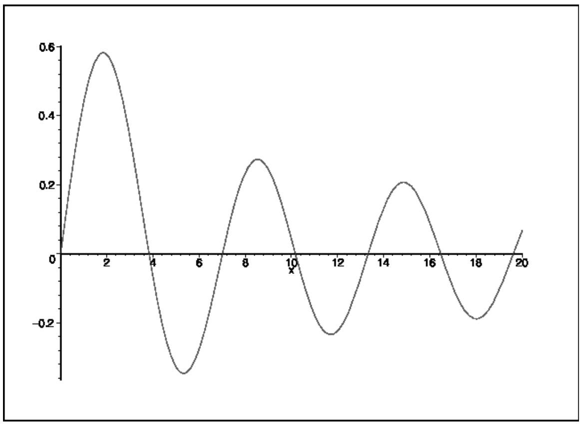
$$H(r) = AJ_1(\lambda r) + BY_1(\lambda r),$$

where A, B are constants and J_1, Y_1 are Bessel functions.

Now Y_1 is singular at $r = 0$, so we must reject it by setting $B = 0$.

To satisfy the boundary condition at $r = a$, we need

$$H(a) = 0 \implies J_1(\lambda a) = 0,$$



disregarding the trivial solution provided by $A = 0$.

The Bessel function $J_1(x)$ looks like this:

Label the zeroes β_n. The first few are:

$$\beta_1 \approx 3.83, \quad \beta_2 \approx 7.02, \quad \beta_3 \approx 10.17.$$

Therefore pick values of λ equal to β_n/a.

Defining $\lambda_n = \beta_n/a$, the general solution is given by

$$v = \sum_{n=1}^{\infty} A_n J_1(\beta_n r/a)\, \mathrm{e}^{-\nu\beta_n^2 t/a^2},$$

for constants A_n.

We can exploit the orthogonality of the Bessel functions to find the A_n. We know that

$$\int_0^a r J_1(\lambda_n r) J_1(\lambda_m r)\, \mathrm{d}r = 0 \quad \text{if} \quad n \neq m.$$

The initial condition is that $v = \Omega r$ at $t = 0$ and so, using the general solution,

$$\Omega r = \sum_{n=1}^{\infty} A_n J_1(\lambda_n r).$$

Multiplying both sides by $r J_1(\lambda_n)$, integrating from 0 to a, and using the orthogonality property above, we find

$$\Omega \int_0^a r^2 J_1(\lambda_n r)\, \mathrm{d}r = A_n \int_0^a r[J_1(\lambda_n r)]^2\, \mathrm{d}r.$$

Thus,

$$A_n = \frac{\Omega \int_0^a r^2 J_1(\lambda_n r)\, \mathrm{d}r}{\int_0^a r[J_1(\lambda_n r)]^2\, \mathrm{d}r}.$$

We can use the particular properties of Bessel functions to simplify this to

$$A_n = -\frac{2a\Omega}{\beta_n J_0(\beta_n)},$$

where J_0 is a Bessel function of zeroth order.

So,

$$v(r,t) = -2a\Omega \sum_{n=1}^{\infty} \frac{J_1(\beta_n r/a)}{\beta_n J_0(\beta_n)}\, \mathrm{e}^{-\nu\beta_n^2 t/a^2},$$

When $t > 0$, the infinite sum is dominated by its first term,

$$v(r,t) = -2a\Omega \frac{J_1(\beta_1 r/a)}{\beta_n J_0(\beta_1)}\, \mathrm{e}^{-\nu\beta_1^2 t/a^2},$$

where $\beta_1 \approx 3.83$.

We can see that v will have dropped to $1/e \approx 0.37$ of its initial value when

$$t = t_e = a^2/(\nu\beta_1^2).$$

So by this time, the swirling fluid has significantly slowed.

Going back to the cup of coffee, if we take $a = 4$cm and $\nu = 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$ (the value for water), we get

$$t_e \approx 2 \text{ minutes},$$

which seems rather long. Experiment suggests that in fact the coffee slows to $1/e$ of its initial value in about 15 seconds. The discrepancy lies in the crucial role played by a boundary layer (see later lectures) of fluid on the bottom of the cup (see Acheson pp. 45, 165, 284).

6. Stokes flow (creeping flow)

We have shown that the steady flow of a Newtonian, incompressible fluid of kinematic viscosity ν and density ρ satisfies the Navier-Stokes equations (written here without a body force),

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -\nabla p/\rho + \nu \nabla^2 \mathbf{u}. \tag{6.1}$$

Stokes flow, sometimes referred to as *creeping flow*, describes the motion of an extremely viscous or sticky fluid such as treacle or golden syrup. Under these conditions, we might expect

$$|\mathbf{u} \cdot \nabla \mathbf{u}| \ll |\nabla^2 \mathbf{u}|,$$

since the flow is very sluggish and so the magnitude of the velocity at any point is small. Therefore, since the left hand side is quadratic in the small velocity, we would expect it to be smaller than the term on the right hand side as a reasonable approximation.

Equations of Stokes flow

Under the aforementioned conditions, we may approximate the fluid motion using the *Stokes equations*

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u}. \tag{6.2}$$

We may derive these equations more formally as follows. Suppose that characteristic length and velocity scales are L and U respectively. For example, we may wish to model a bacterium swimming at speed U, where the length of the bacterium is L. Then, we nondimensionalize (i.e. remove all dimensions from the problem) by writing

$$u = U\,u^*, \quad v = U\,v^*, \quad p = \frac{\mu U}{L}\,p^*, \quad x = L\,x^*, \quad y = L\,y^*.$$

Then all quantities with an asterisk superscript are dimensionless. In component form, the momentum and continuity equations for a 2D steady flow with no body force acting are:

$$\begin{aligned} uu_x + vu_y &= -\frac{1}{\rho}\,p_x + \nu(u_{xx} + u_{yy}), \\ uv_x + vv_y &= -\frac{1}{\rho}\,p_y + \nu(v_{xx} + v_{yy}), \\ u_x + v_y &= 0. \end{aligned}$$

Substituting in, we obtain (using the chain rule to convert the derivatives):

$$\begin{aligned}
\frac{U^2}{L}\left(u^* u^*_{x^*} + v^* u^*_{y^*}\right) &= -\frac{\nu U}{L^2}\, p^*_{x^*} + \frac{\nu U}{L^2}(u^*_{x^*x^*} + u^*_{y^*y^*}), \\
\frac{U^2}{L}\left(u^* v^*_{x^*} + v^* v^*_{y^*}\right) &= -\frac{\nu U}{L^2}\, p^*_{y^*} + \frac{\nu U}{L^2}(v^*_{x^*x^*} + v^*_{y^*y^*}), \\
u^*_{x^*} + v^*_{y^*} &= 0.
\end{aligned}$$

Multiplying by L/U^2 and dropping the asterisks for convenience, we have

$$\begin{aligned}
uu_x + vu_y &= \frac{\nu}{UL}\left(-p_x + u_{xx} + u_{yy} \right), \\
uv_x + vv_y &= \frac{\nu}{UL}\left(-p_y + v_{xx} + v_{yy} \right), \\
u_x + v_y &= 0.
\end{aligned}$$

Defining the Reynolds number R such that

$$R = \frac{UL}{\nu} \tag{6.3}$$

and multiplying through by R, we have

$$\begin{aligned}
R(uu_x + vu_y) &= -p_x + u_{xx} + u_{yy}, \\
R(uv_x + vv_y) &= -p_y + v_{xx} + v_{yy}, \\
u_x + v_y &= 0.
\end{aligned}$$

Formally taking the limit $R \to 0$, we derive the Stokes equations of creeping fluid motion:

$$\begin{aligned}
0 &= -p_x + u_{xx} + u_{yy}, \\
0 &= -p_y + v_{xx} + v_{yy}, \\
u_x + v_y &= 0.
\end{aligned}$$

Or, in vector form, as we had above,

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u}.$$

The key step is taking the limit $R \to 0$, which means we are considering flows for which the Reynolds number is small enough for the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ to be neglected. By the definition of R,

$$R = \frac{UL}{\nu}.$$

This means that for $R \ll 1$, we need U to be small, or L to be small, or for ν to be large. So Stokes flow arises if

- The characteristic velocity scale is sufficiently small (e.g., a bubble moving through treacle)
- The characteristic length scale is sufficiently small (e.g., a swimming spermatozoa)
- The kinematic viscosity, ν, is large (e.g., golden syrup)

If the scales U, L, and ν combine to make R small, we may use the Stokes equations (6.2) to approximate the fluid flow. The great advantage of doing this lies in the fact that the Stokes equations are *linear*. In general, linear problems are easier to solve than nonlinear ones.

Alternative forms of the Stokes equations

Remembering the identity

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}),$$

we may re-write the Stokes equations (6.2) like this,

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla p = -\mu \nabla \times \boldsymbol{\omega} \tag{6.4}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity.

Taking the curl of the second of equations (6.4), we find

$$\mathbf{0} = \nabla \times (\nabla \times \boldsymbol{\omega}),$$

and hence we derive the equations of vorticity transport for a Stokes flow,

$$\nabla \cdot \boldsymbol{\omega} = 0, \qquad \mathbf{0} = \nabla^2 \boldsymbol{\omega}. \tag{6.5}$$

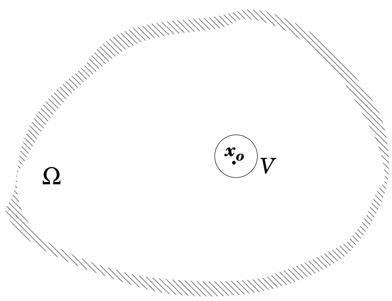
Instead, taking the divergence of the second of equations (6.4), we see that, for a Stokes flow, p satisfies Laplace's equation,

$$\nabla^2 p = 0, \tag{6.6}$$

and so the pressure is a *harmonic function*.

Points of maximum/minimum pressure

An interesting consequence of this is as follows. Consider Stokes flow in a closed region, Ω, say. Then,



since p is a harmonic function, its maxima and minima can *only* occur on the boundary of Ω.

Extrema of harmonic functions

Consider a harmonic function $\phi(\mathbf{x})$ satisfying, by definition, Laplace's equation

$$\nabla^2 \phi = 0$$

at all points $\mathbf{x}$ inside some bounded region D with boundary curve C.

[Sketch region D]

Green's third identity in two-dimensions states that, at a point $\mathbf{x}_0$ in D,

$$\phi(\mathbf{x}_0) = \frac{1}{2\pi} \int_C \ln r \, \nabla \phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \mathrm{d}l(\mathbf{x}) + \frac{1}{2\pi} \int_C \frac{1}{r^2} (\mathbf{x}_0 - \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \phi(\mathbf{x}) \, \mathrm{d}l(\mathbf{x}), \tag{6.7}$$

where $\mathbf{n}(\mathbf{x})$ is the *unit normal* at a point $\mathbf{x}$ on the curve C, and $\mathrm{d}l(\mathbf{x})$ is the increment of arc length along C at the point $\mathbf{x}$.

Choose C to be a circle of radius a centred about the point $\mathbf{x}_0$. Then (6.7) gives

$$\phi(\mathbf{x}_0) = \frac{\ln a}{2\pi} \int_C \nabla \phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \mathrm{d}l(\mathbf{x}) + \frac{1}{2\pi a^2} \int_C (\mathbf{x}_0 - \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \phi(\mathbf{x}) \, \mathrm{d}l(\mathbf{x}).$$

Using the divergence theorem on the first integral,

$$\phi(\mathbf{x}_0) = \frac{\ln a}{2\pi} \int_D \nabla^2 \phi(\mathbf{x}) \, \mathrm{d}S(\mathbf{x}) + \frac{1}{2\pi a^2} \int_C (\mathbf{x}_0 - \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \phi(\mathbf{x}) \, \mathrm{d}l(\mathbf{x}),$$

where $\mathrm{d}S$ is an element of surface area. But $\nabla^2\phi = 0$. Also, since C is a circle, $\mathbf{n} = (\mathbf{x}_0 - \mathbf{x})/a$. Thus

$$(\mathbf{x}_0 - \mathbf{x}) \cdot \mathbf{n} = \frac{1}{a}(\mathbf{x}_0 - \mathbf{x}) \cdot (\mathbf{x}_0 - \mathbf{x}) = \frac{a^2}{a} = a.$$

Hence we are left with

$$\phi(\mathbf{x}_0) = \frac{1}{2\pi a}\int_C \phi(\mathbf{x})\,\mathrm{d}l(\mathbf{x}). \tag{6.8}$$

This is *Gauss' mean value theorem* for harmonic functions in two-dimensions. It states that ϕ at a point $\mathbf{x}_0$ is the average of the ϕ values over a surrounding circle of radius a.

PROPOSITION: Maxima or minima of harmonic functions in a bounded region can only occur on the boundary.

Proof

Assume that an extremum is located at $\mathbf{x}_0$ inside the domain of interest. Using (6.8), it is clear that there must be at least one point on any surrounding circle (lying within the domain) where the value of ϕ is higher or lower than $\phi(\mathbf{x}_0)$ in order for the mean value to be $\phi(\mathbf{x}_0)$. This contradicts the original assumption. Hence extrema can only arise on the boundary.

To see this, suppose that a maximum or minimum of p exists at a point $\mathbf{x}_0$ within Ω. If we integrate equation (6.6) over any small volume V of surface area S with outward unit normal $\mathbf{n}$ around this point, we find

$$0 = \int_V \nabla^2 p\,\mathrm{d}V \quad = \quad \int_S \nabla p \cdot \mathbf{n}\,\mathrm{d}S \quad \text{(by the divergence theorem.)} \tag{6.9}$$

Clearly $\nabla p \cdot \mathbf{n}$ cannot be single-signed over the arbitrary surface S. It follows that p must be increasing or decreasing arbitrarily close to the point $\mathbf{x}_0$, contradicting the supposition that it is either a maximum or a minimum. Therefore the points of maximum or minimum pressure in a closed-region Stokes flow can only occur on the boundary. $\diamond$

Nifty move: This observation hints at a more profound characteristic of Stokes flows. In fact it is possible to describe a Stokes flow *entirely* by the values the various flow variables assume on the boundary. This leads to a very efficient means of computing Stokes flows, known as the *boundary integral method*. For more information on this advanced procedure, see Pozrikidis, 'Boundary Integral and Singularity Methods', Cambridge (1992).

Two-dimensional Stokes flow

If the flow in question is two-dimensional with velocity field $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$, we may introduce a stream function defined by

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}.$$

Calculating the vorticity, we obtain $\boldsymbol{\omega} = \nabla \times \mathbf{u} = -\nabla^2\psi\,\mathbf{k}$. Then, by equation (6.5), since the unit vector $\mathbf{k}$ is constant, we find that the stream function satisfies the *biharmonic* equation

$$\nabla^4\psi = 0. \tag{6.10}$$

We will make use of this feature of two-dimensional Stokes flows later when we study flow in corners.

Uniqueness theorem for Stokes flow

Consider Stokes flow in a region V bounded by a closed surface S with unit outward normal $\mathbf{n}$. If $\mathbf{u}$ is prescribed on the boundary S, then the equations of Stokes flow have at most one solution.

Proof

Suppose that there exist velocity fields $\mathbf{u}$ and $\mathbf{u}'$ satisfying

$$\nabla \cdot \mathbf{u} = 0, \qquad \mu\nabla^2\mathbf{u} = \nabla p$$

and

$$\nabla \cdot \mathbf{u}' = 0, \qquad \mu\nabla^2\mathbf{u}' = \nabla p'$$

with $\mathbf{u} = \mathbf{u}'$ on the boundary S.

Let e_{ij}, e'_{ij} be the stress tensors associated with $(\mathbf{u},\, p)$ and $(\mathbf{u}',\, p')$ respectively.

Then,

$$\begin{aligned}
\int_V (e_{ij} - e'_{ij})e_{ij}\,\mathrm{d}V &= \frac{1}{2}\int_V \Big[\frac{\partial}{\partial x_j}(u_i - u'_i) + \frac{\partial}{\partial x_i}(u_j - u'_j)\Big]e_{ij}\,\mathrm{d}V \\
\text{(since } e_{ij} \text{ symmetric)} &= \int_V \frac{1}{2}\Big[\frac{\partial}{\partial x_j}(u_i - u'_i)\Big]e_{ij} + \frac{1}{2}\Big[\frac{\partial}{\partial x_j}(u_i - u'_i)\Big]e_{ij}\,\mathrm{d}V \\
&= \int_V e_{ij}\frac{\partial}{\partial x_j}(u_i - u'_i)\,\mathrm{d}V \\
&= \int_V \frac{\partial}{\partial x_j}[(u_i - u'_i)e_{ij}]\,\mathrm{d}V - \int_V (u_i - u'_i)\frac{\partial e_{ij}}{\partial x_j}\,\mathrm{d}V \\
\text{(by divergence theorem)} &= \int_S (u_i - u'_i)e_{ij}n_j\,\mathrm{d}S - \int_V (u_i - u'_i)\frac{\partial e_{ij}}{\partial x_j}\,\mathrm{d}V \\
\text{(since } u_i = u'_i \text{ on } S) &= -\int_V (u_i - u'_i)\frac{\partial e_{ij}}{\partial x_j}\,\mathrm{d}V.
\end{aligned}$$

$$\begin{aligned}
&= -\int_V (u_i - u'_i)\frac{1}{2}\Big[\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{\partial}{\partial x_i}\Big(\frac{\partial u_j}{\partial x_j}\Big)\Big]\,\mathrm{d}V \\
\text{(by continuity)} &= -\int_V (u_i - u'_i)\frac{1}{2}\frac{\partial^2 u_i}{\partial x_j^2}\,\mathrm{d}V \\
(\clubsuit)\quad \text{(since } \mu\nabla^2\mathbf{u} = \nabla p) &= -\int_V (u_i - u'_i)\frac{1}{2\mu}\frac{\partial p}{\partial x_i}\,\mathrm{d}V \\
\text{(by continuity again)} &= -\frac{1}{2\mu}\int_V \frac{\partial}{\partial x_i}[p(u_i - u'_i)]\,\mathrm{d}V \\
\text{(by divergence theorem)} &= -\frac{1}{2\mu}\int_S p(u_i - u'_i)n_i\,\mathrm{d}S \\
\text{(since } u_i = u'_i \text{ on } S) &= 0.
\end{aligned}$$

Hence,

$$\int_V (e_{ij} - e'_{ij})e_{ij}\,\mathrm{d}V = 0. \tag{6.11}$$

Moreover, by a similar line of reasoning,

$$\int_V (e'_{ij} - e_{ij})e'_{ij}\,\mathrm{d}V = 0. \tag{6.12}$$

Adding (6.11) and (6.12),

$$\int_V (e'_{ij} - e_{ij})^2\,\mathrm{d}V = 0.$$

Therefore $e_{ij} = e'_{ij}$ and so $\mathbf{u}$ and $\mathbf{u}'$ differ only by a constant. However, we know that $\mathbf{u} = \mathbf{u}'$ on S and so it follows that $\mathbf{u} = \mathbf{u}'$ *everywhere* in V. QED $\diamond$.

Note the crucial step ($\clubsuit$) in the above proof. It is here we use the fact that we are dealing with a Stokes flow. The proof does not work for full Navier-Stokes flow.

Minimum dissipation theorem for Stokes flow

The rate of viscous dissipation in a Stokes flow is less than or equal to that in *any other* incompressible flow satisfying the same boundary conditions.

Proof

Let $\mathbf{u}$, p satisfy the Stokes equations (6.2) in a bounded region V with $\mathbf{u}$ prescribed on the boundary surface S. In addition, let $\mathbf{u}'$, p' satisfy $\nabla \cdot \mathbf{u}' = 0$, but assume that

$$\mu \nabla^2 \mathbf{u}' \neq \nabla p',$$

i.e. $(\mathbf{u}', p)$ does *not* represent a Stokes flow.

From above, we know that,

$$\int_V (e_{ij} - e'_{ij}) e_{ij} \, \mathrm{d}V = 0,$$

but now

$$\int_V (e'_{ij} - e_{ij}) e'_{ij} \, \mathrm{d}V \neq 0 \qquad \text{since} \quad \mu \nabla^2 \mathbf{u}' \neq \nabla p'.$$

Recall that the viscous dissipation for the flow $(\mathbf{u}', \ p')$ is defined as

$$\Phi' = 2\mu \int_V e'_{ij} e'_{ij} \, \mathrm{d}V.$$

We may re-write this in the following way.

$$\begin{aligned} \Phi' &= 2\mu \int_V (e'_{ij} - e_{ij})(e'_{ij} - e_{ij}) + 2(e'_{ij} - e_{ij}) e_{ij} + e_{ij} e_{ij} \, \mathrm{d}V \\ \left(\text{since } \textstyle\int_V (e_{ij} - e'_{ij}) e_{ij} \, \mathrm{d}V = 0\right) &= 2\mu \int_V (e'_{ij} - e_{ij})^2 + e_{ij}^2 \, \mathrm{d}V \end{aligned}$$

Clearly

$$\Phi' \geq 0.$$

The viscous dissipation for flow $(\mathbf{u}, \ p)$ is

$$\Phi = 2\mu \int_V e_{ij} e_{ij} \, \mathrm{d}V = 2\mu \int_V e_{ij}^2 \, \mathrm{d}V.$$

Hence,

$$\Phi' - \Phi = 2\mu \int_V (e'_{ij} - e_{ij})^2 \, \mathrm{d}V \geq 0.$$

QED $\diamond$

NOTES

1. Navier-Stokes flows always dissipate more energy than Stokes flows.

2. Since plane (PPF) and circular Poiseuille flow (CPF) happen to satisfy the Stokes equations, they are therefore minimum dissipation flows.

Reversibility of Stokes flow

One very striking feature of creeping flows is that they are *reversible*. Including a body force $\mathbf{F}$, the equations of motion are

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{0} = -\nabla p + \rho \mathbf{F} + \mu \nabla^2 \mathbf{u}, \tag{6.13}$$

so if we change $\mathbf{u}$ to $-\mathbf{u}$, p to $-p$ and $\mathbf{F}$ to $-\mathbf{F}$, we get another equally valid solution. If the boundary conditions involve $\mathbf{u}$ we must also switch the sign in these. This property is not valid for flows at non-zero Reynolds number, as the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ scuppers this sign-reversal process.

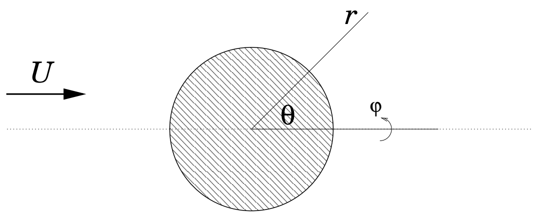
It follows that if we reverse the velocity on the boundary of a Stokes flow, we also reverse the flow velocity throughout the fluid.

PUZZLE: There is an interesting consequence of this feature of Stokes flows. A solid sphere moves slowly through a viscous fluid parallel to a plane wall. Does the sphere feel any force repelling it away from or attracting it towards the wall?

Answer: No; it can't. It if did, when we reversed the flow it would naturally feel the same force in the opposite direction. But the symmetry of the problem makes this ridiculous. Therefore the force acting on the sphere perpendicular to the wall must be zero.

Creeping flow past a solid sphere

This is the classical Stokes flow problem. Consider fluid flowing at low Reynolds number past a solid sphere of radius a, which is fixed inside an unbounded fluid.



Fix axes in the sphere and adopt spherical polar coordinates (r, θ, ϕ). The oncoming fluid moves at speed U and the entire flow is taken to be steady.

For this problem, it is convenient to work with the Stokes equations in the form

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \times (\nabla \times \boldsymbol{\omega}) = \mathbf{0},$$

since $\nabla^2 \boldsymbol{\omega} = \nabla \times (\nabla \times \boldsymbol{\omega})$.

The fluid velocity

$$\mathbf{u} = u\,\hat{\mathbf{r}} + v\,\hat{\boldsymbol{\theta}} + w\,\hat{\boldsymbol{\phi}}.$$

Exploiting the axisymmetry of the sphere, we can assume that there are no changes in ϕ,

$$\frac{\partial}{\partial \phi} \equiv 0, \qquad \text{and also} \quad w \equiv 0.$$

We can therefore introduce a stream function ψ (see earlier section on these), defined by

$$u = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \qquad v = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}.$$

In spherical polars, for vector function $\mathbf{F} = F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_\phi\,\hat{\boldsymbol{\phi}}$,

$$\nabla \times \mathbf{F} = \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(F_\phi \sin\theta)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(r F_\phi)\,\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r F_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}}.$$

So the vorticity,

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \Big\{ -\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\Big) - \frac{1}{r}\frac{\partial}{\partial\theta}\Big(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\Big)\Big\}\,\hat{\boldsymbol{\phi}},$$

or,

$$\boldsymbol{\omega} = \omega_\phi\,\hat{\boldsymbol{\phi}},$$

where

$$\omega_\phi = -\frac{D^2\psi}{r\sin\theta}, \qquad D^2\psi \equiv \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\Big).$$

Now,

$$\begin{aligned}\nabla\times\boldsymbol{\omega} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\omega_\phi\sin\theta)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(r\omega_\phi)\,\hat{\boldsymbol{\theta}}\\ &= -\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(D^2\psi)\,\hat{\mathbf{r}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(D^2\psi)\,\hat{\boldsymbol{\theta}}.\end{aligned}$$

But this has the same form as

$$\mathbf{u} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\,\hat{\mathbf{r}} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\,\hat{\boldsymbol{\theta}}.$$

So, straightaway we can write down

$$\begin{aligned}\nabla\times(\nabla\times\boldsymbol{\omega}) &= \nabla\times\Big\{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(-D^2\psi)\,\hat{\mathbf{r}} - \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(-D^2\psi)\,\hat{\boldsymbol{\theta}}\Big\}\\ &= -\frac{1}{r\sin\theta}D^2(-D^2\psi)\,\hat{\boldsymbol{\phi}}.\end{aligned}$$

Hence

$$\nabla\times(\nabla\times\boldsymbol{\omega}) = \mathbf{0} \implies D^2(D^2\psi) = 0 \implies D^4\psi = 0.$$

The boundary conditions are

$$\mathbf{u} = 0 \quad \text{on} \quad r = a, \qquad \mathbf{u} \to \text{free stream velocity} \quad \text{as} \quad r\to\infty.$$

i.e., the latter requires

$$u \to U\cos\theta, \qquad v \to -U\sin\theta \quad \text{as} \quad r\to\infty.$$

These mean that

$$\psi \to \frac{1}{2}Ur^2\sin^2\theta \quad \text{as} \quad r\to\infty, \tag{6.14}$$

and

$$\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial\theta} = 0 \quad \text{on} \quad r = a.$$

These boundary conditions suggest that we seek a solution of the form

$$\psi = f(r)\sin^2\theta.$$

In this case, the boundary conditions become

$$f = f' = 0 \quad \text{on} \quad r = a.$$

In this case,

$$\begin{aligned}D^2\psi &= \Big\{\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big)\Big\}f(r)\sin^2\theta\\ &= \Big(f'' - 2\frac{f}{r^2}\Big)\sin^2\theta\\ &= H(r)\sin^2\theta, \quad \text{where} \quad H(r) = f'' - 2\frac{f}{r^2}.\end{aligned}$$

So,

$$\begin{aligned} D^4\psi &= \Big(H'' - 2\frac{H}{r^2}\Big)\sin^2\theta \\ &= \Big[\Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r^2}\Big)^2 f\Big]\sin^2\theta. \\ &= 0. \end{aligned}$$

We try solutions of the form $f = r^n$. Thus,

$$\begin{aligned} \Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r^2}\Big)^2 r^n &= 0 \\ \text{i.e.} \quad \Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r^2}\Big)\Big(r^{n-2}[n(n-1)-2]\Big) &= 0 \\ \text{so} \quad r^{n-4}[(n-2)(n-3)-2][n^2-n-2] &= 0 \\ \implies (n^2-5n+4)(n^2-n-2) &= 0 \\ \text{so} \quad (n-4)(n-2)(n-1)(n+1) &= 0. \end{aligned}$$

Therefore $n = 1,\ 2,\ 4$ or -1. Hence

$$f = \frac{A}{r} + Br^2 + Cr + Dr^4,$$

for constants $A,\ B,\ C,\ D$.

The infinity condition (6.14) implies that

$$B = \frac{U}{2}, \qquad D = 0,$$

while

$$\begin{aligned} \frac{\partial\psi}{\partial\theta}\Big|_{r=a} = 0 &\implies \frac{A}{a} + \frac{Ua^2}{2} + Ca = 0, \\ \frac{\partial\psi}{\partial r}\Big|_{r=a} = 0 &\implies -\frac{A}{a^2} + Ua + C = 0, \end{aligned}$$

giving

$$A = \frac{Ua^3}{4}, \qquad C = -\frac{3}{4}Ua.$$

Thus,

$$\psi = \frac{1}{2}Ur^2\sin^2\theta\Big(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\Big),$$

or

$$\psi = \frac{1}{2}Ur^2\sin^2\theta\Big(1 - \frac{a}{r}\Big)^2\Big(1 + \frac{a}{2r}\Big).$$

Now we know ψ, we can compute the velocity and vorticity components:

$$\begin{aligned} r\sin\theta\,\omega_\phi &= -\Big(f'' - \frac{f}{r^2}\Big)\sin^2\theta \quad (= -D^2\psi) \\ &= -\frac{3}{2}\frac{Ua}{r}\sin^2\theta \\ \implies \omega_\phi &= -\frac{3Ua}{2r^2}\sin\theta. \end{aligned}$$

Also,

$$u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = U\cos\theta\Big(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\Big),$$

$$v \;=\; -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \;=\; -U\sin\theta\Big(1-\frac{3a}{4r}-\frac{a^3}{4r^3}\Big).$$

To calculate the fluid pressure, we make use of the alternative form of the momentum equation

$$\begin{aligned}\nabla p &= -\mu\nabla\times\boldsymbol{\omega}, \\ &= -\mu\Big\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\omega_\phi\sin\theta)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(r\omega_\phi)\,\hat{\boldsymbol{\theta}}\Big\}.\end{aligned}$$

So,

$$\begin{aligned}\frac{1}{\mu}\frac{\partial p}{\partial r} &= -\frac{1}{r\sin\theta}\Big(-\frac{3Ua}{r^2}\sin\theta\cos\theta\Big) \\ \Longrightarrow\ \frac{p}{\mu} &= -\frac{3Ua}{2r^2}\cos\theta+g(\theta), \quad \text{arbitrary } g,\end{aligned}$$

and,

$$\begin{aligned}\frac{1}{\mu}\frac{1}{r}\frac{\partial p}{\partial\theta} &= \frac{1}{r}\frac{\partial}{\partial r}(r\omega_\phi) \\ &= \frac{3Ua}{r^3}\sin\theta \\ \Longrightarrow\ \frac{p}{\mu} &= -\frac{3Ua}{2r^2}\cos\theta+f(r), \quad \text{arbitrary } f.\end{aligned}$$

Combining these,

$$p = -\frac{3\mu Ua}{2r^2}\cos\theta + \text{const.}$$

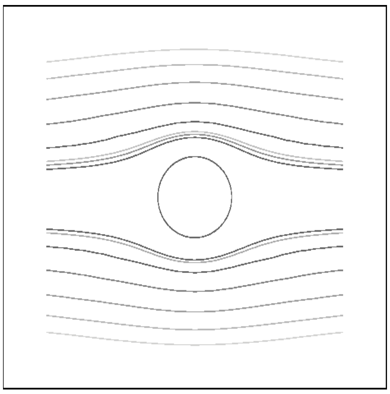
Letting $r\to\infty$, $p\to p_\infty$, its value at infinity. So,

$$p = -\frac{3Ua}{2r^2}\cos\theta + p_\infty.$$

NOTES:

1. The solution for ψ is symmetric about $\theta=\pi/2$. Therefore the streamlines are symmetric fore and aft the sphere.

Exercise: Use Maple to plot the streamlines around the sphere.



2. If we impose a velocity $-U$ on the solution, we can obtain results for a sphere moving through a fluid at rest.

$$\begin{aligned}\text{So}\quad \Psi &= \psi-\frac{1}{2}Ur^2\sin^2\theta \\ &= -\frac{3}{4}Uar\sin^2\theta\Big(1-\frac{a^2}{3r^2}\Big).\end{aligned}$$

Exercise: Use Maple to plot the streamlines for this flow.

Drag on a moving sphere

We can use the previous calculation to compute the drag on a solid sphere moving through a viscous fluid at low Reynolds number.

By symmetry, the drag force will only have a component in the direction of motion. Recalling the definition of the stress tensor,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

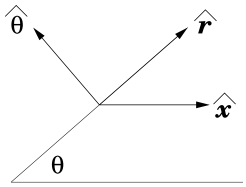
$$\begin{aligned} F_i = \sigma_{ij} n_j &= -pn_i + \left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\} n_j \\ &= -pn_i - \mu(\mathbf{n} \times \boldsymbol{\omega})_i \qquad \text{(see Problem Sheet 2)}. \end{aligned}$$

Here the subscript i means the ith Cartesian component.

But $\mathbf{n} = \hat{\mathbf{r}}$ and so

$$\begin{aligned} F_i = \sigma_{ij} n_j &= -pn_i - \mu([1,0,0] \times [0,0,\omega_\phi])_i \\ &= -pn_i - \mu[0, -\omega_\phi, 0]_i. \end{aligned}$$

To get the desired component, we need to take the dot product of $\mathbf{F}$ with $\mathbf{i}$, the unit vector pointing along the line of motion. Denote $\mathbf{i}$ as $\hat{\mathbf{x}} = (1,0,0)$ in a Cartesian set of axes. Then we get



$$\begin{aligned} (\mathbf{F} \cdot \hat{\mathbf{x}}) &= F_i\, \hat{x}_i = \sigma_{ij} n_j \hat{x}_i \\ &= -p\, n_i \hat{x}_i - \mu\, [0, -\omega_\phi, 0]_i\, \hat{x}_i \\ &= -p\cos\theta - \mu\, \omega_\phi \sin\theta \\ &= -(p_\infty - \frac{3Ua}{2r^2}\cos\theta)\cos\theta - \mu\, \omega_\phi \sin\theta. \end{aligned}$$

Thus, on $r = a$,

$$\mathbf{F} \cdot \hat{\mathbf{x}}|_{r=a} = -p_\infty \cos\theta + \frac{3\mu U}{2a}.$$

Therefore, the total force in the $\hat{\mathbf{x}}$ direction is:

$$\begin{aligned} & \int_{sphere} \Big(\frac{3\mu U}{2a} - p_\infty \cos\theta\Big)\, \mathrm{d}S \\ = \;& 4\pi a^2 \frac{3\mu U}{2a} - p_\infty \int_0^{2\pi}\int_0^{\pi} a^2 \sin\theta\cos\theta\, \mathrm{d}\theta\, \mathrm{d}\phi \end{aligned} \tag{6.15}$$

since the surface element $\mathrm{d}S = a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ in spherical polar coordinates. So the total x force is

$$4\pi a^2 \frac{3\mu U}{2a} - 2\pi\, p_\infty \int_0^{\pi} a^2 \sin\theta\cos\theta\, \mathrm{d}\theta$$

The integral in the second term is zero, and so the total drag on the sphere,

$$F_{tot} = 6\mu\pi a U.$$

Taking the Reynolds number to be $R = Ua/\nu$, we can define a dimensionless drag coefficient

$$C_D = \frac{F_{tot}}{\frac{1}{2}\rho U^2 \, \pi a^2} = \frac{12}{R}.$$

The agreement with experiment is good for $R < 1$ and very good for $R < \frac{1}{2}$.

Consistency analysis

Is the above solution for the sphere everywhere a consistent approximation to the correct solution of the full Navier-Stokes equations when $R \ll 1$? In other words, are the approximations leading to the Stokes equations valid for all values of r?

To answer this question, we examine the order of magnitude of the terms.

We have

$$\begin{aligned} u &= \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} = U\cos\theta\Big(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\Big), \\ v &= -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} = -U\sin\theta\Big(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\Big), \end{aligned}$$

and so,

$$\begin{aligned} u\frac{\partial u}{\partial r} &\sim O\Big(\frac{U^2 a}{r^2}\Big) \quad \text{for large r}, \\ \nu\frac{\partial^2 u}{\partial r^2} &\sim O\Big(\nu\frac{Ua}{r^3}\Big) \quad \text{for large r}. \end{aligned}$$

Recall that we neglected the nonlinear terms, $\mathbf{u}\cdot\nabla\mathbf{u}$, from the Navier-Stokes equations. So,

$$\begin{aligned} \frac{\text{terms neglected}}{\text{terms retained}} &\sim O\Big(\frac{U^2 a/r^2}{\nu U a/r^3}\Big) \\ &= O\Big(\frac{Ur}{\nu}\Big) \\ &= O\Big(\frac{Rr}{a}\Big). \end{aligned}$$

i.e., for any fixed Reynolds number, R, no matter how small, we can still find r sufficiently large such that the terms neglected are as important, or more important, than the retained viscous terms.

So, although the Stokes solution is consistent near the sphere for $0 < R \ll 1$, it **breaks down** as a valid approximation when

$$r = O\Big(\frac{a}{R}\Big).$$

This is a major problem, but at least it's not as bad as the equivalent flow past a circular cylinder, where the Stokes solution doesn't work at all (see Problem Sheet 4)!

The difficulty in the far-field was partially tied up by Oseen in 1910, and later more fully by Proudman & Pearson in 1957.

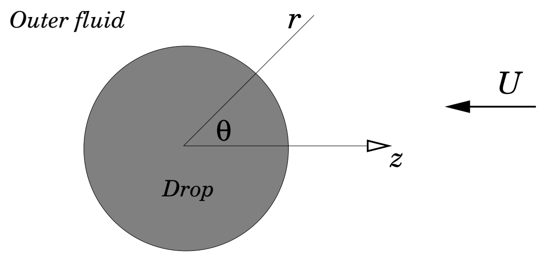
Motion of a liquid drop

Consider a spherical liquid drop of radius a and viscosity μ_d moving at speed U through another fluid of viscosity μ_o at low Reynolds number. The outer fluid cannot mix with that in the drop.

The axes are fixed in the moving drop, so the outer fluid appears to be flowing towards it at speed U.

For this problem we have to consider the flows inside and outside of the drop. In both cases, we assume that the conditions of Stokes flow are satisfied. Thus, the equations of motion are

$$\nabla\cdot\mathbf{u} = 0, \qquad \nabla\times(\nabla\times\boldsymbol{\omega}) = \mathbf{0}.$$



As for the solid sphere calculation above, we assume the flows are axisymmetric,

$$\frac{\partial}{\partial\phi} \equiv 0, \qquad \text{and also} \quad w \equiv 0.$$

We can therefore introduce a stream function ψ, defined by

$$u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

for velocity field

$$\mathbf{u} = u\,\hat{\mathbf{r}} + v\,\hat{\boldsymbol{\theta}}.$$

Following the results for the solid sphere, the equation of motion reduces to

$$D^4\psi = 0, \quad \text{where} \quad D^2\psi \equiv \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\Big).$$

Outside the drop

Label the stream function outside the drop ψ_o.

At infinity the flow consists of a uniform stream moving at speed U in the $-z$ direction. Thus,

$$\psi_o \to -\frac{1}{2}Ur^2\sin^2\theta \quad \text{as} \quad r\to\infty \qquad \text{(cf. solid sphere)}.$$

This suggests we look for a solution of the form

$$\psi_o = f(r)\sin^2\theta.$$

Then, from before,

$$D^4\psi = \Big[\Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r^2}\Big)^2 f\Big]\sin^2\theta = 0.$$

As above, we try solutions of the form $f = r^n$, finding that

$$f = \frac{A}{r} + Br^2 + Cr + Dr^4,$$

for constants $A,\ B,\ C,\ D$.

The infinity condition (6.14) implies that

$$B = -\frac{U}{2}, \qquad D = 0,$$

and so

$$f = \frac{A}{r} - \frac{1}{2}Ur^2 + Cr.$$

Before we can find A and C we need to consider the motion within the drop.

Inside the drop

Label the stream function inside the drop ψ_d.

Since the two fluids are immiscible, there must be no flow across the fluid-drop interface. Thus we require the normal velocities

$$u_d = u_o = 0 \quad \text{on} \quad r = a \implies \psi_d = \psi_o = 0 \quad \text{on} \quad r = a. \tag{6.16}$$

Also we will need the tangential velocities to be continous at the interface:

$$v_d = v_o = 0 \quad \text{on} \quad r = a \implies \frac{\partial \psi_d}{\partial r} = \frac{\partial \psi_o}{\partial r} \quad \text{on} \quad r = a. \tag{6.17}$$

In addition, we need the tangential stress at the surface to be continuous (see previous discussion on fluid-fluid boundary conditions). Thus we want

$$2\mu_d\, e^d_{r\theta} = 2\mu_o\, e^o_{r\theta} \quad \text{on} \quad r = a \implies \kappa \frac{\partial}{\partial r}\Big(\frac{1}{r^2}\frac{\partial \psi_d}{\partial r}\Big) = \frac{\partial}{\partial r}\Big(\frac{1}{r^2}\frac{\partial \psi_o}{\partial r}\Big), \tag{6.18}$$

where

$$\kappa = \mu_d/\mu_o \quad \text{is the viscosity ratio,}$$

since

$$e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\Big(\frac{v}{r}\Big) + \frac{1}{2r}\frac{\partial u}{\partial \theta}$$

in spherical polars, and u is everywhere zero on the drop's surface.

The form of these conditions at the interface suggest that inside the drop we also look for a solution of the form

$$\psi_d = f(r)\sin^2\theta.$$

Thus,

$$f = \frac{E}{r} + Fr^2 + Gr + Hr^4,$$

for constants $E,\ F,\ G,\ H$.

Now, using the definition of ψ, we have

$$u = \frac{2\cos\theta f(r)}{r^2}, \qquad v = -\frac{1}{r}\sin\theta f'(r).$$

Therefore, to avoid a singular velocity at $r = 0$, we need $E = G = 0$.

Condition (6.16) implies that

$$Fa^2 + Ha^4 = \frac{A}{a} - \frac{1}{2}Ua^2 + Ca = 0,$$

while condition (6.17) demands

$$f_d'(a) = f_o'(a) \implies 2aF + 4a^3H = -\frac{A}{a^2} - Ua + C,$$

and condition (6.18) requires

$$\kappa\Big[\frac{1}{r^2}f_d'\Big]' = \Big[\frac{1}{r^2}f_o'\Big]' \quad \text{on} \quad r = a$$

which implies

$$\kappa\Big(-\frac{2F}{a^2} + 4H\Big) = \frac{4A}{a^5} + \frac{U}{a^2} - \frac{2C}{a^3}.$$

Solving for the various constants we find

$$A = -\frac{a^3 U\kappa}{4(1+\kappa)}, \quad C = \frac{aU(2+3\kappa)}{4(1+\kappa)}, \quad F = \frac{U}{4(1+\kappa)}, \quad H = -\frac{U}{4a^2(1+\kappa)}.$$

Thus, we have

$$\begin{aligned} \psi_o &= -\frac{Ur^2\sin^2\theta}{2}\left[1 - \frac{a(2+3\kappa)}{2r(1+\kappa)} + \frac{a^3\kappa}{2r^3(1+\kappa)}\right] \quad \text{outside the drop,} \\ \psi_d &= \frac{Ur^2\sin^2\theta}{4(1+\kappa)}\left[1 - \frac{r^2}{a^2}\right] \qquad\qquad\qquad \text{in the drop.} \end{aligned}$$

The flow inside the drop is called *Hill's spherical vortex*.

Following a similar procedure to that for the rigid sphere, we can compute the pressure within each fluid and evaluate the drag on the drop as it moves with the fluid. Doing this calculation, we find that the drag

$$D = 4\pi\mu U a\left(\frac{1+\frac{3}{2}\kappa}{1+\kappa}\right).$$

Exercise: Verify the above formula for the drag on the moving drop.

As the fluid in the drop becomes more and more viscous, we would expect this result to approach that for the sphere. Indeed, taking the limit $\kappa \to \infty$, we obtain

$$\lim_{\kappa\to\infty} D_{drop} = 6\pi\mu U a = D_{sphere}.$$

Two-dimensional Stokes flow: Complex variable formulation

In two dimensions, the equations of Stokes flow can be reformulated and solved using complex variable methods. We first review some complex variable theory before discussing the formulation.

Complex Variable Theory

NOTATION: Ww begin by noting that some authors[7] use the notation

$$\overline{f}(z) \equiv \overline{f(\overline{z})}$$

DEFINITION: A function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x+iy$ if u and v have continuous partial derivatives u_x, u_y, v_x, v_y and satisfy the Cauchy-Riemann equations,

$$u_x = v_y, \qquad u_y = -v_x. \tag{6.19}$$

For a differentiable function, $f(z) = u + iv$, we have

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = u_x + iv_x = v_y - iu_y. \tag{6.20}$$

DEFINITION: A function $f(z)$ is **analytic** at a point z_0 if it is differentiable at z_0.

Example: The function $f(z) = \sin(z)$ is analytic in the entire complex z plane. Breaking it up into real and imaginary parts we have

$$f(z) = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\sinh y\cos x.$$

[7] See for example Tanveer & Vasconcelos (1995) Time-evolving bubbles in two-dimensional Stokes flow, *J. Fluid Mech.*, **301**, 325-344

So $u = \sin x \cosh y$ and $v = \sinh y \cos x$ and we have

$$u_x = \cos x \cosh y = v_y = \cosh y \cos x,$$

$$u_y = \sin x \sinh y = -v_x = \sin x \sinh y,$$

so that the Cauchy-Riemann equations are satisfied everywhere in the plane.

Example: The function $f(z) = \overline{z}$ is not analytic anywhere in the plane. In this case $u = x$ and $v = -y$ and we see that

$$u_x = 1 \neq v_y = -1,$$

so that the Cauchy-Riemann equations are not satisfied for any z.

Wirtinger calculus

Noting that use can be made of $\overline{z}$, we move on to the concept of a *Wirtinger derivative*. The two Wirtinger derivatives are defined to be

$$\frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{6.21}$$

$$\frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{6.22}$$

We can see how these arise by using the usual chain rule from calculus. Since $z = x + iy$ and $\overline{z} = x - iy$, we have that

$$x = \frac{1}{2}(z + \overline{z}), \qquad y = \frac{1}{2}(\overline{z} - z)i. \tag{6.23}$$

If we think of z and $\overline{z}$ as being independent, then the chain rule gives

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The Wirtinger derivatives can be shown to satisfy the usual rules of differentiation. For example, the product rule holds (see Problem Sheet). Let's consider a couple of examples of its use.

Example: Consider the complex function $f(z) = cz$, where c is in $\mathbb{C}$. Then we have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial c(x+iy)}{\partial x} - i\frac{\partial c(x+iy)}{\partial y}\right) = \frac{1}{2}(c - i(ic)) = c,$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial c(x+iy)}{\partial x} + i\frac{\partial c(x+iy)}{\partial y}\right) = \frac{1}{2}(c + i(ic)) = 0.$$

Exercise: For the function $f(z) = c\overline{z}$, where c in $\mathbb{C}$, show that

$$\frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial \overline{z}} = c.$$

Example: Consider the complex function $f(z) = 1/z$. Then, since

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2},$$

we have

$$\frac{\partial f}{\partial z}=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{x-iy}{x^2+y^2}\right)-i\frac{\partial}{\partial y}\left(\frac{x-iy}{x^2+y^2}\right)\right]=\frac{y^2-x^2+2ixy}{(x^2+y^2)^2}=-\frac{\overline{z}^2}{|z|^2}=-\frac{1}{z^2}.$$

and

$$\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{x-iy}{x^2+y^2}\right)+i\frac{\partial}{\partial y}\left(\frac{x-iy}{x^2+y^2}\right)\right]=0.$$

So we get what we expect to get.

These examples have illustrated a general truth: we may perform Wirtinger derivatives just as with normal partial differentiation. When differentiating with respect to z we treat $\overline{z}$ as constant, and when differentiating with respect to $\overline{z}$ we treat z as constant.

Analytic functions

If a complex function f satisfies

$$\frac{\partial f}{\partial \overline{z}}=0, \tag{6.24}$$

then we say that $f=f(z)$, i.e. f is independent of $\overline{z}$. Any function which satisfies (6.24) is analytic. To see this we note that if (6.24) holds then $f=f(z)=u+iv$ for some u, v. Using (6.22) we have

$$\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x}+i\frac{\partial(u+iv)}{\partial y}\right)=\frac{1}{2}(u_x-v_y)+\frac{1}{2}i(v_x+u_y)=0.$$

Thus, equating real and imaginary parts we see that the Cauchy-Riemann equations hold and the function is analytic.

Example: The function $f(z)=z$ is analytic. Evidently

$$\frac{\partial f}{\partial \overline{z}}=0,$$

since $\overline{z}$ does not appear in f.

Example: The function $f(z)=1/z$ is analytic, except at $z=0$. From the previous example,

$$\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{x-iy}{x^2+y^2}\right)+i\frac{\partial}{\partial y}\left(\frac{x-iy}{x^2+y^2}\right)\right]=0.$$

Note that the derivatives with respect to x and y inside the square brackets are not-defined at $x=y=0$ (i.e. z=0) and so f is not analytic at this point.

The following propositions will be useful:

PROPOSITION 1: For an analytic function $f(z)=u+iv$, we have

$$\frac{\partial}{\partial \overline{z}}\overline{f(z)}=\overline{f'(z)}. \tag{6.25}$$

Proof: Using (6.22) we have

$$\frac{\partial}{\partial \overline{z}}\overline{f(z)}=\frac{\partial}{\partial \overline{z}}(u-iv)=\frac{1}{2}\left[\frac{\partial}{\partial x}(u-iv)+i\frac{\partial}{\partial y}(u-iv)\right]=u_x-iv_x=\overline{u_x+iv_x}=\overline{f'(z)}.$$

Note that we have made use of the Cauchy-Riemann equations (6.19) in the penultimate step, and (6.20) in the ultimate step.

PROPOSITION 2: For an analytic function $f(z)=u+iv$, we have

$$\frac{\partial}{\partial z}\overline{f(z)}=0. \tag{6.26}$$

Proof: This follows on applying definition (6.21) to $\overline{f(z)} = u - iv$.

PROPOSITION 3: Laplace's equation may be expressed as

$$\nabla^2\psi = 4\frac{\partial^2\psi}{\partial z\partial\overline{z}}. \tag{6.27}$$

Proof: We factorise the differential operator and then simply apply the definitions (6.21) and (6.22). Specifically, we write

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi,$$

and then

$$\nabla^2\psi = 4\frac{\partial}{\partial z}\frac{\partial}{\partial\overline{z}}\,\psi,$$

as required.

Stokes flow: Complex variable formulation

In this section, we demonstrate how two-dimensional Stokes flow problems (including those with free boundaries) may be formulated using complex variable methods. The following references give further examples of the complex variable formulation for Stokes flow in a variety of contexts, and include two reference texts giving further details on the formulation.

1. Langlois (1964), Slow viscous flow, Macmillan.

2. Ockendon, H. & Ockendon, J. R., *Viscous flow*, Cambridge University Press.

3. Richardson, S. (1968), Two-dimensional bubbles in slow viscous flows, *J. Fluid Mech.*, **33**, 476-493.

4. Jeong, J-T. & Moffat, H. K. (1992) Free-surface cusps associated with flow at low Reynolds number, *J. Fluid Mech.*, **241**, 1-22.

5. Tanveer & Vasconcelos (1995) Time-evolving bubbles in two-dimensional Stokes flow, *J. Fluid Mech.*, **301**, 325-344.

6. Higley, Siegel & Booty (2012) Semi-analytical solutions for two-dimensional elastic capsules in Stokes flow, *Proc. Roy Soc. Lond. A*.

We return to two-dimensional Stokes flow. The general solution of the biharmonic equation $\nabla^4\psi = 0$ is given by

$$\psi = \mathrm{Im}[\,\overline{z}\,f(z) + g(z)\,]. \tag{6.28}$$

Here the functions $f(z)$ and $g(z)$ are analytic except possibly at a number of isolated points.

To find the velocity field, let's first define

$$F = \overline{z}\,f(z) + g(z). \tag{6.29}$$

Now we know that

$$\psi = \mathrm{Im}[F] = \frac{1}{2i}\left(F - \overline{F}\right).$$

We note that

$$\frac{\partial F}{\partial z} = \overline{z}f'(z) + g'(z), \qquad \frac{\partial F}{\partial\overline{z}} = f(z).$$

Also, $\overline{F} = z\,\overline{f(z)} + \overline{g(z)}$ so that

$$\frac{\partial\overline{F}}{\partial z} = \overline{f(z)}, \qquad \frac{\partial\overline{F}}{\partial\overline{z}} = z\overline{f'(z)} + \overline{g'(z)},$$

where we have used (6.25) and (6.26). Now, using definition (6.22),

$$\frac{\partial\psi}{\partial\overline{z}} = \frac{1}{2}\left(\psi_x + i\psi_y\right) = \frac{i}{2}\left(u + iv\right),$$

and so

$$u + iv = -2i\frac{\partial\psi}{\partial\overline{z}}.$$

Now,

$$\frac{\partial\psi}{\partial\overline{z}} = \frac{1}{2i}\left(\frac{\partial F}{\partial\overline{z}} - \frac{\partial\overline{F}}{\partial\overline{z}}\right) = \frac{1}{2i}\left(f(z) - z\,\overline{f'(z)} - \overline{g'(z)}\right).$$

So the velocity field is given by

$$u + iv = -f(z) + z\,\overline{f'(z)} + \overline{g'(z)}. \tag{6.30}$$

To obtain an expression for the vorticity, we first note that

$$-\omega = \nabla^2\psi = 4\frac{\partial^2\psi}{\partial z\partial\overline{z}}.$$

From above we have

$$\frac{\partial\psi}{\partial\overline{z}} = \frac{1}{2i}\left(f(z) - z\,\overline{f'(z)} - \overline{g'(z)}\right), \qquad \frac{\partial^2\psi}{\partial z\partial\overline{z}} = \frac{1}{2i}\left(f'(z) - \overline{f'(z)}\right) = \mathrm{Im}[f'(z)],$$

using the Proposition 1 (6.25). Thus

$$\omega = -\nabla^2\psi = -4\frac{\partial^2\psi}{\partial z\partial\overline{z}} = -4\mathrm{Im}[f'(z)].$$

To obtain an expression for the pressure, we work with the Stokes momentum equation **in the absence of any body forces**, namely

$$\mathbf{0} = -\nabla p + \mu\nabla^2\mathbf{u}.$$

The x-component of this is

$$0 = -p_x + \mu(u_{xx} + u_{yy}) = -p_x + \mu(\psi_{xxy} + \psi_{yyy}) = -p_x + \mu\frac{\partial\nabla^2\psi}{\partial y} = -p_x - \mu\omega_y,$$

since $\omega = -\nabla^2\psi$. Thus $p_x = -\mu\omega_y$. Working with the y component of the momentum equation, we derive similarly $p_y = \mu\omega_x$. Thus p and $\mu\omega$ satisfy the Cauchy-Riemann equations,

$$p_x = -\mu\omega_y, \qquad p_y = \mu\omega_x.$$

Consequently the function $p - i\mu\omega$ is analytic. Now, from above, $\omega = -4\mathrm{Im}[f'(z)]$. So,

$$\omega_y = -4\frac{\partial}{\partial y}\mathrm{Im}[f'(z)] = -4\frac{\partial}{\partial x}\mathrm{Re}[f'(z)]$$

by applying the Cauchy-Riemann equations for the analytic function $f'(z)$. Similarly,

$$\omega_x = -4\frac{\partial}{\partial x}\mathrm{Im}[f'(z)] = 4\frac{\partial}{\partial y}\mathrm{Re}[f'(z)].$$

Hence

$$p_x = 4\mu\frac{\partial}{\partial x}\mathrm{Re}[f'(z)], \qquad p_y = 4\mu\frac{\partial}{\partial y}\mathrm{Re}[f'(z)].$$

Integrating both of these we deduce that

$$p = 4\mu\mathrm{Re}[f'(z)] + \text{const},$$

and we may set the constant to zero for the reason discussed in the question.

Summary

The velocity field is given by

$$u + iv = -f(z) + z\,\overline{f'(z)} + \overline{g'(z)}. \tag{6.31}$$

In the absence of body forces, we've shown that the fluid pressure is given by

$$p = 4\mu \mathrm{Re}[f'(z)], \tag{6.32}$$

The fluid vorticity is given by

$$\omega = -4\mathrm{Im}[f'(z)]. \tag{6.33}$$

Thus

$$p/\mu - i\omega = 4f'(z). \tag{6.34}$$

Fluid stress

Let us first consider the rate of deformation tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

So

$$e_{11} = u_x, \qquad e_{12} = \frac{1}{2}(u_y + v_x).$$

Now,

$$\frac{1}{2}\left(\partial_x + i\,\partial_y\right)(u + iv) = \frac{1}{2}[u_x + i(v_x + u_y) - v_y].$$

Using continuity, this becomes

$$\frac{1}{2}\left(\partial_x + i\,\partial_y\right)(u + iv) = u_x + \frac{i}{2}(u_y + v_x) = e_{11} + ie_{12}.$$

The left hand side is, using (6.22),

$$\frac{1}{2}\left(\partial_x + i\,\partial_y\right)(u + iv) = \frac{\partial}{\partial \overline{z}}(u + iv).$$

But given the form of the velocity field (6.31), we have

$$\frac{\partial}{\partial \overline{z}}(u + iv) = \frac{\partial}{\partial \overline{z}}\Big(-f(z) + z\,\overline{f'(z)} + \overline{g'(z)}\Big) = z\overline{f''(z)} + \overline{g''(z)}$$

using Proposition 1 above, (6.25). Hence

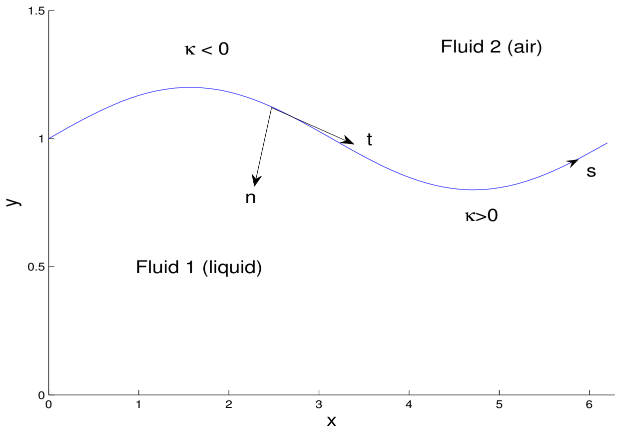
$$e_{11} + ie_{12} = z\overline{f''(z)} + \overline{g''(z)}. \tag{6.35}$$

The same result is given in Higley *et al.* (2012). Note that these authors also write the stress balance at a fluid interface in a simple way using the preceding result.

Stress boundary condition at a free surface

The boundary condition at an interface between two fluids is given by (3.26), written here in the constant surface tension case,

$$\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) \cdot \mathbf{n} = \kappa\gamma\mathbf{n}, \tag{6.36}$$



where $\mathbf{n}$ points into fluid 1, as illustrated in figure 1. So, at a capillary surface between a fluid (fluid 1 in 6.36) and air (fluid 2 in 6.36), we have the condition

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \gamma \kappa \mathbf{n} \tag{6.37}$$

where we have dropped the superscript label 1. Here γ is surface tension and κ is the surface curvature, whose sign is indicated in figure 1 and in the sketch below.
Assuming a Newtonian fluid, this condition is

$$-p\mathbf{n} + 2\mu \mathbf{e} \cdot \mathbf{n} = \gamma \kappa \mathbf{n}. \tag{6.38}$$

Let us describe the surface by $z(s) = x(s) + iy(s)$. The normal vector (pointing downwards and into the liquid, as in the sketch above) is $\mathbf{n} = (n_1, n_2) = (y_s, -x_s)$ which we may compactly describe using the complex representation

$$n_1 + in_2 = -iz_s \equiv N, \tag{6.39}$$

where the subscript means a partial derivative with respect to s. Note that the definition (6.39) is the negative of that used by Tanveer & Vasconcelos (1995 - see reference above), who defined their unit normal to point outside their bubble. Next we note that

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{12} & e_{22} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{12} & -e_{11} \end{pmatrix} \cdot \begin{pmatrix} y_s \\ -x_s \end{pmatrix} = \begin{pmatrix} y_s e_{11} - x_s e_{12} \\ x_s e_{11} + y_s e_{12} \end{pmatrix},$$

where we have used the fact that $e_{11} + e_{22} = 0$ since the flow is incompressible (see earlier in these notes).

Dot product

Suppose $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. We may express the dot product $\mathbf{a} \cdot \mathbf{b}$ using a complex representation, writing $A = a_1 + ia_2$ and $B = b_1 + ib_2$, as

$$\mathbf{a} \cdot \mathbf{b} = \text{Re}[A\overline{B}] = \text{Re}[(a_1 + ia_2)(b_1 - ib_2)] = a_1 b_1 + a_2 b_2.$$

Here we write the dot product $\mathbf{e} \cdot \mathbf{n}$ in the complex form

$$\mathbf{e} \cdot \mathbf{n} \equiv (e_{11} + ie_{12})\overline{(-iz_s)} = i(e_{11} + ie_{12})\overline{z_s}, \tag{6.40}$$

since

$$i(e_{11} + ie_{12})\overline{z_s} = i(e_{11} + ie_{12})(x_s - iy_s) = (y_s e_{11} - x_s e_{12}) + (x_s e_{11} + y_s e_{12})i.$$

Then the stress condition (6.37) may be written

$$-pN + 2\mu(e_{11} + ie_{12})\overline{N} = \gamma \kappa N, \tag{6.41}$$

where $N = -iz_s$. Similar to Tanveer & Vasconcelos (1995), we write $\kappa = \theta_s$, where θ is the angle between the tangent to the surface and the x-axis. Then

$$\tan\theta = y_s/x_s.$$

Differentiating with respect to s,

$$\theta_s \sec^2\theta = \frac{x_s y_{ss} - y_s x_{ss}}{x_s^2}.$$

But

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{y_s^2}{x_s^2} = \frac{x_s^2 + y_s^2}{x_s^2} = \frac{1}{x_s^2},$$

where we have used the fact that

$$x_s^2 + y_s^2 = 1.$$

Hence

$$\theta_s = x_s y_{ss} - y_s x_{ss}.$$

A further consequence of this is that $x_s x_{ss} + y_x y_{ss} = 0$. It follows that

$$z_{ss}\overline{z_s} = (x_{ss} + iy_{ss})(x_s - iy_s) = (x_s x_{ss} + y_s y_{ss}) + i(x_s y_{ss} - y_s x_s s) = i(x_s y_{ss} - y_s x_{ss}).$$

Thus

$$\kappa = \theta_s = x_s y_{ss} - y_s x_{ss} = -iz_{ss}\overline{z_s}$$

Consequently

$$\kappa N = (-iz_{ss}\overline{z_s})(-iz_s) = -z_{ss}$$

since $|z_s|^2 = x_s^2 + y_s^2 = 1$.

Following Tanveer & Vasconcelos (1995) we introduce the function

$$H(z, \overline{z}) = f(z) + z\overline{f'(z)} + \overline{g'(z)}.$$

Then

$$\frac{\partial H}{\partial z} = f'(z) + \overline{f'(z)}, \qquad \frac{\partial H}{\partial \overline{z}} = z\overline{f''(z)} + \overline{g''(z)}.$$

using Propositions 1 and 2 (6.25) and (6.26). So

$$\frac{\partial H}{\partial z} = f'(z) + \overline{f'(z)} = 2\mathrm{Re}[f'(z)] = p/(2\mu)$$

by (6.32), and

$$\frac{\partial H}{\partial \overline{z}} = z\overline{f''(z)} + \overline{g''(z)} = e_{11} + ie_{12}$$

by (6.35).

Therefore (6.41) becomes

$$2\mu\frac{\partial H}{\partial z} iz_s - 2\mu\frac{\partial H}{\partial \overline{z}}\overline{iz_s} = -\gamma z_{ss},$$

using (6.40). Tidying up,

$$\frac{\partial H}{\partial z} z_s + \frac{\partial H}{\partial \overline{z}}\overline{z_s} = \frac{1}{2} i\tau z_{ss}, \tag{6.42}$$

where $\tau = \gamma/\mu$. This is the same as equation (21) of Tanveer & Vasconcelos (1995), with allowance made for our different sign of the normal vector N.

The left hand side of (6.42) is simply

$$\frac{\mathrm{d}H}{\mathrm{d}s}$$

So, integrating (6.42) with respect to s, we obtain the statement of the surface stress condition (6.38) in complex variables,

$$f(z) + z\overline{f'(z)} + \overline{g'(z)} = \frac{1}{2} i\tau z_s, \tag{6.43}$$

where $z(s)$ is the surface. Using (6.31) we may rewrite this as

$$u + iv = \frac{1}{2} i\tau z_s - 2f(z), \tag{6.44}$$

which provides a formula for the surface velocity – see Tanveer & Vasconcelos (1995) equation (3.7). It should be emphasised that equation (6.44) has been derived *in the absence of body forces*.

Some simple flows

Let's consider some simple choices for f and g to see what flows they produce.

Example 1: Let $f = 0$ and $g = z$; then

$$\psi = \text{Im}[z] = y, \qquad u = \psi_y = 1, \quad v = -\psi_x = 0.$$

This corresponds to a *uniform stream* moving in the x direction.

Example 2: Let $f = 0$ and $g = z^2$; then

$$\psi = \text{Im}[z^2] = 2xy, \qquad u = \psi_y = 2x, \quad v = -\psi_x = -2y.$$

This is *stagnation-point flow* toward a solid wall at $y = 0$. Sketch the streamlines for this flow as an exercise. The vorticity for this flow is zero,

$$\omega = -4\text{Im}[f'(z)] = -4\text{Im}[0] = 0,$$

and so the flow is *irrotational*. The pressure in the fluid is

$$p = 4\mu\text{Re}[f'(z)] = 4\mu\text{Re}[0] = 0.$$

So the pressure is constant throughout the fluid.

Example 3: Let $f = iz$ and $g = -iz^2$; then

$$\psi = \text{Im}[i\bar{z}z - iz^2] = \text{Im}\left[i|z|^2 - i(x^2 - y^2 + 2ixy)\right] = |z|^2 - x^2 + y^2 = 2y^2,$$

and so

$$u = \psi_y = 4y, \quad v = -\psi_x = 0.$$

This corresponds to *simple shear flow* in the x direction. The vorticity for this flow is constant,

$$\omega = -4\text{Im}[f'(z)] = -4\text{Im}[i] = -4.$$

The pressure is

$$p = 4\mu\text{Re}[f'(z)] = 4\mu\text{Re}[i] = 0.$$

So the pressure is constant throughout the fluid.

Example 4: For Poiseuille flow, for example, $\psi = y - y^3/3$. We can construct $\psi = y^3$ by taking $f(z) = 3z^2/4$ and $g(z) = -z^3/4$. Then

$$\begin{aligned}\bar{z}(3z^2/4) - z^3/4 = (z^2/4)[3\bar{z} - z] = (z^2/4)[2x - 4iy] = (x^2 - y^2 + 2ixy)(2x - 4iy)/4 \\ = \text{real part} + i(-4x^2y + 4x^2y + 4y^3)/4 = \text{real part} + iy^3.\end{aligned}$$

So

$$\psi = \text{Im}[\bar{z}(3z^2/4) - z^3/4] = y^3.$$

Example 5: Let $f = iz$ and $g = 0$; then

$$\psi = \text{Im}[i\bar{z}z] = \text{Im}\left[i|z|^2\right] = x^2 + y^2.$$

and so

$$u = \psi_y = 2y, \quad v = -\psi_x = -2x.$$

This corresponds to *solid body rotation* about the z axis. The streamlines are concentric circles, given by

$$\psi = x^2 + y^2 = \text{const.}$$

Classical mechanics tells us that if a solid body rotates at angular velocity $2\Omega\mathbf{k}$, the velocity at a point $\mathbf{x}$ in the body is given by

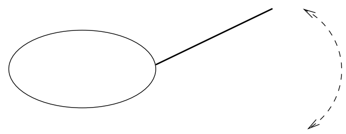
$$(2\Omega\mathbf{k}) \times \mathbf{x} = 2\Omega(-y, x, 0),$$

as above.

Note: The expression (6.28) ensures that the biharmonic equation is satisfied and thus the equations of Stokes flow are satisfied. It only remains to satisfy any boundary conditions which may obtain in a particular problem. We attempt to choose the functions $f(z)$ and $g(z)$ to satisfy these boundary conditions. Unfortunately in practice this is often rather difficult!

Low Reynolds number swimming

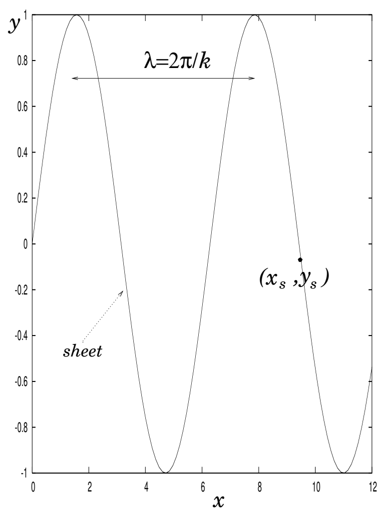
The reversibility of Stokes flows has an interesting consequence with regard to the swimming of very small organisms, such as a spermatozoa. Since the lengthscale of such organisms is very small, the Reynolds number is likely to be very small and the conditions of Stokes flow apply.



If this idealised little spermatozoa swishes its tail up and down as shown, since the flow around it is reversible, it will never get anywhere!

We will look at an idealised model of a swimming motion which overcomes the reversibility problem.

Motion of a thin flexible sheet



A thin flexible sheet ‘swims’ in a viscous fluid under conditions of Stokes flow. A point on the sheet (x_s, y_s) moves according to

$$x_s = x, \qquad y_s = a\sin(kx - \omega t),$$

i.e. the wave moves from left to right with speed $c = \omega/k$. The wavelength $\lambda = 2\pi/k$.

Assume the fluid above the sheet ($y \geq y_s$) is in two-dimensional motion, satisfying the equation

$$\nabla^4 \psi = 0,$$

where

$$u = \psi_y, \qquad v = -\psi_x.$$

The boundary condition at the sheet is that $\mathbf{u} = \mathbf{u}_s$, where

$$\mathbf{u}_s = (\dot{x}_s,\ \dot{y}_s) = (0, -\omega a \cos(kx - \omega t)\ ).$$

As $y \to \infty$, we require $\mathbf{u}$ to be bounded.

Non-dimensionalize by writing

$$x = x^*/k, \quad y = y^*/k, \quad t = t^*/\omega, \quad \psi = k\psi^*/(a\omega).$$

Then $\nabla^4 \psi = 0$ becomes

$$k^3 a\, \omega \Big(\frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} \Big)^2 \psi^* = 0.$$

and the boundary conditions,

$$u^* = \frac{\partial \psi^*}{\partial y^*} = 0, \quad -v^* = \frac{\partial \psi^*}{\partial x^*} = \cos(x^* - t^*) \quad \text{on} \quad y^* = ka\sin(x^* - t^*).$$

From now on we will assume that

$$\epsilon = ka \ll 1.$$

Dropping all the asterisks for convenience, and setting $X = (x - t)$, the boundary conditions are

$$\frac{\partial \psi}{\partial y}(x, \epsilon \sin X) = 0, \qquad \frac{\partial \psi}{\partial x}(x, \epsilon \sin X) = \cos X.$$

Expanding in Taylor series,

$$\frac{\partial \psi}{\partial y}(x, \epsilon \sin X) = \frac{\partial \psi}{\partial y}(x, 0) + \epsilon \sin X \frac{\partial^2 \psi}{\partial y^2}(x, 0) + \cdots = 0,$$

and

$$\frac{\partial \psi}{\partial x}(x, \epsilon \sin X) = \frac{\partial \psi}{\partial x}(x, 0) + \epsilon \sin X \frac{\partial^2 \psi}{\partial x \partial y}(x, 0) + \cdots = \cos X.$$

In view of these boundary conditions, we seek a solution in the form of the expansion

$$\psi = \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots,$$

and so we obtain

$$\nabla^4 \psi_0 + \epsilon \Big[\nabla^4 \psi_1 \Big] + \cdots + \epsilon^n \Big[\nabla^4 \psi_n \Big] + \cdots = 0.$$

The boundary conditions become

$$\frac{\partial}{\partial y} \Big\{ \psi_0 + \epsilon \psi_1 + \cdots \Big\} + \epsilon \sin X \frac{\partial^2}{\partial y^2} \Big\{ \psi_0 + \epsilon \psi_1 + \cdots \Big\} + \cdots = 0,$$

and

$$\frac{\partial}{\partial x} \Big\{ \psi_0 + \epsilon \psi_1 + \cdots \Big\} + \epsilon \sin X \frac{\partial^2}{\partial x \partial y} \Big\{ \psi_0 + \epsilon \psi_1 + \cdots \Big\} + \cdots = \cos X.$$

Taking the limit $\epsilon \to 0$ yields all the terms in the equations of size $O(\epsilon^0)$:

$$\nabla^4 \psi_0 = 0,$$

with conditions

$$\frac{\partial \psi_0}{\partial x} = \cos X, \qquad \frac{\partial \psi_0}{\partial y} = 0$$

on $y = 0$.

At the next 'order' we pick out all terms of size $O(\epsilon^1)$, namely,

$$\nabla^4 \psi_1 = 0,$$

with conditions

$$\frac{\partial \psi_1}{\partial x} + \sin X \frac{\partial^2 \psi_0}{\partial x \partial y} = 0, \quad \frac{\partial \psi_1}{\partial y} + \sin X \frac{\partial^2 \psi_0}{\partial y^2} = 0$$

on $y = 0$.

We can continue in this vein for all orders $O(\epsilon^n)$, $n = 2, 3, 4, \ldots$ as far as we like.

We look for a solution to the $O(\epsilon^0)$ problem by trying

$$\psi_0 = f_0(y) \sin X.$$

Then, since

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x} \frac{\partial}{\partial X} = \frac{\partial}{\partial X},$$

we get:

$$\nabla^2 \psi_0 = (f_0'' - f_0) \sin X = F(y) \sin X, \text{ say.}$$

So,

$$\begin{aligned} \nabla^4 \psi_0 &= \nabla^2(\nabla^2 \psi_0) = \nabla^2(F(y) \sin X) = (F'' - F) \sin X, \\ &= \Big[(f_0'' - f_0)'' - (f_0'' - f_0)\Big] \sin X, \\ &= [f_0^{(iv)} - 2f_0'' + f_0] \sin X. \end{aligned}$$

So

$$f_0^{(iv)} - 2f_0'' + f_0 = 0.$$

Writing $f_0 = \mathrm{e}^{my}$, we have

$$m^4 - 2m^2 + 1 = (m^2 - 1)^2 = 0$$
$$\implies m = 1, 1, -1, -1.$$

Accordingly,

$$\psi_0 = \Big\{(A_0 + B_0\, y)\mathrm{e}^y + (C_0 + D_0\, y)\mathrm{e}^{-y}\Big\} \sin X.$$

We need $A_0 = B_0 = 0$ to satisfy the boundedness as $y \to \infty$ condition, so

$$\psi_0 = (C_0 + D_0\, y)\mathrm{e}^{-y} \sin X.$$

But, on $y = 0$,

$$\frac{\partial \psi_0}{\partial x} = \cos X \implies C_0 = 1.$$

And, again on $y = 0$,

$$\frac{\partial \psi_0}{\partial y} = 0 \implies -C_0 + D_0 = 0 \implies D_0 = 1.$$

Therefore the solution is

$$\psi_0 = (1 + y)\mathrm{e}^{-y} \sin X.$$

The problem for ψ_1 now becomes

$$\nabla^4 \psi_1 = 0,$$

with conditions on $y = 0$,

$$\frac{\partial \psi_1}{\partial x} = 0, \quad \frac{\partial \psi_1}{\partial y} = -\sin X.(-\sin X) = \sin^2 X = \frac{1}{2}(1 - \cos 2X).$$

The boundary conditions suggest that we look for a solution of the form

$$\psi_1 = f_{11}(y) + f_{12}(y) \cos 2X.$$

Then

$$\nabla^2 \psi_1 = f_{11}'' + (f_{12}'' - 4f_{12}) \cos 2X.$$

So,

$$\begin{aligned}
\nabla^4 \psi_1 &= f_{11}^{(iv)} + (f_{12}^{(iv)} - 4f_{12}'') \cos 2X - 4(f_{12}'' - 4f_{12}) \cos 2X \\
&= f_{11}^{(iv)} + (f_{12}^{(iv)} - 8f_{12}'' - 16f_{12}) \cos 2X \\
&= 0.
\end{aligned}$$

We therefore require

$$f_{11}^{(iv)} = 0 \quad \text{and} \quad f_{12}^{(iv)} - 8f_{12}'' - 16f_{12} = 0.$$

Thus,

$$f_{11} = A_{11}y^3 + B_{11}y^2 + C_{11}y + D_{11}.$$

Trying $f_{12} = \mathrm{e}^{my}$, we obtain the auxiliary equation

$$(m^2 - 4)^2 = 0 \implies m = 2, 2, -2, -2,$$

and so

$$f_{12} = (A_{12} + B_{12}y)\, \mathrm{e}^{2y} + (C_{12} + D_{12}y)\, \mathrm{e}^{-2y}.$$

Now,

$$\frac{\partial \psi_1}{\partial x} = 0 \quad \text{on} \quad y = 0 \implies f_{12}(0) = 0,$$

and

$$\frac{\partial \psi_1}{\partial y} = \frac{1}{2}(1 - \cos 2X) \quad \text{on} \quad y = 0 \implies f_{11}'(0) = \frac{1}{2}, \quad f_{12}'(0) = -\frac{1}{2}.$$

Hence $C_{11} = 1/2$ and, because the velocities are bounded as $y \to \infty$, we need $A_{11} = B_{11} = 0$.

For f_{12} we shall also require $A_{12} = B_{12} = 0$ in order for the velocities to be bounded at infinity. Then

$$f_{12}(0) = 0 \implies C_{12} = 0.$$

Also,

$$f_{12}'(0) = -\frac{1}{2} \implies D_{12} = -\frac{1}{2}.$$

Therefore

$$\psi_1 \;=\; f_{11}(y) + f_{12}(y)\cos 2X \;=\; \frac{1}{2}y + D_{11} - \frac{1}{2}y\,\mathrm{e}^{-2y}\cos 2X.$$

We can take $D_{11} = 0$ without loss of generality (since it is just adding a constant to a streamfunction). So,

$$\psi_1 = \frac{1}{2}y(1 - \mathrm{e}^{-2y}\cos 2X).$$

Hence

$$\begin{aligned}\psi \;&=\; \psi_0 + \epsilon\psi_1 + \cdots \\ &=\; (1+y)\mathrm{e}^{-y}\sin X + \frac{1}{2}\epsilon y(1 - \mathrm{e}^{-2y}\cos 2X) + \cdots\,.\end{aligned}$$

Thus, the horizontal velocity component,

$$\begin{aligned}u \;&=\; \psi_y = -y\mathrm{e}^{-y}\sin X + \epsilon\Big(\frac{1}{2} + (y - \frac{1}{2})\,e^{-2y}\cos 2X\Big) + \cdots \\ &=\; -y\mathrm{e}^{-y}\sin(x-t) + \epsilon\Big(\frac{1}{2} + (y - \frac{1}{2})\,e^{-2y}\cos 2[x-t]\Big) + \cdots\,.\end{aligned}$$

In dimensional form,

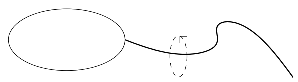
$$u = -\omega kay\mathrm{e}^{-ky}\sin(kx - \omega t) + \frac{\epsilon\omega a}{2}\Big(1 + (2ky-1)\,e^{-2ky}\cos 2[kx - \omega t]\Big) + \cdots\,.$$

So we see that a steady horizontal component of velocity is generated:

$$u \;=\; \frac{\epsilon a\omega}{2} = \epsilon^2\frac{\omega}{2k} \qquad (\text{since }\; \epsilon = ka) \qquad = \frac{1}{2}\epsilon^2 c,$$

where $c = \omega/k$ is the wave speed. If we take a set of axes moving at this streaming velocity, the sheet 'swims' from right to left.

In real life, similar 'non-symmetric' motions have to be adopted at low Reynolds numbers to avoid the reversibility problem.



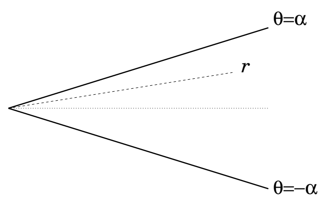
A more realistic picture of the spermatozoa has it wiggling its tail in a rotary motion, sending helical waves down to the tip. This overcomes the problems with reversibility and allows it to move forwards. Bacteria, which often have several tails (or *flagellae*) adopt similar non-symmetric strategies for swimming.

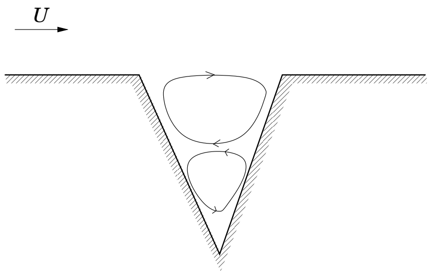
Flow in corners - Moffatt eddies

Now we will look at slow flow of viscous fluid in a sharp corner or crevice of angle 2α. The fluid satisifes no slip and no normal flow at $\theta = \pm\alpha$, and can be thought of as driven by some agency a distance from the corner $r = 0$. For example, uniform flow over a sharp indentation will drive a flow consisting of eddies inside the crevice.

The flow is two-dimensional and satisfies the Stokes equations. We use a stream function, ψ, defined by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v = -\frac{\partial\psi}{\partial r},$$





for velocity field $\mathbf{u} = u\,\hat{\mathbf{r}} + v\,\hat{\boldsymbol{\theta}}$.

The equation of motion is $\nabla^4\psi = 0$, and so

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2}\right)^2 \psi = 0. \tag{6.45}$$

We seek a class of solutions of the form

$$\psi = r^\lambda f(\theta), \qquad \lambda \neq 0$$

where f and λ are to be determined.

Substituting into (6.45), we find

$$\nabla^2\psi = r^{\lambda-2}\Big\{[\lambda(\lambda-1)+\lambda]f + f''\Big\} = r^{\lambda-2}\Big\{\lambda^2 f + f''\Big\} = r^\beta F(\theta), \text{ say.}$$

and so

$$\nabla^4\psi = \nabla^2(\nabla^2\psi) = r^{\beta-2}\Big\{\beta^2 F + F''\Big\} = r^{\beta-2}\Big\{\beta^2[\lambda^2 f + f''] + [\lambda^2 f'' + f^{(iv)}]\Big\}.$$

Thus, to satisfy (6.45), we will need

$$\beta^2\lambda^2 f + (\lambda^2+\beta^2)f'' + f^{(iv)} = 0.$$

Thus, since $\beta = \lambda - 2$,

$$(\lambda-2)^2\lambda^2 f + [\lambda^2 + (\lambda-2)^2]f'' + f^{(iv)} = 0.$$

Seeking a solution of the form $f \propto \mathrm{e}^{m\theta}$, we find

$$m^4 + 2(\lambda^2 - 2\lambda + 2)m^2 + \lambda^2(\lambda-2)^2 = (m^2+\lambda^2)(m^2 + [\lambda-2]^2) = 0$$

$$\implies m = \lambda i,\ -\lambda i,\ (\lambda-2)i,\ -(\lambda-2)i.$$

Hence, assuming $\lambda \neq 1$ and $\lambda \neq 2$,

$$f(\theta) = A\cos\lambda\theta + B\sin\lambda\theta + C\cos(\lambda-2)\theta + D\sin(\lambda-2)\theta.$$

If we now concentrate on solutions which are symmetric about $\theta = 0$, we can set $B = D = 0$ to leave

$$f(\theta) = A\cos\lambda\theta + C\cos(\lambda-2)\theta.$$

The boundary conditions are $u = v = 0$ at $\theta = \pm\alpha$, which translate into

$$\psi = \frac{\partial\psi}{\partial\theta} = 0 \quad \text{on} \quad \theta = \pm\alpha$$

and thus

$$f = f' = 0 \quad \text{on} \quad \theta = \pm\alpha.$$

Hence,

$$A \cos \lambda\alpha + C \cos(\lambda - 2)\alpha = 0,$$
$$A\lambda \sin \lambda\alpha + C(\lambda - 2) \sin(\lambda - 2)\alpha = 0,$$

or,

$$\begin{pmatrix} \cos \lambda\alpha & \cos(\lambda - 2)\alpha \\ \lambda \sin \lambda\alpha & (\lambda - 2) \sin(\lambda - 2)\alpha \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \mathbf{0} \qquad (6.46)$$

So, for non-trivial solutions, we need the determinant of the matrix to vanish:

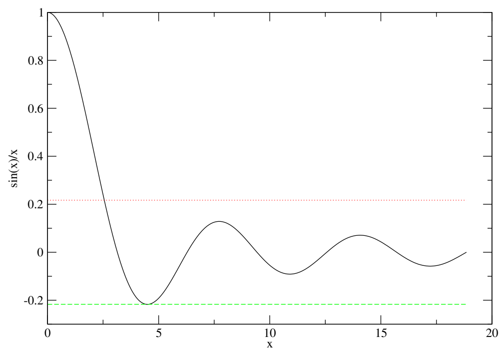
$$\lambda \tan \lambda\alpha = (\lambda - 2) \tan(\lambda - 2)\alpha.$$

Exercise: Show that this can be re-written as

$$\frac{\sin x}{x} = -\frac{\sin 2\alpha}{2\alpha}. \qquad (6.47)$$

where $x = 2(\lambda - 1)\alpha$.

Fixing 2α, we can read $\mathcal{B} = (2\alpha)^{-1} \sin 2\alpha$ from the graph (upper dotted line). For the example shown we get $\mathcal{B} = 0.217$. Then we find where the solid curve corresponds to $-\mathcal{B} = -0.217$. This is shown as the lower dashed line.



If we choose $2\alpha = 2.55$, we get $\mathcal{B} = 0.217$, and -0.217 just touches the global minimum of the solid curve, as shown. So if $\alpha < 2.55$, there will be no real solutions. In degrees, this means that there are no real solutions if

$$2\alpha < 146.3°.$$

Let's consider the two possibilities in turn:

i) $2\alpha > 146.3°$.

In this case, (6.47) has real solutions λ. Thus the θ velocity component

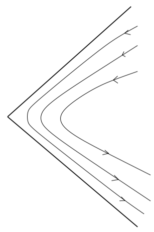
$$v = -\lambda r^{\lambda-1} f(\theta)$$

is single signed for all r and so the fluid moves in and out of the corner something like this:

ii) $2\alpha < 146.3°$.

Now the solutions λ are complex, so write $\lambda = p + iq$. In general A and C will also be complex. Since the equation of motion $\nabla^4\psi = 0$ is linear, using an overbar to denote a complex conjugate, both

$$r^\lambda f(\theta) \quad \text{and} \quad \overline{r^\lambda f(\theta)},$$



are clearly solutions to the equation of motion $\nabla^4\psi = 0$, and since this equation is linear we can add them together, so

$$\psi = \frac{1}{2}\Big[r^\lambda f(\theta) + \overline{r^\lambda f(\theta)}\Big] = \Re\{r^\lambda f(\theta)\}$$

is a solution.

Suppose we fix $\theta = 0$, so we are focussing on the centre line. Then $f(0)$ is just some complex number, call it c. Since $\lambda = p + iq$, the azimuthal velocity component

$$v|_{\theta=0} \;=\; \Re\{-\lambda\, c\, r^{\lambda-1}\} \;=\; \Re\{-(p + i\,q)\, c\, r^{(p-1)+iq}\}.$$

Writing

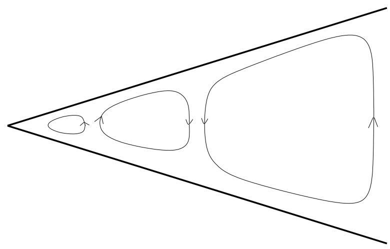
$$r^{iq} = \mathrm{e}^{iq\log r} = \cos(q\log r) + i\sin(q\log r),$$

we have

$$v|_{\theta=0} = a\, r^{p-1}\cos(q\log r + \delta), \tag{6.48}$$

for real constants a, δ, which we can find in terms of p, q and c.

So the azimuthal velocity can now change sign with r and the corner flow must consist of re-circulating eddies, something like this:



Intriguingly, it is clear from (6.48) that there must be an *infinite* number of these eddies, as v will repeatedly change sign as $r \to 0$.

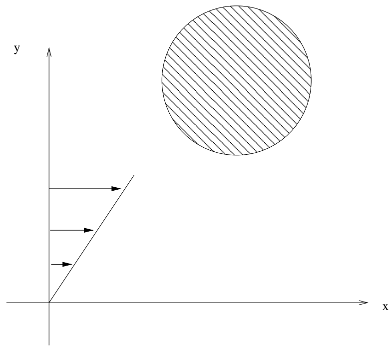
These eddies have been visualized in experiments, although the ones very close to the corner are far too weak to be observed in practice.

Spinning of a white blood cell

White blood cells are approximately spherical and are carried along in the blood along with platelets and red blood cells. In the smaller blood cells, such as the venules, capillaries, and arterioles, the Reynolds number is small and we can approximate the motion using Stokes flow theory.

We will perform an idealised calculation to compute the rate of spinning and velocity of transit of a white blood cell in blood flow in a capillary. We will model this scenario by consider the spinning of a circular particle in an unbounded shear flow, as shown in the diagram. The background shear flow is given by

$$u = ky, \qquad v = 0.$$



The circle is of radius a and its centre has velocity $\mathbf{u} = U\mathbf{i}$ and angular velocity $\Omega\mathbf{k}$. We will seek a solution in polar coordinates with origin at the centre of the circle. We introduce the streamfunction ψ defined so that

$$u = \psi_\theta / r, \qquad v = -\psi_r.$$

Then the Stokes flow around the circle satisfies the biharmonic equation

$$\nabla^4 \psi = 0. \tag{6.49}$$

Relative to the coordinate frame fixed in the centre of the circle, the background shear flow is given by

$$u = k(y - c), \qquad v = 0,$$

where c is the height above the x axis shown in the figure above. In the sequel we take $k = 1$. Accordingly, far from the circle the flow should settle down to this background state, and so we require

$$\psi \sim \frac{1}{2}(y-c)^2 = \frac{1}{2}(r\sin\theta - c)^2 = \frac{1}{2}r^2\sin^2\theta - cr\sin\theta + \frac{1}{2}c^2.$$

as $r \to \infty$. Noting that

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right),$$

so that

$$\psi \sim -\frac{1}{4}r^2\cos 2\theta + \frac{1}{4}r^2 - cr\sin\theta + \frac{1}{2}c^2, \tag{6.50}$$

as $r \to \infty$, we try a solution in the form

$$\psi = f(r)\cos 2\theta + g(r)\sin\theta + h(r).$$

Substituting into (6.49) we find the three differential equations for f, g, and h:

$$\begin{aligned}
2f^{(iv)} + \frac{4}{r}f''' - \frac{18}{r^2}f'' + \frac{18}{r^3}f' &= 0, \\
g^{(iv)} + \frac{2}{r}g''' - \frac{3}{r^2}g'' + \frac{3}{r^3}g' - \frac{3}{r^4}g &= 0, \\
h^{(iv)} + \frac{2}{r}h''' - \frac{1}{r^2}h'' + \frac{1}{r^3}h' &= 0,
\end{aligned}$$

all of which are of Euler type, and we can seek a solution of the form $f \propto r^n$ and so on. We find the general solutions

$$\begin{aligned}
f &= f_1 + f_2 r^{-2} + f_3 r^2 + f_4 r^4, \\
g &= g_1 r^{-1} + g_2 r + g_3 r\log r + g_4 r^3, \\
h &= h_1 + h_2 \log r + h_3 r^2 + h_4 r^2 \log r.
\end{aligned}$$

The matching condition at infinity requires

$$h_1 = \frac{1}{2}c^2, \qquad h_3 = \frac{1}{4}, \qquad h_4 = 0, \qquad g_2 = -c, \qquad g_4 = 0, \qquad f_3 = -\frac{1}{4}, \qquad f_4 = 0,$$

and so

$$\begin{aligned} f &= f_1 + \frac{f_2}{r^2} - \frac{1}{4}r^2, \\ g &= \frac{g_1}{r} - cr + g_3 r \log r, \\ h &= \frac{1}{2}c^2 + h_2 \log r + \frac{1}{4}r^2. \end{aligned}$$

Next we insist that the horizontal velocity of the cell is U and that its angular velocity is Ω, so the cell velocity is

$$\mathbf{u} = U\,\mathbf{i} + \mathbf{r} \times \Omega\,\mathbf{k} = (U - y\Omega, x\Omega).$$

In polar coordinates,

$$u_r = U\cos\theta - y\Omega\,\cos\theta + x\Omega\sin\theta = U\cos\theta + \frac{1}{2}a\Omega(\sin 2\theta - \sin 2\theta) = U\cos\theta,$$

$$u_\theta = -U\sin\theta + a\Omega.$$

Satisfying these conditions, we find

$$\begin{aligned} f &= -\frac{1}{4}r^2 - \frac{a^4}{4r^2} + \frac{a^2}{2}, \\ g &= \frac{a^2(U+c)}{(1+2\log a)r} - cr + \frac{2(U+c)\,r\log r}{1+2\log a}, \\ h &= \frac{1}{2}c^2 - a^2\left(\Omega + \frac{1}{2}\right)\log r + \frac{1}{4}r^2. \end{aligned}$$

It should be noted that the term proportional to $r\log r$ in $g(r)$ is of concern with regard to the matching condition at infinity (6.50). See the comment on Stokes' paradox below.

The force on the cell is given by

$$\mathbf{F} = \oint \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}l$$

Now,

$$\boldsymbol{\sigma}\cdot\mathbf{n} = -p\mathbf{n} + 2\mu\,\mathbf{e}\cdot\mathbf{n},$$

and

$$\mathbf{e}\cdot\mathbf{n} = (\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n})\,\mathbf{n} + (\mathbf{t}\cdot\mathbf{e}\cdot\mathbf{n})\,\mathbf{t}.$$

On a no-slip surface, $\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n} = 0$ (see the Problem Sheets), and so on the cylinder boundary,

$$\boldsymbol{\sigma}\cdot\mathbf{n} = -p\mathbf{n} + 2\mu\,(\mathbf{t}\cdot\mathbf{e}\cdot\mathbf{n})\,\mathbf{t} = -p\mathbf{n} + 2\mu\,e_{r\theta}\,\mathbf{t}.$$

Using the formula for $e_{r\theta}$,

$$e_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right),$$

we have

$$2e_{r\theta} = -r\frac{\partial}{\partial r}\left(\frac{\psi_r}{r}\right) + \frac{1}{r^2}\psi_{\theta\theta} = -\psi_{rr} + \psi_r/r + \psi_{\theta\theta}/r^2.$$

To compute the pressure we note that since $\mathbf{u} = \nabla\times\psi\mathbf{k}$, then

$$\nabla p = \mu\nabla^2\mathbf{u} = -\mu\nabla\times\nabla\times\mathbf{u} = \mu\nabla\times(\nabla^2\psi\mathbf{k}),$$

and so

$$\nabla p = \mu\Big((\nabla^2\psi)_\theta/r,\ -(\nabla^2\psi)_r\Big).$$

We can integrate to find p using the known formula for ψ. Then the total force is

$$\mathbf{F} = -a\int_0^{2\pi} p\mathbf{n}\,\mathrm{d}\theta + \mu a\int_0^{2\pi} 2e_{r\theta}\mathbf{t}\,\mathrm{d}\theta.$$

We find

$$\mathbf{F} = \frac{8\pi\mu(U+c)}{1+2\log a}\,\mathbf{i}.$$

The torque on the cell is given by

$$T = \oint \mathbf{x}\times\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}l = 2\mu a^2\int_0^{2\pi} e_{r\theta}\,\mathrm{d}\theta$$

$$= \mu a^2\int_0^{2\pi}\left[-r\frac{\partial}{\partial r}\left(\frac{\psi_r}{r}\right)+\frac{1}{r^2}\psi_{\theta\theta}\right]\mathrm{d}\theta = \mu a^2\int_0^{2\pi}[-\psi_{rr}+\psi_r/r+\psi_{\theta\theta}/r^2]\,\mathrm{d}\theta.$$

Substituting the result for ψ, we find

$$T = -2\pi a^2\mu(1+2\,\Omega),$$

which agrees with formula (72a) of Chwang & Wu (Journal of Fluid Mechanics, **67**, 1975).

Evidently, for a force free cell, we should choose

$$c = -U,$$

which corresponds to a symmetric shear flow above and below the cell. For a torque-free cell, we should choose

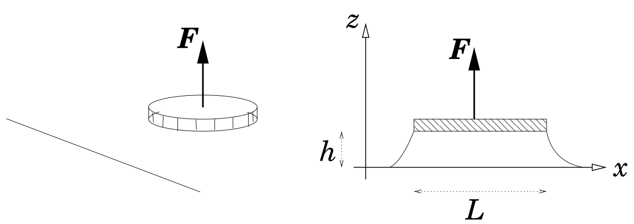
$$\Omega = \frac{1}{2},$$

so that the cell rotates at a rate equal to half the vorticity of the background shear flow.

Stokes paradox: It is important to note that the above calculation is only valid in the case $c=-U$ when there is zero force on the cell. In this case the term proportional to $r\log r$ in $g(r)$ vanishes. Otherwise this term will grow at infinity and compromise the matching condition (6.50). This is Stokes' paradox, namely that it is not possible to write down a consistent solution for unbounded Stokes flow past a cylinder when the fluid imparts a force on the cylinder. However, as we have seen, a solution is possible if the cylinder convects in the fluid with zero force. Contrast the solution for Stokes flow past a sphere, where the paradox does not arise.

7. Lubrication theory

Lubrication theory is the study of thin films of fluid. The key feature of such problems is that one dimension is much larger than the other. For example, you can get a coin to "stick" to a surface by putting a small amount of water between the two. A surprisingly large force is required to pull the coin away from the table. Compared to the diameter of the coin, the height of the film of water is very small ($h \ll L$ in the figure below) and as such is amenable to lubrication analysis. We shall deal with this problem later.



To start the analysis, we write down the Navier-Stokes equations,

$$\nabla\cdot\mathbf{u}=0,\qquad \mathbf{u}\cdot\nabla\mathbf{u} \;=\; -\nabla p/\rho+\nu\nabla^2\mathbf{u}$$

and make some simplifications, using the fact that we are dealing with a thin film of fluid.

Specifically, we work on the assumption that if h and L are typical horizontal (x, y) and vertical (z) dimensions respectively, and if

$$\delta = \frac{h}{L} \quad \text{then} \quad \delta \ll 1.$$

Let U be a typical horizontal flow speed. Then, since $\mathbf{u} = \mathbf{0}$ at the solid walls $z = 0$ and $z = h$, u and v must change by an amount of order U as z varies over a distance h. We can estimate sizes of the terms in the equations using these type of arguments. In the film we expect

$$\frac{\partial u}{\partial z} \sim \frac{U}{h}, \qquad \frac{\partial^2 u}{\partial z^2} \sim \frac{U}{h^2}, \qquad \frac{\partial u}{\partial x} \sim \frac{U}{L}, \qquad \frac{\partial^2 u}{\partial x^2} \sim \frac{U}{L^2},$$

with similar estimates for v and its derivatives.

Now, the continuity equation

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \implies \frac{\partial w}{\partial z} \sim O\Big(\frac{U}{L}\Big) \implies w \sim O\Big(\frac{Uh}{L}\Big).$$

So we can estimate

$$\mathbf{u} \cdot \nabla \mathbf{u} \sim O\Big(\frac{U^2}{L}\Big)\mathbf{i} + O\Big(\frac{U^2}{L}\Big)\mathbf{j} + O\Big(\frac{U^2 h}{L^2}\Big)\mathbf{k},$$

and

$$\nabla^2 \mathbf{u} \approx \frac{\partial^2 \mathbf{u}}{\partial z^2} \sim O\Big(\frac{U}{h^2}\Big)\mathbf{i} + O\Big(\frac{U}{h^2}\Big)\mathbf{j} + O\Big(\delta\frac{U}{h^2}\Big)\mathbf{k}.$$

These arguments suggest that we can assume

$$|\mathbf{u} \cdot \nabla \mathbf{u}| \ll |\nabla^2 \mathbf{u}| \quad \text{if} \quad \delta^2\Big(\frac{UL}{\nu}\Big) \ll 1.$$

Defining the Reynolds number $R = UL/\nu$ as usual, and introducing the *modified* Reynolds number

$$R_m = \delta^2 R,$$

this means that if $R_m \ll 1$, we may neglect the nonlinear terms and approximate the thin film flow with the *lubrication equations*:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2}, \quad 0 = -\frac{\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial z^2}, \quad 0 = -\frac{\partial p}{\partial z} + \mu\frac{\partial^2 w}{\partial z^2}, \tag{7.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{7.2}$$

Note that we do **not** require $R \ll 1$ for this approximation; only $R_m \ll 1$.

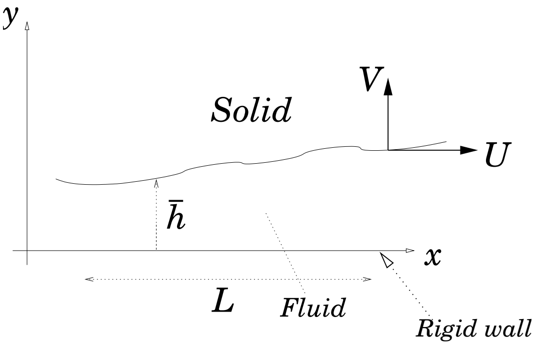
Applications

The applications of the lubrication approximation are manifest. Some examples:

1. Flow in a slider-bearing (see below).

2. Blood flow in slowly-tapering arteries. In some situations, a lubrication model is appropriate.

3. Peeling of the retina away from the choroid during retinal detachment, when part of the vitreous humour in the back of the eye has liquefied. This is a topic of current research.



4. Modelling the thin film of fluid drawn over the eye during a blink. A lubrication model predicts that this film will break-apart after approximately 15 seconds. (Try not to blink for longer than that!) This is also the a subject of current research.

Formal derivation of the lubrication equations

In a typical configuration, such as that shown above, we have two lengthscales, $\overline{h}$ and L (and generally speaking we are interested in the case $\overline{h} \ll L$). It is therefore sensible to non-dimensionalize by setting

$$x = LX \quad \text{and} \quad y = \overline{h}Y.$$

If the solid wall is moving with horizontal speed U, this suggests we take $u = U\hat{U}$. For an order of magnitude balance of terms in the continuity equation, we expect

$$\frac{v}{\overline{h}} \sim \frac{U}{L} \implies v \sim O\Big(U\frac{\overline{h}}{L}\Big),$$

and so this suggests we write $v = U(\overline{h}/L)\hat{V}$.

A typical scale for the pressure comes from considering an order of magnitude balance between the pressure gradient and the largest occuring term in the momentum equation. In other words, since the layer is thin, assuming

$$\frac{\partial p}{\partial x} \sim \mu\frac{\partial^2 u}{\partial y^2},$$

we estimate

$$\frac{p}{L} \sim \mu\frac{U}{h^2} \implies p \sim \mu\frac{UL}{\overline{h}^2},$$

suggesting that we write $p = \mu(UL/\overline{h}^2)\hat{p}$.

Assuming steady flow (so $\partial/\partial t \equiv 0$), we non-dimensionalize using these expressions, i.e.,

$$x = LX, \qquad y = \overline{h}Y$$

$$u = U\hat{U}, \qquad v = U(\overline{h}/L)\hat{V}, \qquad p = \mu(UL/\overline{h}^2)\hat{p}.$$

Substituting into the two-dimensional Navier-Stokes equations we find

$$\begin{aligned} R_m\Big[\hat{U}\hat{U}_X + \hat{V}\hat{U}_Y\Big] &= -\hat{p}_X + \hat{U}_{YY} + \delta^2\hat{U}_{XX}, \\ R_m\Big[\hat{U}\hat{V}_X + \hat{V}\hat{V}_Y\Big] &= -\hat{p}_Y + \delta^2\hat{V}_{YY} + \delta^4\hat{V}_{XX}, \\ \hat{U}_X + \hat{V}_Y &= 0, \end{aligned}$$

where $\delta = \overline{h}/L$, and the *modified Reynolds number*

$$R_m = \delta^2 R, \qquad R = \frac{UL}{\nu}.$$

In the limit $\delta \to 0$ *and* $R_m \to 0$, we obtain the a reduced set of equations,

$$\begin{aligned} 0 &= -\hat{p}_X + \hat{U}_{YY}, \\ 0 &= -\hat{p}_Y \\ \hat{U}_X + \hat{V}_Y &= 0. \end{aligned}$$

Thus, for a lubricating flow, the above analysis shows that we can use an approximate form of the Navier-Stokes equations, given by

$$0 = -p_x + \mu\, u_{yy}, \tag{7.3}$$
$$0 = -p_y \tag{7.4}$$
$$u_x + v_y = 0, \tag{7.5}$$

with boundary conditions

$$u = v = 0 \quad \text{on} \quad y = 0; \qquad u = U, \quad v = V, \quad \text{on} \quad y = h(x). \tag{7.6}$$

The approximation is valid in the limit $\delta \to 0$ and $R_m \to 0$.

We notice that $p = p(x)$, according to (7.4), and so we can integrate (7.3) twice with respect to y to obtain

$$u = \frac{1}{2\mu}\Big[y^2 - yh\Big]p_x + \frac{Uy}{h}.$$

Note that the first term in this expression corresponds locally to plane Poiseuille flow while the second represents plane Couette flow.

The vertical velocity component, v, comes from integrating the continuity equation as follows.

$$v \;=\; -\int_0^y \frac{\partial u}{\partial x}\,\mathrm{d}y \quad (\text{so } v = 0 \text{ on } y = 0 \text{ is satisfied}).$$

For the next bit, we will need to use the following result (see Problem Sheet 4),

$$\frac{\partial}{\partial x}\int_0^{h(x)} f(x,y)\,\mathrm{d}y = \int_0^{h(x)} \frac{\partial f}{\partial x}\,\mathrm{d}y + f(x,h)\frac{\mathrm{d}h}{\mathrm{d}x}.$$

Demanding that $v = V$ on $y = h$ we find

$$\begin{aligned} V &= -\int_0^{h(x)} \frac{\partial u}{\partial x}\,\mathrm{d}y, \\ \text{thus} \quad -V &= \frac{\partial}{\partial x}\int_0^{h(x)} u\,\mathrm{d}y - U\frac{\mathrm{d}h}{\mathrm{d}x}. \end{aligned} \tag{7.7}$$

Meanwhile, the flux

$$\begin{aligned} \int_0^{h(x)} u\,\mathrm{d}y &= \int_0^h \left\{\frac{1}{2\mu}\Big[y^2 - yh\Big]p_x + \frac{Uy}{h}\right\}\mathrm{d}y, \\ &= \frac{1}{2\mu}p_x\Big[-\frac{h^3}{6}\Big] + \frac{Uh}{2} \\ \text{and so} \quad \frac{\partial}{\partial x}\int_0^{h(x)} u\,\mathrm{d}y &= \frac{\partial}{\partial x}\Big(\frac{1}{2\mu}p_x\Big[-\frac{h^3}{6}\Big]\Big) + \frac{U}{2}\frac{\mathrm{d}h}{\mathrm{d}x}. \end{aligned} \tag{7.8}$$

Combining (7.8) with (7.7), we obtain the **Reynolds lubrication equation**:

$$\frac{1}{12\mu}\frac{\partial}{\partial x}\Big(h^3\frac{\partial p}{\partial x}\Big) = -\frac{1}{2}U\frac{\mathrm{d}h}{\mathrm{d}x} + V. \tag{7.9}$$

This equation involves a second order derivative in x and so two boundary conditions are required. The height function h is normally given. We usually solve (7.9) subject to

$$\begin{aligned} p = p_1 \quad &\text{at} \quad x = x_1, \\ p = p_2 \quad &\text{at} \quad x = x_2. \end{aligned}$$

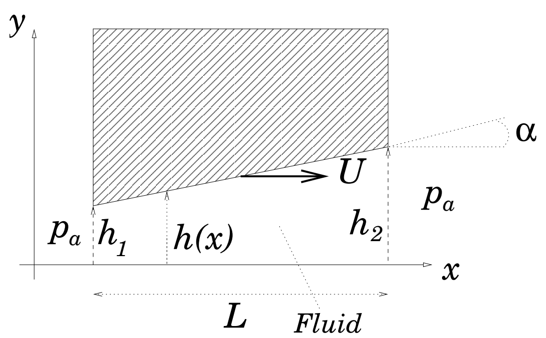
NOTE: The lubrication equation (7.9) also applies when U and V are functions of time, t, provided that

$$\frac{\partial}{\partial t} \ll \nu \frac{\partial^2}{\partial y^2}.$$

We will make use of this fact later.

Flow in a slider bearing

When two solid bodies slide against each other, the frictional resistance and normal reaction force are comparable in magnitude. However, if there is a thin film of fluid in between the bodies, the tangential resistance may be a lot smaller than the normal, pressure force. A good example of this is a slider bearing moving over a solid wall at $y = 0$ with fluid in between (see diagram).



Assume that the angle $\alpha \ll 1$. The pressure on either side of the bearing is atmospheric, which we are at liberty to set to zero. For convenience we change variable from x to h and so, since $h \approx \alpha x + const$, $\mathrm{d}/\mathrm{d}x = \alpha \mathrm{d}/\mathrm{d}h$. Then the Reynolds Lubrication Equation becomes:

$$\frac{1}{12\mu}\alpha^2 \frac{\mathrm{d}}{\mathrm{d}h}\left\{h^3 \frac{\mathrm{d}p}{\mathrm{d}h}\right\} = -\frac{1}{2}U\alpha.$$

Integrating,

$$p = \frac{6\mu U}{\alpha h} + F - \frac{E}{2h^2}, \qquad \text{const} \quad E, F.$$

Choosing E, F such that $p = 0$ at $h = h_1,\ h_2$, we find

$$E = \frac{12\mu}{\alpha}\frac{U h_1 h_2}{h_1 + h_2}, \quad F = -\frac{6\mu U}{\alpha}\frac{1}{(h_1 + h_2)}. \tag{7.10}$$

Then,

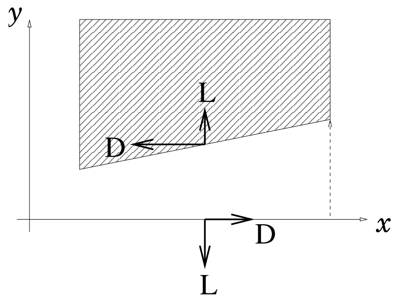
$$p = \frac{6\mu U (h_2 - h)(h - h_1)}{h^2 \alpha (h_1 + h_2)} > 0 \quad \text{and} \quad U > 0 \quad \text{for} \quad h_1 < h < h_2.$$

If the system is closed and the forces at both ends are negligible, then a force balance shows that the lift and drag forces on the bearing are equal and opposite to those on the wall, as shown:

For simplicity, we therefore compute D and L on the lower wall (LW).

Drag force:

$$D = \int \sigma_{12}\,\mathrm{d}x = \mu \int_{LW} \left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}_{y=0} \mathrm{d}x = \int_{LW} \mu \frac{\partial u}{\partial y}\Big|_{y=0} \mathrm{d}x.$$



Now,

$$\frac{\partial u}{\partial y} = -\frac{h}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{U}{h},$$

$$\implies D = \mu\frac{U}{\alpha}\int_{h_1}^{h_2}\Big\{\frac{4}{h} - \frac{6h_1h_2}{(h_1+h_2)^2h^2}\Big\}\,\mathrm{d}h.$$

And so,

$$D = \mu\frac{U}{\alpha}\Big[4\log\Big(\frac{h_2}{h_1}\Big) - \frac{6(h_1-h_2)}{h_1+h_2}\Big].$$

Now the lift force, L:

$$\begin{aligned} L &= \int -\sigma_{22}\,\mathrm{d}x = -\int_{LW}\Big\{-p+\mu\{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\}_{y=0}\Big\}\,\mathrm{d}x \\ \text{(since } \tfrac{\partial v}{\partial y} = -\tfrac{\partial u}{\partial x} = 0 \text{ on } y=0) &= \int_{h_1}^{h_2}\frac{p}{\alpha}\,\mathrm{d}h \\ &= \frac{6\mu U}{\alpha^2}\Big\{\log\Big(\frac{h_2}{h_1}\Big) - \frac{2(h_2-h_1)}{h_1+h_2}\Big\}. \end{aligned}$$

But $h_2 - h_1$ is of $O(\alpha)$. Therefore,

$$\frac{D}{L} = \frac{\alpha}{3}\Big\{1+O(\alpha)\Big\}$$

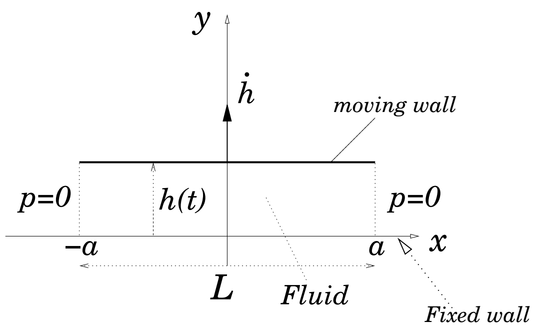
$$\implies \frac{L}{D} \sim 1/\alpha \quad \text{and so} \quad L \gg D \quad \text{if} \quad \alpha \ll 1.$$

In other words, the normal force is much larger than the tangential force and the bearing can withstand heavy loads.

Adhesive problems

We now return to the coin problem mentioned in the introduction to this section. An object sitting on top of a thin film of fluid on a table can be surprisingly tricky to pick up. We have already seen how a thin fluid layer can lead to normal forces which are much larger than tangential ones.

Consider two parallel plane walls, with one of the walls moving away from the other as shown:



We assume that
$$\frac{\partial}{\partial t} \ll \nu \frac{\partial^2}{\partial y^2},$$
so the flow is steady.

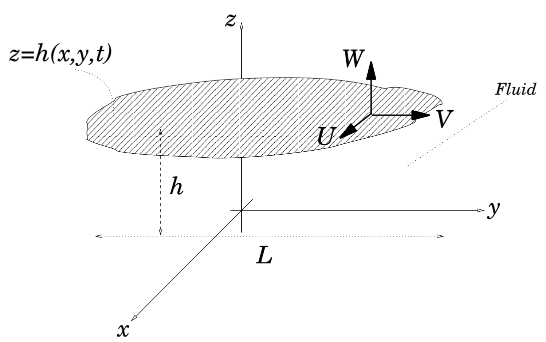
We need to solve the Reynolds lubrication equation,
$$\frac{1}{12\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) = -\frac{1}{2}U\frac{\mathrm{d}h}{\mathrm{d}x} + V,$$
together with the boundary conditions
$$u = v = 0 \quad \text{on} \quad y = 0; \qquad u = 0, \quad v = \dot{h} \quad \text{on} \quad y = h(t).$$
In this case, then, $U = 0$, $V = \dot{h}$ and h is independent of x. Thus (7.11) becomes
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h^3\frac{\mathrm{d}p}{\mathrm{d}x}\right) = 12\mu\dot{h},$$
and so, on integration, we have,
$$p = \frac{6\mu}{h^3}\dot{h}(x^2 - a^2).$$
We want to calculate the force, F, resisting the motion. As for the slider bearing, we assume the forces at the ends are negligible so the force exerted by the fluid on the stationary wall is equal and opposite to that exerted on the moving wall. We can thus compute the resistance by calculating minus the force exerted on the lower wall. We know that
$$\begin{aligned} F &= -\int_{-a}^{a} \sigma_{22}|_{y=0}\,\mathrm{d}x, \quad (\text{since } v_y|_{y=0} = 0 \text{ from cty}) \quad = \int_{-a}^{a} p\,dx \\ &= \frac{6\mu\dot{h}}{h^3}\int_{-a}^{a}(x^2 - a^2)\,\mathrm{d}x \quad = -\frac{8\mu a^3\dot{h}}{h^3}. \end{aligned}$$
Note that force is negative so the fluid is trying to pull the moving wall back downwards.

The most interesting thing about this result is that $F \sim O(h^{-3})$ as $h \to 0$, which implies a huge resistance force from a very thin film!

An adhesive problem in 3-D

Consider a similar problem to the above, but this time in three dimensions. This is closer to the coin problem mentioned in the introduction.

A (possibly curved) surface sits on a thin film of fluid as shown in the diagram.



We will assume that the horizontal velocity components of the surface are of similar size, i.e. $U \sim V$. As usual we assume $h \ll L$ and the modified Reynolds number $R_m \ll 1$.

Furthermore, we expect changes in the z direction to dwarf those in the x and y directions, by analogy with the two-dimensional case,

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y} \sim O\Big(\frac{1}{L}\Big) \ll \frac{\partial}{\partial z} \sim O\Big(\frac{1}{h}\Big).$$

It seems sensible to define

$$u = U\hat{u}, \qquad v = V\hat{v}, \qquad w = \frac{Uh}{L}\hat{w},$$

where the vertical velocity scale comes from the continuity equation $u_x + v_y + w_z = 0$, which we use to obtain the estimate

$$w \sim O\big(U\frac{h}{L}\big).$$

As before, the pressure scale is fixed by the expectation that

$$\frac{\partial p}{\partial x} \sim \mu\,\frac{\partial^2 u}{\partial y^2},$$

which suggests that $p \sim \mu UL/h^2$, so we can write $p = (\mu UL/h^2)\hat{p}$.

Following exactly the same procedure as in the 2-D case, we non-dimensionalize the equations and take the limits $\delta \to 0$ and $R_m \to 0$. Restoring the dimensions to the reduced set of equations, we have,

$$\begin{aligned} 0 &= -p_x + \mu\, u_{zz}, && (7.11)\\ 0 &= -p_y + \mu\, v_{zz}, && (7.12)\\ 0 &= -p_z \\ u_x + v_y + w_z &= 0, \end{aligned}$$

with boundary conditions

$$u = v = w = 0 \quad \text{on} \quad y = 0; \qquad u = U, \quad v = V, \quad w = W \quad \text{on} \quad y = h(x, y, t).$$

Note: As for the two-dimensional problem, there is no vertical pressure gradient.

Since $p = p(x, y, t)$, we can integrate (7.11, 7.12) with respect to z twice. We find

$$u = \frac{1}{2\mu} p_x \Big\{ z^2 - zh \Big\} + \frac{Uz}{h},$$

$$v = \frac{1}{2\mu} p_y \Big\{ z^2 - zh \Big\} + \frac{Vz}{h}.$$

Again, we use the continuity equation to find the vertical velocity component:

$$-w = \int_0^z u_x + v_y \,\mathrm{d}z.$$

Thus, at $z = h$,

$$\begin{aligned} -W &= \int_0^h u_x + v_y \,\mathrm{d}z \\ \text{so} \quad -W &= \frac{\partial}{\partial x}\int_0^h u\,\mathrm{d}z + \frac{\partial}{\partial y}\int_0^h v\,\mathrm{d}z - Uh_x - Vh_y. \end{aligned}$$

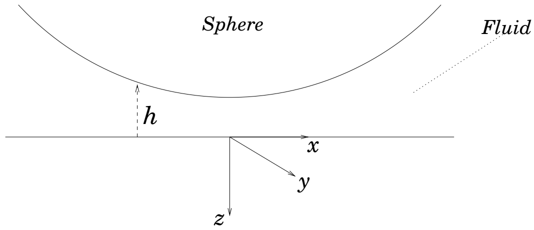
Using the above results for u and v, and integrating, we obtain the **three-dimensional** version of the Reynolds lubrication equation:

$$\frac{\partial}{\partial x}\Big[h^3 \frac{\partial p}{\partial x}\Big] + \frac{\partial}{\partial y}\Big[h^3 \frac{\partial p}{\partial y}\Big] = 6\mu\Big\{ - Uh_x - Vh_y + 2W \Big\} \qquad (7.13)$$

We make use of this equation in the following example.

Impact of a sphere on a plane

A sphere falls slowly through a viscous fluid towards a plane wall. Close to the moment of impact, it squeezes a thin layer of fluid between itself and the flat surface as shown.



We assume that the sphere falls slowly enough to justify a steady lubrication analysis. In other words, we assume that $\partial/\partial t \ll \nu\partial^2/\partial z^2$.

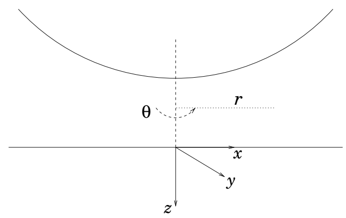
On the surface of the sphere $u = v = 0$ and $w = \dot{h}$, so $U = V = 0$ and $W = \dot{h}$. In this case, Reynolds lubrication equation states that

$$\frac{\partial}{\partial x}\left[\frac{h^3}{12}\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{h^3}{12}\frac{\partial p}{\partial y}\right] = \mu W.$$

Alternatively, we may write this as

$$\nabla \cdot \left[\frac{h^3}{12}\nabla p\right] = \mu W. \tag{7.14}$$

Appealing to the symmetry of the problem, we expect that $p = p(r)$, where $r^2 = x^2 + y^2$. It is therefore more sensible to discuss the problem in cylindrical polar coordinates.

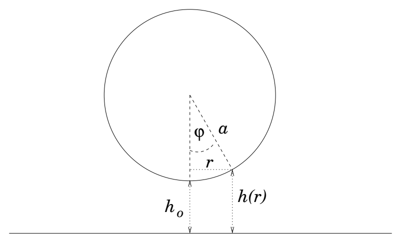


Recalling that in these coordinates, $\nabla \cdot (A\hat{\mathbf{r}}) = (1/r)\partial(rA)/\partial r$, equation (7.14) becomes

$$\begin{aligned} \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[rh^3\frac{\mathrm{d}p}{\mathrm{d}r}\right] &= 12\mu W, \\ \text{(integrating)} \quad rh^3 p_r &= 6\mu W r^2 + c. \end{aligned} \tag{7.15}$$

But we must choose $c = 0$ else, since h is finite when $r \to 0$, $p_r \to \infty$ as $r \to 0$. So $p_r = 6\mu W r/h^3$.

Suppose that the sphere's radius is a. For the lubrication approximation to hold, we need $h \ll a$. This will be true when r is small.



Near $r = 0$, the angle $\phi \ll 1$ and

$$h = h_0 = a - a\cos\phi \approx \frac{1}{2}a\phi^2, \qquad \frac{r}{a} = \sin\phi \approx \phi.$$

$$\text{So} \quad h - h_0 = \frac{r^2}{2a} \quad \text{approximately.}$$

To calculate the force on the sphere, we note that the force in the z direction on the wall, F, is equal and opposite to that on the sphere. Integrating by parts,

$$F \;=\; \int_0^\infty 2\pi r p(r)\,\mathrm{d}r = \Big[\pi r^2 p(r)\Big]_0^\infty - \int_0^\infty \pi r^2 \frac{\mathrm{d}p}{\mathrm{d}r}\,\mathrm{d}r.$$

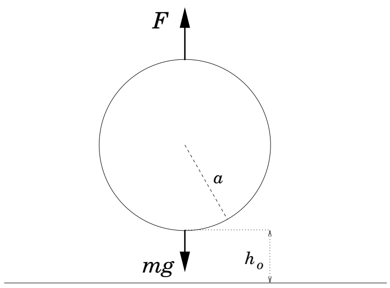
Now, within the lubrication limit, we found above that $h = h_0 + r^2/2a$. Thus, when r is large, but *not large enough* to destroy the validity of the lubrication approximation, we have $h \sim O(r^2)$. Using the result above for the pressure, it follows that

$$p_r \sim O\Big(\frac{r}{h^3}\Big) \sim O\Big(\frac{1}{r^5}\Big) \quad \text{for large } r.$$

We can thus set $\Big[\pi r^2 p(r)\Big]_0^\infty = 0$ and deduce that

$$\begin{aligned}
F \;&=\; \int_0^\infty \pi r^2 \frac{\mathrm{d}p}{\mathrm{d}r}\,\mathrm{d}r = -6\mu\pi W \int_0^\infty \frac{r^3}{[h_0 + r^2/2a]^3}\,\mathrm{d}r \\
(\text{setting } r^2 = 2ah_0\,\xi) \;&=\; -\frac{12\mu\pi a^2 W}{h_o} \int_0^\infty \frac{\xi}{(1+\xi)^3}\,\mathrm{d}\xi \\
\text{So} \quad F \;&=\; -\frac{6\mu\pi a^2 W}{h_o} \quad \Big(\text{since} \quad \int_0^\infty \frac{\xi}{(1+\xi)^3}\,\mathrm{d}\xi = \frac{1}{2}\Big).
\end{aligned}$$

An interesting consequence of this is that the sphere never reaches the wall!



Taking $W = \dot{h}_0$, the equation of motion of the sphere is

$$m\ddot{h}_0 = -6\mu\pi a^2 \frac{\dot{h}_0}{h_0} - mg.$$

This can be written as

$$\ddot{h}_0 + \kappa \frac{\mathrm{d}}{\mathrm{d}t}(\log h_0) = -g$$

for positive constant κ. Integrating once, we find

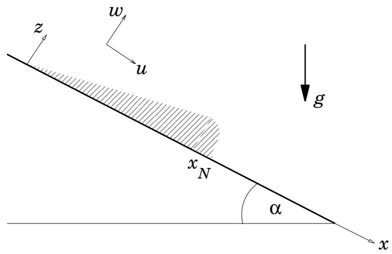
$$\dot{h}_0 + \kappa \log\Big(\frac{h_0}{H_0}\Big) = -gt + \dot{H}_0,$$

on taking $h_0 = H_0$, $\dot{h}_0 = \dot{H}_0$ at $t = 0$.

As $t \to \infty$, the right hand side approaches negative infinity. Since $\dot{h} > 0$ for all time, this must be balanced by $h_0 \to 0$ as $t \to \infty$ on the left hand side. So it takes an infinite amount of time for h_0 to reduce to zero and for the sphere to reach the wall.

Flow of a thin film down a sloped wall

A thin film of viscous fluid spreads along a plane wall which is inclined at an angle α to the horizontal, as shown in the diagram. The nose of the film is at $x_N(t)$.



Starting with the two-dimensional lubrication equations (7.3, 7.4, 7.5), we need to add body force terms to account for gravity. We therefore write

$$\begin{aligned} 0 &= -p_x + \mu\, u_{zz} + \rho g \sin\alpha, && (7.16)\\ 0 &= -p_z - \rho g \cos\alpha, \\ u_x + w_z &= 0, \end{aligned}$$

where z is now playing the role of the vertical coordinate and w the vertical fluid speed.

Integrating the second equation,

$$p = -\rho g z \cos\alpha + f(x,t),$$

for some function f.

At the free surface $z = h(x,t)$,

i) the pressure must be atmospheric, p_a, so

$$p = \rho g[h - z]\cos\alpha + p_a,$$

ii) the tangential stress must equal zero. Since we are implicitly assuming a rather slender film, the normal vector points almost vertically. Consistent with the approximations already made in deriving the lubrication equations, we take the normal to point exactly vertically[8] and require

$$2\mu e_{xz} = \mu\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big) = 0$$

as the zero tangential stress condition. Note that the tangential direction is almost in the x direction, and is taken exactly in the x direction to within a consistent approximation.

But in the lubrication approximation,

$$\frac{\partial u}{\partial z} \gg \frac{\partial w}{\partial x},$$

and so we need $\partial u/\partial z = 0$ on $z = h$.

Substituting for the pressure, (7.16) is now,

$$\nu u_{zz} = -g\sin\alpha + g h_x \cos\alpha, \qquad (\nu = \mu/\rho).$$

[8]This is clearly not true near the nose of the film where the surface is highly curved. As a first estimate, we ignore this fact and proceed regardless. The results turn out to be surprisingly accurate, a not uncommon feature of lubrication-type analyses. More accurate calculations can be made taking into account the curvature at the nose, but these are beyond the scope of the current course.

Again, appealing to the fact that the film is thin, we can drop h_x as it is expected to be small. This leaves

$$\nu u_{zz} = -g \sin\alpha,$$

which is easily integrated. Applying no slip at $z = 0$ and the zero tangential stress condition at $z = h$ we therefore find

$$u = \frac{g \sin\alpha}{\nu}(hz - \frac{1}{2}z^2).$$

As usual, we can find the vertical velocity component by integrating the continuity equation:

$$\begin{aligned} w_z &= -\frac{g \sin\alpha}{\nu} h_x z, \\ \text{(using } w = 0 \text{ at } z = 0) \implies \quad w &= -\frac{g \sin\alpha}{2\nu} h_x z^2. \end{aligned}$$

Lastly we need a *kinematic condition* at the free surface $z = h$ (see the section on boundary conditions earlier in the course). In fact, we need:

$$\begin{aligned} \frac{\mathrm{D}}{\mathrm{D}t}(z - h) &= 0, \\ \text{i.e.} \qquad w &= \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{on} \quad z = h(x,t). \end{aligned}$$

Substituting for u and w, evaluated at $z = h$, we obtain

$$-\frac{g \sin\alpha}{2\nu}\frac{\partial h}{\partial x}h^2 = \frac{\partial h}{\partial t} + \frac{gh^2 \sin\alpha}{2\nu}\frac{\partial h}{\partial x}.$$

Rearranging, we have the evolution equation for h,

$$\frac{\partial h}{\partial t} + \frac{g \sin\alpha}{\nu}h^2\frac{\partial h}{\partial x} = 0. \tag{7.17}$$

This equation has a solution

$$h = \Big(\frac{\nu}{g \sin\alpha}\Big)^{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{t^{\frac{1}{2}}},$$

which is only expected to be valid at large times.

Since the total volume of the film,

$$\int_0^{x_N(t)} h(x,t)\,\mathrm{d}x = A, \text{ say},$$

must be constant, we find

$$\begin{aligned} A &= \int_0^{x_N(t)} h(x,t)\,\mathrm{d}x = \int_0^{x_N} \Big(\frac{\nu}{g \sin\alpha}\Big)^{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{t^{\frac{1}{2}}}\,\mathrm{d}x & (7.18) \\ &= \frac{2}{3}\Big(\frac{\nu}{g \sin\alpha}\Big)^{\frac{1}{2}} \frac{x_N^{\frac{3}{2}}}{t^{\frac{1}{2}}}. & (7.19) \end{aligned}$$

Therefore,

$$x_N(t) = \Big(9A^2\frac{g \sin\alpha}{4\nu}\Big)^{1/3} t^{1/3}.$$

So the speed of the nose is given by

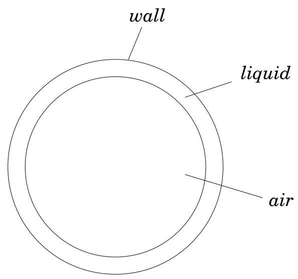
$$\frac{\mathrm{d}x_N}{\mathrm{d}t} \propto t^{-2/3}.$$

So the front of the film spreads more slowly as time increases.

NOTE: Despite neglecting effects associated with the curvature around the nose of the spreading film, the prediction for the rate of spread agrees well with experiments (see Huppert (1986), *Journal of Fluid Mechanics*, vol. 173, pp. 557-594. This journal is available in the Library).

7.5 Limitations of lubrication theory

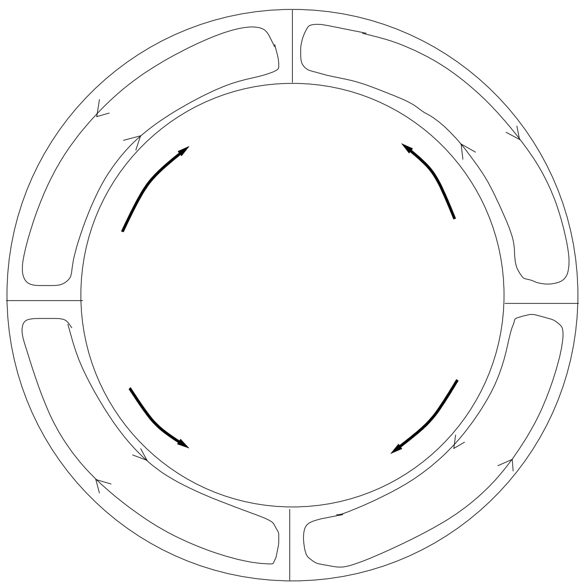
We have seen above that lubrication theory has its limitations and may become invalid in part of the flow field. It is interesting to note, however, that certain flow features such as eddies are not necessarily precluded by the restrictions placed upon a lubrication analysis. For example, consider an annular liquid layer at rest on the inside of a cylindrical tube as shown.



In practice such a fluid layer would become longitutdinally unstable along the tube axis due to the Rayleigh instability and break up to form a sequence of liquid collars. Disregarding this issue, we permit the surface tension at the air-liquid interface to vary so that

$$\gamma = \cos 2\theta.$$

The resulting Marangoni stress will drive a flow in the liquid layer. Solving the lubrication equations in the liquid layer, we find that the flow looks like this[9].



The thick-lined arrows indicate the direction of the driving Marangoni force $\gamma_\theta = -2\sin 2\theta$. Notice that four viscous eddies are established circling the interior of the film.

The presence of the eddies might seem surprising at first. How can the fluid turn a corner and not contravence the restrictions of the lubrication analysis? This stipulates that while the streamwise velocity (here the azimuthal component) u is $O(1)$, the vertical component (here the radial component) should be $O(\delta)$. Yet as an eddy turns a corner, u is zero and v is comparatively large.

[9]See L. Band, *PhD thesis*, University of Nottingham, 2007

Careful thought shows that the restrictions are not in fact contravened. All that is happening is that, as θ increases, u passes through zero. Provided u still varies slowly with respect to θ, the requirements of lubrication theory are still met.

8. Boundary layers

The Reynolds number, defined as $R = UL/\nu$, for typical velocity and lengthscales U, L, appears in the non-dimensional form of the Navier-Stokes equations:

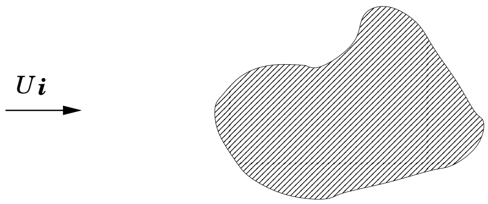
$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -\nabla p + \frac{1}{R}\nabla^2 \mathbf{u}. \tag{8.1}$$

When R is very small we have *Stokes flow*, which we have already discussed.

What happens when R is very large? Letting the Reynolds number in equations (8.1) tend to infinity, it would appear that we can just drop the viscous term $\nabla^2\mathbf{u}/R$ on the grounds that it is very small. This would leave us with Euler's equations governing an inviscid fluid. In dimensional form these are

$$\nabla \cdot \mathbf{u} = 0, \qquad \rho\mathbf{u} \cdot \nabla \mathbf{u} \;=\; -\nabla p.$$

Suppose we ask the question: *what is the drag on an object placed in a high Reynolds number flow?*



From section ??, D'Alembert's Paradox tells us that the drag on the object is precisely zero! But this is clearly at odds with experience. Experiments of high Reynolds number flows past a circular cylinder, for example, predict a drag on the cylinder which is most certainly not zero.

What is going on?

Notion of a boundary layer

Consider the following model problem

$$\epsilon f'' + f' = a, \qquad f = f(y), \tag{8.2}$$

with boundary conditions

$$f(0) = 0, \quad f(1) = 1.$$

The constant a is positive and the parameter ϵ is assumed to be very small.

The exact solution satisfying the boundary conditions is

$$f = ay + \frac{(1-a)}{(1-\mathrm{e}^{-1/\epsilon})}(1 - \mathrm{e}^{-y/\epsilon}).$$

For $y = 0(1)$, i.e. not small,

$$f \sim ay + 1 - a = f_0(y), \text{ say.}$$

This can be thought of as an outer, 'inviscid' solution, valid in a region away from $y = 0$. It can be seen to be the solution of equation (8.2) if we naïvely set ϵ to zero on the grounds that it is small. Crucially, however, it satisfies the condition at $y = 1$ but *not* that at $y = 0$. So it is not the complete solution.

Suppose we now focus on what happens when y is small. To this end, we introduce the change of variables $y = \epsilon Y$, so that if Y is of order one, y will be small. The equation becomes

$$f_{YY} + f_Y = \epsilon a.$$

Since $\epsilon \ll 1$, to a first approximation we may set the right hand side to zero (as we did to obtain $f_0(y)$ above), leaving

$$f_{YY} + f_Y = 0.$$

Integrating this equation yields

$$\begin{aligned} f &= A(1 - \mathrm{e}^{-Y}), \qquad \text{for constant } A \\ &= A(1 - \mathrm{e}^{-y/\epsilon}) \\ &= f_i\,, \ \text{say.} \end{aligned}$$

It is noticeable that this function satisfies the condition at $y = Y = 0$ but *not* that at $y = 1$.

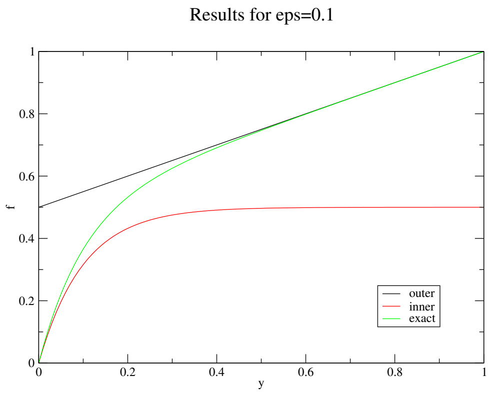
To summarize, we have an inner solution f_i, which satisfies the boundary condition at $y = 0$, and an outer solution, f_o, which satisfies the boundary condition at $y = 1$. We can tie these solutions together by using the following matching procedure

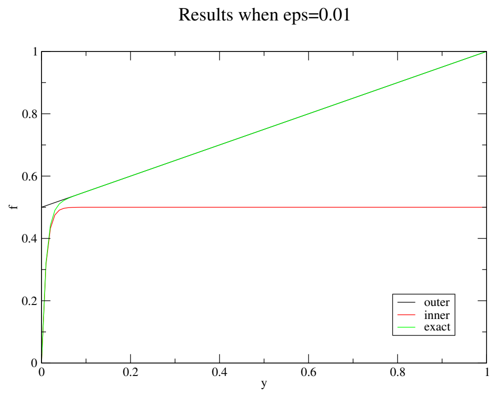
$$\lim_{Y \to \infty} f_i = \lim_{y \to 0} f_0.$$

We then find that $A = (1 - a)$, so now

$$f_i = (1 - a)(1 - \mathrm{e}^{-Y}).$$

The following graphs illustrate these solutions when $\epsilon = 0.1$ and $\epsilon = 0.01$.





Clearly the inner solution only performs well near $y = 0$, while the outer works well away from $y = 0$. The agreement between the approximate inner and outer solutions and the exact solution improves as $\epsilon \to 0$.

The key feature of the above toy problem is the clear development of a region (here near $y = 0$) in which there is a rapid change of the solution. The outer solution has only a slight slope. However, near $y = 0$, the inner solution takes over and exhibits a *dramatic change over a very short distance*. In this very thin region, it is **not** appropriate to drop the term

$$\epsilon f''$$

in the equation. The thin region near $y = 0$ is an example of a *boundary layer*.

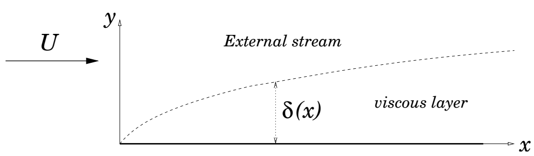
Something similar is happening with the high speed flow problem discussed above. There we may indeed neglect the terms

$$\frac{1}{R}\nabla^2\mathbf{u}$$

over most of the flow field. However, very close to the surface of the body there is a boundary layer in which this term is no longer negligible.

High Reynolds number flow past a flat plate

Consider the classical problem of fluid streaming at high speed past a flat plate:



The flow is taken to be steady. The external stream consists of inviscid flow moving with velocity $\mathbf{u} = U\,\mathbf{i}$, with U constant.

Let L be a characteristic horizontal lengthscale. If the plate is finite, L can be chosen to be its length. The Reynolds number is

$$R = \frac{UL}{\nu} \gg 1,$$

and so is assumed to be very large.

Following the above discussion, we anticipate that the viscous term $R^{-1}\nabla^2\mathbf{u}$ will only be important in a very thin boundary layer of height $\delta(x)$ close to the plate.

We aim to derive a set of approximate equations valid in the boundary layer. We will first do this using an argument based on the estimation of the order of magnitude of all the terms. Later we will go through a more formal derivation.

The continuity equation states

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Since u varies from zero at the plate (no-slip) to U in the external stream, we can say that

$$u \sim U, \quad \text{typically}.$$

Changes in the boundary layer are expected to occur on a lengthscale of typical size L parallel to the plate and δ perpendicular to it (i.e. there is much more rapid change in the y direction than the x direction). We can therefore estimate

$$\frac{\partial}{\partial x} \sim \frac{1}{L}, \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}.$$

If V is a typical size for the vertical velocity v, a balance of terms in the continuity equation suggests that

$$\frac{U}{L} \sim \frac{V}{\delta} \implies V \sim \frac{\delta}{L} U.$$

Now consider the x-momentum Navier-Stokes equation

$$uu_x + vu_y = -\frac{1}{\rho} p_x + \nu[u_{xx} + u_{yy}].$$

Following the above arguments, we can estimate

$$\frac{\partial^2}{\partial x^2} \sim \frac{1}{L^2}, \qquad \frac{\partial^2}{\partial y^2} \sim \frac{1}{\delta^2}.$$

So, we expect u_{yy} to be a lot bigger than u_{xx}. In fact

$$u_{yy} \sim \frac{U}{\delta^2}.$$

Also, we estimate that

$$uu_x \sim \frac{U^2}{L}.$$

The next step is crucial and is due to Prandtl (1905).

Prandtl's Hypothesis

This states that the inertial term, uu_x, should balance the viscous one, νu_{yy}, inside the boundary layer, i.e.

$$uu_x \sim \nu u_{yy} \implies \frac{U^2}{L} \sim \nu \frac{U}{\delta^2}.$$

Simplifying, we obtain the following estimate of the boundary layer thickness,

$$\frac{\delta}{L} \sim \left(\frac{\nu}{UL}\right)^{\frac{1}{2}},$$

which can be re-stated as

$$\delta \sim LR^{-1/2}.$$

So viscous effects are only expected to be important in a layer of thickness $R^{-1/2}$. Since R is very large, this layer is very thin.

Continuing, we can estimate

$$uv_y \sim U\frac{V}{\delta} = U\frac{U\delta/L}{\delta} = \frac{U^2}{L} \sim uu_x,$$

and so these two terms balance.

If we choose a scale for the pressure so that

$$\frac{1}{\rho}p_x \sim uu_x \implies p \sim \rho U^2,$$

then, in summary, the boxed terms in the x-momentum equation are comparable for large R,

$$\boxed{uu_x} + \boxed{vu_y} = \boxed{-\tfrac{1}{\rho}p_x} + \nu(u_{xx} + \boxed{u_{yy}}).$$

So, retaining the dominant terms, the x-momentum equation reduces to

$$uu_x + vu_y = -\frac{1}{\rho}p_x + \nu u_{yy}.$$

In the y-momentum equation,

$$uv_x + vv_y = -\frac{1}{\rho}p_y + \nu(v_{xx} + v_{yy}),$$

we have the estimates

$$p_y \sim \rho\frac{U^2}{\delta}, \quad uv_x \sim vv_y \sim U\frac{\delta}{L}\frac{U}{L} = \frac{\delta U^2}{L^2} \ll \rho\frac{U^2}{\delta}.$$

$$\nu v_{xx} \sim \nu\frac{V}{L^2} = \nu\frac{\delta}{L}\frac{U}{L^2} = \frac{1}{R}\frac{\delta}{L}\frac{U^2}{L} \ll \rho\frac{U^2}{\delta}, \quad \nu v_{yy} \sim \frac{V}{\delta^2} = \left(\frac{1}{R\delta^2}\right)\delta U^2 \sim \delta U^2 \ll \rho\frac{U^2}{\delta}.$$

In the above, we have used the fact that $R = UL/\nu$ and the Prandtl hypothesis that $R\delta^2 \sim 1$.

Therefore, for this equation, only the boxed term is important

$$uv_x + vv_y = \boxed{-\tfrac{1}{\rho}p_y} + \nu(u_{xx} + u_{yy}),$$

i.e., the y-momentum reduces to just

$$p_y = 0.$$

This states that the pressure does not vary across the boundary layer.

Collecting everything together, we have the Prandtl **boundary layer** equations

$$\begin{aligned} uu_x + vu_y &= -\frac{1}{\rho}p_x + \nu u_{yy}, \\ p_y &= 0, \\ u_x + v_y &= 0. \end{aligned} \tag{8.3}$$

The second of these implies that $p = p(x)$.

The boundary conditions include no-slip at the flat plate. Also, we must match to the external stream velocity $U\,\mathbf{i}$. Thus,

$$u = v = 0 \quad \text{on} \quad y = 0; \qquad u \to U \quad \text{as} \quad y \to \infty.$$

No condition on v can be imposed as $y \to \infty$ since the term v_{yy} has been neglected.

The above all follows through if U is a function of x, that is if the flow at infinity (the external stream) is non-uniform.

The special feature of the boundary layer flow, namely that $p_y = 0$, allows us to find the pressure gradient term p_x by relating it to the external stream velocity. If we let $y \to \infty$ in equation (8.3), since $u \to U(x)$ in general, we obtain

$$UU_x + 0 = -\frac{1}{\rho}p_x + 0.$$

The first zero follows since $u_y \to \partial U/\partial y = 0$ and the second for a similar reason.

Hence,

$$p_x = -\rho U U_x,$$

and we can write the x-momentum boundary layer equation (8.3) as

$$uu_x + vu_y = UU_x + \nu u_{yy}.$$

Formal derivation of the boundary layer equations

We begin by *non-dimensionalizing the Navier-Stokes equations*. This means that we introduce length, velocity and pressure scales with which to define *dimensionless variables*. Proceeding, since U and L provide typical velocity and length scales for the current problem, we may write

$$x = Lx^*, \quad y = Ly^*, \quad u = Uu^*, \quad v = Uv^*, \quad p = \rho U^2 p^*.$$

Here a $*$ denotes a dimensionless variable. Note that we have chosen ρU^2 as a suitable scale for the pressure. Check for yourself that ρU^2 has dimensions of pressure.

We now enter these definitions into the Navier-Stokes equations. Let's just consider the x-momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

So, we obtain,

$$\frac{U^2}{L}\Big(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\Big) = -\frac{U^2}{L}\frac{\partial p^*}{\partial x^*} + \frac{\nu U}{L^2}\Big(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\Big).$$

Dividing by U^2 and multiplying by L we find

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{UL}\Big(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\Big).$$

To save writing, we now drop the asterisks and write

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big),$$

where

$$R = \frac{UL}{\nu}$$

is the Reynolds number.

A similar procedure follows for all of the momentum equations, and for the continuity equation. Finally, we have the two-dimensional Navier-Stokes equations in dimensionless form:

$$\begin{aligned} uu_x + vu_y &= -p_x + \frac{1}{R}\nabla^2 u, \\ uv_x + vv_y &= -p_y + \frac{1}{R}\nabla^2 v, \\ u_x + v_y &= 0, \end{aligned}$$

with $R = UL/\nu$ and where u, v have been scaled on U, p on ρU^2, and x, y on L.

In the boundary layer, we expect

$$\frac{\partial}{\partial y} \sim R^{-1/2} \quad \text{for} \quad y \ll 1.$$

Define new variables $(X, Y) = (x, R^{1/2}y)$ and let

$$\begin{aligned} u &= u_0(X,Y) + R^{-1/2}u_1(X,Y) + \cdots \\ v &= R^{-1/2}\{v_0(X,Y) + R^{-1/2}v_1(X,Y) + \cdots\} \\ p &= p_0(X,Y) + R^{-1/2}p_1(X,Y) + \cdots \end{aligned} \tag{8.4}$$

Subsituting these into the Navier-Stokes equations, we obtain

$$\begin{aligned} u_0\frac{\partial u_0}{\partial X} + v_0\frac{\partial u_0}{\partial Y} &= -\frac{\partial p_0}{\partial X} + \frac{\partial^2 u_0}{\partial Y^2} + O\Big(R^{-1/2}\Big) \\ 0 &= -\frac{\partial p_0}{\partial Y} + O\Big(R^{-1/2}\Big) \\ \frac{\partial u_0}{\partial X} + \frac{\partial v_0}{\partial Y} &= O\Big(R^{-1/2}\Big) \end{aligned}$$

Now let $R \to \infty$ to give

$$\begin{aligned} u_0\frac{\partial u_0}{\partial X} + v_0\frac{\partial u_0}{\partial Y} &= -\frac{\partial p_0}{\partial X} + \frac{\partial^2 u_0}{\partial Y^2}, \\ 0 &= \frac{\partial p_0}{\partial Y}, \\ \frac{\partial u_0}{\partial X} + \frac{\partial v_0}{\partial Y} &= 0. \end{aligned}$$

These are the boundary layer equations in dimensionless form. They can be viewed as the leading order asympotic approximation to the Navier-Stokes equations when $R \to \infty$, $y \sim R^{-1/2}$.

Restoring the dimensions, we have

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \\ 0 &= \frac{\partial p}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned}$$

Since the flow is two-dimensional, it may be convenient to write the boundary layer equations in terms of a stream function ψ, where

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

We find:

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = -\frac{1}{\rho}p_x + \nu\psi_{yyy},$$
$$p_y = 0.$$

Boundary conditions

i) No slip requires

$$u = v = 0 \quad \text{on} \quad y = 0,$$

or, equivalently,

$$\psi = \psi_y = 0 \quad \text{on} \quad y = 0.$$

2) Outside the layer, u matches to the external stream velocity, so

$$u \to U(x) \quad \text{as} \quad y \to \infty, \qquad \text{so} \qquad \psi_y \to U \quad \text{as} \quad y \to \infty,$$

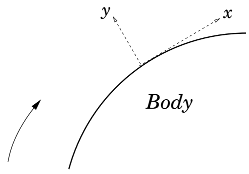
which leads to

$$-p_x = UU_x,$$

where the external stream velocity U is calculated from inviscid theory.

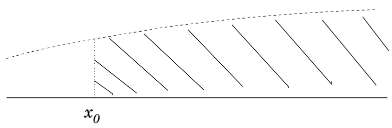
Notes

1. The equations apply on a curved surface, with x, y measuring distance along and normal to the surface respectively.

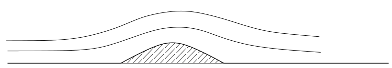


2. The stream function version of the equations are 3rd order in y. Therefore we can only apply 3 boundary conditions on ψ. The vertical velocity $v = \partial\psi/\partial x$ is fixed on solving the equations.

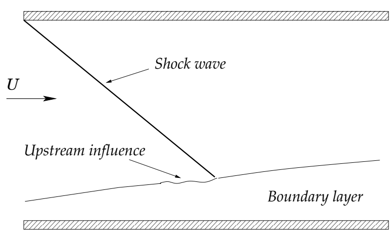
3. The equations are parabolic in x. This means that if we specify $\psi(= f(x), \text{say})$ at $x = \overline{x}$, then ψ is determined for $x > \overline{x}$. There can be no *upstream influence*. This means that the flow at location $x = x_1$ has no effect on that at $x = x_0$ if $x_0 < x_1$. So, if we introduce a small disturbance (say an imperfection in the wall) at $x = x_0$, only the shaded region downstream is affected.



This opens up many interesting questions about boundary layer flows. How, for example, does a boundary layer negotiate a small bump on the wall? Common experience suggests that the streamlines will start to move upwards as they near the bump on the upstream side.



But note 3 says that this is not allowed since the bump may not affect the flow upstream. Famously, some experimentalists showed that upstream effects may occur when a shock wave impinges on a boundary layer in a supersonic flow.



Attempts to understand how such upstream influence might be possible in a boundary layer led to the development of *triple-deck* theory (Stewartson, Messiter 1969). Briefly, the boundary layer gets divided up into three separate superposed decks or layers, and different sets of equations apply in each layer.

Crucially, the lower deck interacts with the upper deck, permitting information to travel upstream. For more information, see Acheson.

Blasius' solution

The solution of the boundary layer equations for high speed flow past a flat plate was given by Blasius in 1908. We take the case when the external stream, U, is constant, $= U_0$, say.

Since $\partial U_0/\partial x \equiv 0$, there is no streamwise pressure gradient and the boundary layer problem to be solved is as follows

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{8.5}$$

with conditions for $x > 0$ (i.e. at the plate),

$$u = v = 0 \quad \text{on} \quad y = 0; \qquad u \to U_0 \quad \text{as} \quad y \to \infty.$$

We seek a *similarity solution* to this problem. To motivate this, consider briefly the flow when the flat plate is of finite length, L.

From the above arguments, at any x station in the boundary layer the flow cannot be affected by what is happening further downstream. In other words, the oncoming flow is unaware that the plate is of length L. Any solution we try to construct should therefore reflect this property.

We used a scaling argument to show that the typical boundary layer thickness is

$$\delta \sim LR^{-1/2} = \left(\frac{\nu L}{U}\right)^{1/2}, \quad \text{since} \quad R = \frac{UL}{\nu}.$$

At a general x station, we expect δ to depend on ν, U and x (but not L). Since

$$[\nu] = \frac{L^2}{T}, \qquad [U] = \frac{L}{T}, \qquad [x] = L$$

then, since $[\delta] = L$, it must appear as the combination

$$\delta \sim \left(\frac{\nu x}{U}\right)^{1/2}$$

which is independent of L.

Now, inside the boundary layer, δ provides a typical scale for y so

$$y \sim \delta \sim \left(\frac{\nu x}{U}\right)^{1/2}$$

i.e. $y \sim x^{1/2}$.

This motivates us to try looking for a solution which is a function of the order one variable

$$\eta = \left(\frac{U}{\nu x}\right)^{1/2} y.$$

We then write $u = Uf'(\eta)$ for the horizontal velocity. Note that,

$$u = Uf'\left(\left[\frac{U}{2\nu x}\right]^{1/2}\right)$$

does not involve L. This agrees with our expectation that the solution should not know where the end of the plate is.

For the infinite plate problem, we seek a solution of the same form. We write the stream function as

$$\psi(x,y) = (2\nu U_0 x)^{1/2}\,\overline{\psi}(x,\eta),$$

where

$$\eta = \Big(\frac{U_0}{2\nu x}\Big)^{1/2} y.$$

The factor of 2 is included purely for convenience. This is another example of a similarity solution, as we shall see below.

Note that

$$\frac{\partial\eta}{\partial x} = -\frac{\eta}{2x}, \quad \frac{\partial\eta}{\partial y} = \Big(\frac{U_0}{2\nu x}\Big)^{1/2}.$$

The velocity components are

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\eta}\Big(\frac{U_0}{2\nu x}\Big)^{1/2} = U_0\frac{\partial\overline{\psi}}{\partial\eta}.$$

$$\begin{aligned} v &= -\frac{\partial\psi}{\partial x} = -\Big[\frac{\partial}{\partial x} + \frac{\partial}{\partial\eta}\Big]\Big[(2\nu U_0 x)^{1/2}\overline{\psi}\Big] \\ &= -(2\nu U_0 x)^{1/2}\Big[\frac{\partial\overline{\psi}}{\partial x} + \frac{\overline{\psi}}{2x} - \frac{\eta}{2x}\frac{\partial\overline{\psi}}{\partial\eta}\Big] \end{aligned}$$

Furthermore,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y} = \Big[\frac{U_0}{2\nu x}\Big]^{1/2} U_0\frac{\partial^2\overline{\psi}}{\partial\eta^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = \Big[\frac{U_0}{2\nu x}\Big] U_0\frac{\partial^3\overline{\psi}}{\partial\eta^3}.$$

The x-momentum equation (8.5) becomes

$$\frac{\partial^2\overline{\psi}}{\partial\eta^3} + \overline{\psi}\,\frac{\partial^2\overline{\psi}}{\partial\eta^2} = 2x\Big[\frac{\partial\overline{\psi}}{\partial\eta}\frac{\partial^2\overline{\psi}}{\partial\eta\partial x} - \frac{\partial^2\overline{\psi}}{\partial\eta^2}\frac{\partial\overline{\psi}}{\partial x}\Big].$$

The boundary conditions are

$$\psi = \frac{\partial\psi}{\partial\eta} = 0 \quad \text{on} \quad \eta = 0 \qquad [u = v = 0 \quad \text{on} \quad y = 0],$$

and $u \to U_0$ as $y \to \infty$ which becomes

$$\frac{\partial\psi}{\partial\eta} \to 1 \quad \text{as} \quad \eta \to \infty.$$

The boundary conditions suggest that we look for a solution which is purely a function of η. Thus, letting $\overline{\psi} = f(\eta)$, we find that f satisfies

$$f''' + ff'' = 0,$$

with conditions

$$f = f' = 0 \quad \text{on} \quad \eta = 0, \quad f' \to 1 \quad \text{as} \quad \eta \to \infty.$$

NOTES

1. f is called the Blasius function. The existence and uniqueness of f was proved by Weyl in 1942.
2. f must be calculated numerically.

3. In terms of f,

$$u = U_0 f'(\eta).$$

Numerical calculation of the Blasius function

In order to solve the system for $f(\eta)$, we integrate the equation forwards in η using a technique called 'Runge-Kutta' integration. In this procedure, the equation is discretized to produce a rule to step the solution forward a small distance in η. Thus, starting at $\eta = 0$ we step the solution forward in η with the aim of satisfying the condition at infinity.

Shooting and matching

Since the equation for f is third order, we need three values at $\eta = 0$ to step forward from. We know $f(0)$ and $f'(0)$, but we don't know $f''(0)$. We can guess this latter value, step forward from $\eta = 0$ and continue stepping until η is large enough for us to have reached 'infinity', in other words far enough for the large η behaviour of f to have become apparent. We can then keep adjusting our initial guess for $f''(0)$ and shooting forwards until we obtain the desired behaviour for f, i.e.

$$f' \to 1 \quad \text{as} \quad \eta \to \infty.$$

We can do the adjusting of the value of $f''(0)$ using Newton-Raphson iteration, for example.

A neat trick

In fact it turns out that we can get away with only having to do one integration over η. To see this, suppose we integrate the system

$$F''' + FF'' = 0,$$

with conditions

$$F = F' = 0 \quad \text{and} \quad F''(0) = 1 \quad \text{on} \quad \eta = 0.$$

This produces a solution $F(\eta)$ with the property

$$F' \to \kappa \quad \text{as} \quad \eta \to \infty,$$

for some constant κ, which will not in general (unless we're really lucky) equal one.

Now let $f(\eta) = aF(a\eta)$ for constant a. Then f satisfies

$$f''' + ff'' = 0,$$

and

$$f(0) = f'(0) = 0.$$

Furthermore,

$$f'(\infty) = a^2 F'(\infty) = a^2 \kappa.$$

So if we choose $a = \kappa^{-1/2}$, then $f(\eta) = \kappa^{-1/2} F(\kappa^{-1/2}\, \eta)$ satisfies

$$f''' + ff'' = 0; \qquad f(0) = f'(0) = 0, \quad f' \to 1 \quad \text{as} \quad \eta \to \infty.$$

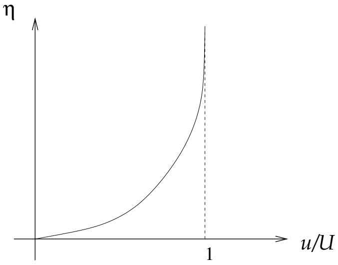
So all we need to do is to solve the system for F, and we get the solution, f, to the Blasius problem straightaway by taking

$$f(\eta) = \kappa^{-1/2} F(\kappa^{-1/2}\, \eta)$$

The computed solution to the Blasius problem looks like this:

The correct value of f'' at the plate is $f''(0) \approx 0.4696$.

Shear stress on the plate



Suppose we wish to calculate the drag in the x direction on a portion of the plate of length L. To do this, we will need to know

$$\sigma_{12}\big|_{y=0} = \mu\frac{\partial u}{\partial y}\Big|_{y=0} = \mu\Big(\frac{U_0}{2\nu x}\Big)^{1/2} f''(0).$$

Then the drag

$$D = \int_0^L \mu\frac{\partial u}{\partial y}\Big|_{y=0} \mathrm{d}x = 2\mu\Big(\frac{U_0}{2\nu}\Big)^{1/2} f''(0)L^{1/2}.$$

From numerical calculation, we have $f''(0) = 0.4696$ and so we predict

$$D = 0.664\mu\Big(\frac{U_0}{\nu}\Big)^{1/2} L^{1/2}.$$

This result agrees well with experimental observations.

Displacement thickness

The displacement thickness provides a measure of the thickness of the boundary layer in terms of the amount by which the oncoming streamlines have been displaced.

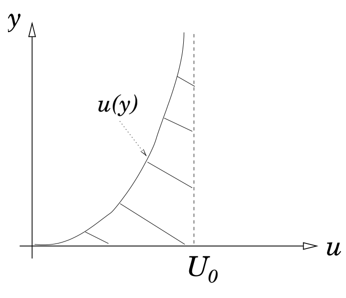
It is defined to be

$$\delta_1 = \int_0^\infty \Big(1 - \frac{u}{U_0}\Big)\,\mathrm{d}y,$$

where $u = U_0 f'$.

Physical interpretation

As stated above, the displacement thickness is a measure of the upwards displacement of the streamlines of the flow by the boundary layer.



The deficit in flux from a purely uniform stream is (shaded area)

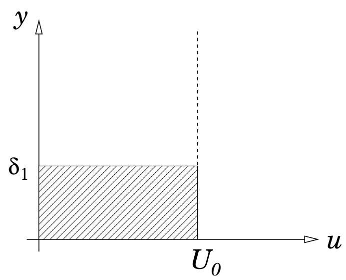
$$\int_0^\infty \Big(U_0 - u\Big)\,\mathrm{d}y = U_0\delta_1,$$

where δ_1 is the corresponding ‘thickness’ of the uniform stream needed for the same flux level.

The shaded areas in these two figures are the same.

Rearranging, we have

$$\delta_1 = \int_0^\infty \Big(1 - \frac{u}{U_0}\Big)\,\mathrm{d}y,$$



for the displacement thickness.

In addition, we may define the 'momentum thickness'

$$\theta_1 = \int_0^\infty \frac{u}{U_0}\Big(1 - \frac{u}{U_0}\Big)\,\mathrm{d}y.$$

It is sometimes possible to calculate both the displacement and momentum thicknesses in closed form.

For the flat plate, numerical integration yields

$$\delta_1 = 1.72\Big(\frac{\nu x}{U_0}\Big)^{1/2}.$$

Example: For a Boeing 747, typically $\delta_1 = 0.1cm$.

Asymptotic form of f

We know that $f' \to 1$ as $\eta \to \infty$, but can we say anything more about the manner in which f' approaches this unit value for large η?

Since $f' \to 1$ as $\eta \to \infty$, we have $f \to \eta - \overline{\eta}$ as $\eta \to \infty$ for constant $\overline{\eta}$. Suppose we now write

$$f \to \eta - \overline{\eta} + g(\eta) \quad \text{as} \quad \eta \to \infty,$$

where $g \ll 1$. We aim to determine the function g.

Substituting into the equation $f''' + ff'' = 0$ we have

$$g''' + (\eta - \overline{\eta})g'' + gg'' = 0.$$

But, since g is small, we can neglect the nonlinear term and just take

$$g''' + (\eta - \overline{\eta})g'' = 0.$$

We need solutions to this equation to satsify $g \to 0$ as $\eta \to \infty$. Two of the possible solutions are $g =$ const, and $g = \lambda\eta$, for constant λ. Clearly both of these must be rejected as they don't satisfy the boundary conditions. To find the third solution, let $\psi = g''$, then

$$\phi' + (\eta - \overline{\eta})\phi = 0.$$

Hence,

$$\phi = A\mathrm{e}^{-(\eta-\overline{\eta})^2/2}, \qquad \text{constant} \quad A.$$

So, as $\eta \to \infty$, we have

$$f \sim \eta - \overline{\eta} + A\mathrm{e}^{-(\eta-\overline{\eta})^2/2}.$$

In fact, had we not neglected the nonlinear terms above, we would find that, more precisely the constant A should be replaced by

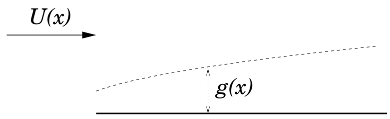
$$-\frac{0.331}{(\eta - \overline{\eta})^2}.$$

Falkner-Skan solutions

The Blasius solution is a special case of a broader class of similarity solutions to the boundary layer equations known as Falkner-Skan solutions.

To motivate these, consider a general external stream velocity $U(x)$, which is assumed to be given. We seek solutions of the boundary layer equations of the form

$$\psi = U(x)g(x)f(\eta), \quad \text{where} \quad \eta = \frac{y}{g(x)}.$$



This form is a more general version of that presented for the Blasius solution. By analogy with that solution, $g(x)$ here represents the 'thickness' of the layer.

The boundary layer equations to be solved are

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = UU_x + \nu\psi_{yyy},$$

with

$$\psi = \psi_y = 0 \quad \text{on} \quad y = 0; \quad \psi_y \to U(x) \quad \text{as} \quad y \to \infty.$$

Now,

$$u = \psi_y = g(x)U(x)\eta_y f_\eta = Uf',$$

$$-v = \psi_x = (Ug)_x f - \frac{\eta U g g_x f'}{g}, \qquad \text{since} \quad \eta_x = -\eta g_x/g,$$

$$\psi_{yy} = \frac{U}{g}f'', \quad \psi_{yyy} = \frac{U}{g^2}f''',$$

$$\psi_{xy} = \frac{1}{g}\Big[(Ug)_x f' - Ug_x(\eta f'' + f')\Big].$$

With these, the x momentum equation becomes

$$f''' + \alpha f f'' + \beta(1 - f'^2) = 0,$$

where

$$\begin{aligned} \nu\alpha &= g(Ug)_x, \\ \nu\beta &= g^2 U_x. \end{aligned}$$

f will satisfy an ordinary differential equation provided that α and β are constants. Otherwise the equation depends on x and η. So henceforth, we assume α and β are constant.

Consider the expression

$$\nu(2\alpha - \beta) = 2g(Ug)_x - g^2 U_x = (g^2 U)_x.$$

If we first suppose that $2\alpha = \beta$, then $(g^2U)_x = 0$, in which case $g^2U = \text{constant} = C$, say. Then

$$\frac{g^2 U_x}{g^2 U} = \frac{2\alpha}{C} \implies U = Be^{2\alpha x/C}, \qquad \text{for} \quad \text{constant} \quad B.$$

Such forms of $U(x)$ are of no particular physical interest.

Instead, then, assume that $2\alpha \neq \beta$. In this case

$$g^2 U = \nu(2\alpha - \beta)(x - x_0),$$

for constant x_0. By a shift of origin we can take $x_0 = 0$. Now,

$$\frac{g^2 U_x}{g^2 U} = \frac{\beta}{(2\alpha - \beta)x} = \frac{n}{x} \implies n = \frac{\beta}{(2\alpha - \beta)},$$

where n is a real number.

Therefore,

$$U = U_0 \Big(\frac{x}{L}\Big)^n,$$

for constant U_0, L. Also,

$$g = \Big[\frac{\nu(2\alpha - \beta)x}{U(x)}\Big]^{1/2} = \Big[\frac{\nu(2\alpha - \beta)L}{U_0}\Big]^{1/2} \Big(\frac{x}{L}\Big)^{\frac{1-n}{2}}.$$

Taking $\alpha = 1$, we have

$$\beta = \frac{2n}{n+1}$$

and the *Falkner-Skan* solutions are given by

$$U = U_0 \Big(\frac{x}{L}\Big)^n, \qquad \eta = \Big(\frac{n+1}{2}\Big)^{1/2} \Big(\frac{U}{\nu x}\Big)^{1/2} y,$$

$$\psi = \Big(\frac{2}{n+1}\Big)^{1/2} (U\nu x)^{1/2} f(\eta),$$

where f satisfies

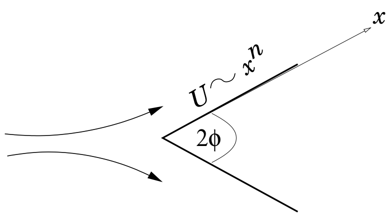
$$f''' + ff'' + \beta(1 - f'^2) = 0,$$

with

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

NOTES

1. External flows with $U \propto x^n$ arise for inviscid flow over a wedge of half-angle $\phi = n\pi/(n+1)$.



2. In the special case $n = 0$ we recover the Blasius solution discussed above.

3. If $\psi = \pi/2$, then $n = 1$ and we have Hiemenz stagnation point flow

Here,

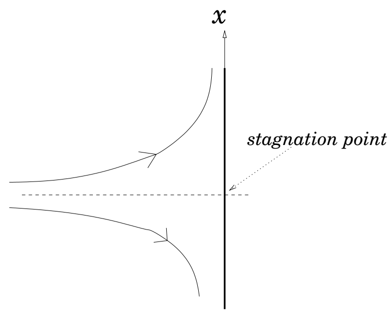
$$f''' + ff'' + \beta(1 - f'^2) = 0,$$

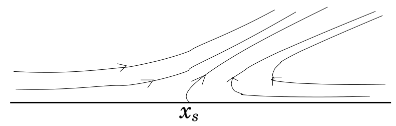
with

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

In this case $g(x)$ is a constant, and so the boundary layer at the wall has constant thickness. This solution was discussed before, when we noted that it formed an *exact* solution of the Navier-Stokes equations.

4. For some values of n there is more than one solution.



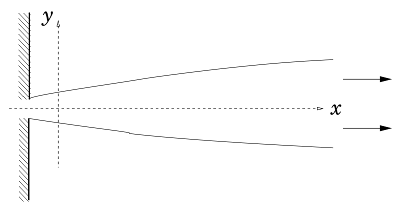


5. If $UU_x < 0$ so that $p_x > 0$ and the pressure gradient is *adverse* (so that the pressure increases as you move downstream), eventually the boundary layer will tend to separate from the wall.

The boundary layer equations break down at the separation point x_s. Here the flow encounters the *Goldstein singularity* and the boundary layer solution cannot be continued for $x > x_s$.

Flow in a viscous jet

Consider the motion of a high speed jet of viscous fluid.



The jet shoots out of a narrow slot in a wall into another fluid which is at rest. The pressure at infinity in the stationary fluid is asummed to be constant, equal to p_0.

We assume that the jet is very thin, so that the variation of velocity across the jet is extremely rapid. Thus

$$\frac{\partial \mathbf{u}}{\partial y} \gg \frac{\partial \mathbf{u}}{\partial x}.$$

Furthermore, we assume that

$$|v| \ll |u|,$$

so that the boundary layer equations are appropriate inside the jet.

The jet is assumed symmetric about $y = 0$. Since the surrounding fluid is at rest, we need $u \to 0$ as we leave the jet. Thus, the equations and boundary conditions are:

$$\begin{aligned} uu_x + vu_y &= -\frac{1}{\rho}p_x + \nu u_{yy}, \\ u_x + v_y &= 0, \end{aligned}$$

with

$$\begin{aligned} &u_y = 0 \quad \text{and} \quad v = 0 \quad \text{on} \quad y = 0 \qquad [\text{symmetry of jet}] \\ &u \to 0 \quad \text{as} \quad y \to \pm\infty \qquad [\text{match to fluid at rest}]. \end{aligned}$$

Note: there is no pressure gradient in the momentum equation.

Introducing a 2-D stream function, ψ, as usual, we seek a similarity solution of the jet equations in the form

$$\psi = x^{\alpha} f(\eta), \quad \text{where} \quad \eta = x^{\beta} y.$$

Thus,

$$u = \psi_y = x^{\alpha+\beta} f'; \quad -v = \psi_x = x^{\alpha-1}\{\alpha f + \beta\eta f'\},$$
$$u_x = x^{\alpha+\beta-1}\{(\alpha+\beta) f' + \beta\eta f''\}; \quad u_y = x^{\alpha+2\beta} f''; \quad u_{yy} = x^{\alpha+3\beta} f'''.$$

Thus, the momentum equation becomes

$$x^{\alpha+\beta} f'.x^{\alpha+\beta-1}[(\alpha+\beta) f' + \beta\eta f''] - x^{\alpha-1}[\alpha f + \beta\eta f'] x^{\alpha+2\beta} f'' = \nu x^{\alpha+3\beta} f'''.$$

Simplifying,

$$x^{\alpha-\beta-1}[(\alpha+\beta) f'^2 - \alpha f f''] = \nu f'''. \tag{8.6}$$

So we need

$$\alpha = \beta + 1. \tag{8.7}$$

To obtain another relation between α and β, we integrate the momentum equation across the jet as follows.

$$\int_{-\infty}^{\infty} u u_x \, \mathrm{d}y + \int_{-\infty}^{\infty} v u_y \, \mathrm{d}y = \nu \Big[u_y \Big]_{-\infty}^{\infty}.$$

Applying the boundary condition that $u \to 0$ as $y \to \pm\infty$, the right hand side vanishes. Now, integrating by parts,

$$\begin{aligned} \int_{-\infty}^{\infty} v u_y \, \mathrm{d}y &= [vu]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u v_y \, \mathrm{d}y \\ \text{(since } u(\pm\infty) = 0) \quad &= -\int_{-\infty}^{\infty} u v_y \, \mathrm{d}y \\ \text{(using cty)} \quad &= \int_{-\infty}^{\infty} u u_x \, \mathrm{d}y. \end{aligned}$$

Using this result, we have

$$2 \int_{-\infty}^{\infty} u u_x \, \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} u^2 \, \mathrm{d}y = 0,$$

and thus,

$$\int_{-\infty}^{\infty} u^2 \, \mathrm{d}y = M,$$

for constant M.

Inserting the form of the similarity solution and noting that, since the integration is performed at a fixed x value,

$$\mathrm{d}\eta = \frac{\partial \eta}{\partial y} \mathrm{d}y,$$

we find

$$x^{2\alpha+\beta} \int_{-\infty}^{\infty} f'^2 \, \mathrm{d}\eta = M, \tag{8.8}$$

and thus we require

$$2\alpha + \beta = 0.$$

Combining this with (8.7), we have

$$\alpha = 1/3, \qquad \beta = -2/3.$$

The momentum equation (8.6) thereby reduces to

$$3\nu f''' + ff'' + f'^2 = 0.$$

This can be integrated to yield,

$$3\nu f'' + ff' = A,$$

for constant A.

Applying the condition $u \to 0$ as $y \to \pm\infty$, we find $A = 0$.

Integrating again,

$$3\nu f' + \frac{1}{2}f^2 = \frac{B^2}{2},$$

for constant B.

Rearranging,

$$6\nu \int \frac{\mathrm{d}f}{B^2 - f^2} = \int \mathrm{d}\eta,$$

The left hand side equals

$$\frac{3\nu}{B} \int \left\{ \frac{1}{B+f} + \frac{1}{B-f} \right\} \mathrm{d}f = \frac{3\nu}{B} \ln\left(\frac{B+f}{B-f}\right).$$

We know that

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

Hence, we have

$$\frac{6\nu}{B} \tanh^{-1}\left(\frac{f}{B}\right) = \eta + C.$$

for constant C. But $f(0) = 0 \implies C = 0$, and so

$$f(\eta) = B \tanh\left(\frac{B\eta}{6\nu}\right).$$

Therefore

$$f'(\eta) = \frac{B^2}{6\nu} \mathrm{sech}^2\left(\frac{B\eta}{6\nu}\right),$$

and so

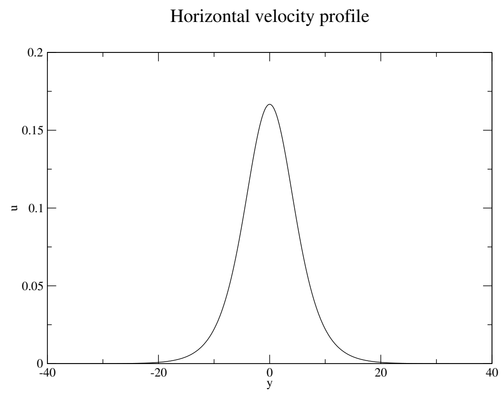
$$\int_{-\infty}^{\infty} f'^2 \, \mathrm{d}\eta = \frac{2}{9} \frac{B^3}{\nu}.$$

Thus, applying (8.8), we can relate B to M,

$$B = \left(\frac{9M\nu}{2}\right)^{\frac{1}{3}}.$$

Finally, the horizontal velocity

$$u = \frac{B^2}{6\nu x^{1/3}} \mathrm{sech}^2\left(\frac{By}{6\nu x^{2/3}}\right)$$



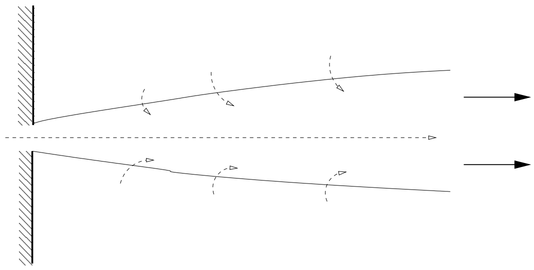
and the vertical velocity

$$
\begin{aligned}
v &= -x^{-2/3}\{\frac{1}{3}f - \frac{2}{3}\eta f'\} \\
&= -\frac{B}{3}x^{-2/3}\Big\{\tanh\Big(\frac{By}{6\nu x^{2/3}}\Big) - \frac{By}{3\nu x^{2/3}}\mathrm{sech}^2\Big(\frac{By}{6\nu x^{2/3}}\Big)\Big\}
\end{aligned}
$$

We observe that at a fixed x station, as $y \to \infty$, $u \to 0$, while

$$v \to -\frac{B}{3}x^{-2/3}.$$

So, fluid is being entrained into the edges of the jet:



The 'width' of the jet corresponds to where the argument of sech^2 is $O(1)$; that is, where the velocity components are significant in magnitude. Thus, where

$$y \sim O\Big(\frac{6\nu}{B}x^{2/3}\Big).$$

Thus the jet thickness grows like $x^{2/3}$ as we move downstream.

9. Hydrodynamic stability

We have calculated quite a wide variety of solutions to the Navier-Stokes equations. We noted that at low Reynolds numbers the flow is always unique. However, in general, there may be more than one solution to the Navier-Stokes equations at a any given finite Reynolds number. On this point, it is instructive to draw an analogy with the solution of some simple algebraic equations.

Model problems

(*i*) Consider the following simple quadratic equation to be solved for u:

$$a - (u - u_0)^2 = 0, \qquad a = R - R_c.$$

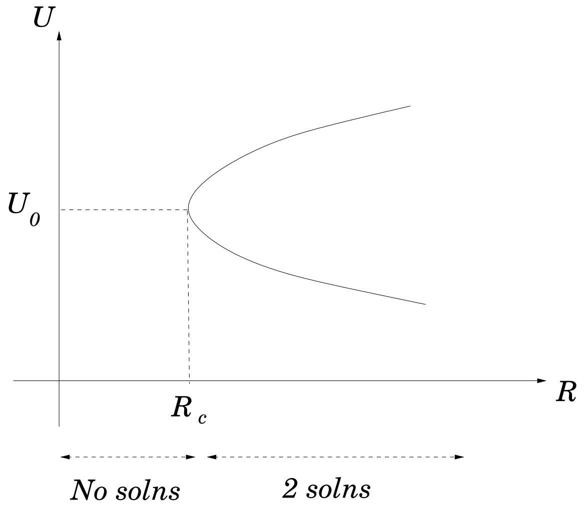
The solution is

$$u = u_0 \pm (R - R_c)^{1/2}.$$

There are three possible cases:

- $R < R_c$: No solutions for u
- $R = R_c$: One solution, $u = u_0$
- $R > R_c$: Two solutions, $u = u_0 \pm (R - R_c)^{1/2}$

If we plot u versus R, we have:



As R increases from zero, U is said to go through a *saddle-node* bifurcation as R passes through the critical value R_c. At this point, the number of possible solutions jumps from none to two. A fluid flow may undergo similar bifurcations as the Reynolds number increases.

(*ii*) Now consider the following cubic equation for u:

$$au - u^3 = 0, \qquad a = R - R_c.$$

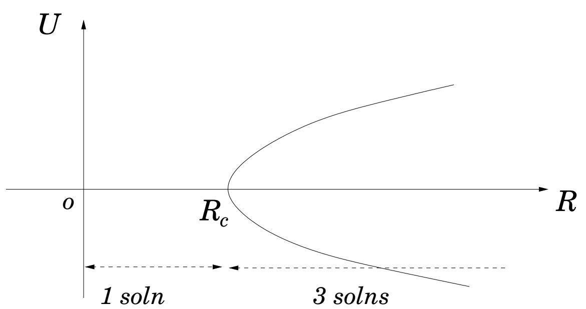
The solution is

$$u = 0, \qquad u = \pm(R - R_c)^{1/2}.$$

There are two cases to consider:

- $R \leq R_c$: The unique solution is $u = 0$
- $R > R_c$: There are three solutions: $u = 0$ and $u = \pm(R - R_c)^{1/2}$.

If we plot u versus R, we have:



As R increases from zero, U is said to go through a *pitchfork* bifurcation as R passes through the critical value R_c. At this point, the number of possible solutions jumps from one to three. A fluid flow may undergo similar bifurcations as the Reynolds number increases.

In the event of multiple solutions to the Navier-Stokes equations, the question arises:

Which solution would you observe in a laboratory experiment?

To answer this question, we need to assess a solution's stability.

Stability theory

The basic idea is to take a solution of the equations and *perturb* it a little bit. Physically, we can think of adding a small disturbance into a flow and observing whether the flow settles back down to its original state, or if it changes dramatically into a different kind of flow altogether.

We will only consider linear stability. This means that we will only consider infinitesimal disturbances.

We call the flow whose stability is to be assessed the *basic flow* or *basic state*.

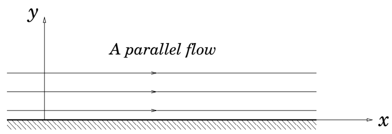
As usual, we take the flow to be governed by the Navier-Stokes equations, written here in dimensionless form:

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{R}\nabla^2 \mathbf{u}, \tag{9.1}$$

for a suitably defined Reynolds number.

Stability of steady parallel flows

A parallel flow past a solid wall is one whose streamlines are parallel to that wall. Thus, plane Poiseuille flow is an example of a parallel flow, while flow in a Blasius boundary layer is not.



The basic state is

$$\mathbf{u} = U(y)\mathbf{i} = \mathbf{U}, \qquad p = p_B,$$

which satisfies the equations

$$\nabla \cdot \mathbf{U} = 0, \qquad 0 = -\nabla p_B + \frac{1}{R}\nabla^2 \mathbf{U} \tag{9.2}$$

To this flow we add a small time-dependent disturbance, writing the new, perturbed flow as

$$\mathbf{u} = \mathbf{U} + \epsilon \hat{\mathbf{u}}, \quad \text{where} \quad \hat{\mathbf{u}} = \hat{u}(x,y,t)\mathbf{i} + \hat{v}(x,y,t)\mathbf{j}, \tag{9.3}$$

and $\epsilon \ll 1$.

We perturb the pressure field in a similar way, writing

$$p = P_B + \epsilon \hat{p}.$$

Substituting this, together with (9.3), into the Navier-Stokes equations (9.1), we obtain

$$\nabla \cdot \mathbf{U} + \epsilon \nabla \cdot \hat{\mathbf{u}} = 0,$$

$$\epsilon \mathbf{u}_t + \mathbf{U} \cdot \nabla \mathbf{U} + \epsilon \hat{\mathbf{u}} \cdot \nabla \mathbf{U} + \epsilon \mathbf{U} \cdot \nabla \hat{\mathbf{u}} + \epsilon^2 \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} = -\nabla P_B - \epsilon \nabla \hat{p} + \frac{1}{R}\nabla^2 \mathbf{U} + \epsilon \frac{1}{R}\nabla^2 \hat{\mathbf{u}}.$$

But, from (9.2),

$$\nabla \cdot \mathbf{U} = 0, \qquad 0 = -\nabla p_B + \frac{1}{R}\nabla^2 \mathbf{U}$$

So,

$$\epsilon\mathbf{u}_t + \mathbf{U}\cdot\nabla\mathbf{U} + \epsilon\hat{\mathbf{u}}\cdot\nabla\mathbf{U} + \epsilon\mathbf{U}\cdot\nabla\hat{\mathbf{u}} + \epsilon^2\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}} = -\epsilon\nabla\hat{p} + \epsilon\frac{1}{R}\nabla^2\hat{\mathbf{u}}.$$

Also, $\mathbf{U}\cdot\nabla\mathbf{U} = \mathbf{0}$ since $\mathbf{U}(y)$ represents a unidirectional flow (see previous work on unidirectional flows). Neglecting the term of $O(\epsilon^2)$ as being smaller than the others, we are left with

$$\nabla\cdot\hat{\mathbf{u}} = 0, \qquad \hat{\mathbf{u}}_t + \hat{\mathbf{u}}\cdot\nabla\mathbf{U} + \mathbf{U}\cdot\nabla\hat{\mathbf{u}} = -\nabla\hat{p} + \frac{1}{R}\nabla^2\hat{\mathbf{u}}. \tag{9.4}$$

It is usual at this stage to assume that the disturbances $\hat{u}$, $\hat{v}$ manifest themselves in the form of travelling waves. We therefore write

$$\hat{u} = \Re\{v(y)\mathrm{e}^{ik(x-ct)}\}, \qquad \hat{v} = \Re\{v(y)\mathrm{e}^{ik(x-ct)}\}, \tag{9.5}$$

where $\Re$ means "take the real part".

In general,

the functions $u(y)$ and $v(y)$ are complex

the parameter k is real

the complex wave speed $c = c_r + ic_i$ is a complex number.

Since $ik(x-ct) = ik(x-c_rt) + kc_it$,

$$\Re\{\mathrm{e}^{ik(x-ct)}\} = \Re\{\mathrm{e}^{ik(x-c_rt)}\mathrm{e}^{kc_it}\} = \mathrm{e}^{kc_it}\ \cos(k[x-c_rt]).$$

So the forms (9.5) represent travelling waves moving with speed c_r of amplitude e^{kc_it}. Always $k > 0$, so it follows that if $c_i > 0$, the amplitude of the wave will grow, the disturbance will get bigger. Eventually, the disturbance will grow so large, it will completely disrupt the basic flow.

To summarise,

$$\begin{array}{lll} \text{if} & c_i > 0 & \text{the flow is linearly unstable;} \\ \text{if} & c_i < 0 & \text{the flow is linearly stable;} \\ \text{if} & c_i = 0 & \text{the flow is neutrally stable;} \end{array}$$

as then the wave is either exponentially growing, exponentially decaying, or remaining constant in amplitude.

The disturbances $\hat{u}$ and $\hat{v}$ are called *modes* and this whole procedure is referred to as the *method of normal modes*.

We write the pressure disturbance in a similar way,

$$\hat{p} = \Re\{P(y)\,\mathrm{e}^{ik(x-ct)}\}.$$

The inviscid limit $R\to\infty$. Rayleigh's criterion

We first consider equation (9.4) in the limit $R\to\infty$, and assume that it is all right to drop the final term,

$$\nabla\cdot\hat{\mathbf{u}} = 0, \qquad \hat{\mathbf{u}}_t + \hat{\mathbf{u}}\cdot\nabla\mathbf{U} + \mathbf{U}\cdot\nabla\hat{\mathbf{u}} = -\nabla\hat{P}.$$

Substituting in the normal modes, we have, writing the equations in component form,

$$\begin{aligned} -ik(c-U)u + U'v &= -ikP, \\ -ik(c-U)v &= -P', \\ iku + v' &= 0. \end{aligned}$$

It is convenient to try to eliminate the pressure from these equations. Differentiating the first with respect to y, multiplying the second by $-ik$ and adding, we obtain

$$-ik(c-U)(u'-ikv)+U''v+U'(iku+v')=0.$$

Then, using the third equation to note that $iku+v'=0$ and to write $u'=-v''/ik$, we have

$$-ik(c-U)(-v''+k^2v)+ikU''v=0,$$

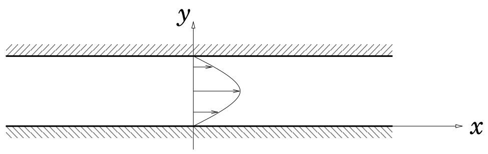
and so

$$v''+\Big\{\frac{U''}{(c-U)}-k^2\Big\}v=0.$$

This is *Rayleigh's equation*.

Example 1

Consider now the specific problem of parallel flow in a channel $0\le y\le 1$:



In this case, $U(y)=y(1-y)$ and we need the conditions $u=v=0$ on $y=0,\ 1$. Therefore we have

$$v''+\Big\{\frac{U''}{(c-U)}-k^2\Big\}v=0,\qquad \text{with}\qquad v=0\quad\text{on}\quad y=0,\ 1. \tag{9.6}$$

This is an eigenvalue problem for c. That is, given a particular disturbance of frequency k, we seek values of c such that this problem has a solution.

Physically, we might think of a flexible ribbon being placed within the channel. The ribbon is vibrated so that it produces fluctuations of fixed wavenumber k. We are interested in the response of the fluid to this disturbance.

As above, we write $c=c_i+ic_i$ and say that the flow is stable if $c_i<0$ and unstable if $c_i>0$.

The system (9.6) can be solved numerically. However, some analytical progress is still possible. If we multiply (9.6) by the complex conjugate of v, namely $\overline{v}$, and then integrate over the channel width, we get

$$\int_0^1 \overline{v}\,v''\,\mathrm{d}y+\int_0^1\Big\{\frac{U''}{(c-U)}-k^2\Big\}|v|^2\,\mathrm{d}y=0.$$

Integrating by parts,

$$\Big[\overline{v}\,v'\Big]_0^1-\int_0^1|v'|^2\,\mathrm{d}y+\int_0^1\Big\{\frac{U''}{(c-U)}-k^2\Big\}|v|^2\,\mathrm{d}y=0.$$

But $v(0)=v(1)=0$, so

$$-\int_0^1|v'|^2\,\mathrm{d}y+\int_0^1\Big\{\frac{U''}{(c-U)}-k^2\Big\}|v|^2\,\mathrm{d}y=0. \tag{9.7}$$

Now

$$\frac{1}{c-U}=\frac{1}{c_r+ic_i-U}=\frac{c_r-U-ic_i}{|c-U|^2},$$

and so the imaginary part of (9.7) gives,

$$c_i \int_0^1 \frac{U''|v|^2}{|c-U|^2}\,\mathrm{d}y = 0.$$

So if $c_i > 0$, the only way for this equation to hold is if U'' changes sign at some point in the interval $[0, 1]$. So U must have an inflection point in that interval. This gives a necessary but *not* a sufficient condition for instability of the flow.

It is known as *Rayleigh's inflection point theorem.*

General size R: The Orr-Sommerfeld equation.

Returning to equations (9.4),

$$\nabla \cdot \hat{\mathbf{u}} = 0, \qquad \hat{\mathbf{u}}_t + \hat{\mathbf{u}} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \hat{\mathbf{u}} = -\nabla \hat{p} + \frac{1}{R}\nabla^2 \hat{\mathbf{u}}. \tag{9.8}$$

we now consider $O(1)$ values of R. This means we can no longer neglect the final term.

As before we use the method of normal modes, writing

$$\hat{u} = \Re\{u(y)\mathrm{e}^{ik(x-ct)}\}, \qquad \hat{v} = \Re\{v(y)\mathrm{e}^{ik(x-ct)}\}, \qquad \hat{p} = \Re\{P(y)\,\mathrm{e}^{ik(x-ct)}\}.$$

Therefore we have

$$\begin{aligned} iku + v' &= 0, & (9.9)\\ -ikcu + ikUu + vU' &= -ikp + \frac{1}{R}[-k^2u + u''] & (9.10)\\ -ikcv + ikUv &= -p' + \frac{1}{R}[-k^2v + v'']. & (9.11)\end{aligned}$$

Multiplying (9.10) by ik and using (9.9) to write $iku = -v'$, we find

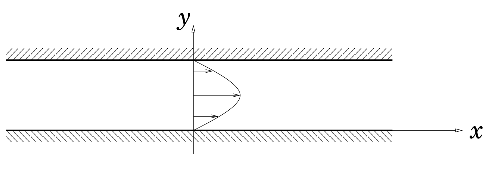
$$-ikcv' + ikUv' - ikU'v = -k^2p + \frac{1}{R}[-k^2v' + v'''].$$

Differentiating this with respect to y and then eliminating p' between the result and (9.11) we obtain,

$$\frac{1}{ikR}\Big[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\Big]^2 v = (U-c)\Big(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\Big)v - U''v. \tag{9.12}$$

This is the *Orr-Sommerfeld equation.*

Consider now the channel flow of Example 1 at finite R.



The flow is driven by a constant negative pressure gradient $\mathrm{d}p/\mathrm{d}x = -G$, with $G > 0$ and the basic solution is $U(y) = y(1-y)$. We can now recognise this as Plane Poiseuille Flow discussed above in the Exact Solutions section.

The disturbance flow is governed by the Orr-Sommerfeld equation (9.12) with boundary conditions

$$u = v = 0 \quad \text{on} \quad y = 0,\ 1.$$

Using equation (9.9) these can be re-expressed as

$$v = v' = 0 \quad \text{on} \quad y = 0,\ 1.$$

For this example we know that $U = y(1-y)$. We now select a wavenumber k (so we restrict attention to disturbances of this wavenumber) and a Reynolds number, R. Then we have to solve the problem

$$\frac{1}{ikR}[v'''' - 2k^2v'' + k^4v] = (y - y^2 - c)(v'' - k^2v) + 2v. \tag{9.13}$$

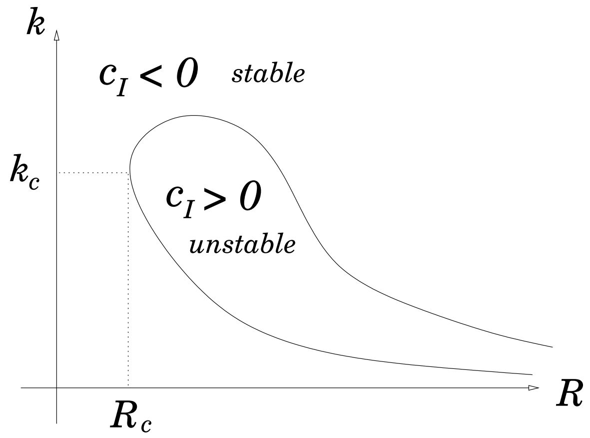
with

$$v = v' = 0 \quad \text{on} \quad y = 0,\ 1,$$

for c. As before, writing $c = c_r + ic_i$,

$$\begin{array}{lll} \text{if} & c_i > 0 & \text{the flow is linearly unstable;} \\ \text{if} & c_i < 0 & \text{the flow is linearly stable;} \\ \text{if} & c_i = 0 & \text{the flow is neutrally stable.} \end{array}$$

Solving for c is generally a numerical task for a computer. Computing c in this way for many values of k and R we can plot the *neutral curve*, which delineates the boundary between stable and unstable solutions. For the current example the neutral curve looks like this:



Every point inside the hoop corresponds to a value of c with positive imaginary part, i.e. an unstable flow. Every point outside has $c_i < 0$, i.e. a stable flow. On the curve itself, $c_i = 0$. Modes corresponding to values of c lying on the curve are known as *neutral modes*.

NOTE: Both of the tails of the hoop asymptote to the R axis as $R \to \infty$.

As we can see from the picture, there is a critical Reynolds number beneath which there are no unstable modes. It is computed to be

$$R_c \approx 5774.$$

The corresponding wavenumber at the very nose of the neutral curve is found to be

$$k_c = 102.$$

A surprise

We have just looked at the stability of flow in a channel. Suppose we instead look at that in a circular pipe. In cylindrical polars, the basic flow is now,

$$U = 1 - r^2,$$

which is the circular Poiseuille flow we calculated earlier in the course.

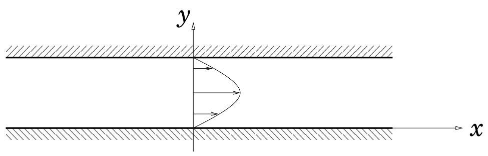
Expressing the disturbance flow as

$$\hat{\mathbf{u}} = \hat{u}\hat{\mathbf{r}} + \hat{w}\mathbf{k},$$

(assuming there is no flow component in the azimuthal (i.e. θ) direction), we can write down the corresponding form of the Orr-Sommerfeld equation for cylindrical polars, substitute in for U, pick a wavenumber k and Reynolds number R and solve for c. Somewhat surprisingly we find that c_i is always negative. In other words, the flow is stable at any value of the Reynolds number. As was mentioned earlier, this contradicts experiments which show that the flow becomes turbulent at a critical Reynolds number. The resolution of this problem is still not complete, but it is generally believed that the turbulence results from perturbations of *finite* rather than infinitesimal amplitude.

We now summarize known stability results for the classic flows discussed in this course.

Plane Poiseuille flow



Flow in a channel driven by a constant pressure gradient.

This flow (see above) is linearly **unstable** at sufficiently large Reynolds numbers

Plane Couette Flow

Flow in a channel driven by the motion of the upper wall.

This flow (see above) is linearly **stable** at any Reynolds number.

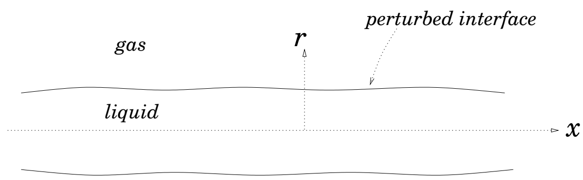
Circular Poiseuille flow

Flow in a circular pipe driven by a constant pressure gradient.

This flow (see above) is linearly **stable** at any Reynolds number.

Capillary instability of a liquid jet

Lord Rayleigh (1879) conducted a famous study of the stability of a jet of liquid shooting into an ambient gas.



Fix a frame of reference moving with the jet, so that in this frame the jet appears stationary. We assume that the flow in the liquid is incompressible and *irrotational*. This means that we can model it using Laplace's equation:

$$\nabla^2 \phi = 0 \tag{9.14}$$

with $\mathbf{u} = \nabla\phi$ the fluid velocity. We assume that the flow is axisymmetric.

In the basic state, the jet is perfectly cylindrical, of radius a. Fix cylindrical polars in the jet as shown in the diagram.

At the interface between the liquid and the gas there is a pressure jump due to *surface tension* γ, which is given by

$$P - p_0 = 2\,\gamma\,\kappa \tag{9.15}$$

where p_0 is the pressure in the gas (e.g. atmospheric pressure) and P is the pressure in the liquid. κ is the *mean curvature* of the interface. According to geometry, if the surface is given by $r = f(x)$, then

$$\kappa = \frac{1}{2}\,\frac{1 + f'^2 - f f''}{f(1 + f'^2)^{3/2}}.$$

So, for the basic flow, $f = a$ and $\kappa = 1/(2a)$.

To assess the jet's stability, we perturb the interface so that now

$$r = f(x,t) = a + \epsilon a_1(t)\cos kx,$$

for some small parameter ϵ. We assume that we can expand the potential like this:

$$\phi = \phi_0 + \epsilon\phi_1 = \phi_0 + \epsilon g(r,t)\cos kx.$$

But $\phi_0 = 0$ since the fluid is stationary in the moving frame.

Equation (9.14) gives (since $\partial/\partial\theta \equiv 0$)

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_1}{\partial r} \;=\; g'' + \frac{g'}{r} - k^2 g \;=\; 0.$$

So,

$$r^2 g'' + r g' - k^2 r^2 g = 0. \tag{9.16}$$

This is a modified form of *Bessel's equation*. Recall that Bessel's equation of zeroth order looks like this:

$$r^2 g'' + r g' + r^2 g = 0.$$

Equation (9.16) is called the modified Bessel's equation of zeroth order. It has linearly independent solutions $I_0(kr)$, $K_0(kr)$. We can thus write the solution as

$$g(r,t) = \alpha(t)\,I_0(kr) + \beta(t)\,K_0(kr).$$

In fact $K_0(r) \to \infty$ as $r \to 0$, so to get a solution which is regular on the axis, we need $\beta = 0$, leaving

$$\phi_1 = \alpha\,I_0(kr)\;\cos kx.$$

At the interface we have the *kinematic condition* (see section 3.4)

$$\frac{D}{\mathrm{D}t}(r - f) = 0 \quad \text{on} \quad r = f.$$

This states the the interface at $r = f(x,t)$ must move along with the liquid. Expanding,

$$\frac{D}{\mathrm{D}t}(r - f) \;=\; \frac{\mathrm{D}r}{\mathrm{D}t} - \frac{\mathrm{D}f}{\mathrm{D}t} \;=\; u - \frac{\partial f}{\partial t} - w\frac{\partial f}{\partial x}, \tag{9.17}$$

on $r = f$. But $\mathbf{u} = \nabla\phi$ and so

$$u = \frac{\partial\phi}{\partial r} = \epsilon\alpha k I_0'(kr)\;\cos kx, \qquad w = \frac{\partial\phi}{\partial x} = -\epsilon\alpha k I_0(kr)\;\sin kx.$$

Therefore, on $r = f$, (9.17) becomes

$$\epsilon\Big[\alpha k I_0'(ka) - \frac{\mathrm{d}a_1}{\mathrm{d}t}\Big] \cos kx + O(\epsilon^2) = 0,$$

giving

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \alpha k I_0'(ka).$$

To find α, we need to consider the pressure jump at the surface, given by (9.15). To find the pressure in the liquid we apply Bernoulli's equation (see Hydrodynamics I, II) in its unsteady version:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} = c(t) \tag{9.18}$$

for some function $c(t)$.

Assuming we can write the pressure as

$$p = P + \epsilon\hat{p}(r,t)\,\cos kx$$

then (9.18) gives

$$\frac{\partial}{\partial t}\Big(\epsilon\alpha I_0(ka)\cos kx\Big) + \frac{1}{\rho}(P + \epsilon\hat{p}(a,t)\cos kx) + O(\epsilon^2) = c(t)$$

on $r = f$. So at $O(\epsilon)$,

$$\rho I_0(ka)\,\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \hat{p}(a,t) = 0,$$

which gives an equation for α in terms of the interfacial pressure.

From above,

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \alpha k I_0'(ka).$$

So

$$\frac{\mathrm{d}^2 a_1}{\mathrm{d}t^2} = -\Big(\frac{k I_0'(ka)}{\rho I_0(ka)}\Big)\hat{p}(a,t).$$

Now, since $r = f = a + \epsilon a_1 \cos kx$, the curvature

$$\begin{aligned} 2\kappa &= \frac{1 + f'^2 - f f''}{f(1+f'^2)^{3/2}} = \frac{1}{a} - \epsilon\Big(\frac{a_1}{a^2} - k^2 a_1\Big)\cos kx + O(\epsilon^2) \\ &= \frac{1}{a} - \epsilon\frac{a_1}{a^2}\Big(1 - a^2k^2\Big)\cos kx + \cdots . \end{aligned} \tag{9.19}$$

But the pressure jump

$$\begin{aligned} p - p_0 &= (P - p_0) + \epsilon\hat{p}(a,t)\cos kx \\ &= 2\gamma\,\kappa = \gamma\Big[\frac{1}{a} - \epsilon\frac{a_1}{a^2}(1 - a^2k^2)\cos kx\Big]. \end{aligned}$$

So $P = p_0 + \gamma/a$ and

$$\hat{p}(a,t) = -\gamma\frac{a_1}{a^2}(1 - a^2k^2).$$

Combining this with the previous result,

$$-\Big(\frac{k I_0'(ka)}{\rho I_0(ka)}\Big)\hat{p}(a,t) = \frac{\mathrm{d}^2 a_1}{\mathrm{d}t^2},$$

we have, finally, setting $\hat{k} = ak$,

$$\frac{\mathrm{d}^2 a_1}{\mathrm{d}t^2} - \left(\frac{\gamma}{\rho a^3}\right)\frac{\hat{k} I_0'(\hat{k})}{I_0(\hat{k})} a_1 (1 - \hat{k}^2) = 0,$$

or

$$\frac{\mathrm{d}^2 a_1}{\mathrm{d}t^2} - \sigma^2 a_1 = 0, \qquad \text{with} \qquad \sigma^2 = \left(\frac{\gamma}{\rho a^3}\right)\frac{\hat{k} I_0'(\hat{k})}{I_0(\hat{k})}(1 - \hat{k}^2)$$

This equation tells us how the amplitude a_1 of the disturbance to the jet varies in time.

The solution is

$$a_1 = c_1 \exp(\sigma t) + c_2 \exp(-\sigma t).$$

If $k > 1/a$, σ is imaginary and the solution is bounded.

If $k < 1/a$, σ is real and the solution grows indefinitely.

Thus, the jet is unstable if the wavenumber of the disturbance

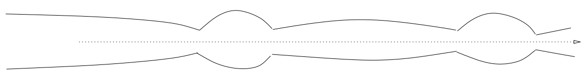
$$k < \frac{1}{a}$$

or if the wavelength $\lambda = 2\pi/k$,

$$\lambda > 1/a.$$

So disturbances of large enough wavelength destabilize the interface.

NOTE: If a suitable wavenumber destabilizes the jet, the amplitude of the disturbance will keep on growing. Eventually it will grow so large that the above theory is invalidated. Remember that we had to insist that ϵ is *small*. At this stage the disturbance interacts *nonlinearly* with the basic flow and disrupts it drastically. Eventually, the jet pinches into a sequence of satellite drops like this:



Appendix A: Invariants of a matrix under rotation

In section 2(b) we noted that it is possible to rotate from one set of axes to a second in which the strain tensor is a diagonal matrix with non-zero entries along its diagonal and zeros everywhere else. Under this action, the trace of the matrix is invariant. In this appendix, we will see why this is the case.

First, we investigate rotations of a vector. Define a transformation matrix $\mathbf{R}$ such that its action on a general vector $\mathbf{x}$ is to map it to another vector $\mathbf{y}$ *of the same length*. Accordingly,

$$\mathbf{y}^T\mathbf{y} = \mathbf{x}^T\mathbf{x}.$$

Furthermore,

$$\mathbf{y}^T\mathbf{y} = (\mathbf{R}\mathbf{x})^T(\mathbf{R}\mathbf{x}) = \mathbf{x}^T\mathbf{R}^T\mathbf{R}\mathbf{x} = \mathbf{x}^T\mathbf{x}.$$

So,

$$\mathbf{x}^T(\mathbf{R}^T\mathbf{R} - \mathbf{I})\mathbf{x} = \mathbf{0}.$$

Since $\mathbf{x}$ is arbitrary, we deduce that

$$\mathbf{R}^T\mathbf{R} = \mathbf{I}.$$

In other words, $\mathbf{R}$ is an orthonal matrix whose transpose is equal to its inverse. Also, since the length of the vector does not change, $\mathbf{R}$ represents a pure rotation of that vector through some determined angle.

Now, if a matrix $\mathbf{A}$ has components a'_{ij} written with respect to one Cartesian frame, and $\mathbf{A}'$ has components a'_{ij} has written with respect to a second frame rotated with respect to the first, it can be shown that

$$a'_{ij} = R_{ki}R_{lj}a_{kl}.$$

Or,

$$a'_{ij} = R^T_{ik}R_{lj}a_{kl},$$

and so,

$$\mathbf{A}' = \mathbf{R}^T\mathbf{A}\mathbf{R}.$$

Consider now the determinant

$$\det(\mathbf{A}' - \lambda\mathbf{I}).$$

We can see that

$$\det(\mathbf{A}' - \lambda\mathbf{I}) = \det(\mathbf{R}^T\mathbf{A}\mathbf{R} - \lambda\mathbf{R}^T\mathbf{R}) = \det(\mathbf{R}^T[\mathbf{A} - \lambda\mathbf{I}]\mathbf{R}) = \det(\mathbf{A} - \lambda\mathbf{I}),$$

since $\det\mathbf{R} = 1$. Thus,

$$\det(\mathbf{A}' - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I}),$$

and the characteristic equation,

$$\det(\mathbf{A} - \lambda\mathbf{I})$$

is independent of the choice of coordinate system. Expanding the equation, we have

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

where the coefficients I_1, I_2 and I_3 must also be independent of the choice of coordinate system. We can I_1, I_2, I_3 the invariants of the matrix $\mathbf{A}$.

To work out what I_1, I_2, I_3 are, we exploit the fact that we can rotate to any coordinate system we like without changing their values. Rotating to a system in which $\mathbf{A}$ is diagonal, we find easily that

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \\ &= \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3.\end{aligned}$$

Hence,

$$\begin{aligned}I_1 &= \lambda_1 + \lambda_2 + \lambda_3 \\ I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \frac{1}{2}\left\{(\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\right\} \\ I_3 &= \lambda_1\lambda_2\lambda_3.\end{aligned}$$

Or,

$$I_1 = \mathrm{tr}\mathbf{A}, \qquad I_2 = \frac{1}{2}\left\{(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right\}, \qquad I_3 = \det\mathbf{A}.$$

So, since $I_1 = \mathrm{tr}\mathbf{A}$, the trace of a matrix $\mathbf{A}$ is invariant under rotation. Similarly, the determinant of a matrix $\mathbf{A}$ is invariant under rotation, as is

$$\frac{1}{2}\left\{(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right\}.$$

Appendix B: The momentum integral equation

The momentum integral equation (3.31) is derived as follows. First, we state Newton's second law for a fixed volume of constant density fluid $V(t)$,

$$\rho \int_V \frac{\partial \mathbf{u}}{\partial t}\, \mathrm{d}V + \rho \int_V \mathbf{u} \cdot \nabla \mathbf{u}\, \mathrm{d}V = \int_V \rho \mathbf{F}\, \mathrm{d}V + \int_V \nabla \cdot \boldsymbol{\sigma}\, \mathrm{d}V.$$

Note that[10]

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot (\mathbf{u}\mathbf{u}),$$

since $\nabla \cdot \mathbf{u} = 0$. Using the divergence theorem then, we have

$$\rho \int_V \frac{\partial \mathbf{u}}{\partial t}\, \mathrm{d}V + \int_S \rho\, \mathbf{u}(\mathbf{u} \cdot \mathbf{n})\, \mathrm{d}S = \int_V \rho \mathbf{F}\, \mathrm{d}V + \int_S \boldsymbol{\sigma} \cdot \mathbf{n}\, \mathrm{d}S.$$

Thus,

$$\int_V \frac{\partial}{\partial t}(\rho\, \mathbf{u})\, \mathrm{d}V = -\int_S \rho\, \mathbf{u}(\mathbf{u} \cdot \mathbf{n})\, \mathrm{d}S + \int_V \rho \mathbf{F}\, \mathrm{d}V + \int_S \boldsymbol{\sigma} \cdot \mathbf{n}\, \mathrm{d}S,$$

as in (3.31).

Appendix C: The boundary layer equations in Lagrangian coordinates.

Although the equations of fluid motion are more cumbersome when expressed in Lagrangian coordinates (see section 3.2.1 for an introduction to these coordinates), a landmark advance in fluid mechanics was made by van Dommelen & Shen (J. Comp. Phys, 1980, **38**) in the analysis of an impulsively started cylinder by first expressing the boundary layer equations in terms of Lagrangian coordinates.

The boundary layer momentum equation in Eulerian (x, y, t) coordinates is

$$u_t + u u_x + v u_y = -p_x + u_{yy}. \tag{9.20}$$

In Lagrangian coordinates (ξ, η, t) [here we take the vector $\mathbf{a} = (\xi, \eta)$ in section 3.2.1] the left hand side simplifies to u_t. To convert the right hand side, we note that (see Mead, J. Comp. Phys, 2004, **200**),

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}, \tag{9.21}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}, \tag{9.22}$$

Alternatively

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \\ \frac{\partial}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

So

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \\ \frac{\partial}{\partial \eta} \end{pmatrix}.$$

[10] Note that $\mathbf{u}\mathbf{u}$ is an example of a *dyadic product*. In index notation, it means the matrix $u_i u_j$.

Now

$$\mathbf{J}^{-1} = J \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ & \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{pmatrix},$$

where the Jacobian

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}.$$

For an incompressible flow (see equation 3.9) we have that

$$J = 1.$$

So,

$$\frac{\partial}{\partial x} = \left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right), \qquad \frac{\partial}{\partial y} = \left(-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\right). \tag{9.23}$$

So

$$u_y = (-x_\eta u_\xi + x_\xi u_\eta),$$

and

$$u_{yy} = (-x_\eta (u_y)_\xi + x_\xi (u_y)_\eta).$$

The algebra is rather tedious and we simply present the result below. Next we need

$$p_x = y_\eta p_\xi - y_\xi p_\eta.$$

Therefore the boundary layer equation (9.20) in Lagrangian coordinates is:

$$\begin{aligned} u_t &= (y_\eta p_\xi - y_\xi p_\eta) \\ &+\Big(x_\xi^2 u_{\eta\eta} - 2x_\xi x_\eta u_{\xi\eta} + x_\eta^2 u_{\xi\xi} - x_\xi u_\xi x_{\eta\eta} + (x_x i u_\eta + x_\eta u_\xi) x_{\xi\eta} - x_\eta u_\eta u_{\xi\xi}\Big), \end{aligned} \tag{9.24}$$

where the term in small brackets on the right hand side is the pressure gradient, and the large brackets on the right hand side contain the viscous term.

The continuity equation transforms as follows:

$$u_x + v_y = \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial t}\right) = \frac{1}{J}\frac{\partial J}{\partial t}$$

from (3.14). As noted above, $J = 1$ for incompressible flow giving $u_x + v_y = 0$.

Example: Flow past an impulsively started sphere

In this case, the inviscid flow around the cylinder is such that

$$-p_x = \sin x \cos x,$$

and the boundary layer equation is (see van Dommelen & Shen, J. Comp Phys, **38**, 1980, eq. 7),

$$u_t = \frac{1}{2}\sin 2x + x_\xi^2 u_{\eta\eta} - 2x_\xi x_\eta u_{\xi\eta} + x_\eta^2 u_{\xi\xi} - x_\xi u_\xi x_{\eta\eta} + (x_\xi u_\eta + x_\eta u_\xi) x_{\xi\eta} - x_\eta u_\eta u_{\xi\xi}. \tag{9.25}$$