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SCHEMATIC LEARNING OF THE ADDITION AND MULTIPLICATION TABLES — STICKS AS CONCRETE MANIPULATIVES

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The soul never thinks without an image.
Aristotle

Abstract. The numbers 1, 2, ... , 20 are represented in the form of arrangements of horizontal and vertical lines and, when materialized, these lines are replaced by flat, longitudinal, rectangular sticks each having two sides dyed in two different colors. Sharp individuality of these arrangements is excellent for quick recognition of the numbers they represent. The way of arranging emphasizes the relation of the numbers 1, 2, ... , 10 to five and ten and this "ten fingers" model is basic, both conceptually and operationally, for our approach to schematic learning of the arithmetic tables. In case of addition and subtraction, the chosen structures of the arrangements reflect clearly "crossings the five and ten lines", serving efficently as illustrations (and explanations) of these methods.

The suggested designs of pictured products $m \times n$ are easily seen as m groups of n sticks and, in the same time, as groups of tens and ones. Wall maps of these designs might be used in the class, letting the pupil have them to fall back on and so helping him/her form gradually a store of mental images related to the multiplication table.

The use of space holders is also suggested to help the child compose the symbolic codes which immediately follow manipulative activities. Thus, a one-to-one correspondence between manipulative, reflective and symbolic operations is established, what also makes them connected in a child's mind.

1. Introduction. The addition table consists of all relations $k + m = n$, where k and m take values in the set $\{1, 2, \dots, 9\}$ and the multiplication table consists of all relations $k \cdot m = n$, where k and m take values in the set $\{2, 3, \dots, 9\}$ (ignoring the trivial cases of the summand "0" in the former and of the factors "0" and "1" in the latter table). Using the commutative laws, the number of relations also reduces to 45 in the former and to 36 in the latter case.

The addition table is usually left to look after itself, but the multiplication table always recives some attention (and in bygone days it was memorized by frequent repetition).

The sets of relations $n - m = k$ and $n : m = k$ go together with these two tables. Though hardly ever seen being grouped into table cards, these relations constitute the subtraction and the division table respectively. Having learnt the multiplication table, we know immediately all relations from the division table, what is not the case with the subtraction, where the "serach for" the result of some differences necessarily involves a short quick calculation. It is easy to see a logical reason for it.

In the multiplication table, when we know n the factors k and m are uniquely determined except in the cases

$$12 = 2\cdot 6 = 3\cdot 4,\quad 16 = 2\cdot 8 = 4\cdot 4,\quad 18 = 2\cdot 9 = 3\cdot 6,\quad 24 = 3\cdot 8 = 4\cdot 6,$$

where two pairs of factors exist (among one-digit numbers). In the addition table, when $n = 10$, there exist five diferent pairs of summands

$$10 = 5+5 = 4+6 = 3+7 = 2+8 = 1+9,$$

and for $n = 8$, 9, 11, 12 four pairs, $n = 6$, 7, 13, 14 three pairs, $n = 4$, 5, 15, 16 two pairs and the summands are uniquely determined only when $n = 2$, 3, 17, 18.

For example, when dividing, on hearing "63", the pair $(7, 9)$ is "ready" and as soon as we know the divisor, the answer follows immediately. Not quite so, when subtracting, on hearing "11", we do not try to make "ready" four different pairs and after hearing the subtrahend, we perform a quick calculation.

Our aim in this paper is to produce designs of arrangements of sticks which form regular patterns, the shape of which projects their number at a glance. Then, these heaps of sticks will also be used to "materialize" (express in concrete form) the arithmetic operations which are carried out by rearranging and which are supposed to be followed by writing of corresponding mathematical expressions. Thus, a one-to-one correspondence between manipulative and perceptive activities on one side and formal activities on the other side is attained and visual representations and symbolism are strictly linked. Of course, we are confined to the frames of the addition and multiplication tables and our main objective is their schematic learning, which, in the present context, is to mean an intelligent learning. (See [4] for the role of such learning.)

2. Number Arrangements. When we gather four objects together, then such a collection materializes our abstract idea, or better to say, mental image of the number "4". Drawing a group of four points or circles, we also materialze the same idea that way. As known from the Papyrus Rhind, in the ancient Egyptian arithmetic, the following pictures (hieroglyphs):

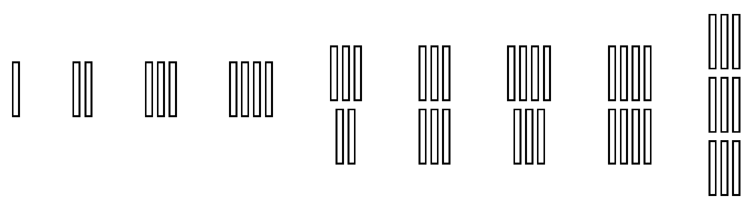


Fig. 1

were used to denote the numbers 1, 2, ..., 9. Notice that all these pictured heaps of sticks have easily recognizable shape and that "5" is seen as "$3+2$", "6" as "$3+3$", "7" as "$4+3$", "8" as "$4+4$" and "9" as "$3+3+3$". Several systems of number pictures (Busse, Hentchel, Born etc.) are also known and they were designed in 19th century, as a result of following the Comenian programme of learning which starts with the "pictured surroundings" (orbis pictus) to provide so meaning and evidence.

When we plan to materialize the idea of a number by means of concrete materials, we might well gather a heap of jettons. And comparing with number-pictures, the advantage of such heaps is at least in the possibility of changing them easily by adding or removing some jettons.

Spreading linearly, sticks have an advantage over jettons that, when put one upon another, they can be still arranged to form patterns casting regularity of their structures.

Well, now we turn to the description of concrete manipulatives following an evidently very ancient idea and respecting the child's instinct of manipulation. Our didactical set contains twenty light, flat, longitudinal rectangular sticks each having its two sides dyed in two colors (say, blue and red). In this "colorless" paper, two sides of a stick will be portrayed by an "empty" (white) rectangle and a "full" (black) one.

Fig. 2

Now our plan is to use the arrangements of these sticks (or pictures of such arrangements) with the intention to materialize mental and formal operations related to the process of forming of the addition and multiplication tables.

3. Standard Arrangements. Each of the following expressions

$$111-74, \quad 8+11+2+16, \quad 3\cdot 7+16, \quad 37$$

represents the same number and if asked "which one" the right answer would be "that one the expression represents". This rightness is based on the fact that they are numerically equivalent or, as we also say it, they have the same numerical value. It is something else to ask which of these expressions is more informative, that is to say, which of them stimulates more easily the intuitive representation of the number that it stands for. Then, we are uniquely determined to choose "37" because it evokes the representation of 3 tens plus 7 ones. That presentation of a number in the form of the sum of ones, tens, hundreds etc. is the basis of the decimal system, and thinking in images, such groupings evoke the clearest spacial representations of numbers (due to cultural enviroment and schooling, but in some past civilizations quite different systems were used, [5]).

For that matter, the calculation is nothing more than a sequence of transitions from an expression to a simpler, numerically equivalent one until the decimal notation is obtained. The effect of such notation explains our readiness to answer the question like "which number is represented by the expression $3\cdot 7+16$", saying "37".

Representing a number by means of an arrangement of sticks, jettons, pips or some other objects, the shape, as a global property of such arrangements, is significant, and we could say in principle, that with more regularity more structural distinction is present. As a matter of fact, "nice" shapes would reflect sharply the number of object building such an arrangement.

The dominoes are a common example of a design, where the number of spots on each piece can be seen at once.

Now we proceed advocating for the grouping of sticks as shown in Fig. 3 and Fig. 4.

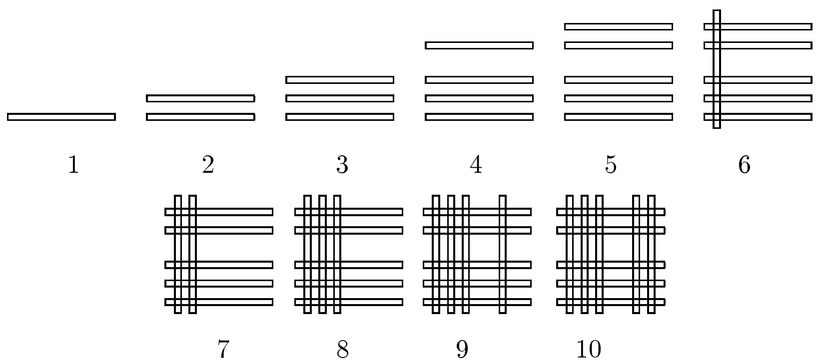


Fig. 3

Let us observe that "3" is privileged as the largest number of objects a child can see at once and what is an opinion maintained by many teachers. Furthermore, "4" is seen as "3+1", "5" as "3+2", "6" as "5+1", "7" as "5+2", "8" as "5+3", "9" as "5+4" and "10" as "5+5". The way of grouping emphasizes the relation of each number to five and ten. A connexion between this grouping and the fingers on one hand together with those, in a crossing position, on the other hand is evident. And as it is everybody's personal experiance, the fingers are the first piece of arithmetic apparatus to be used.

The numbers 11, ... , 20 are represented so that they are seen as $10+1, \dots, 10+10$.

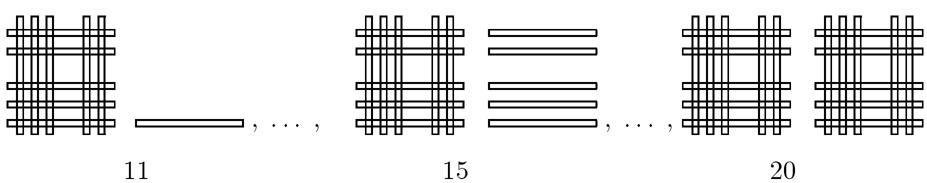


Fig. 4

The heaps of sticks obtained by this grouping will be called *standard arrangements*.

Sharp individuality of these patterns is excellent for quick recognition. The activity of arranging should go parallelly with the exercises of counting and then, the children easily visualize each number up to 20.

There are, of course, many important little things that we shall avoid to tell here and which would be more appropriate to a related workbook.

4. Arranging instead of Counting. When counting elements of a set belonging to the natural environment, we, in our thoughts, sort them one after another associating to each the name of a number pronounced in the innner speech or loudly. To the counting as a mental operation, the activity of composing of standard arrangements is correspondent as a manipulative operation. The order of moves by which a stick is put to its position in the heap is displayed in Fig. 5.

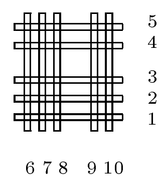


Fig. 5

and the removing goes in the oposite order. Thus, such arrangings and removings are correspondent to the counting forwards and backwards and, it is essential, that this activity is cultivated when performed in the class.

Suppose the pupil has learnt to compose the standard heaps and to recognize them at a glance. Then, the effect of their regularity has to be emphasized in a proper way. For instance, the pupil is asked to guess the number of sticks in a chaotic heap and then he/she is required to arrange them standardly. With comparing of two numbers, the result of guessing is evaluated and such activities could spring up an interesting competition in the class.

Two standardly arranged heaps put one next to the other, as Fig. 6 represents it,

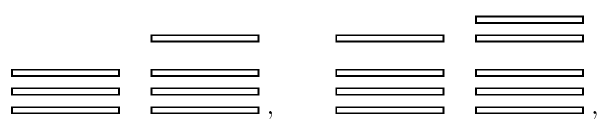


Fig. 6

form together one heap which will not be considered as very chaotic, although the regularity of its "parts" does not improve much the accuracy of guessing. In this case, rearranging and forming of the standard heaps (as shown in Fig. 7)

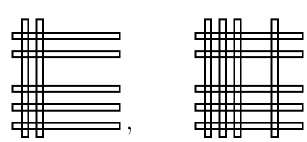


Fig. 7

is a manipulative activity which can be considered as a prelude to the addition.

Mere rearranging is also an activity within which the principle of invariance of number is expressed manipulatively.

From the eighteenth century and the Tillich's bricks on, several systems of manipulatives have been developed: the Chelsea bead-bars, the "Welbent" bead-bars, the Montessori numerical rods (see [3]), the Cuisenaire's numbers in colors (see [2]) etc., not to mention all varieties of abacus already existing in the ancient times. In spite of that, we try with an innovation in this paper and the advantages, when existing, are exactly those which will be recognized by the readers.

5. Beginnings in Addition. From the very beginning of the teaching of addition, a variety of word problems is involved to help the pupil link the symbolic code of mathematics to the reality of his/her everyday world. Thus the meaning of this operation rises from many specific actions expressed by so many verbs and, therefore, there exists no concrete material to take place of them. That is why this teaching theme is usually divided into the following steps:

(I) *Recognition of situations* in natural or pictured surroundings to which we react adding.

(Using the set theoretical language, we can describe such situations as beeing the collections of two disjoint sets whose cardinatlities are known.)

(II) *Composition of the sum* as an expression formed by the use of the plus sign.

(For example, writing or pronouncing of a sum like "$7+5$".)

(III) *Equating* of the sum to its decimal notation.

(For example, writing or pronouncing of an equation like "$7+5=12$".)

When reduced to the step (III), the addition is a formal operation carried out by the procedures which reduce each sum to those ones in the addition table. And when a formal work like memorizing of tables is planed to be induced, it is wise to limit the types of apparatus.

Working with sticks, a situation to which we react adding will be shaped as two standardly arranged heaps set down one beside the other.

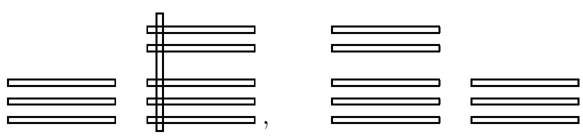


Fig. 8

In Fig. 8, the pairs of heaps represent the sums "$3+6$" and "$5+3$". To the finding of sums, the arranging into one heap is correspondent, when all sticks of the second heap are placed to the position of the first one. The following arrangements:

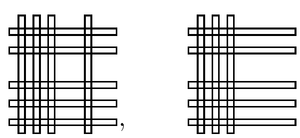
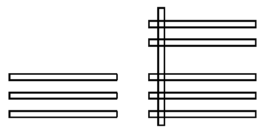


Fig. 9

illustrate it in the case of pairs of heaps given in Fig. 8.

When the child "makes up a sum" with concrete material, the space holders can be exploited to help him/her write the proper equations and so to get him/her relate manipulative and reflective activities.

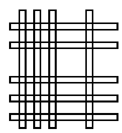
For example, we start with the picture:



$\_\_\_ + \_\_\_$

Fig. 10

When the sum "$3 + 6$" is written, we suggest the arranging into one heap, adding the equality sign and another space holder (Fig. 11).



$\underline{\ 3\ } + \underline{\ 6\ } = \_\_\_\ .$

Fig. 11

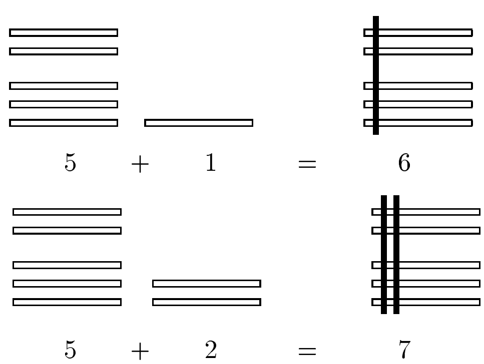
Recognizing the heap, the child completes the equation, obtaining so "$3 + 6 = 9$".

The following additions are easy

$$5 + 1 = 6$$
$$5 + 2 = 7$$
$$5 + 3 = 8$$
$$5 + 4 = 9$$
$$5 + 5 = 10.$$

This significant role of the summand "5" is also displayed by arranging (and is accentuated by turning of the opposite side of the sticks beeing moved in Fig. 12).

(before) (after)



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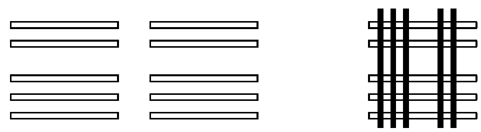


5 + 5 = 10

Fig. 12

The easiness of these additions is also felt when working with sticks: the whole second heap can be lifted and put over the first one. In the harder cases, the heap representing the number “5” is formed first, and then the rest of the sticks is put over. Let us illustrate these two steps:

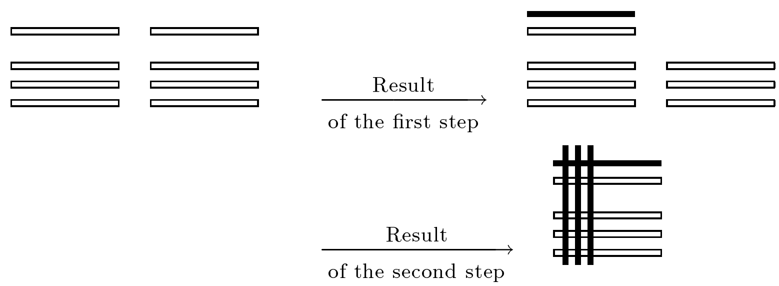


Fig. 13

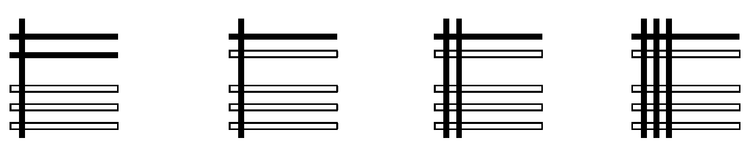
and the corresponding formal operations are

$$\begin{array}{c} 1 \quad 3 \\ \overset{+}{\bigvee} \\ 4+4=5+3=8. \end{array}$$

where the first step is the completing of the five.

With the sums not exceeding ten, there exist only four harder cases and they deserve to be selected and pictured (Fig. 14).

6 6 7 8



$3+3$ $4+2$ $4+3$ $4+4$

Fig. 14

6. Inverse Addition and Decomposition into Summands. Knowing the sum of two numbers and one of them, the other one is found by subtraction. But this operation, when worded in this way: “What number must be added to 7 to make 10”, is known as the inverse addition.

Building up the addition table some specific cases of the inverse addition are necessary involved and we have to give a clear evidence of them.

What number must be added to	We see it is
3 to make 5	2
4 to make 5	1
5 to make 10	5
6 to make 10	4
7 to make 10	3
8 to make 10	2
9 to make 10	1

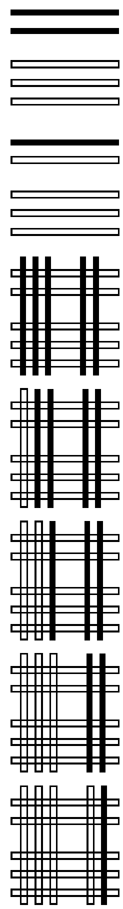


Fig. 15

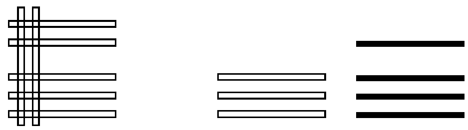
Hereupon we expect the child to form mental representations based on the pictures in Fig. 15 and, after some practice, to give the answers easily.

In the previous case, the given sum was 5 or 10. Now we turn to those cases where the given sum is 2, 3, 4, 6, 7, 8 and 9.

Working with sticks, the child decomposes the arrangements representing the given number (sum) until the heap representing the given summand is obtained.

The removed sticks are simultaneously arranged into a new heap turning the red side up. For example, when the number “7” is given and its summand “3”, the result of this manipulation is diplayed in the following figure.

(before) (after)



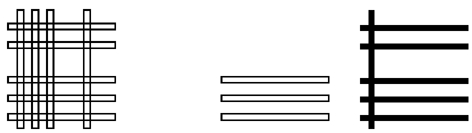
7 = 3 + 4

Fig. 16

Such a manipulation has to be followed by the corresponding equation.

In the case of the number “9” and its summand “3”, we have

(before) (after)

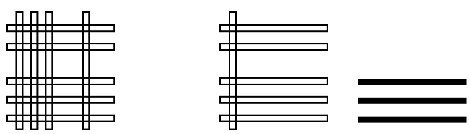


9 = 3 + 6

Fig. 17

Children should be helped to discover the easier way

(before) (after)

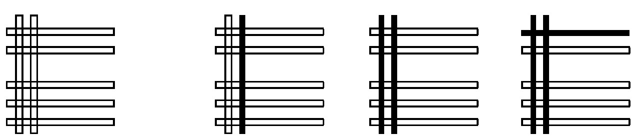


9 = 6 + 3

Fig. 18

where the decomposing stops as soon as the “red” three appears. This way, in which we express the commutative law, is also felt as manipulatively easier one.

These exercises can be summarized using the two color cards (as the following one is)

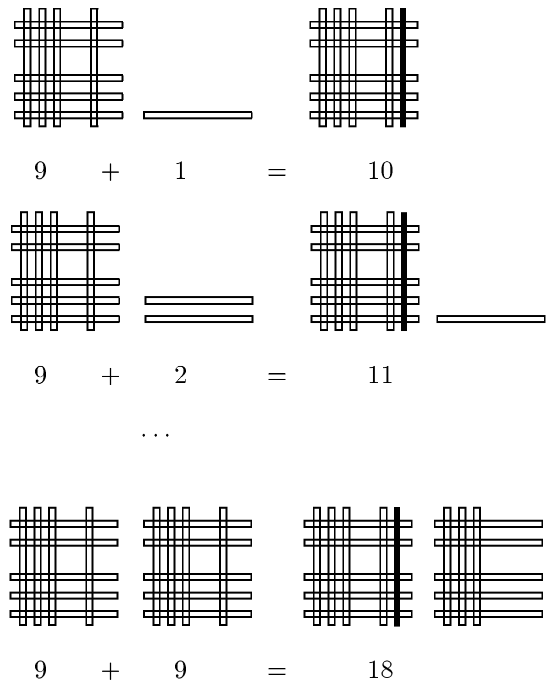


$7 = 6 + \mathbf{1} = 5 + \mathbf{2} = 4 + \mathbf{3}$

Fig. 19

which would illustrate all these decompositions.

7. Two Easy Cases of Adding up over Ten. The sums of the form $9+a$, $a = 1, 2, \dots, 9$ are easily memorized and, working with sticks, the rearrangement consists of a single move by which we place just one stick taken from the "a" heap.



$9+1=10$
$9+2=11$
$9+3=12$

$9+4=13$
$9+5=14$
$9+6=15$

$9+7=16$
$9+8=17$
$9+9=18$

Fig. 20

In addition to its standard arrangement "10" is also easily recognizable when set up as "$5+5$":

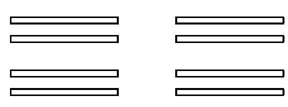
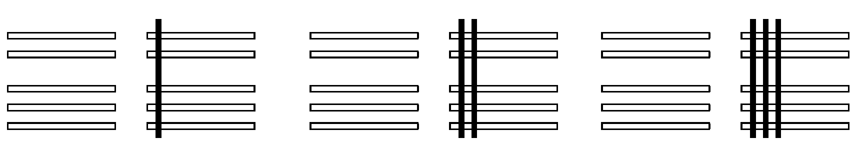
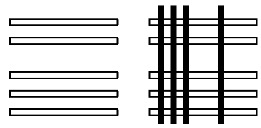


Fig. 21

Without any arranging, the sums $5+a$, $a = 6, 7, 8, 9$ are immediately seen. Put $a = 5+b$, $b = 1, 2, 3, 4$ and form the "b" heap turning its sticks to the red side in order to have an invariant "blue" ten:



$5+6=11$ $5+7=12$ $5+8=13$



$5 + 9 = 14$

Fig. 22

Notice that "$5+6$" is seen as "$10+1$", "$5+7$" as "$10+2$", "$5+8$" as "$10+3$" and "$5+9$" as "$10+4$".

As it might be expected, the easiest sums involving "crossing the ten" are those of the form $10 + a$, where the settings representing these sums and the standard arrangements representing their decimal notations coincide.

8. Addition Involving "Crossing the 10-line". When the sum of the two one-digit numbers exceeds ten, then such addition is said to involve "crossing the 10-line". In this case, the second summand breaks up so that a part of it, taken together with the first summand completes ten.

In view of the commutative law, we might suppose that the first summand is equal or greater than the second one.

As an example, take the sum "$8 + 7$":

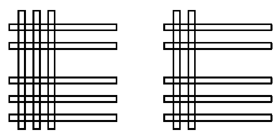


Fig. 23

Removing the sticks from the second heap, they are turned and placed to complete the first heap which represents the number "10":

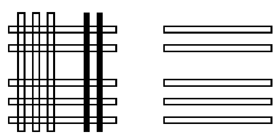


Fig. 24

Working with apparatus this summing goes with such an ease that it is likely to become a mere amusement. That is why the manipulative activities have to be followed by formal operations which make this method clearly split into intermediate steps.

Going back to the previous example, as soon as the two heaps are arranged (Fig. 23), the expression "$8 + 7$" has to be written. Then, following the standard arrangement (Fig. 24), the second summand is decomposed

$$\begin{array}{c} \mathbf{2} \quad 5 \\ \overset{+}{\vee} \\ 8 + 7 \end{array}$$

and (after the oral "$8+2=10$") the formal work completes as follows

$$\begin{array}{c} \mathbf{2} \quad 5 \\ \stackrel{+}{\bigvee} \\ 8+7=10+5=15. \end{array}$$

For the sum "$7+5$", we have

(before) (after)

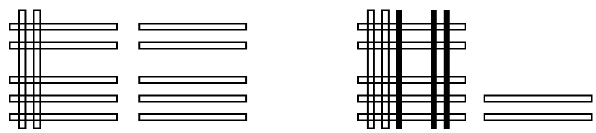


Fig. 25

and formally

$$\begin{array}{c} \mathbf{3} \quad 2 \\ \stackrel{+}{\bigvee} \\ 7+5=10+2=12. \end{array}$$

Working with this kind of apparatus, the pupil forms gradually visual images of standard arrangements what helps him/her to perform the operations in the absence of any concrete material. While the child is working all these exrecises, the teacher can dose a help using space holders and, directing the activities, he/she should use as few words as possible.

When both summands are equal or greater than five, there is another example of addition whose easiness is quite imposing. The sum of the numbers $5+a$ and $5+b$ is $10+a+b$, so that the addition of the smaller than five numbers a and b is left. In the arrangements, "5" is represented with the blue side of sticks being up and a, b with red.

In the following figure the sum "$8+7$" is represented

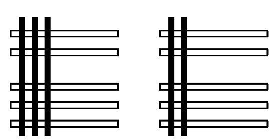
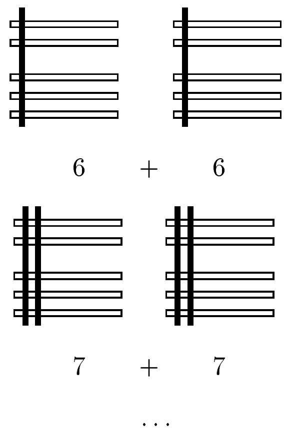


Fig. 26

which is also seen as "blue" ten plus "$3+2$" red sticks.

Such a way of adding is especially effective when the summands are equal:



$6+6=12$

$7+7=14$
$8+8=16$
$9+9=18$

Fig. 27

and these pictures also serve as an explanation of the easiness with which such sums are memorized.

9. Beginnings in Subtraction. Everything what was said about addition, in the section 5, is equally valid for subtraction. Namely, this theme is also divided into steps: recognition of situation to which we react subtracting, composition of the difference and equating to decimal notation. Staying whithin the frame of subtraction table, we will also use sticks to materialize the corresponding mental operations.

The most straightforward idea of materialization of the difference "$n-m$" is "taking away": from n take m. Manipulatively, this means that the sticks are taken away from the "n" heap and arranged until the "m" heap is obtained (with the red side of sticks being up). In the case of the difference "$9-5$", we have:

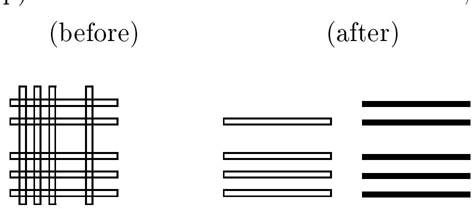
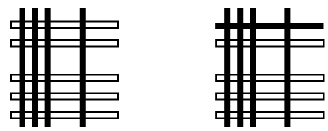


Fig. 28

The remaining sticks stand for "$n-m$" and, in the example we follow, such heap represents the number "4".

Observe some disadvantages of this modeling of the difference. The "n" heap that we start with suggests in no way the difference "$n-m$". After decomposing, these two heaps suggest this difference, but the minuend they represent is not standardly arranged.

A better idea is to start with the heap composed of sticks in two colors. For example, the difference "$9-5$" can be represented in two ways

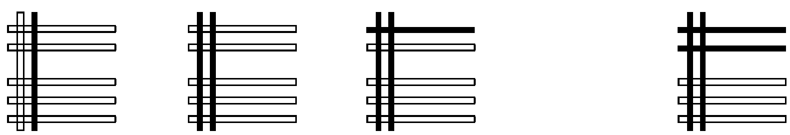


$9-5$ $\qquad$ $9-\mathbf{5}$

Fig. 29

where the first way suggests the inverse addition and the second one the direct subtraction. Both these methods are equally important and they have to be equally practicized. At the very beginning a "blue" and a "red" subtrahend might be written to correspond in color with sticks.

To illustrate, consider the differences where the minuend is less than 10 and where "the 5-line is crossed".



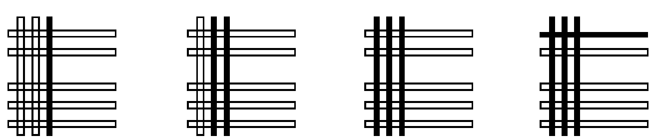
$7-6=\mathbf{1}$ $\qquad$ $7-5=\mathbf{2}$ $\qquad$ $7-4=\mathbf{1}+\mathbf{2}=\mathbf{3}$ $\qquad$ $7-3=\mathbf{2}+\mathbf{2}=\mathbf{4}$

$7-\mathbf{1}=6$ $\qquad$ $7-\mathbf{2}=5$ $\qquad$ $7-\mathbf{3}=7-\mathbf{2}-\mathbf{1}=4$ $\qquad$ $7-\mathbf{4}=7-\mathbf{2}-\mathbf{2}=3$

Fig. 30

Notice how the pattern of these arrangements serves very well as an explanation how to "cross the 5-line", with the red sticks split into a horizontal and a vertical group.

Similarly, we have:



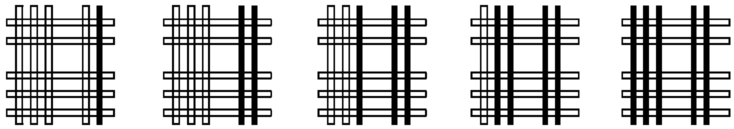
$8-7=\mathbf{1}$ $\qquad$ $8-6=\mathbf{2}$ $\qquad$ $8-5=\mathbf{3}$ $\qquad$ $8-4=\mathbf{1}+\mathbf{3}=\mathbf{4}$

$8-\mathbf{1}=7$ $\qquad$ $8-\mathbf{2}=6$ $\qquad$ $8-\mathbf{3}=5$ $\qquad$ $8-\mathbf{4}=8-\mathbf{3}-\mathbf{1}=4$

Fig. 31

and together with $6-2$, $6-4$, $6-3$, they are all possible differences which involve "crossing of the 5-line".

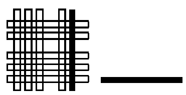
The following easy differences are involved when “the 10-line is crossed”:



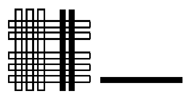
$10-9=\mathbf{1}$ $10-\mathbf{1}=9$ $10-8=\mathbf{2}$ $10-\mathbf{2}=8$ $10-7=\mathbf{3}$ $10-\mathbf{3}=7$ $10-6=\mathbf{4}$ $10-\mathbf{4}=6$ $10-5=\mathbf{5}$ $10-\mathbf{5}=5$

Fig. 32

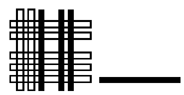
10. Subtraction Involving “Crossing the 10-line”. When the minuend is greater than ten and the difference less than ten, then such subtraction is said to involve “crossing the 10-line”. Now we proceed with illustrating of all possible cases of these differences:



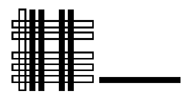
$11-9=\mathbf{1}+\mathbf{1}=\mathbf{2}$
$11-\mathbf{2}=11-\mathbf{1}-\mathbf{1}=9$



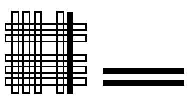
$11-8=\mathbf{2}+\mathbf{1}=\mathbf{3}$
$11-\mathbf{3}=11-\mathbf{1}-\mathbf{2}=8$



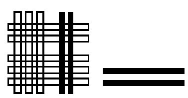
$11-7=\mathbf{3}+\mathbf{1}=\mathbf{4}$
$11-\mathbf{4}=11-\mathbf{1}-\mathbf{3}=7$



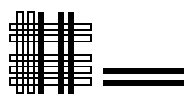
$11-6=\mathbf{4}+\mathbf{1}=\mathbf{5}$
$11-\mathbf{5}=11-\mathbf{1}-\mathbf{4}=6$



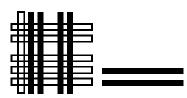
$12-9=\mathbf{1}+\mathbf{2}=\mathbf{3}$
$12-\mathbf{3}=12-\mathbf{2}-\mathbf{1}=9$



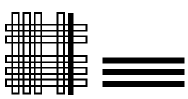
$12-8=\mathbf{2}+\mathbf{2}=\mathbf{4}$
$12-\mathbf{4}=12-\mathbf{2}-\mathbf{2}=8$



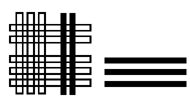
$12-7=\mathbf{3}+\mathbf{2}=\mathbf{5}$
$12-\mathbf{5}=12-\mathbf{2}-\mathbf{3}=7$



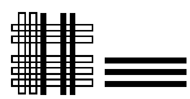
$12-6=\mathbf{4}+\mathbf{2}=\mathbf{6}$
$12-\mathbf{6}=12-\mathbf{2}-\mathbf{4}=6$



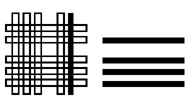
$13-9=\mathbf{1}+\mathbf{3}=\mathbf{4}$
$13-\mathbf{4}=13-\mathbf{3}-\mathbf{1}=9$



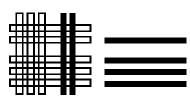
$13-8=\mathbf{2}+\mathbf{3}=\mathbf{5}$
$13-\mathbf{5}=13-\mathbf{3}-\mathbf{2}=8$



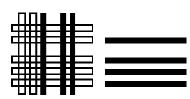
$13-7=\mathbf{3}+\mathbf{3}=\mathbf{6}$
$13-\mathbf{6}=13-\mathbf{3}-\mathbf{3}=7$



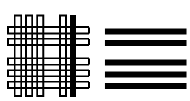
$14-9=\mathbf{1}+\mathbf{4}=\mathbf{5}$
$14-\mathbf{5}=14-\mathbf{4}-\mathbf{1}=9$



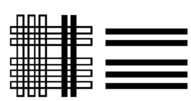
$14-8=\mathbf{2}+\mathbf{4}=\mathbf{6}$
$14-\mathbf{6}=14-\mathbf{4}-\mathbf{2}=8$



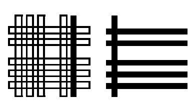
$14-7=\mathbf{3}+\mathbf{4}=\mathbf{7}$
$14-\mathbf{7}=14-\mathbf{4}-\mathbf{3}=7$



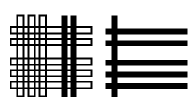
$15-9=\mathbf{1}+\mathbf{5}=\mathbf{6}$
$15-\mathbf{6}=15-\mathbf{5}-\mathbf{1}=9$



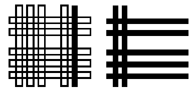
$15-8=\mathbf{2}+\mathbf{5}=\mathbf{7}$
$15-\mathbf{7}=15-\mathbf{5}-\mathbf{2}=8$



$16-9=\mathbf{1}+\mathbf{6}=\mathbf{7}$
$16-\mathbf{7}=16-\mathbf{6}-\mathbf{1}=9$

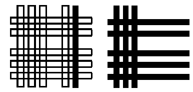


$16-8=\mathbf{2}+\mathbf{6}=\mathbf{8}$
$16-\mathbf{8}=16-\mathbf{6}-\mathbf{2}=8$



$17-9=\mathbf{1}+\mathbf{7}=\mathbf{8}$

$17-\mathbf{8}=17-\mathbf{7}-\mathbf{1}=9$



$18-9=\mathbf{1}+\mathbf{8}=\mathbf{9}$

$18-\mathbf{9}=18-\mathbf{8}-\mathbf{1}=9$

Fig. 33

For the sake of equality of two methods, the differences

$$11-5,\quad 11-\mathbf{6},\quad 13-6,\quad 13-\mathbf{7},\quad 15-7,\quad 15-\mathbf{8},\quad 17-8,\quad 17-\mathbf{9},$$

might also be included.

We all are inclined to calculate the difference "$11-9$" by inverse addition and the difference "$11-2$" just as $11-\mathbf{2}$.

Which way of calculation is easier depends, of course, on the individual inclinations (often spontaneously formed). Nevertheless, the following could be said: *the inverse addition is easier whenever the subtrahend is greater than the half of minuend and the direct subtraction is easier whenever the subtrahend is smaller than the half of minuend.*

11. Multiplication. In the multiplication table the values of products go up to 81 and manipulation with so many sticks would mean facing of drudgery. When starting with multiplication, the pupil will usually have learnt addition and subtraction up to 100. If he/she has been using standard arrangements, then his/her corresponding visual representations are very well developed. For him/her, the following arrangement:

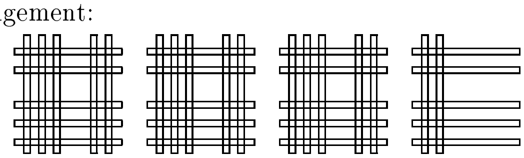


Fig. 34

would mean three tens and seven ones, that is he/she would immediately see the number "37" being set up. Thus, pictured sticks should be used instead of concrete sticks, when their number surpasses a reasonable limit.

Now our objective is to design pictured arrangements which will represent the products

$$\begin{aligned} &n\cdot 5, \quad n=1,\dots,9\\ &n\cdot 6, \quad n=1,\dots,6\\ &n\cdot 7, \quad n=1,\dots,7\\ &n\cdot 8, \quad n=1,\dots,8\\ &n\cdot 9, \quad n=1,\dots,9 \end{aligned}$$

To describe the idea how these pictorized representations are formed, take an example. For, say, $6 \cdot 7$ we design an arrangement which is easily seen as six groups of seven sticks and, in the same time, as four groups of ten and one group of two sticks. In addition to the standard arrangements, the numbers 6, 7, 8 and 9 will also be represented by a suitable arrangement for each of them, designated by $1 \cdot 6$, $1 \cdot 7$, $1 \cdot 8$ and $1 \cdot 9$ respectively. The series of such drawings, sized up properly, might be used as $n \cdot 6$, $n \cdot 7$, $n \cdot 8$ and $n \cdot 9$ maps kept on walls in the class. Contacts with these pictorial arrangements will surely enrich the arithmetic imagery of a child in this matter which is often exposed to a drill of learning by rote.

First we start with the "$n \cdot 5$" series.

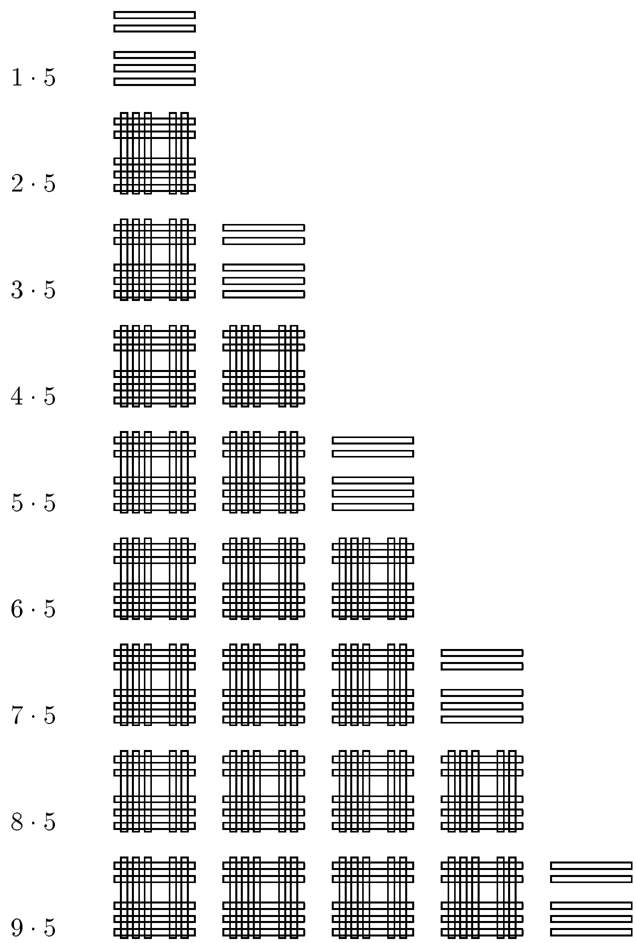
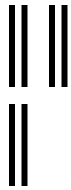


Fig. 35

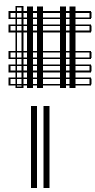
The results of multiplications are supposed to be seen from the pictures.

Modeling a suitable "red" six, the "$n \cdot 6$" series is as follows:

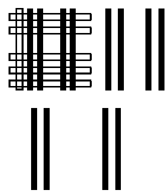
$1 \cdot 6$



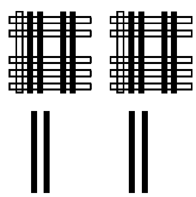
$2 \cdot 6$



$3 \cdot 6$



$4 \cdot 6$



$5 \cdot 6$ is omitted since $6 \cdot 5$ has already been represented.

$6 \cdot 6$

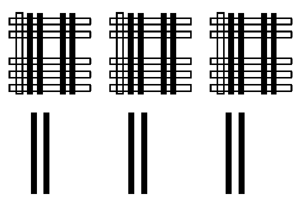
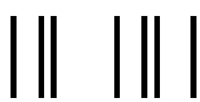


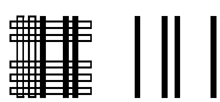
Fig. 36

Notice that the pictures for $1 \cdot 6$ and $2 \cdot 6$ are constituent parts.

$1 \cdot 7$



$2 \cdot 7$



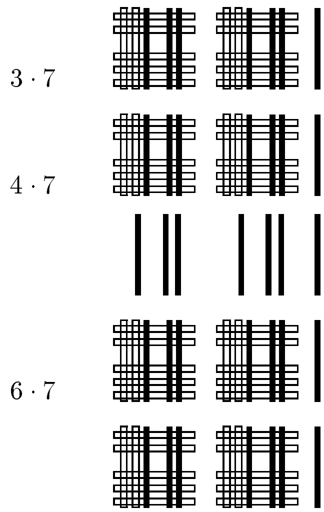
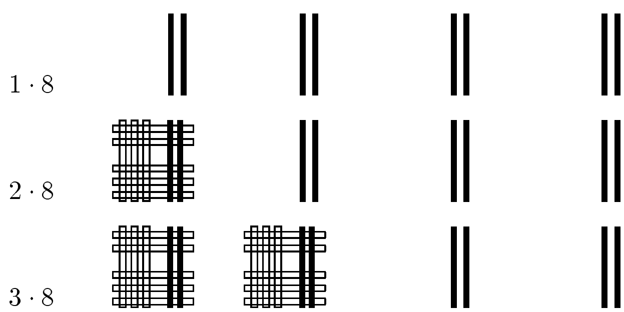


Fig. 37

Notice also here that the pictures for $1 \cdot 7$, $2 \cdot 7$ and $3 \cdot 7$ are constituent.



etc.

Fig. 38

When n runs, the product $n \cdot 9$ is equal to 9, 18, 27, 36, 45, 54, 63, 72, 81. How the figures of these decimal notations are related to n is shown by this table

tens:	0	1	2	3	4	5	6	7	8
ones:	9	8	7	6	5	4	3	2	1

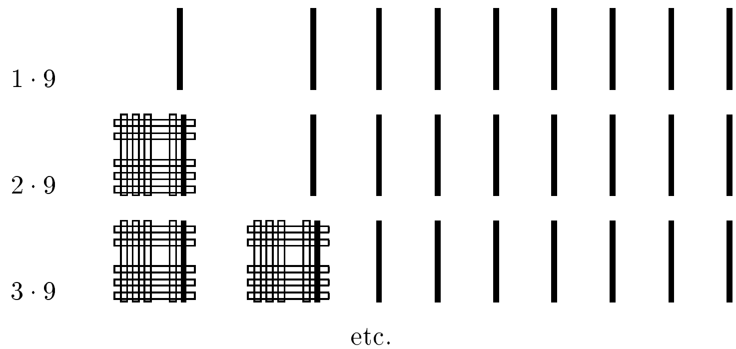


Fig. 39

12. Some Remarks. Instead of sticks, many other kinds of concrete objects might be used. To preserve the manipulations and their outcomes described in this paper, standard arrangements of such objects have to be designed following our "ten fingers" model. For instance, square jettons (with sides in different colors) have to be standardly arranged as follows

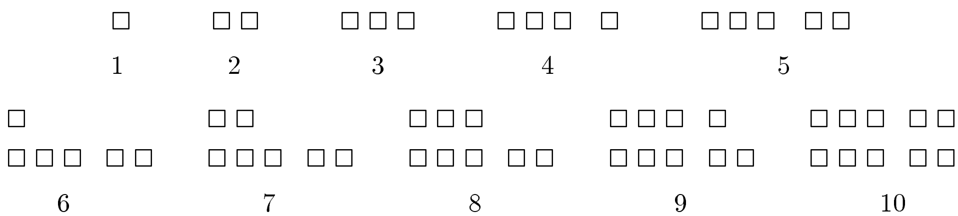


Fig. 40

Besides being more compact entities, the arrangements of sticks, pictured as line segments, are easier to be drawn than the ones composed of jettons.

Working with number pictures, the sticks should be drawn as straight line segments. When sticks have not been used as manipulatives, weaving a story, the teacher can relate such segments and their arrangements to some real world objects and their groups.

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