Chapter 6

Mixing

SUMMARY: In environmental flows, mixing is important because it brings into contact fluid parcels with different origins, dilutes their properties, and promotes chemical reactions. Some preliminary considerations aim at defining mixing and quantifying its efficiency. The present chapter then explores several situations in which a shear flow generates mixing, with or without the impeding action of density stratification.

6.1 The Nature of Mixing

Because mixing is an everyday word, most readily associated with our daily food preparations and home washing machines, it tends to lack a sharp definition. It is nonetheless accompanied by an extensive body of knowledge mostly stemming from chemical engineering. Indeed, a frequent preoccupation among chemical engineers is the design of swirling flows that maximize mixing followed by the determination of the energy required to stir the contents of a chemical reactor in a certain amount of time. The study of mixing also pertains to environmental fluid mechanics because it is the process by which fluid parcels with different origins, and possibly different contaminant loadings, come into contact, dilute their properties into one another, and promote transformative chemical reactions.

A first approach to the study of mixing is the consideration of its geometrical aspects, that is, the topology of streamlines in the flow (Ottino, 1989). The questions then concern the folding of fluid trajectories and rates of deformation of a cloud of tracked fluid particles. Mixing can viewed as a three-dimensional "shuffling process" (Brothman *et al.*, 1945) during which there is an increasing probability over time that a fluid parcel of a given size will pass through a certain region. Following this line of thought and carefully distinguishing between stirring and mixing[1], Eckart

[1] According to Eckart (1948), *stirring* should be used only to refer to the kinematic stretching and folding of lines of fluid particles whereas *mixing* is the process that erases differences (gradients)

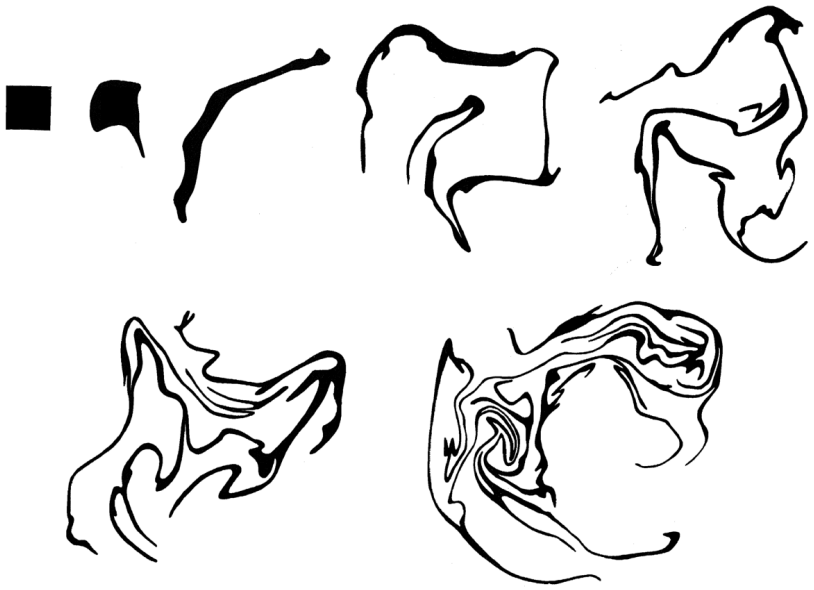

Figure 6.1: Temporal deformation of a patch of butanol floating on water in a rotating vessel. [From Welander, 1955]

(1948) calculated "mixing times" of temperature and salt anomalies in the ocean without resorting to any dynamical description of the underlying processes.

Welander (1955) offers the following description for stirring in two-dimensions. An initial square element with side short compared to the length scale of the deforming flow field is subjected to translation, rotation and shearing. As time passes, the deformation of the element becomes significant, and the element gradually acquires an irregular shape. Increasingly long and thin bands are formed (Figure 6.1). Whenever the band lies at right angle to the flow, folds develop at points of maximum velocity. Folds also appear where the velocity gradient is nearly opposite to the velocity direction. As the band folds, it becomes increasingly elongated and thin while not occupying much more overall space. Within a finite amount of time, the situation appears chaotic.

One reasonable assumption is that the rate of stretching of a band segment of length ℓ grows proportionally to ℓ itself,

$$\frac{d\ell}{dt} \;=\; \alpha\, \ell, \tag{6.1}$$

with the coefficient of proportionality α depending on the features of the flow field. Considering a stationary turbulence field, Batchelor (1952) took α as a constant over time, with ℓ then increasing exponentially but never reaching an infinite value

under the subsequent action of molecular diffusion.

in a finite time. Welander (1955), in contrast, argued that an infinite length could be reached in a finite time. His argument based on properties of stretching in high-curvature folds leads to the expression

$$\alpha \;=\; \frac{V}{L}\,\frac{\ell}{\ell_0}\;, \tag{6.2}$$

in which L and V are respectively the length and velocity scales of the flow field, and ℓ_0 the initial length of the segment. Equation (6.1) then includes a factor ℓ^2 on the right and leads to a solution that reaches an infinite value in a finite time:

$$\ell(t) \;=\; \frac{\ell_0}{1 - Vt/L}\;. \tag{6.3}$$

Naturally, dissipation would act to smear fluid properties before a band of fluid becomes infinitely long and thin. The point is that mixing time can be as short as L/V.

Besides the consideration of deforming bands, an alternative approach is to explore the probability of the position of a particle in the flow. For this, Taylor (1921) considered a point source emitting particles, such as smoke particles from a chimney, into a stationary and homogeneous fluid with zero mean velocity. With $\vec{x}(t)$ the position of the particle at time t after its release from the origin [$\vec{x}(0) = 0$], the following variance is defined:

$$R^2(t) \;=\; <\vec{x}(t)\cdot\vec{x}(t)>\;, \tag{6.4}$$

in which $< ... >$ indicates an average over turbulent fluctuations and $\cdot$ represents the scalar product of the two vectors. The variable $\vec{x}(t)$ is a random variable with no mean, which corresponds to a trajectory with random turns. Although the particle may occasionally move in a direction that takes it back toward its origin, it is unlikely that it will ever return there, and the more time has elapsed, the further away from the origin it is likely to be. The correlation $R^2(t)$ measures the square of the distance from the origin by time t, regardless of direction, and the vigor of mixing is measured by the rate at which $R^2(t)$ grows with time. So, we take its time derivative:

$$\frac{dR^2}{dt} \;=\; 2\;<\vec{x}\cdot\frac{d\vec{x}}{dt}>\;. \tag{6.5}$$

Since the time derivative of position is velocity, the preceding equation may also be successively expressed as

$$\begin{aligned}
\frac{dR^2}{dt} \;&=\; 2\;<\left(\int_0^t \vec{u}(t')dt'\right)\cdot\vec{u}(t)> \\
&=\; 2\int_0^t <\vec{u}(t')\cdot\vec{u}(t)>\;dt' \\
&=\; 2\int_0^t <\vec{u}(t-\tau)\cdot\vec{u}(t)>\;d\tau\;,
\end{aligned} \tag{6.6}$$

in which $\vec{u} = d\vec{x}/dt$ is the vector velocity in the turbulent flow, which is a random variable, too. The second expression was obtained by switching the order of operation between time integration and averaging over turbulent fluctuations, and the change of variable $\tau = t - t'$ was made to arrive at the third expression.

If the mixing fluid is in a state of stationary, homogeneous and isotropic turbulence, then the dimensionless velocity auto-correlation over time,

$$S^2(\tau) \;=\; \frac{<\vec{u}(t)\cdot\vec{u}(t-\tau)>}{<\vec{u}(t)\cdot\vec{u}(t)>} \;=\; \frac{<\vec{u}(t)\cdot\vec{u}(t-\tau)>}{u_*^2}\;, \tag{6.7}$$

is independent of time t, position and direction and can only be a function of the time delay τ. The velocity variance u_*^2 is a constant measuring the turbulence intensity of the flow. Equation (6.6) can then be cast as

$$\frac{dR^2}{dt} \;=\; 2u_*^2 \int_0^t S^2(\tau)\; d\tau. \tag{6.8}$$

Two limiting cases are of interest. For short times ($\tau \to 0$), the velocity cannot be significantly decorrelated because acceleration must remain finite. Hence, $S^2(\tau) \simeq 1$ for short τ, and

$$\frac{dR^2}{dt} \;=\; 2u_*^2\; t, \tag{6.9}$$

of which the time integration yields:

$$R(t) \;=\; u_*\; t. \tag{6.10}$$

Thus, the spread of the particle position increases as the first power of time. In essence, the particle is propelled in the direction of its initial velocity.

But this initial trajectory will not last long because velocity fluctuations will soon redirect the particle and force it to meander. At long times, the velocity auto-correlation vanishes (Figure 6.2), and the relevant quantity becomes the integration of $S^2(t)$ from $t = 0$ to $t = \infty$, which defines a correlation time T of the turbulent flow,

$$T \;=\; \int_0^\infty S^2(t)\; dt, \tag{6.11}$$

interpreted as the memory time of the turbulent flow. In that limit, the variance of the particle position becomes

$$R^2(t) \;=\; 2u_*^2\; T\; t, \tag{6.12}$$

of which the square root is

$$R(t) \;=\; u_*\; \sqrt{2Tt}\;. \tag{6.13}$$

At long times, the variance increases as the pace of the square root of time. The situation is similar to random walk.

Chaotic advection (Aref, 1984).

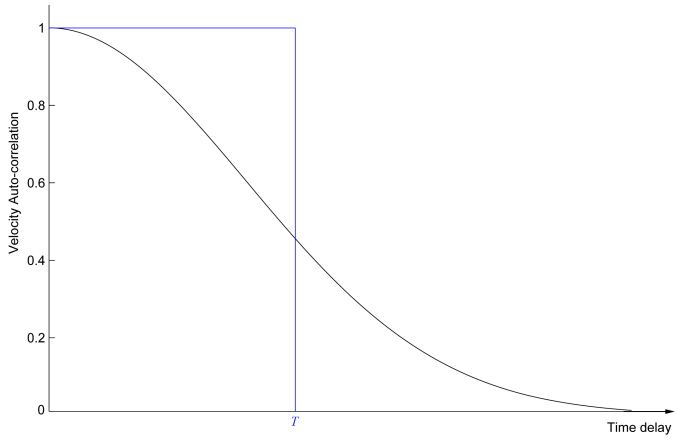

Figure 6.2: Typical velocity auto-correlation in a turbulent flow. With no time delay, velocity is perfectly correlated to itself, and the auto-correlation value is 1. As time delay increases, the auto-correlation diminishes until it vanishes. The time T defined in the text is the time scale over which the autocorrelation diminishes. The area of the rectangle is equal to the area under the curve.

6.2 Mixing by Shear

Phenomenological description of mixing in wall turbulence. Mixing length concept (Prandtl, 1925). How it is been used in practice.

6.3 Mixing in the Presence of Stratification

As seen in Sections 5.1 and 5.2, a velocity shear can destabilize a vertical stratification. In such a case, kinetic energy is supplied by the flow and partly consumed by an increase in potential energy, which is required to overcome buoyancy forces and raise the center of gravity of the system. Whereas the analysis of the previous chapter dealt with the early, linear stages of the process, we turn our attention here to the end result, that of the mixed state. Because a detailed analysis of a highly complex and turbulent regime is not possible, we first consider the energetics of the mixing.

To elucidate the energetics of vertical mixing in the presence of a density difference, consider the following problem: Initially, the system is comprised of two layers of fluid of thicknesses H_1 and H_2 with different densities, ρ_1 and ρ_2, and different velocities, U_1 and U_2. Then, we assume that mixing occurs, leaving a single layer of homogenized fluid of density ρ, flowing at a uniform velocity U over the entire thickness $H = H_1 + H_2$ of the system (Figure 6.3). Conservation of momentum

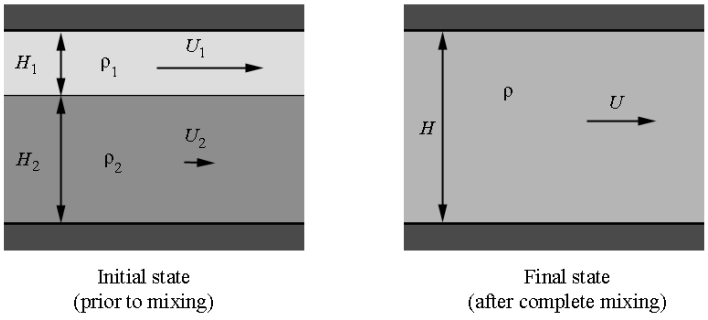

Figure 6.3: Mixing of a two-layer stratified fluid with velocity shear. Rising of denser fluid and lowering of lighter fluid require work against buoyancy forces and are possible only if there is a sufficient supply of kinetic energy. Kinetic energy is released from the flow when the velocity shear is reduced.

and heat requires that the new velocity U be the weighted average of the initial velocities U_1 and U_2, and the new density ρ be the weighted average of the initial densities ρ_1 and ρ_2:

$$U = \frac{H_1 U_1 + H_2 U_2}{H_1 + H_2} \tag{6.14}$$

$$\rho = \frac{H_1 \rho_1 + H_2 \rho_2}{H_1 + H_2} \; . \tag{6.15}$$

Per unit area of the system, the kinetic energy prior to the mixing event was (by invoking the Boussinesq approximation, which allows us to replace the actual densities by the reference density ρ_0):

$$\begin{aligned} KE_{\text{initial}} &= \int_0^H \frac{1}{2} \rho_0 u^2(z) \, dz \\ &= \frac{1}{2} \rho_0 U_2^2 H_2 \; + \; \frac{1}{2} \rho_0 U_1^2 H_1 \\ &= \frac{1}{2} \rho_0 (H_1 U_1^2 + H_2 U_2^2) \end{aligned} \tag{6.16}$$

and has become afterwards

$$\begin{aligned} KE_{\text{final}} &= \frac{1}{2} \rho_0 U^2 H \\ &= \frac{1}{2} \rho_0 \frac{(H_1 U_1 + H_2 U_2)^2}{H_1 + H_2}. \end{aligned} \tag{6.17}$$

There has been a drop in kinetic energy:

$$\begin{aligned} KE_{\text{drop}} &= KE_{\text{initial}} - KE_{\text{final}} \\ &= \frac{1}{2}\,\rho_0\,\frac{U_1^2 H_1 H_2 \;+\; U_2^2 H_1 H_2 \;-\; 2U_1 U_2 H_1 H_2}{H_1 + H_2} \\ &= \frac{1}{2}\,\rho_0\,\frac{H_1 H_2}{H_1 + H_2}\,(U_1 - U_2)^2, \end{aligned} \tag{6.18}$$

which is always positive. (This is because the square of an average is always less than the average of the squares.)

Similarly, the potential energy was

$$\begin{aligned} PE_{\text{initial}} &= \int_0^H \rho(z) g z \; dz \\ &= \int_0^{H_2} \rho_2 g z \; dz \;+\; \int_{H_2}^{H_1+H_2} \rho_1 g z \; dz \\ &= \frac{1}{2}\,\rho_2 g H_2^2 \;+\; \frac{1}{2}\,\rho_1 g (2H_1 H_2 + H_1^2) \end{aligned} \tag{6.19}$$

and has become

$$\begin{aligned} PE_{\text{final}} &= \int_0^H \rho g z \; dz \\ &= \rho g\,\frac{H^2}{2} \\ &= \frac{1}{2}(\rho_1 H_1 + \rho_2 H_2) g\,(H_1 + H_2), \end{aligned} \tag{6.20}$$

causing a gain of potential energy:

$$\begin{aligned} PE_{\text{gain}} &= PE_{\text{final}} - PE_{\text{initial}} \\ &= \frac{1}{2}(\rho_2 - \rho_1) g H_1 H_2. \end{aligned} \tag{6.21}$$

Note how the densities were approximated by the reference density ρ_0 in the expressions of the kinetic energy, where small variations in density do not matter, but not in the expressions of the potential energy, where the small density variations make the whole difference.

Physically, the potential-energy level has been raised because cold fluid has been elevated and light fluid lowered, all against gravity, while kinetic energy has dropped. Naturally, if the kinetic-energy release (6.21) exceeds the potential-energy gain (6.24), mixing will take place spontaneously. Therefore, the system will undergo spontaneous mixing if the following inequality is satisfied

$$\frac{1}{2}\ \rho_0\ \frac{H_1 H_2}{H_1+H_2}\ (U_1-U_2)^2\ >\ \frac{1}{2}(\rho_2-\rho_1)gH_1H_2$$

which can be reduced to

$$\rho_0(U_1-U_2)^2\ >\ gH(\rho_2-\rho_1). \tag{6.22}$$

In reality, a sizeable fraction of the kinetic energy released by the mixing creates turbulence and, hence, is dissipated by friction. Only the remainder of the energy release serves to increase the potential energy. Laboratory experiments (*reference needed here*) indicate that this remainder is about 30%. Thus, the preceding inequality must be amended as follows:

$$(0.3)\rho_0\ (U_1-U_2)^2\ >\ gH\ (\rho_2-\rho_1), \tag{6.23}$$

i.e., if

$$\frac{gH\Delta\rho}{\rho_0\Delta U^2}\ <\ 0.3, \tag{6.24}$$

where $\Delta U = |U_1 - U_2|$ is the absolute velocity difference and $\Delta\rho = \rho_2 - \rho_1$ is the density excess of the lower layer with respect to the upper layer. If we further replace the density difference by its equivalent temperatrure difference, $\Delta\rho = \alpha\rho_0\Delta T$, the same criterion becomes:

$$\frac{\alpha gH\Delta T}{\Delta U^2}\ <\ 0.3, \tag{6.25}$$

Hence, mixing occurs whenever the fluid is sufficiently shallow under given density and velocity differences or, put another way, whenever the velocity shear is sufficiently large or the density (temperature) difference is sufficiently weak.

In the event that inequality (6.28) is not satisfied, there is not enough energy available in the velocity shear for complete mixing. Mixing is then confined to an intermediate height, say h, that marginally obeys (6.28):

$$h\ =\ 0.3\ \frac{\rho_0\Delta U^2}{g\Delta\rho}\ =\ 0.3\ \frac{\Delta U^2}{\alpha g\Delta T}\ . \tag{6.26}$$

In practice, it is helpful to codify the preceding considerations by introducing a dimensionless number, called the *Richardson number*:

$$Ri\ =\ \frac{\alpha gH\Delta T}{\Delta U^2}\ , \tag{6.27}$$

where H is the fluid depth under consideration (either total depth or depth of a turbulent layer, whichever the context dictates is the most appropriate), ΔT is a measure of the vertical temperature difference across the depth H, and ΔU a measure of the velocity shear across that same layer. Physics tell us that:

- If $Ri\ <\ 0.3$, mixing occurs and engulfs the entire thickness H of the fluid.
- If $Ri\ >\ 0.3$, mixing occurs but does not extend over the entire thickness of the domain; it is confined to the thickness h given by (6.29).

The time scale over which mixing occurs, is given by:

$$\begin{aligned} \text{mixing time} &\simeq \frac{\text{thickness of mixing}}{\text{velocity of overturn}} \\ &= \frac{\Delta U^2/\alpha g \Delta T}{\Delta U} = \frac{\Delta U}{\alpha g \Delta T} \end{aligned} \tag{6.28}$$

and is usually quite fast.

6.4 Entrainment

Text of section. Turner (1973)

6.5 Mixed-Layer Modeling

Effect of rotation on mixing. Pollard, Rhines and Thompson (1973). Price's models. Examples of applications.

Problems

6-1. Problem statement

6-2.

6-3.

6-4.

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