A SYNTHESIS OF DIFFERENT PSYCHOLOGICAL LEARNING THEORIES? -Piaget and Vygotsky

Bettina Dahl

Norwegian Center for Mathematics Education
Norwegian University of Science and Technology, Trondheim

Bdahls(at)math.ntnu.no
@ is replaced by (at) to stop the automatic garnering of email addresses by Spam factories - Editor

This paper begins by describing a study that showed that when interviewing high-achieving high school pupils about their cognitive learning, single pupils can say things that are supported by different theories. The paper then discusses how Piaget and Vygotsky commented on each other's work. Then the paper continues with a discussion of the epistemological and ontological basis of constructivism and activity theory, respectively. It is then discussed if accepting a learning theory implies an ontological commitment. Then the paper discusses various ways of overcoming the differences between the theories, and here the concepts of synthesis, grand theory, and complementarity are discussed. It is concluded that in terms of the various theories within the psychology of learning mathematics a concept of "odd complementarity" might be useful, for now.

1. INTRODUCTION

This paper summarizes some of the discussion in a recently finished Ph.D. study in mathematics education (Dahl 2002). The thesis was about how ten high-achieving high school pupils from Denmark and England explain in their own words how they learn a mathematical concept that is new to them. The pupils' metacognition made it possible for them to explain their learning in their own words and most of the time it was quite easy to identify some theoretical notions which reflects what the pupils said. The study used various and different psychological theories of learning mathematics. A result of the study was that it seems that the pupils each have their own way of learning, however, there are also similarities. Also, seemingly contradictory theories are

often seen referred to within one single pupil. This latter observation made it necessary to discuss to what extent various different learning theories can complement each other or if it is possible to make a synthesis among various theories.

I will therefore in Section 2 discuss if the basis of the theories of Piaget and Vygotsky are different and, if so, whether this means that they cannot somehow be "united". Following this, Section 3 will discuss if various learning theories imply an ontological commitment and if, in principle, a synthesis is possible. The subsequent sections discuss various ways of solving the problem of mutual exclusive theories and duality between theories. Among other things, the concepts of *complementarity* and *grand theory* will be discussed.

2. PIAGET AND VYGOTSKY ON LANGUAGE AND THE INDIVIDUAL-SOCIAL RELATION

Piaget (1896-1980) and Vygotsky (1896-1934) belong to two different traditions. Briefly stated, Piaget belongs to the constructivism perspective that sees learning as construction, and Vygotsky to the activity theory perspective that sees learning as appropriation. Lerman states that "Vygotsky's and Piaget's programs have fundamentally different orientations, the former placing the social life as primary and the latter placing the individual as primary … the assumption of complementarity leads to incoherence" (Lerman, 1996, p. 133). Lerman does thus not even think that these theories could be complementary. I agree with Lerman in that if one sees the Piaget perspective and the Vygotsky perspective as two bodies of knowledge that are built up as a mathematical-logical formal system with different basis, then the theories are mutually exclusive. But then one might ask, what is the basis of these theories, and does it matter if they are different?

Vygotsky and Piaget's work is spread out on a great number of areas and books. I will focus on one of the main areas, namely the question of language and its importance for thoughts and learning. There will also be a discussion of the relationship between the individual and the social. Below are some rather long quotations from both Vygotsky and Piaget when they discuss each other's argumentation on this topic. The purpose is, besides telling what they think, to show the "tone" between them.

2.1 The debate about language and egocentric speech

In one of his main books, "Thought and Language" which was published posthumously in 1934, Vygotsky writes about Piaget "Psychology owes a great deal to Jean Piaget. It is not an exaggeration to say that he revolutionized the study of child language and thought" (Vygotsky, 1962, p. 9). However, Vygotsky also has some criticism of Piaget around the concept of egocentrism[1] and egocentric speech. According to Vygotsky (1962, p. 14-15), Piaget's observations made him conclude that children's speech can only fall into two groups, the egocentric and the socialized. The difference between them is mainly in their function as in egocentric speech, the child does only talk about himself and has no interest in others and expects no answers. Socialized speech attempts an exchange with others. According to Vygotsky, Piaget's experiments showed that most of the talk of preschool children is egocentric but as the child approaches school age, egocentric speech atrophies. In contrast to Piaget's view, Vygotsky states that his experiments suggest that egocentric speech has a very specific function (Vygotsky, 1962, p. 16). He writes:

> *In order to determine what causes egocentric talk, what circumstances provoke it, we organized the children's activities in much the same way Piaget did, but we added a series of frustrations and difficulties. For instance, when a child was getting ready to draw, he could suddenly find that there was no paper, or no pencil of the color he needed. In other words, by obstructing his free activity we made him face problems. We found that in these difficult situations the coefficient of egocentric speech almost doubled, in comparison with Piaget's normal figure for the same age and also in comparison with our figure for children not facing these problems. The child would try to grasp and to remedy the situation in talking to himself: "Where's the pencil? I need a blue pencil. Never mind, I'll draw with the red one and wet it with water; it will become dark and look like blue. … Our findings indicate that egocentric speech does not long remain a mere accompaniment to the child's activity. Besides being a means of expression and of release of tension, it soon becomes an instrument of thought in the proper sense - in seeking and planning the solution of a problem.*
> *(Vygotsky, 1962, p. 16)*

Thus, to Vygotsky egocentric speech, besides its communicative role, has an important role as a thinking-tool and as a tool to solve problems. According to Vygotsky, Piaget sees the development of thought as "gradual socialization of deeply intimate, personal, autistic mental

[1] *The notion of egocentrism in Piaget's work is "quite unrelated to the common meaning of the term, hypertrophy of the consciousness of self. Cognitive egocentrism, as I have tried to make clear, stems from a lack of differentiation between one's own point of view and the other possible ones, and not at all from an individualism that precedes relations with others" (Piaget, 1962, p. 4).*

states. Even social speech is represented as following, not preceding, egocentric speech" (Vygotsky, 1962, p. 18). But following Vygotsky, the order is different:

> *Thus our schema of development - first social, then egocentric, then inner speech - contrast both with the traditional behaviorist schema - vocal speech, whisper, inner speech - and with Piaget's sequence - from nonverbal autistic thought through egocentric thought and speech to socialized speech and logical thinking. In our conception, the true direction of the development of thinking is not from the individual to the socialized, but from the social to the individual.*
> *(Vygotsky, 1962, p. 19-20)*

To Vygotsky, inner speech "is not the interior aspect of external speech - it is a function in itself. It still remains speech, i.e., thought connected with words. But while in external speech thought is embodied in words, in inner speech words die as they bring forth thought. Inner speech is to a large extent thinking in pure meanings" (Vygotsky, 1962, p. 149). Thus to Vygotsky a main and basic criticism of Piaget is that thinking develops from the social level to the individual, while it is opposite for Piaget. Furthermore egocentric speech has a main role in problem-solving.

Vygotsky's book "Thought and Language" was first published in 1934 in Russian but the book was suppressed in the Soviet Union from 1936-1956.[2] It was not until 1957 that an English translation was begun on Luria's initiative (Vygotsky, 1962, p. xi). This translation was published in 1962 and it was not until then that Piaget actually read Vygotsky's critique of Piaget's work from 1923-24. Following this Piaget wrote in 1962 a "Comment" on Vygotsky's critique of him. Piaget begins by stating:

> *It is not without sadness that an author discovers, twenty-five years after its publication, the work of a colleague who has died in the meantime, when that work contains so many points of immediate interest to him which should have been discussed personally and in detail. Although my friend A. Luria kept me up to date concerning Vygotsky's sympathetic and yet critical position with respect to my work, I was never able to read his writings or to meet him in person, and in reading his book today, I regret this profoundly, for we could have come to an understanding on a number of points. … on certain points I find myself more in agreement with Vygotsky that I would have been in 1934, while on other points I believe I now have better arguments for answering him.*
> *(Piaget, 1962, p. 1)*

[2] *The reason for the surpression was that "he would not brook either materialist reductionism or mentalism, nor the easy Cartesian dualism" (Vygotsky, 1962, p. vi). The fact, that Vygotsky's work was forbidden seem to be owing to him not being a "real" Marxist, however: "From the Marxist ideological perspective, he is celebrated as the man who recognized the historical determination of man's consciousness and intellect. But looking at Vygotsky's place in world psychology, his position transcends either the usual functionalism of the Dewey-James variety or the conventional historical materialism of Marxist ideology. Vygotsky is an original. It is a disservice to him … to find his significance solely in developing Soviet conceptions of man" (Vygotsky, 1962, p. vi).*

As a response to Vygotsky's critique of Piaget's previous view of egocentric speech, Piaget has a rather long preamble, but then continues:

> *This long preamble has seemed necessary to bring out how much I respect Vygotsky's position on the issue of egocentric speech, even though I cannot agree with him on all points. First, Vygotsky did realize that a real problem was involved, and not merely a question of statistics. Second, he himself verified the facts in question, instead of suppressing them through the artifices of measuring; and his observations on the frequency of egocentric speech in children when their activity is blocked and on the decrease of such speech during the period when inner speech begins to form are of very great interest. In the third place, he proposed a new hypothesis: that egocentric speech is the point of departure for the development of inner speech, which is found at a later stage of development, and that this intercrossed language can serve both autistic ends and logical thinking. I find myself in complete agreement with these hypotheses. On the other hand, what I think Vygotsky still failed to appreciate fully is egocentrism itself as the main obstacle to the co-ordination of viewpoints and to co-operation. … In brief, when Vygotsky concludes that the early function of language must be that of global communication and that later speech becomes differentiated into egocentric and communicative proper, I believe I agree with him. But when he maintains that these two linguistic forms are equally socialized and differ only in function, I cannot go along with him because the word socialization becomes ambiguous in this context: if an individual A mistakenly believes that an individual B thinks the way A does, and if he does not manage to understand the difference between the two points of view, this is, to be sure, social behavior in the sense that there is contact between the two, but I call such behavior unadapted from the point of view of intellectual co-operation. … As far as I know I have never spoken of speech 'not meant for others'; this would have been misleading, for I have always recognized that the child thinks he is talking to others and is making himself understood. My view is simply that in egocentric speech the child talks for himself.*
> *(Piaget, 1962, pp. 7-8)*

Above, Piaget declares that he agrees completely with Vygotsky in that for instance egocentric speech is the point of departure for the development of inner speech and that it is this inner speech that can serve logical thinking. Vygotsky emphasises that language is not just a means of expression; it is an instrument of thought. Whether Piaget's expression "serve logical thinking" is the same as Vygotsky's "instrument of thought" is, however, not certain since Piaget's expression seem to grant language slightly less significance for the development of thought than Vygotsky's. However, the difference is small and might be hair-splitting. A place where they do disagree is, according to Piaget, that Vygotsky still failed to understand that egocentrism itself could be a main obstacle for learning. This means that language can also hamper learning. Regarding if egocentric speech is "for others" or not, there seem to have been some kind of misunderstanding between the two; both seem to think that egocentric speech is social. About the correct sequence of 'egocentric speech', 'inner speech', and 'socialized speech', they do not agree. But this does not stop either of them from showing the mutual admiration they have for each other.

2.2 The debate about individual and the social

Another central issue about the difference between Piaget and Vygotsky is the discussion of the role of the individual and the role of the social in learning. Piaget (1969) writes that there is a doubleness in the teaching, and he states that on the one side is the rising individual and on the other side the social, intellectual, and moral values that the educator tries to convey. His method's aim is to make children try to approach the grownup stage not through overtaking readymade reasons and rules for the right action, but by capturing it through own force, self-regulation, and personal experiences (Piaget, 1969, p. 132). Piaget does thus not say that learning is social, only that the *individual* himself and by himself takes over some of the surrounding world's knowledge. Also social-constructivism might be seen as such an attempt.

According to the culture-historical school and activity theory, to which Vygotsky belongs, learning is a question of appropriation of the culturally created surroundings. In line with Vygotsky, Leontiev[3] says: "The child is not adapted to the world of human objects and phenomena surrounding him, but takes it to himself, i.e. appropriates it. … This is a process which has as its result reproduction in the individual of the historical formation of human qualities, abilities and characteristics of behavior" (Eriksen, 1993, p. 43). Knowledge and concepts are not seen as constructions that are created by the single individual, but it is historical and cultural founded mental artefacts and phenomena, which the individual takes up and makes his own. So far, there seems to be quite agreement between Piaget and Vygotsky in the sense that what the individual must learn is a social product of past generations. However, Eriksen (1993, p. 45) writes that basically Leontiev argues that learning is an active process from the point of view of the child, and the child can with social support of the surroundings reproduce culturally accumulated knowledge. Eriksen (1993, p. 45) quotes Leontiev for saying:

> *This process takes place in the child's activity in relation to objects and phenomena of the surrounding world in which are embodied the achievements of mankind. Such activity, however, cannot be developed by the child himself, it develops in practical and verbal intercourse with people surrounding him, in combined activity with them; when the aim of such activity is specifically to transmit to the child certain knowledge, skills and habits then we say that the child learns, the adult teaches.*

[3] *The quotes of Leontiev come from: Leontiev, A. N. (1963, pp. 72-75) 'Principles of Mental Development and the Problem of Intellectual Backwardness' In B. Simon (eds.) Educational Psychology in the U.S.S.R. (London, Routledge & Kegan Paul). The original source could not be found.*

The main difference in Piaget and Vygotsky is therefore if the ability to learn, construct, is inborn or if other people are necessary for the learning process. The answer to the question of whether Piaget and Vygotsky are different is Yes. However, the difference is not about, as formulated by Cole and Wertsch (1996), a primacy of individual primacy of individual psychogenesis versus sociogenesis of mind, but, briefly stated, more that Piaget sees the individual as *the* source of learning, and that children learn by continuous interaction and experience with their environment, the egocentric speech is valuable for logical thinking but it can also obscure the meaning. Instead Vygotsky emphasises that one cannot learn without the verbal interaction and activity with others. The dualism is thus still there, but not as distinct as seems at first. According to Vejleskov (1998, p. 117), some has suggested to build a bridge between the two, by naming in co-constructivism. Furthermore Vejleskov quotes Bruner for, at a Piaget-Vygotsky congress in 1996, having said that Piaget owes us an explanation of how the self-regulation is taking place and Vygotsky owes us an explanation to why we do not all become a copy of the socio-cultural context in which we grow up. Another pair of authors who describe the difference between Vygotsky and Piaget are Cole and Wertsch who argue that

> *For Vygotsky, like Piaget, the relationship between the individual and the social is necessarily relational. However, by placing cultural mediation at the center of adult cognition and the process of cognitive development, social origins take on a special importance in Vygotsky's theories that is less symmetrical than Piaget's notion of social equilibration as 'resulting from the interplay of the operations that enter into all cooperation'. For Vygotsky and cultural-historical theorists more generally, the social world does have primacy over the individual in a very special sense. Society is the bearer of the cultural heritage without which the development of mind is impossible.*
> *(Cole & Wertsch, 1996)*

The difference between Piaget and Vygotsky are thus not that one puts an emphasis on the individual side and the other on the social side, but instead that one, Piaget, seem to balance the two aspects more equally than the other. Also in the Vygotskian school, knowledge comes from the outside, as a transition whereas Piaget talks about man's innate capability of learning. This has resemblances with the ancient philosophical debate about the brain. This is a debate about whether we as humans are born with a blank slate or we are born with certain capabilities, or somewhere in between. In terms of mathematics, Wynn (1992) made an investigation of 5,000 4-6 months old babies. She demonstrated that well before babies can talk, they understand the concepts of addition and subtraction. Using puppets, she shows, for instance, one puppet to a baby. A curtain is then drawn to hide the puppet while the baby sees another puppet go behind

the curtain, which is then opened. Sometimes a third puppet is inserted through a secret door and the study shows that the babies look longer at what they see when the outcome is incorrect – as if they wonder. This might suggest that humans are born with a capacity to do mathematics and that when we learn mathematics we, so to speak, realize what we already know deep down in our brain. This could be seen to support constructivism at least at the basic levels of mathematics. One might argue that for higher level of mathematics, language might be important, which is for instance argued by Dowling: "In my terms playing with teddy bears in not a discourse to the extent that its principles are always context-dependent and so non-explicit. School mathematics, on the other hand, is more discursive, because its principles are comparatively explicit and context independent" (Dowling, 1998, p. 97). Also Dahl (1996a&b) argues that learning mathematics has a linguistic aspect when she talks about learning mathematics as language-games transitions.

However, the dualism still exists. It has resemblance with the traditional actor-structure dualism also seen in sociology and philosophy. Whether of not this dualism can be overcome, will be discussed below. But first I will discuss if it is possible to somehow "synthesize" the two theories without violating that they each, on the level of ontology, are completely different. In other words, does a learning theory automatically imply that one has to "buy" the theory's epistemology and ontology? If not, it might be less problematic to synthesize the theories.

3. DOES A LEARNING THEORY IMPLY AN ONTOLOGICAL COMMITMENT?

When discussing theories about learning, one basically operates with three levels - ontology (the nature of reality), epistemology (the nature of knowledge), and the "learning theory" level (how one learns). This section will discuss constructivism, with Piaget, and activity theory, with Vygotsky, in relation to these three levels.

In terms of the ontological level, the background of constructivism is "non-realism" which means that there is no reality that exist independently of human thinking while the philosophical background of activity theory is dialectical materialism, which is a particular type of metaphysical realism that stands for that everything that exist does have physical characteristics, but there are also several levels in reality such as the physical, the organic, the conscious, the

socio-economical, etc. (Lübcke, 1993). On the epistemological level, constructivism states that knowledge is in the heads of persons and the thinking subject has no alternative but to construct what he knows on the basis of his own experience. Contrary to this is the activity theory that emphasises that knowledge is created in a negotiation/interaction among people and that people appropriate knowledge.

One problem in creating a "synthesis" between Piaget and Vygotsky is that *if* we for instance "buy" constructivism on the level of "learning theory" do we *then* have to follow it all the way back to an ontology of non-realism? *If* this is the case, *then* a synthesis becomes impossible as Vygotsky rests on a branch of realism. To answer this question I will discuss the connection between ontology and epistemology for Piaget and Vygotsky's works:

For Piaget: Constructivism claims that we *always* and *only* learn through constructing. *If* this is the case, *then*, I will argue, it must follow that this is how we learn regardless of how we are being taught and regardless of the nature of reality. Piaget does not separate the level of epistemology from the level of learning theory in his work. He states that his genetic epistemology deals with both the formation of and the meaning of knowledge (Piaget, 1970, p. 12). Furthermore, he writes:

> *From the empiricist point of view, a 'discovery' is new for the person who makes it, but what is discovered was already in existence in external reality and there is therefore no construction of new realities. … By contrast, for the genetic epistemologist, knowledge results from continuous construction, since in each act of understanding, some degree of invention is involved; in development, the passage from one stage to the next is always characterized by the formation of new structures which did not exist before, either in the external world or in the subject's mind.*
> *(Piaget, 1970, p. 77)*

The above quote does not show a rejection of realism as what Piaget here discusses is epistemology, and what he says is that *knowledge* does not exist beforehand in the external world; but he does not say that the external world does not exist independently of man.

Hence, being a constructivist on the level of epistemology is not synonymous with having non-realism as one's ontology.

For Vygotsky: (1) *If* Vygotsky is right in saying that learning activity cannot be developed by the child himself but only though social interaction with other people, *then* external reality must exist as we must assume that it is in reality that these other human beings resides. Furthermore, since learning in Vygotsky's view is transmission of knowledge, one could argue that knowledge must be "somewhere" outside the individual *before* the individual has learnt it

and thus that there is something that exists independently of an individual's construction of it. Furthermore, Vygotsky says himself: "Once we acknowledge the historical character of verbal thought, we must consider it subject to all the premises of historical materialism" (Vygotsky, 1962, p. 51). One of the premises of historical materialism is realism. (2) If reality does exist independently of us, then it does not automatically imply that knowledge is of a certain kind or that the way one gains knowledge of the world is through interacting with this reality. A Platonist might argue, that knowledge is in an imaginary world.

The conclusion on this discussion must be that being a constructivist on the level of epistemology and/or learning theory *does not imply* an ontological commitment, whereas being a Vygotskian on the level of epistemology or learning theory *does imply* an ontological commitment to realism. One could therefore conclude that to create a synthesis of Piaget and Vygotsky on the level of learning theory one must:

- Include realism on the level of ontology as Vygotsky's theory is indispensable without it, but Piaget's epistemology and learning theory does still "work" in a realist world.

A conclusion is thus that a synthesis seems less difficult to create as they do not necessarily have a different ontological basis. However, as the theories stand today, the dualism still exists. From the "tone" between Piaget and Vygotsky, their critique of each other never seemed personal, but was carried by mutual respect and, probably, driven by a desire for finding the truth.

4. GIDDENS' ATTEMPT TO SOLVE A DUALITY IN SOCIAL SCIENCE

It is not only within psychology that one sees dualism between theories. Giddens describes several of what he calls dilemmas, and one of them seems related to the above mentioned between Piaget and Vygotsky:

> *One dilemma concerns human action and social structure. It is: How far are we creative human actors, actively controlling the conditions of our own lives? Or is most of what we do the result of general social forces outside our control? This issue has always divided, and continues to divide, sociologists. Symbolic interactionism stresses the active, creative components of human behaviour. The other three (eds.: Functionalism, Structuralism, and Marxism) emphasize the constraining nature of social influences on our actions.*
> *(Giddens, 1993, p. 718)*

Overcoming such a duality might be a first step in finding an overall grand theory of the field. In this connection also Skinner (2000) writes about several attempts on finding a grand theory in the human sciences towards the end of the 20th century. Skinner (2000, p. 3) quotes a book written by the sociologist Mills in 1959 where Mills stated his skepticism of the goal that the human sciences should seek a grand theory, and thus construct a systematic theory of 'the nature of man and society'. According to Skinner, "this hostility towards the construction of abstract and normative theories of human nature and conduct was an attitude he [Mills] shared with most of the leading practitioners not merely of sociology but of all the human sciences in the English-speaking world at that time" (Skinner, 2000, p. 3). Particularly for psychology, Skinner writes that:

> *even more vociferous doubts about the normative presuppositions of positivism have been voiced of recent years by the psychologists. To perceive all human behaviour in lawlike, causal terms … presupposes that the question to ask about abnormal behaviour must always be what malfunction is prompting it. But this it to overlook the possibility that the behaviour in question may be strategic, a way of trying to cope with the world. And this oversight … has the effect of reducing the agents involved to objects of manipulation when they deserve to be treated as subjects of consciousness.*
> *(Skinner, 2000, p. 9)*

Skinner later argues that all the sceptical stands against creating a grand theory actually contribute to a return of grand theory. He argues as follows:

> *Although they [the sceptics] have given reasons for repudiating the activity of theorising, they have of course been engaged in theorising at the same time. There is no denying that Foucault has articulated a general view about the nature of knowledge, that Wittgenstein presents us with an abstract account of meaning and understanding, that Feyerabend has a preferred and almost Popperian method of judging scientific hypotheses, and even that Derrida presupposes the possibility of constructing interpretation when he tells us that our next task should be that of deconstructing them. … We next need to note that, during the past two decades, there has also been an unashamed return to the deliberate construction of precisely those grand theories of human nature and conduct which Wright Mills and his generation had hoped to outlaw by from any central place in the human sciences. This can be seen most obviously in the case of moral and political philosophy. … One has been a renewed willingness directly to address the most pressing evaluative issues of the day. As a result, such topics as the justice of war, the social causes of famine … all these and many other kindred questions of obvious urgency have again become the staples of philosophical debate. But the other and even more startling development has been a return to Grand Theory in the most traditional and architectonic style, the style employed by the great normative systembuilders of earlier centuries. Moral and political philosophers have ceased to be in the least shy of telling us that their task is that of helping us to understand how best to live our lives.*
> *(Skinner, 2000, pp. 12-14)*

In relation to the dualism of individual-system between Piaget and Vygotsky, Giddens tried in the area of sociology to abolish this classical dualism and create a grand theory. Giddens puts the actor in the centre and with the notion of structure-duality he tries to transgress the sociology's traditional structure/actor dualism. With this notion Giddens wants to emphasize that social system's structural characteristics at once is a medium for and a result of the individual actors actions:

> *Structure is not to be equated with constraints but is always both constraining and enabling. This, of course, does not prevent the structure of properties of social systems from stretching away, in time and space, beyond the control of the individual actor. Nor does it compromise the possibility that actors' own theories of the social systems which they help to constitute and reconstitute in their activities may reify those systems.*
> *(Giddens, 1986, p. 25)*

Giddens' theory has however been criticized for being so abstract that it could not be employed in empirical research in practice (Gregson, 1989).

5. BOHR'S ATTEMPT TO SOLVE A DUALITY IN NATURAL SCIENCE

We could also look at what physicists do faced with the problem of what light is. Some theories state that light is a wave (which means a field spread out in a large space), others that it a particle (which means that the substance is limited to a very little volume). Which one is it? The theories are mutually exclusive, but still physicists use both, they exists side by side. What physics do is to use the theory that "fits" the given problem they are solving. Furthermore: "Niels Bohr's Principle of Complementarity … states that each description excludes the other, but both are necessary - they complement each other" (Marshall & Zohar, 1997, p. 101). Russell seems to discuss something similar when he describes Einstein's general theory of relativity. According to Russell, Einstein's theory does, *inter alia*, lead to the conclusion that

> *the universe is finite but unbounded, like the surface of a sphere, but in three dimensions. All this involves non-Euclidean geometry, and is apt to seem mysterious to those whose imagination is obstinately Euclidean. … Professor Milne holds that there is no need to regard space as non-Euclidean, and that the geometry we adopt can be decided entirely by motives of convenience. The difference between different geometries, according to him, is a difference in language, not in what is described.*
> *(Russell, 1948, p. 34)*

Russell does here argue for that fundamental different approaches can be used to describe the same thing. This remark might be seen as being very surprising considering that non-Euclidean geometry per definition is any geometry which denies one of Euclid's five basic postulates (Euclid, 1959; 1st edition about 300 BC); in practice the Parallel Postulate. If this approach is possible for natural scientists, it perhaps ought to be possible for psychologists or researchers in education as well. It is a paradox, but Marshall and Zohar quote the physicists Feynman for saying: "A paradox is not a conflict within reality. It is a conflict between reality and your feeling of what reality should be like" (Marshall & Zohar, 1997, p. 387). The idea of complementarity is, however, also criticized by Marshall and Zohar who argued that Bohr's idea of complementarity rests on ideas of the old worldview. In my view, old ideas are not wrong just because they are old. Old-fashioned and outdated is not the same. What is important must be which view is true.

In relation to Bohr's idea of the range of application of the Principle of Complementarity, Marshall and Zohar writes that:

> *Bohr himself applied his Principle of Complementarity widely in fields outside physics. … thought and action, subjectivity and objectivity, feeling and reasoning, male and female, the truths and values of one culture and those of another. Physics and philosophers of Bohr's generation liked this way of thinking because it rested within the dualist either/or paradigm of the old world view and required no revolution in thinking. … To accept that light is both a wave and a particle, is one of the creative leaps quantum physics calls upon us to make. Applied in other fields, both/and thinking requires us to see that there may be two or more mutually contradictory ways of doing something, or of looking at something, all which are valid. Seeing the truth of all tells us something more profound about the situation.*
> *(Marshall & Zohar, 1997, p. 102)*

Bohr did therefore not only use the concept of complementarity within a context of quantum physics. Also Cole and Wertsch argues within the area of psychology that "There is little doubt in our view that there is still much to be learned from both Piaget and Vygotsky, and in many cases the strengths of one theorist complement the weakness of the other" (Cole & Wertsch, 1996). Also Piaget himself used a concept of complementarity: "I shall begin by making a distinction between two aspects of thinking that are different, although complementary" (Piaget, 1970, p. 14). Researchers in mathematics education also use the concept of complementarity. For instance does Vithal (1999) discuss the connections between mathematics education and democratic society and hence the relation between democracy and authority, which according to Vithal, is best understood and explained with reference to the idea of complementarity. And Sfard (1991, p. 4) writes that "operational and structural conceptions of the same mathematical

notion are not mutual exclusive. Although ostensibly incompatible … they are in fact *complementary*". In that sense one could argue that the concept of complementarity has a more general application into the range of mathematics education. Also in the psychological debate about nature-nurture, Bates et al. (1998) argue that "all reasonable scholars today agree that genes and environment interact to determine complex cognitive outcome".

6. SO WHAT DO WE DO NOW?

If one wants to use various, sometimes contradictory, theories, it seems that one has three options: (1) Find/invent the unifying grand theory, (2) use a concept of complementarity, or (3) follow the recommendation of Marshall and Zohar (1997, p. 102) and accept a both/and thinking and seeing the truth of all. I will now discuss these options.

6.1 A grand theory of the psychology of learning mathematics?

Vygotsky thought that psychology ought not to be divided into different schools; he states:

> *As long as we lack a generally accepted system incorporating all the available psychological knowledge, any important factual discovery inevitably leads to the creation of a new theory to fit the newly observed facts.*
> *(Vygotsky, 1962, p. 10)*

He can therefore be interpreted as talking about a grand theory. To look for a unifying theory seems to be in line with Descartes' dream. Descartes finds reason to be the method to unify all sciences. According to Davis and Hersh:

> *The vision of Descartes became the new spirit. Two generations later, the mathematician and philosopher Leibniz talked about the 'characteristica universalis'. This was the dream of a universal method whereby all human problems, whether of science, law, or politics, could be worked out rationally, systematically, by logical computation. In our generation, the visions of Descartes and Leibniz are implemented on every hand.*
> *(David & Hersh, 1988, pp. 7-8)*

We can therefore ask if this reason is still the method, or do we need an additional vision? Milne (quoted above) stated that the different descriptions of reality are just different languages. In line with this view, we have to find the unifying language, as this could be the appropriate tool. As an

example, mathematics took a big step forward in its development after Newton and Leibniz, separately, had developed differential and integral calculus towards the end of the 17th century. They had created/discovered a very useful tool that was further developed by the next generations of mathematicians (Andersen, 1978, p. 48). Hence we might need an appropriate language and/or tool to be able to find the grand theory.

We could seek inspiration from areas such as brain research or physics. The former has the later years made huge discoveries (see for instance Gade, 1997) and, as written above, some researchers in physics talk about the principle of complementarity, but others seem to be looking for a grand theory. In relation to the latter: "Physicists seek a theory that will unify all known forces of nature" (Nozick, 2001, p. 161). Hawking writes that "we might be near finding a complete theory that would describe the universe and everything in it" (Hawking, 1994, p. 29). Deutsch writes that: "quantum physical investigations of shadows and light have extraordinary consequences, and to explain these demand not only new physical laws but also a new level of description. It first and foremost reveals the existence of parallel universes" (Deutsch, 1998, pp. 32-33). I do not want to go any deeper into the discussion of parallel universes but only draw a conclusion from these quotes, namely that physics scientists expect to find a grand theory and that new research in physics suggests not only new laws, or theories; but more radical changes of ways of thinking and describing.

A grand theory of psychology of learning mathematics might therefore exist, but to my knowledge it has not been found/invented yet. To find it requires not only considerations as the ones in Section 3, input from other areas such as the latest brain research, but also that we find some new language, and a new ground on which to built the theory. The extent, to which one believes that a unifying theory exists, also rests on the modern way of thinking. Modernism refers to a long and dominating cultural tradition that *inter alia* had as characteristics:

> *the ideal of a complete and scientific explanation of physical and social reality. Though this might not in practice be possible, it remains an intelligible ideal. … there is thus a 'grand narrative' which we have subscribed to, namely, the 'enlightenment' view that reason, in the light of systematically researched evidence, will provide the solution to the various problems we are confronted with.*
> *(Pring, 2000, p. 110)*

Also the positivist tradition seems to be in favour of grand theories. Pring argues, when he discusses positivism, as follows:

> *First, there can be no clear logical distinction between research into physical phenomena and research into social institutions and structures. Society can be studied scientifically. There are social facts, just as there are physical facts. People, despite their individuality, fall into types or groups, and general statements can be made about these types. Such generalizations can be verified. Gradually a theoretical picture can be built up which relates types to social structures, such that to explain why certain people act in the way they do one refers to the social structures which could be said to cause that kind of behaviour. Such social explanations contradict those which seek to explain behaviour in terms of personal choice or individual psychology. Of course, one cannot deny that there is some personal choice, but, first, such choice will be exercised within parameters determined by the social facts, and, second, typical behaviors are what are being explained - there can always be exceptions. … Second, the positivist spirit requires a clear distinction between the aims and values of education, on the one hand, and the means of reaching those ends, on the other. Matters of value are not open to empirical enquiry (and are thus outside the bounds of meaningful discussion) whereas the means of realizing those values are. Researchers are required to show how certain ends might be reached, not to say what those ends ought to be.*
> *(Pring, 2000, pp. 93-94)*

Furthermore, positivism can be seen as a guard against dominating suppressing ideas: “Those who now decry the positivist agenda need to remember the spirit and motives which drove it. There was a deep suspicion of those explanations, without evidence to support them and not open therefore to counter argument, which sustained the social order as it was, despite the obvious injustices and evils” (Pring, 2000, p. 90).

6.2 Complementarity in the psychology of learning mathematics?

When discussing the possibility of using the concept of complementarity in this work, one needs to discuss two things. (1) There might be qualitative differences in the nature of physics and the nature of psychology of learning mathematics, which means that even if a grand theory exists in physics, it does not mean it exists in the psychology of learning. (2) How does a concept of complementarity influence our logic?

In relation to question (1) of whether the psychical problem of light and the problem of a psychological learning theory are different:

First, the former is a natural science phenomena, the other a psychological phenomena. For instance Berger and Luckmann argues that social phenomena are not as the physical, and that human reality is a socially constructed reality (Berger & Luckmann, 1984, pp. 210-211). Furthermore according to, among others, Skinner there has within the philosophy of science been a critique of the positivist account of what constitutes an explanation:

> *the widespread reaction against the assumption that the natural sciences offer an adequate or even a relevant model for the practice of the social disciplines. The clearest reflection of this growing doubt has been the revival of the suggestion that the explanation of human behaviour and the explanation of natural events are logically distinct undertakings, and thus that the positivist contention that all successful explanation must conform to the same deductive model must be fundamentally misconceived. From many different directions the cry has instead gone up for the development of a hermeneutic approach to the human sciences.*
> *(Skinner, 2000, p. 6)*

However, following Descartes and the modern dream, there is something that unites *all* sciences.

Second, in the theory of light, the two views of light are mutually exclusive as a certain thing cannot be a particle and a wave at the same time. However, according to Heisenberg, the dualism here is not problematic, as we know from the mathematical formulation of the theory that there cannot arise contradictions. By a simple transformation one can rewrite the equation of motion for the co-ordinates and the momenta of the particles to make it look like a wave equation for an ordinary 3-dimensional matter wave. "Therefore, this possibility of playing with different complementary pictures has its analogy in the different transformations of the mathematical scheme; it does not lead to any difficulties in the Copenhagen interpretation of quantum theory" (Heisenberg, 2000, pp. 18-19). This means that since both theories of light build on the same basis and language, which is mathematics, it is unproblematic to say that they can complement each other. I would therefore call such an incidence for *even complementarity* to denote that both, compared to reality and the general mathematical knowledge, are equals. But the various psychological learning theories do not share such a common ground; at least not on the level of epistemology. Therefore, "even complementarity" is not possible here. Instead I will call for a term of *odd complementarity* to denote that neither theory is completed, but they might not be equally *dis*-completed. In other words, I call for choosing a small preference for either the hen (or the egg) and then subsequently state that the egg (or the hen) is indispensable compliments.

In relation to (2), how this influences our logic, one can argue that perhaps one does not need to have problems with having two different theories complement each other. Mathematics itself it not a foolproof consistent system. Gödel's Theorem from 1931 set out to prove if it is possible to formulate a rich or interesting mathematical system that could contain the proofs of all its own truths: "Gödel proved that any consistent logical or mathematical 'formal system' rich enough to contain the natural numbers (1, 2, 3 …) would also contain a statement that could be neither proved nor disproved from within the system itself " (Marshall & Zohar, 1997, p. 176). On could argue, that if this is the case for mathematics, which obviously works, then why not for

social sciences. We also know from the double-slit experiment that *one* photon can enter *two* different holes at the same time (unless we observe it) (Gribbin, 1984, pp. 163-171); perhaps unbelievable, but yet true. This does not eliminate the concept of truth/false; it merely teaches us more about the truth and makes us understand the truth better.

If we can accept that a photon can be two places at the same time, perhaps we can accept to use two different theories in a sort of (odd) complementarity until we might find/invent the grand theory. At least in cases where the two different theories are not that different. The war on theories is then not on one *or* the other, but more on which is primary and which is secondary. Furthermore, if one is a modernist and still in favour of the principle of complementarity, one needs to include the concept of odd complementarity; otherwise one is inconsistent. Mellin-Olsen (1989, p. 18) furthermore argues that the relationship between Vygotsky and Piaget can be interpreted as being dialectical. It is not either-or. Instead it is about, while teaching, to have these two theories in one's mind (as well as other theories) and then balance wisely.

6.3 Everything is true?

Marshall and Zohar argue for accepting "both/and" thinking and accept the truth of all explanations. This view seems postmodern, which according to Pring is a questioning of the modern premises. Pring describes postmodernism as follows:

> *Rival disputes about what is to count as a rational view of the world cannot be settled by appeal to reason. There is no 'meta-narrative' of rationality to which we can appeal and which will bring a certain unity to this diversity. … There is no grand narrative which legitimate one set of values rather than another or one way of organising knowledge rather than another. Therefore we need to come to terms with pluralism, not simply in recognizing the diverse modes of rationality and of perspective. Is not reason, too, a social construct?*
> *(Pring, 2000, pp. 110-111)*

On could state here, that saying that there is no grand theory is in itself a grand theory. And that following the postmodern way of arguing one might end up with accepting any explanation. In line with Marshall and Zohar is Eisner who stated that "there is no single legitimate way to make sense of the world. ... Insofar as our understanding of the world is our own making, what we consider true is also the product of our own making" (Eisner, 1993, p. 54). A critique here is formulated by Collin who argues that "the social items that are claimed to generate social facts must themselves be understood to be generated by other social items, and so on *ad infinitum"*

(Collin, 1997, p. 78). I could also argue that the view of Eisner is internally illogical as it with certainty rejects "objective universal truth", only to replace it with a new universal truth, namely that the universal truth does not exist. In connection with this argument, Nozick argues (2001, p. 15) that he feels uncomfortable with this kind of quick refutation of relativism (i.e.: that if the relativist position, that all truth is relative, itself is nonrelative, then it is false; and if it is not a general position and instead says that all other truth except itself are relative; and then what makes it so special). Nozick (2001, p. 16) instead defines the 'relaxed relativism' as "the relativist granting that some statement is nonrelative, namely, the statement of the relativist position itself (along with its consequences)". He continues: "This makes it look as though relativism about truth is a coherent position. … To say that relativism about truth is a coherent position is not to say that it is the *correct* position" (Nozick, 2001, pp. 16-17). Nozick also argues that the 'weak absolutist' can hold that some truths are relative (Nozick, 2001, pp. 20 & 65). Thus relativism does not undercut itself if we take into consideration its domain of application. Nozick then introduces the concept of 'alterability': "the relativity of a truth is not the same as its alterability. Even if it is a nonrelative truth that my pen is on my desk, that is a fact easily changed. Whereas if it is merely a relative truth that New York City is adjacent to the Atlantic Ocean or that capitalism outproduces socialism, these are not facts that are changed easily" (Nozick, 2001, p. 23). Following this line of reasoning, I would argue that even if relativism about truth is a true position, it does not change the fact that there are ways of working with mathematics, or setting in which we work, that are "unhelpful" (or more helpful) if the desired "output" of the activities is that the pupils should have learnt certain things. These facts are not easily changed unless one can genetically change the nature of man. Thus, even talking Nozick's argumentation into consideration, the truth about how to learn mathematics might still exist.

I would also like to follow Phillips when he argues that truth exists independently of us but we can never reach it. Objectivity and truth are thus not synonyms, but through criticism we can approach truth and the, at any time, most rational theory is thus the most objective (Phillips, 1993, p. 61). This is in line with Popper's view that we can never verify a theory but only falsify it, he says: "*we do justify our preferences by an appeal to the idea of truth*: truth plays the role of a regulative idea. *We test for truth*, by eliminating falsehood" (Popper, 1979, pp. 29-30). Kuhn might here pose the counter argument that with this type of falsification one is still within the same paradigm and progress is caused by paradigm change (Kuhn, 1970, pp. 52-66). On the

other hand, Hollis argues that "the difference is a matter of degree of entrenchment, with normal science more willing to question its core theories than Kuhn recognised" (Hollis, 1994, p. 88). Thus we can never reach truth, but this does not mean that any version of reality is as good as any other. I would argue in line with Pring: "The acceptance of a reality independent of the researcher does not contradict the possibility of many interpretations of that reality" (Pring, 2000, p. 114).

I would also like to criticize Marshall and Zohar for not being ambitious enough for natural sciences. Giving up on finding a grand theory on psychological level is, in my view, and being inspired from a discussion of Hawking, the same as looking away from the fact that even though the human brain is subject to Heisenberg's Uncertainty Principle,[4] and therefore has an element of quantum mechanical randomness, it is little energies that are transformed in the brain, so the quantum-mechanical uncertainty has only a minor effect. The real reason why we cannot (now) predict human actions has more to do with that it is too difficult. According to Hawking, we already know the basic physics laws that govern the brain's activity, and they are rather simple, but it is too difficult to solve the equations when there are more than a few particles involved. Even in the simpler Newtonian theory of gravitation one can only solve the equations exactly if there are no more than two particles present. For three or more particles one has to rely on approximations and the difficulties rises by the number of particles. The human brain contains approximately 10^{26} particles, which is far too many for us to ever solve the equations and predict the brain's behaviour (Hawking, 1994, pp. 120-121). Or as Hawking puts it elsewhere: "Although in principle we know the equations that govern the whole of biology, we have not been able to reduce the study of human biology to a branch of applied mathematics" (Hawking, 1994, p. 43). I would argue, that if the real problem was that of solving equations, it would just be a matter of time until we invent larger enough computers. The essence of the problem of a grand theory of psychology, I would argue, is instead whether it at all is possible to predict

[4] *This principle is basically that "The Uncertainty Principle asserts that it must always be so; we must always content ourselves with partial truth and ambiguity when dealing with fundamental physical reality. A particle was always thought to have both position and momentum. A given particle should always be somewhere (have a location) and is always travelling at a certain speed. But we can never know both. If we measure, or focus on, the position, the momentum becomes unfixed; if we measure the momentum, we lose the position" (Marshall & Zohar, 1997, pp. 182-184). Davies writes in an introduction that "This unpredictability of quantum systems does not imply anarchy, however. Quantum mechanics still enables the relative probabilities of the alternatives to be specified precisely" (Heisenberg, 2000, p. x). Furthermore: "what the uncertainty principle tells us is that, according to the fundamental equations of quantum mechanics, there is no such thing as an electron that possesses both a precise momentum and a precise position. … quantum theory cuts free from the determinacy of classical ideas. To Newton, it would be possible to predict the entire course of the future if we knew the position and momentum of every particle in the universe" (Gribbin, 1984, p. 157).*

human behaviour. If one cannot (always) predict human behaviour in particularly a learning situation (neither the human actions nor the brain behaviour) then a complete and all-including grand theory is impossible. I would draw on Hawking who talks about free will, and writes "The ultimate objective test of free will would seem to be: can one predict the behaviour of the organism? If one can, then it clearly doesn't have free will but is predetermined. On the other hand, if one cannot predict the behaviour, one could take that as an operational definition that the organism has free will" (Hawking, 1994, p. 120). Hence, I would argue, that the question of a grand theory in psychology might boil down to the (theological) question of whether we as human have a free will.

I would therefore argue that the option, given by Marshall and Zohar, of thinking of "all as truth" is partly inconsistent and partly not necessary (if we do not have a free will); at least in the longer run where I will expect science to know more. Even if we as humans do have a free will, it does not rule out that we can get *more* understanding of how we act and learn as the quantum mechanical uncertainty only has a *minor* effect; thus, it is not anarchy either.

7. CONCLUSION: PRAGMATISM FOR NOW: ODD COMPLEMENTARITY

As Hawking writes: "However, it is too difficult to think up a whole theory of everything all at one go. … What we do instead is to look for partial theories that will describe situations in which certain interactions can be ignored or approximated in a simple manner" (Hawking, 1994, p. 46). Whether there is a grand theory of the psychology of learning mathematics, and whether we in that case can find it, does not solve the immediate problem of today, namely that we do not know it *yet.* We only have partial theories so far. So for this thesis I will settle with Bohr's Principle of Complementarity, in the sense of Odd Complementarity. This is also well connected with the discussion in the beginning of this paper, where it became clear that Piaget and Vygotsky each have great admiration and respect for each other. Also the Ph.D. study showed that the theories seem to complement each other in some way.

8. BIBLIOGRAPHY

Andersen, K. M. (previously Pedersen) (1978) *Træk af den matematiske analyses udvikling i 1600-tallet* (Aarhus, Institut for de eksakte videnskabers historie).

Bates, E.; Elman, J.; Johnson, M.; Karmiloff-Smith, A.; Parisi, D.; Plunkett, K. (1998) 'Innateness and emergenitism' In W. Bechtel and G. Graham (eds.) *A Companion to Cognitive Science* (Oxford, Basil Blackwood).

Berger, P. L. & Luckmann, T. (1984) *The Social Construction of Reality: A Treatise in the Sociology of Knowledge* (New York, Penguin).

Cole, M. & Wertsch, J. V. (1996) *Beyond the individual-Social Antimony in Discussions of Piaget and Vygotsky* http://www.massey.ac.nz/~alock//virtual/colevyg.htm

Collin, F. (1997) *Social Reality* (London, Routledge).

Dahl, B. (1996a) 'Learning Mathematics as Language-Games Transitions' *Aalborg University, Institute for Electronic Systems, Department of Mathematics and Computer Science, R-96-2047.*

Dahl, B. (1996b) 'Læring som sprogspilsoverskridelse' *Nordic Studies in Mathematics Education* 4 1, pp. 7-24.

Dahl, B. (2002) 'A focus group study of Danish and English high-achieving high school pupils of mathematics: What can we learn from their verbalised explanations of how they learn mathematics?' Ph.D. thesis, IMFUFA, Roskilde University, Denmark.

Davis, P. J. & Hersh, R. (1988) *Descartes' Dream: The World According to Mathematics* (London, Penguin).

Deutsch, D. (1998) *The Fabric of Reality* (London, Penguin).

Eisner, E. (1993) 'Objectivity in Educational Research' In M. Hammersley (eds.) *Educational Research - Current Issues* (London, The Open University) pp. 49-56.

Eriksen, D. B. (1993) *Personlige og sociale sider ved elevernes tilegnelse af faglig viden og kunnen i folkeskolens matematikundervisning* (Copenhagen, Danmarks Lærerhøjskole, Afdeling for matematik; now The Danish University of Education).

Euclid (1959) *Euklids Elementer* (Copenhagen, Gyldendal).

Gade, A. (1997) *Hjerneprocesser - Kognition og neurovidenskab* (Copenhagen, Frydenlund).

Giddens, A. (1986) *The Constitution of Society. Outline of the Theory of Structuration* (Cambridge, Polity).

Giddens, A. (1993) *Sociology* (Cambridge, Polity).

Gregson, N. (1989) 'On the (ir)relevance of Structuration Theory to empirical research' In D.

Held & J. Thompson *Social Theory of Modern Society - Anthony Giddens and his Critics* (Cambridge, Cambridge University), pp. 235-248.

Gribbin, J. (1984) *In search of Schrödinger's cat: The startling world of quantum physics explained* (Londong, Wildwood House).

Hawking, S. W. (1994) *Black Holes and Baby Universes and other essays* (London, Bantam Books).

Heisenberg, W. (2000) *Physics and Philosophy: The Revolution in Modern Science* (London, Penguin Classics).

Hollis, M. (1994) *The philosophy of social science - an introduction* (Cambridge, Cambridge University).

Kuhn, T. (1970) *The Structure of Scientific Revolutions* (second edition) (Chicago, The University of Chicago).

Lerman, S. (1996) 'Intersubjectivity in Mathematics Learning: A Challenge to the Radical Constructivist Paradigm?' *Journal for Research in Mathematics Education* 27 2, pp. 211-223.

Lübcke, P. (1993) *Politikens filosofi leksikon* (Copenhagen, Politiken).

Marshall, I. & Zohar, D. (1997) *Who's Afraid of Schrödinger's Cat?* (London, Bloomsbury)

Mellin-Olsen, S. (1989) *Kunnskapsformidling. Virksomhetsteoretiske perspektiver* (Rådal, Caspar).

Nozick, R. (2001) *Invariances - The Structure of the Objective World* (Cambridge Massachusetts, Harvard University).

Phillips, D. C. (1993) 'Subjectivity and Objectivity: An Objective Inquiry' In M. Hammersley (eds.) *Educational Research - Current Issues* (London, The Open University) pp. 57-72.

Piaget, J. (1962) *Comments on Vygotsky's critical remarks concerning 'The Language and Thought of the Child', and 'Judgment and Reasoning in the Child'* (Cambridge Massachusetts, The M.I.T.).

Piaget, J. (1969) *Psykologi og pædagogik* (Copenhagen, Hans Reitzel).

Piaget, J. (1970) *Genetic Epistemology* (New York, Columbia University).

Popper, K. (1979) *Objective Knowledge - An Evolutionary Approach* (Oxford, Clarendon).

Pring, R. (2000) *Philosophy of Educational Research* (London, Continuum).

Russell, B. (1948) *Human Knowledge - Its Scope and Limits* (London, George Allen & Unwin).

Sfard, A. (1991) 'On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin' *Educational Studies in Mathematics* 22, pp. 1-36.

Skinner, Q. (eds.) (2000) *The Return of Grand Theory in the Human Sciences* (Cambridge, Cambridge University - Canto edition).

Vejleskov, H. (1998) 'Teorier om kognitiv udvikling som inspiration for pædagogikken' In N. J. Bisgaard (eds.) *Pædagogiske teorier* (Værløse, Billesø & Baltzer).

Vithal, R. (1999) 'Democracy and Authority: A Complementarity in Mathematics Education?' *ZDM, Zentralblatt für Didaktik der Mathematik, International Reviews on Mathematical Education* 31 1.

Vygotsky, L. S. (1962) *Thought and Language* (Cambridge Massachusetts, The M.I.T.).

Wynn, K. (1992) 'Addition and subtraction by human infants' *Nature 358* pp. 749-750.