The mathematical work of

Maryam Mirzakhani

The mathematical work of

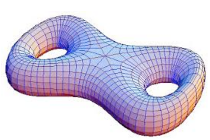
Maryam Mirzakhani

Selecta

- Simple loops on X
- Complex geodesics in $\mathcal{M}_g$
- Earthquakes

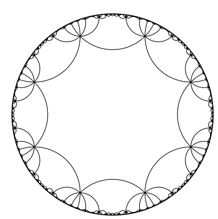
The Setting

X = $\mathbb{H}/\Gamma$



Riemann surface
of genus g≥2

hyperbolic metric



$\mathbb{H}$

Moduli space of Riemann surfaces

$\mathcal{M}_g$ = {isomorphism classes of X of genus g}

---

2 sides of moduli space

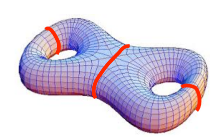
$\mathcal{M}_g$ = (i) complex variety, $\dim_{\mathbb{C}} \mathcal{M}_g$ = 3g-3

= (ii) symplectic orbifold

Symplectic

$(\mathcal{M}_g,\omega) \Leftarrow$ hyperbolic geometry of X

$$\omega = \sum_{1}^{3g-3} d\ell_i \wedge d\tau_i$$



Fenchel-Nielsen length-twist coordinates; Wolpert

Complex structure on $\mathcal{M}_g$: inherited from X

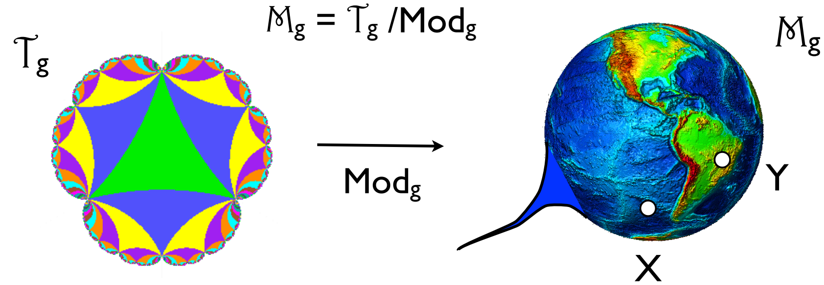
$T^*_X \mathcal{M}_g = Q(X)$ = {holomorphic quadratic differentials
$q = q(z)\, dz^2$ on X }

Complex structure ⇒ *Teichmüller metric on* $\mathcal{M}_g$

$$||q|| = \int_X |q(z)|\, |dz|^2 = \text{area}(X,|q|)$$

= Kobayashi metric (Royden)

$\mathcal{M}_g$ is totally inhomogeneous

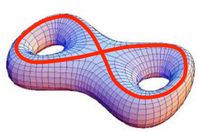


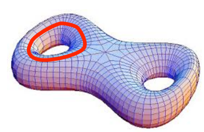
- $\text{Aut}(\mathcal{T}_g) = \text{Mod}_g$
- $Q(X) \simeq Q(Y) \Rightarrow X \simeq Y$

unlike S^n or $\mathbb{H}^n$ or $K\backslash G/\Gamma$

Work of Mirzakhani: I

Simple loops in X





Classical

(2004)

#Closed(X,L) ~ e^L/L

(prime number theorem, 1940s)

Theorem - Mirzakhani

#Simple(X,L) ~ $C_X L^{6g-6}$

Proof: Integration over $\mathcal{M}_g$ and hyperbolic dissection

---

⇒ *New proof of Witten conjecture*

Intersection numbers on moduli space:

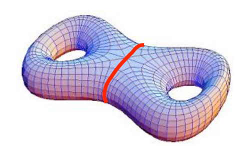
$$\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} c_1(L_1)^{d_1} \cdots c_1(L_n)^{d_n}$$

⇒ solution of KdV equations / Virasoro algebra.

Kontsevich, 1992

⇒ *Topological statistics*

E.g., probability a random simple loop in genus 2 separates is 1/7.



Mirzakhani's volume formulas

Moduli of surfaces with geodesic boundary:

$P_{g,n}(L_1,\ldots,L_n) = \mathrm{Vol}\ \mathcal{M}_{g,n}(L_1,\ldots,L_n) = \int \omega^N$

= polynomial with coefficients in $\mathbb{Q}[\pi]$.

ex: $P_{1,1}(L) = (1/24)(L^2 + 4\pi^2)$

Previously only $P_{g,n}(0,\ldots,0)$ was known.

[Coefficients $\Rightarrow$ statistics and characteristic numbers]

Work of Mirzakhani: II

Complex geodesics in $\mathcal{M}_g$

Real geodesic: (local) isometry $f : \mathbb{R} \to \mathcal{M}_g$

Complex geodesic: *holomorphic* isometry
$F : \mathbb{H} \to \mathcal{M}_g$

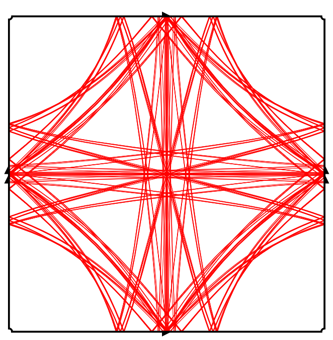
Abundance:
There exists complex geodesics through every $p \in \mathcal{M}_g$, in every possible direction.

(Teichmüller disks)

Behavior of a real geodesic

* Usually $\overline{f(\mathbb{R})}$ is dense in $\mathcal{M}_g$; ----

* But sometimes, $\overline{f(\mathbb{R})}$ can be a fractal cobweb.... ....defying classification.



Behavior of a complex geodesic

2D cobweb?

Theorem - Mirzakhani & coworkers

$V = \overline{F(\mathbb{H})}$ is always an algebraic subvariety of $\mathcal{M}_g$.

(E.g. genus 2, V = Teichmüller curve, Hilbert modular surface or whole space $\mathcal{M}_2$.)

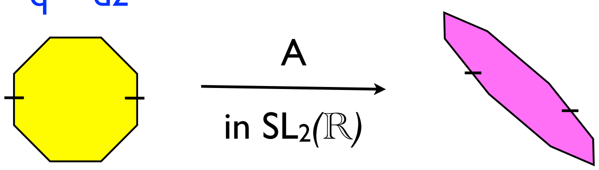
Dynamics over moduli space

$Q\mathcal{M}_g \circlearrowleft SL_2(\mathbb{R})$
$\downarrow$
$\mathcal{M}_g$

complex geodesic =
projection of orbit $SL_2(\mathbb{R}) \cdot (X,q)$

$X = P/\sim$
$q = dz^2$

$A \cdot (X,q) = (A(P)/\sim, dz^2)$



$P \subset \mathbb{C}$ $\quad$ $A(P) \subset \mathbb{C}$

Proof that $V = \overline{F(\mathbb{H})}$ is an algebraic subvariety of $\mathcal{M}_g$.

I. Eskin & Mirzakhani :

All ergodic $SL_2(\mathbb{R})$-invariant measures in $Q\mathcal{M}_g$ come from special analytic varieties $A \subset Q\mathcal{M}_g$.

II. E & M & Mohammadi:

All $SL_2(\mathbb{R})$ orbit closures come from such A.

III. Filip:

Any such $A \subset Q\mathcal{M}_g$ is an algebraic subvariety defined over a number field.

(and A projects to V.) ■

Ramifications:
Beyond Homogeneous Spaces

$U \circlearrowleft G/\Gamma$ $\quad$ G Lie group
Γ lattice
$U \subset G$ subgroup

Margulis, Ratner, et al

$\overline{SL_2(\mathbb{R})x} = Hx \subset G/\Gamma$

$SL_2(\mathbb{R}) \circlearrowleft Q\mathcal{M}_g = Q\mathcal{T}_g/\mathrm{Mod}_g$

Mirzakhani

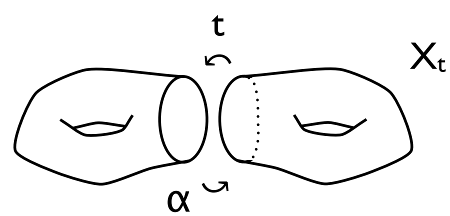
$\overline{SL_2(\mathbb{R})x} = A \subset Q\mathcal{T}_g/\mathrm{Mod}_g$

Rich theory of homogeneous dynamics
resonates in
highly inhomogeneous world of moduli spaces

Work of Mirzakhani: III

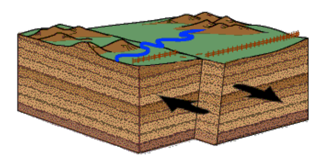
Classical:
simple loop α, $t \in \mathbb{R} \Rightarrow$

X_t = *twist* $(t\,\alpha, X)$



Earthquakes

twist(λ, X), λ in $\mathcal{ML}_g$



(Thurston)

= Hamiltonian flows on $\mathcal{M}_g$ generated by function $X \to \mathrm{length}(X, \lambda)$

Earthquake dynamics over $\mathcal{M}_g$

$\{\text{unit length } (X,\lambda)\} = L_1\mathcal{M}_g$ ↺ *earthquake flow*

$\downarrow$

$\mathcal{M}_g$

Theorem - Mirzakhani

Thurston's earthquake flow is ergodic and mixing.

Previously, not even one dense earthquake path was known!

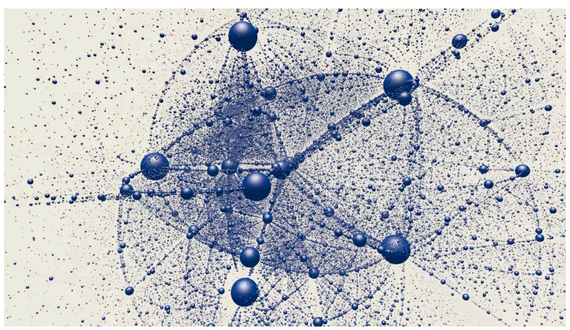
Proof:

Symplectic - Holomorphic Bridge

earthquake flow ↺		horocycle flow ↺ $\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$
$L_1\mathcal{M}_g \downarrow \mathcal{M}_g$	⟺ measurable isomorphism	$Q_1\mathcal{M}_g \downarrow \mathcal{M}_g$
symplectic		complex

(⇒ earthquake flow is ergodic)

Work of Mirzakhani: Scope and perspectives



$\mathcal{ML}$

(Dumas)

breadth of methods integrated into a transformative research program

many developments still unfolding