

Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus

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December 13, 2010

1 Introduction

In this paper, we investigate the geometric properties of hyperbolic surfaces by studying the lengths of simple closed geodesics. The moduli space $\mathcal{M}_{g,n}$ of complete hyperbolic surfaces of genus $g \geq 2$ with n punctures, is equipped with a natural notion of measure, which is induced by the *Weil-Petersson* symplectic form $\omega_{g,n}$ (§2). By a theorem of Wolpert, this form is the symplectic form of a Kähler noncomplete metric on the moduli space $\mathcal{M}_{g,n}$. We describe the relationship between the behavior of lengths of simple closed geodesics on a hyperbolic surface and properties of the moduli space of such surfaces. First, we study the asymptotic behavior of Weil-Petersson volume $V_{g,n}$ of the moduli spaces of hyperbolic surfaces of genus g with n punctures as $g \rightarrow \infty$. Then discuss some geometric properties of a random hyperbolic surface with respect to the Weil-Petersson measure as $g \rightarrow \infty$.

Notation. For any function $F : \mathcal{M}_g \rightarrow \mathbb{R}$, let

$$\mathbb{E}_{X \sim wp}^g(F(X)) = \frac{\int_{\mathcal{M}_g} F(X) dX}{V_g},$$

where the integral is taken with respect to the Weil-Petersson volume form. Also,

$$\text{Prob}_{wp}^g(F(X) \leq C) = \mathbb{E}_{X \sim wp}^g(G(X)),$$

where $G(X) = 1$ iff $F(X) \leq C$ and $G(X) = 0$ otherwise.

In this paper, $f_1(g) \asymp f_2(g)$ means that there exists a constant $C > 0$ independent of g such that

$$\frac{1}{C}f_2(g) \leq f_1(g) \leq Cf_2(g).$$

Similarly, $f_1(g) = O(f_2(g))$ means there exists a constant $C > 0$ independent of g such that

$$f_1(g) \leq Cf_2(g).$$

*partially supported by an NSF grant.

1.1

Moduli spaces of hyperbolic surfaces with geodesic boundary components. The universal cover of $\mathcal{M}_{g,n}$ is the Teichmüller space $\mathcal{T}_{g,n}$. Every isotopy class of a closed curve on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ contains a unique closed geodesic. Given a homotopy class of a closed curve α on a topological surface $S_{g,n}$ of genus g with n marked points and $X \in \mathcal{T}_{g,n}$, let $\ell_\alpha(X)$ be the length of the unique geodesic in the homotopy class of α on X . This defines a length function ℓ_α on the Teichmüller space $\mathcal{T}_{g,n}$.

When studying the behavior of hyperbolic length functions, it proves fruitful to consider more generally bordered hyperbolic surfaces with geodesic boundary components. Given $L = (L_1, \dots, L_n) \in \mathbb{R}_+^n$, we consider the Teichmüller space $\mathcal{T}_{g,n}(L)$ of hyperbolic structures with geodesic boundary components of length L_1, \dots, L_n . Note that a geodesic of length zero is the same as a puncture. In fact, the space $\mathcal{T}_{g,n}(L)$ is naturally equipped with a symplectic form ω_{wp} . The Weil-Petersson volume $V_{g,n}(L)$ of $\mathcal{M}_{g,n}(L_1, \dots, L_n)$ is a polynomial in L_1^2, \dots, L_n^2 of degree $3g - 3 + n$ and $V_{g,n} = V_{g,n}(0, \dots, 0)$. We will show that in order to get bounds on the integrals of geometric functions over \mathcal{M}_g we need to understand the asymptotics of the polynomial $V_{g,n}(L)$ as $g \rightarrow \infty$.

1.2

New results. Here we discuss the main results obtained in this paper:

I): Asymptotic behavior of Weil-Petersson volumes. Peter Zograf has developed a fast algorithm for calculating the volume polynomials, and made several conjectures on the basis of the numerical data obtained by his algorithm [Z2].

Conjecture 1.1 (Zograf). *For any fixed $n \geq 0$*

$$V_{g,n} = (4\pi^2)^{2g+n-3} (2g-3+n)! \frac{1}{\sqrt{g\pi}} \left(1 + \frac{c_n}{g} + O\left(\frac{1}{g^2}\right) \right)$$

as $g \rightarrow \infty$.

Here

$$V_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \omega_{g,n}^{3g-3+n} / (3g-3+n)!.$$

In §3, we show :

Theorem 1.2. *For any $n \geq 0$:*

$$\frac{V_{g,n+1}}{2gV_{g,n}} = 4\pi^2 + O\left(\frac{1}{g}\right),$$

and

$$\frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O\left(\frac{1}{g}\right)$$

as $g \rightarrow \infty$.

These estimates imply that there exists $M > 0$ such that

$$g^{-M} \leq \frac{V_{g,n}}{(4\pi^2)^{2g+n-3}(2g-3+n)!} \leq g^M. \quad (1.1)$$

In order to prove this theorem, we discuss the asymptotics of all the coefficients of the volume polynomials $V_{g,n}(L)$ (see Theorem 2.3); there are

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \cdot \omega^{3g-3+n-|\mathbf{d}|},$$

where for $1 \leq i \leq n$, $\psi_i \in H^2(\mathcal{M}_{g,n}, \mathbb{Q})$ is the first Chern class of the tautological line bundle corresponding to the i -th puncture on $X \in \mathcal{M}_{g,n}$ (§2), and $|\mathbf{d}| = d_1 + \dots + d_n$.

In §3 we apply known recursive formulas for these numbers and obtain some basic estimates for the intersection pairings of ψ_i classes on $\overline{\mathcal{M}}_{g,n}$ as $g \rightarrow \infty$.

II): Geometric behavior of surfaces of high genus. In section §4, we prove that as $g \rightarrow \infty$ the followings hold:

- In §4.2, we show that the probability that a random Riemann surface has a short non-separating simple closed geodesic is asymptotically positive. More precisely, let $\ell_{sys}(X)$ denote the length of the shortest simple closed geodesic on X . Then for any small (but fixed) $\epsilon > 0$, as $g \rightarrow \infty$

$$\text{Prob}_{wp}^g(\ell_{sys}(X) < \epsilon) \asymp \epsilon^2.$$

- However, *separating* simple closed geodesics tend to be much longer §4.3. Let $\ell_{sys}^s(X)$ denote the length of the shortest *separating* simple closed geodesic on X . we show that

$$\text{Prob}_{wp}^g(\ell_{sys}^s(X) < m_1 \log(g)) = O(\log(g)g^{(m_1/2-1)}),$$

and

$$\mathbb{E}_{X \sim wp}^g(\ell_{sys}^s(X)) \asymp \log(g)$$

as $g \rightarrow \infty$.

- Similarly, using the asymptotics of $V_{g,n}(L)$ we get bounds for the expected length of the shortest simple closed geodesic of a given combinatorial type. In particular, the shortest simple closed geodesic separating the surface into two roughly equal areas has length at least linear in g . Moreover, in §4.5 we show that the Cheeger constant $h(X)$ of a random Riemann surfaces $X \in \mathcal{M}_g$ is bounded from below by a universal constant. More precisely, as $g \rightarrow \infty$

$$\text{Prob}_{wp}^g\left(h(X) \leq \frac{\ln(2)}{\pi + \ln(2)}\right) \rightarrow 0.$$

By Cheeger's theorem the smallest positive eigenvalue of the Laplacian on a generic point X is $\geq \frac{1}{4}C_h^2$, where $C_h = \frac{\ln(2)}{\pi + \ln(2)}$.

- Finally, we show that a generic hyperbolic surface in \mathcal{M}_g has a small diameter, with a large embedded ball §4.6. More precisely, as $g \rightarrow \infty$

$$\text{Prob}_{wp}^g(\text{diam}(X) \geq C_d \log(g)) \rightarrow 0,$$

and

$$\mathbb{E}_{X \sim wp}^g(\sqrt{\text{diam}(X)}) \asymp \sqrt{\log(g)}.$$

Also,

$$\text{Prob}_{wp}^g(\text{Emb}(X) \leq C_E \log(g)) \rightarrow 0,$$

and

$$\mathbb{E}_{X \sim wp}^g(\text{Emb}(X)) \asymp \log(g)$$

where $\text{Emb}(X)$ is the radius of the largest embedded ball in X . Here $C_E = \frac{1}{3}$, and $C_d = 5$.

We remark that none of the constants in these statements are sharp. However, in this paper for the sake of simplicity, we chose the simpler proofs which would give weaker constants.

1.3

Our main tool is the close relationship between the Weil-Petersson geometry of $\mathcal{M}_{g,n}$ and the lengths of simple closed geodesics on surfaces in \mathcal{M}_g . Here we discuss one application of this relationship.

Let $\mathcal{S}_{g,n}$ denote the set of homotopy classes of non-trivial simple closed curves on a topological surface $S_{g,n}$ of genus g with n marked points. For any $\gamma \in \mathcal{S}_{g,n}$, let $S_{g,n} - \gamma$ denote the surface obtained by cutting the surface $S_{g,n}$ along γ . Given $\alpha_1, \alpha_2 \in \mathcal{S}_g$, we say $\alpha_1 \sim \alpha_2$ if α_1 and α_2 are of the same *type*; that is $S_g - \alpha_1$ is homeomorphic to $S_g - \alpha_2$. Given a connected simple closed curve $\gamma \in \mathcal{S}_g$ and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ define $f_\gamma : \mathcal{T}_g \rightarrow \mathbb{R}_+$ by

$$f_\gamma(X) = \sum_{\alpha \sim \gamma} f(\ell_\alpha(X)).$$

We have

$$\int_{\mathcal{M}_g} f_\gamma(X) dX = \int_0^\infty f(t) t V_{g,2}(t, t) dt. \quad (1.2)$$

For the general case of this formula see Theorem 2.2. The main idea is that the decomposition of the surface along γ gives rise to a description of a cover of \mathcal{M}_g in terms of moduli spaces corresponding to simpler surfaces.

1.4

Notes and remarks.

- A recursive formula for the Weil-Petersson volume of the moduli space of punctured spheres was obtained by Zograf [Z1]. Moreover, Zograf and Manin have obtained generating functions for the Weil-Petersson volume of $\mathcal{M}_{g,n}$ [MZ]. See also [KMZ]. The following exact asymptotic formula was proved in [MZ].

Theorem 1.3. *There exists $C > 0$ such that for any fixed $g \geq 0$*

$$V_{g,n} = n! C^n n^{(5g-7)/2} (a_g + O(1/n)), \quad (1.3)$$

as $n \rightarrow \infty$.

- In [Gr], it is shown that for a fixed $n > 0$ there are $c_1, c_2 > 0$ such that

$$c_2^g (2g)! < \text{Vol}(\mathcal{M}_{g,n}) < c_1^g (2g)!.$$

This result was extended to the case of $n = 0$ in [ST]. Note that these estimates do not give any information about the growth of

$$B_{g,n} = V_{g,n}/V_{g-1,n+2}$$

and

$$C_{g,n} = V_{g,n+1}/(2gV_{g,n})$$

when $g \rightarrow \infty$.

- Penner has developed a different method for calculating the Weil-Petersson volume of the moduli spaces of curves with marked points by using decorated Teichmüller theory [Pe].
- In [BM] Brooks, and Makover developed a method for the study of *typical* Riemann surfaces with large genus by using trivalent graphs. In this model the expected value of the systole of a random Riemann surface turns out to be bounded (independent of the genus) [MM]. See also [Ga]. We will see in this note that a random Riemann surface with respect to the Weil-Petersson volume form has similar features. However, it is not clear how the measure induced by their model is related to the measure induced by the Weil-Petersson volume.
- The distribution of hyperbolic surfaces of genus g produced randomly by gluing Riemann surfaces with long geodesic boundary components is closely related to the volume form induced by ω on $\mathcal{M}_{g,n}$. See [M3] for details.

Questions.

- It would be useful to know the asymptotics of

$$\frac{V_{g,n(g)}}{V_{g-1,n(g)+2}},$$

where $n(g) \rightarrow \infty$ as $g \rightarrow \infty$. Note that by Theorem 1.3 and Theorem 1.2, we know the asymptotics of $V_g/V_{g-1,2}$ and $V_{1,2g-4}/V_{0,2g-2}$. However, we don't know much about the behavior of the sequence

$$V_g, V_{g-1,2}, \dots, V_{0,2g}$$

as $g \rightarrow \infty$.

- As in Theorem 2.3, when $n = 1$ the volume polynomial can be written as

$$V_{g,1}(L) = \sum_{k=0}^{3g-2} \frac{a_{g,k}}{(2k+1)!} L^{2k},$$

where $a_{g,k}$ are rational multiples of powers of π . It would be helpful to understand the asymptotics of $a_{g,k}/a_{g,k+1}$ for an arbitrary k (which can grow with g). Note that $a_{g,0} = V_{g,1}$. In Theorem 3.5(a), we show that for given $i \geq 0$

$$\lim_{g \rightarrow \infty} \frac{a_{g,i+1}}{a_{g,i}} = 1.$$

On the other hand, it is known that [IZ]

$$\frac{\int \psi_1^{3g-2}}{\mathcal{M}_g} = \frac{1}{24g!},$$

and hence

$$\frac{a_{g,3g-2}}{a_{g,0}} \rightarrow 0,$$

as $g \rightarrow \infty$.

- The results obtained in this paper are only small steps towards understanding the geometry of random hyperbolic surfaces of large genus. Many interesting questions about such random surfaces are open. Investigating geometric properties of random Riemann surfaces could shed some light on the asymptotics geometry of \mathcal{M}_g as $g \rightarrow \infty$, see [CP], [T], and [Hu] for some results in this direction.

Acknowledgement. I would like to thank P. Zograf for many helpful and illuminating discussions regarding the growth of Weil-Petersson volumes. I am also grateful to Rick Schoen and Jan Vondrak.

2 Background and notation

In this section, we recall definitions and known results about the geometry of hyperbolic surfaces and properties of their moduli spaces. For more details see [M2], [Bu] and [W3].

2.1

Teichmüller Space. A point in the *Teichmüller space* $\mathcal{T}(S)$ is a complete hyperbolic surface X equipped with a diffeomorphism $f : S \rightarrow X$. The map f provides a *marking* on X by S . Two marked surfaces $f : S \rightarrow X$ and $g : S \rightarrow Y$ define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1} : Y \rightarrow X$ is isotopic to a conformal map. When ∂S is nonempty, consider hyperbolic Riemann surfaces homeomorphic to S with geodesic boundary components of fixed length. Let $A = \partial S$ and $L = (L_\alpha)_{\alpha \in A} \in \mathbb{R}_+^{|A|}$. A point $X \in \mathcal{T}_{g,n}(L)$ is a marked hyperbolic surface with geodesic boundary components such that for each boundary component $\beta \in \partial S$, we have

$$\ell_\beta(X) = L_\beta.$$

By convention, a geodesic of length zero is a cusp and we have

$$\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0, \dots, 0).$$

Let $\text{Mod}(S)$ denote the mapping class group of S , or the group of isotopy classes of orientation preserving self homeomorphisms of S leaving each boundary component setwise fixed. The mapping class group $\text{Mod}_{g,n} = \text{Mod}(S_{g,n})$ acts on $\mathcal{T}_{g,n}(L)$ by changing the marking. The quotient space

$$\mathcal{M}_{g,n}(L) = \mathcal{M}(S_{g,n}, \ell_{\beta_i} = L_i) = \mathcal{T}_{g,n}(L_1, \dots, L_n) / \text{Mod}_{g,n}$$

is the moduli space of Riemann surfaces homeomorphic to $S_{g,n}$ with n boundary components of length $\ell_{\beta_i} = L_i$. Also, we have

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \dots, 0).$$

By work of Goldman [Go], the space $\mathcal{T}_{g,n}(L_1, \dots, L_n)$ carries a natural symplectic form invariant under the action of the mapping class group. This symplectic form is called the *Weil-Petersson symplectic form*, and denoted by ω or ω_{wp} . This symplectic form is in fact the Kähler form of a Kähler metric [IT].

By work of Wolpert, over Teichmüller space the Weil-Petersson symplectic structure has a simple form in Fenchel-Nielsen coordinates [W1].

The Fenchel-Nielsen coordinates. A *pants decomposition* of S is a set of disjoint simple closed curves which decompose the surface into pairs of pants. Fix a system of pants decomposition of $S_{g,n}$, $\mathcal{P} = \{\alpha_i\}_{i=1}^k$, where $k = 3g - 3 + n$. For a marked hyperbolic surface $X \in \mathcal{T}_{g,n}(L)$, the *Fenchel-Nielsen coordinates* associated with \mathcal{P} , $\{\ell_{\alpha_1}(X), \dots, \ell_{\alpha_k}(X), \tau_{\alpha_1}(X), \dots, \tau_{\alpha_k}(X)\}$, consists of the set of lengths of all geodesics used in the decomposition and the set of the *twisting* parameters used to glue the pieces. We have an isomorphism

$$\mathcal{T}_{g,n}(L) \cong \mathbb{R}_+^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

by the map

$$X \rightarrow (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)).$$

See [Bu] for more details.

Theorem 2.1 (Wolpert). *The Weil-Petersson symplectic form is given by*

$$\omega_{wp} = \sum_{i=1}^k d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

By Theorem 2.1 the natural twisting around α is the Hamiltonian flow of the length function of α .

2.2

Integrating geometric functions over moduli spaces. Here, we discuss a method for integrating certain geometric functions over $\mathcal{M}_{g,n}(L)$. Let $Y \in \mathcal{T}_{g,n}$. For a simple closed curve γ on $S_{g,n}$, let $[\gamma]$ denote the homotopy class of γ and let $\ell_\gamma(Y)$ denote the hyperbolic length of the geodesic representative of $[\gamma]$ on Y . To each simple closed curve γ on $S_{g,n}$, we associate the set

$$\mathcal{O}_\gamma = \{[\alpha] \mid \alpha \in \text{Mod}_{g,n} \cdot \gamma\}$$

of homotopy classes of simple closed curves in the $\text{Mod}_{g,n}$ -orbit of γ on $X \in \mathcal{M}_{g,n}$. Given a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and a multicurve γ on $S_{g,n}$ define

$$f_\gamma : \mathcal{M}_{g,n} \rightarrow \mathbb{R}$$

by

$$f_\gamma(X) = \sum_{[\alpha] \in \mathcal{O}_\gamma} f(\ell_\alpha(X)). \quad (2.1)$$

Let $S_{g,n}(\gamma)$ be the result of cutting the surface $S_{g,n}$ along γ ; that is $S_{g,n}(\gamma) \cong S_{g,n} - U_\gamma$, where U_γ is an open neighborhood of γ homeomorphic to $\gamma \times (0, 1)$. Thus $S_{g,n}(\gamma)$ is a possibly disconnected compact surface with $n + 2$ boundary components. We define $\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)$ to be the moduli space of Riemann surfaces homeomorphic to $S_{g,n}(\gamma)$ such that the lengths of the 2 boundary components corresponding to γ are equal to t . Then we have ([M2]):

Theorem 2.2. *For any multicurve $\gamma = \sum_{i=1}^k c_i \gamma_i$, the integral of f_γ over $\mathcal{M}_{g,n}(L)$ with respect to the Weil-Petersson volume form is given by*

$$\int_{\mathcal{M}_{g,n}(L)} f_\gamma(X) dX = \frac{2^{-M(\gamma)}}{|\text{Sym}(\gamma)|} \int_{\mathbf{x} \in \mathbb{R}_+^k} f(|\mathbf{x}|) V_{g,n}(\Gamma, \mathbf{x}, \beta, L) \mathbf{x} \cdot d\mathbf{x},$$

where $\Gamma = (\gamma_1, \dots, \gamma_k)$, $|\mathbf{x}| = \sum_{i=1}^k c_i x_i$, $\mathbf{x} \cdot d\mathbf{x} = x_1 \cdots x_k \cdot dx_1 \wedge \cdots \wedge dx_k$, and

$$M(\gamma) = |\{i \mid \gamma_i \text{ separates off a one-handle from } S_{g,n}\}|.$$

Given a multicurve $\gamma = \sum_{i=1}^k c_i \gamma_i$, the symmetry group of γ , $\text{Sym}(\gamma)$, is defined by

$$\text{Sym}(\gamma) = \text{Stab}(\gamma) / \cap_{i=1}^k \text{Stab}(\gamma_i).$$

Recall that given $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}_+^k$, $V_{g,n}(\Gamma, \mathbf{x}, \beta, L)$ is defined by

$$V_{g,n}(\Gamma, \mathbf{x}, \beta, L) = \text{Vol}(\mathcal{M}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x}, \ell_\beta = L)).$$

Also,

$$V_{g,n}(\Gamma, \mathbf{x}, \beta, L) = \prod_{i=1}^s V_{g_i, n_i}(\ell_{A_i}),$$

where

$$S_{g,n}(\gamma) = \bigcup_{i=1}^s S_i, \quad (2.2)$$

$S_i \cong S_{g_i, n_i}$, and $A_i = \partial S_i$.

By Theorem 2.2 integrating f_γ , even for a compact Riemann surface, reduces to the calculation of volumes of moduli spaces of bordered Riemann surfaces.

Remark. Let $g \in \text{Sym}(\gamma)$, where $\gamma = \sum_{i=1}^k c_i \gamma_i$. Then $g(\gamma_i) = \gamma_j$ implies that $c_i = c_j$.

2.3

Connection with the intersection pairings of tautological line bundles.

The moduli space $\mathcal{M}_{g,n}$ is endowed with natural cohomology classes. When $n > 0$, there are n tautological line bundles defined on $\overline{\mathcal{M}}_{g,n}$ as follows. We can define \mathcal{L}_i in the orbifold sense whose fiber at the point $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space of C at x_i . Then $\psi_i = c_1(\mathcal{L}_i) \in H_2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Note that although the complex curve C may have nodes, x_i never coincides with the singular points. See [HM] and [AC] for more details.

In [M1], we use the symplectic geometry of moduli spaces of bordered Riemann surfaces to relate these intersection pairings to the volume polynomials. This method allows us to read off the intersection numbers of tautological line bundles from the volume polynomials:

Theorem 2.3. *In terms of the above notation,*

$$\text{Vol}(\mathcal{M}_{g,n}(L_1, \dots, L_n)) = \sum_{|\mathbf{d}| \leq 3g-3+n} C_g(\mathbf{d}) L_1^{2d_1} \dots L_n^{2d_n},$$

where $\mathbf{d} = (d_1, \dots, d_n)$, and $C_g(\mathbf{d})$ is equal to

$$\frac{2^{m(g,n)|\mathbf{d}|}}{2^{|\mathbf{d}|} |\mathbf{d}|! (3g-3+n-|\mathbf{d}|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \cdot \omega^{3g-3+n-|\mathbf{d}|}.$$

Here $m(g,n) = \delta(g-1) \times \delta(n-1)$, $\mathbf{d}! = \prod_{i=1}^n d_i!$, and $|\mathbf{d}| = \sum_{i=1}^n d_i$.

Remark. We warn the reader that there are some small differences in the normalization of the Weil-Petersson volume form in the literature; in this paper,

$$V_{g,n} = V_{g,n}(0, \dots, 0) = \frac{1}{(3g-3+n)!} \int_{\mathcal{M}_{g,n}} \omega^{3g-3+n}$$

which is slightly different from the notation used in [Z2] and [ST]. Also, in [Z1] the Weil-Petersson Kähler form is 1/2 the imaginary part of the Weil-Petersson pairing, while here the factor 1/2 does not appear. So our answers are different by a power of 2.

3 Asymptotic behavior of Weil-Petersson volumes

In this section, we study the asymptotics of $V_{g,n}(\mathbf{L}) = \text{Vol}(\mathcal{M}_{g,n}(L_1, \dots, L_n))$ as $g \rightarrow \infty$.

Notation. For $\mathbf{d} = (d_1, \dots, d_n)$ with $d_i \in \mathbb{N} \cup \{0\}$ and $|\mathbf{d}| = d_1 + \dots + d_n \leq 3g-3+n$, let $d_0 = 3g-3-|\mathbf{d}|$ and define

$$\begin{aligned} [\prod_{i=1}^n \tau_{d_i}]_{g,n} &= \frac{\prod_{i=1}^n (2d_i+1)! 2^{|\mathbf{d}|}}{\prod_{i=0}^n d_i!} \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \omega^{d_0} = \\ &= \frac{\prod_{i=1}^n (2d_i+1)! 2^{2|\mathbf{d}|} (2\pi^2)^{d_0}}{d_0!} \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \kappa_1^{d_0}, \end{aligned}$$

where $\kappa_1 = \omega/(2\pi^2)$ is the first Mumford class on $\overline{\mathcal{M}}_{g,n}$ [AC]. By Theorem 2.3 for $L = (L_1, \dots, L_n)$ we have:

$$V_{g,n}(2L) = \sum_{|\mathbf{d}| \leq 3g-3+n} [\tau_{d_1}, \dots, \tau_{d_n}]_{g,n} \frac{L_1^{2d_1}}{(2d_1+1)!} \dots \frac{L_n^{2d_n}}{(2d_n+1)!}. \quad (3.1)$$

3.1

Recursive formulas for the intersection pairings. Given $\mathbf{d} = (d_1, \dots, d_n)$ with $|\mathbf{d}| \leq 3g-3+n$, the following recursive formulas hold:

I.

$$\begin{aligned} [\tau_0 \tau_1 \prod_{i=1}^n \tau_{d_i}]_{g,n+2} &= [\tau_0^4 \prod_{i=1}^n \tau_{d_i}]_{g-1,n+4} + \\ &+ \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ \{1,\dots,n\}=I \amalg J}} [\tau_0^2 \prod_{i \in I} \tau_{d_i}]_{g_1,|I|+2} \cdot [\tau_0^2 \prod_{i \in J} \tau_{d_i}]_{g_2,|J|+2}, \end{aligned}$$

II.

$$(2g-2+n) [\prod_{i=1}^n \tau_{d_i}]_{g,n} = \frac{1}{2} \sum_{L=0}^{3g-3+n} (-1)^L (L+1) \frac{\pi^{2L}}{(2L+3)!} [\tau_{L+1} \prod_{i=1}^n \tau_{d_i}]_{g,n+1}.$$

III. Let $a_0 = 1/2$, and for $n \geq 1$,

$$a_n = \zeta(2n)(1 - 2^{1-2n}).$$

Then we have

$$[\tau_{d_1}, \dots, \tau_{d_n}]_{g,n} = \sum_{j=2}^n \mathcal{A}_{\mathbf{d}}^j + \frac{1}{2} \mathcal{B}_{\mathbf{d}} + \frac{1}{2} \mathcal{C}_{\mathbf{d}},$$

where

$$\mathcal{A}_{\mathbf{d}}^j = \sum_{L=0}^{d_0} (2d_j + 1) a_L [\tau_{d_1+d_j+L-1}, \prod_{i \neq 1, j} \tau_{d_i}]_{g, n-1}, \quad (3.2)$$

$$\mathcal{B}_{\mathbf{d}} = \sum_{L=0}^{d_0} \sum_{k_1+k_2=L+d_1-2} a_L [\tau_{k_1} \tau_{k_2} \prod_{i \neq 1} \tau_{d_i}]_{g-1, n+1}, \quad (3.3)$$

and

$$\mathcal{C}_{\mathbf{d}} = \sum_{\substack{I \sqcup J = \{2, \dots, n\} \\ 0 \leq g' \leq g}} \sum_{L=0}^{d_0} \sum_{k_1+k_2=L+d_1-2} a_L [\tau_{k_1} \prod_{i \in I} \tau_{d_i}]_{g', |I|+1} \times [\tau_{k_2} \prod_{i \in J} \tau_{d_i}]_{g-g', |J|+1}. \quad (3.4)$$

References.

- For results on the relationship between the Weil-Petersson volumes and the intersections of ψ -classes on $\overline{\mathcal{M}}_{g,n}$ see [Wi] and [AC]. An explicit formula for the volumes in terms of the intersection of ψ -classes was developed in [KMZ].
- Formula (I) is a special case of Proposition 3.3 in [LX1].
- For different proofs of (II) see [DN] and [LX1]. The proof presented in [DN] uses the properties of moduli spaces of hyperbolic surfaces with cone points.
- For a proof of (III) see [M2]; in view of Theorem 2.3, (III) can be interpreted as a recursive formula for the volume of $\mathcal{M}_{g,n}(L)$ in terms of volumes of moduli spaces of Riemann surfaces that we get by removing a pair of pants containing at least one boundary component of $S_{g,n}$. See also [Mc] and [LX2].
- If $d_1 + \dots + d_n = 3g - 3 + n$, (III) gives rise to a recursive formula for the intersection pairings of ψ_i classes which is the same as the Virasoro constraints for a point. See also [MS]. For different proofs and discussions related to these relations see [Wi], [Ko], [OP], [M1], [KL], and [EO].

Remarks.

- In terms of the volume polynomials equation **(II)** can be written as ([DN]):

$$\frac{\partial V_{g,n+1}}{\partial L}(L, 2\pi i) = 2\pi i(2g - 2 + n)V_{g,n}(L).$$

When $n = 0$,

$$V_{g,1}(2\pi i) = 0,$$

and

$$\frac{\partial V_{g,1}}{\partial L}(2\pi i) = 2\pi i(2g - 2)V_g. \quad (3.5)$$

- Note that **(III)** applies only when $n > 0$. In the case of $n = 0$, (3.5) allows us to prove necessary estimates for the growth of $V_{g,0}$.
- Although **(III)** has been described in purely combinatorial terms, it is closely related to the topology of different types of pairs of pants in a surface.
- In this paper, we are mainly interested in the intersection pairings only containing κ_1 and ψ_i classes. For generalizations of **(III)** to the case of higher Mumford's κ classes see [LX1] and [E].
- We will show that n is fixed and $g \rightarrow \infty$ both terms $\mathcal{A}_{\mathbf{d}}$, and $\mathcal{B}_{\mathbf{d}}$ in **(III)** contribute to $V_{g,n} = [\tau_0, \dots, \tau_0]_g$. More precisely, for $\mathbf{d} = (0, \dots, 0)$

$$\frac{\mathcal{B}_{\mathbf{d}}}{\mathcal{A}_{\mathbf{d}}} \asymp 1.$$

On the other hand, for $\mathbf{d} = (0, \dots, 0)$ the contribution of $\mathcal{C}_{\mathbf{d}}$ in **(III)** is negligible. More precisely, we will see that $\frac{\mathcal{C}_{\mathbf{d}}}{V_{g,n}} = O(1/g)$.

3.2

Basic estimates for the intersection pairings. The main advantage of using **(III)** is that all the coefficients are positive. Moreover, it is easy to check that

$$a_n = \zeta(2n)(1 - 2^{1-2n}) = \frac{1}{(2n-1)!} \int_0^\infty \frac{t^{2n-1}}{1+e^t} dt.$$

Hence,

$$a_{n+1} - a_n = \int_0^\infty \frac{1}{(1+e^t)^2} \left(\frac{t^{2n+1}}{(2n+1)!} + \frac{t^{2n}}{2n!} \right) dt.$$

As a result, we have:

Lemma 3.1. *In terms of the above notation, $\{a_n\}_{n=1}^\infty$ is an increasing sequence. Moreover, $\lim_{n \rightarrow \infty} a_n = 1$, and*

$$a_{n+1} - a_n \asymp 1/2^{2n}. \quad (3.6)$$

Using this observation and (3.1) one can prove the following general estimates:

Lemma 3.2. *In terms of the above notation, the following estimates hold:*

1.

$$[\tau_{d_1}, \tau_0, \dots, \tau_0]_{g,n} \leq [\tau_0, \dots, \tau_0]_{g,n} = V_{g,n},$$

and in case of $\mathbf{d} = (1, 0, \dots, 0)$ we have

$$[\tau_1, \tau_0, \dots, \tau_0]_{g,n} \asymp [\tau_0, \dots, \tau_0]_{g,n}.$$

2. More generally,

$$[\tau_{d_1}, \dots, \tau_{d_n}]_{g,n} \leq (2d_1 + 1) \cdots (2d_n + 1) V_{g,n},$$

and

$$V_{g,n}(2L_1, \dots, 2L_n) \leq e^L V_{g,n}, \quad (3.7)$$

where $L = L_1 + \dots + L_n$.

3. for any $g, n \geq 0$,

$$V_{g,n+2} \geq V_{g-1,n+4}, \quad \text{and} \quad V_{g,n+1} > \frac{(2g-2+n)}{b} V_{g,n}, \quad (3.8)$$

where $b = \sum_{L=0}^{\infty} \pi^{2L} (L+1) / 2(2L+3)!$.

Proof. Part (1) and (2) follow by comparing the contributions of $\mathcal{A}_{\mathbf{d}}$, $\mathcal{B}_{\mathbf{d}}$, and $\mathcal{C}_{\mathbf{d}}$ for (d_1, d_2, \dots, d_n) , $(d_1, 0, \dots, 0)$ and $(0, \dots, 0)$ in **(III)**. See (3.2), (3.3), and (3.4.) Then (3.1) implies (3.7).

Moreover, since

$$[\tau_1, \tau_0, \dots, \tau_0] \leq V_{g,n}, \quad \text{and} \quad [\tau_0, \dots, \tau_0]_{g'} \geq 0$$

equation **(I)** for $\mathbf{d} = (0)$ implies that for any $n \geq 0$, $V_{g,n+2} \geq V_{g-1,n+4}$. Similarly, since

$$[\tau_{L+1}, \tau_0, \dots, \tau_0] \leq V_{g,n+1},$$

equation **(II)** for $\mathbf{d} = 0$ implies $bV_{g,n+1} > 2(2g-2+n)V_{g,n}$. \square

Remarks.

- A stronger lower bound for $\frac{V_{g,n+1}}{(2g-2+n)V_{g,n}}$ was obtained in [ST]. But in this paper, we will use only (3.8).
- We will show that as $g \rightarrow \infty$ the first inequality of (3.8) is asymptotically sharp. However, (1.3) implies that when g is fixed and n is large this inequality is far from being sharp; in fact, given $g \geq 1$ as $n \rightarrow \infty$

$$V_{g,n+2} \asymp \sqrt{n} V_{g-1,n+4}.$$

3.3

It is crucial for the applications in §4 to understand the behaviour of the polynomial $V_{g,1}(L)$ as $g \rightarrow \infty$. We know that in general $V_{g,1}(L) \leq e^{\frac{L}{2}} V_{g,1}$. In view of (3.1), the estimates we prove in Theorem 3.5 (a), imply that if $L \ll g$

$$V_{g,1}(L) \asymp e^{\frac{L}{2}} V_{g,1}.$$

To simplify the notation, let

$$[\mathbf{x}]_{g,n} := [\tau_{x_1}, \dots, \tau_{x_n}]_{g,n},$$

where $\mathbf{x} = (x_1, \dots, x_n)$. Also, given $\mathbf{b} = (b_1, \dots, b_l)$ and $\mathbf{c} = (c_1, \dots, c_m)$, let

$$\mathbf{b} \oplus \mathbf{c} = (b_1 + c_1 - 1, b_2, \dots, b_l, c_2, \dots, c_m).$$

The following lemma plays an important role in the proof of Theorem 1.2.

Lemma 3.3. *In terms of the above notation, for $\mathbf{x} = (x_1, \dots, x_l)$, and $\mathbf{y} = (y_1, \dots, y_m)$, we have*

$$\sum_{\substack{g_1+g_2+1=g \\ g_2+1 \geq g_1 \geq 0}} [\mathbf{x}]_{g_1,l} \times [\mathbf{y}]_{g_2+1,m} \leq C[\mathbf{x} \oplus \mathbf{y}]_{g-1,n-1}, \quad (3.9)$$

where $n = l + m$, and $C > 0$ is a constant independent of g , n , \mathbf{x} , and \mathbf{y} .

Note that $[x_1, \dots, x_n]_{g,n} > 0$ if and only if $x_1 + \dots + x_n \leq 3g - 3 + n$. We remark that if at least one term on the left hand side of (3.9) is non-zero then $x_1 + y_1 - 1 + x_2 + \dots + x_l + y_2 + \dots + y_m \leq 3(g-1) - 3 + (n-1)$ and hence $[\mathbf{x} \oplus \mathbf{y}]_{g-1,n-1} > 0$.

Sketch of proof. The proof is by induction on $2g - 2 + n$. The main idea is using (III) for \mathbf{x} and $\mathbf{x} \oplus \mathbf{y}$. First, we can choose C such that 3.9 holds for $2g - 2 + n \leq 2$.

Expand $[\mathbf{x} \oplus \mathbf{y}]_{g-1,n-1}$ and all the terms including x'_i 's in (3.9) using the recursive relation (III).

Roughly speaking, since all the terms in equation (III) and (3.9) are positive, it is enough to check that after expanding both sides every term on the left hand side has a corresponding term on the right hand side.

Here we check this for the terms in \mathcal{B} defined by (3.3). By the definition, in $\mathcal{B}_{\mathbf{x}}$ for $\mathbf{x} = (x_1, \dots, x_l)$ the coefficient of a term $[k_1, k_2, x_3, \dots, x_l]_{g'}$ with $g' = g_1 - 1$ and $k_1 + k_2 = L + x_1 - 2$ is equal to a_L . Now using the induction hypothesis for $\mathbf{x}' = (k_1, k_2, x_3, \dots, x_l)$ and \mathbf{y} implies that the contribution of the terms corresponding to \mathbf{x}' is $\leq Ca_L[\mathbf{x}' \oplus \mathbf{y}]_{g-2,n}$. On the other hand, when we apply (III) to $[\mathbf{x} \oplus \mathbf{y}]_{g-1,n-1}$ the coefficient of the term $[\mathbf{x}' \oplus \mathbf{y}]_{g-2,n}$ in \mathcal{B} is equal to $a_{L'}$ where $L' = k_1 + y_1 - 1 + k_2 - (x_1 + y_1 - 1) + 2 = k_1 + k_2 - x_1 + 2 = L$. \square

By applying this result to $m = \ell = 1$, $\mathbf{x} = (1)$, and $\mathbf{y} = (0)$, Lemma 3.2(a) implies:

Corollary 3.4. *As $g \rightarrow \infty$*

$$\sum_{i=1}^{g-1} V_{i,1} V_{g-i,1} = O(V_{g-1,1}) = O\left(\frac{V_g}{g}\right). \quad (3.10)$$

Remark. Similarly, by induction on $2g + n$ and using **III** one can show that, for $d \geq 1$

$$\sum_{a_1+a_2=d} [a_1, a_2, x_1, \dots, x_n] \leq [d-1, 0, x_1, \dots, x_n]$$

and

$$[m, x_1, \dots, x_{n-1}]_{g,n} \leq [m-1, 0, 0, x_1, \dots, x_{n-1}]_{g-1, n+2}.$$

We skip the proofs since we won't need these inequalities in this paper. Now we can prove the main result of this section:

Theorem 3.5. *Let $n \geq 0$.*

- **a):** *For any $k \in \mathbb{N}$*

$$\frac{[\tau_k, \tau_0, \dots, \tau_0]_{g, n+1}}{V_{g, n+1}} = 1 + O(1/g),$$

as $g \rightarrow \infty$.

- **b):**

$$\frac{V_{g, n+1}}{2gV_{g, n}} = 4\pi^2 + O(1/g),$$

- **c):**

$$\frac{V_{g, n}}{V_{g-1, n+2}} = 1 + O(1/g).$$

Remark.

- These estimates are consistent with the conjectures on the growth of Weil-Petersson volumes in [Z2]; we remark that the statements had been predicted by Peter Zograf.
- Following the ideas used in the proof, one can show that

$$\frac{V_{g, n+1}}{2gV_{g, n}} = 4\pi^2 + \frac{a_{1, n}}{g} + \dots + \frac{a_{k, n}}{g^k} + O\left(\frac{1}{g^{k+1}}\right),$$

and

$$\frac{V_{g, n}}{V_{g-1, n+2}} = 1 + \frac{b_{1, n}}{g} + \dots + \frac{b_{k, n}}{g^k} + O\left(\frac{1}{g^{k+1}}\right).$$

However, in general it is not easy to calculate $a_{i, n}$ and $b'_{i, n}$ s.

Proof of Theorem 3.5. Fix $n \geq 0$. Applying Lemma 3.3 for $(1, \dots, 0)$ and $(0, \dots, 0)$ implies that as $g \rightarrow \infty$

$$\sum_{\substack{g_1+g_2=g \\ \{1, \dots, n\} = I_1 \sqcup I_2}} V_{g_1, |I_1|+1} \times V_{g_2, |I_2|+1} = O(V_{g-1, n+1}) \quad (3.11)$$

$$\sum_{\substack{g_1+g_2=g \\ \{1, \dots, n\} = I_1 \sqcup I_2}} V_{g_1, |I_1|+2} \times V_{g_2, |I_2|+2} = O(V_{g-1, n+3}). \quad (3.12)$$

First, we assume that $n \geq 1$. Then by the inequalities of (3.8)

$$V_{g-1, n+2} = O\left(\frac{V_{g, n+1}}{g}\right), \quad \text{and} \quad V_{g, n} = O\left(\frac{V_{g, n+1}}{g}\right).$$

and from (3.11) we get that

$$\sum_{\substack{g_1+g_2=g \\ \{1, \dots, n\} = I_1 \sqcup I_2}} V_{g_1, |I_1|+1} \times V_{g_2, |I_2|+1} = O(V_{g-1, n+1}) = O\left(\frac{V_{g, n+1}}{g}\right).$$

Now by comparing the contributions of $\mathcal{A}_{\mathbf{d}}$, $\mathcal{B}_{\mathbf{d}}$, and $\mathcal{C}_{\mathbf{d}}$ for $\mathbf{d} = (k, 0, \dots, 0)$ and $(0, \dots, 0)$ in (III), and Lemma 3.1 we get

$$\left| \frac{[\tau_k, \tau_0, \dots, \tau_0]_{g, n+1}}{V_{g, n+1}} - 1 \right| \leq c_0 \frac{k^2}{g}, \quad (3.13)$$

where c_0 is a universal constant independent of g and k .

We use the following elementary observation to prove (b) for $n \geq 1$:

Elementary fact. Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of real numbers and $\{k_g\}_{g=1}^{\infty}$ be an increasing sequence of positive integers. Assume that for $g \geq 1$, and $i \in \mathbb{N}$, $0 \leq c_{g,i} \leq c_i$, and $\lim_{g \rightarrow \infty} c_{g,i} = c_i$. If $\sum_{i=1}^{\infty} |c_i r_i| < \infty$, then

$$\lim_{g \rightarrow \infty} \sum_{i=1}^{k_g} r_i c_{g,i} = \sum_{i=1}^{\infty} r_i c_i. \quad (3.14)$$

Now, let

$$r_i = (-1)^i \frac{\pi^{2i}(i+1)}{(2i+3)!}, \quad k_g = 3g - 3 + n, \quad c_i = 1 \quad \text{and} \quad c_{g,i} = \frac{[\tau_{i+1} \tau_0 \dots \tau_0]_{g, n}}{V_{g, n+1}}.$$

By (3.14), and (II) for $\mathbf{d} = 0$ we get

$$\lim_{g \rightarrow \infty} \frac{2(2g-2+n)V_{g, n}}{V_{g, n+1}} = \frac{1}{3!} - \frac{2\pi^2}{5!} + \dots + (-1)^L (L+1) \frac{\pi^{2L}}{(2L+3)!} + \dots = \frac{1}{2\pi^2}.$$

In fact, similarly (3.13) implies that

$$\frac{2(2g-2+n)V_{g, n}}{V_{g, n+1}} = \frac{1}{2\pi^2} + O\left(\frac{1}{g}\right).$$

On the other hand, from **(I)** and (3.12) we get that for $n \geq 2$:

$$\lim_{g \rightarrow \infty} \frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O(1/g).$$

Now it is easy to check that

$$V_{g,1} = \frac{1}{g} V_{g,2} \left(\frac{1}{4\pi^2} (1 - O(1/g)) \right), \quad V_{g-1,3} = \frac{1}{g} V_{g-1,4} \left(\frac{1}{4\pi^2} (1 - O(1/g)) \right)$$

and

$$V_{g,2} = V_{g-1,4} (1 + O(1/g))$$

imply

$$\frac{V_{g,1}}{V_{g-1,3}} = 1 + O(1/g).$$

In other words, **(b)** for $n = 1$ and $n = 2$ proves **(c)** for $n = 1$.

We remark that (3.5) implies **(b)** for $n = 0$. Finally **(b)** for $n = 0$ and $n = 1$ implies **(c)** for $n = 0$. \square

It is easy to check that if $\{k_n\}$, is a bounded sequence $|k_n| < c$

$$\frac{1}{m} g^{-c} < \prod_{n=1}^g \left(1 + \frac{k_n}{n} \right) < m g^c,$$

where m is independent of g . Hence, Theorem 3.5 implies the following

Corollary 3.6. *There exists $M > 0$ such that:*

$$g^{-M} \mathcal{F}_{g,n} < V_{g,n} < g^M \mathcal{F}_{g,n}, \quad (3.15)$$

where

$$\mathcal{F}_{g,n} = (4\pi^2)^{2g+n-3} (2g-3+n)! \frac{1}{\sqrt{g\pi}}.$$

Finally, (3.15) implies that

$$\sum_{i=r+1}^{g/2} V_{i,1} \times V_{g-i,1} \asymp \frac{V_g}{g^{2r+1}}. \quad (3.16)$$

A simple calculation shows that

$$\sum_{\substack{g_0+g_1=g+1-k, \\ r \leq g_0 \leq g_1}} e^{Cg_0} g_0 \frac{\mathcal{F}_{g_0,k} \mathcal{F}_{g_1,k}}{\mathcal{F}_g} = O\left(\frac{1}{g^r}\right), \quad (3.17)$$

where $C = 2 \ln(2)$. Therefore, we get the following estimate which will be used in the next section:

Corollary 3.7. *Let $k \geq 0$, $1 > \beta > 0$, and $c > 0$. Then as $g \rightarrow \infty$*

$$\sum_{g_0+g_1=g+1-k} e^{Cg_0+cg_0^\beta} g_0 V_{g_0,k} V_{g_1,k} = O\left(\frac{V_g}{g}\right),$$

where $C = 2 \ln(2)$.

4 Random Riemann surfaces of high genus

In this section, we apply the asymptotic estimates on the volume polynomials to study the geometric properties of random hyperbolic surfaces; in particular, we are interested in the length of the shortest simple closed geodesic of a given combinatorial type, diameter and the Cheeger constant of a random surface. See [BM] for more in the case of random hyperbolic surfaces constructed by random trivalent graphs.

4.1

Notation. Recall that the mapping class group $\text{Mod}_{g,n}$ acts naturally on the set $\mathcal{S}_{g,n}$ of isotopy classes of simple closed curves on $S_{g,n}$: Two simple closed curves α_1 and α_2 are of the same *type* if and only if there exists $g \in \text{Mod}_{g,n}$ such that $g \cdot \alpha_1 = \alpha_2$. The type of a simple closed curve is determined by the topology of $S_{g,n} - \alpha$, the surface that we get by cutting $S_{g,n}$ along α .

Let

$$\mathcal{S}_k^m = \{\gamma = \gamma_1 + \dots + \gamma_k \mid \gamma_i \in \mathcal{S}_{g,n} \text{ distinct } S - \gamma = S_1 \cup S_2, |\chi(S_1)| = m\}.$$

Note that each \mathcal{S}_k^m is invariant under the action of the mapping class group. To simplify the notation, let $\tilde{\gamma}_0$ be a non-separating simple closed curve on S_g , and $\tilde{\gamma}_i$ be a separating simple closed curve on S_g such that

$$S_g - \tilde{\gamma}_i = S_{i,1} \cup S_{g-i,1}.$$

That is $\tilde{\gamma}_i \in \mathcal{S}_1^{2i-1}$, and $\tilde{\gamma}_0 \in \mathcal{S}_1^{2g-2}$.

Consider the counting function

$$N_\alpha(\cdot, \cdot) : \mathbb{R}_+ \times \mathcal{M}_g \rightarrow \mathbb{R}_+$$

defined by

$$N_\alpha(L, X) = |\{\gamma \mid \gamma \in \alpha \cdot \text{Mod}_g, \ell_\gamma(X) \leq L\}|.$$

Let

$$F_i^L(X) = |\{\gamma \mid \gamma \in \mathcal{O}_{\tilde{\gamma}_i}, \ell_\gamma(X) \leq L\}|. \quad (4.1)$$

Using Theorem 2.2 and the estimates proved in §3, we can show that if L is fixed

$$\int_{\mathcal{M}_g} F_0^L(X) dX = \int_0^L tV_{g-1,2}(t, t) dt \asymp e^L L^2 V_g,$$

but

$$\int_{\mathcal{M}_g} F_1^L(X) dX \asymp \frac{e^{L/2} L^3 V_g}{g},$$

as $g \rightarrow \infty$. Similar estimates hold when L is *much smaller* than g . We remark that since the number of closed geodesics of length $\leq L$ on $X \in \mathcal{M}_g$ is at most

$e^{L+6}(g-1)$ (see Lemma 6.6.4 in [Bu]), we can not expect the similar bounds to hold in general. However, in general we have

$$\int_{\mathcal{M}_g} F_0^L(X) dX = O(e^L L^2 V_g), \text{ and } \int_{\mathcal{M}_g} F_1^L(X) dX = O\left(\frac{e^{L/2} L^3 V_g}{g}\right). \quad (4.2)$$

4.2

Systoles and injectivity radius. Let

$$\mathcal{M}_{g,n}^\epsilon = \{X \mid \exists \gamma, \ell_\gamma(X) \leq \epsilon\} \subset \mathcal{M}_{g,n}.$$

The set $\mathcal{M}_{g,n} - \mathcal{M}_{g,n}^\epsilon$ of hyperbolic surfaces with lengths of closed geodesics bounded below by a constant $\epsilon > 0$ is a compact subset of the moduli space $\mathcal{M}_{g,n}$.

Theorem 4.1. *Let $n \geq 0$. There exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$*

$$\text{Vol}_{wp}(\mathcal{M}_{g,n}^\epsilon) \asymp \epsilon^2 \text{Vol}_{wp}(\mathcal{M}_{g,n})$$

as $g \rightarrow \infty$.

Proof. Here we sketch the proof for the case of $n = 0$. Fix ϵ such that no two simple closed geodesics of length $\leq \epsilon$ could meet. Consider the function

$$F^\epsilon(X) = N(\epsilon, X) = F_0^\epsilon(X) + \dots + F_{g/2}^\epsilon(X),$$

as defined in (4.1). Then by Theorem 2.2, we have

$$\begin{aligned} \text{Vol}_{wp}(\mathcal{M}_g^\epsilon) &\leq \int_{\mathcal{M}_g} F^\epsilon(X) dX \leq \\ &\leq \sum_{i=1}^{g/2} \int_0^\epsilon t \text{Vol}_{wp}(\mathcal{M}(S_g - \gamma_i, t, t)) dt + \int_0^\epsilon t \text{Vol}_{wp}(\mathcal{M}_{g-1,2}(t, t)) dt \end{aligned}$$

On the other hand, by (3.7) we know that if t is small enough for $i \geq 1$,

$$\text{Vol}_{wp}(\mathcal{M}(S_g - \gamma_i, t, t)) \leq 2V_{i,1} \times V_{g-i,1},$$

and

$$\text{Vol}_{wp}(\mathcal{M}_{g-1,2}(t, t)) \leq 2V_{g-1,2}.$$

Hence, when ϵ is small (independent of g), from (3.16) and (3.10) we get

$$\text{Vol}_{wp}(\mathcal{M}_g^\epsilon) = O\left(\epsilon^2 \left(\sum_{i=1}^{g/2} V_{i,1} V_{g-i,1} + V_{g-1,2}\right)\right) = O(\epsilon^2 V_g).$$

Next, we prove that the volume of the locus with a *non-separating* short simple closed geodesic of length $\leq \epsilon$ is asymptotically positive.

Since $\int_{\mathcal{M}_g} F_0^\epsilon(X) dX \asymp \epsilon^2 V_g$ in order to get a lower bound $\text{Vol}_{wp}(\mathcal{M}_{g,n}^\epsilon)$ we need to prove an upper bound for the volume of the locus where $F_0^\epsilon(X) \geq k$ for $k \geq 2$. In fact

$$\int_{\mathcal{M}_g} F_0^\epsilon(X) = \sum_{k=1}^{\infty} \text{Vol}_{wp}(\{X | F_0^\epsilon \geq k\}).$$

Note that by Lemma 3.2

$$V_{g-1,n+4} \leq V_{g,n+2},$$

and if $\sum_{i=1}^s (2g_i - 2 + k_i) = 2g - 2$ with $s \geq 2$, and $k_i \geq 2$ for $1 \leq i \leq s$

$$\prod V_{g_i, k_i} = O\left(\frac{V_g}{g^2}\right). \quad (4.3)$$

Let

$$\mathcal{U} = \{X | \exists \gamma_1, \dots, \gamma_l \in \text{Mod} \cdot \gamma_0, \ell_{\gamma_i}(X) \leq \epsilon, S - \cup \gamma_i \text{ is disconnected}\} \subset \mathcal{M}_g.$$

Since $F_0^\epsilon(X) \leq 3g - 3$, from (4.3)

$$\int_{\mathcal{U}} F_0^\epsilon(X) dX \leq O\left(\frac{\epsilon^4 V_g}{g}\right).$$

On the other hand, by using the same argument for

$$F_{0,k}^\epsilon(X) = |\{\{\gamma_1, \dots, \gamma_k\} | \gamma_i \text{ non-separating } \ell_{\gamma_i}(X) \leq \epsilon\}|$$

and applying Theorem 2.2, we get

$$\text{Vol}_{wp}(\{X | F_{0,k}^\epsilon \geq 1\} - \mathcal{U}) = \text{Vol}_{wp}(X | F^\epsilon \geq k) - \mathcal{U} \leq c \frac{\epsilon^{2k} e^{\epsilon k}}{k!},$$

where c is a constant independent of g and k . Therefore if $\epsilon > 0$ is small enough

$$\sum_{k=2}^{\infty} \text{Vol}_{wp}(\{X | F_0^\epsilon(X) \geq k\}) \leq \epsilon^4 \text{Vol}_{wp}(\mathcal{M}_g)$$

which implies the result. □

Let

$$f(X) = \sum_{\ell_\alpha(X) \leq 1} \frac{1}{\ell_\alpha(X)}.$$

Then using Theorem 2.2

$$\int_{\mathcal{M}_g} f(X) dX = \int_0^1 V_{g-1,2}(t, t) dt + \sum_{i=1}^{g/2} \int_0^1 V_{g-i,1}(t) V_{i,1}(t) dt \asymp V_g$$

and hence Theorem 4.1 implies that :

Corollary 4.2. *As $g \rightarrow \infty$*

$$\int_{\mathcal{M}_g} \frac{1}{\ell_{sys}(X)} dX \asymp V_g.$$

4.3

Behavior of separating simple closed geodesics. By [SS] there exists a positive constant $C > 0$ such that every closed surface X of genus $g \geq 2$, $\ell_{sys}^s(X) \leq C \log(g)$. We show that as $g \rightarrow \infty$ $\ell_{sys}^s(X)$ is generically at least of $(2 - \epsilon) \log(g)$. Moreover generically if a separating curve γ satisfies $\ell_\gamma(X) < C \log(g)$ then $S_g - \gamma = S_{g_1} \cup S_{g_2}$ with $g_1 = O(1)$.

Theorem 4.3. *Let $0 < m < 2$ then*

$$\text{Prob}_{wp}^g(\ell_{sys}^s(X) < m \log(g)) = O(\log(g)g^{(m/2-1)}),$$

and

$$\mathbb{E}_{X \sim wp}^g(\ell_{sys}^s(X)) \asymp \log(g)$$

as $g \rightarrow \infty$.

Proof. Note that

$$\text{Prob}_{wp}^g(\ell_{sys}^s(X) < L) \leq \frac{e^{L/2}L^3}{g} + \sum_{i=2}^{g/2} \frac{e^L V_{i,1} \times V_{g-i,1}}{V_g}. \quad (4.4)$$

On the other hand, by (3.16)

$$\sum_{i=2}^{g/2} e^L V_{i,1} \times V_{g-i,1} = O\left(\frac{V_g}{g^3}\right)$$

which implies the result. \square

4.4

Injectivity radius and embedded balls. Let $\text{Inj}(x)$ denote the injectivity radius of $x \in X$. We show that on a generic $X \in \mathcal{M}_g$ almost every point $x \in X$ has $\text{Inj}(x) \geq \frac{1}{6} \log(g)$. By the definition of the injectivity radius, corresponding to each x , there exists a simple closed curve γ_x of length $\leq 2 \text{Inj}(x)$ such that the distance of x from the geodesic representative of γ_x is at most $2 \text{Inj}(x)$. Also, let $N(L, X)$ be the number of simple closed geodesics of length $\leq L$ on X . Then

- It is easy to see from (4.2) and (4.4) that as $g \rightarrow \infty$

$$\text{Prob}^g(\{X \mid N(\log(g)/3, X) \geq g^{1/3+1/4}\}) = O(g^{-1/4}). \quad (4.5)$$

- A simple calculation shows that given a simple closed geodesic γ of length $\leq \log(g)/3$ the volume of the locus on X with $\gamma_x = \gamma$ is at most $g^{1/3} \log(g)$.

Therefore, for a generic point in $X \in \mathcal{M}_g$ (defined by (4.5))

$$\text{Vol}(\{x \in X \mid \text{Inj}(x) \leq \frac{1}{6} \log(g)\}) = O(g^{11/12} \log(g)).$$

Hence, we have:

Theorem 4.4. As $g \rightarrow \infty$

$$\begin{aligned} \text{Prob}_{wp}^g(\text{Emb}(X) \leq C_E \log(g)) &\rightarrow 0, \\ \mathbb{E}_{X \sim wp}^g(\text{Emb}(X)) &\asymp \log(g), \end{aligned}$$

where $C_E = \frac{1}{3}$.

4.5

Cheeger constants and isoperimetric inequalities. Recall that the Cheeger constant of X is defined by

$$h(X) = \inf \frac{\ell(A)}{\min\{\text{Area}(X_1), \text{Area}(X_2)\}}$$

where the infimum is taken over all smooth 1-dimensional submanifolds of X which divide it into two disjoint submanifolds X_1 and X_2 such that $X - A = X_1 \cup X_2$ and $A \subset \partial(X_1) \cap \partial(X_2)$.

We remark that:

- In fact, by an observation due to Yau, we may restrict A to a family of curves for which X_1 and X_2 are connected. See [Bu].
- By a result of Cheng [C],

$$h(X) \leq 1 + \frac{16\pi^2}{\text{diam}(X)}.$$

Therefore, there is an upper bound for the Cheeger constant which tends to 1 as $g(X) \rightarrow \infty$. See also §III and §X in [Ch].

Given $i \leq g$

$$H_i(X) = \inf \frac{\ell_\alpha(X)}{\min\{\text{Area}(X_1), \text{Area}(X_2)\}}$$

where $\alpha = \cup_{j=1}^s \alpha_j$ is a union of simple closed *geodesics* on X with $X - \alpha = X_1 \cup X_2$, and X_1 and X_2 are connected subsurfaces of X such that $|\chi(X_1)| = i \leq |\chi(X_2)|$. Here $|\chi(X_1)| = 2g_1 - 2 + s$. Now we define the *geodesic* Cheeger constant of X by

$$H(X) = \min_{i \leq g} H_i(X).$$

In general, by the definition

$$H(X) \geq h(X),$$

but the inequality is not sharp. However, using the following basic properties of perimeter minimizers allows us to get a lower bound for $h(X)$ in terms of $H(X)$.

Recall that in a compact hyperbolic surface, there exists a perimeter minimizer among regions of prescribed area bounded by embedded rectifiable curves; it consists of curves of equal constant curvature. Moreover, by a result of Adams and Morgan [AM]:

Theorem 4.5. For given area $0 < A < 4\pi g$, a perimeter-minimizing system of embedded rectifiable curves bounding a region R of area A consists of a set of curves of one of the following four types :

1. a circle,
2. horocycles around cusps,
3. two neighboring curves at constant distance from a geodesic, bounding an annulus or complement,
4. geodesics or single neighboring curves.

All curves in the set have the same constant curvature.

In fact in the case of a circle or neighboring curves h is strictly bigger than 1. On the other hand, by a simple calculation (see Lemma 2.3 [AM]) if a neighboring curve of length L and curvature κ at distance s from a geodesic of length ℓ , enclosing area A , then

$$A = \ell \sinh(s), \quad L = \ell \cosh(s), \quad \text{and} \quad \kappa = \tanh(s).$$

Therefore by using basic isoperimetric inequalities for hyperbolic surfaces, we have:

Proposition 4.6. Let $X \in \mathcal{M}_g$ be a hyperbolic surface of genus g . Then

$$h(X) \geq \frac{H(X)}{H(X) + 1}$$

Now, we can show:

Theorem 4.7. As $g \rightarrow \infty$

$$\text{Prob}_{wp}^g \left(h(X) \leq \frac{\ln(2)}{\pi + \ln(2)} \right) \rightarrow 0,$$

and

$$\int_{\mathcal{M}_g} \frac{1}{h(X)} dX \asymp V_g. \tag{4.6}$$

Let

$$\mathcal{W}_k^{2m-1}(L) = \text{Vol}_{wp}(\{X \in \mathcal{M}_g \mid \exists \gamma \in \mathcal{S}_k^{2m-1}, \ell_\gamma(X) \leq L\}).$$

Let $\gamma_k^{2m-1} = \gamma_1 + \dots + \gamma_k \in \mathcal{S}_k^{2m-1}$ (see §4.1). By Theorem 2.2 for $N_{\gamma_k^{2m-1}}(X, L)$ we have:

$$\mathcal{W}_k^{2m-1}(L) \leq e^L V_{m,1} \times V_{g-m,1} \times \int_{L_1 + \dots + L_k \leq L} \frac{1}{k!} L_1 \cdots L_k dL_1 \cdots dL_k,$$

and

$$\mathcal{W}^{2m-1}(L) \leq e^L V_{m,1} \times V_{g-m,1} \sum_{k=1}^{2m} \int_{L_1+\dots+L_k \leq L} \frac{1}{k!} L_1 \cdots L_k dL_1 \cdots dL_k.$$

On the other hand, since

$$\int_{L_1+\dots+L_s \leq L} L_1 \cdots L_s dL_1 \cdots dL_s = \frac{L^{2s}}{(2s)!},$$

and

$$\sum_{s=1}^{\infty} \frac{L^{2s}}{s!(2s)!} = O(e^{\frac{3}{2}L^{2/3}})$$

we get

$$\mathcal{W}^{2m-1}(L) \leq e^{L+\frac{3}{2}L^{2/3}} V_{m,1} \times V_{g-m,1}. \quad (4.7)$$

As before, let

$$H_k(X) = \inf \frac{\ell_\alpha(X)}{\pi k}$$

where $\alpha = \cup_i \alpha_i$ is a union of simple closed *geodesics* on X with $X - \alpha = X_1 \cup X_2$, and X_1 and X_2 are connected subsurfaces of X such that $|\chi(X_1)| = k < |\chi(X_2)|$. Recall that by Lemma 3.2, for $n \geq 0$

$$V_{g-1, n+4} \leq V_{g, n+2}.$$

Hence, from (4.7) we get:

Lemma 4.8. *Let $m = 2m_1 - 2 + n_1 \leq 2g - 2$, where $n_1 \in \{0, 1\}$. Then*

$$\text{Vol}_{wp}(\{X | H_m(X) \leq C\}) = O(m e^{\pi \cdot m \cdot C + cm^{2/3}} V_{m_1+n_1-1, 2-n_1} V_{2g-2-m+n_1, 2-n_1}),$$

where c is a constant independent of g .

Proof of Theorem 4.7. Lemma 4.8 and Corollary 3.7 imply that as $g \rightarrow \infty$

$$\text{Prob}_{wp}^g \left(H(X) \leq \frac{\ln(2)}{\pi} \right) \rightarrow 0.$$

Therefore, in view of Proposition 4.6, we get the result. Corollary 4.2 implies the second part of the theorem. \square

Moreover, we have:

Theorem 4.9. *Let s_g be a sequence such that $\lim_{g \rightarrow \infty} \frac{s_g}{g} = 0$. Given $M > 0$,*

$$\text{Prob}_{wp}^g(H_{s_g}(X) \leq M) \rightarrow 0,$$

as $g \rightarrow \infty$.

4.6

Diameter. It is known that the diameter of a Riemannian manifold of constant curvature -1 satisfies:

$$\text{diam}(X) \leq 2(r_0 + \frac{1}{h} \log(\frac{\text{Vol}(X)}{2B(r_0)})), \quad (4.8)$$

where $r_0 > 0$ and $B(r_0)$ is the infimum of the volume of a ball of radius r_0 in X . Using this result, we get:

Theorem 4.10. *As $g \rightarrow \infty$*

$$\text{Prob}_{wp}^g(\text{diam}(X) \geq C_d \log(g)) \rightarrow 0,$$

and

$$\mathbb{E}_{X \sim wp}^g(\sqrt{\text{diam}(X)}) \asymp \sqrt{\log(g)},$$

where $C_d = 5$.

Proof. From the proof of Theorem 4.1

$$\text{Prob}^g(\{X \mid \exists \gamma \ell_\gamma(X) \leq \frac{1}{\log(g)}\}) = O(\frac{1}{\log(g)^2}).$$

Therefore the first part of this theorem is a direct consequence of (4.8) and Theorem 4.7. Next we need to prove that

$$\mathbb{E}_{X \sim wp}^g(\sqrt{\text{diam}(X)}) = O(\sqrt{\log(g)}).$$

Since (as in Corollary 4.2)

$$\int_{\mathcal{M}_g} |\log(\ell_{sys}(X))| dX \asymp V_g$$

we have

$$\int_{\mathcal{M}_g} \sqrt{\frac{|\log(\ell_{sys}(X))|}{h(X)}} dX \asymp V_g.$$

Now the second part follows from Corollary 4.2 and (4.6). □

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