Operations on Power Series Related to Taylor Series

In this problem, we perform elementary operations on Taylor series – term by term differentiation and integration – to obtain new examples of power series for which we know their sum. Suppose that a function f has a power series representation of the form:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x-c)^n$$

convergent on the interval $(c-R, c+R)$ for some R. The results we use in this example are:

- (Differentiation) Given f as above, $f'(x)$ has a power series expansion obtained by by differentiating each term in the expansion of $f(x)$:

$$f'(x) = a_1 + a_2(x-c) + 2a_3(x-c) + \cdots = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

- (Integration) Given f as above, $\int f(x)\,dx$ has a power series expansion obtained by by integrating each term in the expansion of $f(x)$:

$$\int f(x)\,dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

  for some constant C depending on the choice of antiderivative of f.

Questions:

1. Find a power series representation for the function $f(x) = \arctan(5x)$. (Note: $\arctan x$ is the inverse function to $\tan x$.)
2. Use power series to approximate

$$\int_0^1 \sin(x^2)\,dx$$

   (Note: $\sin(x^2)$ is a function whose antiderivative is not an elementary function.)

Solution:

For question (1), we know that $\arctan x$ has a simple derivative: $\dfrac{1}{1+x^2}$, which then has a power series representation similar to that of $\dfrac{1}{1-x}$, where we subsitute $-x^2$ for x. Hence:

$$\frac{d}{dx}\arctan(5x) = \frac{5}{1+25x^2} = 5\sum_{n=0}^{\infty}(-25x^2)^n = \sum_{n=0}^{\infty}(-1)^n 5^{2n+1} x^{2n},$$

where the second equality above follows from the familiar geometric series representation for $\dfrac{1}{1-x}$. The last equality presents a cleaner final form after straightforward algebraic simplification. Thus to obtain a power series expression for $\arctan x$ we may integrate this power series expression term by term. This gives:

$$\arctan(5x) = C + 5x - \frac{5^3}{3}x^3 + \cdots = C + \sum_{n=0}^{\infty}(-1)^n \frac{5^{2n+1}}{2n+1}x^{2n+1},$$

and we may solve for C by comparing both sides of the equality for any value of x. Choosing $x = 0$, we see that $\arctan(x) = 0$ and all non-constant terms of the power series are 0, hence $C = 0$ as well.

For question (2), we have seen that $sin(x)$ has a power series expansion:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Using a change of variable (replacing x by x^2 in the power series above), we have the power series expansion

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

Now taking the indefinite integral of both sides, we obtain a power series representation for the antiderivative of $\sin(x^2)$:

$$\int \sin(x^2)\,dx = \frac{1}{3}x^3 - \frac{1}{7}\frac{x^7}{3!} + \frac{1}{11}\frac{x^{10}}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3}\frac{x^{4n+3}}{(2n+1)!}.$$

The power series expression is valid for any real number x since the power series for $\sin(x)$, and hence $\sin(x^2)$ converged for all x.

To approximate the definite integral, we may use as many terms of the series as we like. For example, using only the first non-zero term would give:

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3}x^3 \Big|_{x=0}^{x=1} = \frac{1}{3}.$$

The first two non-zero terms gives:

$$\int_1^2 \sin(x^2)\,dx \approx \left(\frac{1}{3}x^3 - \frac{1}{7}\frac{x^7}{3!}\right)\Bigg|_{x=0}^{x=1} = \left(\frac{1}{3} - \frac{1}{42}\right) = \frac{13}{42}.$$

Using a numerical integration on a computer-algebra system, we find that the answer is approximately .31026... while $13/42 = .309524$. We can improve this estimate by using more terms in the power series.