CONTRIBUTIONS TO DIFFERENTIAL GEOMETRY OF SPACELIKE CURVES IN LORENTZIAN PLANE $\mathbb{L}^2$

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Abstract. In this work, first the differential equation characterizing position vector of spacelike curve is obtained in Lorentzian plane $\mathbb{L}^2$. Then the special curves mentioned above are studied in Lorentzian plane $\mathbb{L}^2$. Finally some characterizations of these special curves are given in $\mathbb{L}^2$.

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1. Introduction

The theory of degenerate submanifolds is recently treated by the researchers and so some of classical differential geometry topics are extended to Lorentzian manifolds. For instance in [13], author deeply studies theory of the curves and surfaces and also presents mathematical principles about theory of Relativitiy. Also, T. Ikawa presents some characterizations of the theory of curves in an indefinite-Riemannian manifold [7].

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There are lots of interesting and important problems in the theory of curves at differential geometry. One of the interesting problems is the problem of characterization of a regular curve in the theory of curves in the Euclidean and Minkowski spaces, see, [3, 6].

Special curves are obtained under some definitions such as Smarandache curves, spherical indicatrices, and curves of constant breadth, and etc.: Smarandache curves are regular curves whose position vectors are obtained by the Frenet frame vectors on another regular curve. These curves were firstly introduced by Turgut and Yılmaz in [14]. Then many researches occured about the different characterizations of Smarandache curves in Euclidean and Minkowsi spaces, see [1, 4]. Spherical indicatrix is the locus of a point whose position vector is equal to the unit tangent T, the principal normal vector N, and the principal binormal vector B at any point of a given curve in the space. Spherical indicatrices have been studied in many works as special curves [15, 16]. As you know, spherical indicatrix turns into circular indicatrix in the plane. Curves of constant breadth were firstly introduced by Euler as another special curve [5]. Then in chronological order, it was studied in [2, 10, 12, 17].

Our motivation was to see the corresponding results of the special curves mentioned above in Lorentzian plane $\mathbb{L}^2$. As much as we look at the classical differential geometry literature of the works in Lorentzian plane $\mathbb{L}^2$, the works were rare, see, [7, 9, 11, 18]. First, we obtain the differential equation characterizing position vector of spacelike curve in Lorentzian plane $\mathbb{L}^2$. Then we study the **special curves in** $\mathbb{L}^2$. We give some characterizations of these special spacelike curves in $\mathbb{L}^2$.

2. Preliminaries

Let $\mathbb{L}^2$ be the Lorentzian plane with metric

$$g = dx_1^2 - dx_2^2, \tag{1}$$

where x_1 and x_2 are rectangular coordinate system. A vector r of $\mathbb{L}^2$ is said to be spacelike if $g(r, r) > 0$, or $r = 0$, timelike if $g(r, r) < 0$ and null if $g(r, r) = 0$ for $r \neq 0$ [8].

A curve x is a smooth mapping

$$x : I \to \mathbb{L}^2,$$

from an open interval I onto $\mathbb{L}^2$. Let s be an arbitrary parameter of x, then we denote the orthogonal coordinate representation of x as $x=(x_1(s), x_2(s))$ and also the vector

$$\frac{dx}{ds} = (\frac{dx_1}{ds}, \frac{dx_2}{ds}) = T \tag{2}$$

is called the tangent vector field of the curve $x = x(s)$. If tangent vector field of $x(s)$ is a spacelike, timelike or null then, the curve $x(s)$ is called spacelike, timelike or null, respectively [7].

In the rest of the paper, we will consider spacelike curves. While the tangent vector field T is spacelike, N is timelike. We can have the arclength parameter s and have the Frenet formula

$$\begin{bmatrix} T' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \cdot \begin{bmatrix} T \\ N \end{bmatrix}, \tag{3}$$

where $\kappa = \kappa(s)$ is the curvature of the unit speed curve $x = x(s)$ [11]. The vector field N is called the normal vector field of the curve $x(s)$. Note that since $\langle T, N \rangle = 0$, T is a timelike vector, and also N spacelike vector. Given $\phi(s)$ is the slope angle of the curve, then as in [8], we have

$$\frac{d\phi}{ds} = \kappa(s). \tag{4}$$

3. Position vector of a curve in $\mathbb{L}^2$

Let $\alpha = \alpha(s)$ be an unit speed spacelike curve on the plane $\mathbb{L}^2$. Then we can write position vector of $\alpha(s)$ with respect to Frenet frame as

$$X = X(s) = \lambda_1 T + \lambda_2 N, \tag{5}$$

where λ_1 and λ_2 are arbitrary functions of the arc length parameter s. Differentiating (5) and using Frenet equations we have a system of ordinary differential equations as follows:

$$\begin{cases} \dfrac{d\lambda_1}{ds} - \lambda_2\kappa - 1 = 0, \\ \\ \dfrac{d\lambda_2}{ds} + \lambda_1\kappa = 0. \end{cases} \tag{6}$$

Using $(6)_1$ in $(6)_2$ we obtain

$$\frac{d}{ds}[\frac{1}{\kappa}(\frac{d\lambda_1}{ds}-1)]+\lambda_1\kappa=0. \tag{7}$$

So, according to λ_1, the equation (7) is the differential equation of second order which is a characterization for the curve $x = x(s)$.

Using change of variable

$$\theta=\int_0^s \kappa ds \tag{8}$$

in (7), we arrive at

$$-\frac{d^2\lambda_1}{d\theta^2}+\lambda_1=\frac{d\rho}{d\theta}, \tag{9}$$

where $\kappa=\frac{1}{\rho}$.

By the method of variation of parameters and differential solution of (9) we have

$$\lambda_1=e^{\theta}\left[A-\int_0^{\theta}\kappa e^{\theta}d\theta\right]+e^{-\theta}\left[B+\int_0^{\theta}\kappa e^{-\theta}d\theta\right], \tag{10}$$

where $A, B\in\mathbb{R}$. Rewriting the change of variable (9) into (10), we get

$$\lambda_1=e^{\int_0^{\theta}\kappa ds}\left[A-\int_0^{\theta}\kappa e^{\theta}d\theta\right]+e^{-\int_0^{\theta}\kappa ds}\left[B+\int_0^{\theta}\kappa e^{-\theta}d\theta\right]. \tag{11}$$

Denoting differentiation of the equation (11) as $\frac{d\lambda_1}{ds}=\xi(s)$, and using (6), then we have

$$\lambda_2=\frac{1}{\kappa}\left[\xi(s)-1\right]. \tag{12}$$

According to the above expression we can give the following theorem:

Theorem 3.1. Let $\alpha=\alpha(s)$ be an arbitrary unit speed spacelike curve in Lorentzian plane $\mathbb{L}^2$, then position vector of the curve $\alpha=\alpha(s)$ with respect to the Frenet frame can be composed by

$$X=X(s)=(e^{\theta}\left[A-\int_0^{\theta}\kappa e^{\theta}d\theta\right]+e^{-\theta}\left[B+\int_0^{\theta}\kappa e^{-\theta}d\theta\right])T$$

$$+(\frac{1}{\kappa}\left[\xi(s)-1\right])N.$$

Theorem 3.2. Let $\alpha = \alpha(s)$ be an arbitrary unit speed spacelike curve in Lorentzian plane $\mathbb{L}^2$. Position vector and curvature of the curve satisfy the differential equations of third order as follow

$$\frac{d}{ds}\left[\frac{1}{\kappa}\frac{d^2\alpha}{ds^2}\right] + \kappa\frac{d\alpha}{ds} = 0.$$

Proof. Let $\alpha = \alpha(s)$ be an arbitrary unit speed spacelike curve in Lorentzian plane $\mathbb{L}^2$. Then Frenet derivative formula holds $(3)_1$ in $(3)_2$, we easily have

$$\frac{d}{ds}\left[\frac{1}{\kappa}\frac{dT}{ds}\right] + \kappa T = 0. \tag{13}$$

Let $\dfrac{d\alpha}{ds} = T = \dot{\alpha}$. So, expression of (13) can be written as follows:

$$\frac{d}{ds}\left[\frac{1}{\kappa}\frac{d^2\alpha}{ds^2}\right] + \kappa\frac{d\alpha}{ds} = 0. \tag{14}$$

Hence, the proof is complete.

Let us solve the equation (13) with respect to T. Here we know ,

$$T = (t_1, t_2) = (\dot{\alpha}_1, \dot{\alpha}_2).$$

Using the change of variable (8) in (14) according to the parameter of T, we get

$$\frac{d^2 t_1}{d\theta^2} + \theta = 0, \quad \frac{d^2 t_2}{d\theta^2} + \theta = 0, \tag{15}$$

solving the equations in (15), we obtain

$$\begin{cases} t_1 = \psi_1 \cos\theta + \psi_2 \sin\theta, \\ \\ t_2 = \psi_3 \cos\theta + \psi_4 \sin\theta, \end{cases} \tag{16}$$

where $\psi_i \in \mathbb{R}$ for $1 \leq i \leq 4$.

4. Special curves in $\mathbb{L}^2$

In this section we will study some special spacelike curves such as Smarandache curves, circular indicatrices, and curves of constant breadth in Lorentzian plane $\mathbb{L}^2$.

4.1. **Smarandache curves.** A regular curve in Lorentzian plane $\mathbb{L}^2$ whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve. We will study $TN-$Smarandache curve as the only Smarandache curve of Lorentzian plane $\mathbb{L}^2$.

Definition 4.1 ($TN-$Smarandache curves). Let $\alpha = \alpha(s)$ be a unit speed spacelike curve in $\mathbb{L}^2$ and $\{T^\alpha, N^\alpha\}$ be its moving Frenet frame. The curve $\alpha = \alpha(s)$ is said to be $TN-$Smarandache curve whose form is

$$\beta(s^*) = \frac{-1}{\sqrt{2}}\left(T^\alpha - N^\alpha\right). \tag{17}$$

We can investigate Frenet invariants of $TN-$Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (17) with respect to s gives us

$$\dot{\beta} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = -\frac{1}{\sqrt{2}}\left(\kappa^\alpha N^\alpha + \kappa^\alpha T^\alpha\right). \tag{18}$$

Rearranging of this expression, we get

$$T_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(\kappa^\alpha N^\alpha + \kappa^\alpha T^\alpha\right). \tag{19}$$

By (18), we have

$$\frac{ds^*}{ds} = \kappa^\alpha. \tag{20}$$

Hence using (19) and (20), we find the tangent vector of the curve β as follows:

$$T_\beta = \frac{\left(T^\alpha + N^\alpha\right)}{\sqrt{2}}. \tag{21}$$

Differentiating (21) with respect to s, we have

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = -\frac{\left(\kappa^\alpha T^\alpha + \kappa^\alpha N^\alpha\right)}{\sqrt{2}}. \tag{22}$$

Substituting (20) in (22), we obtain

$$T_\beta^{'} = \frac{-\left(T^\alpha + N^\alpha\right)}{\sqrt{2}}.$$

The curvature and principal normal vector field of the curve β are, respectively,

$$\left\|T_\beta^{'}\right\| = \kappa_\beta = \sqrt{\frac{(T^\alpha)^2 - (N^\alpha)^2}{2}} = 0, \tag{23}$$

and

$$N_\beta = \frac{-\left(T^\alpha + N^\alpha\right)}{\sqrt{(T^\alpha)^2 - (N^\alpha)^2}} \to \infty. \tag{24}$$

Thus the expressions in (23) and (24) simply mean that the Smarandache curve of a spacelike plane curve is a straight line.

4.2. **Circular indicatrices.** Circular indicatrix is the locus of a point whose position vector is equal to the unit tangent T or the principal normal vector N at any point of a given curve in the plane. We will characterize tangent and normal circular indicatrices of spacelike curves in Lorentzian plane $\mathbb{L}^2$.

Tangent circular indicatrices of spacelike curves in $\mathbb{L}^2$

Let $\varepsilon = \varepsilon(s)$ be a spacelike curve in Lorentzian plane $\mathbb{L}^2$. If we translate of the first vector field of the Frenet frame to the center of the unit Lorentzian circle S^1, then we have the tangent circular indicatrix $\delta = \delta(s_\delta)$ in Lorentzian plane $\mathbb{L}^2$.

Here we shall denote differentiation according to s by a dash and differentiation according to s_δ by a dot. By the Frenet frame, we obtain the tangent vector of $\delta = \delta(s_\delta)$ as

$$\delta' = \frac{d\delta}{ds_\delta}\frac{ds_\delta}{ds} = T_\delta \frac{ds_\delta}{ds} = \kappa N, \tag{25}$$

where

$$T_\delta = N, \text{ and } \frac{ds_\delta}{ds} = \kappa(s). \tag{26}$$

From (25), we have

$$\dot{T_\delta} = \kappa' N - \kappa^2 T,$$

and also we arrive at

$$\kappa_\delta = \left\| \dot{T_\delta} \right\| = \sqrt{|(\kappa')^2 + \kappa^4|}. \tag{27}$$

Thus we have the principal normal of the curve $\delta = \delta(s_\delta)$ as

$$N_\delta = \frac{-\kappa^2 T + \kappa' N}{\sqrt{\kappa'^2 + \kappa^4}}. \tag{28}$$

Principal normal circular indicatrices of spacelike curves in $\mathbb{L}^2$

Let $\varepsilon = \varepsilon(s)$ be a spacelike curve in Lorentzian plane $\mathbb{L}^2$. If we translate of the second vector field of Frenet frame to the center of the unit circle S^1, then we have the principal normal circular indicatrix $\phi = \phi(s_\phi)$ in Lorentzian plane $\mathbb{L}^2$.

We shall follow similar procedure of the tangent circular indicatrix, to determine relations among Fenet apparatus of circular indicatrices of with Frenet apparatus of curve $\varepsilon = \varepsilon(s)$. The differentiation of the principal normal circular indicatrix is as follows

$$\phi' = \frac{d\phi}{ds_\phi}\frac{ds_\phi}{ds} = -\kappa T,$$

where

$$T_\phi = -T, \text{ and } \frac{ds_\phi}{ds} = \kappa(s). \tag{29}$$

Differentiating (29), we obtain

$$T_\phi' = \dot{T}_\phi \frac{ds_\phi}{ds} = -\kappa N, \tag{30}$$

or in another words,

$$\dot{T}_\phi = -N. \tag{31}$$

Using (30) and (31) we have the first curvature and the principal normal vector of the principal normal circular indicatrix $\phi = \phi(s_\phi)$ as

$$\kappa_\phi = \left\| \dot{T}_\phi \right\| = 1, \text{ and } N_\phi = -N.$$

4.3. **Curves of constant breadth.** Let $\varphi = \varphi(s)$ and $\varphi^* = \varphi^*(s)$ be simple closed spacelike curves in Lorentzian plane $\mathbb{L}^2$. These curves will be denoted by C and C^*. Any point p of the curve lying in the normal plane meets the curve at a single point q except p. We call the point q as the opposite point of p. We consider curves in the class Γ as in Fujivara (1914) having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve.

A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to the Frenet frame by the following

$$\varphi^* = \varphi + m_1 T + m_2 N, \tag{32}$$

where φ and φ^* are opposite points and $m_i(s), 1 \leq i \leq 2$ are arbitrary functions of s.

The vector

$$d = \varphi^* - \varphi$$

is called "the distance vector" between the opposite points of C and C^*.

Differentiating (32) and considering the Frenet derivative formulae (3), we have

$$\frac{d\varphi^*}{ds} = T^* \frac{ds^*}{ds} = T + \frac{dm_1}{ds} T + m_1 \kappa N + \frac{dm_2}{ds} N - m_2 \kappa T.$$

Since

$$\frac{d\varphi}{ds} = T \text{ and } \frac{d\varphi^*}{ds^*} = T^*,$$

and using Frenet derivative formulas, we get

$$T^* \frac{ds^*}{ds} = (1 + \frac{dm_1}{ds} - m_2\kappa)T + (m_1\kappa + \frac{dm_2}{ds})N. \tag{33}$$

From the definition of curve of constant breadth, there is the relation

$$T^* = -T, \tag{34}$$

between the tangent vectors of the curves, and also using (34) in (33), we obtain

$$\frac{ds^*}{ds} = m_2\kappa - \frac{dm_1}{ds} - 1, \text{ and } m_1\kappa + \frac{dm_2}{ds} = 0. \tag{35}$$

Let θ be the angle between the tangent vector T at a point $\alpha(s)$ of an oval and a fixed direction, then we have

$$\frac{ds}{d\theta} = \rho = \frac{1}{\kappa}, \text{ and } \frac{ds^*}{d\theta} = \rho^* = \frac{1}{\kappa^*}. \tag{36}$$

Using (36) in (35), the equation (35) becomes as

$$\begin{cases} m_2 - \dfrac{dm_1}{d\theta} = \rho + \rho^* = f(\theta), \\ \\ \dfrac{dm_2}{d\theta} = -m_1. \end{cases} \tag{37}$$

Eliminating m_1 in (37), we obtain the linear differential equation of the second order as

$$\frac{d^2m_2}{d\theta^2} + m_2 = f(\theta). \tag{38}$$

By general solution of the equation (38) we find

$$m_2 = \sin\theta\left(\int_0^\theta f(t)e^t dt\right) + \varepsilon_2 - \cos\theta\left(\int_0^\theta f(t)e^{-t}dt\right) + \varepsilon_1.$$

Using $m_1 = -\dfrac{dm_2}{d\theta}$ in (37) we obtain the value of m_1 as

$$m_1 = -\cos\theta\left(\int_0^\theta f(t)\cos t dt + \varepsilon_2\right) - \sin\theta\left(\int_0^\theta f(t)\sin t dt + \varepsilon_1\right).$$

Hence using (32) the position vector of the curve $\vec{\varphi}^*$ is given as follows

$$\varphi^* = \varphi + [-\cos\theta\left(\int_0^\theta f(t)\cos t dt + \varepsilon_2\right) - \sin\theta\left(\int_0^\theta f(t)\sin t dt + \varepsilon_1\right)]T$$

$$+[\sin\theta\left(\int_0^\theta f(t)e^t dt\right) + \varepsilon_2 - \cos\theta\left(\int_0^\theta f(t)e^{-t} dt\right) + \varepsilon_1]N.$$

If the distance between opposite points of C and C^* is constant, then we can write that

$$\|\vec{\varphi}^* - \vec{\varphi}\| = m_1^2 - m_2^2 = const., \tag{39}$$

and differentiating (39), then we have

$$m_1\frac{dm_1}{d\theta} - m_2\frac{dm_2}{d\theta} = 0. \tag{40}$$

Taking the system (37) and (40) together into consideration, we obtain

$$m_1\left(\frac{dm_1}{d\theta} + m_2\right) = 0. \tag{41}$$

From here, we arrive at

$$m_1 = 0 \text{ or } \frac{dm_1}{d\theta} = -m_2. \tag{42}$$

According to the expression (42), we will study cases below:

Case 1: If

$$m_1 = 0,$$

: then from (37) we find that

$$f(\theta) = const. \text{ and } m_2 = const.$$

If

$$m_1 \neq 0 = const. \text{ and also } \frac{dm_1}{d\theta} = -m_2,$$

then we obtain

$$m_2 = 0.$$

If

$$m_1 = k_1, (k_1 \in R),$$

then the equation (32) turns into

$$\varphi^* = \varphi + k_1 T.$$

Case 2: If

$$\frac{dm_1}{d\theta} = -m_2,$$

then from (37) we have

$$f(\theta) = 0, \text{ and } m_1 = -\int_0^\theta m_2 d\theta\ .$$

If

$$\frac{dm_1}{d\theta} = m_2 \neq 0 = k_2 = const.,$$

then from (37) we obtain

$$f(\theta) = 0, \text{ and } m_1 = 0$$

Hence the equation (32) becomes as follows:

$$\varphi^* = \varphi + k_2 N.$$

References

[1] Ahmad T. Ali, Special Smarandache Curves in the Euclidean Space, Math Combin. Book Ser., 2(2), 30-36, (2010).

[2] Wilhelm Blaschke, Konvexe Bereichee Gegebener Konstanter Breite und Kleinsten Inhalt, Math. Annalen, B., 76, 504-513, (1915).

[3] Bang-Yen Chen, When Does the Position Vector of a Space Curve Always Lie in its Rectifying Plane?, Amer. Math. Monthly, 110, 147-152, (2003).

[4] Muhammed Çetin, Yılmaz Tunçer and Mustafa Kemal Karacan, Smarandache Curves According to Bishop Frame in Euclidean 3-Space, Gen. Math. Notes, 20, 50-56, (2014).

[5] Leonhard Euler, De Curvis Triangularibus, Acta Acad. Petropol., 1778, 3-30, (1780).

[6] S. Izumiya, N. Takeuchi, New Special Curves and Developable Surfaces, Turkish J. Math., 28(2), 531-537, (2004).

[7] Toshihiko Ikawa, On Curves and Submanifolds in an Indefinite-Riemannian Manifold, Tsukuba J. Math., 9, 353-371, (1985).

[8] Toshihiko Ikawa, Euler-Savary's Formula on Minkowski Geometry, Balkan J. Geo. Appl., 8(2), 31-36, (2003).

[9] Mustafa Kemal Karacan, Bahaddin Bükçü, Parallel Curve (offset) in Lorentzian Plane, Erciyes Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 24 (1-2), 334- 345, (2008).

[10] Ömer Köse, On Space Curves of Constant Breadth, Doğa Tr. J. Math., 10(1), 11-14, (1986).

[11] Rafael Lopez, The Theorem of Schur in the Minkowski Plane, J. Geom and Phys., 61, 342–346, (2011).

[12] Abdullah Mağden, Ömer Köse, On the Curves of Constant Breadth in Space, Tr. J. of Mathematics, 21(3), 277-284, (1997).

[13] Barrett O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[14] Melih Turgut, Süha Yılmaz, Smarandache Curves in Minkowski Space-time, Int J. Math. Comb., 3, 51-55, (2008).

[15] Süha Yılmaz, Melih Turgut, A New Version of Bishop Frame and an Application to Spherical Images, J Math Anal Appl., 371, 764-776, (2010).

[16] Süha Yılmaz, Spherical Indicators of Curves and Characterizations of Some Special Curves in Four Dimensional Lorentzian Space L^2, PhD Dissertation, Dokuz Eylul University, 2001.

[17] Süha Yılmaz, Ümit Ziya Savcı, and Yasin Ünlütürk, Spacelike Curves of Constant Breadth in Minkowski 3-Space E_1^3, Math Combin Book Ser., 4, 1-6, (2014).

[18] Süha Yılmaz, Notes on the Curves in Loretzian Plane L^2, Int. J. Math. Comb., 1, 38-41, (2009).