Logic meets number theory in o-minimality

The work of PETERZIL, PILA, STARCHENKO, and WILKIE

Logic Colloquium, Vienna 2014

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...plus four more papers!

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1. A *variety* $V \subseteq \mathbb{C}^n$ is the zero set of finitely many polynomials $P_1, \ldots, P_m \in \mathbb{C}[X_1, \ldots, X_n]$:

   $$V = \left\{ x \in \mathbb{C}^n : P_1(x) = \cdots = P_m(x) = 0 \right\}.$$

   Hypersurface = zero set of a single polynomial.
2. The varieties form the closed sets of a topology on $\mathbb{C}^n$, the ZARISKI *topology.* Below: dense = ZARISKI-dense.
3. A variety V is *irreducible* if it is not the union of two varieties properly contained in V.
4. Every variety V is the union of a finite number of irreducible varieties; this decomposition is unique if one removes those subsets that are contained in another one, and the elements of this unique decomposition are called *irreducible components* of V.

Diophantine Geometry studies how the *geometric* features of a variety $V \subseteq \mathbb{C}^n$ interact with its *diophantine* properties. For example, given an "interesting" set $K \subseteq \mathbb{C}$, how does the geometry of V influence the structure of $V(K) := K^n \cap V$?

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If V is a *special* variety and $X \subseteq V$ is a variety which contains a dense set of *special* points, then X, too, has to be *special.* (Whatever "special" means.)

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PILA-ZANNIER found a general procedure using o-minimality to prove instances of the "general principle." This allowed PILA to give the first unconditional proof (not relying, e.g., on the RIEMANN Hypothesis) of ANDRÉ-OORT for the case $V = \mathbb{C}^n$.

Here is an archetypical example of the "general principle." Put

$$\mathbb{U} := \{z \in \mathbb{C} : z^n = 1 \text{ for some } n \geqslant 1\} \qquad \text{(roots of unity).}$$

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Theorem (Laurent, 1984)

Let $X \subseteq (\mathbb{C}^\times)^n$ be irreducible. If $X(\mathbb{U})$ is dense in X, then X is defined by equations

$$X_1^{\alpha_1} \cdots X_n^{\alpha_n} = b \qquad (\alpha_1, \ldots, \alpha_n \in \mathbb{Z},\ b \in \mathbb{U}).$$

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This is an instance of the Manin-Mumford Conjecture (= Raynaud's Theorem). The Pila-Zannier method (extended by Peterzil-Starchenko) gives (yet) another proof.

The main idea

- We have an analytic surjection

$$e\colon \mathbb{C}^n \to (\mathbb{C}^\times)^n, \qquad e(z_1, \dots, z_n) = (e^{2\pi i z_1}, \dots, e^{2\pi i z_n}).$$

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Then with $\widetilde{e} := e \restriction D$, we still have

$$\zeta \in \mathbb{U}^n \iff \zeta = \widetilde{e}(z) \text{ for some } z \in D \cap \mathbb{Q}^n.$$

The method of PILA-ZANNIER

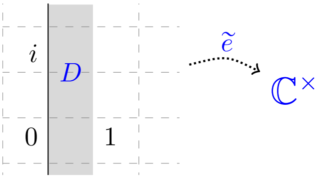
e is “logically” badly behaved (its kernel is $\mathbb{Z}^n$), but $\tilde{e}$ and thus

$$\tilde{X} := \tilde{e}^{-1}(X)$$

are definable in the o-minimal structure

$$\big(\mathbb{R}; <, 0, 1, +, \times, \exp, \sin\restriction[0, 2\pi]\big),$$

with $\tilde{X}(\mathbb{Q}) = \tilde{e}^{-1}\big(X(\mathbb{U})\big)$.



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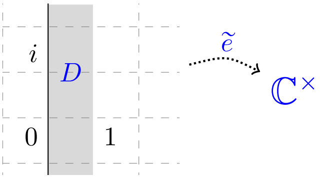

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(Definability in an o-minimal structure is obvious in this case, but by far non-obvious in many other applications of the PILA-ZANNIER method → PETERZIL-STARCHENKO.)

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Below, "definable" means "definable in $\boldsymbol{R}$, possibly with parameters." A map $f: S \to R^n$, where $S \subseteq R^m$, is called definable if its graph $\Gamma(f) \subseteq R^{m+n}$ is.

Definition

$$\boldsymbol{R} \text{ is } \textbf{o-minimal} \quad :\Longleftrightarrow \left\{ \begin{array}{l} \text{all definable subsets of } \mathbb{R} \text{ are finite} \\ \text{unions of singletons and (open)} \\ \text{intervals} \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{l} \text{all definable subsets of } \mathbb{R} \text{ have} \\ \text{finitely many connected compo-} \\ \text{nents} \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{l} \text{the definable subsets of } \mathbb{R} \text{ are} \\ \text{those that are already definable in} \\ \text{the reduct } (\mathbb{R}; <) \text{ of } \boldsymbol{R} \end{array} \right.$$

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Although the o-minimality axiom only refers to definable subsets of $\mathbb{R}$, it implies finiteness properties for the definable subsets of $\mathbb{R}^n$ for arbitrary $n \geqslant 1$.

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  - *piecewise differentiable* up to some fixed finite order;
  - the *finite* fibers have uniformly bounded cardinality.
- *Dimension* of definable sets is very well-behaved, e.g., invariant under definable bijections: no space-filling curves.

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Example

If $S \subseteq \mathbb{R}^n$ is definable, then so is its closure

$$\mathrm{cl}(S) = \big\{x \in \mathbb{R}^n : \forall \varepsilon > 0\, \exists y \in S : \,|x - y| < \varepsilon\big\}.$$

(Here we assume that $\boldsymbol{R}$ expands $(\mathbb{R}; <, 0, 1, +, \times)$.)

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Each o-minimal structure $\boldsymbol{R}$ gives rise to a self-contained universe for a kind of “tame topology” (no pathologies) as envisaged by GROTHENDIECK (1980s).

O-minimal structures: examples

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TARSKI, 1940s

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van den Dries, 1986

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WILKIE, 1991

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O-minimal structures: examples

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semialgebraic sets

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The citation for the Karp Prize 2014 mentions ...

- J. PILA, O-minimality and the André-Oort conjecture for $\mathbb{C}^n$
  Ann. of Math. **173** (2011), 1779–1840.

- J. PILA and A. J. WILKIE, The rational points of a definable set
  Duke Math. J. **133** (2006), 591–616.

- Y. PETERZIL and S. STARCHENKO, Uniform definability of the Weierstrass $\wp$-functions and generalized tori of dimension one
  Selecta Math. (N.S.) **10** (2004), 525–550.

- ______, Definability of restricted theta functions and families of abelian varieties
  Duke Math. J. **162** (2013), 731–765.

...plus four more papers!

O-minimal structures: diophantine properties

Fix an o-minimal expansion $\boldsymbol{R} = (\mathbb{R}; <, 0, 1, +, \times, \dots)$ of $\mathbb{R}_{\text{alg}}$.

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These developments culminated in the theorem of PILA-WILKIE (2006):

Definable sets which are sufficiently “transcendental” contain few rational points.

O-minimal structures: diophantine properties

Notation

Given non-zero coprime $a, b \in \mathbb{Z}$ define the **height** of $x = \frac{a}{b}$ by $H(x) := \max\{|a|, |b|\}$, and set $H(0) := 0$.

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Given $X \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}$, put

$$X(\mathbb{Q}, t) \ := \ \left\{x \in X \cap \mathbb{Q}^n : H(x) \leqslant t\right\} \quad \text{(a finite set).}$$

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We'd like to understand the asymptotic behavior of $|X(\mathbb{Q}, t)|$.

Example

$$\left\{\begin{array}{l} X = \Gamma(P) \text{ where } P\colon \mathbb{R}^{n-1} \to \mathbb{R} \\ \text{is a polynomial function with in-} \\ \text{teger coefficients of degree } d \end{array}\right\} \Rightarrow |X(\mathbb{Q}, t)| \sim C t^{2(n-1)/d}$$

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When does $|X(\mathbb{Q}, t)|$ grow sub-polynomially as $t \to \infty$?

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When does $|X(\mathbb{Q},t)|$ grow sub-polynomially as $t \to \infty$?

Given $X \subseteq \mathbb{R}^n$ we let

$$X^{\mathrm{alg}} := \left\{\begin{array}{l} \text{union of all infinite connected semial-} \\ \text{gebraic subsets of } X \end{array}\right\}$$ *algebraic part of X*

$$X^{\mathrm{trans}} := X \setminus X^{\mathrm{alg}}$$ *transcendental part of X.*

(A caveat: even if X is definable, then X^{alg} in general is not.)

Theorem (Pila-Wilkie, 2006)

Let $X \subseteq \mathbb{R}^n$ be definable. Then for each $\varepsilon > 0$ there is some $t_0 = t_0(\varepsilon)$ such that

$$|X^{\mathrm{trans}}(\mathbb{Q}, t)| \leqslant t^{\varepsilon} \qquad \text{for all } t \geqslant t_0.$$

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Remark

- The theorem continues to hold if given $d \geqslant 1$, we replace

$$\begin{aligned} \mathbb{Q} &\rightsquigarrow \text{set of algebraic numbers of degree} \leqslant d \\ H &\rightsquigarrow \text{a suitable height function on } \mathbb{Q}^{\mathrm{alg}}. \end{aligned}$$

(Pila, 2009)

Two crucial ingredients in the proof:

- a parametrization theorem for bounded definable sets by maps with *bounded* derivatives (generalizing YOMDIN and GROMOV);
- a result about covering the rational points of such parametrized sets by few algebraic hypersurfaces (no definability assumptions here).

O-minimal structures: diophantine properties

Theorem I (parametrization)

Let $X \subseteq (0,1)^n$ be definable, non-empty, and $p \in \mathbb{N}$. There is a finite set Φ of definable maps $\phi\colon (0,1)^{\dim X} \to (0,1)^n$ such that

- the union of the images of the $\phi \in \Phi$ equals X;
- each $\phi \in \Phi$ is C^p with $||\phi^{(\alpha)}|| \leqslant 1$ for $|\alpha| \leqslant p$.

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Theorem II (covering by hypersurfaces)

Let $m < n$ and d be given. Then there are $p \in \mathbb{N}$ and $\varepsilon, C > 0$ with the following properties:

- if $\phi\colon (0,1)^m \to \mathbb{R}^n$ is C^p with image X such that $||\phi^{(\alpha)}|| \leqslant 1$ for $|\alpha| \leqslant p$, then for each t, $X(\mathbb{Q}, t)$ is contained in the union of Ct^{ε} hypersurfaces of degree d;
- $\varepsilon = \varepsilon(m,n,d) \to 0$ as $d \to \infty$.

Toy case of the PILA-WILKIE Theorem:

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Let $\varepsilon > 0$ be given. Choose d so that $\varepsilon(1,2,d) \leqslant \varepsilon$, and then choose C, p as in Theorem II. Let Φ be a parametrization of X as in Theorem I. Then $X(\mathbb{Q},t)$ is contained in the union of $C_1 t^{\varepsilon}$ hypersurfaces of degree d, where $C_1 := C \cdot |\Phi|$.

O-minimal structures: diophantine properties

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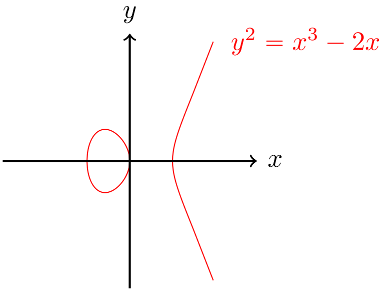
The hypersurfaces H of degree d come in a definable (in fact, semialgebraic) family. So for each such H, either $X \cap H$ is infinite (and then $X \cap H \subseteq X^{\mathrm{alg}}$) or finite of uniformly bounded size (by o-minimality). This allows us to count $X^{\mathrm{trans}}(\mathbb{Q},t)$. $\square$

Elliptic curves: viewed algebraically

PILA's Theorem deals with *elliptic curves*:

$$y^2 = x^3 + ax + b \qquad \text{where } a, b \in \mathbb{C},\ 4a^3 \neq -27b^2.$$



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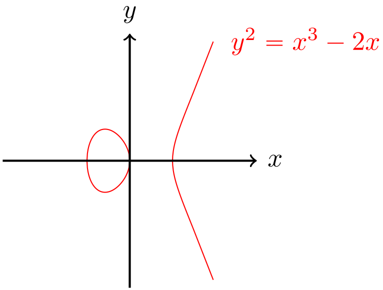

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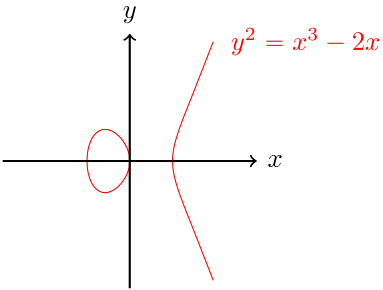

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It is natural to add a "point 0 at infinity":

$$E = \left\{(x, y) \in \mathbb{C}^2 : y^2 = x^3 + ax + b\right\} \cup \{0\}$$

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Let E be an elliptic curve. Then E can be made into an abelian group with 0 as its identity element.

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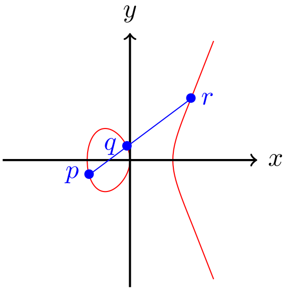

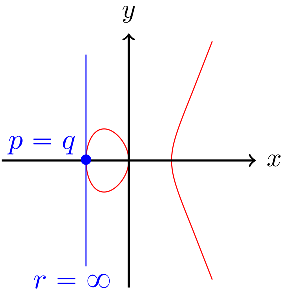

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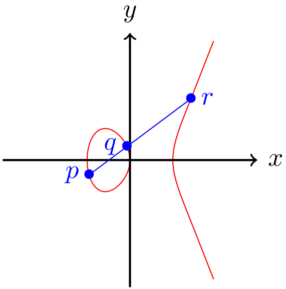

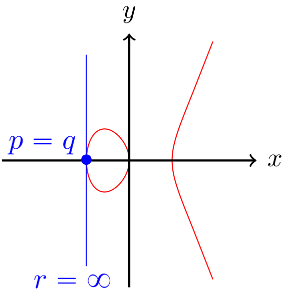

This makes E into an *algebraic* group (the group operations are given by rational functions of the coordinates).

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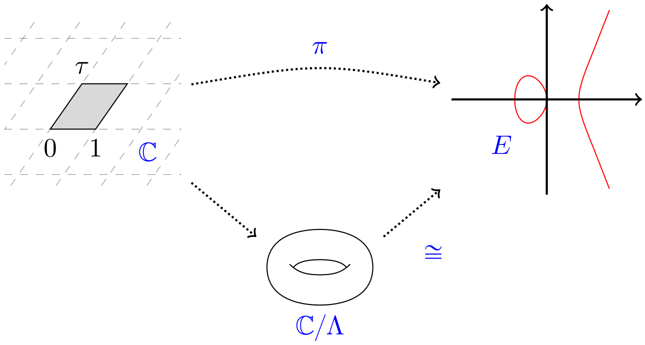

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The j-invariant (19th century: Klein ...)

There exists a holomorphic surjection $j\colon \mathbb{H} \to \mathbb{C}$ such that

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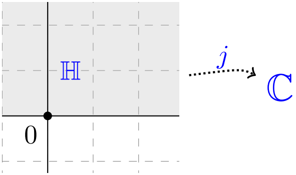
(So the “correct” parameter space is $\mathbb{C} \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$.)

PILA's Theorem

This is an analogue of LAURENT's Theorem with



$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$$
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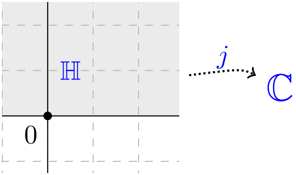

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Theorem (PILA, 2011)

Let $X \subseteq \mathbb{C}^n$ be an irreducible variety. If X contains a dense set of special points, then X is special.

Of course, we need to define what "special" should mean.

Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ ($\tau \in \mathbb{H}$).

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Why are these points special?

One possible explanation from algebraic number theory:

- A finite field extension K of $\mathbb{Q}$ has abelian GALOIS group $\Longleftrightarrow K \subseteq \mathbb{Q}(\zeta)$ for some $\zeta \in \mathbb{U}$. (KRONECKER-WEBER)
- If E_τ has CM, all abelian extensions of $\mathbb{Q}(\tau)$ can similarly be constructed from $\mathbb{U}$, $j(\tau)$, and torsion points of E_τ.

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While e is a group morphism, j is not. ($\mathbb{H}$ is not even a group!) Still, there are algebraic relations between $j(\tau)$ and $j(n\tau)$:

The nth classical modular polynomial ($n \in \mathbb{N}^{\geqslant 1}$)

There is an irreducible $\Phi_n \in \mathbb{Z}[X, Y]$ such that for $x, y \in \mathbb{C}$:

$$\Phi_n(x, y) = 0 \quad \iff \quad x = j(\tau),\ y = j(n\tau) \text{ for some } \tau \in \mathbb{H}.$$

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For example,

$$\begin{aligned}\Phi_2 = X^3 - X^2Y^2 + 1488X^2Y - 162000X^2 + 1488XY^2 + \\ 40773375XY + 8748000000X + Y^3 - 162000Y^2 + \\ 8748000000Y - 157464000000000\end{aligned}$$

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Definition

A variety $V \subseteq \mathbb{C}^n$ is *special* if it is an irreducible component of a variety defined by equations

$$\Phi_n(x_i, x_j) = 0 \quad \text{and} \quad x_i = a \text{ where } a \in \mathbb{C} \text{ is special.}$$

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Further developments

Many recent results employ and further develop ingredients from this circle of ideas:

- in number theory: ULLMOV-YAFAEV, KLINGER-ULLMO-YAFAEV, PILA-TSIMERMAN, MASSER-ZANNIER, HABEGGER-PILA ...
- in logic: PETERZIL-STARCHENKO, FREITAG-SCANLON, BIANCONI, THOMAS, JONES-THOMAS, JONES-THOMAS-WILKIE, BAYS-KIRBY-WILKIE, CLUCKERS-COMTE-LOESER, ...

> *"It is now widely accepted that a new method has emerged in this subject."*

If you want to know more

SCANLON's excellent surveys:

- *O-minimality as an approach to the André-Oort conjecture*, Panor. Synth., to appear.
- *Counting special points: logic, Diophantine geometry, and transcendence theory,* Bull. Amer. Math. Soc. **49** (2012), 51–71.
- *A proof of the André-Oort conjecture via mathematical logic* [*after Pila, Wilkie and Zannier*], Séminaire Bourbaki: Vol. 2010/2011, Astérisque **348** (2012), 299–315.