SMOOTH INFINITESIMAL ANALYSIS BASED MODEL OF MULTIDIMENSIONAL GEOMETRY

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Abstract. In this work a new approach to multidimensional geometry based on smooth infinitesimal analysis (SIA) is proposed. An embedded surface in this multidimensional geometry will look different for the external and internal observers: from the outside it will look like a composition of infinitesimal segments, while from the inside like a set of points equipped by a metric. The geometry is elastic. Embedded surfaces possess dual metric: internal and external. They can change their form in the bulk without changing the internal metric.

INTRODUCTION

Traditionally, under the term "space" we imply a set of zero-size points on which a metric can be defined. Under the term "dimensionality" we understand a minimal number of real numbers needed to describe this set unequally. An elementary geometrical object "point" itself has no dimensionality and is the same for all dimensions [1]. This understanding of dimensions tells nothing about their true nature. Why do compositions of points have different number of dimensions?

Einstein's full theory of space-time, called General Relativity can be extended easily to higher space dimensions. This fact is a good argument in favor of the multidimensional science concept. Modern physics is not truly multidimensional – we don't know how universes of different dimensionalities (I mean here the number of large dimensions) and different physical parameters can be embedded one into another. In order to create multidimensional physics we firstly should create multidimensional geometry [2-4].

This will be done in the next section.

DUAL METRIC MODEL OF MULTIDIMENSIONAL GEOMETRY

The concept of multidimensional geometry itself has a dualistic meaning: each surface may be embedded into a higher dimensional bulk and at the same time it may contain lower dimensional surfaces embedded in it. Multidimensional geometry is tightly connected with the basic rules of human perception and depends on how we explain the terms "dimension", "embeddance" and "space".

In this work a new approach to multidimensional geometry based on smooth infinitesimal analysis (SIA) is proposed. An embedded surface must be considered from two sighting points, namely, for internal and external observers. For the internal observer we have a picture we are used to (for example, the space-time we are living in), but for the external observer the picture is quite different (when we try to imagine a 2-dimensional surface we act like external observers).

According to this approach n-dimensional spaces and surfaces are composed of n-dimensional elementary objects "point-connections." The number of dimensions of a manifold depends on how its points are connected. So, an n-dimensional object "point-connection" has a dual nature: in addition to being a point of a manifold, it plays a role of connection within a certain set of points of a manifold. In other words, an n-dimensional "point-connection" has two elements: first – a "point" to be connected, and second – a "connection" which connects the "points."

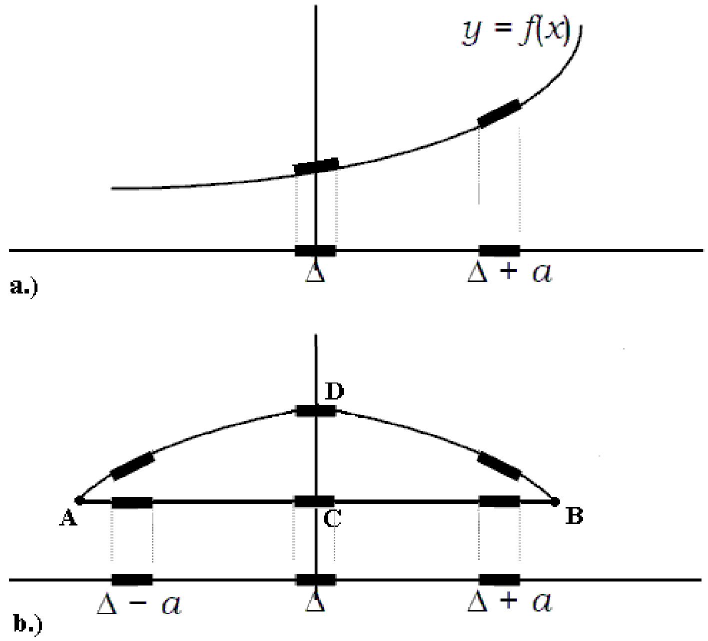

FIGURE 1. Infinitesimal segments and their contribution into the length of a curve.

Smooth infinitesimal analysis is a mathematically rigorous reformulation of the calculus in terms of infinitesimals. It views all functions as being continuous and incapable of being expressed in terms of discrete entities [5]. The *nilsquare* or *nilpotent* infinitesimals are numbers ε where $\varepsilon^2 = 0$ is true, but $\varepsilon = 0$ need not be true at the same time. In SIA every function whose domain is R, the real numbers extended by infinitesimals, is continuous and infinitely differentiable. Intuitively, smooth infinitesimal analysis can be interpreted as describing a world in which lines are made out of infinitesimally small segments, not out of points. These segments can be thought of as being long enough to have a definite direction, but not long enough to be curved.

The standard point of view postulates that lines are made of points. This point of view and SIA are complementary and give us a basis for a new multidimensional geometry: each manifold in this geometry will look different from the points of view of external and internal observers. From the point of view of an external observer it will be a set of infinitesimal segments and from the inside – a set of points equipped by a metric.

Another interesting feature of SIA is its **elasticity**: different segments make different contributions into the length of a curve, depending on the angle between a segment and the OX axis. The curve ADB may be considered as a result of stretching of the curve ACB. Infinitesimal segments have no length but they may be stretched (See Fig. 1).

An infinitesimal segment cannot be considered as a separate entity, it can exist only as a part of the line: we'll call it a **connection**. One and only one point of the manifold will correspond to each infinitesimal segment. But there may be an infinite number of connections passing through the selected point. From the point of view of the external observer each manifold may be represented as a set of connections which connect the points of the manifold. A holistic manifold (our Universe-like) will be composed from holistic elements – "point-connections". Proceeding from general considerations, we will use closed connections, because they are suitable for both finite and infinite manifolds. In the case of isotropic and continuous manifolds connections will have spherical form.

Speaking formally, for any set of points X we can define a function $c: X \times X \times X \rightarrow \{0,1\}$ such that $c(x_0,x,y)=1$ means that points x, y are connected by the connection corresponding to the point x_0. In this case we will call the points x, y *directly* *connected.* We say that points x, y are not directly connected if $c(x_0,x,y)=0$ for $\forall x_0 \in X$. We call two points x, y *indirectly connected* if there exists a succession of points $\{x_1, x_2, \ldots, x_{n+1}\} \in X$ such that every two subsequent points are directly connected and $x_1=x$, $x_{n+1}=y$. It is supposed that $c(x_0,x,y)=c(x_0,y,x)$ and $c(x_0,x,x)=c(x, x_0,x_0)$ for $\forall x_0,x,y \in X$ (symmetry) and each pair of points x, y $\in X$ may be connected (connectivity). We can define a metric $\rho(x,y)$ on X as a number n, where (n+1) is a minimal number of connections needed to connect points x, y:

$$\rho(x, y) = \min n, \text{ where } x_1=x,\ x_{n+2}=y \qquad (1)$$

If we have two sets X, Y and $Y \subset X$, where Y is a subset of X, and a function c_{int}: $Y \times Y \times Y \to \{0,1\}$ describes the structure of "point-connections" of Y, when for $\forall x,y \in Y$ we can define two metrics: internal $\rho_{int}(x, y)$, derived from c_{int} and external $\rho_{ext}(x, y)$ which depends on $c(x,x,x)$.

We see that structures composed of holistic elements – "point-connections" have a metric embedded in them: from the inside the metric has a discrete character and it will be continuous from the outside.

Figure 2 shows 1, 2, 3 – dimensional point-connections and how they form 1, 2, 3 - dimensional spaces (each space is shown as a discrete set of points only for clarity; the model implies a continuous set of points). We can see that a 1-dimensional point-connection is a combination of a point and a 0-dimensional connection – two infinitesimal segments; a 2-dimensional point-connection is a combination of a point and a 1-dimensional connection – points connected by it form a circle which, in turn, can be decomposed into 0-dimensional connections; a 3-dimensional point-connection is a combination of a point and a 2-dimensional connection – points connected by it form a sphere which can be decomposed into 1-dimensional connections. By analogy, an n-dimensional point-connection is a combination of a point and an (n-1)-dimensional connection – points connected by it form an (n-1)-dimensional sphere which can be decomposed into a set of (n-2)-dimensional connections.

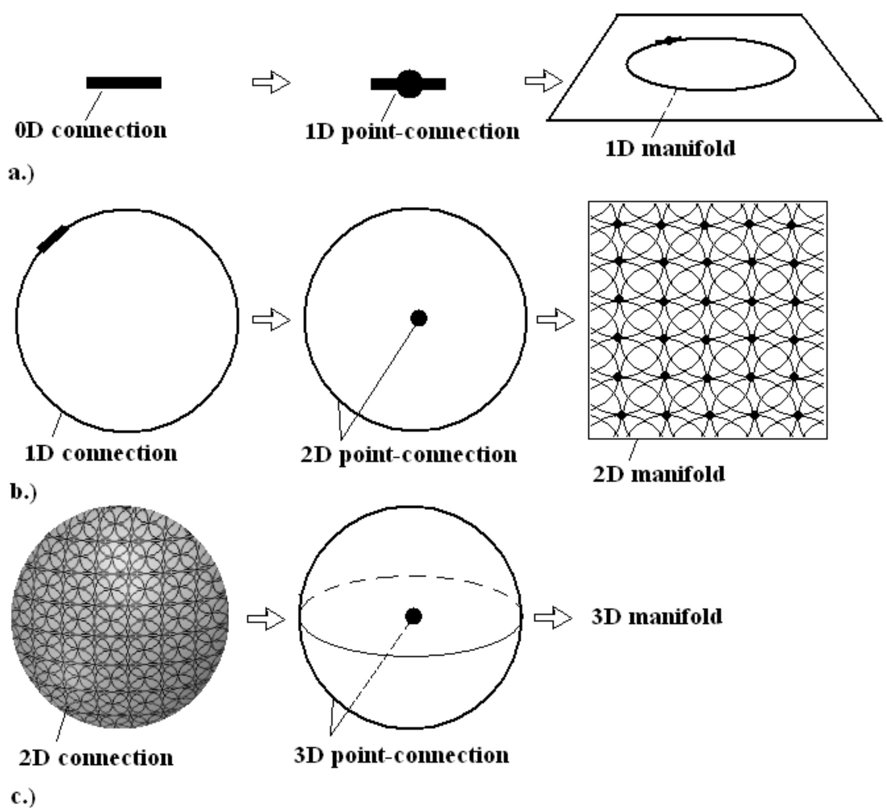

FIGURE 2. Structure of 1, 2, 3 - dimensional point-connections and spaces.

We see that each connection itself may have an internal structure: it is also to decompose into subconnections. The process of decomposition of a manifold into subconnections will stop when all subconnections will consist only of 0-dimensional connections. Under dimensionality of a manifold we will understand the number of different levels of subconnections encountered during the process of decomposition including the first (connections themselves) and the last (0-dimensional subconnections) levels.

Let's consider a simple case: a two-dimensional plane imbedded into Euclidean space R^3. We can apply to this plane two different transformations which don't differ from the traditional point of view:

1. Expansion: increases k times the distance between every two points on the plane, but radii of the connections stay the same;
2. Stretching: proportionally increases both the distance and radii of the connections (from the sighting point of the external observer).

We can see that the points connected before deformation stay connected after stretching, but they may become unconnected after expansion (See Fig. 3). An example of an expansion is what happens in our Universe after the Big Bang. The standard Big Bang cosmology assumes that the Universe began expanding from the state that was very hot, very small, and very highly curved. This inflationary model agrees very well with observations. Stretching is not observable for the internal observer because it doesn't change the structure of the embedded surface: the internal metric $\rho_{int}(x,y)$ defined by the equation (1) doesn't change after stretching. But the external one does; $\rho_{ext}(x,y) \approx k \cdot \rho_{int}(x,y)$ after stretching, because the external metric depends on the different set of connections-the connections of the bulk. After expansion $\rho_{int}(x,y)=\rho_{ext}(x,y)$.

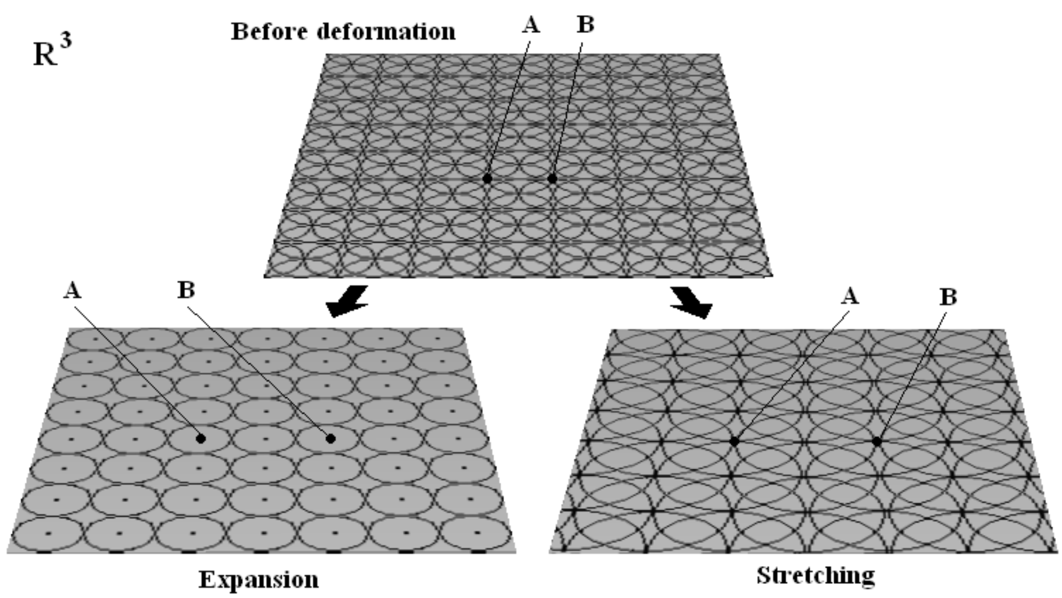

FIGURE 3. Expansion and stretching of a two-dimensional plane.

Clearly any multiple of an infinitesimal is also an infinitesimal. For set Δ of infinitesimals and any positive real number k we can define a function f: $\Delta \rightarrow \Delta$ as $f(x)=k*x$. This function translates Δ into Δ, but $f(x) \neq x$ if $x \neq 0$ and $k \neq 1$. We'll call this function stretching of Δ with parameter k.

The embedded surface may be treated in two ways: as a subset of the points of the bulk or as a set of lower dimensional connections added to the bulk. During expansion and stretching points of the embedded manifold change their positions the same way, but the connections don't.

As was shown, n-dimensional connections (n>1) are composed of 1-dimensional connections – circles. These circles according to SIA are made of infinitesimally small straight lines. Stretching of such a circle may be understood as stretching of these infinitesimal segments (1-dimensional connections introduced earlier are the analogs of those segments). The internal observer doesn't feel the deformations if they act like stretching or compressing because any set Δ of infinitesimals the infinitesimal segments are associated with will be translated into itself. But stretching of Δ changes the its relative position in respect to the bulk.

The proposed model shows that the classical approach based on the notion of a limit and smooth infinitesimal analysis are rather complementary than conflicting: if the former describes the world from the sighting point of the internal observer, the latter is more suitable when investigating an embedded surfaces from the sighting point of the external observer.

Obviously we will have two different metrics for the embedded surface - internal and external. In the context of the proposed geometrical model an embedded surface may change its form in the bulk, undergo vibrations, but its internal structure stays unaware of these changes if they act as stretching or compressing. In other words, such geometry is an elastic one. It is just like inflating a balloon with a pattern on it: during the inflation process everything grows bigger and bigger, but when the air is out everything is restored.

The same considerations are equally applicable to any smooth (n-1)-dimensional (n>1) surface embedded into an n-dimensional Euclidean space R^n and any diffeomorphic transformation φ of the surface. For any smooth curve $\gamma(t)$ in the embedded manifold from A to B with $\gamma(0)$=A and $\gamma(1)$=B the following formulas are valid.

Before deformation (the internal and external metrics are the same):

$$l_{int} = l_{ext} = \int_0^1 |\gamma'(t)|\, dt = l_0 , \qquad (2)$$

where l_0 is the length of the curve before deformation, l_{ext} is the length of the curve measured from R^n, l_{int} is the length of the curve measured from the embedded manifold taking into account its Riemannian metric.

For any diffeomorphic transformation φ of the surface, we can define k (t) –the coefficient of transformation of the embedded manifold along the curve $\gamma(t)$

$$\text{k (t)} = \lim_{t' \to t} l_{ext}(\tilde{\gamma}(t), \ \tilde{\gamma}(t'))/\, l_{ext}(\gamma(t), \gamma(t')) = |\ \tilde{\gamma}'(t))| \,/\, |\gamma'(t)|, \qquad (3)$$

where $\tilde{\gamma}(t)$ is the curve $\gamma(t)$ after deformation and t, $t' \in [0,1]$.

After expansion:

$$l_{int} = l_{ext} = \int_0^1 |\gamma_{\exp}'(t)|\, dt = \int_0^1 k(t) \cdot |\gamma'(t)|\, dt . \qquad (4)$$

After stretching (the internal structure of the embedded manifold stays the same):

$$l_{ext} = \int_0^1 |\gamma_{\exp}'(t)|\, dt = \int_0^1 k(t)\cdot|\gamma'(t)|\, dt\,, \qquad (5)$$

$$l_{int} = \int_0^1 (1/k(t))\cdot|\gamma_{str}'(t)|\, dt = \int_0^1 |\gamma'(t)|\, dt = l_0, \qquad (6)$$

where $k(t)=1+k_{str}(t)$, $k_{str}(t)$ – the coefficient of stretching, $\gamma_{exp}(t)$ - $\gamma(t)$ after expansion, $\gamma_{str}(t)$ - $\gamma(t)$ after stretching.

From formulas (2) - (6) we see the relation between the internal and external metrics after expansion and stretching. If an applied transformation is a combination of expansion and stretching, the task becomes more complicated and needs a more complex solution.

CONCLUSION

The new model of multidimensional geometry based on smooth infinitesimal analysis has been proposed. The proposed geometry has four features, which distinguish it from the existing geometries:

a. It is holistic. Space is represented as interweaving of connections; each point exists only in the context of the background space, which may be understood as indivisible whole just like our Universe is.

b. It is really multidimensional. Point-connections of different dimensionality have different topology.

c. It is elastic. Embedded surfaces possess dual metric: internal and external. They can change their form in the bulk without changing the internal metric;

d. Structures composed of holistic elements – "point-connections" have a metric embedded in them: from the inside the metric has a discrete character and it will be continuous from the outside.

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