THE REMEZ ALGORITHM

This section describes how to design linear-phase FIR filters based on the Chebyshev (or minimax) error criterion. The minimization of the Chebyshev norm is useful because it permits the user to explicitly specify band-edges and relative error sizes in each band. We will see that linear-phase FIR filters that minimize a Chebyshev error criterion can be found with the *Remez* algorithm or by linear programming techniques. Both these methods are iterative numerical algorithms and can be used for very general functions $D(\omega)$ and $W(\omega)$ (although many implementations work only for piecewise linear functions). The Remez algorithm is not as general as the linear programming approach, but it is very robust, converges very rapidly to the optimal solution, and is widely used.

Parks and McClellan proposed the use of the Remez algorithm for FIR filter design and made programs available [5, 6, 9, 15]. Many texts describe the Parks-McClellan (PM) algorithm in detail [7, 8, 11, 14].

The Remez algorithm can be use to design all four types of linear-phase filters (I,II,III,IV), but for convenience only the design of type I filters will be described here. Note that the weighted error function is given by

$$E(\omega) = W(\omega)\ (A(\omega) - D(\omega)). \tag{1}$$

The amplitude response of a type I FIR filter is given by

$$A(\omega) = \sum_{n=0}^{M} a(n)\cos(n\omega). \tag{2}$$

The Chebyshev design problem can formulated as follows. Given:

N : filter length

$D(\omega)$: desired (real-valued) amplitude function

$W(\omega)$: nonnegative weighting function,

find the linear-phase filter that minimizes the weighted Chebyshev error, defined by

$$\boxed{||E(\omega)||_\infty = \max_{\omega \in [0,\pi]} |W(\omega)\ (A(\omega) - D(\omega))|.} \qquad (3)$$

The solution to this problem is called the best weighted Chebyshev approximation to $D(\omega)$. Because it minimizes the maximum value of the error, it is also called the *minimax* solution.

The Remez algorithm for computing the best Chebyshev solution uses the *alternation theorem*. This theorem characterizes the best Chebyshev solution.

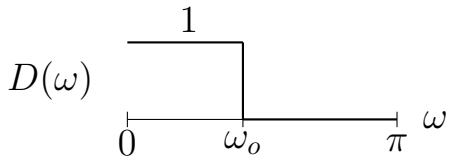
For low-pass filter design via the PM algorithm, the functions $D(\omega)$ and $W(\omega)$ are usually defined as

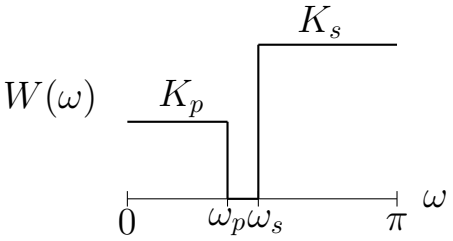
$$D(\omega) = \begin{cases} 1 & 0 < \omega < \omega_o \qquad \text{(pass-band)} \\ 0 & \omega_o < \omega < \pi \qquad \text{(stop-band)} \end{cases} \tag{4}$$

and

$$W(\omega) = \begin{cases} K_p & 0 \leq \omega \leq \omega_p \\ 0 & \omega_p < \omega < \omega_s \\ K_s & \omega_s \leq \omega \leq \pi \end{cases} \tag{5}$$

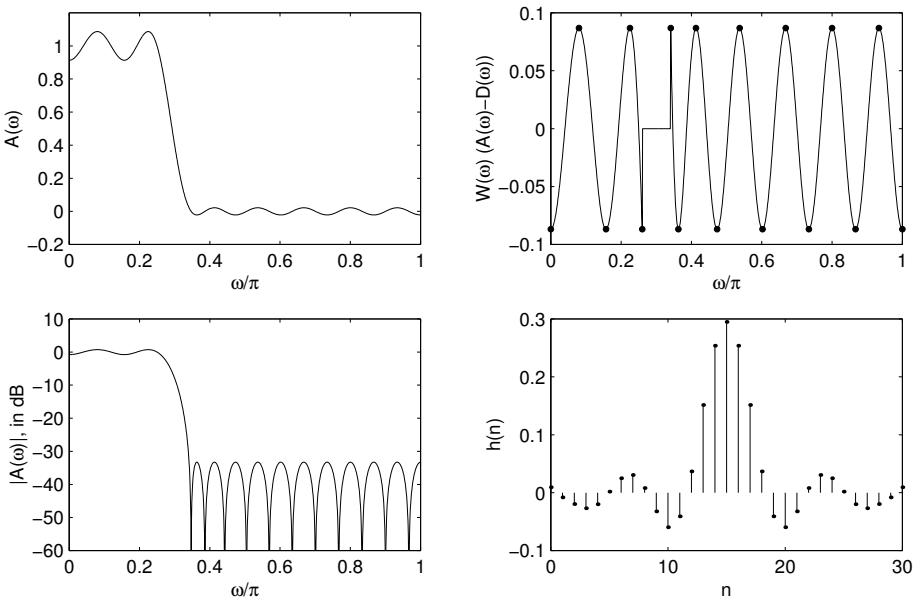
where $0 < \omega_p < \omega_o < \omega_s < \pi$.





For low-pass filters so obtained, the deviations δ_p and δ_s satisfy the relation $\delta_p/\delta_s = K_s/K_p$. For example, consider the design of a symmetric low-pass filter of length $N = 31$, with $K_p = 1$, $K_s = 4$ and $\omega_p = 0.26\pi$, $\omega_s = 0.34\pi$. The best Chebyshev solution and its weighted error function are illustrated in the figure. The

maximum errors in the pass-band and stop-band are $\delta_p = 0.0892$ and $\delta_s = 0.0223$ respectively. The circular marks in the figure indicate the extremal points of the alternation theorem.



The alternation theorem states that $|E(\omega)|$ attains its maximum value at a minimum of $M+2$ points — and that the weighted error function alternates sign on at least $M+2$ of those points. For convenience, we will define R to be $M+2$.

$$\boxed{R := M+2} \tag{6}$$

The alternation can be stated specifically as follows.
If $A(\omega)$ is given by (2), then a necessary and sufficient condition that $A(\omega)$ be the unique minimizer of (3) is that there exist at least $R := M+2$ extremal points $\omega_1, \ldots, \omega_R$ (in order: $0 \leq \omega_1 < \omega_2 < \cdots < \omega_R \leq \pi$), such that

$$\boxed{E(\omega_i) = c \cdot (-1)^i \, ||E(\omega)||_\infty \quad \text{for} \quad i = 1, \ldots, R} \tag{7}$$

where c is either 1 or -1.
Consequently, the weighted error functions of best Chebyshev solutions exhibit an *equiripple* behavior.

To understand the Remez exchange algorithm, first note that (7) can be written as follows. Let δ represents $c \cdot ||E(\omega)||_\infty$. Then (7) becomes

$$W(\omega_i)\,(A(\omega_i) - D(\omega_i)) = (-1)^i\,\delta \tag{8}$$

$$A(\omega_i) - D(\omega_i) = \frac{(-1)^i\,\delta}{W(\omega_i)} \tag{9}$$

or

$$\sum_{k=0}^{M} a(k)\cos(k\omega_i) - \frac{(-1)^i\delta}{W(\omega_i)} = D(\omega_i) \quad \text{for} \quad i = 1, \ldots, R. \tag{10}$$

If the set of extremal points in the alternation theorem were known in advance, then the solution could be found by solving the system of equations (10). The system (10) represents an interpolation problem, which in matrix form becomes

$$\begin{bmatrix} 1 & \cos\omega_1 & \cdots & \cos M\omega_1 & 1/W(\omega_1) \\ 1 & \cos\omega_2 & \cdots & \cos M\omega_2 & -1/W(\omega_2) \\ \vdots & & & & \vdots \\ & & & & \\ 1 & \cos\omega_R & \cdots & \cos M\omega_R & -(-1)^R/W(\omega_R) \end{bmatrix} \begin{bmatrix} a(0) \\ a(1) \\ \vdots \\ a(M) \\ \delta \end{bmatrix} = \begin{bmatrix} D(\omega_1) \\ D(\omega_2) \\ \vdots \\ \\ D(\omega_R) \end{bmatrix} \tag{11}$$

This is not a typical interpolation problem. There are $M+1$ coefficients $a(n)$, but there are $R = M+2$ interpolation points. However, the value δ is taken as a variable in this interpolation problem, so that there are in fact a total R variables and R equations. Provided the points ω_k are distinct, there will be a unique solution to this linear system of equations.

The problem of obtaining the filter that minimizes the Chebyshev criteria then becomes one of finding the correct set of points over which to solve the interpolation problem (10). In the process, one obtains δ.

The Remez exchange algorithm proceeds by iteratively

1. solving the interpolation problem (11) over a specified set of R points (the *reference set*), and

2. updating the reference set (by an exchange procedure).

The initial reference set can be taken to be R points uniformly spaced in $[0, \pi]$ (excluding regions where $W(\omega)$ is zero). Convergence is achieved when

$$\frac{||E_k(\omega)||_\infty - |\delta_k|}{||E_k(\omega)||_\infty} < \epsilon, \tag{12}$$

where ϵ is a small number (like 10^{-6}) indicating the numerical accuracy desired.

In practice, the algorithm is implemented by discretizing the frequency variable ω. Typically a uniform array of points

$$\omega_k = \frac{\pi}{L}\, k \qquad 0 \le k \le L \tag{13}$$

is used.

The initialization step

The algorithm can be initialized by selecting any R frequency points between 0 and π for which the weighting function $W(\omega)$ is not zero. For example, one can choose the initial frequencies to be uniformly spaced over the region where $W(\omega) > 0$.

The interpolation step

The interpolation step requires solving the linear system (11). It can be treated as a general linear system, however, there do exist fast algorithms for solving the system (11).

Updating the reference set

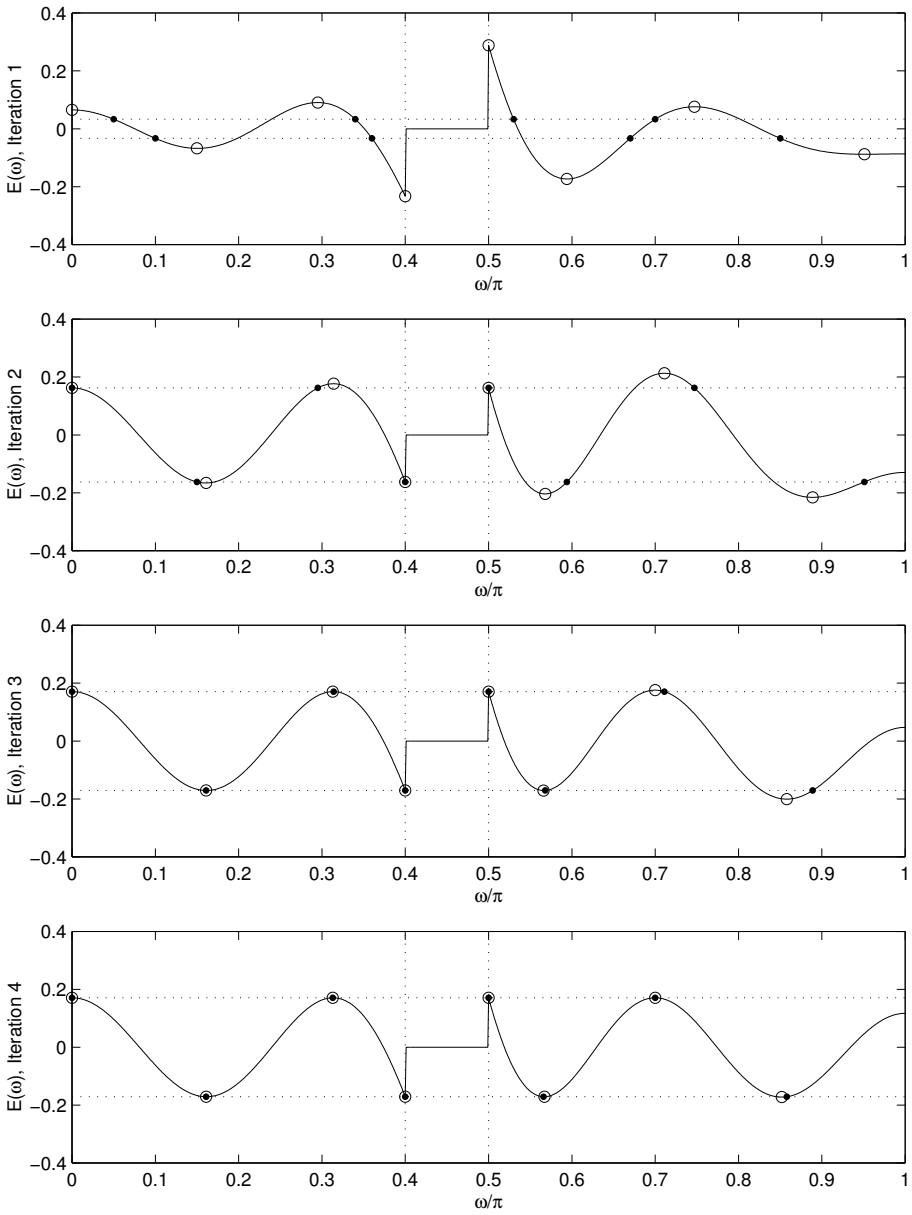
After the interpolation step is performed, the weighted error function is computed, and a new reference set $\omega_1, \ldots, \omega_R$ is found such that the satisfy the following *update criteria*.

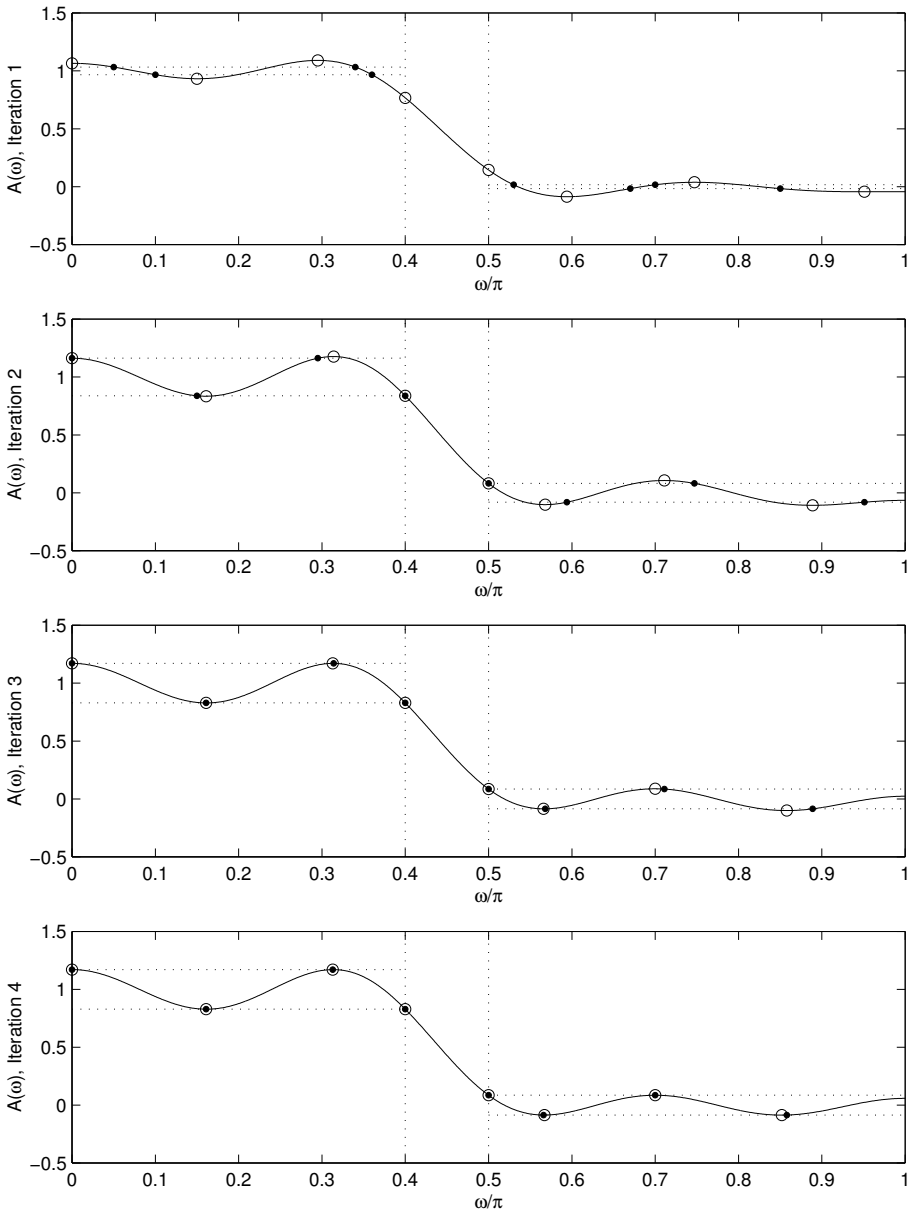
UPDATE CRITERIA

1. The current weighted error function $E(\omega)$ alternates sign on the new reference set.
2. $|E(\omega_i)| \geq |\delta|$ for each point ω_i of the new reference set.
3. $|E(\omega_i)| > |\delta|$ for at least one point ω_i of the new reference set.

Generally, the new reference set is found by taking the set of local minima and maxima of $E(\omega)$ that exceed the current value of δ, and taking a subset of this set that satisfies the alternation property. The procedure to update the set of interpolation points (the reference set) is best illustrated by an example. In the following figures, the design of a type I FIR filter of length 13 is shown, with the pass-band and stop-band edges at $\omega_p = 0.4\pi$ and $\omega_s = 0.5\pi$, and weights $K_p = 1$, $K_s = 2$.

The solid circular marks indicate the reference set for iteration k. The open circular marks indicate the new reference set. They are the local minima and maxima of the current error function, chosen so that (1) the sign of the error alternates from point to point, and (2) the error function at the new points is not lower than the current value of $|\delta|$. Note that the value of $|\delta|$ is shown by a dotted line.


E(ω), Iteration 1
E(ω), Iteration 2
E(ω), Iteration 3
E(ω), Iteration 4
ω/π
0.4
0.2
0
−0.2
−0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

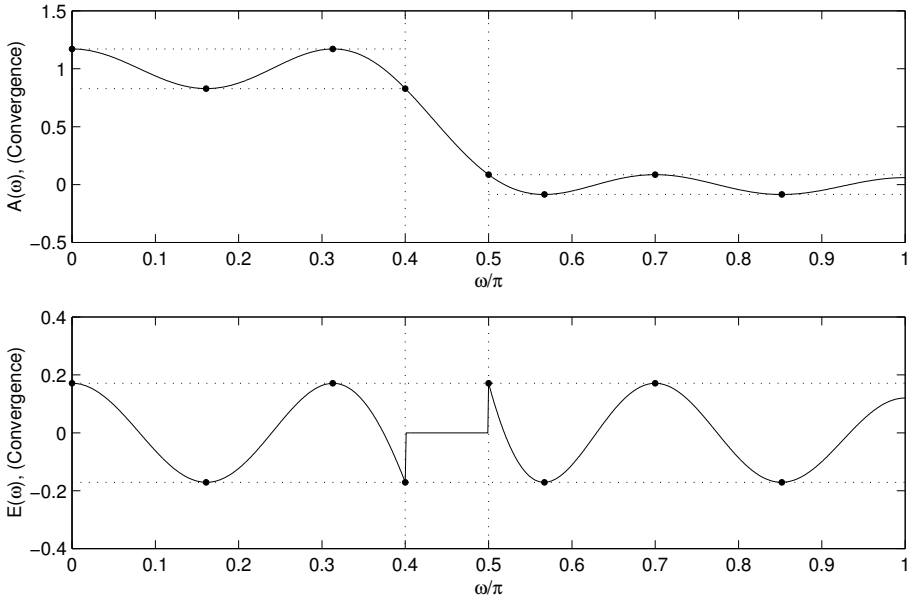


The value of $|\delta|$ and the value of the maximum value of the weighted error $||E(\omega)||_\infty$ are given in the table for each iteration k. It can be seen in the table that $|\delta|$ increases from iteration to iteration. In fact, it is always the case that $|\delta|$ is monotonically increasing (until convergence is achieved). The table also shows that for this example, $||E_k(\omega)||_\infty$ decreases from iteration to iteration, but this

is not always the case. For some examples, it will increase on some iterations, but that does not affect the over all convergence. Once $\delta_k = ||E_k(\omega)||_\infty$, then the algorithm is finished.

iteration k	δ_k	$\|\|E_k(\omega)\|\|_\infty$
1	0.03298702	0.28817888
2	0.16230703	0.21563341
3	0.17078465	0.20058324
4	0.17095620	0.17228714
5	0.17096130	0.17096130

The following figure shows the function $A(\omega)$ and $E(\omega)$ after the Remez algorithm has converged. It is clear from the figures that the algorithm converges very rapidly.



To make the Remez algorithm clear, we show how it can be implemented in Matlab, using the length 13 filter as an example, with $\omega_p = 0.4\pi$ and $\omega_s = 0.5\pi$ and weights $K_p = 1$, $K_s = 2$.
Before the program can begin, we must define the filter specifications and set up the discretized versions of ω, $W(\omega)$ and $D(\omega)$. The result of the comparison command yields a vector of ones and zeros, so `W` and `D` can be built using them.

```
N  = 13;                          % filter length
Kp = 1;                           % pass-band weight
Ks = 2;                           % stop-band weight
wp = 0.4*pi;                      % pass-band edge
ws = 0.5*pi;                      % stop-band edge
wo = (wp+ws)/2;                   % cut-off freq.
L  = 1000;                        % grid size
w  = [0:L]'*pi/L;                 % frequency
W  = Kp*(w<=wp) + Ks*(w>=ws);     % weight function
D  = (w<=wo);                     % desired function
```

The first part of the program defines M, R. For this example, $M = 6$, $R = 8$.

```
M = (N-1)/2;
R = M + 2;       % R = size of reference set
```

Next, we initialize the reference set. We can do this by specifying the indices `k` in the vector `w`. They can be chosen randomly, with the stipulation that $W(\omega)$ is not zero at any of the points. For example:

```
% initialize reference set
k = [51 101 341 361 531 671 701 851];
```

The actual 8 frequency values can be obtained by `w(k)/pi`,

```
>> w(k)'/pi
   0.0500  0.1000  0.3400  0.3600  0.5300  0.6700  0.7000  0.8500
```

Iteration 1: Iteration 1 can now begin, by solving the interpolation problem to obtain $a(n)$ and δ.

```
m   = 0:M;
s   = (-1).^(1:R)';
x   = [cos(w(k)*m), s./W(k)] \ D(k);
a   = x(1:M+1);
del = x(M+2);
```

The value of δ is:

```
>> del
   0.0330
```

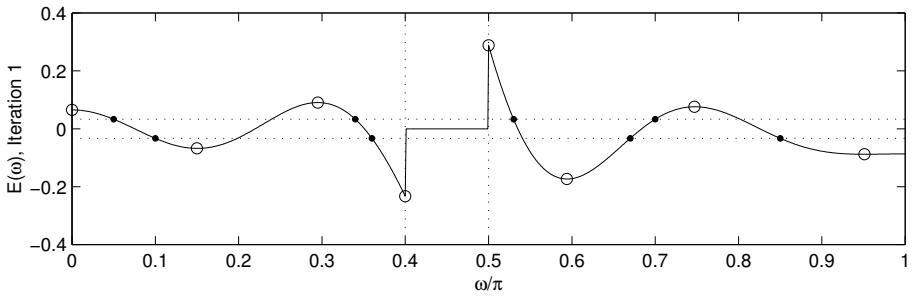
We can compute the frequency response amplitude $A(\omega)$ on the uniformly discretized frequency ω_k using the previously developed Matlab function `firamp`.

```
h   = [a(M+1:-1:2); 2*a(1); a(2:M+1)]/2;
A   = firamp(h,1,L)';
err = (A-D).*W;
```

We can verify that for the interpolation points, the error function $E(\omega)$ has a constant value but alternates sign:

```
>> [w(k)/pi err(k)]

    0.0500    0.0330
    0.1000   -0.0330
    0.3400    0.0330
    0.3600   -0.0330
    0.5300    0.0330
    0.6700   -0.0330
    0.7000    0.0330
    0.8500   -0.0330
```

The error function $E(\omega)$ is plotted in the figure labeled 'Iteration 1,' repeated here for convenience.



In this figure, the dashed line indicates the value of δ, and it is clear that $E(\omega)$ passes through $(-1)^i \cdot \delta$ at the interpolation points, which are indicated by the solid circles.

To update the reference set, we first find the local minima and maxima of the error function that exceed $|\delta|$ in absolute value. This can be done with the Matlab function `locmax` (see the appendix). This function returns the *indices* of a vector corresponding to the local maxima. The variable `newk` represents the indices of `k`, and `errk` represents the value of $E(\omega)$ at `w(k)`.

```
newk = sort([locmax(err); locmax(-err)]);
errk = (A(newk)-D(newk)).*W(newk);
```

The new frequency points and the value of $E(\omega)$ at these points are:

```
>> [w(newk)/pi errk]

         0    0.0652
    0.1500   -0.0677
    0.2950    0.0906
    0.4000   -0.2330
    0.5000    0.2882
    0.5940   -0.1734
    0.7470    0.0760
    0.9510   -0.0880
    1.0000   -0.0867
```

There are two problems with this set of frequency points.

1. There are more than $R = 8$ of them.

2. The function $E(\omega)$ does not alternate sign on them. (The last two points have the same sign.)

Therefore, one of these frequencies must be dropped from this set. This is done with a sub-program called `etap` which returns a set of indices. (See the appendix.)

```
v    = etap(errk);   % ensure the alternation property
newk = newk(v);
errk = errk(v);
```

The new `newk` and `errk` are:

```
>> [w(newk)/pi errk]

         0    0.0652
    0.1500   -0.0677
    0.2950    0.0906
    0.4000   -0.2330
    0.5000    0.2882
    0.5940   -0.1734
    0.7470    0.0760
    0.9510   -0.0880
```

They are indicated in the figure by open circles. Now there are the correct number of points ($R = 8$), they alternate in sign, and the are all greater than the current value of $|\delta| = 0.0330$. So we can replace the reference set by the new one

```
k = newk;
```

and go on to the second iteration.

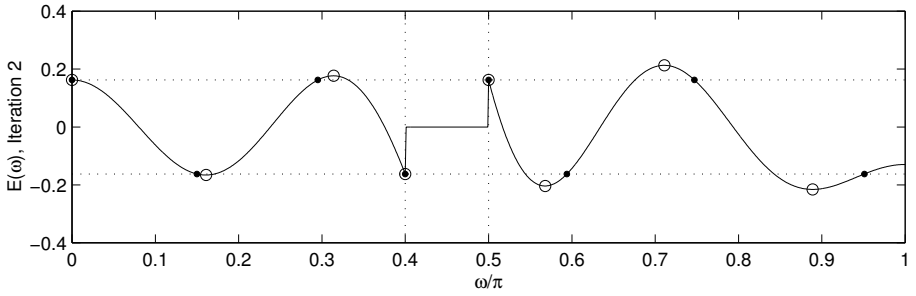
Iteration 2: Iteration 2 begins by solving the interpolation problem with the updated reference points and finding the local minima and maxima of the resulting function $E(\omega)$. This is done with the same commands used for iteration 1:

```
x    = [cos(w(k)*m), s./W(k)] \ D(k);
a    = x(1:M+1);
del  = x(M+2);
h    = [a(M+1:-1:2); 2*a(1); a(2:M+1)]/2;
A    = firamp(h,1,L)';
err  = (A-D).*W;
newk = sort([locmax(err); locmax(-err)]);
errk = (A(newk)-D(newk)).*W(newk);
```

The new value of δ and the value of the $E(\omega)$ on the new local minima and maxima are:

```
>> del

    0.1623

>> [w(newk)/pi errk]

         0    0.1623
    0.1610   -0.1659
    0.3140    0.1768
    0.4000   -0.1623
    0.5000    0.1623
    0.5680   -0.2038
    0.7110    0.2129
    0.8890   -0.2156
    1.0000   -0.1295
```

The new error function $E(\omega)$ is plotted in the figure labeled 'Iteration 2,' repeated here for convenience.



It can be seen that at the last value of the new set, $\omega = \pi$, $|E(\omega)|$ is not greater than the current $|\delta| = 0.1623$, so it can be deleted from the set. This can be done with the following Matlab commands.

```
SN   = 1e-8;
v    = abs(errk) >= (abs(del)-SN);
newk = newk(v);
errk = errk(v);
```

The vector v are those indices corresponding to points where $|E(\omega_k)|$ is equal to or greater than the current value of $|\delta|$. It is necessary to subtract a small number (SN) from $|\delta|$ to address the comparison of two equal numbers in finite precision arithmetic. The corrected set of points is

```
>> [w(newk)/pi errk]

         0    0.1623
    0.1610   -0.1659
    0.3140    0.1768
    0.4000   -0.1623
    0.5000    0.1623
    0.5680   -0.2038
    0.7110    0.2129
    0.8890   -0.2156
```

They are shown as open circles in the figure. This set of points satisfy the conditions for the update of the reference set, so we can replace the reference set by the new one

```
k = newk;
```

and go on to the next iteration.

Putting these commands together we get a program for designing type 1 FIR filters that minimize the Chebyshev criterion.

```
function [h,del] = fircheb(N,D,W)
% h = fircheb(N,D,W)
% weighted Chebyshev design of Type I FIR filters
%
%   h : length-N impulse response
%   N : length of filter (odd)
%   D : ideal response   (uniform grid)
%   W : weight function  (uniform grid)
%   need: length(D) == length(W)
```

```
%
% % Example
%   N = 31;  Kp = 1; Ks = 4;
%   wp = 0.26*pi;  ws = 0.34*pi; wo = 0.3*pi;
%   L = 1000;
%   w = [0:L]*pi/L;
%   W = Kp*(w<=wp) + Ks*(w>=ws);
%   D = (w<=wo);
%   h = fircheb(N,D,W);

% subprograms: locmax.m, etap.m

W = W(:);
D = D(:);
L = length(W)-1;

SN = 1e-8;              % small number for stopping criteria, etc
M = (N-1)/2;
R = M + 2;              % R = size of reference set

% initialize reference set (approx equally spaced where W>0)
f = find(W>SN);
k = f(round(linspace(1,length(f),R)));

w = [0:L]'*pi/L;
m = 0:M;
s = (-1).^(1:R)';       % signs

while 1
   % --------------- Solve Interpolation Problem ---------------------
   x = [cos(w(k)*m), s./W(k)] \ D(k);
   a = x(1:M+1); % cosine coefficients
   del = x(M+2); % delta
   h = [a(M+1:-1:2); 2*a(1); a(2:M+1)]/2;
   A = firamp(h,1,L)';
   err = (A-D).*W; % weighted error

   % --------------- Update Reference Set ----------------------------
   newk = sort([locmax(err); locmax(-err)]);
   errk = (A(newk)-D(newk)).*W(newk);

   % remove frequencies where the weighted error is less than delta
   v = abs(errk) >= (abs(del)-SN);
   newk = newk(v);
   errk = errk(v);

   % ensure the alternation property
   v = etap(errk);
   newk = newk(v);
   errk = errk(v);
```

```
   % if newk is too large, remove points until size is correct
   while length(newk) > R
      if abs(errk(1)) < abs(errk(length(newk)))
         newk(1) = [];
      else
         newk(length(newk)) = [];
      end
   end

   % --------------- Check Convergence ------------------------------
   if (max(errk)-abs(del))/abs(del) < SN
      disp('I have converged.')
      break
   end
   k = newk;
end

del = abs(del);
h = [a(M+1:-1:2); 2*a(1); a(2:M+1)]/2;
```

The program also checks if the reference set is too large. This can happen in the general case when there are more than enough points that satisfy the update criteria. In this case, we just remove the end points until it is of the correct size.

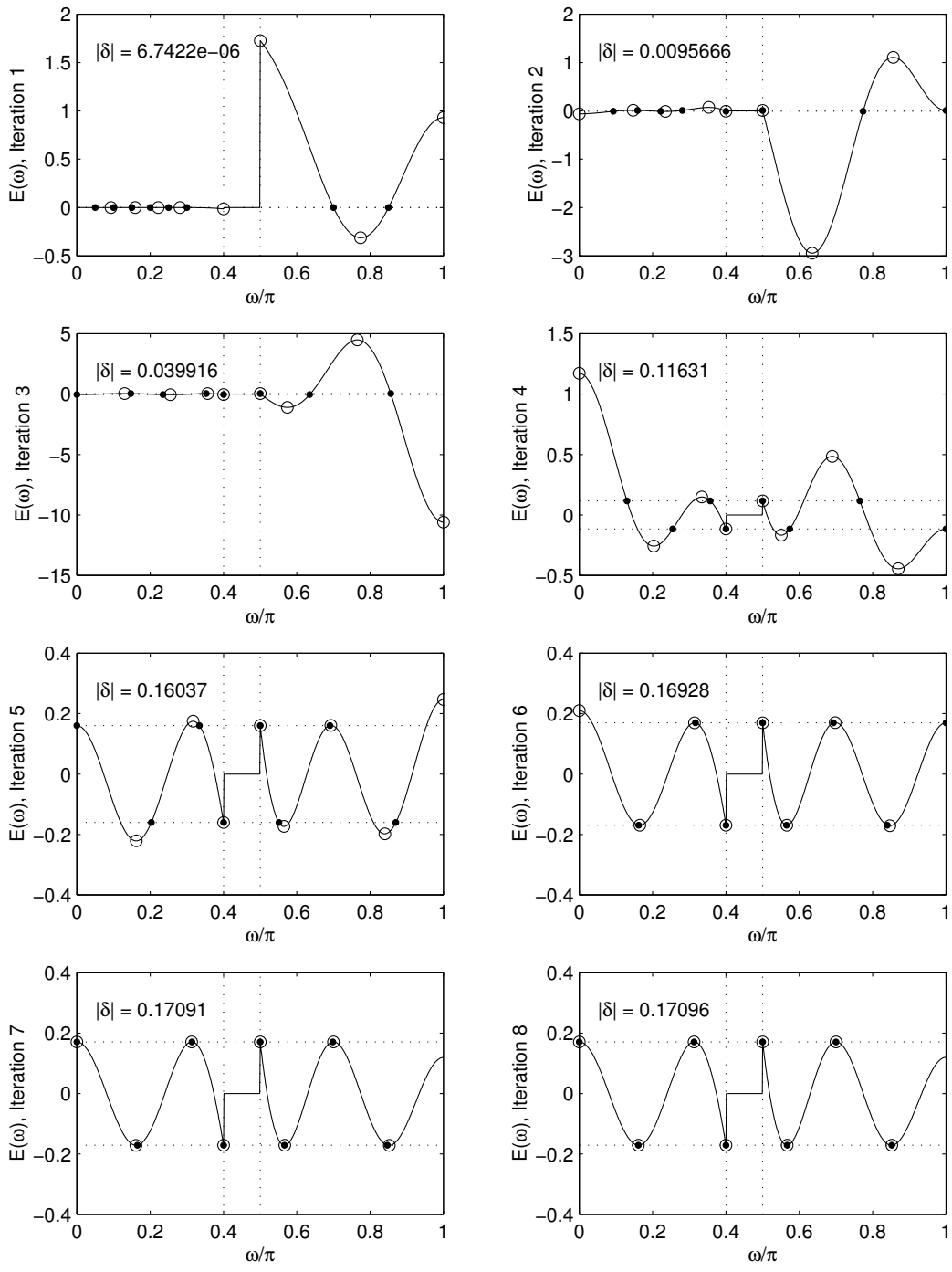
The stopping criteria

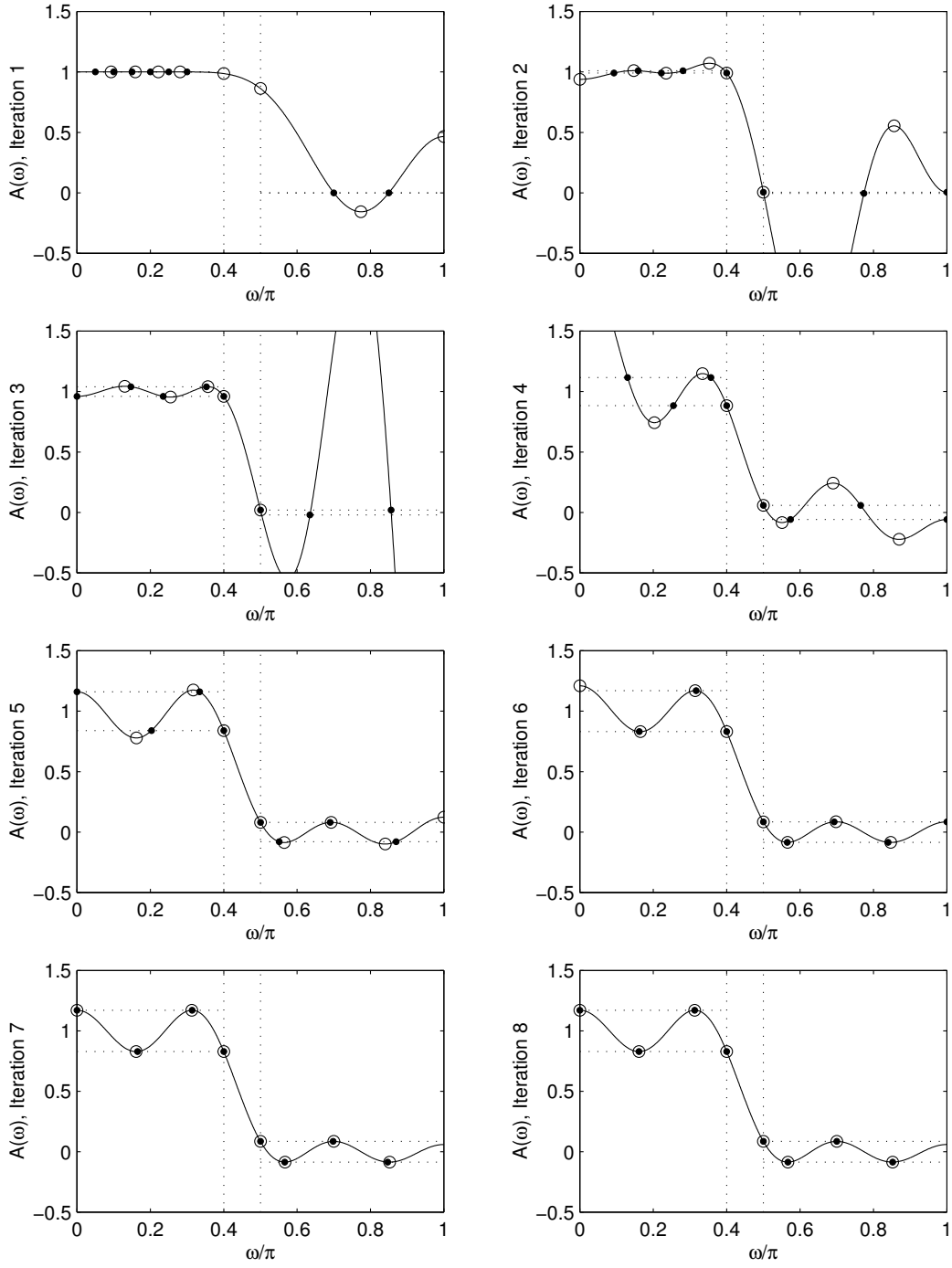
$$\frac{||E_k(\omega)||_\infty - |\delta_k|}{||E_k(\omega)||_\infty} < \epsilon, \tag{14}$$

measures how far the current solution is from the optimal solution because $|\delta_k|$ is always increasing and because for the optimal solution $||E(\omega)||_\infty$ can not be greater than $||E_k(\omega)||_\infty$ for any iteration k.

The Matlab function `remez` that is part of the Signal Processing Toolbox also implements the Remez algorithm for FIR filter design.

The Remez algorithm has very robust convergence properties. Even if the initial interpolation points are very different from the extremal points of the optimal solution, the Remez algorithm converges rapidly. In the following example, the design of the length 13 filter in the previous example is repeated, but with initial interpolation points that are more different from the final extremal points. In particular, the number of interpolation points in the pass-band and stop-band respectively is 6 and 2 at iteration 1, but the Remez algorithm ends up with 4 in each band which is correct for this example.


|δ| = 6.7422e−06
E(ω), Iteration 1
|δ| = 0.0095666
E(ω), Iteration 2
|δ| = 0.039916
E(ω), Iteration 3
|δ| = 0.11631
E(ω), Iteration 4
|δ| = 0.16037
E(ω), Iteration 5
|δ| = 0.16928
E(ω), Iteration 6
|δ| = 0.17091
E(ω), Iteration 7
|δ| = 0.17096
E(ω), Iteration 8
ω/π


A(ω), Iteration 1
A(ω), Iteration 2
A(ω), Iteration 3
A(ω), Iteration 4
A(ω), Iteration 5
A(ω), Iteration 6
A(ω), Iteration 7
A(ω), Iteration 8
ω/π
1.5
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1

While the PM algorithm is applicable for the approximation of arbitrary responses $D(\omega)$, the low-pass case has received particular attention [3, 4, 10, 13]. In the design of low-pass filters via the PM algorithm, there are five parameters of interest:

N : filter length
ω_p : pass-band edge
ω_s : stop-band edge
δ_p : maximum pass-band error
δ_s : maximum stop-band error

Their values are not independent — any four determines the fifth. Formulas for predicting the required filter length for a given set of specifications make this clear. Kaiser developed the following approximate relation for estimating the filter length for meeting the specifications,

$$N \approx \frac{-20\log_{10}(\sqrt{\delta_p\delta_s}) - 13}{14.6\Delta F} + 1 \tag{15}$$

where $\Delta F = (\omega_s - \omega_p)/(2\pi)$. Herrmann et al. [3] gave a somewhat more accurate formula

$$N \approx \frac{D_\infty(\delta_p, \delta_s) - f(\delta_p, \delta_s)(\Delta F)^2}{\Delta F} + 1 \tag{16}$$

where

$$\begin{aligned} D_\infty(\delta_p, \delta_s) =& (0.005309(\log_{10}\delta_p)^2 + 0.07114\log_{10}\delta_p - 0.4761)\log_{10}\delta_s \\ & - (0.00266(\log_{10}\delta_p)^2 + 0.5941\log_{10}\delta_p + 0.4278), \end{aligned} \tag{17}$$

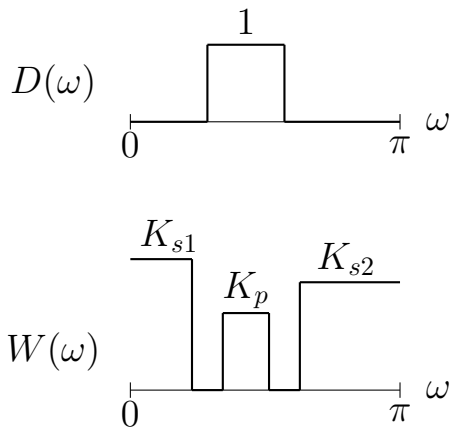
$$f(\delta_p, \delta_s) = 11.01217 + 0.51244(\log_{10}\delta_p - \log_{10}\delta_s). \qquad (18)$$

These formulas assume that $\delta_s < \delta_p$. If otherwise, then one must interchange δ_p and δ_s. This formula is implemented in the Matlab function `remezord` in the Signal Processing Toolbox.

To use the PM algorithm for low-pass filter design, the user specifies $N, \omega_p, \omega_s, \delta_p/\delta_s$. The PM algorithm can be modified so that the user specifies other parameters sets however [17]. For example, with one modification, the user specifies $N, \omega_p, \delta_p, \delta_s$; or similarly, $N, \omega_s, \delta_p, \delta_s$. With a second modification, the user specifies $N, \omega_p, \omega_p, \delta_p$; or similarly, $N, \omega_s, \omega_p, \delta_s$. The necessary modifications are simple.

Note that both (15) and (16) state that the filter length N and the transition width ΔF are inversely proportional (for (16), as ΔF goes to 0). This is in contrast to the corresponding relation for maximally flat symmetric filters (covered in another section). For equiripple filters with fixed δ_p and δ_s, ΔF diminishes like $1/N$; while for maximally-flat filters, ΔF diminishes like $1/\sqrt{N}$.

Band-pass filters that minimize the Chebyshev error can be designed with functions $D(\omega)$ and $W(\omega)$ that have the following form.



It should be noted that when a zero weighted transition band is used ($W(\omega) = 0$ for some interval in $(0, \pi)$), then the best Chebyshev solution may have undesirable behavior in those bands: large peaks may occur. Although this does not occur in low-pass filter design, it is well documented in band-pass filter design, in particular, when one transition band is much larger than the other. For example, consider the design of band-pass filter with the following specifications

$$\begin{array}{rcll} -\delta_{s1} \leq & A(\omega) & \leq \delta_{s1}, & 0 \leq \omega \leq \omega_{s1} \\ 1-\delta_p \leq & A(\omega) & \leq 1+\delta_p, & \omega_{p1} \leq \omega \leq \omega_{p2} \\ -\delta_{s2} \leq & A(\omega) & \leq \delta_{s2}, & \omega_{s2} \leq \omega \leq \pi \end{array} \tag{19}$$

where

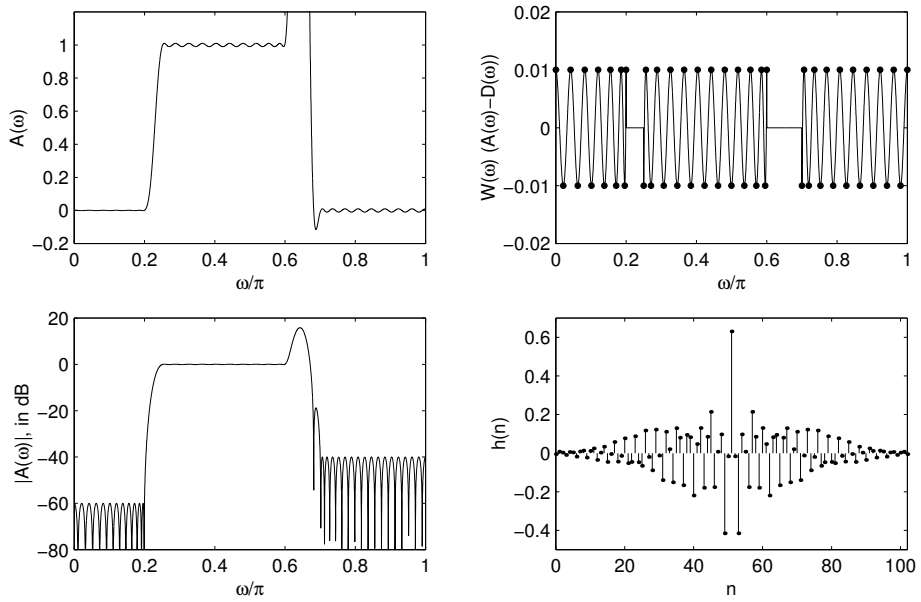
$$\delta_{s1} = 0.001, \quad \delta_p = 0.01, \quad \delta_{s2} = 0.01, \tag{20}$$

$$\omega_{s1} = 0.20\pi, \quad \omega_{p1} = 0.25\pi, \quad \omega_{p2} = 0.60\pi, \quad \omega_{s2} = 0.70\pi. \tag{21}$$

Then we can take $K_{s1} = 10$, and $K_p = K_{s2} = 1$. Using `fircheb` we can obtain a filter of minimal length that satisfies these specifications.

```
N   = 103;
ws1 = 0.20*pi;
wp1 = 0.25*pi;
wp2 = 0.60*pi;
ws2 = 0.70*pi;
Ks1 = 10;
Kp  = 1;
Ks2 = 1;
wo  = (ws1+wp1)/2;
w1  = (wp1+ws2)/2;
L   = 4000;
w   = [0:L]*pi/L;
W   = Ks1*(w<=ws1) + Kp*((w>=wp1)&(w<=wp2)) + Ks2*(w>=ws2);
D   = (wp1<=w)&(w<=wp2);
h   = fircheb(N,D,W);
```

It is of length 103 and illustrated in the following figure.

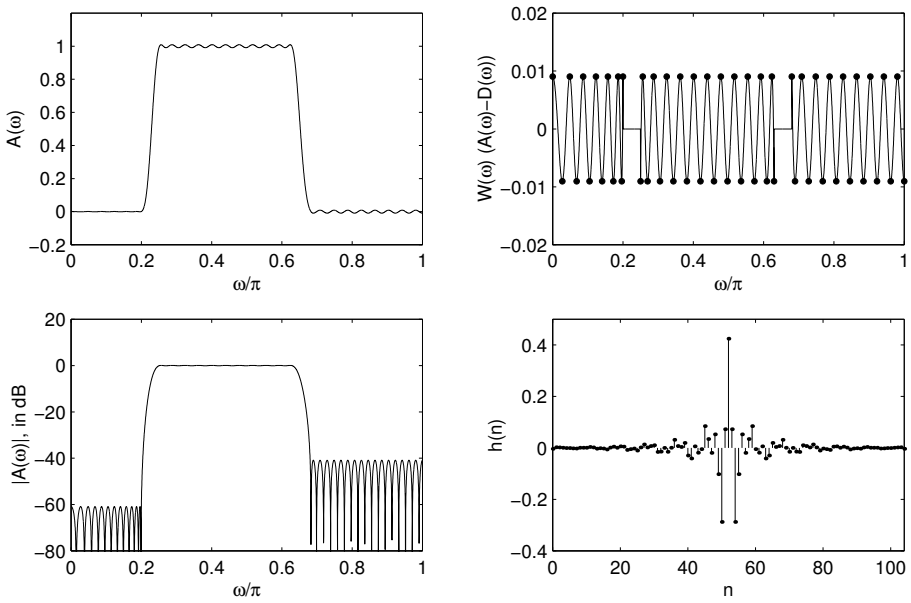


It is clear from the figure that the behavior in the transition is unacceptable. This can occur because the transition-band is not

included in the criteria (the weight function $W(\omega)$ is zero there). Various simple techniques based on modifying the weight function $W(\omega)$ and/or the desired amplitude $D(\omega)$ effectively eliminate this problem. A simple method makes a transition band narrower [12]. For this band-pass example, if the second transition band is changed to

$$\omega_{p2} = 0.63\pi, \quad \omega_{s2} = 0.68\pi, \tag{22}$$

which is *narrower* than the original specification, then the filter length needs to be increased to only 105 to meet the specifications.



Another technique fills in the transition band by a weight function and desired function that join the two bands by a line. A third technique sets the weight function equal to a small constant in the transition region [16]. Grenez describes how to include some additional appropriate constraints into the Parks-McClellan program [2]. Another approach uses linear programming techniques [18].

If it is desired that the filter have a null at a specified frequency, then the Remez algorithm can still be used, but with modified functions $D(\omega)$ and $A(\omega)$.

If the filter $h(n)$ of length N has a null at a specified frequency ω_n,

$$A(\omega_n) = 0, \tag{23}$$

then the transfer function $H(z)$ can be factored as

$$H(z) = H_1(z)\, H_2(z) \tag{24}$$

where $h_2(n)$ is of length $N_2 = N - 2$ and $h_1(n)$ is of length 3 and has the transfer function:

$$\begin{aligned} H_1(z) &= (z^{-1} - e^{j\omega_n})\,(z^{-1} - e^{-j\omega_n}) && (25)\\ &= z^{-2} - z^{-1}\,(e^{j\omega_n} + e^{-j\omega_n}) + 1 && (26)\\ &= z^{-2} - 2z^{-1}\,\cos(\omega_n) + 1 && (27)\\ &= z^{-1}\,\left(z^{-1} - 2\cos(\omega_n) + z\right) && (28) \end{aligned}$$

so the impulse response of is

$$h_1 = [1,\ \ -2\,\cos(\omega_n),\ \ 1] \tag{29}$$

and the frequency response is

$$\begin{aligned} H_1(e^{j\omega}) &= e^{-j\omega}\left(e^{j\omega} - 2\cos(\omega_n) + e^{-j\omega}\right) && (30)\\ &= e^{-j\omega}\,(2\cos(\omega) - 2\cos(\omega_n)) && (31)\\ &= e^{-j\omega} A_1(\omega) && (32) \end{aligned}$$

where

$$\boxed{A_1(\omega) = 2\left(\cos(\omega) - \cos(\omega_n)\right).} \tag{33}$$

Note that $H_2(e^{j\omega})$ can be written as

$$H_2(e^{j\omega}) = e^{-jM_2\omega}\, A_2(\omega) \tag{34}$$

where $M_2 = (N_2 - 1)/2$. Then the amplitude response of $h(n)$ can be written as the product

$$\boxed{A(\omega) = A_1(\omega)\, A_2(\omega).} \tag{35}$$

The weighted error function $E(\omega)$ can be written as

$$E(\omega) = W(\omega)\left(A(\omega) - D(\omega)\right) \tag{36}$$

$$E(\omega) = W(\omega)\left(A_1(\omega)A_2(\omega) - D(\omega)\right) \tag{37}$$

$$E(\omega) = W(\omega)\, A_1(\omega)\, \left(A_2(\omega) - \frac{D(\omega)}{A_1(\omega)}\right) \tag{38}$$

$$|E(\omega)| = W_2(\omega)\, |A_2(\omega) - D_2(\omega)| \tag{39}$$

where

$$\boxed{W_2(\omega) = |W(\omega)\, A_1(\omega)|} \tag{40}$$

$$\boxed{D_2(\omega) = \frac{D(\omega)}{A_1(\omega)}.} \tag{41}$$

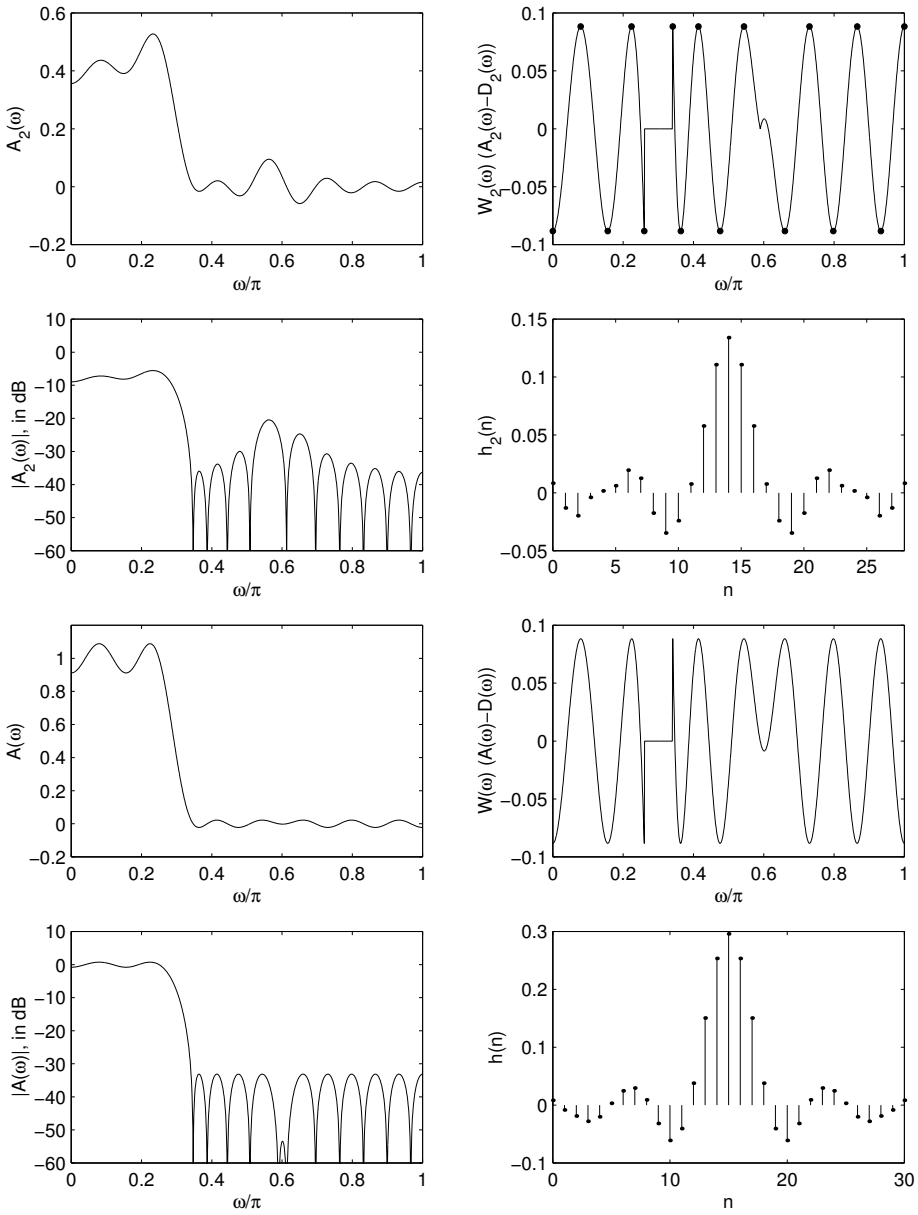
Then the sought filter $h(n)$ of length N can be found by first finding the filter of length $N_2 = N - 2$ that minimizes the Chebyshev error with weight function $W_2(\omega)$ and desired amplitude response $D_2(\omega)$, and by then convolving it with $h_1(n)$

$$h(n) = h_1(n) * h_2(n). \tag{42}$$

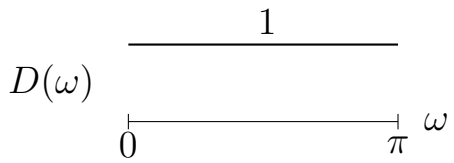
This procedure is implemented in the following Matlab code, for the design of a length 31 type I FIR filter with $K_p = 1$, $K_s = 4$ and $\omega_p = 0.26\pi$, $\omega_s = 0.34\pi$.

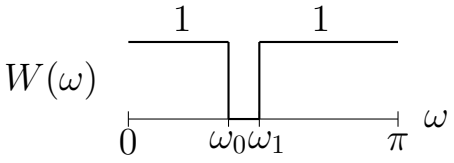
```
% CHEBYSHEV LOWPASS FILTER with SPECIFIED NULL

% filter length
N  = 31;
N2 = N - 2;

% set band-edges and weighting
wp = 0.26*pi;
ws = 0.34*pi;
wo = 0.30*pi;
Kp = 1;
Ks = 4;

% set null
wn = 0.59*pi;

L  = 1000;
w  = [0:L]*pi/L;
W  = Kp*(w<=wp) + Ks*(w>=ws);
D  = (w<=wo);

A1 = 2*(cos(w)-cos(wn));
W2 = W.*abs(A1);
D2 = D./A1;

h2 = fircheb(N2,D2,W2);
h1 = [1 -2*cos(wn) 1];
h  = conv(h2,h1);
```

For this example, both the filters $h_2(n)$ and $h(n)$ are illustrated in the following figures. Note that while $A_2(\omega)$ does not have an equi-ripple behavior, the *error function* $E(\omega)$ does.


A2(ω)
ω/π
W2(ω) (A2(ω)−D2(ω))
ω/π
|A2(ω)|, in dB
ω/π
h2(n)
n
A(ω)
ω/π
W(ω) (A(ω)−D(ω))
ω/π
|A(ω)|, in dB
ω/π
h(n)
n

Consider the design of a notch filter using the Chebyshev error criteria. As in the square-error design of a notch filter, one way to pose the design problem is to let the desired amplitude response be a constant, but to delete an interval around the notch frequency ω_n from the error function by setting the weighting function $W(\omega)$ to 0 there. The desired amplitude $D(\omega)$ and weight function $W(\omega)$ will be as shown.





As in the design of a low-pass filter with a specified null in the stop-band, the transfer function of a notch filter with a null at the frequency ω_n can be written as

$$H(z) = H_1(z)\, H_2(z) \tag{43}$$

where $H_1(z)$ contains only the null. To design a notch filter with a double zero at ω_n, $H_1(z)$ will be a length 5 filter with the transfer function

$$H_1(z) = (z^{-1} - e^{j\omega_n})^2\, (z^{-1} - e^{-j\omega_n})^2 \tag{44}$$

$$= z^{-2}\, \left(z^{-1} - 2\cos(\omega_n) + z\right)^2 . \tag{45}$$

Then

$$H_1(e^{j\omega}) = e^{-2j\omega} A_1(\omega) \tag{46}$$

where

$$A_1(\omega) = 4\left(\cos(\omega) - \cos(\omega_n)\right)^2. \tag{47}$$

$H_2(z)$ will then be of length $N_2 = N - 4$ and can be written as

$$H_2(z) = e^{-jM_2\omega}\, A_2(\omega) \tag{48}$$

where $M_2 = (N_2 - 1)/2$. The amplitude response of $h(n)$ can be written as the product

$$A(\omega) = A_1(\omega)\, A_2(\omega). \tag{49}$$

Then the weighted error function $|E(\omega)|$ can be written as

$$|E(\omega)| = W_2(\omega)\, |A_2(\omega) - D_2(\omega)| \tag{50}$$

where

$$W_2(\omega) = |W(\omega)\, A_1(\omega)|, \qquad D_2(\omega) = \frac{D(\omega)}{A_1(\omega)}. \tag{51}$$

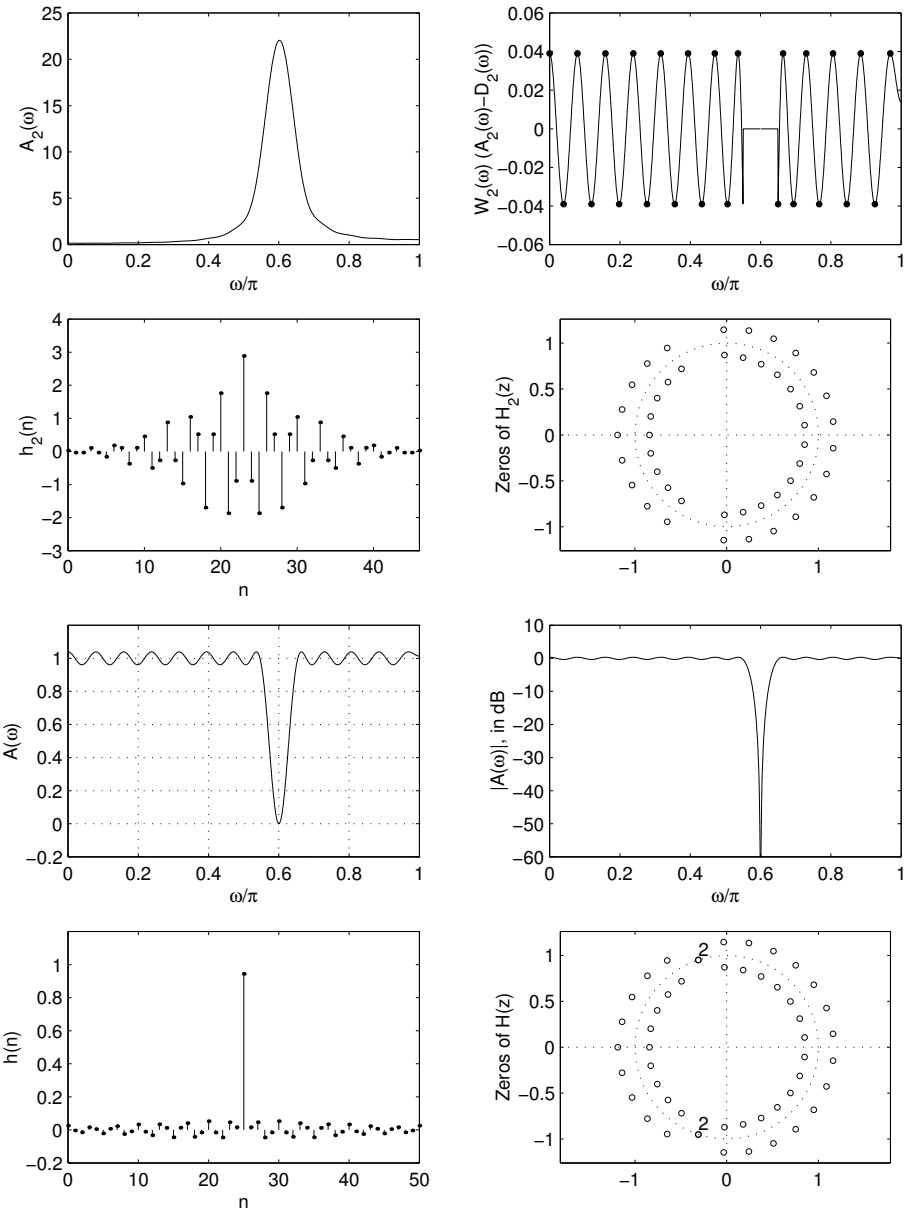
As in the previous example, we first find the filter $h_2(n)$ of length $N_2 = N - 4$ that minimizes the Chebyshev error with weight function $W_2(\omega)$ and desired amplitude response $D_2(\omega)$, and then convolve it with $h_1(n)$.

Design example. The following Matlab code designs a length 51 Type 1 FIR notch filter according to the Chebyshev criterion with notch frequency $\omega_n = 0.6\pi$, and $\omega_0 = 0.55\pi$, $\omega_1 = 0.65\pi$.

```
% CHEBYSHEV NOTCH FILTER

% filter length
```

```
N  = 51;
N2 = N - 4;

% set band-edges
w0 = 0.55*pi;
w1 = 0.65*pi;

% set notch frequency
wn = 0.60*pi;

L  = 1000;
w  = [0:L]*pi/L;
W  = (w<=w0) + (w>=w1);
D  = ones(size(w));

A1 = 4*(cos(w)-cos(wn)).^2;
W2 = W.*abs(A1);
D2 = D./A1;

h1 = conv([1 -2*cos(wn) 1],[1 -2*cos(wn) 1]);
h2 = fircheb(N2,D2,W2);
h  = conv(h2,h1);
```

The filter $h(n)$ is illustrated in the following figure.


A2(ω)
ω/π
W2(ω) (A2(ω)−D2(ω))
ω/π
h2(n)
n
Zeros of H2(z)
A(ω)
ω/π
|A(ω)|, in dB
ω/π
h(n)
n
Zeros of H(z)
2
2

REMEZ ALGORITHM: ADVANTAGES AND DISADVANTAGES

- Produces linear-phase FIR filters that minimize the weighted Chebyshev error.
- Provides explicit control of band edges and relative ripple sizes.
- Efficient algorithm, always converges.
- Allows the use of a frequency dependent weighting function.
- Suitable for arbitrary $D(\omega)$ and $W(\omega)$.
- Does not allow arbitrary linear constraints.

A locmax.m

```
function maxind = locmax(x)
% function maxind = locmax(x)
% finds indices of local maxima of data x

x = x(:);
n = length(x);
maxind = find(x > [x(1)-1;x(1:n-1)] & x > [x(2:n);x(n)-1]);
```

B etap.m

```
function v = etap(E)
%
% v = etap(E);
% Ensuring The Alternation Property
% E : error vector
% v : index of original E

if size(E,1) > 1
   a = 1;
else
   a = 0;
```

```
end
j = 1;
xe = E(1);
xv = 1;
for k = 2:length(E)
   if sign(E(k)) == sign(xe)
      if abs(E(k)) > abs(xe)
         xe = E(k);
         xv = k;
      end
   else
      v(j) = xv;
      j = j + 1;
      xe = E(k);
      xv = k;
   end
end
v(j) = xv;
if a == 1
   v = v(:);
end
```

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