Handout #1: Mathematical Reasoning

1 Propositional Logic

A **proposition** is a mathematical statement that it is either true or false; that is, a statement whose certainty or falsity can be ascertained; we call this the "truth value" of the statement. Thus, a proposition can have only one two truth values: it can be either true, denoted by T, or it can be false, denoted by F.

For example, the statement

"n^2 is even"

is either true or false depending on the value that n takes on in the set of integers.

On the other hand, the statement

"This sentence is false"

is not a proposition (Why?).

The truth values of a proposition, P, can be displayed in tabular form as follows:

P
T
F

This is an example of a "truth table."

Given a proposition P, its **negation**, denoted by $\neg P$, is defined by the truth table

P	$\neg P$
T	F
F	T

Example 1.1. *Give the negation of the statement*

"All students in this class are male."

Answer: "There is at least one student in this class who is not male."
□

Example 1.2. *Give the negation of the statement*

"If n is even, the n^2 is even."

Answer: "There exist and integer n such that n is even and n^2 is not even." □

The previous two examples illustrate the concept of a **quantifier** in mathematical statements. Expressions like "there is," "there exists," "all," "for all," and "for some" are examples of quantifiers.

Quantifiers modify statements of the form $P(x)$ which depend on a variable (in this case x) which can take on values in a set under consideration. For example,

$P(n)$: "n^2 is even"

can be modified as follows

"There exists and integer, n, such that n^2 is even,"

or

"For all integers, n, n^2 is even."

The first statement is true, but the second one is false.

The first statement can be written in more compact notation as follows:

"$(\exists n \in \mathbb{Z})$ (n^2 is even)."

The symbol $\exists$ reads "there exists," "there is" or "for some." It is called the **existential** quantifier.

The second statement can be written as

"$(\forall n \in \mathbb{Z})$ (n^2 is even)."

The symbol $\forall$ reads "for all " or "for every." It is called the **universal** quantifier.

The negation of the statement

"$(\forall n \in \mathbb{Z})$ (n^2 is even)"

is the statement

"$(\exists n \in \mathbb{Z})$ (n^2 is not even)."

In symbols, if we let $P(n)$ denote "n^2 is even," we write

$$\neg(\forall n \in \mathbb{Z})P(n)) \equiv (\exists n \in \mathbb{Z})\neg P(n)).$$

The symbol $\equiv$ is read: "is equivalent to."

Definition 1.3. *Two mathematical statements are equivalent if they have the same truth tables. If P and Q are equivalent, we write*

$$P \equiv Q$$

or

$$P \iff Q.$$

Thus, the truth table for the statement $P \equiv Q$ is

P	Q	$P \equiv Q$
T	T	T
T	F	F
F	T	F
F	F	T

Thus, $P \equiv Q$ is true only when the propositions P and Q have the same truth values. Logical equivalence, $\equiv$, is an example of a logical connector. In the next section we will see more examples of logical connectors.

2 Logical Connectors

Most mathematical statements are made up of several propositions. Propositions can be put together in various ways and following certain rules that prescribe the truth values of the composite statements. In this section we describe how this is done.

2.1 Conjunctions

The conjunction of two propositions, P and Q, is the statement $P \wedge Q$ defined by the truth table:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$P \wedge Q$ reads "P and Q." Thus, the conjunction, $P \wedge Q$ is true only when both P and Q are true.

2.2 Disjunctions

The disjunction of two propositions, P and Q, is the statement $P \vee Q$ defined by the truth table:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

$P \vee Q$ reads "P or Q." Thus, the disjunction, $P \vee Q$ is false only when both P and Q are false. In other words $P \vee Q$ is true when either one of the statements, P and Q, is true.

2.3 Implications

The implication of two propositions, P and Q, also known as material implication or truth–functional conditional, is the statement $P \Rightarrow Q$ defined by the truth table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \Rightarrow Q$ reads "P implies Q," or "If P, then Q." Other ways to translate the material implication $P \Rightarrow Q$ in Mathematics are: "Q follows from P;" "Q, if P;" "P only if Q;" "Q is necessary for P;" and "P is sufficient for Q." The statement P in the material implication $P \Rightarrow Q$ is referred to as the hypothesis or antecedent, while Q is called the conclusion or consequent of the implication.

Material implication are very important in Mathematics. Most mathematical theorems are of the form: "If some set of hypotheses hold true, then a conclusion must necessarily follow." Thus, in order to prove a mathematical statement, it is important to understand the import of the truth–functional definition given in the truth–table: $P \Rightarrow Q$ is false only when P is true and Q is false. Thus, in order to establish the validity of a theorem of the form "If P, then Q," one must show that if P is true, then Q must be true. We will explore this idea further in the Section 4 on page 7 of these notes where we discuss methods for proving mathematical statements.

3 Equivalences and Rules of Reasoning

We have already seen the definition of logical equivalence; namely, two mathematical statements are equivalent if they have the same truth tables. If P and Q are equivalent, we write

$$P \equiv Q$$

or

$$P \iff Q.$$

The statement $P \iff Q$ is also read "P if and only if Q," or "P iff Q," or "P is necessary and sufficient for Q."

The truth table for the statement $P \iff Q$ is

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 3.1. *Show that the statements $P \Rightarrow Q$ and $\neg P \vee Q$ are equivalent.*

> ***Solution**:* The strategy here is to set up a truth table that displays all the truth value for the statements and shows that they are the same.
>
> | P | Q | $\neg P$ | $P \Rightarrow Q$ | $\neg P \vee Q$ |
> |---|---|---|---|---|
> | T | T | F | T | T |
> | T | F | F | F | F |
> | F | T | T | T | T |
> | F | F | T | T | T |
>
> Since the last two columns have the same truth values, the two statements are equivalent. That is,
>
> $$(P \Rightarrow Q) \equiv (\neg P \vee Q)$$
>
> or
>
> $$(P \Rightarrow Q) \iff (\neg P \vee Q)$$
>
> □

Alternatively, we can also solve the previous example by computing the truth values (or truth table) for the statement $(P \Rightarrow Q) \iff (\neg P \vee Q)$.

Example 3.2. *Compute the truth table for* $(P \Rightarrow Q) \iff (\neg P \vee Q)$.

Solution: We complete the following true table using the definition of logical equivalence.

P	Q	$\neg P$	$P \Rightarrow Q$	$\neg P \vee Q$	$(P \Rightarrow Q) \iff (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Observe that all the truth values under $(P \Rightarrow Q) \iff (\neg P \vee Q)$ are T.
□

Definition 3.3. *A statement for which all the truth values are T is called a **tautology**.*

Example 3.4. *Determine if the implications $P \Rightarrow Q$ and $Q \Rightarrow P$ are equivalent.*

Solution: Consider the truth table

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Observe that the last two columns have different truth values. Therefore, the two implications are not equivalent. □

The implication $Q \Rightarrow P$ is called the **converse** of the implication $P \Rightarrow Q$. The previous example shows that an implication and its converse are not equivalent.

Fact 3.5. *Two statements P and Q are equivalent if and only if $P \Rightarrow Q$ and $Q \Rightarrow P$ both hold true.*

Proof. We show that the statements $P \iff Q$ and $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ are equivalent. To do so, consider the following truth table

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$P \iff Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

□

Example 3.6. *Show that the implications $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are equivalent.*

Solution: To prove this compute the truth table

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

□

The implication $\neg Q \Rightarrow \neg P$ is called the **contrapositive** of the implication $P \Rightarrow Q$. The previous example shows that an implication and its contrapositive are equivalent.

4 Methods of Proof

The goal of this section is to develop techniques for proving mathematical statements.

4.1 Direct Methods

We have already seen one way of proving a mathematical statement of the form

Theorem. If P, then Q.

Based of the fact that the implication $P \Rightarrow Q$ is false only when P is true and Q is false, the idea behind the method of proof that we discussed was to assume that P is true and then to proceed, through a chain of logical deductions, to conclude that Q is true. Here is the outline of the argument:

Proof: Suppose that P is true.

Then, ...

$\vdots$

Therefore, Q

□

This is an example of a **direct method** of proof. In the following section we discuss **indirect methods** of proof.

Example 4.1. Prove the statement: *If n is even, then n^2 is even.*

Proof: Assume that the integer n is even. Then, $n = 2k$, for some integer k. It then follows that

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2),$$

which shows that n^2 is even. □

4.2 Indirect Methods

4.2.1 Proving the Contrapositive

The idea behind this method of proof comes from the fact that the implication

$$\neg Q \Rightarrow \neg P$$

is equivalent to the implication

$$P \Rightarrow Q.$$

Thus, in order to prove $P \Rightarrow Q$, it suffices to prove $\neg Q \Rightarrow \neg P$. Here is the outline of the argument:

> **Theorem.** *If P, then Q*
>
> *Proof.* Suppose that $\neg Q$ is true; i.e., assume that Q is false.
>
> Then, ...
>
> $\vdots$
>
> Hence, $\neg P$.
>
> Consequently, $\neg Q \Rightarrow \neg P$ is true; therefore, $P \Rightarrow Q$ is true.
>
> □

Example 4.2. *Prove that n^2 is even implies that n is even.*

Proof. Suppose that n is not even. Then $n = 2k + 1$ for some integer k. It then follows that

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which shows that n^2 is not even. Thus, n is not even implies that n^2 is not even, and therefore the contrapositive is true; namely, n^2 is even implies that n is even □

4.2.2 Proof by Contradiction or *Reductio ad Absurdum*

A **contradiction** is a statement of the form $R \wedge \neg R$; that is, a statement which is always false. The idea behind *Reductio ad Absurdum* comes from the fact that the implication $(P \wedge \neg Q) \Rightarrow (R \wedge \neg R)$ is equivalent to the implication $P \Rightarrow Q$. To see why this is the case, consider the following truth table:

P	Q	$\neg Q$	$P \wedge \neg Q$	$R \wedge \neg R$	$(P \wedge \neg Q) \Rightarrow (R \wedge \neg R)$	$P \Rightarrow Q$
T	T	F	F	F	T	T
T	F	T	T	F	F	F
F	T	F	F	F	T	T
F	F	T	F	F	T	T

Here is the outline of an argument by contradiction:

Theorem. *If P, then Q*

Proof. Assume that P is true. Arguing by contradiction, suppose also that Q is false.

Then, ...

$\vdots$

R

$\vdots$

$\neg R$

Hence, $R \wedge \neg R$.

This is absurd. Consequently, it must be that case that Q is true. Therefore, $P \Rightarrow Q$ is true.

□

The argument outlined above can also be used to prove a statement of the form

Theorem. *P is true.*

Proof. Arguing by contradiction, assume that $\neg P$ is true; that is, suppose that P is false.

Then, ...

$\vdots$

R

$\vdots$

$\neg R$

Hence, $R \wedge \neg R$.

This is absurd. Consequently, it must be that case that P is true.

$\square$

In order to verify the validity of this argument, consider the truth table:

P	$\neg P$	$R \wedge \neg R$	$\neg P \Rightarrow (R \wedge \neg R)$
T	F	F	T
F	T	F	F

Observe that the columns for P and $\neg P \Rightarrow (R \wedge \neg R)$ have the same truth values; therefore, P and $\neg P \Rightarrow (R \wedge \neg R)$ are equivalent; in symbols,

$$(\neg P \Rightarrow (R \wedge \neg R)) \iff P$$

Before we present the next example, recall a few facts about prime numbers.

Definition 4.3. *A natural number $p > 1$ is said to be a* ***prime number****, or a* ***prime****, is its only (positive) divisors are 1 and p.*
If an integer is not a prime, then it is said to be a ***composite*** *number.*

Theorem 4.4 (Fundamental Theorem of Arithmetic)**.** *Every natural number $n > 1$ can be written as a product of primes. This representation of n is unique up to the order of the prime factors.*

Theorem 4.5. *There are infinitely many prime numbers.*[1]

[1]The first proof of this theorem is usually ascribed to Euclid, a Greek mathematician who lived circa 330 B.C.

Proof. Arguing by contradiction, suppose that there are only a finite number of prime numbers. Denote them by

$$p_1, p_2, p_3, \ldots, p_r.$$

Then, this list contains all the prime numbers.

Form the number

$$q = p_1 p_2 p_3 \cdots p_r + 1.$$

Then, q is not a prime since it is not in the list of primes given above. Thus, q is composite. Let p be a prime divisor of q. Then p is not on the list of primes; for if it was in the list, then p would divide $q - p_1 p_2 p_3 \cdots p_r = 1$. This is impossible. Hence, p is a prime which is not on the list; but this is absurd since the list

$$p_1, p_2, p_3, \ldots, p_r$$

is assumed to contain all the primes. This contradiction shows that there must be infinitely many primes. $\square$