NESTED DERIVATIVES: A SIMPLE METHOD FOR COMPUTING SERIES EXPANSIONS OF INVERSE FUNCTIONS.

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ABSTRACT. We give an algorithm to compute the series expansion for the inverse of a given function. The algorithm is extremely easy to implement and gives the first N terms of the series. We show several examples of its application in calculating the inverses of some special functions.

1. INTRODUCTION

"One must **always** invert."
Carl G. J. Jacobi

The existence of series expansions for inverses of analytic functions is a well-known result of complex analysis [17]. The standard inverse function theorem, a proof of which can be found, for example, in [12], is

Theorem 1. *Let $h(x)$ be analytic for $|x - x_0| < R$ where $h'(x_0) \neq 0$. Then $z = h(x)$ has an analytic inverse $x = H(z)$ in some ε-neighborhood of $z_0 = h(x_0)$.*

In the case when $x_0 = z_0 = 0$, $|h(x)| \leq M$ for $|x| < R$, and $h'(0) = a$, R. M. Redheffer [25] has shown that it is enough to take $\varepsilon = \frac{1}{4}\frac{(aR)^2}{M}$.

However, the *procedure* to obtain the actual series is usually very difficult to implement in practice. Under the conditions of Theorem 1, the two standard methods to compute the coefficients b_n of

$$h^{-1}(z) = H(z) = \sum_{n \geq 0} b_n (z - z_0)^n$$

are reversion of series [16], [26], [33], and Lagrange's theorem. The first requires one to expand $h(x)$ around x_0

$$h(x) = \sum_{n \geq 0} a_n (x - x_0)^n$$

and then solve for b_n in the equation

$$z = \sum_{n \geq 0} a_n \left[\sum_{n \geq 0} b_n (z - z_0)^n - x_0 \right]^n$$

Date: October 23, 2002.
1991 *Mathematics Subject Classification.* Primary 40-04, 33F10; Secondary 41A58.
Key words and phrases. Inverse function, Series expansion, Inversion Formula, Lagrange's Reversion Theorem, Special function, Symbolic Computation.

by equating powers of z and taking into account that $a_0 = z_0$ and $b_0 = x_0$. This method is especially useful if all that is known about $h(x)$ are the first few a_n . When $x_0 = z_0 = 0$, $a_1 = a$, it was shown by E. T. Whittaker [34] that

$$b_1 = \frac{1}{a}, \quad b_2 = -\frac{a_2}{a^3}, \quad b_3 = \frac{1}{3!a^5}\begin{vmatrix} 3a_2 & a \\ 6a_3 & 4a_2 \end{vmatrix}, \quad \cdots$$

$$b_n = \frac{(-1)^{n-1}}{n!a^{2n-1}}\begin{vmatrix} na_2 & a & 0 & 0 & \cdots \\ 2na_3 & (n+1)a_2 & 2a & 0 & \cdots \\ 3na_4 & (2n+1)a_3 & (n+2)a_2 & 3a & \cdots \\ 4na_5 & (3n+1)a_4 & 2(n+1)a_3 & (n+3)a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

In Example 10, we show how to get the b_n in term of the a_n using our method.

Maple [1] can reverse the power series of $h(x)$, provided $h(x)$ is not too complicated, by using the command:

$$> \text{Order} := N + 1;$$
$$> \text{solve (series (} h(x) \text{ , } x = x_0 \text{ , } N + 1) = z \text{ , } x);$$

where N is the number of terms wanted. Fast algorithms of order $(n \log n)^{3/2}$ for reversion of series have been analyzed by Brent and Kung [5], [6]. The multivariate case has been studied by several authors [4], [8], [14], [21] and Wright [35] has studied the connection between reversion of power series and "rooted trees".

The second and more direct method is Lagrange's inversion formula [1]

$$b_n = \frac{1}{n!}\,\frac{d^{n-1}}{dx^{n-1}}\left\{\left[\frac{x - x_0}{h(x) - z_0}\right]^n\right\}\Bigg|_{x=x_0}. \tag{1.1}$$

Unfortunately, more direct doesn't necessarily mean easier, and except for some simple cases (1.1) it is extremely complicated for practical applications. The q-analog[2] of (1.1) has been studied by various authors [2], [18], [19], [20] and a unified approach to both the regular and q-analog formulas have been obtained by Krattenthaler [23]. There has also been a great deal of attention to the asymptotic expansion of inverses [27], [28], [31], [32],.

In this paper, we present a simple, easy to implement method for computing the series expansion for the inverse of any function satisfying the conditions of Theorem 1, although the method is especially powerful when $h(x)$ has the form

$$h(x) = \int_a^x g(x)dx$$

and $g(x)$ is some function simpler than $h(x)$. Since this is the case for many special functions, we will present several such examples. The paper is organized as follows:

In section 2 we define a sequence of functions $\mathfrak{D}^n[f](x)$, obtained from a given one $f(x)$, that we call "nested derivatives", for reasons which will be clear from the definition. We give a computer code for generating the nested derivatives and examples of how $\mathfrak{D}^n[f](x)$ look for some elementary functions. Section 3 shows

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[1] We employ the computer system Maple 6.

[2] A q-analog is a mathematical expression parametrized by the quantity q which generalizes a known expression and reduces to it in the limit $q \to 1^+$.

how to compute the nested derivatives by using generating functions. We present some examples and compare the results with those obtained in Section 1.

Section 4 contains our main result of the use of nested derivatives to compute power series of inverses. We test our result with some known results and we apply the method for obtaining expansions for the inverse of the Error Function, the Incomplete Gamma Function, the Sine Integral, and other special functions.

2. Definitions

Definition 1. *We define the nested derivative* $\mathfrak{D}^n[f](x)$ *by the following recursion:*

$$\mathfrak{D}^0[f](x) \equiv 1$$

$$\mathfrak{D}^n[f](x) = \frac{d}{dx}\left[f(x) \times \mathfrak{D}^{n-1}[f](x)\right], \quad n \geq 1. \tag{2.1}$$

Proposition 1. $\mathfrak{D}^n[f](x)$ *satisfies the following basic properties:*

1. *For* $n \geq 1$, $\mathfrak{D}^n[\kappa] \equiv 0$, κ *constant.*
2. *For* $n \geq 0$, $\mathfrak{D}^n[\kappa f](x) = \kappa^n \mathfrak{D}^n[f](x)$, κ *constant.*
3. *For* $n \geq 1$, $\mathfrak{D}^n[f](x)$ *has the following integral representation:*

$$\mathfrak{D}^n[f](x) = \frac{1}{(2\pi i)^n} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_n} \frac{f(z_n)}{(z_n - x)^2} \prod_{k=1}^{n-1} \frac{f(z_k)}{(z_k - z_{k+1})^2} dz_n \dots dz_1,$$

where C_k *is a small loop around* x.

Proof. 1) and 2) follow immediately from the definition of $\mathfrak{D}^n[f](x)$.

3) We use induction on n. For $n = 1$ the result follows from Cauchy's formula:

$$\mathfrak{D}^1[f](x) = \frac{df}{dx} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z_1)}{(z_1 - x)^2} dz_1.$$

Assuming the result is true for n and using (2.1)

$$\begin{aligned}
\mathfrak{D}^{n+1}[f](x) = \frac{d}{dx}\left[f(x) \times \mathfrak{D}^n[f](x)\right] &= \frac{1}{2\pi i} \oint_{C_{n+1}} \frac{f(z_{n+1})\mathfrak{D}^n[f](z_{n+1})}{(z_{n+1} - x)^2} dz_{n+1} \\
&= \frac{1}{(2\pi i)^{n+1}} \oint_{C_1} \cdots \oint_{C_{n+1}} \frac{f(z_{n+1})}{(z_{n+1} - x)^2} \frac{f(z_n)}{(z_n - z_{n+1})^2} \prod_{k=1}^{n-1} \frac{f(z_k)}{(z_k - z_{k+1})^2} dz_{n+1} \dots dz_1 \\
&= \frac{1}{(2\pi i)^{n+1}} \oint_{C_1} \cdots \oint_{C_{n+1}} \frac{f(z_{n+1})}{(z_{n+1} - x)^2} \prod_{k=1}^{n} \frac{f(z_k)}{(z_k - z_{k+1})^2} dz_{n+1} \dots dz_1.
\end{aligned}$$

□

Algorithm 1. *The* $\mathfrak{D}$ *algorithm.*

The following Maple procedure implements the recurrence relation (2.1). We define $d(k)=\mathfrak{D}^{k}[f](x)$. N is the number of terms desired.

$$
\begin{aligned}
&> d(0) := 1;\\
&> \texttt{for}\ k\ \ \texttt{from}\ \ 0\ \texttt{to}\ \ N\ \texttt{do:}\\
&> d(k+1):\ = \texttt{simplify}\ (\ \texttt{diff}\ (\ f\ \ *\ d(k),\ x\ ):\\
&> \texttt{print}\ (\ k+1,\ d(k+1)\ ):\\
&> \texttt{od:}
\end{aligned}
\tag{2.2}
$$

Example 1. $f(x)=x$

$$
\begin{aligned}
\mathfrak{D}^{1}[f](x) &= 1\\
\mathfrak{D}^{2}[f](x) &= 1\\
&\ \vdots\\
\mathfrak{D}^{n}[f](x) &= 1.
\end{aligned}
$$

Example 2. $f(x)=x^{r},\quad r\neq 1$

$$
\begin{aligned}
\mathfrak{D}^{1}[f](x) &= rx^{r-1}\\
\mathfrak{D}^{2}[f](x) &= r(2r-1)x^{2(r-1)}\\
\mathfrak{D}^{3}[f](x) &= r(2r-1)(3r-2)x^{3(r-1)}\\
&\ \vdots\\
\mathfrak{D}^{n}[f](x) &= \prod_{j=1}^{n}\left[jr-(j-1)\right]x^{n(r-1)}\\
&= (r-1)^{n}\frac{\Gamma\left(n+1+\frac{1}{r-1}\right)}{\Gamma\left(1+\frac{1}{r-1}\right)}x^{n(r-1)}.
\end{aligned}
$$

Notice that when $r=\frac{k}{k+1},\quad k=1,2,\ldots$ *the sequence is finite*

$$
\mathfrak{D}^{n}[f](x)=\begin{cases}\frac{k!}{(k-n)!\ (k+1)^{n}}x^{-\frac{n}{k+1}}, & 1\leq n\leq k\\ 0, & n\geq k+1\end{cases}.
$$

Example 3. $f(x)=e^{rx}$

$$
\begin{aligned}
\mathfrak{D}^{1}[f](x) &= re^{rx}\\
\mathfrak{D}^{2}[f](x) &= 2r^{2}e^{2rx}\\
\mathfrak{D}^{3}[f](x) &= 6r^{3}e^{3rx}\\
&\ \vdots\\
\mathfrak{D}^{n}[f](x) &= n!r^{n}e^{nrx}.
\end{aligned}
$$

3. Generating functions

Generating functions provide a valuable method for computing sequences of functions defined by an iterative process; we will use them to calculate $\mathfrak{D}^{n}[f](x)$. In the sequel, we shall implicitly assume that the generating function series converges in some small disc around $z = 0$.

Theorem 2. *Given* $h(x) = \int \frac{1}{f(x)} dx$, *its inverse* $H(x) = h^{-1}(x)$, *and the exponential generating function* $G(x,z) = \sum_{n\geq 0} \mathfrak{D}^n[f](x) \frac{z^n}{n!}$, *we have*

$$G(x,z) = \frac{1}{f(x)} (f \circ H) [z + h(x)] . \tag{3.1}$$

Proof. Taking (2.1) into account

$$\begin{aligned}
\frac{\partial}{\partial x} [f(x)G(x,z)] &= \sum_{n\geq 0} \frac{d}{dx} [f(x) \times \mathfrak{D}^n[f](x)] \frac{z^n}{n!} \\
&= \sum_{n\geq 0} \mathfrak{D}^{n+1}[f](x) \frac{z^n}{n!} = \sum_{n\geq 1} \mathfrak{D}^n[f](x) \frac{z^{n-1}}{(n-1)!} \\
&= \frac{\partial}{\partial z} \sum_{n\geq 0} \mathfrak{D}^n[f](x) \frac{z^n}{n!} = \frac{\partial}{\partial z} G(x,z).
\end{aligned}$$

Hence, the generating function satisfies the PDE

$$\frac{\partial (f \times G)}{\partial x} = \frac{\partial G}{\partial z}$$

with general solution

$$G(x,z) = \frac{1}{f(x)} g [z + h(x)]$$

where $g(z)$ is an arbitrary analytic function. Invoking the boundary condition $G(x,0) = \mathfrak{D}^0[f](x) = 1$, gives

$$\frac{1}{f(x)} g [h(x)] = 1$$

and therefore

$$f(x) = g[h(x)].$$

If we take $x = H(w)$ then

$$f [H(w)] = g \{h [H(w)]\} = g(w)$$

and the theorem follows. □

Example 4. $f(x) = x$.

Here $h(x) = \int \frac{1}{x} dx = \ln(x)$, $H(x) = e^x$, *and from (3.1) it follows that*

$$G(x,z) = \frac{1}{x} \exp [z + \ln(x)] = e^z.$$

We could obtain the same result from Example 1 by summing the series

$$G(x,z) = \sum_{n\geq 0} 1 \frac{z^n}{n!} = e^z.$$

Example 5. $f(x) = x^r, \quad r \neq 1.$

Now $h(x) = \int x^{-r} dx = \frac{x^{1-r}}{1-r}, \quad H(x) = [(1-r)x]^{\frac{1}{1-r}}$ *, and we get*

$$G(x,z) = x^{-r} \left\{ \left[(1-r) \left(z + \frac{x^{1-r}}{1-r} \right) \right]^{\frac{1}{1-r}} \right\}^r = \left[\frac{(1-r)z + x^{1-r}}{x^{1-r}} \right]^{\frac{r}{1-r}}$$
$$= \left[1 + (1-r)x^{r-1} z \right]^{\frac{r}{1-r}}.$$

Expanding in series around $z = 0$*, we recover the result from Example 2.*

If $\frac{r}{1-r} = k$*, i.e.* $r = \frac{k}{k+1}, \quad k = 0, 1, \ldots,$ *then* $G(x,z)$ *is a polynomial of degree* k *in* z *and hence*

$$\mathfrak{D}^n [f] (x) = 0, \quad n \geq k+1$$

as we have already observed in Example 2.

Given the particular form of the function $h(x)$ in Theorem 2, we can get alternative expressions for (3.1) which sometimes are easier to employ.

Corollary 1. *i)*

$$G(x,z) = \frac{1}{f(x)} H' [z + h(x)] \tag{3.2}$$

ii)

$$G(x,z) = \frac{d}{dx} H [z + h(x)] .$$

Proof. i) By definition $(h \circ H)(x) = x$, so

$$h' [H(x)] H'(x) = 1$$

but since $h(x) = \int \frac{1}{f(t)} dt$, we get

$$\frac{1}{f [H(x)]} H'(x) = 1$$

or

$$(f \circ H)(x) = H'(x)$$

and therefore

$$G(x,z) = \frac{1}{f(x)} (f \circ H) [z + h(x)]$$
$$= \frac{1}{f(x)} H' [z + h(x)] .$$

ii) We have

$$\frac{d}{dx} H [z + h(x)] = H' [z + h(x)] h'(x)$$
$$= H' [z + h(x)] \frac{1}{f(x)}$$

and the conclusion follows from part (i).

□

4. Applications

We now state our Main Result.

Theorem 3. *Given* $h(x) = \int\limits_a^x \frac{1}{f(t)} dt$, *with* $f(a) \neq 0, \pm\infty$, *and its inverse* $H(x) = h^{-1}(x)$, *we have*

$$H(z) = a + f(a) \sum_{n \geq 1} \mathfrak{D}^{n-1}[f]\,(a) \frac{z^n}{n!} \tag{4.1}$$

where $|z| < \varepsilon$, *for some* $\varepsilon > 0$.

Proof. We first observe that since $h(a) = 0$, we have $H(0) = a$, and from (3.2)

$$G(a, z) = \frac{1}{f(a)} H'\left[z + h(a)\right] = \frac{1}{f(a)} H'(z)$$

where

$$G(a, z) = \sum_{n \geq 0} \mathfrak{D}^n[f]\,(a) \frac{z^n}{n!}.$$

Hence,

$$\begin{aligned} H(z) &= H(0) + \int\limits_0^z f(a) \sum_{n \geq 0} \mathfrak{D}^n[f]\,(a) \frac{t^n}{n!} dt \\ &= a + f(a) \sum_{n \geq 0} \mathfrak{D}^n[f]\,(a) \frac{z^{n+1}}{(n+1)!} \\ &= a + f(a) \sum_{n \geq 1} \mathfrak{D}^{n-1}[f]\,(a) \frac{z^n}{n!}. \end{aligned}$$

□

Example 6. $f(x) = e^{-x}, \quad a = 0.$

We have $f(0) = 1$,

$$\begin{aligned} h(x) &= \int\limits_0^x e^t dt = e^x - 1 \\ H(x) &= \ln(x+1) \end{aligned}$$

and from Example (3)

$$\mathfrak{D}^n[f]\,(0) = (-1)^n n!.$$

Hence, from (4.1) we get the familiar formula

$$\ln(z+1) = \sum_{n \geq 1} (-1)^{n-1} \frac{z^n}{n}.$$

Example 7. $f(x) = x^2 + 1, \quad a = 0.$

Now $f(0) = 1$,

$$\begin{aligned} h(x) &= \int\limits_0^x \frac{1}{t^2+1} dt = \arctan(x) \\ H(x) &= \tan(x) \end{aligned}$$

and (4.1) implies

$$\tan(z) = \sum_{n\geq 1} \mathfrak{D}^{n-1}[x^2+1]\,(0) \frac{z^n}{n!}.$$

Therefore,

$$\mathfrak{D}^{2k+1}[x^2+1]\,(0) = 0, \quad k \geq 0$$

$$\mathfrak{D}^{2k}[x^2+1]\,(0) = \frac{2}{k+1} 4^k \left(4^{k+1}-1\right) \left|B_{2(k+1)}\right|, \quad k \geq 1 \tag{4.2}$$

where B_k are the Bernoulli numbers [1].

Remark 1. *From Example 2, we recall that*

$$\mathfrak{D}^n[x^2]\,(x) = (n+1)!x^n$$

and consequently

$$\mathfrak{D}^n[x^2]\,(0) = 0, \quad n \geq 1. \tag{4.3}$$

Comparing (4.2) and (4.3) we can see the highly nonlinear behavior of the nested derivatives, since even the addition of 1 to $f(x)$ creates a completely different sequence of values, far more complex than the original.

Lets now start testing our result on some classical functions.

Example 8. $f(x) = \sqrt{1-p^2\sin^2(x)}, \quad 0 \leq p \leq 1, \quad a = 0.$

We have, $f(0) = 1$ and

$$h(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1-p^2\sin^2(\theta)}} = F(p;\phi)$$
$$H(p;x) = \mathrm{am}(p;x)$$

where $F(p;\phi)$ is the incomplete elliptic integral of the first kind, and $\mathrm{am}(p;x)$ is the elliptic amplitude [29]

$$\mathrm{am}(p;x) = \arcsin[\mathrm{sn}(p;x)] = \arccos[cn(p;x)] = \arcsin\left[\frac{\sqrt{1-dn^2(p;x)}}{p}\right]$$

with $\mathrm{sn}(p;x)$, $cn(p;x)$, and $dn(p;x)$ denoting the Jacobian elliptic functions.

Computing $\mathfrak{D}^n[f]\,(0)$ with (2.2) gives:

$$\mathfrak{D}^{2k+1}[f]\,(0) = 0, \quad k \geq 0$$
$$\mathfrak{D}^{2k}[f]\,(0) = (-1)^k p^2 Q_k(p), \quad k \geq 1$$

where $Q_k(p)$ is a polynomial of degree $2(k-1)$ of the form

$$Q_k(p) = p^{2(k-1)} + \cdots + 2^{2(k-1)}.$$

The first few $Q_k(p)$ are:

$$Q_1(p) = 1$$
$$Q_2(p) = p^2 + 4$$
$$Q_3(p) = p^4 + 44p^2 + 16$$
$$Q_4(p) = p^6 + 408p^4 + 912p^2 + 64$$
$$Q_5(p) = p^8 + 3688p^6 + 307682p^4 + 15808p^2 + 256$$

and (4.1) implies

$$am(p;x) = z - p^2\frac{z^3}{3!} + p^2(p^2+4)\frac{z^5}{5!} - p^2(p^4+44p^2+16)\frac{z^7}{7!} + \cdots \tag{4.4}$$

in agreement with the known expansions for $am(p;x)$ [11].

Example 9. $f(x) = e^{-x}(x+1)^{-1}, \quad a = 0, \quad f(a) = 1.$

Here

$$h(x) = xe^x$$
$$H(x) = LW(x)$$

where by $LW(x)$ *we denote the Lambert W Function* [9], [10], [22]. *In this case (2.2) gives*

$$\mathfrak{D}^1[f]\,(0) = -2$$
$$\mathfrak{D}^2[f]\,(0) = 9$$
$$\mathfrak{D}^3[f]\,(0) = -64$$
$$\vdots$$
$$\mathfrak{D}^n[f]\,(0) = [-(n+1)]^n\,.$$

From (4.1) we conclude

$$LW(z) = \sum_{n\geq 1}(-1)^{n-1}n^{n-1}\frac{z^n}{n!}.$$

Example 10. *Lets now derive a well known result* [1] *about reversion of series. If we take*

$$h(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots$$

where $a_1 \neq 0$, *then*

$$f(x) = \frac{1}{h'(x)} = \frac{1}{a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + 7a_7x^6 + \cdots}$$

$a = 0$, $f(0) = \frac{1}{a_1}$, *and from (2.2) we get*

$$\mathfrak{D}^1[f]\,(0) = -2\frac{a_2}{(a_1)^2}$$

$$\mathfrak{D}^2[f]\,(0) = 6\frac{2\,(a_2)^2 - a_1a_3}{(a_1)^4}$$

$$\mathfrak{D}^3[f]\,(0) = 24\frac{5a_1a_2a_3 - (a_1)^2\,a_4 - 5(a_2)^3}{(a_1)^6}$$

$$\mathfrak{D}^4[f]\,(0) = 120\frac{6(a_1)^2a_2a_3 + 3\,(a_1a_3)^2 + 14(a_2)^4 - (a_1)^3a_5 - 21a_1(a_2)^2a_3}{(a_1)^8}.$$

Hence,

$$H(z) = \frac{1}{a_1}z - \frac{a_2}{(a_1)^3}z^2 + \frac{2\,(a_2)^2 - a_1a_3}{(a_1)^5}z^3 + \frac{5a_1a_2a_3 - (a_1)^2\,a_4 - 5(a_2)^3}{(a_1)^7}z^4$$
$$+ \frac{6(a_1)^2a_2a_3 + 3\,(a_1a_3)^2 + 14(a_2)^4 - (a_1)^3a_5 - 21a_1(a_2)^2a_3}{(a_1)^9}z^5 + \cdots b_nz^n + \cdots\,.$$

Remark 2. *A derivation of the explicit formula for the n^{th} term is given by Morse and Feshbach* [24, Part 1 pp. 411–413],

$$b_n = \frac{1}{n\,(a_1)^n} \sum_{s,t,u,\ldots} (-1)^{s+t+u+\cdots} \frac{n(n+1)\cdots(n-1+s+t+u+\cdots)}{s!t!u!\cdots} \left(\frac{a_2}{a_1}\right)^s \left(\frac{a_3}{a_1}\right)^t \cdots$$

$$s+2t+3u+\cdots = n-1.$$

Example 11. *The Error Function,* $\mathrm{erf}(x)$.

We now have

$$h(x) = \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$f(x) = \frac{\sqrt{\pi}}{2} e^{x^2}, \quad a = 0, \quad f(a) = \frac{\sqrt{\pi}}{2}$$

and (2.2) gives

$$\mathfrak{D}^n[f](0) = \begin{cases} 0, & n = 2k+1, \quad k \geq 0 \\ \left(\frac{\sqrt{\pi}}{2}\right)^n A_k, & n = 2k, \quad k \geq 0 \end{cases}$$

where

$$A_0 = 1, \quad A_1 = 2, \quad A_2 = 28, \quad A_3 = 1016, \quad A_4 = 69904$$
$$A_5 = 7796768, \quad A_6 = 1282366912, \quad A_7 = 291885678464.$$

From (4.1) we get

$$H(z) = \sum_{n \geq 0} A_n \left(\frac{\sqrt{\pi}}{2}\right)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}.$$

Our result agrees with other authors calculations previously published [3], [7], [13], [15], [30].

We will now extend (4.1) to a more general result.

Corollary 2. *Given* $h(x) = \int_a^x \frac{1}{f(t)} dt$, $z_0 = h(b)$, *with* $f(b) \neq 0, \pm\infty$, *and its inverse* $H(x) = h^{-1}(x)$, *we have*

$$H(z) = b + f(b) \sum_{n \geq 1} \mathfrak{D}^{n-1}[f](b) \frac{(z-z_0)^n}{n!} \tag{4.5}$$

where $|z - z_0| < \varepsilon$, *for some* $\varepsilon > 0$.

Proof. Lets consider the function

$$u(x) = h(x) - z_0$$

which satisfies $u(b) = 0$, and its inverse $U(x) = u^{-1}(x)$. Since $f(b) \neq 0, \pm\infty$, we can apply (4.1) to $u(x)$ and conclude that

$$U(z) = b + f(b) \sum_{n \geq 1} \mathfrak{D}^{n-1}[f](b) \frac{z^n}{n!}.$$

All that is left is to see the relation between $U(z)$ and $H(z)$.

Suppose that $u(x) = y$. Then

$$\begin{aligned} y = u(x) &= h(x) - z_0 \\ h(x) &= y + z_0 \\ x &= H(y + z_0) \end{aligned}$$

and therefore

$$U(y) = H(y + z_0)$$

or

$$H(z) = U(z - z_0)$$

and (4.5) follows. □

And now we take one step into the unknown.

Example 12. *The Incomplete Gamma Function, $\gamma(\nu; x)$.*
We have

$$\begin{aligned} h(\nu; x) &= \gamma(\nu; x) = \int_0^x e^{-t} t^{\nu-1} dt, \quad \nu > 0, \quad x \geq 0 \\ f(\nu; x) &= e^x x^{1-\nu}, \quad a = 0. \end{aligned}$$

Since

$$f(\nu; a) = \begin{cases} 0, & 0 < \nu < 1 \\ \infty, & \nu > 1 \end{cases}$$

we can't apply (4.1). Choosing $b = 1$, $z_0(\nu) = \gamma(\nu; 1)$, $f(\nu; b) = e$, we conclude from (4.5)

$$H(\nu; z) = 1 + e \sum_{n \geq 1} \mathfrak{D}^{n-1}[f](1) \frac{[z - z_0(\nu)]^n}{n!}.$$

We use (2.2) to compute the first few $\mathfrak{D}^n[f](1)$ and obtain

$$\mathfrak{D}^n[f](1) = e^n Q_n(\nu)$$

where $Q_n(\nu)$ is a polynomial of degree n

$$\begin{aligned} Q_1(\nu) &= 2 - \nu \\ Q_2(\nu) &= 7 - 7\nu + 2\nu^2 \\ Q_3(\nu) &= 36 - 53\nu + 29\nu^2 - 6\nu^3 \\ Q_4(\nu) &= 245 - 474\nu + 375\nu^2 - 146\nu^3 + 24\nu^4 \\ Q_5(\nu) &= 2076 - 4967\nu + 5104\nu^2 - 2847\nu^3 + 874\nu^4 - 120\nu^5 \end{aligned}$$

and we can write

$$H(\nu; z) = 1 + \sum_{n \geq 1} e^n Q_{n-1}(\nu) \frac{[z - z_0(\nu)]^n}{n!}.$$

Example 13. *The Sine Integral,* $\mathrm{Si}(x)$.

In this case

$$h(x) = \mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt$$
$$f(x) = \frac{x}{\sin(x)}, \quad a = 0.$$

For this example, $f(a)$ is well defined but to simplify the calculations we choose $b = \frac{\pi}{2}, \quad z_0 = \mathrm{Si}(\frac{\pi}{2}) \simeq 1.37076216$, *and then*

$$f(b) = \frac{\pi}{2}$$
$$\mathfrak{D}^n[f]\left(\frac{\pi}{2}\right) = Q_n(\pi)$$

where $Q_n(x)$ is once again a polynomial

$$Q_1(x) = 1$$
$$Q_2(x) = 1 + \frac{1}{4}x^2$$
$$Q_3(x) = 1 + \frac{7}{4}x^2$$
$$Q_4(x) = 1 + 8x^2 + \frac{9}{16}x^4$$
$$Q_5(x) = 1 + \frac{61}{2}x^2 + \frac{159}{16}x^4$$
$$Q_6(x) = 1 + \frac{423}{4}x^2 + \frac{1671}{16}x^4 + \frac{225}{64}x^6.$$

and from (4.5) we obtain

$$H(z) = \frac{\pi}{2} + \frac{\pi}{2}\sum_{n\geq 1} Q_n(\pi)\frac{(z-z_0)^n}{n!}.$$

Example 14. *The Logarithm Integral,* $\mathrm{li}(x)$.

From the definition

$$h(x) = \mathrm{li}(x) = \int_0^x \frac{1}{\ln(t)} dt$$
$$f(x) = \ln(x), \quad a = 0.$$

In this case $f(a) = -\infty$, so we must choose b. A natural candidate is $b = e$, and then

$$f(b) = 1, \quad z_0 = \mathrm{li}(e) \simeq 1.895117816$$
$$\mathfrak{D}^n[f]\,(e) = e^{-n}A_n$$

with

$$A_1 = 1, \quad A_2 = 0, \quad A_3 = -1, \quad A_4 = 2, \quad A_5 = 1$$
$$A_6 = -26, \quad A_7 = 99, \quad A_8 = 90, \quad A_9 = -3627$$

and we have

$$H(z) = e + \sum_{n\geq 1} A_n \frac{(z-z_0)^n}{n!}.$$

Example 15. *The Incomplete Beta Function,* $B(\nu,\mu;x)$.

By definition

$$h(\nu,\mu;x) = B(\nu,\mu;x) = \int_0^x t^{\nu-1}(1-t)^{\mu-1}dt, \quad 0 \leq x < 1$$

and hence

$$f(\nu,\mu;x) = x^{1-\nu}(1-x)^{1-\mu}, \quad a = 0.$$

To avoid the possible singularity at $x = 0$, *we consider* $b = \frac{1}{2}$ *and therefore*

$$f(\nu,\mu;b) = \frac{1}{4}2^{\nu+\mu}$$

$$z_0(\nu,\mu) = B(\nu,\mu;\frac{1}{2}).$$

The $\mathfrak{D}$ *algorithm now gives*

$$\mathfrak{D}^n[f]\left(\frac{1}{2}\right) = 2^{n(\nu+\mu-1)}Q_n(\nu,\mu)$$

with $Q_n(\nu,\mu)$ *a multivariate polynomial of degree* n

$$Q_1(\nu,\mu) = \mu - \nu$$

$$Q_2(\nu,\mu) = -2 + \nu - 4\nu\mu + \mu + 2\mu^2 + 2\nu^2$$

$$Q_3(\nu,\mu) = (\mu-\nu)(6\mu^2 - 12\nu\mu + 7\mu - 12 + 6\nu^2 + 7\nu)$$

$$\begin{aligned} Q_4(\nu,\mu) = {} & 16 - 46\nu\mu^2 - 46\nu^2\mu - 63\mu^2 - 22\mu + 154\nu\mu - 96\nu\mu^3 - 96\nu^3\mu + \\ & 144\nu^2\mu^2 - 22\nu - 63\nu^2 + 24\nu^4 + 46\nu^3 + 24\mu^4 + 46\mu^3 \end{aligned}$$

$$\begin{aligned} Q_5(\nu,\mu) = {} & (\mu-\nu)(120\mu^4 + 326\mu^3 - 480\nu\mu^3 + 720\nu^2\mu^2 - 323\mu^2 - 326\nu\mu^2 \\ & - 362\mu - 480\nu^3\mu + 1154\nu\mu - 326\nu^2\mu - 323\nu^2 + 240 + 120\nu^4 \\ & + 326\nu^3 - 362\nu). \end{aligned}$$

Conclusion 1. *We have presented a simple method for computing the series expansion for the inverses of functions and given a Maple procedure to generate the coefficients in these expansions. We showed several examples of the method applied to elementary and special functions, and stated the first few terms of the series in each case.*

Acknowledgement 1. *I wish to express my gratitude to the referees for their extremely valuable suggestions and comments on previous versions of this work.*

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