FIR Filter Design via Spectral Factorization and Convex Optimization

Shao-Po Wu, **Stephen Boyd** (Stanford)
& Lieven Vandenberghe (UCLA)

UCSB 10/24/97

Outline

- Convex optimization & interior-point methods
- FIR filters & magnitude specs
- Spectral factorization
- Examples
  - lowpass filter design
  - minimax logarithmic (dB) approximation
  - third-octave equalization
  - antenna array pattern design
- Spectral factorization methods
- Discretization

Convex optimization problems

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0, \ldots, f_L(x) \leq 0, \\ & Ax = b \end{array}$$

- $x \in \mathbf{R}^n$ is optimization variable
- f_i are **convex**: for $0 \leq \lambda \leq 1$,
  $f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f_i(y)$
- examples: linear & (convex) quadratic programs

(roughly speaking,)

Convex optimization problems are fundamentally tractable

- computation time is small, grows gracefully with problem size and required accuracy
- large problems solved quickly in practice
- what “solve” means:
  - find **global** optimum within a given tolerance, or,
  - find **proof** (certificate) of infeasibility

Interior-point methods

- handle linear and **nonlinear** convex problems
- based on Newton's method applied to 'barrier' functions that trap x in **interior** of feasible region (hence the name IP)
- worst-case complexity theory: # Newton steps $\sim \sqrt{\text{problem size}}$
- in practice: # Newton steps between 5 & 50 (!)
- can exploit problem structure (sparsity, state equations) to reduce computation per Newton step
- 1000s variables, 10000s constraints feasible on PC

FIR filter

Finite impulse response (FIR) filter of order n:

$$y(t) = \sum_{k=0}^{n-1} h(k)u(t-k)$$

$h = \big(h(0), h(1), \ldots, h(n-1)\big) \in \mathbf{R}^n$ are the filter **coefficients**

Frequency response $H : [0, \pi] \to \mathbf{C}$,

$$H(\omega) = h(0) + h(1)e^{-j\omega} + \cdots + h(n-1)e^{-j(n-1)\omega}$$

Filter magnitude specs

magnitude spec:

$$L(\omega) \leq |H(\omega)| \leq U(\omega), \quad \omega \in [0, \pi]$$

$L, U : [0, \pi] \to \mathbf{R}_+$ given bounds; can take $L(\omega) = 0$, $U(\omega) = \infty$

- arises in many applications (audio, spectrum shaping, ...)
- upper bounds are convex in h; lower bounds are not

Magnitude filter design problem involves magnitude specs

Classical example: lowpass filter design

lowpass filter with maximum stopband attenuation:

$$\begin{array}{ll} \text{minimize} & \delta_2 \\ \text{subject to} & 1/\delta_1 \leq |H(\omega)| \leq \delta_1, \quad \omega \in [0, \omega_\mathrm{p}] \\ & |H(\omega)| \leq \delta_2, \quad \omega \in [\omega_\mathrm{s}, \pi] \end{array}$$

- variables: $h \in \mathbf{R}^n$ (filter coefficients), $\delta_2 \in \mathbf{R}$ (stopband attenuation)
- parameters: $\delta_1 \in \mathbf{R}$ (logarithmic passband ripple), n (order), ω_p (passband frequency), ω_s (stopband frequency)

magnitude filter design problems are **nonconvex**

- can get trapped in local minima
- cannot unambiguously determine feasibility

by change of variables, can formulate as **convex** problem

- can **efficiently** compute **global** solution
- unambiguously determine feasibility
- get absolute limit of performance

Autocorrelation coefficients

autocorrelation coefficients of filter:

$$r(t) = \sum_{k=-n+1}^{n-1} h(k)h(k+t), \quad t \in \mathbf{Z}$$

- $r(t) = r(-t)$; $r(t) = 0$ for $t \geq n$
- suffices to specify $r = \big(r(0), \ldots, r(n-1)\big) \in \mathbf{R}^n$

Fourier transform of r is

$$R(\omega) = r(0) + \sum_{k=1}^{n-1} r(k)\left(e^{jk\omega} + e^{-jk\omega}\right) = |H(\omega)|^2$$

Magnitude spec via r

magnitude spec can be expressed as

$$L(\omega)^2 \leq R(\omega) \leq U(\omega)^2, \quad \omega \in [0, \pi]$$

- for each ω, **linear inequality** in r
- hence magnitude spec is **convex** constraint in r

must add: r is the autocorrelation coefficients of **some** $h \in \mathbf{R}^n$.

Spectral factorization theorem

$$R(\omega) = r(0) + \sum_{k=0}^{n-1} r(k) \left(e^{jk\omega} + e^{-jk\omega} \right)$$

admits the representation

$$R(\omega) = \left| \sum_{k=0}^{n-1} h(k) e^{-jk\omega} \right|^2$$

if and only if

$$R(\omega) \geq 0, \quad \omega \in [0, \pi]$$

- spectral factorization condition is **convex** constraint in r
- many ways to find spectral factor h given r

Lowpass filter design (again)

with variables r and $\tilde{\delta}_2$, problem becomes

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \leq R(\omega) \leq \tilde{\delta}_1, \quad \omega \in [0, \omega_{\rm p}] \\ & R(\omega) \leq \tilde{\delta}_2, \quad \omega \in [\omega_{\rm s}, \pi] \\ & R(\omega) \geq 0, \quad \omega \in [0, \pi] \end{array}$$

($\tilde{\delta}_i$ corresponds to δ_i^2 in original problem)

- a **convex** problem in r and $\tilde{\delta}_2$
- hence, can be **efficiently, globally** solved

Variations

- minimize ripple $\tilde{\delta}_1$ in dB (nonlinear convex problem)
- minimize order n (quasiconvex problem)
- minimize stopband $\omega_{\rm p}$ (quasiconvex problem)
- multiple stop & pass bands

these can be efficiently, globally solved

Minimax logarithmic (dB) approximation

given desired frequency response magnitude $D : [0, \pi] \to \mathbf{R}_+$, find

$$h = \mathrm{argmin} \max_{\omega \in [0,\pi]} \left| \log |H(\omega)| - \log D(\omega) \right|$$

reformulate as

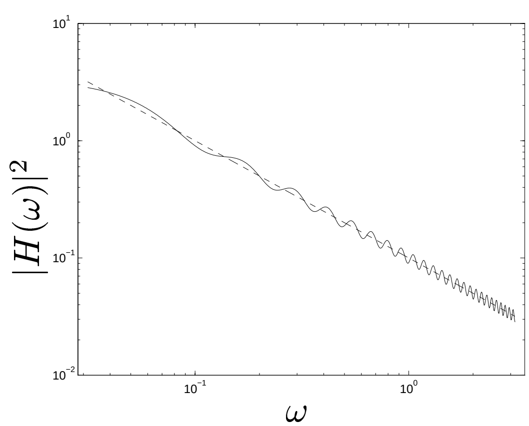
$$\begin{array}{ll} \text{minimize} & \tau \\ \text{subject to} & 1/\tau \le R(\omega)/D(\omega)^2 \le \tau, \quad \omega \in [0, \pi] \end{array}$$

(constraint implies $R(\omega) \ge 0$ for $\omega \in [0, \pi]$)

- a **convex** problem in $r \in \mathbf{R}^n$ and $\tau \in \mathbf{R}$
- hence efficiently, globally solved

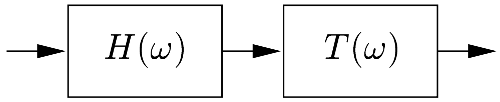
Example: $1/f$ (pink noise) filter

minimax dB fit over $0.01\pi \le \omega \le \pi$, $D(\omega) = 1/\sqrt{\omega}$



for 50-tap filter, optimal fit is ± 0.5dB

Equalization



- given system frequency response $T : [0, \pi] \to \mathbf{C}$
- design FIR equalizer H
- so equalized freq response TH has desired properties

Third-octave equalization

K third-octave frequency intervals in $[0, \pi]$:

$$[\Omega_1, \Omega_2], \quad \dots \quad [\Omega_K, \Omega_{K+1}], \qquad \Omega_k = 2^{(k-1)/3}\Omega_1$$

gain of equalized system in kth band is

$$g_k = \left(\frac{1}{\Omega_{k+1} - \Omega_k} \int_{\Omega_k}^{\Omega_{k+1}} |TH(\omega)|^2 \, d\omega \right)^{1/2}$$

third-octave equalization: choose H so $g_k \approx 1$
(gives good results for audio perception)

formulate third-octave equalization problem as

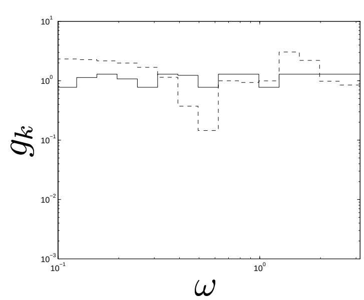
$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & 1/\alpha \leq \frac{1}{\Omega_{k+1}-\Omega_k} \int_{\Omega_k}^{\Omega_{k+1}} R(\omega)|T(\omega)|^2 \, d\omega \leq \alpha, \quad k = 1, \ldots, K, \\ & R(\omega) \geq 0, \quad \omega \in [0, \pi] \end{array}$$

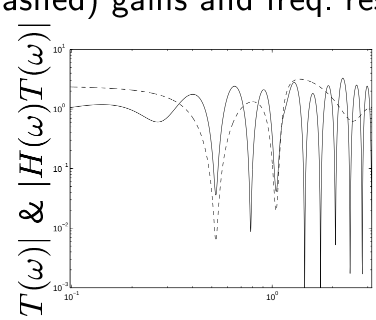
- nonlinear **convex** problem in r, α
- hence efficiently, globally solved

Third-octave equalization example

- $n = 20$; 15 third-octave bands from $\Omega_1 = 0.031\pi$ to $\Omega_{16} = \pi$
- constraint $|H(\omega)| \leq 10$ for all ω

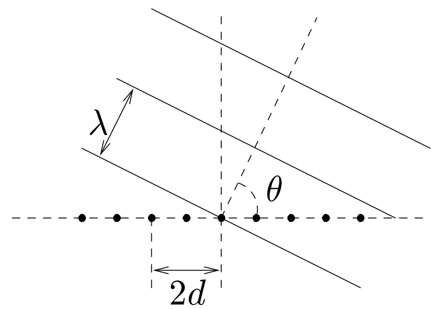
equalized (solid) and unequalized (dashed) gains and freq. response:





- gains equalized to ± 2dB
- deep notch in T near $\omega = 0.5$ makes constraint $|H| \leq 10$ active

Antenna array magnitude pattern design



- n isotropic antenna elements with spacing d
- plane harmonic wave incident from angle θ
- frequency ω, wavelength λ
- element outputs linearly combined with complex weights w_i

Antenna array magnitude pattern design

far-field pattern $G : [0, 2\pi) \to \mathbf{C}$:

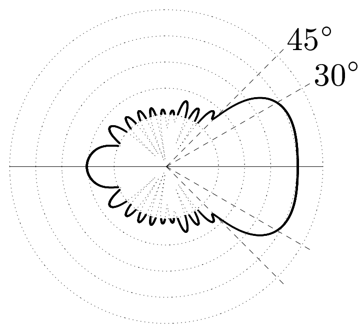
$$G(\theta) = \sum_{k=0}^{n-1} w(k) \exp\left(j \frac{2\pi k d}{\lambda} \cos\theta \right)$$

- design variables: $w \in \mathbf{C}^n$
- magnitude spec: $L(\theta) \leq |G(\theta)| \leq U(\theta)$, $\theta \in [0, 2\pi)$

- can convert to FIR filter problem with complex coefficients
- hence, same techniques work . . .

Antenna array pattern design example

- 12 elements, spacing $d = 0.45\lambda$
- allowed ripple ± 2dB in $\pm 30°$ beam
- minimize max of $|G|$ outside $\pm 45°$



10dB divisions; -19dB sidelobe attenuation achieved

Extensions

some other specifications/problems that are convex in r:

- bound on size of filter coefficients:

$$r(0) = \sum_i h(i)^2 \leq M$$

- bounds on log-magnitude slope:

$$a \leq \frac{d|H|}{d\omega} \frac{\omega}{|H(\omega)|} = (1/2) \frac{dR}{d\omega} \frac{\omega}{R(\omega)} \leq b$$

- multi-system magnitude equalization
- magnitude design of infinite impulse response (IIR) filters

Spectral factorization methods

given $T(z) = r(0) + \sum_{k=1}^{n-1} r(k)(z^k + z^{-k})$ with

$$T(e^{j\omega}) \geq 0, \qquad \omega \in [0, \pi]$$

find $S(z) = h(0) + h(1)z^{-1} + \cdots + h(n-1)z^{-(n-1)}$ such that

$$T(z) = S(z)S(z^{-1})$$

methods:

- compute roots of T inside unit disk
- Cholesky factorization of banded Toeplitz matrix
- solution of algebraic Riccati equation
- Newton's method
- Fast Fourier transform

Discretization

constraints in problems above are **semi-infinite**:
have a constraint for each $\omega \in [0, \pi]$

discretization: replace $[0, \pi]$ by finite set, e.g., $\omega_i = i\pi/m$, $i = 0, \ldots, m$

example: discretized max attenuation lowpass filter:

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \leq R(\omega_i) \leq \tilde{\delta}_1, \quad \omega_i \in [0, \omega_{\rm p}] \\ & R(\omega_i) \leq \tilde{\delta}_2, \quad \omega_i \in [\omega_{\rm s}, \pi] \\ & R(\omega_i) \geq 0, \quad i = 0, \ldots, m \end{array}$$

... a linear program in r and $\tilde{\delta}_2$

Discretization (cont'd)

- works very well in practice; common rule of thumb $m \approx 15n$
- can add appropriate, small 'safety factor' to ensure $R(\omega) \geq 0$ between sampled frequencies
- is basis for sophisticated methods (e.g., exchange)

Conclusions

- magnitude filter design problems can be reformulated as **convex optimization problems**
- hence, efficiently solved by new interior-point methods
- autocorrelation coefficients are designed; filter coefficients are obtained via spectral factorization
- can handle many useful extensions, e.g., minimum-order, minimax dB designs

References

- Wu, Boyd, Vandenberghe, FIR filter design via spectral factorization and convex optimization
- Boyd, Vandenberghe, Introduction to convex optimization with engineering applications (course notes 1997)
- Herrman and Schüssler, Design of nonrecursive digital filters with minimum-phase (Electronics Letters 1970)
- Mian and Nainer (1982), Kamp and Wellekens (1983), Chen and Parks (1986), Samueli (1988), ...