Mirzakhani's Starting Point

Caroline Series

THE UNIVERSITY OF WARWICK

Maryam Mirzakhani, Stanford University

Fields medal 2014

Myself

Born in Oxford.

Father a physicist, lecturer at Oxford.

Oxford High School for Girls.

Somerville College, BA mathematics.

PhD Harvard under George Mackey.

Postdocs in Berkeley and Cambridge.

Warwick University since 1979.

Worked for some years with Joan Birman (Columbia).

Myself and Women in Maths

As a graduate student about 1974, I went to a meeting of Boston Women in Maths. The Boston group became part of the USA based *Association for Women in Mathematics*.

In 1986, I was part of a small group of European women who founded a similar organisation in Europe, *European Women in Mathematics*.

Last year, the then President of the IMU, Ingrid Daubechies, led an initiative to form an *International Women in Maths section* of the IMU website. This is an international repository of information about and for female mathematicians.

We were absolutely delighted that the launch of the website coincided with the announcement that *Maryam Mirzakhani* had become the first woman ever to be awarded the Fields medal.

Mirzakhani's Starting Point

Born in Tehran, Iran.

High school in Tehran: A special technological school for talented students.

Gold medals at two International Mathematical Olympiads.

Sharif University of Technology, Tehran, BSc Mathematics.

PhD at Harvard under Curt McMullen, 2004.

Clay Fellowship and Assistant Professor at Princeton.

Professor at Stanford University since 2008.

I am going to be talking about the starting point of the remarkable work in Maryam's PhD thesis.

McShane's Identity

$$\sum_{\gamma} \frac{1}{1 + e^{\ell(\gamma)}} = 1/2$$

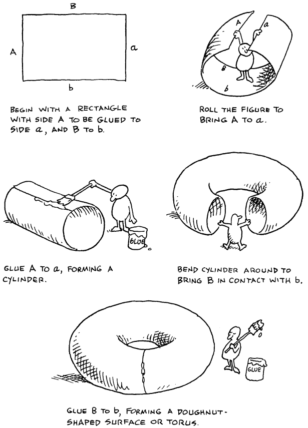
where the sum is over all simple closed curves on a hyperbolic once punctured torus and $\ell(\gamma)$ is the length of the geodesic representative in the homotopy class of γ.

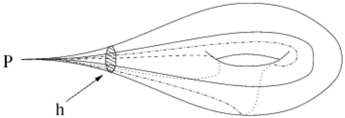
Greg McShane, Univ. de Grenoble
Ph D Warwick 1992:
A remarkable formula for lengths of curves on surfaces

Note: A curve is called *simple* if it has no self-intersections.

The meaning of McShane's identity: the surface

A flat (Euclidean) torus is one which looks everywhere like a piece of Euclidean plane. It is made by gluing up the sides of a parallelogram or rectangle.





Likewise a hyperbolic torus is one which looks everywhere like a piece of hyperbolic plane. Such a torus has to have one missing point, a puncture or 'cusp'. The curve h is called a "horocycle". Horocycles are important in what we do.

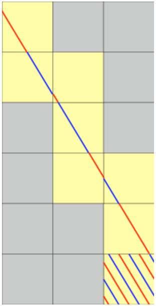
The hyperbolic plane

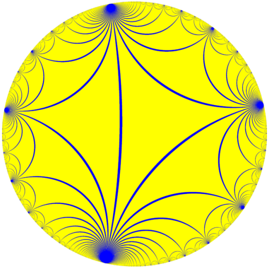
A surface with constant negative curvature is called *hyperbolic*. There is a unique simply connected surface of constant negative curvature -1. It can be represented as either the unit disk $\mathbb{D}$ with metric $ds = |dz|/2(1-|z|^2)$, or the upper half plane $\mathbb{H}$ with metric $ds = |dz|/(Imz)^2$. We will use both models. Both are called 'the hyperbolic plane'. The main points to remember are:

- The boundary is infinitely far from the 'middle' – either $i \in \mathbb{H}$ or $O \in \mathbb{D}$.
- Geodesics are represented by circular arcs perpendicular to the boundary. In $\mathbb{H}$, vertical straight lines are a special case.
- Distances are exponentially distorted as you move towards the boundary. If points P, Q are hyperbolic distance 1 apart and distance K from the 'middle' then their Euclidean distance is $O(e^{-K})$.

The meaning of McShane's identity: the curves

The Euclidean plane is the universal cover of the flat torus. Every closed geodesic loop on the torus is the projection of a line of rational slope on the plane. All parallel lines project to the a family of homotopic geodesics of equal length.





The universal cover of a hyperbolic once punctured torus is the hyperbolic plane. Homotopy classes of closed simple curves are the same as on the flat torus, but there is only one closed geodesic in each class. Its length is denoted $\ell(\gamma)$.

Plan of talk

- Proof of McShane's identity.
- The Birman-Series Theorem.
- What Mirzakhani did with McShane's identity.
- Some consquences.
- Some of her further results.

Recall: we want to prove that

$$\sum_{\gamma} \frac{1}{1+e^{\ell(\gamma)}} = 1/2$$

where the sum is over all simple closed curves on a hyperbolic once punctured torus T and $\ell(\gamma)$ is the length of the geodesic representative in the homotopy class of γ.

Proof of McShane's identity: Vertical geodesics

On the punctured torus T, consider 'vertical' geodesics which emanate from the cusp and cut a horocycle H round the cusp perpendicularly. Typically, such a geodesic cuts itself, forming a loop γ in one of the countably many homotopy classes of closed curves we met above.

Now cut T open along the geodesic representative of γ. This makes a sphere with two holes of equal length $\ell(\gamma)$ and a puncture (cusp). On each side, there are 4 special vertical geodesics which spiral into these two boundary curves, one pair from each side.

Any vertical geodesic α which meets H between a spiralling pair, cuts itself creating a curve homotopic to γ. Such a curve α is not simple.

Proof of McShane's identity: The gaps

We can normalise so that H has length 1 and the part of the surface 'above' H has area 4. It can be calculated that the distance along H between the two spiralling curves is $1/(1 + e^{\ell(\gamma)})$.

For each homotopy class of simple closed curves on T, we have found a 'gap' along H such that no vertical geodesic from the puncture which meets H in the gap can be simple.[1] Moreover any vertical curve meeting the gap forms a loop homotopic to γ, so gaps corresponding to distinct γ are disjoint.

There is a second gap corresponding to γ, coming from the vertical curves which spiral around γ in the opposite direction.

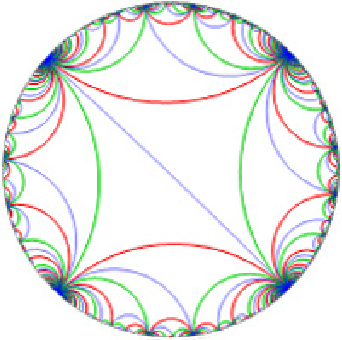
We have used up all the possible homotopy classes of loops, so any other vertical line α from the cusp which does not cut H in one of the gaps must be simple.

[1] Modulo the one exceptional curve from cusp to itself mentioned above.

Proof of McShane's identity: The Birman-S. theorem

We claim that the set of all points in H which are outside the gaps has measure zero. This is a special case of the *Birman-S. theorem*, which states that the set of all points on a hyperbolic surface covered by a simple curve has Hausdorff dimension zero.

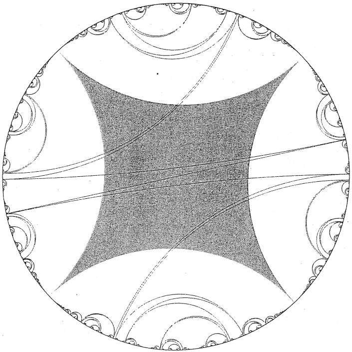
The proof relies on looking carefully at how a simple geodesic cuts a fundamental region R which glues up to make the torus. We can take the R to be a quadrilateral with four vertices on the boundary at ∞.



The images of R under the covering group tessellate the hyperbolic plane.

Patterns of crossings.

Any simple geodesic cuts R in a very special pattern, corresponding to the pattern in which lines of a given slope cut a square. There are only 4 basic possible arrangements.



Consider the first n crossings of a vertical geodesic which crosses in the pattern shown. Given the number of strands meeting each side of the square, there is a unique way of joining them up to recover the path up to homotopy. Thus there are at most $O(n)$ possible arrangements.

Proof of McShane's identity: The Birman-S. theorem continued.

No simple geodesic can pass too near the cusp (because if a geodesic goes too near the cusp it must loop round and cut itself). So every time an arc of a geodesic crosses a copy of the fundamental domain in the universal cover, it picks up a definite length $d > 0$. So if we follow a vertical line α from it first crossing of the horocycle H through n segments, its length is at least nd. Suppose the first n segments of two vertical simple geodesics α, α' follow the same pattern. Then the first n sides of the tessellation they cut are the same. The n^{th} side must be at distance at least dn from H, hence has Euclidean radius at most $O(e^{-dn})$. Thus α, α' cut H at distance at most $O(e^{-dn})$ apart.

Proof of McShane's identity: Finishing the proofs.

Proof of the B-S theorem. Let S be the set of points in H met by a simple vertical geodesic. We have just shown that S can be can covered by $O(n)$ intervals each of length $O(e^{-dn})$. Since $ne^{-dn} \to 0$ as $n \to \infty$, this proves that S can be covered by intervals of arbitrarily small size. Hence S has measure zero.

Proof of McShane's identity. Let $H = S \cup NS$ be the partition of H into points covered by simple and non-simple vertical geodesics respectively. We showed that NS is covered by disjoint gaps of length $1/1 + e^{\ell(\gamma)}$, with 2 gaps for each γ.

So NS has total length $2\sum_\gamma 1/1 + e^{\ell(\gamma)}$. Since H has length 1 and S has length 0, we have shown that

$$\sum_\gamma \frac{1}{1 + e^{\ell(\gamma)}} = 1/2.$$

Background: Teichmüller space

The space of all possible hyperbolic structures on an OPT is called *Teichmüller space* $\mathcal{T}$. One way to describe $\mathcal{T}$ uses *Fenchel-Nielsen coodinates*.

Fix a simple geodesic loop γ of length ℓ. Cut T open along γ then glue back the cut open surface with a twist t. The *Fenchel-Nielsen coodinates* of T are (ℓ, t)

Twisting once around γ gives a new torus with coordinates $(\ell, t+\ell)$. It looks the same as before, but curves transverse to γ all change length. This is called a *Dehn twist* D_γ about γ. D_γ is a diffeomorphism of T.

Fix a curve δ transverse to γ so that γ, δ generate $\pi_1(T)$. Then $D_\gamma(\delta) = \gamma\delta$ in $\pi_1(T)$. So D_γ changes the *marking* of T: it induces an isometry of T which changes the labels of the curves.

More background: Moduli space

Teichmüller space describes *marked* tori, where we label all the closed curves (equivalently specify which curves are the generators). If we forget the labelling, we get *moduli space* $\mathcal{M}$. $\mathcal{M}$ is the quotient of $\mathcal{T}$ by the action of the *mapping class group* Mod, which you can think of as the group of possible changes of markings, generated by all possible Dehn twists.

There is a natural volume element $d\ell \wedge dt$ on $\mathcal{T}$ called the *Weil-Petersson volume*. Luckily, this is invariant under the action of Mod, hence independent of the choice of γ. So it induces a volume on moduli space $\mathcal{M}$.

Mirzakhani found a very clever way to calculate $Vol(\mathcal{M})$. The sum in McShane's identity is over *all* simple geodesics so it doesn't depend on the marking. So if you want to find $Vol(\mathcal{M})$ you can integrate over $\mathcal{M}$. Since the RHS of the identity is constant, this would give a formula for $Vol(\mathcal{M})$.

What Mirzakhani did: Setting up the integration

Note that McShane's identity holds on *any* hyperbolic once punctured torus.

The term $1/(1+e^{\ell(\gamma)})$ in McShane's identity depends on γ, but not on the twist t and hence not on the marking of the curves which cross γ. So Mirzakhani integrated over a space $\mathcal{M}^*$ intermediate between $\mathcal{T}$ and $\mathcal{M}$. In this space we record the labelled curve γ but we don't label the transverse curves which cross it. $\mathcal{M}^*$ is the set of pairs (X, γ) where $X \in \mathcal{M}$ and γ is a simple closed curve on T.

Fix $\gamma = \gamma_0$ and let $Stab(\gamma_0)$ be the stabiliser of γ_0 in Mod . $Stab(\gamma_0)$ consists of the powers of the twist D_{γ_0}. D_{γ_0} acts on $\mathcal{T}$ by translation by ℓ.

We can identify $\mathcal{M}^*$ with the subset of $\mathcal{T}$ with Fenchel-Nielsen coordinates $\{(\ell, t) : 0 < \ell < \infty, 0 \leq t < \ell(\gamma)\}$.

We can identify $\mathcal{M}$ with a choice of fundamental domain $\Delta \subset \mathcal{M}^*$ for the action of Mod on $\mathcal{T}$. Then we can choose coset representatives $gStab(\gamma_0) \in \Gamma = \text{Mod}\,/Stab(\gamma_0)$ so that $\mathcal{M}^* = \cup g\Delta$.

What Mirzakhani did: The integration trick

For any simple closed curve on T, set $f(\gamma) = 1/1 + e^{\ell(\gamma)}$. Since a point in $\mathcal{M}^*$ is a pair (X, γ), we see that $f(\gamma)$ is a function on $\mathcal{M}^*$. On the other hand, $\sum_\gamma f(\gamma)$ can be viewed as a function on $\mathcal{M}$. Note that $\sum_\gamma f(\gamma) = \sum_{g \in \Gamma} f(g \cdot \gamma_0)$ where $\Gamma = \text{Mod}\,/Stab(\gamma_0)$.

Identifying $\mathcal{M}$ with $\Delta \subset \mathcal{M}^*$ as before, we have:

$$\begin{aligned}
\int_{\mathcal{M}} \sum_\gamma f(\gamma) &= \int_\Delta \sum_\gamma f(\gamma) = \sum_{g \in \Gamma} \int_\Delta f(g \cdot \gamma_0) = \sum_{g \in \Gamma} \int_{g\Delta} f(\gamma_0) \\
= \int_{\mathcal{M}^*} f(\gamma_0) &= \int_0^\infty \int_{t=0}^{\ell(\gamma_0)} f(\gamma_0) dt d\ell = \int_0^\infty \ell/(1 + e^\ell) d\ell.
\end{aligned}$$

On the other hand, by McShane's identity $\int_{\mathcal{M}} \sum_\gamma f(\gamma) = 1/2Vol(\mathcal{M})$.

So all that remains is to work out $\int_0^\infty \ell/(1 + e^\ell) d\ell$.

The final step!

We want calculate $Vol(\mathcal{M}) = I = \int_0^\infty \ell/(1+e^\ell)d\ell$.

Integrate by parts to get $I = \int_0^\infty log(1+e^{-\ell})d\ell$.
Expand the integrand as a Taylor series to get

$$I = \sum_{n=1}^{\infty} \int_0^\infty (-1)^{n+1}(e^{-n\ell}/n)d\ell = \sum_{n=1}^{\infty}(-1)^{n+1}/n^2 = \pi^2/12,$$

where the last identity is found either by Fourier series or complex analysis.
(The Fourier series for x^2 on $(-\pi, \pi)$ is

$$x^2 = \pi^2/3 - 4(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots).$$

Now substitute $x = 0$.)

Thus we conclude: $Vol(\mathcal{M}) = \pi^2/6$.

Mirzakhani's further development of her idea

- Extended McShane's identity to hyperbolic surfaces of any genus with boundary curves of arbitrary lengths $L_1, \dots, L_n$. Found recursive formulae to calculate $Vol(\mathcal{M})$ for the corresponding moduli spaces; showed it is polynomial in $L_1, \dots, L_n$ with coefficients which rational multiples of powers of π.
- Let X be a hyperbolic surface of genus g. Used the volume integration to show that the number of simple closed curves on X of length at most L is asymptotic to $c_X L^{6g-6}$. (Birman-S methods prove the upper bound, but not the lower.)

  The method actually gives asymptotic formulae for the frequencies of different topological types of simple curves. Eg: There is chance $1/7$ that a random simple closed curve on a genus 2 surface cuts it into two one holed tori.
- Also used the recursive volume formulae to give a completely new proof of the *Kontsevich-Witten* conjecture about intersection numbers between certain line bundles over $\mathcal{M}$.

Other results by Mirzakhani

- Complex geodesics in moduli space are algebraic varieties.
- Classified orbits and ergodic invariant measures of $SL(2, \mathbb{R})$ action on the tangent space to $\mathcal{T}$.
- Applications to billiards.
- Dynamics on moduli space: Thurston's earthquake flow – the flow induced by Dehn twisting along either curves or laminations – is ergodic on the space of laminations as a space over moduli space.

For more information and comment about Maryam, see the IMU WIM webpages (under News, August 2014) and in particular for a good blog about her work see Mathoverflow, linked to the bottom of the IMU news page here.