Taylor Series Expansions

Here are a collection of facts you might want to know about Taylor Series Expansions.

Fact 1 We start first with a theorem:

Taylor's Theorem
Under certain circumstances, a function f which is infinitely differentiable on the open interval (a, x) has a *Taylor series expansion*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

where for each n we have

$$a_n = \frac{f^{(n)}(a)}{n!}$$

and $f^{(n)}(a)$ denotes the n^{th} derivative of f evaluated at a.

This tells you how to get a Taylor series expansion for a function whose derivatives you know.

Fact 2 It is important to know that a Taylor series has what we call a *radius of convergence*, denoted here by R, which is a number such that the series converges (in other words, makes sense) if $|x-a| < R$ and does not converge (does not make sense) if $|x-a| > R$. A way to think about this is that everything makes sense and is good "near enough" a (when we are at most R away from a) and nothing makes sense and nothing works "far away" from a. Some series converge everywhere, like for example the series expansion for e^x.

Fact 3 We now collect theorems about what we can do to Taylor series.

Theorem
If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for all x in $(a-R, a+R)$, then

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1}$$

for all x in $(a-R, a+R)$.

Theorem
If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for all x in $(a-R, a+R)$, then

$$\int f(x)\, dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$$

for all x in $(a-R, a+R)$.

These theorems tell you that you can basically differentiate and integrate a Taylor series in the same way you can differentiate and integrate a polynomial. Note that these are deep theorems; it is not clear that we should be able to do this.

Theorem

Let $p(x)$ be a polynomial and $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for all x in $(a-R, a+R)$. Then we have

$$p(x)f(x) = \sum_{n=0}^{\infty} a_n p(x)(x-a)^n$$

for all x in $(a-R, a+R)$ and

$$f(p(x)) = \sum_{n=0}^{\infty} a_n (p(x)-a)^n$$

for all x such that $p(x)$ is in $(a-R, a+R)$.

Examples

a) We have that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$, so if we use $p(x) = x$, we have

$$\frac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1}$$

for $|x| < 1$.

b) Using $p(x) = \frac{x}{2}$, we get that

$$\frac{x}{2-x} = \frac{\frac{x}{2}}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}}$$

for $|\frac{x}{2}| < 1$ or more simply $|x| < 2$.

Fact 4 Finally here are a few common series expansions:

a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

b) $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for all $|x| < 1$.

c) $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for all $|x| < 1$.

d) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for all x.

e) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all x.