2. Propositional Equivalences

2.1. Tautology/Contradiction/Contingency.

DEFINITION 2.1.1. *A* **tautology** *is a proposition that is always true.*

EXAMPLE 2.1.1. $p \vee \neg p$

DEFINITION 2.1.2. *A* **contradiction** *is a proposition that is always false.*

EXAMPLE 2.1.2. $p \wedge \neg p$

DEFINITION 2.1.3. *A* **contingency** *is a proposition that is neither a tautology nor a contradiction.*

EXAMPLE 2.1.3. $p \vee q \rightarrow \neg r$

Discussion

One of the important techniques used in proving theorems is to replace, or substitute, one proposition by another one that is equivalent to it. In this section we will list some of the basic propositional equivalences and show how they can be used to prove other equivalences.

Let us look at the classic example of a tautology, $p \vee \neg p$. The truth table

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

shows that $p \vee \neg p$ is true no matter the truth value of p.

> [*Side Note.* This tautology, called the *law of excluded middle*, is a direct consequence of our basic assumption that a proposition is a statement that is either true or false. Thus, the logic we will discuss here, so-called Aristotelian logic, might be described as a "2-valued" logic, and it is the logical basis for most of the theory of modern mathematics, at least as it has developed in western culture. There is, however, a consistent logical system, known as constructivist, or intuitionistic, logic which does not assume the law of excluded middle. This results in a 3-valued logic in which one allows for

a third possibility, namely, "other." In this system proving that a statement is "not true" is not the same as proving that it is "false," so that indirect proofs, which we shall soon discuss, would not be valid. If you are tempted to dismiss this concept, you should be aware that there are those who believe that in many ways this type of logic is much closer to the logic used in computer science than Aristotelian logic. You are encouraged to explore this idea: there is plenty of material to be found in your library or through the worldwide web.]

The proposition $p \vee \neg(p \wedge q)$ is also a tautology as the following the truth table illustrates.

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

EXERCISE 2.1.1. *Build a truth table to verify that the proposition* $(p \leftrightarrow q) \wedge (\neg p \wedge q)$ *is a contradiction.*

2.2. Logically Equivalent.

DEFINITION 2.2.1. *Propositions r and s are* **logically equivalent** *if the statement $r \leftrightarrow s$ is a tautology.*

Notation: If r and s are logically equivalent, we write

$$r \Leftrightarrow s.$$

Discussion

A second notation often used to mean statements r and s are logically equivalent is $r \equiv s$. You can determine whether compound propositions r and s are logically equivalent by building a single truth table for both propositions and checking to see that they have exactly the same truth values.

Notice the new symbol $r \Leftrightarrow s$, which is used to denote that r and s are logically equivalent, is defined to mean the statement $r \leftrightarrow s$ is a tautology. In a sense the

symbols $\leftrightarrow$ and $\Leftrightarrow$ convey similar information when used in a sentence. However, $r \Leftrightarrow s$ is generally used to assert that the statement $r \leftrightarrow s$ is, in fact, true while the statement $r \leftrightarrow s$ alone does not imply any particular truth value. The symbol $\Leftrightarrow$ is the preferred shorthand for "is equivalent to."

2.3. Examples.

EXAMPLE 2.3.1. *Show that $(p \rightarrow q) \wedge (q \rightarrow p)$ is logically equivalent to $p \leftrightarrow q$.*

Solution 1. *Show the truth values of both propositions are identical.*

Truth Table:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Solution 2. *Examine every possible case in which the statement $(p \rightarrow q) \wedge (q \rightarrow p)$ may not have the same truth value as $p \leftrightarrow q$*

Case 1. Suppose $(p \rightarrow q) \wedge (q \rightarrow p)$ is false and $p \leftrightarrow q$ is true. There are two possible cases where $(p \rightarrow q) \wedge (q \rightarrow p)$ is false. Namely, $p \rightarrow q$ is false or $q \rightarrow p$ is false (note that this covers the possibility both are false since we use the inclusive "or" on logic).
(a) Assume $p \rightarrow q$ is false. Then p is true and q is false. But if this is the case, the $p \leftrightarrow q$ is false.
(b) Assume $q \rightarrow p$ is false. Then q is true and p is false. But if this is the case, the $p \leftrightarrow q$ is false.
Case 2. Suppose $(p \rightarrow q) \wedge (q \rightarrow p)$ is true and $p \leftrightarrow q$ is false. If the latter is false, the p and q do not have the same truth value and the there are two possible ways this may occur that we address below.
(a) Assume p is true and q is false. Then $p \rightarrow q$ is false, so the conjunction also must be false.
(b) Assume p is false and q is true. Then $q \rightarrow p$ is false, so the conjunction is also false.

We exhausted all the possibilities, so the two propositions must be logically equivalent.

Discussion

This example illustrates an alternative to using truth tables to establish the equivalence of two propositions. An alternative proof is obtained by excluding all possible ways in which the propositions may fail to be equivalent. Here is another example.

EXAMPLE 2.3.2. *Show* $\neg(p \rightarrow q)$ *is equivalent to* $p \wedge \neg q$.

Solution 1. Build a truth table containing each of the statements.

p	q	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Since the truth values for $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are exactly the same for all possible combinations of truth values of p and q, the two propositions are equivalent.

Solution 2. We consider how the two propositions could *fail* to be equivalent. This can happen only if the first is true and the second is false or vice versa.

Case 1. Suppose $\neg(p \rightarrow q)$ is true and $p \wedge \neg q$ is false.
    $\neg(p \rightarrow q)$ would be true if $p \rightarrow q$ is false. Now this only occurs if p is true and q is false. However, if p is true and q is false, then $p \wedge \neg q$ will be true. Hence this case is not possible.
Case 2. Suppose $\neg(p \rightarrow q)$ is false and $p \wedge \neg q$ is true.
    $p \wedge \neg q$ is true only if p is true and q is false. But in this case, $\neg(p \rightarrow q)$ will be true. So this case is not possible either.

Since it is not possible for the two propositions to have different truth values, they must be equivalent.

EXERCISE 2.3.1. *Use a truth table to show that the propositions* $p \leftrightarrow q$ *and* $\neg(p \oplus q)$ *are equivalent.*

EXERCISE 2.3.2. *Use the method of Solution 2 in Example 2.3.2 to show that the propositions* $p \leftrightarrow q$ *and* $\neg(p \oplus q)$ *are equivalent.*

2.4. Important Logical Equivalences. The logical equivalences below are important equivalences that should be memorized.

Identity Laws:	$p \wedge T \Leftrightarrow p$
	$p \vee F \Leftrightarrow p$
Domination Laws:	$p \vee T \Leftrightarrow T$
	$p \wedge F \Leftrightarrow F$
Idempotent Laws:	$p \vee p \Leftrightarrow p$
	$p \wedge p \Leftrightarrow p$
Double Negation Law:	$\neg(\neg p) \Leftrightarrow p$
Commutative Laws:	$p \vee q \Leftrightarrow q \vee p$
	$p \wedge q \Leftrightarrow q \wedge p$
Associative Laws:	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
Distributive Laws:	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws:	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
Absorption Laws:	$p \wedge (p \vee q) \Leftrightarrow p$
	$p \vee (p \wedge q) \Leftrightarrow p$

Implication Law: $(p \to q) \Leftrightarrow (\neg p \vee q)$

Contrapositive Law: $(p \to q) \Leftrightarrow (\neg q \to \neg p)$

Tautology: $p \vee \neg p \Leftrightarrow T$

Contradiction: $p \wedge \neg p \Leftrightarrow F$

Equivalence: $(p \to q) \wedge (q \to p) \Leftrightarrow (p \leftrightarrow q)$

Discussion

Study carefully what each of these equivalences is saying. With the possible exceptions of the De Morgan Laws, they are fairly straight-forward to understand. The main difficulty you might have with these equivalences is remembering their names.

EXAMPLE 2.4.1. *Use the logical equivalences above and substitution to establish the equivalence of the statements in Example 2.3.2.*

Solution.

$$\begin{aligned}
\neg(p \to q) \;\; &\Leftrightarrow \neg(\neg p \vee q) \quad \textit{Implication Law} \\
&\Leftrightarrow \neg\neg p \wedge \neg q \quad \textit{De Morgan's Law} \\
&\Leftrightarrow p \wedge \neg q \qquad \textit{Double Negation Law}
\end{aligned}$$

This method is very similar to simplifying an algebraic expression. You are using the basic equivalences in somewhat the same way you use algebraic rules like $2x - 3x = -x$ or $\dfrac{(x+1)(x-3)}{x-3} = x + 1$.

EXERCISE 2.4.1. *Use the propositional equivalences in the list of important logical equivalences above to prove $[(p \to q) \wedge \neg q] \to \neg p$ is a tautology.*

EXERCISE 2.4.2. *Use truth tables to verify De Morgan's Laws.*

2.5. Simplifying Propositions.

EXAMPLE 2.5.1. *Use the logical equivalences above to show that $\neg(p \vee \neg(p \wedge q))$ is a contradiction.*

Solution.

$$\neg(p \vee \neg(p \wedge q))$$
$$\Leftrightarrow \neg p \wedge \neg(\neg(p \wedge q)) \quad \text{De Morgan's Law}$$
$$\Leftrightarrow \neg p \wedge (p \wedge q) \quad\quad \text{Double Negation Law}$$
$$\Leftrightarrow (\neg p \wedge p) \wedge q \quad\quad \text{Associative Law}$$
$$\Leftrightarrow F \wedge q \quad\quad\quad\quad \text{Contradiction}$$
$$\Leftrightarrow F \quad\quad\quad\quad\quad \text{Domination Law and Commutative Law}$$

EXAMPLE 2.5.2. *Find a simple form for the negation of the proposition "If the sun is shining, then I am going to the ball game."*

Solution. *This proposition is of the form $p \rightarrow q$. As we showed in Example 2.3.2 its negation, $\neg(p \rightarrow q)$, is equivalent to $p \wedge \neg q$. This is the proposition*

"The sun is shining, and I am not going to the ball game."

Discussion

The main thing we should learn from Examples 2.3.2 and 2.5.2 is that the negation of an implication is *not* equivalent to another implication, such as "If the sun is shining, then I am not going to the ball game" or "If the sun is not shining, I am going to the ball game." This may be seen by comparing the corresponding truth tables:

p	q	$p \rightarrow \neg q$	$\neg(p \rightarrow q) \Leftrightarrow (p \wedge \neg q)$	$\neg p \rightarrow q$
T	T	F	F	T
T	F	T	T	T
F	T	T	F	T
F	F	T	F	F

If you were to construct truth tables for all of the other possible implications of the form $r \rightarrow s$, where each of r and s is one of p, $\neg p$, q, or $\neg q$, you will observe that none of these propositions is equivalent to $\neg(p \rightarrow q)$.

The rule $\neg(p \to q) \Leftrightarrow p \wedge \neg q$ should be memorized. One way to memorize this equivalence is to keep in mind that the negation of $p \to q$ is the statement that describes the only case in which $p \to q$ is false.

EXERCISE 2.5.1. *Which of the following are equivalent to $\neg(p \to r) \to \neg q$? There may be more than one or none.*

(1) $\neg(p \to r) \vee q$
(2) $(p \wedge \neg r) \vee q$
(3) $(\neg p \to \neg r) \vee q$
(4) $q \to (p \to r)$
(5) $\neg q \to (\neg p \to \neg r)$
(6) $\neg q \to (\neg p \vee r)$
(7) $\neg q \to \neg(p \to r)$

EXERCISE 2.5.2. *Which of the following is the negation of the statement "If you go to the beach this weekend, then you should bring your books and study"?*

(1) *If you do not go to the beach this weekend, then you should not bring your books and you should not study.*
(2) *If you do not go to the beach this weekend, then you should not bring your books or you should not study.*
(3) *If you do not go to the beach this weekend, then you should bring your books and study.*
(4) *You will not go to the beach this weekend, and you should not bring your books and you should not study.*
(5) *You will not go to the beach this weekend, and you should not bring your books or you should not study.*
(6) *You will go to the beach this weekend, and you should not bring your books and you should not study.*
(7) *You will go to the beach this weekend, and you should not bring your books or you should not study.*

EXERCISE 2.5.3. *p is the statement "I will prove this by cases", q is the statement "There are more than 500 cases," and r is the statement "I can find another way." State the negation of $(\neg r \vee \neg q) \to p$. in simple English. Do not use the expression "It is not the case."*

2.6. Implication.

DEFINITION 2.6.1. *We say the proposition r implies the proposition s and write $r \Rightarrow s$ if $r \to s$ is a tautology.*

This is very similar to the ideas previously discussed regarding the $\Leftrightarrow$ verses $\leftrightarrow$. We use $r \Rightarrow s$ to imply that the statement $r \to s$ is true, while that statement $r \to s$

alone does not imply any particular truth value. The symbol $\Rightarrow$ is often used in proofs as a shorthand for "implies."

EXERCISE 2.6.1. *Prove* $(p \to q) \wedge \neg q \Rightarrow \neg p$.

EXERCISE 2.6.2. *Prove* $p \wedge (p \to q) \to \neg q$ *is a contingency using a truth table.*

EXERCISE 2.6.3. *Prove* $p \to (p \vee q)$ *is a tautology using a verbal argument.*

EXERCISE 2.6.4. *Prove* $(p \wedge q) \to p$ *is a tautology using the table of propositional equivalences.*

EXERCISE 2.6.5. *Prove* $[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$ *using a truth table.*

EXERCISE 2.6.6. *Prove* $[(p \vee q) \wedge \neg p] \Rightarrow q$ *using a verbal argument.*

EXERCISE 2.6.7. *Prove* $(p \wedge q) \to (p \vee q)$ *is a tautology using the table of propositional equivalences.*

2.7. Normal or Canonical Forms.

DEFINITION 2.7.1. *Every compound proposition in the propositional variables* p, q, r, ..., *is uniquely equivalent to a proposition that is formed by taking the disjunction of conjunctions of some combination of the variables* $p, q, r, ...$ *or their negations. This is called the* **disjunctive normal form** *of a proposition.*

Discussion

The *disjunctive normal form* of a compound proposition is a natural and useful choice for representing the proposition from among all equivalent forms, although it may not be the simplest representative. We will find this concept useful when we arrive at the module on Boolean algebra.

2.8. Examples.

EXAMPLE 2.8.1. *Construct a proposition in disjunctive normal form that is true precisely when*

  (1) *p is true and q is false*
      Solution. $p \wedge \neg q$
  (2) *p is true and q is false or when p is true and q is true.*

        Solution. $(p \wedge \neg q) \vee (p \wedge q)$
  (3) *either p is true or q is true, and r is false*
      Solution. $(p \vee q) \wedge \neg r \Leftrightarrow (p \wedge \neg r) \vee (q \wedge \neg r)$ *(Distributive Law)*

(Notice that the second example could be simplified to just p.)

The methods by which we arrived at the disjunctive normal form in these examples may seem a little *ad hoc*. We now demonstrate, through further examples, a sure-fire method for its construction.

2.9. Constructing Disjunctive Normal Forms.

EXAMPLE 2.9.1. *Find the disjunctive normal form for the proposition $p \rightarrow q$.*

Solution. *Construct a truth table for $p \rightarrow q$:*

p	q	$p \rightarrow q$	
T	T	T	$\leftarrow$
T	F	F	
F	T	T	$\leftarrow$
F	F	T	$\leftarrow$

$p \rightarrow q$ is true when either
p is true and q is true, or
p is false and q is true, or
p is false and q is false.
The disjunctive normal form is then

$$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

Discussion

This example shows how a truth table can be used in a systematic way to construct the disjunctive normal forms. Here is another example.

EXAMPLE 2.9.2. *Construct the disjunctive normal form of the proposition*

$$(p \rightarrow q) \wedge \neg r$$

Solution. *Write out the truth table for $(p \rightarrow q) \wedge \neg r$:*

p	q	r	$p \rightarrow q$	$\neg r$	$(p \rightarrow q) \wedge \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	F
F	T	T	T	F	F
T	F	F	F	T	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

The disjunctive normal form will be a disjunction of three conjunctions, one for each row in the truth table that gives the truth value T for $(p \rightarrow q) \wedge \neg r$. These rows have been boxed. In each conjunction we will use p if the truth value of p in that row is T and ¬p if the truth value of p is F, q if the truth value of q in that row is T and ¬q if the truth value of q is F, etc. The disjunctive normal form for $(p \rightarrow q) \wedge \neg r$ is then

$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r),$$

because each of these conjunctions is true only for the combination of truth values of p, q, and r found in the corresponding row. That is, $(p \wedge q \wedge \neg r)$ has truth value T only for the combination of truth values in row 2, $(\neg p \wedge q \wedge \neg r)$ has truth value T only for the combination of truth values in row 6, etc. Their disjunction will be true for precisely the three combinations of truth values of p, q, and r for which $(p \rightarrow q) \wedge \neg r$ is also true.

Terminology. The individual conjunctions that make up the disjunctive normal form are called **minterms**. In the previous example, the disjunctive normal form for the proposition $(p \rightarrow q) \wedge \neg r$ has three minterms, $(p \wedge q \wedge \neg r)$, $(\neg p \wedge q \wedge \neg r)$, and $(\neg p \wedge \neg q \wedge \neg r)$.

2.10. Conjunctive Normal Form. The **conjunctive normal form** of a proposition is another "canonical form" that may occasionally be useful, but not to the same degree as the disjunctive normal form. As the name should suggests after our discussion above, the conjunctive normal form of a proposition is the equivalent form that consists of a "conjunction of disjunctions." It is easily constructed indirectly using disjunctive normal forms by observing that if you negate a disjunctive normal form you get a conjunctive normal form. For example, three applications of De Morgan's Laws gives

$$\neg[(p \wedge \neg q) \vee (\neg p \wedge \neg q)] \Leftrightarrow (\neg p \vee q) \wedge (p \vee q).$$

Thus, if you want to get the conjunctive normal form of a proposition, construct the disjunctive normal form of its *negation* and then negate again and apply De Morgan's Laws.

EXAMPLE 2.10.1. *Find the conjunctive normal form of the proposition* $(p \wedge \neg q) \vee r$.

Solution.

(1) Negate: $\neg[(p \wedge \neg q) \vee r] \Leftrightarrow (\neg p \vee q) \wedge \neg r$.

(2) Find the disjunctive normal form of $(\neg p \vee q) \wedge \neg r$:

p	q	r	$\neg p$	$\neg r$	$\neg p \vee q$	$(\neg p \vee q) \wedge \neg r$
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
F	T	T	T	F	T	F
T	F	F	F	T	F	F
F	T	F	T	T	T	T
F	F	T	T	F	T	F
F	F	F	T	T	T	T

The disjunctive normal form for $(\neg p \vee q) \wedge \neg r$ is
$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r).$$

(3) The conjunctive normal form for $(p \wedge \neg q) \vee r$ is then the negation of this last expression, which, by De Morgan's Laws, is
$$(\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r).$$