Chapter 13

Useful Functions

13.1. Introduction

In this chapter we consider several useful functions from D or $D \times D$ to D that can be exploited to establish new stochastic-process limits from given ones. We concentrate on four basic functions introduced in Section 3.5: composition, supremum, reflection and inverse. Another basic function is addition, but it has already been treated in Sections 12.6, 12.7 and 12.11. Our treatment of useful functions follows Whitt (1980), but the emphasis there was on the J_1 topology, even though the M_1 topology was used in places. In contrast, here the emphasis is on the M_1 and M_2 topologies, although we also give results for the J_1 topology. As in the last chapter, many proofs are omitted. Most of the missing proofs appear in Chapter 7 of the Internet Supplement.

Here is how this chapter is organized: We start in Section 13.2 by considering the composition map, which plays an important role in establishing FCLTs involving a random time change. We consider composition without centering in Section 13.2; then we consider composition with centering in Section 13.3.

In Section 13.4 we study the supremum function, both with and without centering. In Section 13.5 we apply the supremum results to treat the (one-sided one-dimensional) reflection map, which arises in queueing applications. We study the two-sided reflection map in Section 14.8.

We start studying the inverse function in Section 13.6. We study the inverse map without centering in Section 13.6 and with centering in Section 13.7. In Section 13.8 we apply the results for inverse functions to obtain corresponding results for closely related counting functions.

Application of these convergence-preservation results to stochastic-process limits are described in Sections 7.3 and 7.4. Section 7.3 contains FCLT's for counting processes, while Section 7.4 contains FCLT's for renewal-reward processes. When there are heavy-tailed distributions, the M_1 topology plays an important role.

In Chapter 3 of the Internet Supplement we discuss pointwise convergence and its preservation under mappings. The perservation of pointwise convergence focuses on relations for individual sample paths, as in the queueing book by El-Taha and Stidham (1999). From Chapter 3 of the Internet Supplement, we see that a function-space setting is not required for all convergence preservation.

13.2. Composition

This section is devoted to the composition function, mapping (x, y) into $x \circ y$, where

$$(x \circ y)(t) \equiv x(y(t)) \quad \text{for all} \quad t .$$

We have in mind a map from $D^k \times D$ into D^k, where $D^k \equiv D([0,\infty), \mathbb{R}^k)$. The situation is much easier when we consider single times and the map is from $D^k \times \mathbb{R}_+$ to $\mathbb{R}^k$. We can still take advantage of the Skorohod topology on D, though. The following is an elementary, but important, consequence of the local uniform convergence established in Section 12.4.

Proposition 13.2.1. (local uniform convergence) *If*

$$(x_n, t_n) \to (x, t) \quad in \quad (D^k, WM_2) \times \mathbb{R}_+ \ ,$$

where $t \in Disc(x)^c$, *then*

$$x_n(t_n) \to x(t) \quad in \quad \mathbb{R}^k \ .$$

We now consider the composition map as a map from $D^k \times D$ to D, where we allow the domains of x and y to be $\mathbb{R}_+ \equiv [0, \infty)$ and we restrict the range of y to be $\mathbb{R}_+$. However, that is not enough; we need additional regularity conditions to have $x \circ y \in D$.

Example 13.2.1. *The need for a condition on y.* To see that $x \circ y$ need not be in D without additional conditions on y, let $x = I_{[2^{-1},\infty)}$ and $y = 2^{-1} + \sum_{n=1}^{\infty} (-2)^{-n} I_{[2^{-1}-2^{-n}, 2^{-1}-2^{-(n+1)})}$. Then $x, y \in D$, but $x \circ y$ has no limit from the left at $t = 1/2$. ∎

Henceforth in this chapter, unless stipulated otherwise, when $D \equiv D^k$, so that the range of functions is $\mathbb{R}^k$, we let D be endowed with the strong version of the J_1, M_1 or M_2 topology, and simply write J_1, M_1 or M_2. It will be evident that most results also hold with the corresponding weaker product topology.

To ensure that $x \circ y \in D$, we will assume that y is also nondecreasing. We begin by defining subsets of $D \equiv D^k \equiv D([0,\infty), \mathbb{R}^k)$ that we will consider. Let D_0 be the subset of all $x \in D$ with $x^i(0) \geq 0$ for all i. Let $D_\uparrow$ and $D_{\uparrow\uparrow}$ be the subsets of functions in D_0 that are nondecreasing and strictly increasing in each coordinate. Let D_m be the subset of functions x in D_0 for which the coordinate functions x^i are monotone (either increasing or decreasing) for each i. Let C_0, $C_\uparrow$, $C_{\uparrow\uparrow}$ and C_m be the corresponding subsets of C; i.e., $C_0 \equiv C \cap D_0$, $C_\uparrow \equiv C \cap D_\uparrow$, $C_{\uparrow\uparrow} = C \cap D_{\uparrow\uparrow}$, and $C_m = C \cap D_m$.

It is important that all of these subsets are measurable subsets of D with the Borel σ-fields associated with the non-uniform Skorohod topologies, which all coincide with the Kolmogorov σ-field generated by the projection maps; see Theorems 11.5.2 and 11.5.3.

Lemma 13.2.1. (Measurability of C in D) *C is a closed subset of (D, J_1) and so a measurable (but not closed) subset of D with the M_1 and M_2 topologies.*

Recall that a subset of a topological space is a G_δ subset if it is a countably intersection of open subsets. Clearly, a G_δ subset belongs to the Borel σ-field.

Lemma 13.2.2. (measurability of subsets of C) *C_m is a closed subset of C, $C_\uparrow$ is a closed subset of C_m and $C_{\uparrow\uparrow}$ is a G_δ subset of $C_\uparrow$.*

Proof. For the third relation, note that

$$C_{\uparrow\uparrow} = \cap_{p \in Q} \cap_{\substack{q \in Q \\ q > p}} \cap_{i=1}^{k} \{x \in C : x^i(q) - x^i(p) > 0\}$$

where Q is the set of rationals in $\mathbb{R}_+$. ■

Lemma 13.2.3. (measurability of subsets of D) *With any of the non-uniform Skorohod topologies, D_0 is a closed subset of D, D_m is a closed subset of D_0, $D_\uparrow$ is a closed subset of D_m and $D_{\uparrow\uparrow}$ is a G_δ subset of $D_\uparrow$.*

Proof. For the last relation, let $\{t_j\}$ be a countable dense subset of $\mathbb{R}_+$. For each (j,l), let

$$D_{i,j,l} = \{x \in D_\uparrow : x^i \text{ is constant over } [t_j \wedge t_l, t_j \vee t_l]\} \ .$$

Then $D_{i,j,l}$ is a closed subset of $D_\uparrow$ and

$$D_{\uparrow\uparrow} = \cap_{j=1}^{\infty} \cap_{l=1}^{\infty} \cap_{i=1}^{k} (D_\uparrow - D_{i,j,l}) \ ,$$

so that $D_{\uparrow\uparrow}$ is indeed a G_δ subset of $D_\uparrow$. ■

We now return to the composition map in (12.2), stating the condition for $x \circ y \in D$ as a lemma.

Lemma 13.2.4. (criterion for $x \circ y$ to be in D) *For each $x \in D([0,\infty), \mathbb{R}^k)$ and $y \in D_\uparrow([0,\infty), \mathbb{R}_+)$, $x \circ y \in D([0,\infty), \mathbb{R}^k)$.*

A basic result, from pp. 145, 232 of Billingsley (1968), is the following. The continuity part involves the topology of uniform convergence on compact intervals.

Theorem 13.2.1. (continuity of composition at continuous limits) *The composition map from $D^k \times D_\uparrow^1$ to D^k is measurable and continuous at $(x,y) \in C^k \times C_\uparrow^1$.*

Example 13.2.2. *Composition is not continuous everywhere.* To see that the composition on $D^1 \times D_\uparrow^1$ is not continuous in any of the Skorohod topologies, let $x_n = x = I_{[1/2,1]}$, $n \geq 1$, $y(t) = 2^{-1}$ and $y_n(t) = 2^{-1} - n^{-1}$, $0 \leq t \leq 1$. Then $x_n = x$ and $\|y_n - y\| = n^{-1} \to 0$, but $(x_n \circ y_n)(t) = 0$ and $(x \circ y)(t) = 1$, $0 \leq t \leq 1$. ■

Our goal now is to obtain additional positive continuity results under extra conditions. We use the following elementary lemma.

Lemma 13.2.5. *If $y(t) \in Disc(x)$ and y is strictly increasing and continuous at t, then $t \in Disc(x \circ y)$.*

Example 13.2.3. *The need for y to be strictly increasing.* To see the need for the condition that y be strictly increasing at t in Lemma 13.2.5, let $x = I_{[1,\infty)}$ and $y(t) = 1$, $t \geq 0$. Then $(x \circ y)(t) = 1$ for all t, so that $x \circ y$ is continuous. Moreover, if $x_n = x$ and $y_n(t) = 1 - n^{-1}$, $t \geq 0$, $n \geq 1$, then $(x_n \circ y_n)(t) = 0$ for all n and t, so that $x_n \circ y_n$ fails to converge to $x \circ y$ for any t. ■

The following is the J_1 result, taken from Whitt (1980). As indicated before, the proof appears in the Internet Supplement. The first J_1 composition results were established by Silvestrov; see Silvestrov (2000) for an account. See Serfozo (1973, 1975) and Gut (1988) for stochastic-process limits involving composition.

Theorem 13.2.2. (J_1-continuity of composition) *The composition map from $D^k \times D^1_\uparrow$ to D^k taking (x, y) into $(x \circ y)$ is continuous at $(x, y) \in (C^k \times D^1_\uparrow) \cup (D^k \times C^1_{\uparrow\uparrow})$ using the J_1 topology throughout.*

We have a different result for the M topologies:

Theorem 13.2.3. (M-continuity of composition) *If $(x_n, y_n) \to (x, y)$ in $D^k \times D^1_\uparrow$ and $(x, y) \in (D^k \times C^1_{\uparrow\uparrow}) \cup (C^k_m \times D^1_\uparrow)$, then $x_n \circ y_n \to x \circ y$ in D^k, where the topology throughout is M_1 or M_2.*

In most applications we have $(x, y) \in D^k \times C^1_{\uparrow\uparrow}$, as is illustrated by the next section. That part of the M conditions is the same as for J_1. The mode of convergence in Theorem 13.2.3 for $y_n \to y$ does not matter, because on $D^1_\uparrow$, convergence in the M_1 and M_2 topologies coincides with pointwise convergence on a dense subset of $[0, \infty)$, including 0; see Corollary 12.5.1.

It is easy to see that composition cannot in general yield convergence in a stronger topology, because $x \circ y = x$ and $x_n \circ y_n = x_n$, $n \geq 1$, when $y_n = y = e$, where $e(t) = t$, $t \geq 0$. Unlike for the J_1 topology, the composition map is in general *not* continuous at $(x, y) \in C \times D^1_\uparrow$ in the M topologies.

Example 13.2.4. *Why the J_1 and M conditions differ.* To see that composition is not continuous at $(x, y) \in C \times D^1_\uparrow$ in the M topologies, let y, y_n, $x = x_n$ be elements of $D([0, \infty), \mathbb{R})$ defined by

$$\begin{aligned} y(0) &= y(.5-) = 0, y(.5) = .25, y(1) = 1, y(t) = t, t > 1, \\ y_n(0) &= y_n(.5 - n^{-1}) = 0, y_n(.5) = .25, y_n(1) = 1, y_n(t) = t, t > 1, \\ x(0) &= x(.25) = x(t) = 0 \quad \text{for} \quad t > 0.25, x(.125) = 1\ , \end{aligned}$$

with the functions defined by linear interpolation elsewhere. Note that y jumps from 0 to 0.25 at 0.5, while y_n increases from 0 to 0.25 linearly over the interval $[2^{-1} - n^{-1}, 2^{-1}]$ for each n. Hence $y_n \to y$ in the M topologies but not in the J topologies. Note that $x(y(t)) = 0$, $t \geq 0$, while $x_n(y_n(2^{-1} - (2n)^{-1})) = x_n(.125) = 1$. Hence $x_n \circ y_n \not\to x \circ y$ as $n \to \infty$ in any of the Skorohod topologies. ∎

We actually prove a more general continuity result, which covers Theorem 13.2.3 as a special case.

Theorem 13.2.4. (more general M-continuity of composition) *Suppose that* $(x_n, y_n) \to (x, y)$ *in* $D^k \times D^1_\uparrow$. *If (i) y is continuous and strictly increasing at t whenever $y(t) \in Disc(x)$ and (ii) x is monotone on $[y(t-), y(t)]$ and $y(t-), y(t) \notin Disc(x)$ whenever $t \in Disc(y)$, then $x_n \circ y_n \to x \circ y$ in D^k, where the topology throughout is M_1 or M_2.*

Theorem 13.2.3 follows easily from Theorem 13.2.4: First, on $D^k \times C^1_\uparrow$, y is continuous, so only condition (i) need be considered; it is satisfied because y is continuous and strictly increasing everywhere. Second on $C^k_m \times D^1_\uparrow$, x is continuous so only condition (ii) need be considered; it is satisfied because x is monotone everywhere. Hence it suffices to prove Theorem 13.2.4, which is done in the Internet Supplement. The general idea in our proof of Theorem 13.2.4 is to work with the characterization of convergence using oscillation functions evaluated at single arguments, exploiting Theorems 12.5.1 (v), 12.5.2 (iv), 12.11.1 (v) and 12.11.2 (iv).

13.3. Composition with Centering

We now consider the composition map with centering. To obtain results, we apply both composition and addition. The results yield sufficient conditions for random sums and other processes transformed by a random time change to satisfy FCLTs, as we show in Section 7.4.

We start by establishing convergence properties of composition plus addition. We state results for the J_1 topology as well as the M_1 and M_2 topologies. As before, let e be the identity map on $[0, \infty)$.

Theorem 13.3.1. (convergence preservation for composition plus addition) *Let x, z and x_n, $n \geq 1$ be elements of D^k; let y, y_n and v_n, $n \geq 1$ be elements of $D^1_\uparrow$; and let $c_n \in \mathbb{R}^k$ for $n \geq 1$. If*

$$(x_n - c_n e, y_n, c_n(y_n - v_n)) \to (x, y, z) \quad in \quad D^k \times D^1_\uparrow \times D^k \ , \tag{3.1}$$

$y \in C^1_{\uparrow\uparrow}$ and

$$Disc(x \circ y) \cap Disc(z) = \phi \ , \tag{3.2}$$

then

$$x_n \circ y_n - c_n v_n \to x \circ y + z \quad in \quad D^k \ , \tag{3.3}$$

where the topology throughout is J_1, M_1 or M_2. If the topology is M_1 or M_2, then instead of (3.2) it suffices for $x^i \circ y$ and z^i to have no common discontinuities with jumps of the opposite sign for $1 \leq i \leq k$.

Proof. Note that

$$x_n \circ y_n - c_n v_n = (x_n - c_n e) \circ y_n + c_n(y_n - v_n) \ .$$

For the M topologies, apply Theorem 13.2.3 for composition, using the condition $y \in C^1_{\uparrow\uparrow}$, and Corollaries 12.7.1 and 12.11.5 for addition with the M_1 and M_2 topologies, respectively. The J_1 result is proved similarly, using Theorem 13.2.2 instead of Theorem 13.2.3. For addition with J_1, use Remark 12.6.2. Use Theorems 12.7.3 and 12.11.6 for the weaker condition for addition to be continuous with the M topologies. ∎

The standard application of Theorem 13.3.1 has $c^i_n \to \infty$ as $n \to \infty$ for each i and $v_n = b_n e$, where $b_n \to b$. We describe that case below.

Corollary 13.3.1. (convergence preservation for composition with linear centering) *Let x, z and x_n, $n \geq 1$, be elements of D^k; let y_n, $n \geq 1$, be elements of $D^1_\uparrow$; let $c_n \in R^k$ and $b_n \in \mathbb{R}^1$ satisfy $|c^i_n| \to \infty$ for each i and $b_n \to b$ as $n \to \infty$. If*

$$(x_n - c_n e, c_n(y_n - b_n e)) \to (x, z) \quad in \quad D^k \times D^k \tag{3.4}$$

and

$$Disc(x \circ be) \cap Disc(z) = \phi \ , \tag{3.5}$$

then

$$(x_n \circ y_n - c_n b_n e) \to x \circ y + z \quad in \quad D^k \ , \tag{3.6}$$

where the topology throughout is J_1, M_1 or M_2. If the topology is M_1 or M_2, then instead of condition (3.5) *it suffices for $x^i \circ be$ and z^i to have no common discontinuities with jumps of opposite sign, $1 \leq i \leq k$.* ∎

Proof. Since $|c^i_n| \to \infty$ as $n \to \infty$ for each i, the limit in (3.4) implies that $\|y_n - b_n e\| \to 0$ as $n \to \infty$. Hence $\|y_n - be\| \to 0$ as $n \to \infty$ and

$$(x_n - c_n e, y_n, c_n(y_n - be)) \to (x, y, z) \quad \text{in} \quad D^k \times D^1_\uparrow \times D^k \ ,$$

where $y = be$. Hence we can apply Theorem 13.3.1 to obtain the desired conclusion. ∎

We now consider an application of the convergence-preservation results above to obtain a FCLT involving a random time change. Specifically, we consider an application of Corollary 13.3.1. Let $(X_n(t), Y_n(t)) : t \geq 0\}$ be random elements of $D^k \times D^1_\uparrow$ for each $n \geq 1$, with one of the topologies under consideration. Let $\mathbf{X}_n$, $\mathbf{Y}_n$ and $\mathbf{Z}_n$ be normalized processes constructed by

$$\begin{aligned} \mathbf{X}_n(t) &\equiv \delta_n^{-1}[X_n(nt) - \mu_n nt], \quad t \geq 0 \\ \mathbf{Y}_n(t) &\equiv \delta_n^{-1}[Y_n(nt) - \lambda_n nt], \quad t \geq 0 \\ \mathbf{Z}_n(t) &\equiv \delta_n^{-1}[(X_n(Y_n(nt)) - \lambda_n \mu_n nt], \quad t \geq 0\,. \end{aligned} \tag{3.7}$$

Corollary 13.3.2. (stochastic consequence with linear centering) *Suppose that (X_n, Y_n) is a random element of $D^k \times D^1_\uparrow$ for each n. If*

$$(\mathbf{X}_n, \mathbf{Y}_n) \Rightarrow (\mathbf{U}, \mathbf{V}) \quad in \quad D^k \times D^1 \tag{3.8}$$

with topology J_1, M_1 or M_2, for the scaled processes $\mathbf{X}_n$, $\mathbf{Y}_n$ in (3.7) with $\delta_n \to \infty$, $n\delta_n^{-1} \to \infty$, $\mu_n \to \mu$ with $\mu^i \neq 0$ for all i and $\lambda_n \to \lambda$, and if

$$P(Disc(\mathbf{U} \circ \lambda \mathbf{e}) \cap Disc(\mathbf{V}) = \phi) = 1\ , \tag{3.9}$$

then

$$\mathbf{Z}_n \Rightarrow \mathbf{U} \circ \lambda \mathbf{e} + \mu \mathbf{V} \quad in \quad D^k \tag{3.10}$$

for $\mathbf{Z}_n$ in (3.7) and the same topology. If the topology is M_1 or M_2, then instead of condition (3.9) it suffices for $\mathbf{U}^i \circ \lambda \mathbf{e}$ and $\mathbf{V}^i$ to almost surely have no common discontinuities with jumps of opposite sign, $1 \leq i \leq k$.

Proof. First, since $\mu_n \to \mu$ as $n \to \infty$ in $\mathbb{R}^k$, from condition (3.8) we obtain

$$(\mathbf{X}_n, \mu_n \mathbf{Y}_n) \Rightarrow (\mathbf{U}, \mu \mathbf{V}) \quad \text{in} \quad D^k \times D^k \tag{3.11}$$

from the continuous mapping theorem. Now apply Corollary 13.3.1 with $c_n = n\delta_n^{-1}\mu_n$, $b_n = \lambda_n$,

$$x_n(t) = \delta_n^{-1} X_n(nt) \quad \text{and} \quad y_n(t) = n^{-1} Y_n(nt)\ .$$

By the Skorohod (1956) representation theorem, there exist versions of the processes such that almost surely

$$(x_n - c_n e, c_n(y_n - b_n e)) \to (x, z) \quad \text{as} \quad n \to \infty$$

where $x = \mathbf{U}$ and $z = \mu \mathbf{V}$. Corollary 13.3.1 then yields

$$(x_n \circ y_n - c_n b_n e) \to x \circ y + z \quad \text{as} \quad n \to \infty \tag{3.12}$$

almost surely in D^k, where $y = \lambda \mathbf{e}$, but the limit process in (3.12) is distributed the same as the limit process in (3.10). The almost sure convergence in (3.12) implies the convergence in distribution in (3.10). ∎

A standard application of Corollary 13.3.2 is to random sums. Then, for each $n \geq 1$, $\{X_n(nt) : t \geq 0\}$ corresponds to a sequence of partial sums; i.e.,

$$X_n(nt) = \sum_{j=1}^{\lfloor nt \rfloor} Z_{n,j}, \quad t \geq 0 ,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x and $\{Z_{n,j} : j \geq 1\}$ is a sequence of random vectors in $\mathbb{R}^k$ for each n. The composition then yields a random sum, i.e.,

$$(x_n \circ y_n)(t) = \delta_n^{-1} X_n(Y_n(nt)) = \delta_n^{-1} \sum_{j=1}^{Y_n(nt)} Z_{n,j} ,$$

so that the limit (3.10) becomes for a random sum. We consider the special case in which the summands $Z_{n,j}$ come from a single IID sequence and the random index $Y_n(t)$ is a renewal process in Section 7.4.

Another application of Corollary 13.3.2 is to establish stochastic-process limits that imply asymptotic validity of sequential stopping rules in stochastic simulations. The asymptotic validity occurs in the limit as the desired volume of the target confidence set decreases. See Chapter 4 of the Internet Supplement.

We now establish a variant of Theorem 13.3.1 with nonlinear centering terms. In the proof we apply continuity of multiplication, which we now establish. By multiplication of x and y in D, we mean $(xy)(t) \equiv x(t)y(t)$ for all t. For the M topologies, the condition on the behavior at common discontinuities is more stringent for multiplication than for addition because of the way signs multiply.

Example 13.3.1. *The need for stronger conditions.* To see the need for stronger conditions with multiplication, let $x_n \equiv -1 + 2I_{[2^{-1}-n^{-1},\infty)}$ and let $y_n \equiv y \equiv -1 + 2I_{[2^{-1},\infty)}$ for $n \geq 2$. Then $x_n \rightarrow y$ in (D, J_1) as $n \rightarrow \infty$, but $x_n y_n = 1 - 2I_{[2^{-1}-n^{-1},2^{-1}]}$, which does not converge to $y^2 = 1$ in any of the Skorohod topologies. ∎

Theorem 13.3.2. (continuity of multiplication) *Suppose that $x_n \rightarrow x$ and $y_n \rightarrow y$ in $D([0,\infty),\mathbb{R})$ with one of the Skorohod topologies J_1, M_1 or M_2. If the topology is J_1, then assume that $Disc(x) \cap Disc(y) = \phi$. If the topology is*

M_1 or M_2, then assume for each $t \in Disc(x) \cap Disc(y)$ that $x(t), x(t-), y(t)$ and $y(t-)$ are all nonnegative and $[x(t) - x(t-)][y(t) - y(t-)] \geq 0$. Then $x_n y_n \to xy$ in $D([0,\infty),\mathbb{R})$ with the same topology, where $(xy)(t) \equiv x(t)y(t)$ for $t \geq 0$.

Proof. For J_1, we can conclude that $(x_n, y_n) \to (x, y)$ in D^2 by the J_1 analog of Theorem 12.6.1; see Remark 12.6.2. It is then easy to show that $x_n y_n \to xy$. Use the fact that $x_n \to x$ implies that $\sup_n\{\|x_n\|\} < \infty$. For M_1, apply the characterization in Theorem 12.5.1 (v). For M_2, apply the characterization in Theorem 12.11.7. ■

Theorem 13.3.3. (convergence preservation for composition with nonlinear centering) *Let x, $x_n \in D^k$, $y, y_n \in D^1_{\uparrow}$, $y \in C_{\uparrow\uparrow}$, x have a continuous derivative $\dot{x}$ and $c_n \to \infty$. If*

$$c_n(x_n - x, y_n - y) \to (u, v) \quad in \quad D^k \times D^1 \tag{3.13}$$

with one of the topologies J_1, M_1 or M_2, where

$$Disc(u \circ y) \cap Disc(v) = \phi \ , \tag{3.14}$$

then

$$c_n(x_n \circ y_n - x \circ y) \to u \circ y + (\dot{x} \circ y)v \quad in \quad D^k \tag{3.15}$$

with the same topology, where

$$[(\dot{x} \circ y)v](t) \equiv [\dot{x}^1(y(t))v(t), \ldots, \dot{x}^k(y(t))v(t)] \, . \tag{3.16}$$

If the topology is M_1 or M_2, then instead of condition (3.14) *it suffices to have $\dot{x}(t) \geq (\leq) 0$ for all t and the functions $u \circ y$ and v to have no common discontinuities with jumps of opposite (common) sign.*

Proof. Note that

$$c_n(x_n \circ y_n - x \circ y) = c_n(x_n - x) \circ y_n + c_n(x \circ y_n - x \circ y) \, ,$$

Given condition (3.13), we obtain

$$[c_n(x_n - x), c_n(y_n - y), y_n] \to [u, v, y] \quad \text{in} \quad D^k \times D^1 \times D^1$$

and then, applying composition, multiplication and addition,

$$[c_n(x_n \circ y_n - x \circ y_n) + (\dot{x} \circ y)c_n(y_n - y)] \to u \circ y + (\dot{x} \circ y)v$$

by virtue of Theorems 13.2.2, 13.2.3 and 13.3.2 and condition (3.14) (or the alternative M-topology condition). Note that

$$\begin{aligned} &\|c_n(x_n \circ y_n - x \circ y) - c_n(x_n \circ y_n - x \circ y_n) - c_n(\dot{x} \circ y)(y_n - y)\| \\ &\quad \leq \|c_n(x \circ y_n - x \circ y) - c_n(\dot{x} \circ y)(y_n - y)\| . \end{aligned} \tag{3.17}$$

However, the term on the right in (3.17) is asymptotically negligible because

$$c_n(x \circ y_n - x \circ y)(t) = c_n \int_{y(t)}^{y_n(t)} \dot{x}(s) ds$$

and

$$\sup_{0 \leq s \leq t} \left| c_n \int_{y(s)}^{y_n(s)} \dot{x}(u) du - \dot{x}(y(s)) c_n (y_n(s) - y(s)) \right| \to 0 \text{ as } n \to \infty ,$$

because $\dot{x}$ is uniformly continuous over bounded intervals and $\|y_n - y\|_t \to 0$ as a consequence of $d(c_n(y_n - y), v) \to 0$. ■

13.4. Supremum

In this section we consider the supremum function, mapping $D \equiv D([0,T], \mathbb{R})$ into itself according to

$$x^{\uparrow}(t) = \sup_{0 \leq s \leq t} x(s), \quad 0 \leq t \leq T . \tag{4.1}$$

We are primarily interested in the supremum function because it is closely related to the reflection map, discussed in the next section. Another motivation is extreme-value theory; see Resnick (1987) and Embrechts et al. (1997).

We have already observed that the map from D to $\mathbb{R}$ taking x into $x^{\uparrow}(t)$ is continuous in the M_2 topology at all $t \in Disc(x)^c$; that is a consequence of Theorem 12.11.7. Now we consider the map from D to D taking x into the function $x^{\uparrow}$ in (4.1).

The supremum function can be thought of as the *nondecreasing majorant*: It is easy to see that

$$x^{\uparrow} = \inf\{y \in D : y \geq x, \ y \text{ nondecreasing}\} ,$$

where $y \geq x$ if $y(t) \geq x(t)$ for all t. If $x \in D_0$, then $x^{\uparrow} \in D_{\uparrow}$.

It is easy to see that the supremum function is Lipschitz in the uniform norm:

Lemma 13.4.1. (Lipschitz property of the supremum function with the uniform norm) *For any* $x_1, x_2 \in D([0,T],\mathbb{R})$,

$$\|x_1^{\uparrow} - x_2^{\uparrow}\| \le \|x_1 - x_2\| \ .$$

As consequences of Lemma 13.4.1, we obtain corresponding Lipschitz properties with the J_1, M_1 and M_2 metrics d_{J_1}, d_s and m_s, here denoted by d_{J_1}, d_{M_1} and d_{M_2}. For the M_1 topology, we use the following result.

Lemma 13.4.2. (inheritance of parametric representations) *For any* $x \in D$, *if* $(u,r) \in \Pi(x)$ $(\Pi_{s,2}(x))$, *then* $(u^{\uparrow}, r) \in \Pi(x^{\uparrow})$ $(\Pi_{s,2}(x))$.

Theorem 13.4.1. (Lipschitz property of the supremum function) *For any* $x_1, x_2 \in D([0,T],\mathbb{R})$,

$$\begin{aligned} d_{J_1}(x_1^{\uparrow}, x_2^{\uparrow}) &\le d_{J_1}(x_1, x_2) \ , \\ d_{M_1}(x_1^{\uparrow}, x_2^{\uparrow}) &\le d_{M_1}(x_1, x_2) \ , \\ d_{M_2}(x_1^{\uparrow}, x_2^{\uparrow}) &\le d_{M_2}(x_1, x_2) \ . \end{aligned}$$

Example 13.4.1. *Convergence preservation fails with pointwise convergence.* It is significant that analogs of Lemma 13.4.1 and Theorem 13.4.1 do not hold for pointwise convergence: Let $x_n = I_{[n^{-1}, 2n^{-1}]}$. Then $x_n(t) \to 0$ as $n \to \infty$ for all t, while $x_n^{\uparrow}(t) \to 1$ as $n \to \infty$ for all $t > 0$. ■

On the other hand, there is a pointwise-convergence analog of Theorem 13.4.1 for a single function; see Section 3.3 of the Internet Supplement.

Moreover, the conclusion in Theorem 13.4.1 can be recast in terms of pointwise convergence: Since $x^{\uparrow}$ is nondecreasing, convergence $x_n^{\uparrow} \to x^{\uparrow}$ in the M topologies is equivalent to pointwise convergence at continuity points of $x^{\uparrow}$, because on $D_{\uparrow}$ the M_1 and M_2 topologies coincide with pointwise convergence on a dense subset of $\mathbb{R}_+$ including 0 and T; see Corollary 12.5.1. Thus the M topologies have not contributed much so far. We obtain more useful convergence-preservation results for the supremum map with the M topologies when we combine supremum with centering. As before, let e be the identity map, i.e., $e(t) = t$, $0 \le t \le T$. The proof is in the Internet Supplement.

Theorem 13.4.2. (convergence preservation with the supremum function and centering) *Suppose that* $c_n(x_n - e) \to y$ *as* $n \to \infty$ *in* $D([0,T],\mathbb{R})$ *with one of the topologies* J_1, M_1 *or* M_2, *where* $c_n \to \infty$.

(a) If the topology is M_1 or M_2, then $c_n(x_n^{\uparrow}-e) \to y$ in the same topology.

(b) If the topology is J_1, then $c_n(x_n^{\uparrow} - e) \to y$ if and only if y has no negative jumps.

Example 13.4.2. *Pointwise convergence is not enough.* To see that a pointwise convergent analog of Theorem 13.4.2 does not hold, let $x_n = c_n^{-1} I_{[n^{-1}, 2n^{-1}]} + e$ where $c_n \to \infty$. Then $c_n(x_n - e)(t) = I_{[n^{-1},2n^{-1}]}(t) \to 0$ as $n \to \infty$ for all $t > 0$, while $x_n^{\uparrow}(t) = c_n^{-1} + t$ and $c_n(x_n^{\uparrow} - e)(t) = 1$ for all n sufficiently large, for $t > 0$. ■

A common case covered by Theorem 13.4.2 is $y \in C$. If $y \in C$, then all modes of convergence in Theorem 13.4.2 reduce to uniform convergence and we have $c_n(x_n^{\uparrow} - e) \to y$ whenever $c_n(x_n - e) \to y$. Since $c_n \to \infty$, under the conditions of Theorem 13.4.2, $\|x_n - e\| \to 0$ as $n \to \infty$. By Theorem 13.4.1, $\|x_n^{\uparrow} - e\| \to 0$ as well.

We use the following lemma in the proof of both Theorem 13.4.2 above and Theorem 13.4.3 below.

Lemma 13.4.3. *If $x \in D([0,T],\mathbb{R})$ and x has no negative jumps, then for any $\epsilon > 0$ there is a $\delta > 0$ such that*

$$v^{-}(x,\delta) \equiv \sup_{\substack{0 \vee (t-\delta) \le t' \le t \\ 0 \le t \le T}} \{x(t') - x(t)\} < \epsilon \ . \tag{4.2}$$

We can easily extend Theorem 13.4.2 to cover a case of nonlinear centering. Recall that $\Lambda \equiv \Lambda([0,T])$ is the set of increasing homeomorphisms of $[0,T]$. We use elements of Λ as the centering term.

Corollary 13.4.1. (convergence preservation with the supremum and nonlinear centering) *Suppose that $c_n(x_n - \lambda_n) \to y$ as in $D([0,T],\mathbb{R})$ with one of the topologies J_1, M_1 or M_2, where $\lambda_n \to \lambda$ with $\lambda, \lambda_n \in \Lambda([0,T])$ and $c_n \to \infty$.*

(a) If the topology is M_1 or M_2, then $c_n(x_n^{\uparrow} - \lambda_n) \to y$ in the same topology.

(b) If the topology is J_1, then $c_n(x_n^{\uparrow} - \lambda_n) \to y$ if and only if y has no negative jumps.

Proof. Given $c_n(x_n - \lambda_n) \to y$, we have $c_n(x_n \circ \lambda_n^{-1} - e) \to y \circ \lambda^{-1}$ by applying Theorems 13.2.2 and 13.2.3. Then Theorem 13.4.2 implies that $c_n(x_n^{\uparrow} \circ \lambda_n^{-1} - e) \to y \circ \lambda^{-1}$ with the limit holding J_1 if and only if $y \circ$

λ^{-1} has no negative jumps. Clearly, $y \circ \lambda^{-1}$ has no negative jumps if and only if y does. Finally, apply Theorems 13.2.2 and 13.2.3 again to get $c_n(x_n^{\uparrow} \circ \lambda_n^{-1} \circ \lambda_n - \lambda_n) \to y \circ \lambda^{-1} \circ \lambda$, which implies the conclusion because $\lambda_n^{-1} \circ \lambda_n = \lambda^{-1} \circ \lambda = e$. ∎

We now obtain joint convergence in the stronger topologies on $D([0,T], \mathbb{R}^2)$ under the condition that the limit function have no negative jumps.

Theorem 13.4.3. (criterion for joint convergence) *Suppose that* $c_n(x_n - e) \to y$ *as* $n \to \infty$ *in* $D([0,T], \mathbb{R})$ *with one of the* J_1, M_1 *or* M_2 *topologies, where* $c_n \to \infty$. *If, in addition,* y *has no negative jumps, then*

$$c_n(x_n - e, x_n^{\uparrow} - e) \to (y, y) \quad as \quad n \to \infty \tag{4.3}$$

in $D([0,T], \mathbb{R}^2)$ *with the strong version of the same topology, i.e., with* SJ_1, SM_1 *or* SM_2.

Since addition is continuous on D^2 with the strong topologies, we obtain the following corollary.

Corollary 13.4.2. *Under the conditions of Theorem 13.4.3,*

$$\|c_n(x_n^{\uparrow} - x_n)\| \to 0 \quad as \quad n \to \infty \, .$$

Example 13.4.3. *The problem with negative jumps.* To see that Corollary 13.4.2 does not hold and the simple direct argument with parametric representations in the proof of Theorem 13.4.3 does not work for Theorem 13.4.2 when there are negative jumps, let $y = -I_{[1/2,1]}$, $c_n = n$ and $c_n(x_n - e) = y$, i.e., $x_n = e + n^{-1}y$. First,

$$c_n(x_n^{\uparrow} - x_n)(1/2) = 1 \quad \text{for all} \quad n \geq 1 \, .$$

We now show what goes wrong with the parametric representations. let $u_n = u$ and $r_n = r$ with

$$u(0) = u(1/3) = 0, \quad u(2/3) = u(1) = -1 \tag{4.4}$$

and

$$r(0) = 0,\ r(1/3) = 1/2 = r(2/3),\ r(1) = 1 \, ,$$

with u and r defined by linear interpolation elsewhere. Then $(u_n', r) \in \Pi(c_n(x_n^{\uparrow} - e))$ for $u_n' = (u + nr)^{\uparrow} - nr$, so that

$$u_n'(0) = u_n'(1/3) = u_n'((2/3) = 0,\ u_n'((2/3) + n^{-1}) = u_n'(1) = -1 \tag{4.5}$$

with u_n' defined by linear interpolation elsewhere. From (4.4) and (4.5), we see that $|u_n'(2/3) - u(2/3)| = 1$ for all n. Thus, to get the positive result, different parametric representations are needed for $c_n(x_n^{\uparrow} - e)$. ∎

We next give an elementary result about the supremum function when the centering is in the other direction, so that x_n must be rapidly decreasing. Convergence $x_n^{\uparrow}(t) \to x(0)$ as $n \to \infty$ is to be expected, but that conclusion can not be drawn if the M_2 convergence in the condition is replaced by pointwise convergence.

Theorem 13.4.4. (convergence preservation with the supremum function when the centering is in the other direction) *Suppose that* $c_n \to \infty$ *and* $x_n + c_n e \to y$ *in* $D([0,T],\mathbb{R},M_2)$*. Then*

$$\|x_n^{\uparrow} - z(y)\| \to 0 \quad as \quad n \to \infty \ ,$$

where $z(y)(t) \equiv y(0)$*,* $0 \le t \le T$*.*

Example 13.4.4. *M_2 convergence cannot be replaced by pointwise convergence.* To see that the M_2 convergence cannot be replaced by pointwise convergence in the condition in Theorem 13.4.4, even to get pointwise convergence in the conclusion, let $x(t) = 0$, $0 \le t \le 1$, and $x_n(t) = I_{[n^{-1},2n^{-1}]}(t) - t$, $0 \le t \le 1$, $n \ge 1$. Then $x_n + e \to x$ pointwise (and not M_2), but $x_n^{\uparrow}(t) \to 1$ as $n \to \infty$ for all $t > 0$.

13.5. One-Dimensional Reflection

Closely related to the supremum function is the one-dimensional (one-sided) reflection mapping, which we have used to construct queueing processes. Indeed, the reflection mapping can be defined in terms of the supremum mapping as

$$\phi(x) \equiv x + (-x \vee 0)^{\uparrow} \ ;$$

i.e.,

$$\phi(x)(t) = x(t) - (\inf\{x(s) : 0 \le s \le t\} \wedge 0) \ , \quad 0 \le t \le T \ , \qquad (5.1)$$

as in (2.5) in Section 5.2.

The Lipschitz property for the supremum function with the uniform topology in Lemma 13.4.1 immediately implies a corresponding result for the reflection map ϕ in (5.1).

Lemma 13.5.1. (Lipschitz property with the uniform metric) *For any* $x_1, x_2 \in D([0,T],\mathbb{R})$,

$$\|\phi(x_1) - \phi(x_2)\| \le 2\|x_1 - x_2\| \ .$$

Proof. By (5.1),

$$\begin{aligned}\|\phi(x_1)-\phi(x_2)\| &\leq \|x_1-x_2\|+\|(-x_1\vee 0)^{\uparrow}-(-x_2\vee 0)^{\uparrow}\| \\ &\leq \|x_1-x_2\|+\|(-x_1\vee 0)-(-x_2\vee 0)\|\leq 2\|x_1-x_2\|\,. \quad \blacksquare\end{aligned}$$

Example 13.5.1. *The bound is tight.* To see that the bound in Lemma 13.5.1 is tight, let $x_1(t)=0$, $0\leq t\leq 1$, and $x_2=-I_{[1/3,1/2)}+I_{[1/2,1]}$ in $D([0,1],\mathbb{R})$. Then $\phi(x_1)=x_1$, while $\phi(x_2)=2I_{[1/2,1]}$, so that $\|x_1-x_2\|=1$ and $\|\phi(x_1)-\phi(x_2)\|=2$.

Unfortunately, however, the Lipschitz property for the reflection map ϕ with the uniform topology does not even imply continuity in all the Skorohod topologies. In particular, ϕ is not continuous in the M_2 topology.

Example 13.5.2. *Continuity fails in M_2.* To see that the reflection map ϕ in (5.1) is not continuous in the M_2 topology, let $x=-I_{[1,2]}$ and

$$x_n(0)=x_n(1-3n^{-1})=x(1-n^{-1})=0$$

and

$$x_n(1-2n^{-1})=x_n(1)=x_n(2)=-1$$

with x_n defined by linear interpolation elsewhere. Then $x_n\to x$ in $D([0,2],\mathbb{R})$, but $\phi(x)(t)=0$, $0\leq t\leq 2$, and $\phi(x_n)(1-n^{-1})=1$, so that $\phi(x_n)\not\to\phi(x)$. This example fails to be a counterexample for the M_1 topology because then $x_n\not\to x$ as $n\to\infty$. ■

We do obtain positive results with the J_1 and M_1 topologies. As before, let d_{J_1} and d_{M_1} be the metrics in equations (3.2) and (3.4) in Section 3.3.. For the J_1 result, we use the following elementary lemma.

Lemma 13.5.2. *For any $x\in D$ and $\lambda\in\Lambda$,*

$$\phi(x)\circ\lambda=\phi(x\circ\lambda)\ .$$

For the M_1 result, we use the following lemma. A fundamental difficulty for treating the more general multidimensional reflection map is that Lemma 13.5.3 below does not extend to the multidimensional reflection map; see Chapter 14.

Lemma 13.5.3. (preservation of parametric representations under reflections) *For any $x\in D$, if $(u,r)\in\Pi(x)$, then $(\phi(u),r)\in\Pi(\phi(x))$.*

Proof. First, $(\phi(u), r)$ is continuous since (u, r) is, by Lemma 13.5.1. It suffices to show that $(\phi(u)(s), r(s)) \in \Gamma_{\phi(x)}$ for all s and that $(\phi(u), r)$ is nondecreasing in the order on $\Gamma_{\phi(x)}$. If $t \in Disc(x^c)$, then by (5.1) $\phi(u)(s) = \phi(x)(t)$ for each s such that $r(s) = t$. It remains to consider $t \in Disc(x)$. There exists an interval $[a, b] \subseteq [0, 1]$ such that $r(s) = t$ for $s \in [a, b]$, $u(a) = x(t-)$ and $u(b) = x(t)$. Moreover, by (5.1), $\phi(u)(a) = \phi(x)(t-)$ and $\phi(u)(b) = \phi(x)(t)$, with $\phi(u)(s)$ moving continuously and monotonically from $\phi(u)(a)$ to $\phi(u)(b)$ as s increases over $[a, b]$. Hence $(\phi(u)(s), r(s)) \in \Gamma_{\phi(x)}$ for all $s \in [0, 1]$ and $(\phi(u), r)$ is nondecreasing in the order on $\Gamma_{\phi(x)}$. ∎

Theorem 13.5.1. (Lipschitz property with the J_1 and M_1 metrics) *For any* $x_1, x_2 \in D([0, T], \mathbb{R})$,

$$d_{J_1}(\phi(x_1), \phi(x_2)) \le 2 d_{J_1}(x_1, x_2)$$

and

$$d_{M_1}(\phi(x_1), \phi(x_2)) \le 2 d_{M_1}(x_1, x_2)) \ ,$$

where ϕ *is the reflection map in* (5.1).

Proof. First, for the J_1 metric, by Lemmas 13.5.2 and 13.5.1,

$$\begin{aligned} d_{J_1}(\phi(x_1), \phi(x_2)) &= \inf_{\lambda \in \Lambda} \{\|\phi(x_1) \circ \lambda - \phi(x_2)\| \vee \|\lambda - e\|\} \\ &= \inf_{\lambda \in \Lambda} \{\|\phi(x_1 \circ \lambda) - \phi(x_2)\| \vee \|\lambda - e\|\} \\ &\le \inf_{\lambda \in \Lambda} \{2\|x_1 \circ \lambda - x_2\| \vee \|\lambda - e\|\} \le 2 d_{J_1}(x_1, x_2) \, . \end{aligned}$$

Turning to M_1, we use Lemma 13.5.3 to conclude that $(\phi(u), r) \in \Pi(\phi(x))$ whenever $(u, r) \in \Pi(x)$. Then, by Lemma 13.5.1,

$$\begin{aligned} d_{M_1}(\phi(x_1), \phi(x_2)) &= \inf_{\substack{(u_i, r_i) \in \Pi(\phi(x_i)) \\ i=1,2}} \{\|u_1 - u_2\| \vee \|r_1 - r_2\|\} \\ &\le \inf_{\substack{(u_i, r_i) \in \Pi(x_i) \\ i=1,2}} \{\|\phi(u_1) - \phi(u_2)\| \vee \|r_1 - r_2\|\} \\ &\le \inf_{\substack{(u_i, r_i) \in \Pi(x_i) \\ i=1,2}} \{2\|u_1 - u_2\| \vee \|r_1 - r_2\|\} \le 2 d_{M_1}(x_1, x_2) \, . \end{aligned}$$

Remark 13.5.1. *The Lipschitz constant.* Example 13.5.1 shows that the bounds in Theorem 13.5.1 are tight; i.e., the Lipschitz constant is 2. ∎

Theorem 13.5.1 covers the standard heavy-traffic regime for one single-server queue when $\rho = 1$, where ρ is the traffic intensity. The next result covers the other cases: $\rho < 1$ and $\rho > 1$. We use the following elementary lemma in the easy case of the uniform metric.

Lemma 13.5.4. *Let d be the metric for the U, J_1, M_1 or M_2 topology. Let $x \vee a : D \to D$ be defined by*

$$(x \vee a)(t) \equiv x(t) \vee a, \quad 0 \leq t \leq T. \tag{5.2}$$

Then, for any $x_1, x_2 \in D$,

$$d(x_1 \vee a(x_1), x_2 \vee a(x_2)) \leq d(x_1, x_2) \ .$$

Theorem 13.5.2. (convergence preservation with centering) *Suppose that $x_n - c_n e \to y$ in $D([0,T], \mathbb{R})$ with the U, J_1, M_1 or M_2 topology.*

(a) If $c_n \to +\infty$, then

$$\phi(x_n) - c_n e \to y + \gamma(y) \quad as \quad n \to \infty \quad in \quad D$$

with the same topology, where

$$\gamma(y)(t) \equiv (-y(0)) \vee 0 = -(y(0) \wedge 0), \quad 0 \leq t \leq T.$$

(b) If $c_n \to -\infty$, $y(0) \leq 0$ and y has no positive jumps, then

$$\|\phi(x_n) - 0e\| \to 0 \quad as \quad n \to \infty \quad in \quad D \ ,$$

where $e(t) = t$, $0 \leq t \leq T$.

Example 13.5.3. *The necessity of the condition on $y(0)$.* To see the need for the condition $y(0) \leq 0$ in Theorem 13.5.2 (b), let $y(t) = 1$, $0 \leq t \leq T$, $c_n = -n$ and $x_n(t) = (c_n e + y)(t) = 1 - nt$ for all t. Then $x_n - c_n e = y$ for all n, but $\phi(x_n)(0) = 1$ and $\phi(x_n)(t) \to 0$ for all $t > 0$.

13.6. Inverse

We now consider the inverse map, which arises in the study of renewal processes, first passage times and extremal processes; see Billingsley (1968), Gut (1988) and Resnick (1987).

It is convenient to consider the inverse map on the subset D_u of x in $D \equiv D([0,\infty), \mathbb{R})$ that are unbounded above and satisfy $x(0) \geq 0$. For $x \in D_u$, let the inverse of x be

$$x^{-1}(t) = \inf\{s \geq 0 : x(s) > t\}, \quad t \geq 0 \ . \tag{6.1}$$

As before, let D_0 be the subset of x in D with $x(0) \geq 0$, and let $D_\uparrow$ and $D_{\uparrow\uparrow}$ be the subsets of nondecreasing and strictly increasing functions in D_0. Let $D_{u,\uparrow} \equiv D_u \cap D_\uparrow$ and $D_{u,\uparrow\uparrow} \equiv D_u \cap D_{\uparrow\uparrow}$. Clearly,

$$D_{u,\uparrow\uparrow} \subseteq D_{u,\uparrow} \subseteq D_u \subseteq D_0 \ .$$

13.6.1. The Standard Topologies

Recall that on $D_\uparrow$ the M_1 and M_2 topologies reduce to pointwise convergence on a dense subset including 0. The following result supplements Lemmas 13.2.1–13.2.3.

Lemma 13.6.1. (measurability of D_u) *Let D have one of the topologies J_1, M_1 or M_2. The subset D_u is a G_δ subset of D_0.*

Proof. Note that

$$D_u = \cap_{n=1}^{\infty}(D_0 - \bar{D}_n) ,$$

where $\bar{D}_n$ is the subset of functions in D_0 bounded above by n. In the non-uniform Skorohod topologies, $\bar{D}_n$ is a closed subset of D_0, so that D_u is a G_δ subset of D_0. ∎

We begin our study of the inverse function by stating some basic results. Our first result shows that the inverse map is closely related to the supremum.

Lemma 13.6.2. (duality) *For any $x \in D_u$, $x^{-1} \in D_{u,\uparrow}$ and $(x^{-1})^{-1} = x^{\uparrow}$.*

Corollary 13.6.1. *For any $x \in D_{u,\uparrow}$, $(x^{-1})^{-1} = x$.*

Remark 13.6.1. *The left-continuous inverse.* As part of Lemma 13.6.2, x^{-1} is right-continuous. In some circumstances it is convenient to work instead with the left-continuous inverse

$$x^{\leftarrow}(t) \equiv \inf\{s \geq 0 : x(s) \geq t\}, \quad t \geq 0 . \tag{6.2}$$

For $x \in D_u$, $x^{\leftarrow}(t) = x^{-1}(t-)$, $t \geq 0$, with $x^{-1}(0-) \equiv 0$. Note that $x^{\leftarrow}$ need not be right-continuous at 0. Indeed, $x^{\leftarrow}(0) > 0 = x^{\leftarrow}(0)$ if and only if $x^{-1}(0) > 0$. If $x^{-1}(0) = 0$, then the completed graphs of x^{-1} in (6.1) and $x^{\leftarrow}$ in (6.2) are identical, which implies that many M_1 and M_2 results for x^{-1} apply directly to $x^{\leftarrow}$ as well under that condition. ∎

The left-continuous inverse has an appealing inverse property not shared by the right-continuous inverse:

Lemma 13.6.3. (inverse relation) *For any $x \in D_{u,\uparrow}$ and $t_1, t_2 \geq 0$,*

$$x^{\leftarrow}(t_1) \leq t_2 \quad \textit{if and only if} \quad x(t_2) \geq t_1 . \tag{6.3}$$

Lemma 13.6.4. *For any $x \in D_{u,\uparrow}$,*

$$\begin{aligned} 0 &\leq (x \circ x^{-1})(t) - t \leq x(x^{-1}(t)) - x(x^{-1}(t)-) , &(6.4)\\ 0 &\leq (x^{-1} \circ x)(t) - t \leq x^{-1}(x(t)) - x^{-1}(x(t)-)) , &(6.5)\\ 0 &\leq (x \circ x^{\leftarrow})(t) - t \leq x(x^{\leftarrow}(t)) - x(x^{\leftarrow}(t)-) , &(6.6)\\ 0 &\leq t - (x^{\leftarrow} \circ x)(t) \leq x^{-1}(x(t)) - x^{\leftarrow}(x(t)) , &(6.7) \end{aligned}$$

where $x(0-)$ is interpreted as 0.

Let $J_t(x)$ be the maximum jump of x over $[0,t]$, i.e.

$$J_t(x) \equiv \sup_{0 \leq s \leq t} \{x(t) - x(t-)\} . \tag{6.8}$$

where again $x(0-) \equiv 0$.

Corollary 13.6.2. *For any $x \in D_{u,\uparrow}$ and $t > 0$,*

$$\|x \circ x^{-1} - e\|_t \leq J_{x^{-1}(t)}(x) \tag{6.9}$$

and

$$\|x^{-1} \circ x - e\|_t \leq J_{x(t)}(x^{-1}) , \tag{6.10}$$

for $J_t(x)$ in (6.8).

Lemma 13.6.5. *Suppose that $x \in D_{u,\uparrow}$. Then $x \in D_{u,\uparrow\uparrow}$ if and only if $x^{-1} \in C_{u,\uparrow}$.*

We now consider the inverse together with composition applied to elements of $\Lambda \equiv \Lambda([0,\infty))$, i.e., to homeomorphisms of $\mathbb{R}_+ \equiv [0,\infty)$. For each $\lambda \in \Lambda$, $\lambda(0) = 0$ and there is an inverse λ^{-1} with λ, $\lambda^{-1} \in C_{\uparrow\uparrow}$ and $\lambda \circ \lambda^{-1} = \lambda^{-1} \circ \lambda = e$.

Lemma 13.6.6. *If $x \in D_{u,\uparrow}$ and λ_1, $\lambda_2 \in \Lambda([0,\infty))$, then*

$$(\lambda_1 \circ x \circ \lambda_2)^{-1} = \lambda_2^{-1} \circ x^{-1} \circ \lambda_1^{-1} .$$

Proof. Note that

$$\begin{aligned} (\lambda_1 \circ x \circ \lambda_2)^{-1}(t) &= \inf\{s \geq 0 : (\lambda_1 \circ x \circ \lambda_2)(s) > t\}\\ &= \inf\{s \geq 0 : (x \circ \lambda_2)(s) > \lambda^{-1}(t)\}\\ &= \inf\{\lambda_2^{-1}(s) \geq 0 : x(s) > \lambda^{-1}(t)\}\\ &= (\lambda_2^{-1} \circ x^{-1} \circ \lambda^{-1})(t) . \quad \blacksquare \end{aligned}$$

We now turn to continuity properties of the inverse map. First we note that the inverse map from (D_u, J_1) to (D_u, J_1) or even from (D_u, U) to (D_u, J_1) is in general not continuous.

Example 13.6.1. *The inverse is not continuous when the range has the J_1 topology.* To see that the inverse map from $(D_{u,\uparrow}, U)$ to $(D_{u,\uparrow}, J_1)$ is not continuous, let $x = 2I_{[0,2]} + eI_{[2,\infty)}$ and

$$x_n = (2 - n^{-1})I_{[0,1)} + (2 + n^{-1})I_{[1,2+n^{-1})} + eI_{[2+n^{-1},\infty)} \ .$$

Then $\|x_n - x\| = n^{-1} \to 0$ and $x_n^{-1} \to x^{-1}$ (M_1), but $x_n^{-1} \not\to x^{-1}$ (J_1). ■

Even for the M_1 topology, there are complications at the left endpoint of the domain $[0, \infty)$.

Example 13.6.2. *Complications at the left endpoint of the domain.* To see that the inverse map from $(D_{u,\uparrow}, U)$ to $(D_{u,\uparrow}, M_1)$ is in general not continuous, let $x(t) = 0$, $0 \le t < 1$, and $x(t) = t$, $t \ge 1$; Let $x_n = t/n$, $0 \le t < 1$ and $x_n(t) = t$, $t \ge 1$. Then $\|x_n - x\|_\infty = n^{-1} \to 0$, but $x_n^{-1}(0) = 0 \not\to 1 = x^{-1}(0)$, so that $x_n^{-1} \not\to x^{-1}$ (M_1). ■

To avoid the problem in Example 13.6.2, we can require that $x^{-1}(0) = 0$. To develop an equivalent condition, let $D_{u,\epsilon}^{\uparrow}$ be the subset of functions x in D_u such that $x(t) = 0$ for $0 \le t \le \epsilon$.

Then let

$$D_u^* \equiv \cap_{n=1}^{\infty} (D_{u,n^{-1}})^c \ . \tag{6.11}$$

Lemma 13.6.7. (measurability of D_u^*) *With the J_1, M_1 or M_2 topology, D_u^* in (6.11) is a G_δ subset of D_u and*

$$D_u^* = \{x \in D_u : x^{-1}(0) = 0\} \ . \tag{6.12}$$

Let $D_{u,\uparrow}^* \equiv D_\uparrow \cap D_u^*$. A key property of $D_{u,\uparrow}^*$, not shared by $D_{u,\uparrow}$ because of the complication at the origin, is that parametric representation (u, r) for x directly serve as parametric representations for x^{-1} when we switch the roles of the components u and r.

Lemma 13.6.8. (switching the roles of u and r) *For $x \in D_{u,\uparrow}^*$, the graph Γ_x serves as the graph of $\Gamma_{x^{-1}}$ with the axes switched. Thus, $(u, r) \in \Pi(x)$ if and only if $(r, u) \in \Pi(x^{-1})$, where $\Pi(x)$ is the set of M_1 parametric representations.*

Corollary 13.6.3. (continuity on (D_u^*, M_1)) *The inverse map from (D_u^*, M_1) to $(D_{u,\uparrow}, M_1)$ is continuous.*

Proof. First apply Theorem 13.4.1 for the supremum. Then apply Lemma 13.6.8. ■

We now generalize Corollary 13.6.3 by only requiring that the limit be in D_u^*. As before, the missing proof is in the Internet Supplement.

Theorem 13.6.1. (measurability and continuity at limits in D_u^*) *The inverse map in* (6.1) *from* (D_u, M_2) *to* $(D_{u,\uparrow}, M_1)$ *is measurable and continuous at* $x \in D_u^*$, *i.e., for which* $x^{-1}(0) = 0$.

Corollary 13.6.4. . (continuity at strictly increasing functions) *The inverse map from* (D_u, M_2) *to* $(D_{u,\uparrow}, U)$ *is continuous at* $x \in D_{u,\uparrow\uparrow}$.

Proof. First, $D_{u,\uparrow\uparrow} \subseteq D_{u,\uparrow}^*$, so that we can apply Theorem 13.6.1 to get $x_n^{-1} \to x^{-1}$ in $(D_{u,\uparrow}, M_1)$. However, by Lemma 13.6.4, $x^{-1} \in C$ when $x \in D_{u,\uparrow\uparrow}$. Hence the M_1 convergence $x_n^{-1} \to x^{-1}$ actually holds in the stronger topology of uniform convergence over compact subsets. ■

13.6.2. The M_1' Topology

For cases in which the condition $x^{-1}(0) = 0$ in Theorem 13.6.1 is not satisfied, we can modify the M_1 and M_2 topologies to obtain convergence, following Puhalskii and Whitt (1997). With these new weaker topologies, which we call M_1' and M_2', we do not require that $x_n(0) \to x(0)$ when $x_n \to x$. We construct the new topologies by extending the graph of each function x by appending the segment $[0, x(0)] \equiv \{\alpha 0 + (1-\alpha)x(0) : 0 \le \alpha \le 1\}$. Let the new graph of $x \in D$ be

$$\begin{aligned} \Gamma_x' \;=\; & \{(z,t) \in \mathbb{R}^k \times [0,\infty) : z = \alpha x(t) + (1-\alpha)x(t-) \\ & \text{for } 0 \le \alpha \le 1 \text{ and } t \ge 0\}\,, \end{aligned} \tag{6.13}$$

where $x(0-) \equiv 0$. Let $\Pi'(x)$ and $\Pi_2'(x)$ be the sets of all M_1 and M_2 parametric representations of Γ_x', defined just as before. We say that $x_n \to x$ in (D, M_1') if there exist parametric representations $(u_n, r_n) \in \Pi'(x_n)$ and $(u, r) \in \Pi'(x)$, where $\Pi'(x)$ is the set of M_1' parametric representations of x, such that

$$\|u_n - u\|_t \vee \|r_n - r\|_t \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad t > 0\,. \tag{6.14}$$

We have a corresponding definition of convergence in (D, M_2') using the parametric representations in $\Pi_2'(x)$ instead of $\Pi'(x)$. With the M_i' topologies, we obtain a cleaner statement than Lemma 13.6.8.

Lemma 13.6.9. (graphs of the inverse with the M_i' topology) *For $x \in D_{u,\uparrow}$, the graph Γ_x' serves as the graph $\Gamma_{x^{-1}}'$ with the axes switched, so that $(u,r) \in \Pi'(x)$ $(\Pi_2'(x))$ if and only if $(r,u) \in \Pi'(x^{-1})$ $(\Pi_2'(x^{-1}))$.*

Thus we get an alternative to Theorem 13.6.1.

Theorem 13.6.2. (continuity in the M_1' topology) *The inverse map in* (6.1) *from (D_u, M_2') to $(D_{u,\uparrow}, M_1')$ is continuous.*

Proof. By the M_2' analog of Theorem 13.4.1, if $x_n \to x$ in (D_u, M_2'), then $x_n^{\uparrow} \to x^{\uparrow}$ in $(D_{u,\uparrow}, M_2')$. Since the M_2' topology coincides with the M_1' topology on $D_{\uparrow}$, we get $x_n^{\uparrow} \to x^{\uparrow}$ in $(D_{u,\uparrow}, M_1')$. By Lemma 13.6.9, we get $(x_n^{\uparrow})^{-1} \to (x^{\uparrow})^{-1}$ in $(D_{u,\uparrow}, M_1')$. That gives the desired result because $(x^{\uparrow})^{-1} = x^{-1}$ for all $x \in D_u$. ■

An alternative approach to the difficulty at the origin besides M_i' topology on $D_u([0,\infty),\mathbb{R})$ is the ordinary M_i topology on $D_u((0,\infty),\mathbb{R})$. The difficulty at the origin goes away if we ignore it entirely, which we can do by making the function domain $(0,\infty)$ for the image of the inverse functions.

In particular, Theorem 13.6.2 implies the following corollary.

Corollary 13.6.5. (continuity when the origin is removed from the domain) *The inverse map in* (6.1) *from $D_u([0,\infty), M_2)$ to $D_{u,\uparrow}((0,\infty), M_1)$ is continuous.*

Proof. Since the M_2' topology is weaker than M_2, if $x_n \to x$ in $D_u([0,\infty), M_2)$, then $x_n \to x$ in $D_u([0,\infty), M_2')$. Apply Theorem 13.6.2 to get $x_n^{-1} \to x^{-1}$ in $D_{u,\uparrow}([0,\infty), M_1')$. That implies $x_n^{-1} \to x^{-1}$ for the restrictions in $D_{\uparrow}([t_1,t_2], M_1)$ for all $t_1, t_2 \in Disc(x^{-1})^c$, which in turn implies that $x_n^{-1} \to x^{-1}$ in $D_{u,\uparrow}((0,\infty), M_1)$. ■

However, in general we cannot work with the inverse on $D_u((0,\infty),\mathbb{R})$.

Example 13.6.3. *Difficulty with the domain* $(0,\infty)$. To see the problem with having the function domain be $(0,\infty)$, let $x = e$ and $x_n(0) = x_n(2n^{-1}) = 0$, $x_n(n^{-1}) = 1$, $x_n(t) = t - 2n^{-1}$, $t \geq 2n^{-1}$, with x_n defined by linear interpolation elsewhere. Then $x_n \to x$ in $D((0,\infty),\mathbb{R},U)$, but $x_n^{-1} \not\to x^{-1} \equiv e$, because $x_n^{-1}(t) \to 1$ as $n \to \infty$ for each t with $0 < t < 1$. ■

We can obtain positive results if all the functions are required to be monotone. The following result is elementary.

Theorem 13.6.3. (equivalent characterizations of convergence for monotone functions) *For x_n, $n \geq 1$, $x \in D_{u,\uparrow}([0,\infty),\mathbb{R})$, the following are equivalent:*

$$x_n \to x \quad in \quad D_{u,\uparrow}((0,\infty),\mathbb{R},M_1)\ ; \tag{6.15}$$

$$x_n \to x \quad in \quad D_{u,\uparrow}([0,\infty),\mathbb{R},M_1')\ ; \tag{6.16}$$

$$x_n(t) \to x(t) \quad for\ all\ t\ in\ a\ dense\ subset\ of\ (0,\infty)\ ; \tag{6.17}$$

$$x_n^{-1} \to x^{-1} \quad in \quad D((0,\infty),\mathbb{R},M_1)\ ; \tag{6.18}$$

$$x_n^{-1} \to x^{-1} \quad in \quad D([0,\infty),\mathbb{R},M_1')\ ; \tag{6.19}$$

$$x_n^{-1}(t) \to x^{-1}(t) \quad for\ all\ t\ in\ a\ dense\ subset\ of\ (0,\infty). \tag{6.20}$$

Example 13.6.4. *The need for monotonicity.* To see the advantage of M_1' on $[0,\infty)$ over M_1 on $(0,\infty)$, let $x(t) = 1$, $t \geq 0$,

$$x_n^1(0) = 0, x_n^1(n^{-1}) = 1 = x_n^1(t), \quad t \geq n^{-1}\ , \tag{6.21}$$

and

$$x_n^2(0) = 0 = x_n^2(2n^{-1}), x_n^2(n^{-1}) = x_n^2(3n^{-1}) = 1 = x_n^2(t), \quad t \geq 3n^{-1}\ , \tag{6.22}$$

with x_n^1 and x_n^2 defined by linear interpolation elsewhere. Then $x_n^1 \to x$ in both $D((0,\infty),\mathbb{R},M_1)$ and in $D([0,\infty),\mathbb{R},M_1')$, but $x_n^2 \to x$ only in $D((0,\infty),\mathbb{R},M_1)$. The monotonicity condition provides the equivalence in Theorem 13.6.3.

13.6.3. First Passage Times

In this final subsection we consider some real-valued functions closely related to the inverse function. Sometimes we are interested in the first passage time to or beyond some specified level. Given any specified level $z \in \mathbb{R}$, the *first passage time* beyond z is the function $\tau_z : D_u \to \mathbb{R}$ defined in terms of the inverse function by

$$\tau_z(x) \equiv x^{-1}(z)\ . \tag{6.23}$$

It is elementary that τ_z has the following two scaling invariance properties: For any $c > 0$,

$$\tau_{cz}(cx) = \tau_z(x) \tag{6.24}$$

and

$$c\tau_z(x \circ ce) = \tau_z(x)\ , \tag{6.25}$$

where e is the identity map, i.e., $e(t) = t$ for $t \geq 0$.

Three functions closely related to the first-passage-time function τ_z are the *overshoot function* $\gamma_z : D_u \to \mathbb{R}$ defined by

$$\gamma_z(x) \equiv x(\tau_z(x)) - z \ , \tag{6.26}$$

the *last-value function* $\lambda_z : D_u \to \mathbb{R}$ defined by

$$\lambda_z(x) \equiv x(\tau_z(x)-) \tag{6.27}$$

and the *final-jump functions* $\delta_z : D_u \to \mathbb{R}$ defined by

$$\delta_z(x) \equiv x(\tau_z(x)) - x(\tau_z(x)-) \ . \tag{6.28}$$

The following continuity properties are elementary, but of course important. It clearly does not suffice to have pointwise convergence.

Theorem 13.6.4. (continuity of first-passage-time functions) *Let x be an element of D_u that is not equal to z throughout the interval $(\tau_z(x)-\epsilon, \tau_z(x))$ for any $\epsilon > 0$. If $x_n \to x$ in (D, M_2), then*

$$(\tau_z(x_n), \gamma_z(x_n), \lambda_z(x_n), \delta_z(x_n)) \to (\tau_z(x), \gamma_z(x), \lambda_z(x), \delta_z(x))$$

as $n \to \infty$ in $\mathbb{R}^4$.

The regularity condition holds almost surely for Lévy processes. Hence we have the following consequence of Theorem 13.6.4, which we apply to queues in Section 9.7.

Theorem 13.6.5. (convergence of first-passage-time functions for Lévy limit processes) *Let X be a Lévy process such that*

$$P(\overline{\lim_{t\to\infty}} \ X(t) = \infty) = 1 \ . \tag{6.29}$$

If $X_n \Rightarrow X$ in (D_u, M_2), then

$$(\tau_z(X_n), \gamma_z(X_n), \lambda_z(X_n), \delta_z(X_n)) \to (\tau_z(X), \gamma_z(X), \lambda_z(X), \delta_z(X))$$

in $\mathbb{R}^4$ for any $z > 0$.

13.7. Inverse with Centering

We continue considering the inverse map, but now with centering. We start by considering linear centering. In particular, we consider when a limit for $c_n(x_n - e)$ implies a limit for $c_n(x_n^{-1} - e)$ when $x_n \in D_u \equiv D_u([0,\infty),\mathbb{R})$ and $c_n \to \infty$. By considering the behavior at one t, it is natural to anticipate that we should have $c_n(x_n^{-1} - e) \to -y$ when $c_n(x_n - e) \to y$. A first step for the M topologies is to apply Theorem 13.4.2, which yields limits for $c_n(x_n^{\uparrow} - e)$. Thus for the M topologies, it suffices to assume that $x_n \in D_{u,\uparrow}$.

For the J_1 topology, however, a different argument is needed to get limits when $y \notin C$, as the following result shows.

Lemma 13.7.1. *Suppose that $x_n \in D_u$, $n \geq 1$, and $c_n \to \infty$. if $c_n(x_n^{\uparrow} - e) \to y$ and $c_n(x_n^{-1} - e) \to -y$ (J_1), then $y \in C$.*

Proof. Since $x_n^{\uparrow} \in D_{u,\uparrow}$, $c_n(x_n^{\uparrow} - e)$ has no negative jumps. Since the topology is J_1 and $c_n(x_n - e) \to y$, y has no negative jumps; e.g., see p. 301 of Jacod and Shiryaev (1987). Similarly, $c_n(x_n^{-1} - e)$ has no negative jumps. Since $c_n(x_n^{-1} - e) \to -y$ (J_1), $-y$ has no negative jumps. ■

The following lemma establishes a necessary condition in any of the topologies.

Lemma 13.7.2. *If $x_n \in D_{u,\uparrow}$, $c_n(x_n - e)(0) \to y(0)$ and $c_n(x_n^{-1} - e)(0) \to -y(0)$, where $c_n \to \infty$, then $y(0) = 0$.*

Proof. Since $x_n \in D_{u,\uparrow}$, $x_n(0) \geq 0$ and $x_n^{-1}(0) \geq 0$. Since $e(0) = 0$, the convergence $c_n(x_n - e)(0) \to y(0)$ implies that $y(0) \geq 0$. Similarly, the convergence $c_n(x_n^{-1} - e)(0) \to -y(0)$ implies that $y(0) \leq 0$. ■

Now we state the main limit theorem for inverse functions with centering.

Theorem 13.7.1. (inverse with linear centering) *Suppose that $c_n(x_n - e) \to y$ as $n \to \infty$ in $D([0,\infty),\mathbb{R})$ with one of the topologies M_2, M_1 or J_1, where $x_n \in D_u$, $c_n \to \infty$ and $y(0) = 0$.*

(a) If the topology is M_2 or M_1, then $c_n(x_n^{-1} - e) \to -y$ as $n \to \infty$ with the same topology.

(b) If the topology is J_1 and if y has no positive jumps, then $c_n(x_n^{-1} - e) \to -y$ as $n \to \infty$.

We can combine Lemma 13.6.6 and Theorem 13.7.1 to obtain the following corollary. Let Λ be the space of homeomorphisms of $\mathbb{R}_+$.

Corollary 13.7.1. *Suppose that* $x_n \in D_{u,\uparrow}$ *and* $\lambda_{1,n},\ \lambda_{2,n} \in \Lambda$, $n \geq 1$. *Let* $c_n \to \infty$ *and* $y(0) = 0$. *Then*

$$c_n(\lambda_{2,n} \circ x_n \circ \lambda_{1,n} - e) \to y \quad in \quad D([0,\infty), \mathbb{R}, M_i) \tag{7.1}$$

if and only if

$$c_n(\lambda_{1,n}^{-1} \circ x_n^{-1} \circ \lambda_{2,n}^{-1} - e) \to -y \quad in \quad D([0,\infty), \mathbb{R}, M_i)\,, \tag{7.2}$$

where the topology in both cases is either M_1 *or* M_2.

We can apply Corollary 13.7.1 to obtain generalizations of Theorem 13.7.1 with nonlinear centering terms. (We obtain a more general result at the end of the section.)

Corollary 13.7.2. (centering functions from Λ) *Suppose that, in addition to the conditions of Corollary 13.7.1,* $\lambda_{i,n} \to \lambda_i$ *as* $n \to \infty$ *for each* i, *where* $\lambda_i \in \Lambda$. *Then*

$$c_n(\lambda_{2,n} \circ x_n - \lambda_{1,n}^{-1}) \to y \circ \lambda_1^{-1} \quad in \quad (D, M_i) \tag{7.3}$$

if and only if

$$c_n(\lambda_{1,n}^{-1} \circ x_n^{-1} - \lambda_{2,n}^{-1}) \to -y \circ \lambda_2^{-1} \quad in \quad (D, M_i)\,. \tag{7.4}$$

Proof. Apply Theorem 13.2.3 with the composition map to show that (7.3) is equivalent to (7.1) and (7.4) is equivalent to (7.2). ∎

We can use Corollary 13.7.1 to obtain the following consequence.

Corollary 13.7.3. *Suppose that* $x_n \in D_u$, $y(0) = 0$, $c_n \to \infty$ *and* $a_n \to a > 0$. *If*

$$c_n(x_n - a_n e) \to y \quad in \quad D$$

with the M_1 *or* M_2 *topology, then*

$$c_n(x_n^{-1} - a_n^{-1}e) \to -a^{-1}y \circ a^{-1}e \quad in \quad D$$

with the same topology.

Proof. Under the condition, $(a_n c_n)(a_n^{-1} x_n - e) \to x$, so that by Corollary 13.7.1, $(a_n c_n)(x_n^{-1} \circ a_n e - e) \to -y$. Now applying the composition map with $a_n^{-1} e$, $a_n c_n (x_n^{-1} - a_n^{-1} e) \to x \circ a^{-1} e$. Dividing by a_n yields the conclusion. ∎

Stochastic limit theorems are not often expressed directly in the form of Corollaries 13.7.1 or 13.7.3. We now state consequences of Corollary 13.7.1 that have more direct applications.

Corollary 13.7.4. *Let $y_n \in D_{u,\uparrow}$ and $\phi_{1,n}, \phi_{2,n} \in \Lambda$, $n \geq 1$; let $u(0) = 0$ and $n/\psi(n) \to \infty$ as $n \to \infty$. Let*

$$w_n(t) \equiv \psi(n)^{-1}[(\phi_{2,n} \circ y_n \circ \phi_{1,n})(nt) - nt], \quad t \geq 0\,, \tag{7.5}$$

and

$$x_n(t) \equiv \psi(n)^{-1}[(\phi_{1,n}^{-1} \circ y_n^{-1} \circ \phi_{2,n}^{-1})(nt) - nt], \quad t \geq 0\,, \tag{7.6}$$

for all $n \geq 1$. Then

$$w_n \to u \quad in \quad D([0,\infty), \mathbb{R}) \tag{7.7}$$

if and only if

$$x_n \to -u \quad in \quad D([0,\infty, \mathbb{R})\ , \tag{7.8}$$

where the topology throughout is either M_1 or M_2.

Proof. Apply Corollary 13.7.1 with $x_n(t) = n^{-1} y_n(t)$, $\lambda_{i,n}(t) = n^{-1}\phi_{i,n}(nt)$ and $c_n = n/\psi(n)$. Then $w_n = c_n(\lambda_{2,n} \circ x_n \circ \lambda_{1,n} - e)$ and $x_n = c_n(\lambda_{1,n}^{-1} \circ x_n^{-1} \circ \lambda_{2,n}^{-1} - e)$. ∎

We now consider the special case of Corollary (13.7.4) in which the homeomorphisms $\phi_{i,n}$ are linear, i.e., $\phi_{i,n} = a_{i,n} e$, $n \geq 1$.

Corollary 13.7.5. *Suppose that $y_n \in D_{u,\uparrow}$, $w(0) = 0$, $a_n \to a > 0$ and $n/\psi(n) \to \infty$ as $n \to \infty$. Let*

$$\tilde{w}_n = \psi(n)^{-1}[y_n(nt) - a_n nt], \quad t \geq 0\,, \tag{7.9}$$

and

$$\tilde{x}_n = \psi(n)^{-1}[y_n^{-1}(nt) - a_n^{-1} nt], \quad t \geq 0\,. \tag{7.10}$$

Then

$$\tilde{w}_n \to w \quad in \quad D([0,\infty), \mathbb{R}) \tag{7.11}$$

if and only if

$$\tilde{x}_n \to a^{-1} w \circ a^{-1} e \quad in \quad D([0,\infty), \mathbb{R})\ , \tag{7.12}$$

where the topology throughout is M_1 or M_2.

Proof. Apply Corollary 13.7.1 with $x_n(t) = n^{-1}y_n(nt)$, $\lambda_{2,n} = a_n^{-1}e$, $\lambda_{1,n} = e$ and $c_n = na_n/\psi(n)$. Then $\tilde{w}_n = c_n(\lambda_{2,n} \circ x_n \circ \lambda_{1,n} - e)$, so that $\tilde{w}_n \to w$ if and only if $c_n(\lambda_{1,n}^{-1} \circ x_n^{-1} \circ \lambda_{2,n}^{-1} - e) \to -w$. However,

$$c_n(\lambda_{1,n}^{-1} \circ x_n^{-1} \circ \lambda_{2,n}^{-1} - e) = a_n\tilde{x}_n \circ a_n e \tag{7.13}$$

and

$$-a_n\tilde{x}_n \circ a_n e \to -w \quad \text{if and only if} \quad \tilde{x}_n \to -a^{-1}w \circ a^{-1}e\,. \quad \blacksquare \tag{7.14}$$

Following Puhalskii (1994), we can generalize Theorem 13.7.1 by allowing nonlinear centering terms. We present several results of this kind.

Theorem 13.7.2. *Suppose that*

$$c_n(x_n - \lambda) \to u \quad as \quad n \to \infty \quad in \quad D$$

with one of the topologies M_2, M_1 or J_1, where $x_n \in D_u$, $u(0) = 0$, u has no positive jumps if the topology is J_1, $\lambda \in \Lambda$ and $c_n \to \infty$. Then

$$c_n(\lambda \circ x_n^{-1} - e) \to -u \circ \lambda^{-1} \quad as \quad n \to \infty \tag{7.15}$$

with the same topology. If, in addition, λ is absolutely continuous with continuous positive derivative $\dot{\lambda}$, then

$$c_n(x_n^{-1} - \lambda^{-1}) \to \frac{-u \circ \lambda^{-1}}{\dot{\lambda} \circ \lambda^{-1}} \quad as \quad n \to \infty\,, \tag{7.16}$$

where $(u/v)(t) \equiv u(t)/v(t)$, $t \geq 0$.

Proof. Apply Theorems 13.2.2 and 13.2.3 with the composition map to get $c_n(x_n \circ \lambda^{-1} - \lambda \circ \lambda^{-1}) \to u \circ \lambda^{-1}$ as in the same topology. Since $\lambda \circ \lambda^{-1} = e$, we can apply Theorem 13.7.1 or Corollary 13.7.1 to get (7.15) with the same topology. Now suppose that λ is absolutely continuous with continuous positive derivative $\dot{\lambda}$. Then

$$\begin{aligned} c_n(\lambda \circ x_n^{-1} - e)(t) &= c_n(\lambda \circ x_n^{-1} - \lambda \circ \lambda^{-1})(t) \\ &= c_n \int_{\lambda^{-1}(t)}^{x_n^{-1}(t)} \dot{\lambda}(s)ds\,. \end{aligned} \tag{7.17}$$

Since $c_n(x_n - \lambda) \to u$, $\|x_n - \lambda\|_t \to 0$ and $\|x_n^{-1} - \lambda^{-1}\|_t \to 0$ as $n \to \infty$ for all t. Since $\dot{\lambda}$ is continuous, it is uniformly continuous over bounded intervals. Hence

$$\sup_{0 \leq s \leq t} \left| c_n \int_{\lambda^{-1}(s)}^{x_n^{-1}(s)} \dot{\lambda}(u)du - \dot{\lambda}(\lambda^{-1}(s))c_n(x_n^{-1}(s) - \lambda^{-1}(s) \right| \to 0\,. \tag{7.18}$$

Then (7.15), (7.17) and (7.18) imply that

$$(\dot{\lambda} \circ \lambda^{-1}) c_n (x_n^{-1} - \lambda^{-1}) \to -u \circ \lambda^{-1} \quad \text{as} \quad n \to \infty \tag{7.19}$$

in the same topology, where $(uv)(t) \equiv u(t)v(t)$ for $u, v \in D$. Finally (7.19) implies (7.16). ■

Corollary 13.7.6. *Suppose that* $x_n \in D_{u,\uparrow}$, $u(0) = 0$, $\lambda \in \Lambda$, λ *is absolutely continuous with continuous positive derivative* $\dot{\lambda}$ *and* $c_n \to \infty$. *Then*

$$c_n(x_n - \lambda) \to u \quad in \quad D \tag{7.20}$$

with one of the topologies M_1 *or* M_2 *if and only if*

$$c_n(x_n^{-1} - \lambda^{-1}) \to \frac{-u \circ \lambda^{-1}}{\dot{\lambda} \circ \lambda^{-1}} \quad in \quad D \tag{7.21}$$

with the same topology.

Proof. The implication (7.20)→(7.21) is directly covered by Theorem 13.7.2. to go the other way, note that $\lambda^{-1} \in \Lambda$ and λ^{-1} is absolutely continuous with continuous positive derivative $1/\dot{\lambda}(\lambda^{-1}(t))$. Moreover, if $v = -(u \circ \lambda^{-1})/\dot{\lambda} \circ \lambda^{-1}$ in (7.21), then $v(0) = 0$ and $-(v \circ \lambda)/(\dot{\lambda^{-1}}) \circ \lambda = u$. ■

We can often apply the basic convergence-preservation results in combination. We can combine Theorems 13.3.1 and 13.7.2 to obtain limits for functions $x_n \circ y_n^{-1}$ and $x_n^{-1} \circ y_n$ with nonlinear centering.

Theorem 13.7.3. (composition plus inverse with centering) *Suppose that* $x_n \in D$, $y_n \in D_u$, $c_n \to \infty$,

$$c_n(x_n - x, y_n - y) \to (u, v) \quad in \quad D \times D \tag{7.22}$$

with one of the J_1, M_1 *or* M_2 *topologies, where* $v(0) = 0$ *and* v *has no positive jumps if the topology is* J_1, $y \in \Lambda$, x *and* y *are absolutely continuous with continuous derivative* $\dot{x}$ *and* $\dot{y}$ *with* $\dot{y} > 0$ *and*

$$Disc(u) \cap Disc(v) = \phi \ . \tag{7.23}$$

Then

$$c_n(x_n \circ y_n^{-1} - x \circ y^{-1}) \to u \circ y^{-1} - \left(\frac{\dot{x} \circ y^{-1}}{\dot{y} \circ y^{-1}}\right)(v \circ y^{-1}) \quad in \quad D\,. \tag{7.24}$$

If the topology is M_1 *or* M_2, *then instead of* (7.23) *it suffices for* u *and* v *to have no common discontinuities with jumps of common (opposite) sign with* $\dot{x}(t) \geq (\leq) 0$ *for all* t.

Proof. The conditions imply that the conditions of Theorem 13.7.2 hold for y_n, so that

$$c_n(y_n^{-1} - y^{-1}) \to -\frac{v \circ y^{-1}}{\dot{y} \circ y^{-1}} \quad \text{in} \quad D\ . \tag{7.25}$$

The conditions then imply that the conditions of Theorem 13.3.3 hold with y_n^{-1} here playing the role of y_n there. We need

$$Disc(u \circ y^{-1}) \cap Disc(v \circ y^{-1}) = \phi \tag{7.26}$$

but that is equivalent to (7.23). With the M topologies, we can apply Theorems 12.7.3 and 12.11.6 to treat addition and Theorem 13.3.2 to treat multiplication. ∎

We now turn to the general first passage times

$$(x_n^{-1} \circ y_n)(t) = \inf\{s \geq 0 : x_n(s) > y_n(t)\}, \quad t \geq 0\,, \tag{7.27}$$

which are elements of D when $x_n \in D_u$ and $y_n \in D_\uparrow$. The following is Puhalskii's (1994) result extended to allow discontinuous limits. For an application to obtain heavy-traffic stochastic-process limits for waiting times directly from corresponding heavy-traffic stochastic-process limits for queue lengths, see Section 5.4 of the Internet Supplement.

Theorem 13.7.4. (Puhalskii's theorem) *Suppose that $x_n \in D_u$, $y_n \in D_\uparrow$, $c_n \to \infty$,*

$$c_n(x_n - x, y_n - y) \to (u, v) \quad \textit{in} \quad D \times D \tag{7.28}$$

with one of the J_1, M_1 or M_2 topologies, where $u(0) = 0$, u has no positive jumps if the topology is J_1,

$$Disc(u \circ x^{-1} \circ y) \cap Disc(v) = \phi\ , \tag{7.29}$$

$y \in C_{\uparrow\uparrow}$ and x is absolutely continuous with a continuous positive derivative $\dot{x}$, then

$$c_n(x_n^{-1} \circ y_n - x^{-1} \circ y) \to \frac{v - u \circ x^{-1} \circ y}{\dot{x} \circ x^{-1} \circ y} \quad \textit{in} \quad D \tag{7.30}$$

with the same topology. If the topology is M_1 or M_2, then instead of condition (7.29) *it suffices for $u \circ x^{-1} \circ y$ and v to have no common discontinuities with jumps of common sign.*

Proof. Since x is absolutely continuous with continuous positive derivative $\dot{x}$, $x \in C_{\uparrow\uparrow}$. Hence the conditions of Theorem 13.7.2 hold, so that

$$c_n(x_n^{-1} - x^{-1}) \to \frac{-u \circ x^{-1}}{\dot{x} \circ x^{-1}} \quad \text{in} \quad D \tag{7.31}$$

with the same topology. We now apply Theorem 13.3.3, noting that x^{-1} has a continuous derivative $1/\dot{x}(x^{-1}(t))$. Condition (7.29) implies condition (3.14) for u in (3.14) equal to $-(u \circ x^{-1})/\dot{x} \circ x^{-1}$. Then (3.15) becomes (7.30). With the M topologies, we can apply Theorems 12.7.3 and 12.11.6. ∎

Remark 13.7.1. *Relating the theorems under extra conditions.* Under extra regularity conditions, we can apply Theorem 13.7.2 to obtain limits for $y_n \circ x_n^{-1}$ from limits for $x_n \circ y_n^{-1}$ provided by Theorem 13.7.3. We need $u(0) = v(0) = 0$, $x, y \in \Lambda$, $x_n, y_n \in D_u$ and both $\dot{x}$ and $\dot{y}$ to be continuous and positive. Since $x_n, y_n \in D_u$, $x_n^{-1}, y_n^{-1} \in D_{u,\uparrow}$. Then $\lambda \equiv x \circ y^{-1} \in \Lambda$ and $(x_n \circ y_n^{-1})^{-1} = y_n \circ x_n^{-1}$. From (7.16) and (7.24), we obtain

$$c_n(y_n \circ x_n^{-1} - y \circ x^{-1}) \to z \tag{7.32}$$

where

$$z = \frac{-1}{\dot{\lambda} \circ \lambda^{-1}} \left(u \circ y^{-1} - \left(\frac{\dot{x} \circ y^{-1}}{\dot{y} \circ y^{-1}} \right) (v \circ y^{-1}) \right) \circ \lambda^{-1} \tag{7.33}$$

for $\lambda = x \circ y^{-1}$. Since $\lambda^{-1} = y \circ x^{-1}$,

$$\dot{\lambda} = \frac{\dot{x} \circ y^{-1}}{\dot{y} \circ y^{-1}}, \quad \dot{\lambda} \circ \lambda^{-1} = \frac{\dot{x} \circ x^{-1}}{\dot{y} \circ x^{-1}} \tag{7.34}$$

and

$$z = -\frac{(\dot{y} \circ x^{-1})}{\dot{x} \circ x^{-1}} (u \circ x^{-1}) + v \circ x^{-1} , \tag{7.35}$$

which coincides with (7.24) with the labels changed, i.e., with (x, y, u, v) replaced by (y, x, v, u).

Similarly, under extra regularity conditions, we can apply Theorem 13.7.2 to obtain limits for $y_n^{-1} \circ x_n$ from limits for $x_n^{-1} \circ y_n$ provided by Theorem 13.7.4. We now need $x_n, y_n \in D_{u,\uparrow}$. We obtain

$$c_n(y_n^{-1} \circ x_n - y^{-1} \circ x) \to z , \tag{7.36}$$

where

$$z = \frac{-1}{\dot{\lambda} \circ \lambda^{-1}} \left(\frac{v - u \circ x^{-1} \circ y}{\dot{x} \circ x^{-1} \circ y} \right) \circ \lambda^{-1} \tag{7.37}$$

for $\lambda = x^{-1} \circ y$. Since $\lambda^{-1} = y^{-1} \circ x$,

$$\dot{\lambda} = \frac{\dot{y}}{\dot{x} \circ x^{-1} \circ y}, \quad \dot{\lambda} \circ \lambda^{-1} = \frac{\dot{y} \circ y^{-1} \circ x}{\dot{x}}, \tag{7.38}$$

and

$$z = \frac{u - v \circ y^{-1} \circ x}{\dot{y} \circ y^{-1} \circ x}, \tag{7.39}$$

which agrees with (7.30) with the labels changed, i.e., with (x, y, u, v) replaced by (y, x, v, u). ■

13.8. Counting Functions

Inverse functions or first-passage-time functions are closely related to counting functions. A counting function is defined in terms of a sequence $\{s_n : n \geq 0\}$ of nondecreasing nonnegative real numbers with $s_0 = 0$. We can think of s_n as the partial sum

$$s_n \equiv x_1 + \cdots + x_n, \quad n \geq 1, \tag{8.1}$$

by simply writing $x_i \equiv s_i - s_{i-1}$, $i \geq 1$. The associated *counting function* $\{c(t) : t \geq 0\}$ is defined by

$$c(t) \equiv \max\{k \geq 0 : s_k \leq t\}, \quad t \geq 0. \tag{8.2}$$

To have $c(t)$ finite for all $t > 0$, we assume that $s_n \to \infty$ as $n \to \infty$. We can reconstruct the sequence $\{s_n\}$ from $\{c(t) : t \geq 0\}$ by

$$s_n = \inf\{t \geq 0 : c(t) \geq n\}, \quad n \geq 0. \tag{8.3}$$

The sequence $\{s_n\}$ and the associated function $\{c(t) : t \geq 0\}$ can serve as sample paths for a stochastic point process on the nonnegative real line. Then there are (countably) infinitely many points with the n^{th} point being located at s_n. The summands x_n are then the intervals between successive points. The most familiar case is when the sequence $\{x_n : n \geq 1\}$ constitutes the possible values from a sequence $\{X_n : n \geq 1\}$ of IID random variables with values in $\mathbb{R}_+$. Then the counting function $\{c(t) : t \geq 0\}$ constitutes a possible sample path of an associated renewal counting process $\{C(t) : t \geq 0\}$; see Section 7.3.

Paralleling Lemma 13.6.3, we have the following basic inverse relation for counting functions.

Lemma 13.8.1. (inverse relation) *For any nonnegative integer n and nonnegative real number t,*

$$s_n \le t \quad \textit{if and only if} \quad c(t) \ge n\,. \tag{8.4}$$

We can put counting functions in the setting of inverse functions on $D_\uparrow$ by letting

$$y(t) \equiv s_{\lfloor t \rfloor}, t \ge 0\,. \tag{8.5}$$

To have $y \in D_\uparrow$, we use the assumption that $s_n \to \infty$ as $n \to \infty$. if all the summands are strictly positive, then

$$y^{-1}(t) = c(t) + 1, \quad t \ge 0\,, \tag{8.6}$$

where y^{-1} is the image of the inverse map in (6.1) applied to y in (8.5). With (8.6), limits for counting functions can be obtained by applying results in the previous two sections.

The connection to the inverse map can also be made when the summands x_i are only nonnegative. To do so, we observe that the counting function c is a time-transformation of y^{-1}. both are right-continuous, but $c(t) < y^{-1}(t)$. In particular, c and y can be expressed in terms of each other.

Lemma 13.8.2. (relation between counting functions and inverse functions) *For y in* (8.5) *and c in* (8.2),

$$\begin{aligned} c(t) &= y^{-1}(y(y^{-1}(t)-)-), \quad t \ge 0\,, && (8.7)\\ c(t) &= y^{-1}(t-) \quad \textit{for all} \quad t \in Disc(c) = Disc(y^{-1})\,, && (8.8)\\ y^{-1}(t) &= c(c^{-1}(c(t)), \quad t \ge 0\,. && (8.9) \end{aligned}$$

The three functions y, y^{-1} and c are depicted for a typical initial segment of a sequence $\{s_n : n \ge 0\}$ in Figure 13.1.

We can apply (8.7)–(8.9) in Lemma 13.8.1 to show that limits for scaled counting functions with centering, are equivalent to limits for scaled inverse functions. We use the fact that the M topologies are not altered by changing to the left limits, because the graph is unchanged. We first consider the case of no centering; afterwards we consider the case of centering. When there is no centering, the M_1 and M_2 topologies coincide and reduce to pointwise convergence on a dense subset of $\mathbb{R}_+$ including 0.

Consider a sequence of counting functions $\{\{c_n(t) : t \ge 0\} : n \ge 1\}$ with associated processes

$$y_n^{-1}(t) \equiv c_n(c_n^{-1}(c_n(t))), \quad t \ge 0\,, \tag{8.10}$$

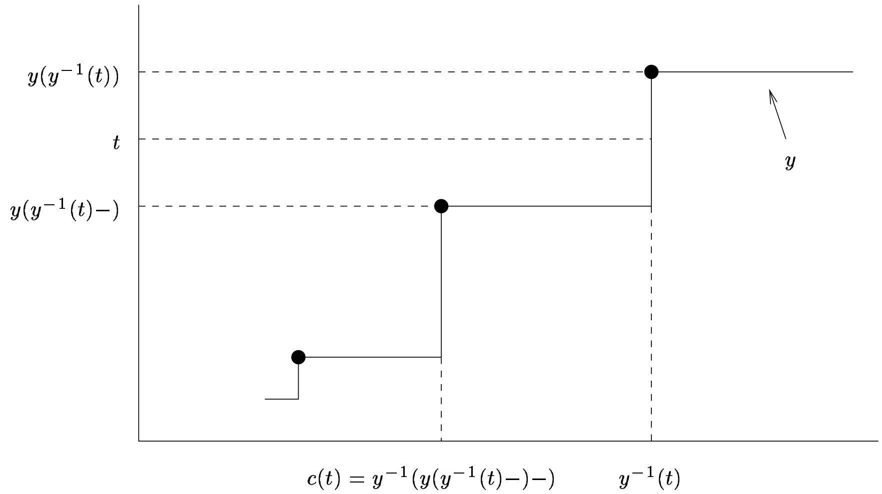

Figure 13.1: The relation between the counting function c and the inverse function y^{-1} for a typical function y.

$y_n = (y_n^{-1})^{-1}$. Form scaled functions by setting

$$\hat{\mathbf{c}}_n(t) = n^{-1}c_n(a_n t) \quad \text{and} \quad \hat{\mathbf{y}}_n(t) = a_n^{-1}y_n(nt), \quad t \ge 0\,, \tag{8.11}$$

where a_n are positive real numbers with $a_n \to \infty$. Note that

$$\hat{\mathbf{c}}_n^{-1}(t) = a_n^{-1}c_n^{-1}(nt) \quad \text{and} \quad \hat{\mathbf{y}}_n^{-1}(t) = n^{-1}y_n(a_n t), \quad t \ge 0\,. \tag{8.12}$$

Theorem 13.8.1. (asymptotic equivalence of limits for scaled processes) *Suppose that $\hat{\mathbf{y}}_n \in D_{u,\uparrow}$, $n \ge 1$, for $\hat{\mathbf{y}}_n$ in* (8.11). *Then any one of the limits $\hat{\mathbf{y}}_n \to \hat{\mathbf{y}}$, $\hat{\mathbf{y}}_n^{-1} \to \hat{\mathbf{y}}^{-1}$, $\hat{\mathbf{c}}_n \to \hat{\mathbf{y}}^{-1}$ or $\hat{\mathbf{c}}_n^{-1} \to \hat{\mathbf{y}}$ in $D_\uparrow([0,\infty),\mathbb{R})$ with the M_2 $(= M_1)$ topology, for $\hat{\mathbf{y}}_n^{-1}$, $\hat{\mathbf{c}}_n$ and $\hat{\mathbf{c}}_n^{-1}$ in* (8.11) *and* (8.12), *implies the others.*

We now apply the results for inverse maps with centering in Section 13.7 to obtain limits for counting functions with centering. Consider a sequence of counting functions $\{\{c_n(t) : t \ge 0\} : n \ge 1\}$ associated with a sequence of nondecreasing sequences of nonnegative numbers $\{\{s_{n,k} : k \ge 0\} : n \ge 1\}$ defined as in (8.2). Let the scaled functions $\hat{\mathbf{c}}_n$, $\hat{\mathbf{y}}_n$, $\hat{\mathbf{c}}_n^{-1}$ and $\hat{\mathbf{y}}_n^{-1}$ be defined as in (8.10)–(8.12).

Theorem 13.8.2. (asymptotic equivalence of counting and inverse functions with centering) *Consider* $\hat{\mathbf{y}}_n$, $\hat{\mathbf{c}}_n$, *and* $\hat{\mathbf{y}}_n^{-1}$ *and* $\hat{\mathbf{c}}_n^{-1}$ *as defined in* (8.11) *and* (8.12). *Suppose that* $\hat{\mathbf{y}}_n \in D_{u,\uparrow}$, $n \geq 1$, $b_n \to \infty$ *and* $\mathbf{z}(0) = 0$. *Then any one of the limits* $b_n(\hat{\mathbf{y}}_n - e) \to \mathbf{z}$, $b_n(\hat{\mathbf{c}}_n - e) \to -\mathbf{z}$, $b_n(\hat{\mathbf{y}}_n^{-1} - e) \to -\mathbf{z}$ *or* $b_n(\hat{\mathbf{c}}_n^{-1} - e) \to \mathbf{z}$ *in* $D([0,\infty),\mathbb{R})$ *with the* M_1 *or* M_2 *topology implies the others with the same topology.*

Corollary 13.8.1. *Consider a sequence of nondecreasing nonnegative sequences* $\{\{s_{n,k} : k \geq 0\} : n \geq 1\}$ *with* $s_{n,0} = 0$ *and* $s_{n,k} \to \infty$ *as* $k \to \infty$ *for all* n. *Let*

$$\mathbf{x}_n(t) = \delta_n^{-1}[s_{n,\lfloor nt \rfloor} - m_n nt], \quad t \geq 0\,,$$

and

$$\mathbf{y}_n(t) = \delta_n^{-1}[c_n(nt) - m_n^{-1} nt], \quad t \geq 0\ ,$$

for $c_n(t)$ *defined as in* (8.2). *Suppose that* $\mathbf{u}(0) = 0$, $\delta_n \to \infty$, $n/\delta_n \to \infty$ *and* $m_n \to m > 0$ *as* $n \to \infty$. *Then* $\mathbf{x}_n \to \mathbf{u}$ *in* $D([0,\infty,\mathbb{R})$ *with the* M_1 *or* M_2 *topology if and only if* $\mathbf{y}_n \to -m^{-1}\mathbf{u} \circ m^{-1}\mathbf{e}$ *in* $D([0,\infty),\mathbb{R})$ *with the same topology.*

Proof. Apply Theorem 13.8.2, letting $\hat{\mathbf{c}}_n(t) = (a_n n)^{-1} c_n(nt)$ for $a_n = m_n^{-1}$ and, necessarily, $\hat{\mathbf{y}}_n(t) = n^{-1} s_{n,\lfloor a_n nt \rfloor}$. Then $b_n(\hat{\mathbf{y}}_n - e) \to \mathbf{z}$ if and only if $b_n(\hat{\mathbf{c}}_n - e) \to -\mathbf{z}$ for $b_n \to \infty$ and $\mathbf{z}(0) = 0$. However, $b_n(\hat{\mathbf{y}}_n - e) \to \mathbf{u} \circ m^{-1}\mathbf{e}$ if and only if $\mathbf{x}_n \to \mathbf{u}$, while $b_n(\hat{\mathbf{c}}_n - e) \to m^{-1}\mathbf{z}$ if and only if $\mathbf{y}_n \to \mathbf{z}$, for $b_n = n/\delta_n \to \infty$. ∎