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FOURIER SERIES EXPANSION OF THE TRANSFER EQUATION IN THE ATMOSPHERE–OCEAN SYSTEM

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Abstract—We consider radiative transfer in a plane-parallel atmosphere bounded by a rough ocean surface. The problem is solved by using a Fourier series decomposition of the radiation field. For the case of a Lambertian surface as a boundary condition, this decomposition is classically achieved by developing the scattering phase matrix in a series of Legendre functions. For the case of a rough ocean surface, we obtain the decomposition by developing both the Fresnel reflection matrix and the wave facet distribution function in Fourier series. This procedure allows us to derive the radiance field for the case of the ruffled ocean surface, with a computation time only a few percent larger than for the case of a Lambertian surface.

INTRODUCTION

Remote sensing from satellites allows frequent observations with a wide coverage of the Earth. In the visible range, the measured upward radiance is very sensitive to atmospheric and surface parameters and may provide information about these parameters. An impressive amount of work has been done to compute the satellite data as a function of these parameters which is obviously necessary to retrieve this information. In these computations, however, the lower boundary condition generally corresponds to a Lambert reflector. This approach is valid for land observations. But simulations of the satellite signal for the case of a bi-directional reflectance are interesting for some land observations (for example, for water or ice surfaces) and for oceanic observations for which the bi-directional reflectance is related to the Fresnel reflection on the sea surface. Moreover, the percent polarization of the radiation field, which may be valuable information, generally is not taken into account in these computations.

Fraser and Walker[1] assumed a simple model of the ocean–atmosphere system (a standard gas on a smooth ocean) and reported the intensity and degree of polarization. For the same case of a smooth sea surface, Dave[2] and Katawar et al[3] conducted computations for more realistic atmospheric models. For a rough sea surface exhibiting the true complexity of the boundary conditions, Raschke,[4] Plass et al[5] and Quenzel and Kaestner[6] solved the problem, but neglected the polarization of diffuse radiation in the atmosphere is well as polarization of the reflected radiation. Ahmad and Fraser[7] and Takashima and Masuda[8] performed complete calculations accounting for the degree of polarization and also presented some limited comparisons.

The difficulty of exact radiative transfer calculations for rough-ocean reflection is mainly numerical. Most radiative transfer calculations are made tractable by using Fourier series decomposition of the radiation field as a function of the azimuth. For the case of a Lambertian ground or of a smooth sea surface, the boundary condition is compatible with this series expansion. On the other hand, this approach is not easy to follow for the case of a rough ocean because of the complexity introduced by the wave slopes. Here, we solve this problem, taking into account radiation-field polarization.

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THEORETICAL BACKGROUND

Formal transfer equation for the s Fourier component

The radiative transfer equation in a plane-parallel finite atmosphere can be written as

$$\mu \frac{\partial \tilde{I}(\delta, \mu, \phi)}{\partial \delta} = \tilde{I}(\delta, \mu, \phi) - \frac{\omega_0}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} \tilde{P}(\delta, \mu, \phi, \mu', \phi') \tilde{I}(\delta, \mu', \phi') \, d\mu' \, d\phi' - \frac{\omega_0}{4\pi} \exp(\delta/\mu_s) \tilde{P}(\delta, \mu, \phi, \mu_s, \phi_s) \tilde{E}_s, \quad (1)$$

where δ is the optical depth, ω_0 the albedo for single scattering, μ the cosine of the zenith angle, ϕ the azimuth angle, the E_s the solar irradiance; the subscript s refers to solar quantities. The components of the four-vector $\tilde{I}$ are the Stokes parameters I, Q, U, and V, with the meridian plane as reference. In Eq. (1), the kernel $\tilde{P}(\delta, \mu, \phi, \mu', \phi')$ is given by

$$\tilde{P}(\mu, \phi, \mu', \phi') = \tilde{L}(-\chi) \tilde{P}(\cos \Theta) \tilde{L}(\chi'), \quad (2)$$

where $\tilde{P}(\cos \Theta)$ is the phase matrix, with the scattering plane as reference and Θ the scattering angle. We have omitted for convenience the dependence of the phase matrix on the optical depth. The matrices $\tilde{L}(-\chi)$ and $\tilde{L}(\chi')$ are required to rotate the meridian planes before and after scattering onto the scattering plane.[9] Here,

$$\tilde{L}(\chi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3)$$

A usual procedure for solving the transfer equation is to consider a Fourier series expansion in azimuth for the radiance. If we consider an atmosphere illuminated by the solar beam [with Stokes parameters $(E_s, 0, 0, 0)$] and with symmetrical boundary conditions with respect to the incident plane, then

$$\begin{bmatrix} I(\delta, \mu, \phi) \\ Q(\delta, \mu, \phi) \\ U(\delta, \mu, \phi) \\ V(\delta, \mu, \phi) \end{bmatrix} = \sum_{s=0}^{\infty} (2 - \delta_{0,s}) \begin{bmatrix} I^s(\delta, \mu) \cos s(\phi - \phi_s) \\ Q^s(\delta, \mu) \cos s(\phi - \phi_s) \\ U^s(\delta, \mu) \sin s(\phi - \phi_s) \\ V_s(\delta, \mu) \sin s(\phi - \phi_s) \end{bmatrix}. \quad (4)$$

If the phase matrix terms are expanded in the same manner in a Fourier series, then the complete problem separates into the following set of independent equations:

$$\mu \frac{\partial \tilde{I}^s(\delta, \mu)}{\partial \delta} = \tilde{I}^s(\delta, \mu) - \frac{\omega_0}{2} \int_{-1}^{+1} \tilde{P}^s(\mu, \mu') \tilde{I}^s(\delta, \mu') \, d\mu' - \frac{\omega_0}{4\pi} \tilde{P}^s(\mu, \mu_s) \exp(\delta/\mu_s) E_s. \quad (5)$$

A major advantage of Fourier series expansion of the transfer equation lies in the simplicity of the corresponding transfer code, since integrations on ϕ and θ are separated. Moreover, only the zeroth term provides answers to interesting problems such as flux computations or radiance estimates in the nadir-viewing direction, as involved in LANDSAT or SPOT observations; according to the reciprocity principle, nadir-radiance for any solar zenith angle θ_s may be derived from radiance at the viewing angle θ_s for null solar zenith angle, which requires only calculation of the zeroth term of the Fourier series.

The $\tilde{P}^s(\mu, u_s)$ term in Eq. (5) may be derived numerically, but a more powerful approach involves the use of circularly polarized states for the light representation. By assuming that the terms of the phase matrix are developed in Legendre polynomials P_l and associated functions P_l^2 in the form

$$\tilde{P}(\cos \Theta) = \begin{bmatrix} \Sigma_{l=0}^{L} \beta_l P_l(\cos \Theta) & \Sigma_{l=2}^{L} \gamma_l P_l^2(\cos \Theta) & 0 & 0 \\ \Sigma_{l=2}^{L} \gamma_l P_l^2(\cos \Theta) & \Sigma_{l=0}^{L} \beta_l P_l(\cos \Theta) & 0 & 0 \\ 0 & 0 & \Sigma_{l=0}^{L} \delta_l P_l(\cos \Theta) & -\Sigma_{l=2}^{L} \epsilon_l P_l^2(\cos \Theta) \\ 0 & 0 & \Sigma_{l=2}^{L} \epsilon_l P_l^2(\cos \Theta) & \Sigma_{l=0}^{L} \delta_l P_l(\cos \Theta) \end{bmatrix}, \quad (6)$$

it may be shown[10–13] that the kernel $\tilde{P}(\mu, \phi, \mu', \phi')$ in Eq. (1) may be expanded into the form

$$\tilde{P}(\mu, \phi, \mu', \phi') = \sum_{s=0}^{L} (2 - \delta_{0,s})[\cos s(\phi - \phi')\, \tilde{P}^s_c(\mu, \mu') + \sin s(\phi - \phi')\, \tilde{P}^s_s(\mu, \mu')], \tag{7}$$

where

$$\tilde{P}^s_c(\mu, \mu') =$$

$$\begin{bmatrix} \Sigma^L_{l=s}\beta_l P^s_l P^{s\prime}_l & \Sigma^L_{l=s}\gamma_l P^s_l R^{s\prime}_l & 0 & 0 \\ \Sigma^L_{l=s}\gamma_l R^s_l P^{s\prime}_l & \Sigma^L_{l=s}(\alpha_l R^s_l R^{s\prime}_l + \zeta_l T^s_l T^{s\prime}_l) & 0 & 0 \\ 0 & 0 & \Sigma^L_{l=s}(\alpha_l T^s_l T^{s\prime}_l + \zeta_l R^s_l R^{s\prime}_l) & -\Sigma^L_{l=s}\epsilon_l R^s_l P^{s\prime}_l \\ 0 & 0 & \Sigma^L_{l=s}\epsilon_l P^s_l R^{s\prime}_l & \Sigma^L_{l=s}\delta_l P^s_l P^{s\prime}_l \end{bmatrix}, \tag{8}$$

and

$$\tilde{P}^s_s(\mu, \mu') =$$

$$\begin{bmatrix} 0 & 0 & \Sigma^L_{l=s}\gamma_l P^s_l T^{s\prime}_l & 0 \\ 0 & 0 & \Sigma^L_{l=s}(\alpha_l R^s_l T^{s\prime}_l + \zeta_l T^s_l R^{s\prime}_l) & -\Sigma^L_{l=s}\epsilon_l T^s_l P^{s\prime}_l \\ -\Sigma^L_{l=s}\gamma_l T^s_l P^{s\prime}_l & -\Sigma^L_{l=s}(\alpha_l T^s_l R^{s\prime}_l + \zeta_l R^s_l T^{s\prime}_l) & 0 & 0 \\ 0 & -\Sigma^L_{l=s}\epsilon_l P^s_l T^{s\prime}_l & 0 & 0 \end{bmatrix}. \tag{9}$$

In these equations, P^s_l, R^s_l, T^s_l stand for $P^s_l(\mu)$, $R^s_l(\mu)$, $T^s_l(\mu)$ and $P^{s\prime}_l$, $R^{s\prime}_l$, $T^{s\prime}_l$ stand for $P^s_l(\mu')$, $R^s_l(\mu')$, $T^s_l(\mu')$; $R^s_l(\mu)$ and $T^s_l(\mu)$ are linear combinations of the generalized Legendre functions $P^s_{2,2}(\mu)$ and $P^2_{2,-2}(\mu)$, which are defined in the Appendix.

When substituting Eqs. (4), (7), (8), and (9) into Eq. (1), it separates immediately into the set of independent Eqs. (5) with

$$\tilde{P}^s(\mu, \mu') =$$

$$\begin{bmatrix} \Sigma^L_{l=s}\beta_l P^s_l P^{s\prime}_l & \Sigma^L_{l=s}\gamma_l P^s_l R^{s\prime}_l & -\Sigma^L_{l=s}\gamma_l P^s_l T^{s\prime}_l & 0 \\ \Sigma^L_{l=s}\gamma_l R^s_l P^{s\prime}_l & \Sigma^L_{l=s}(\alpha_l R^s_l R^{s\prime}_l + \zeta_l T^s_l T^{s\prime}_l) & -\Sigma^L_{l=s}(\alpha_l R^s_l T^{s\prime}_l + \zeta_l T^s_l R^{s\prime}_l) & \Sigma^L_{l=s}\epsilon_l T^s_l P^{s\prime}_l \\ -\Sigma^L_{l=s}\gamma_l T^s_l P^{s\prime}_l & -\Sigma^L_{l=s}(\alpha_l T^s_l R^{s\prime}_l + \zeta_l R^s_l T^{s\prime}_l) & \Sigma^L_{l=s}(\alpha_l T^s_l T^{s\prime}_l + \zeta_l R^s_l R^{s\prime}_l) & -\Sigma^L_{l=s}\epsilon_l R^s_l P^{s\prime}_l \\ 0 & -\Sigma^L_{l=s}\epsilon_l P^s_l T^{s\prime}_l & \Sigma^L_{l=s}\epsilon_l P^s_l R^{s\prime}_l & \Sigma^L_{l=s}\delta_l P^s_l P^{s\prime}_l \end{bmatrix}. \tag{10}$$

The required order L for the developments in Eq. (6) depends mainly on the dimension of the scattering particles. It is known that $L = 2$ for molecular scattering; scattering by terrestrial aerosols typically requires about $L = 48$. The set of coefficients β_l, γ_l, δ_l, and ϵ_l may be computed by using orthogonality relations for Legendre functions and polynomials (see the Appendix for details); α_l and ζ_l are linear combinations of β_l and δ_l as follows:

$$\sum_{l=0}^{L} (\beta_l + \delta_l)P_l(\mu) = \sum_{l=2}^{L} (\alpha_l + \zeta_l)P^l_{2,2}(\mu), \tag{11}$$

$$\sum_{l=0}^{L} (\beta_l - \delta_l)P_l(\mu) = \sum_{l=2}^{L} (\alpha_l - \zeta_l)P^l_{2,-2}(\mu). \tag{12}$$

Boundary condition; reflection matrix for the rough ocean

We now consider the atmosphere–ocean system. The boundary condition corresponding to the radiation scattered from the sea water is routinely accounted for by a Lambertian condition. Since

Fourier-series expansion of the radiation field raises no problem for such a condition, we will ignore this term and will limit ourselves to the surface-reflection problem.

Should the sea surface be horizontal, an incident beam would be reflected in the specular direction and the boundary condition at sea level becomes

$$\tilde{I}(\delta_1, -\cos\omega, \phi) = \tilde{R}(\omega)\tilde{I}(\delta_1, \cos\omega, \phi), \tag{13}$$

where δ_1 is the optical thickness of the atmosphere, ω the incident angle, and $\tilde{R}(\omega)$ the Fresnel matrix, expressed as a function of the complex Fresnel coefficients by[14]

$$\tilde{R}(\omega) = \frac{1}{2}\begin{bmatrix} r_l r_l^* + r_r r_r^* & r_l r_l^* - r_r r_r^* & 0 & 0 \\ r_l r_l^* - r_r r_r^* & r_l r_l^* + r_r r_r^* & 0 & 0 \\ 0 & 0 & r_l r_r^* + r_r r_l^* & r_l r_r^* - r_l r_r^* \\ 0 & 0 & r_l r_r^* - r_r r_l^* & r_l r_r^* + r_l r_r^* \end{bmatrix}. \tag{14}$$

Here, r_l and r_r depend on ω and on the sea water complex refractive index m, according to

$$r_l = \frac{\sqrt{m^2 - \sin^2\omega} - m^2\cos\omega}{\sqrt{m^2 - \sin^2\omega} + m^2\cos\omega}, \tag{15}$$

$$r_r = \frac{\cos\omega - \sqrt{m^2 - \sin^2\omega}}{\cos\omega + \sqrt{m^2 - \sin^2\omega}}. \tag{16}$$

For visible and near-infrared wavelengths, the imaginary part of m is negligible[15] so that $R_{3,4} = R_{4,3} = 0$.

For the case of a rough surface, given an arbitrary observation direction (μ, ϕ) and the downward direction (μ', ϕ'), water facets exist with the normal direction $N(\mu_n, \phi_n)$ such that they can reflect downward radiance towards the observer. The reflection geometry is shown in Fig. 1. According to Eq. (13), the resulting contribution $d\tilde{I}(\delta_1, \mu, \phi)$ in $\tilde{I}(\delta_1, \mu, \phi)$ from the downward radiance $\tilde{I}(\delta_1, \mu', \phi')$ will be given by

$$d\tilde{I}(\delta_1, \mu, \phi) = f(\mu_n, \phi_n)\tilde{L}(-\chi)\tilde{R}(\omega)\tilde{L}(\chi')\tilde{I}(\delta_1, \mu', \phi')\,d\mu'\,d\phi'; \tag{17}$$

$\tilde{L}(-\chi)$ and $\tilde{L}(\chi')$ have been introduced to take into account the required rotations of the meridian planes into the reflection plane, which is no longer a vertical one, and $f(\mu_n, \phi_n)$ stands for the required weighting of $\tilde{R}(\omega)$ by the density of water facets with the convenient inclination.

Using analysis of aerial photographs of the glitter, Cox and Munk[16] investigated the probability distribution of water facet normals. They showed that it is nearly independent of ϕ_n. When

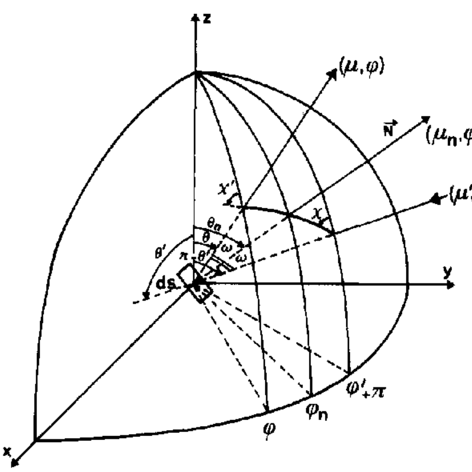

Fig. 1. Geometry of the reflection on a wave facet.

accounting for the transformation of this normal probability distribution into the energetic balance for reflection, it may be shown[16] that $f(\mu_n, \phi_n)$ is given by

$$f(\mu_n, \phi_n) = \frac{1}{4\mu\mu_n^4}\frac{1}{\pi\sigma^2}\exp\left(-\frac{1-\mu_n^2}{\sigma^2\mu_n^2}\right). \tag{18}$$

In Eq. (18), σ is related to the wind speed v by

$$\sigma^2 = 0.003 + 0.0512v. \tag{19}$$

On the other hand, given the directions (μ, ϕ) and (μ', ϕ'), convenient normals **N** are such that

$$\mu_n = \frac{|\mu - \mu'|}{2\cos\omega} \tag{20}$$

and the resulting reflection angle ω is given by

$$\cos 2\omega = -\mu\mu' - \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi - \phi'). \tag{21}$$

Equations (18)–(21) determine the problem. Clearly, $f(\mu_n, \phi_n)$ and $\tilde{R}(\mu, \phi, \mu', \phi')$ [or $\tilde{L}(-\chi)\tilde{R}(\omega)\tilde{L}(\chi')$] depend only on the azimuth difference $(\phi - \phi')$. By introducing the reflection matrix

$$\tilde{M}(\mu, \phi, \mu', \phi') = f(\mu_n, \phi_n)\tilde{R}(\mu, \phi, \mu', \phi'), \tag{22}$$

the boundary condition at sea level is given by

$$\tilde{I}(\delta_1, \mu > 0, \phi) = \tilde{M}(\mu, \phi, \mu_s, \phi_s)E_s\exp(\delta_1/\mu_s) + \int_0^{2\pi}\int_0^{+1}\tilde{M}(\mu, \phi, \mu', \phi')\tilde{I}(\delta_1, \mu', \phi')\,d\mu'\,d\phi'. \tag{23}$$

FOURIER-SERIES EXPANSION OF THE REFLECTION MATRIX

In order to preserve the Fourier-series expansion of the transfer equation, we need developments of the M_{ij} terms in cosine or sine series of the azimuth, according to the parity of the Stokes parameters involved. These developments may be derived directly by numerical methods. Such a solution requires, however, impressive data storage and is very time consuming. On the other hand, we note the similarity between $\tilde{R}(\omega)$ and $\tilde{P}(\cos\Theta)$ when considering $(\pi - 2\omega)$ as the scattering angle Θ. Therefore, Fourier-series expansion of $\tilde{R}(\mu, \phi, \mu', \phi')$ may be achieved in the same manner as for the case of the scattering matrix. Since this term does not depend on the sea-surface roughness, this calculation is only needed once. Then, by expanding the scalar term $f(\mu_n, \phi_n)$ into the Fourier series of $(\phi - \phi')$, the expected developments will be obtained as a mixture of the two developments.

First, in parallel with Eq. (6), we expand the terms of $\tilde{R}(\omega)$ in a series of appropriate Legendre functions of $\Omega = \pi - 2\omega$ into the form

$$\tilde{R}(\omega) = \begin{bmatrix} \Sigma_{l=0}^{L} b_l P_l(\cos\Omega) & \Sigma_{l=2}^{L} g_l P_l^2(\cos\Omega) & 0 & 0 \\ \Sigma_{l=2}^{L} g_l P_l^2(\cos\Omega) & \Sigma_{l=0}^{L} b_l P_l(\cos\Omega) & 0 & 0 \\ 0 & 0 & \Sigma_{l=0}^{L} d_l P_l(\cos\Omega) & -\Sigma_{l=2}^{L} e_l P_l^2(\cos\Omega) \\ 0 & 0 & \Sigma_{l=2}^{L} e_l P_l^2(\cos\Omega) & \Sigma_{l=0}^{L} d_l P_l(\cos\Omega) \end{bmatrix}. \tag{24}$$

Then, Fourier-series expansion of $\tilde{R}(\mu, \phi, \mu', \phi')$, will be given by equations similar to Eqs. (7), (8) and (9), but with a_l, b_l, g_l, d_l, e_l, and z_l, respectively, in place of α_l, β_l, γ_l, δ_l, ϵ_l, and ζ_l, with a_l and z_l derived from combinations of b_l and d_l, similar to those introduced in Eqs. (11) and (12).

All of these coefficients may be obtained by appropriate (e.g., Gaussian) numerical quadratures by using orthogonality relations in the Legendre basis. The only problem is to develop to an adequate order N the Fresnel matrix $\tilde{R}(\omega)$. The convergence of this development is correct for $N = 48$, as is illustrated in Fig. 2 for the term $r_1 = R_{1,1} = R_{2,2}$, and in Fig. 3 for $r_2 = R_{2,1} = R_{1,2}$. Figure 2 shows that slight difficulties occur for grazing incidences, where r_1 increases quickly. However, as a result of the vanishing irradiance for grazing incidences, the resulting error should

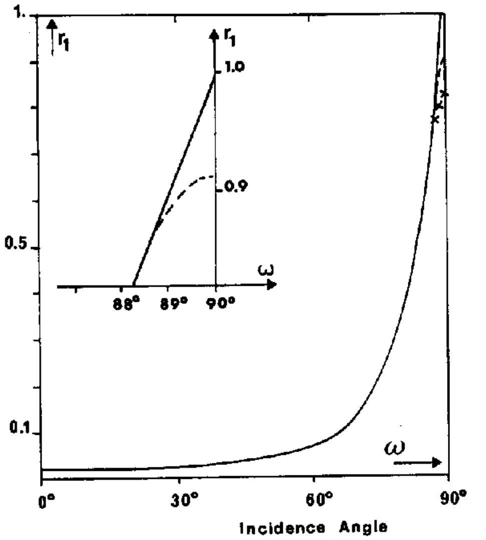

Fig. 2. Development of the reflection coefficient r_1 in series of Legendre polynomials. The exact coefficient (——) is compared with results obtained from a series development of order $N = 24$ (×) and of order $N = 48$ (---). The range of grazing incidence angles is zoomed in the upper corner.

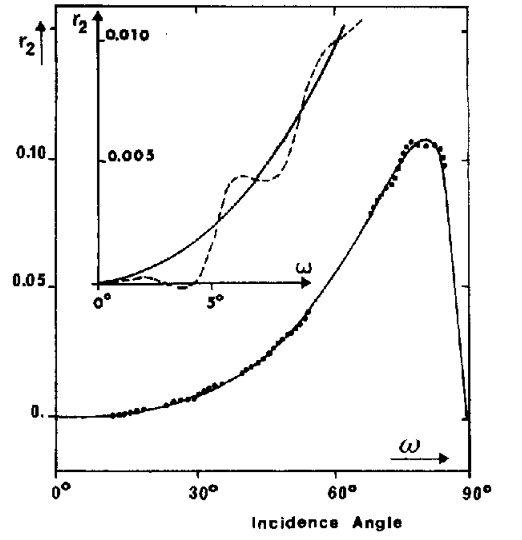

Fig. 3. Development of the reflection coefficient r_2 in series of Legendre polynomials. The exact coefficient (——) is compared with results obtained from a series development of order $N = 48$ (●). The range of near-nadir incidence angles is zoomed in the upper corner for the exact case (——) and the development of order $N = 48$ (---).

be negligible. In Fig. 3, slight oscillations appear in r_2 around the nadir, but they are unimportant. These directions generally correspond to low polarization ratios. Moreover, these oscillations may be smoothed when integrating Eq. (23). Therefore, we write

$$\tilde{R}(\mu, \phi, \mu', \phi') = \sum_{n=0}^{N=48} (2 - \delta_{0,n})[\cos n(\phi - \phi')\tilde{R}_c^n(\mu, \mu') + \sin n(\phi - \phi')\tilde{R}_s^n(\mu, \mu')], \quad (25)$$

where $\tilde{R}_c^n(\mu, \mu')$ and $\tilde{R}_s^n(\mu, \mu')$ are given by Eqs. (8) and (9), respectively, but with $a_l, \ldots, z_l$ in place of $\alpha_l, \ldots, \zeta_l$.

According to the symmetry of $f(\mu_n, \phi_n)$, we can write

$$f(\mu_n, \phi_n) = \sum_{k=0}^{K} (2 - \delta_{0,k}) F_k(\mu, \mu') \cos k(\phi - \phi') \quad (26)$$

with

$$f_k(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} \cos k(\phi - \phi') f(\mu_n, \phi_n)\, d\phi'. \quad (27)$$

Since this function is very sharp around the specular direction, especially for low wind speeds and for grazing incidences, a Gaussian quadrature is no longer valid in Eq. (27), and we used trapezoidal quadratures restricted to intervals $\Delta\phi = \phi - \phi'$ such that $f(\Delta\phi_{max})/f(\Delta\phi = 0) = 10^{-4}$. The integration step was defined by using a dichotomy method with a convergence test at 10^{-4}. For not too large incident and emergent angles, regardless of the wind speed, $K = 96$ provides a fairly good restitution of $f(\mu_n, \phi_n)$, as is shown in Fig. 4 for $\theta = 32.5°$ and $\theta^{*\prime} = 21.3°$ (θ^* will stand for the supplement of θ). But, for grazing angles and small wind speeds, as a result of the sharp feature of the glitter, 1500–2000 terms would be needed to retrieve $f(\mu_n, \phi_n)$ within the planned 10^{-4} accuracy (see for example Fig. 5, where $\theta = \theta^{*\prime} = 88.4°$). Fortunately, the Fourier-series decomposition of the radiation field requires much lower order expansions, so that such extensive developments of $f(\mu_n, \phi_n)$ will prove to be useless when introducing the boundary conditions.

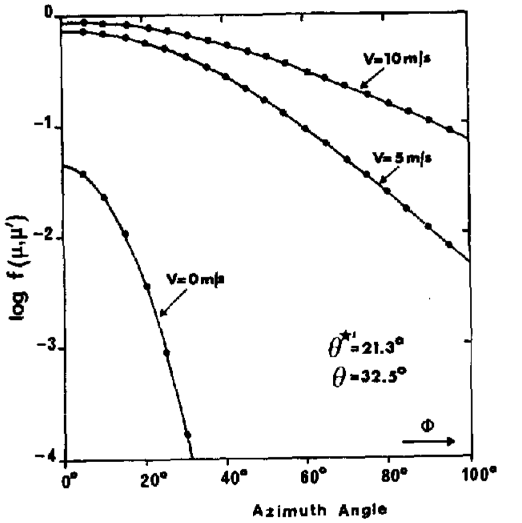

Fig. 4. The function $f(\mu_n, \phi_n)$ is shown as a function of the azimuth difference $\Phi = \phi - \phi'$ for $\theta^{*\prime} = 21.3°$ and $\theta = 32.5°$. For three wind speeds, exact computations (——) are compared with Fourier-series expansion (●) according to Eq. (26).

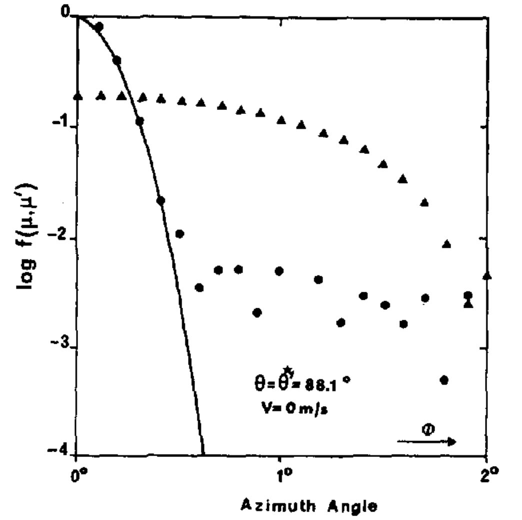

Fig. 5. The same as in Fig. 4 but for a wind speed equal to zero and for grazing angles ($\theta = \theta^{*\prime} = 88.4°$). The probability function is retrieved by using 1624 terms. The 96 (△) and 1000 (●) terms developments are also compared with the exact result.

Finally, we substitute Eqs. (25) and (26) into Eq. (22). Since the series products may be written as

$$\sum_{n=0}^{N}\sum_{k=0}^{K}(2-\delta_{0,n})(2-\delta_{0,k})\cos n(\phi-\phi')\cos k(\phi-\phi')f^k\tilde{R}_c^n = \sum_{p=0}^{K+N}(2-\delta_{0,p})\cos p(\phi-\phi')\,\tilde{M}_c^p \tag{28}$$

$$\sum_{n=1}^{N}\sum_{k=0}^{K}2(2-\delta_{0,k})\sin n(\phi-\phi')\cos k(\phi-\phi')f^k\tilde{R}_s^n = \sum_{p=1}^{K+N}2\sin p(\phi-\phi')\,\tilde{M}_s^p, \tag{29}$$

the reflection matrix will be obtained in the required form

$$\tilde{M}(\mu,\phi,\mu',\phi') = \sum_{p=0}^{P}(2-\delta_{0,p})[\cos p(\phi-\phi')\tilde{M}_c^p(\mu,\mu') + \sin p(\phi-\phi')\tilde{M}_s^p(\mu,\mu')]. \tag{30}$$

By simple rearrangements, $\tilde{M}_c^p(\mu,\mu')$ and $\tilde{M}_s^p(\mu,\mu')$ may be written as

$$\tilde{M}_c^p(\mu,\mu') = f^p(\mu,\mu')\,\tilde{R}_c^0(\mu,\mu') + \sum_{n=1}^{N}[f^{p+n}(\mu,\mu') + f^{|p-n|}(\mu,\mu')]\tilde{R}_c^n(\mu,\mu'), \tag{31}$$

$$\tilde{M}_s^p(\mu,\mu') = \sum_{n=1}^{N}[f^{p+n}(\mu,\mu') - f^{|p-n|}(\mu,\mu')]\tilde{R}_s^n(\mu,\mu'). \tag{32}$$

These 4 × 4 matrices are in the same form as $\tilde{P}_c^n(\mu,\mu')$ and $\tilde{P}_s^n(\mu,\mu')$; they may be partitioned into four 2 × 2 submatrices, with zero submatrices on the trailing diagonal of $\tilde{M}_c^p(\mu,\mu')$ and zero submatrices on the leading diagonal of $\tilde{M}_s^p(\mu,\mu')$. By substituting Eqs. (4) and (30) into Eq. (23), the boundary condition will clearly preserve the separation of the Fourier components and, on account of the particular form of $\tilde{M}_c^p(\mu,\mu')$ and $\tilde{M}_s^p(\mu,\mu')$, the Stokes parameter parity will also be preserved. Finally, the boundary condition may be expressed by

$$\tilde{I}^s(\delta_1,\mu) = \tilde{M}^s(\mu,\mu_s)E_s\exp(\delta_1/\mu_s) + 2\pi\int_{-1}^{0}\tilde{M}^s(\mu,\mu')\tilde{I}^s(\delta_1,\mu')\,d\mu' \tag{33}$$

by writing

$$\tilde{M}^p(\mu,\mu') = \tilde{M}_c^p(\mu,\mu') + \tilde{M}_s^p(\mu,\mu')\tilde{D}, \tag{34}$$

where $\tilde{D} = \mathrm{diag}(1, 1, -1, -1)$.

The boundary condition requires integration of the diffuse downward radiance. An idea about the functions to integrate is given by the plots of Figs. 6 and 7. Two sets of $M^s_{1,1}(\mu, \mu')$ terms are shown as functions of the incident angle θ' for the two viewing zenith angles $\theta = 2.8$ and $84.4°$. The wind speed is 5 m/sec. For nadir observations, the glitter spot is obviously almost independent of the azimuth; therefore, the zeroth Fourier-series term is the main term of the expansion and follows the sharp peak of the sunglint. A major advantage of the Fourier-series expansion is observed for grazing angles. Since the glitter spot is very narrow, the Fourier-series convergence is slow. However, Fig. 7 shows that the behaviour of the $M^s_{1,1}(\mu, \mu')$ terms as functions of the incident zenith angles is smooth enough to apply a classical Gaussian quadrature. To check this statement, we considered an isotropic incident source and computed the reflected radiance for several viewing directions; the wind speed was 5 m/sec. For nine viewing zenith angles, Table 1 shows the exact results, derived from a very accurate trapezoidal quadrature, as well as results obtained by using a Gaussian quadrature with 24 angles. The two results agree within 1%. Although this computation involves only the zeroth term, Fig. 7 shows that the integration problem would be the same for the other terms of the Fourier-series.

Although the Fourier-series expansion of $\tilde{M}(\mu, \phi, \mu', \phi')$ may require very large orders P, the expansion of the scattered part of the radiation field is of order L of the $\tilde{P}(\Theta)$ expansion. Therefore, we limit the analysis to this order L in the Fourier-series expansions of $\tilde{I}(t, \mu, \phi)$ and $\tilde{M}(\mu, \phi, \mu', \phi')$. We let $\tilde{M}^L(\mu, \phi, \mu', \phi')$ stand for the approximate reflection matrix thus obtained. By solving this L-term problem, the resulting error will involve only that part of the radiation field corresponding to light reflected from the direct sunbeam and then directly transmitted through the atmosphere, i.e., the sunglint term. But this term, say $\tilde{I}^{ex}_{tr}(\delta, \mu, \phi)$, may be calculated without any Fourier series expansion, from

$$\tilde{I}^{ex}_{tr}(\delta, \mu, \phi) = [\exp(\delta - \delta_1)/\mu] f(\mu_n, \phi_n) \tilde{L}(-\chi_s) \tilde{R}(\omega_s) E_s \exp(\delta_1/\mu_s), \tag{35}$$

where ω_s and χ_s stand for ω and χ when the incident direction is the sun direction. Since the boundary condition will provide for this light the erroneous counterpart

$$\tilde{I}^{err}_{tr}(\delta, \mu, \phi) = [\exp(\delta - \delta_1)/\mu] \tilde{M}^L(\mu, \phi, \mu_s, \phi_s) E_s \exp(\delta_1/\mu_s), \tag{36}$$

the results of the code must be corrected by $\tilde{I}^{ex}_{tr}(\delta, \mu, \phi) - \tilde{I}^{err}_{tr}(\delta, \mu, \phi)$, the calculation of which raises no particular problem.

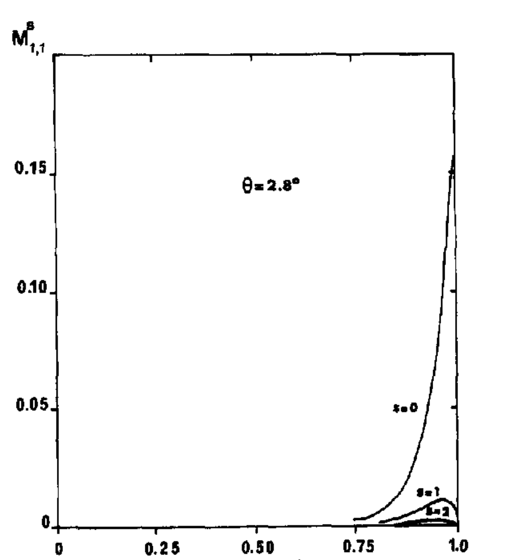

Fig. 6. The Fourier-series terms $M^s_{1,1}(\mu, \mu')$, vs the cosine of the incident zenith angle, for a viewing zenith angle of $\theta = 2.8°$.

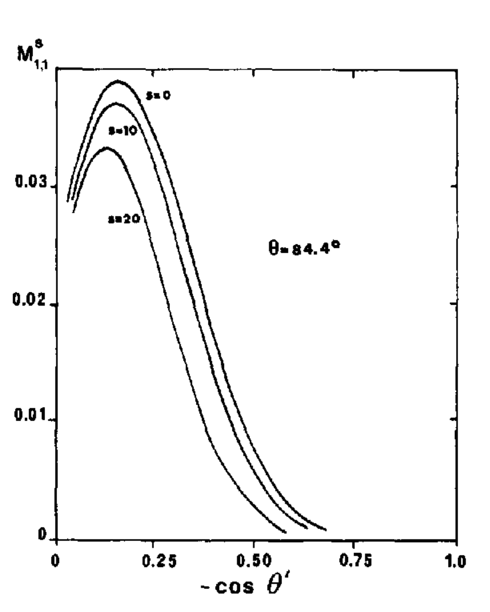

Fig. 7. The same as in Fig. 6 but for a viewing zenith angle of $\theta = 84.4°$.

Table 1. Upward radiances observed at sea level for isotropic incident irradiance. The boundary condition of Eq. (33), for a wind speed of 5 m/sec, is integrated by using a Gaussian quadrature with 24 angles. These results are contrasted with the exact results derived from a suitable trapezoidal method (last column).

$\cos\theta$	I_1	I_2
0.99877	0.02006	0.02008
0.97059	0.02019	0.02020
0.80766	0.02349	0.02349
0.57722	0.04551	0.04552
0.40869	0.09170	0.09176
0.28736	0.1520	0.1522
0.16122	0.2589	0.2594
0.09700	0.3708	0.3716
0.03238	0.8409	0.8368

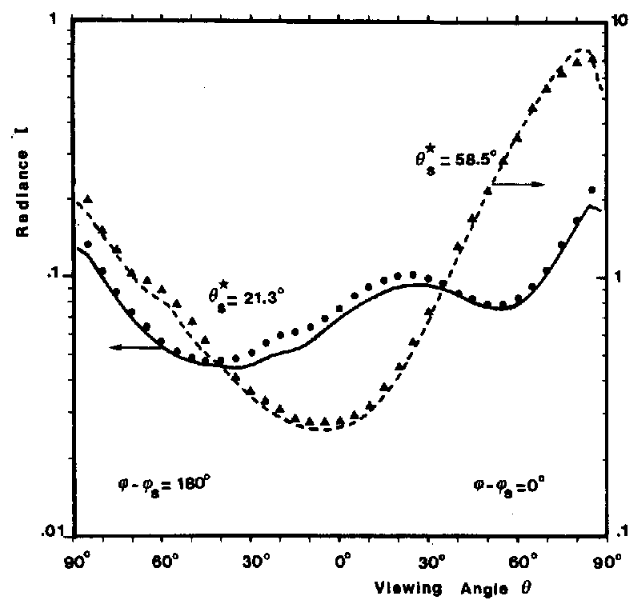

Fig. 8. The radiance $I(\theta)$ leaving the top of the atmosphere–ocean system, as defined by Ahmad and Fraser (see text), has been calculated for solar zenith angles $\theta_s^* - 21.3$ and 58.8° at the wavelength $\lambda = 0.7\,\mu$m. The results correspond to upward directions in the principal plane. Our computations (—— and – – –) are compared with the results reported by Ahmad and Fraser[7] (● and ▲).

APPLICATION

As an example, we will now see how the specified scheme may be used in successive orders of the scattering code. With this method,[17,18] we use the radiative transfer equation in its integral form and estimate the n-times scattered light $\tilde{I}_{(n)}(\delta, \mu, \phi)$ from $\tilde{I}_{(n-1)}(\delta, \mu, \phi)$, with $\tilde{I}_{(1)}(\delta, \mu, \phi)$ given by the known primary scattering from the direct sunbeam. For the case of a black background, the resulting equations for each term of the Fourier series expansion of $\tilde{I}_{(n)}(\delta, \mu, \phi)$ are, therefore,

$$\tilde{I}^s_{(1)}(\delta, \mu > 0) = \frac{1}{4\pi\mu} \int_\delta^{\delta_1} \omega_0(\delta') \exp[-(\delta' - \delta)/\mu] \tilde{P}^s(\mu, \mu_s) E_s \exp(\delta'/\mu_s)\, d\delta'; \tag{37}$$

for $n > 1$,

$$\tilde{I}^s_{(n)}(\delta, \mu > 0) = \frac{1}{2\mu} \int_\delta^{\delta_1} \omega_0(\delta') \exp[-(\delta' - \delta)/\mu] \left[\int_{-1}^{+1} \tilde{P}^s(\mu, \mu') \tilde{I}^s_{(n-1)}(\delta', \mu')\, d\mu' \right] d\delta'. \tag{38}$$

These equations apply to the upward directions ($\mu > 0$); the corresponding equations for the downward directions ($\mu > 0$) are obtained with 0 in place of δ_1 as the integral upper bound. In order to take into account the boundary condition, it is sufficient to keep the previous expressions unchanged for downward radiances and to add to the expressions for upward radiances as follows. In Eq. (37), the light reflected from the direct sunbeam and transmitted to the level considered in the specified direction is proportional to

$$[\exp - (\delta_1 - \delta)/\mu] \tilde{M}^s(\mu, \mu_s) E_s \exp(\delta_1/\mu_s). \tag{39}$$

In Eq. (38), the radiance reflected from $\tilde{I}^s_{(n-1)}(\delta, \mu)$ and transmitted to the level considered in the specified direction is proportional to

$$2\pi [\exp - (\delta_1 - \delta)/\mu] \int_{-1}^{0} \tilde{M}^s(\mu, \mu') \tilde{I}^s_{(n-1)}(\delta_1, \mu')\, d\mu', \tag{40}$$

which involves consideration of one reflection on the sea-surface as equivalent to one scattering event in the (n) enumeration.

The $\tilde{I}^s(\delta, \mu)$ terms are calculated at discrete levels δ_i and for discrete directions θ_j, which are Gaussian points of a Gaussian quadrature of order L, with $L/2$ upward and $L/2$ downward directions. Therefore, the $L(4 \times 4)$ matrices $\tilde{M}^p(\mu, \mu')$ must first be calculated from Eqs. (31) and (32) for $(L/2)^2$ couples (θ_j, θ_k), that is about $4L^3$ terms. Next, the successive order code may be started, and the results are finally corrected for the error in the sunglint term, as was indicated previously.

In order to test the validity of the scheme, the radiance and polarization of the light leaving the top of the atmosphere were calculated for a model of the ocean–atmosphere system close to that used by Ahmad and Fraser.[7] The molecular component was fixed according to the US 62 standard atmosphere. The aerosols were spherical particles with refractive index $m = 1.50 - 0.0i$ and size distribution of the form

$$n(r) = C \text{ for } 0.03 < r < 0.1\,\mu\text{m};\; n(r) = C(0.1/r)^4 \text{ for } 0.1 < r < 5.0\,\mu\text{m}, \tag{41}$$

which were distributed vertically according to Elterman's distribution.[19] A wind speed of 10 m/sec was considered for surface-roughness modelling and calculations were performed for solar zenith angles of 21.3 and 58.5° at a wavelength $\lambda = 700$ nm. Figures 8 and 9 show the resulting radiances and polarization ratios as a function of the zenith viewing angle for upward directions in the principal plane. The agreement with the results of Ahmad and Fraser is quite good and it is worthwhile to note that, compared with calculations for a black background as boundary condition, the computation time increased only by about 2% when taking into account the sea-surface reflection.

CONCLUSIONS

We consider the radiative transfer of polarized light in a plane-parallel atmosphere bounded by a rough ocean surface, with wave slope orientations governed by a distribution function. We use Fourier-series decomposition of the radiation field. In order to preserve the separation of the problem into a set of independent problems for each Fourier component, the Fresnel matrix for reflexion and the distribution function for slope orientations are both decomposed by Fourier-series of the azimuth.

For the Fresnel matrix, this development is derived from a preliminary development in a series of Legendre functions or polynomials of the reflection angle. These series need to be computed only once and only for a few wavelengths because of the weak spectral variation of the sea-water refractive index. The wave-slope distribution depends only on the wind speed. The numerical

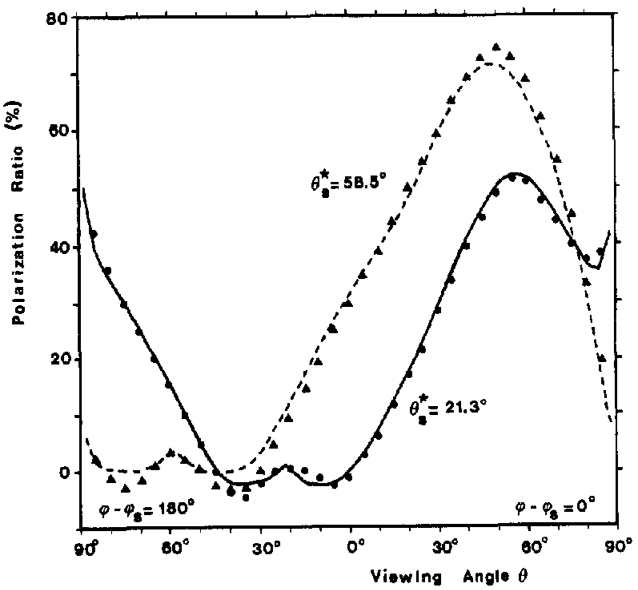

Fig. 9. The same legend as in Fig. 8 but for the polarization ratio.

difficulties encountered in its Fourier-series expansion, which are related to the sharp angular feature around the specular direction, have been investigated. Next, the Fourier-series decomposition of the reflexion matrix corresponding to the boundary condition is obtained as a mixture of the two developments.

Accurate restitution of the reflection matrix generally requires a very large order in the Fourier-series expansion. However, separate calculation of the sunglint term, which raises no particular problem, allows us to solve the rough ocean-problem with Fourier-series expansion of the radiation field of the same order as for the Lambertian boundary condition. The resulting radiation code takes account of reflection on the rough sea-surface with computation times that are only few percent longer than for a black background condition.

Reflection mechanisms, including noticeable polarization effects, are also exhibited by vegetation canopies, as shown by Vanderbilt and Grant,[20] or by natural surfaces, as shown by Coulson et al.[21] The generalization of our previous scheme to such boundary conditions will be examined in the future.

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APPENDIX

Complements about the Phase Matrix Development

The generalized Legendre functions, introduced by Gel'fand and Sapiro,[22] are defined by

$$P^l_{m,n}(\mu) = A^l_{m,n}(1-\mu)^{-(n-m)/2}(1+\mu)^{-(n+m)/2}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}], \tag{A1}$$

where

$$A^l_{m,n} = \frac{(-1)^{l-m}}{2^l(l-m)!}\sqrt{\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}}. \tag{A2}$$

These functions are normalized by $2/(2l+1)$.

The Legendre polynomials correspond to $m = n = 0$ and the associated Legendre functions to $m = 2$ or -2 and $n = 0$. All of these functions can be computed by using recurrence relations.

The set of coefficients β_l, γ_l, δ_l, and ϵ_l may be derived from the $P_{i,j}$ terms of the phase function according to

$$\beta_l = 2/(2l+1) \int_{-1}^{+1} P_{1,1}(\mu) P_l(\mu)\, d\mu, \tag{A3}$$

$$\delta_l = 2/(2l+1) \int_{-1}^{+1} P_{3,3}(\mu) P_l(\mu)\, d\mu, \tag{A4}$$

$$\gamma_l = 2/(2l+1) \int_{-1}^{+1} P_{1,2}(\mu) P_l^2(\mu)\, d\mu, \tag{A5}$$

$$\epsilon_l = 2/(2l+1) \int_{-1}^{+1} P_{3,4}(\mu) P_l^2(\mu)\, d\mu. \tag{A6}$$

The functions R_s^l and T_s^l used in Eq. (8) are given by

$$R_s^l(\mu) = [P_{s,2}^l(\mu) + P_{s,-2}^l(\mu)]/2, \tag{A7}$$

$$T_s^l(\mu) = [P_{s,2}^l(\mu) + P_{s,-2}^l(\mu)]/2. \tag{A8}$$