Math 102
Standard Logical Arguments, Fallacies, and 2-Column Proofs

Standard Logical Arguments:

Recall that we used the truth table method to show that any argument of the following form is valid:

Premise 1:	$p \to q$
Premise 2:	$\sim q$
Conclusion:	$\therefore \sim p$

Now that we know this form is valid, we can use it to demonstrate the validity of an argument without having to go through the truth table method again (we already did it once, which is enough).
The name for this valid argument form is: **The Law of Contraposition**. It gets its name from the fact that the reason behind the validity of the argument is the fact that whenever a conditional statement is true, the related contrapositive statement is also true.
Here is a short list of come common valid logical arguments. Each of them can be verified using the truth table method.

Law of Detachment
$p \to q$
p
$\therefore\ q$

Law of Contraposition
$p \to q$
$\sim q$
$\therefore \sim p$

Law of Syllogism
$p \to q$
$q \to r$
$\therefore\ p \to r$

Disjunctive Syllogism
$p \vee q$
$\sim p$
$\therefore\ q$

Simplification
$p \wedge q$
$\therefore\ p$

Addition
p
$\therefore\ p \vee q$

Logical Fallacies:

- It is vital to realize that not every argument is valid. The following are two common invalid arguments that it is important to be able to recognize and avoid. The truth table method can be used to verify that these are not valid arguments since the related truth tables do not have all T's in their final columns.
- Common invalid arguments are often called **Logical Fallacies** or just **Fallacies**.
- The names of these fallacies come from the fact that their reasoning is based an accidentally assuming that the converse and the inverse are logically equivalent to the related conditional statement (which is false).

The Fallacy of the Converse
$p \to q$
q
$\therefore\ p$

The Fallacy of the Inverse
$p \to q$
$\sim p$
$\therefore \sim q$

Let's practice translating arguments into symbolic form and checking their validity based on the form of the each argument compared to the argument forms listed above.

If my car runs out of gas then I will not make it home.
My car did not run out of gas.
Therefore, I made it home.

If my car runs out of gas then I will not make it home.
My car ran out of gas.
Therefore, I did not make it home.

We will use the following variables to rewrite these arguments in symbolic form: p "my car runs out of gas", q "I do not make it home".

Then the symbolic form of these arguments are:

The Fallacy of the Inverse
$p \to q$
$\sim p$
$\therefore \sim q$

The Law of Detachment
$p \to q$
p
$\therefore\ q$

From the form of these arguments, we conclude that the first argument is invalid, since it is the Fallacy of the Inverse while the second argument is valid, since it is the Law of Detachment.

Two Column Proofs:

As arguments become more complicated, it becomes impractical to use truth tables to check the validity of arguments and there are many arguments that do not exactly fit the form of the basic arguments listed above. For this reason, mathematicians often use formal proofs to verify arguments. Although there are many proof methods, we will limit out attention to a method called **direct proof**. A direct proof is a sequence of true statements such that:

- the statement is a premise that has been assumed to be true, *or*
- the statement is the conclusion of a valid argument form whose premises are statements that are already listed, *or*
- the statement is logically equivalent to one of the statements already listed.
- The last statement in the sequence must follow from previous statements and must be the conclusion statement that we were aiming to prove.

One way of presenting a direct proof is a clear and concise way is to write it as a 2-column proof. In a 2-column proof, the left column contains lists the sequence of *statements* in the proof while the right column gives the justification or *reason* behind that statement (the premise, logical equivalence, or argument it is based on).

Here is a list of logical equivalences that are often used in direct proofs:

Double Negation $p \equiv \sim (\sim p)$

DeMorgan's Laws $\sim (p \wedge q) \equiv \sim p \vee \sim q$
$\sim (p \vee q) \equiv \sim p \wedge \sim q$

Contraposition $p \to q \equiv \sim q \to \sim p$

Conditional to Disjunction $p \to q \equiv \sim p \vee q$

Commutative Laws $p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$

Let's look at some specific examples of using a 2-column proof to verify an argument.

1.
$p \vee r$
$r \to q$
$s \vee \sim q$
$\sim s$
$\therefore p$

Solution:

Statement	**Reason**
1. $s \vee \sim q$	Premise
2. $\sim s$	Premise
3. $\sim q$	1, 2, Disjunctive Syllogism
4. $r \to q$	Premise
5. $\sim r$	3, 4, Law of Contraposition
6. $p \vee r$	Premise
7. p	5, 6, Disjunctive Syllogism

2.
$p \to q$
$\sim q$
$p \vee s$
$\therefore s$

Solution:

Statement	**Reason**
1. $p \to q$	
2. $p \vee s$	
3. $\sim p \to s$	
4. $\sim q \to \sim p$	
5. $\sim q \to s$	
6. $\sim q$	
7. s	

3. Find a 2-column proof for the following argument:

$(q \vee r) \to p$
$\sim p$
$s \to r$
$\therefore \sim s$

Statement	**Reason**
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	